Dynamics of Countable-State Markov Models
Solved Problems

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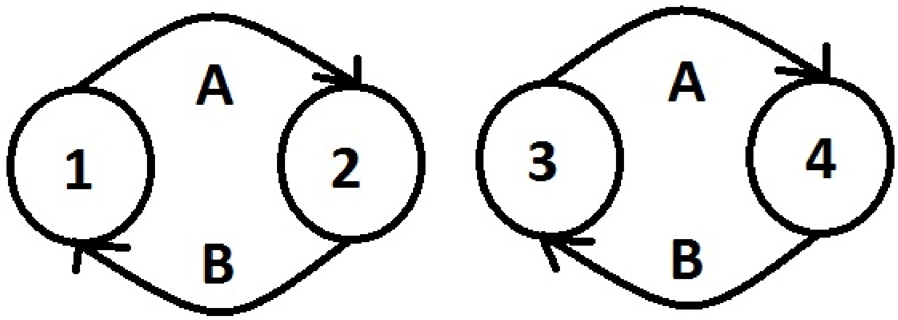
Example 1. Consider the continuous-time Markov process with the transition rate diagram



Solution. The Q matrix is the block-diagonal matrix given by

$$Q = \begin{pmatrix} -A & A & 0 & 0 \\ B & -B & 0 & 0 \\ 0 & 0 & -A & A \\ 0 & 0 & B & -B \end{pmatrix}.$$

This process is not irreducible, but rather the transition rate diagram can be decomposed into two parts. The equilibrium probability distributions are the probability distributions having the form

$$\pi = \left(\lambda \frac{B}{A+B}, \lambda \frac{A}{A+B}, (1-\lambda)\frac{B}{A+B}, (1-\lambda)\frac{A}{A+B}\right)$$

where λ is the probability placed on the subset $\{1, 2\}$.

Example 2. Consider a pipeline consisting of two single-buffer states in series. Model the system as a continuous-time Markov process. Suppose new packets are offered to the first stage according to a rate $L = \lambda$ Poisson process. A new packet is accepted at stage one if the buffer in stage one is empty at the time of arrival. Otherwise, the new packet is lost. If at a fixed time t, there is a packet in stage one and no packet in stage two, then the packet is transfered during $[t, t+h)$ to stage two with probability

$$hM1 + o(h).$$

Similarly, if at time t the second stage has a packet, then the packet leaves the system during $[t, t+h)$ with probability

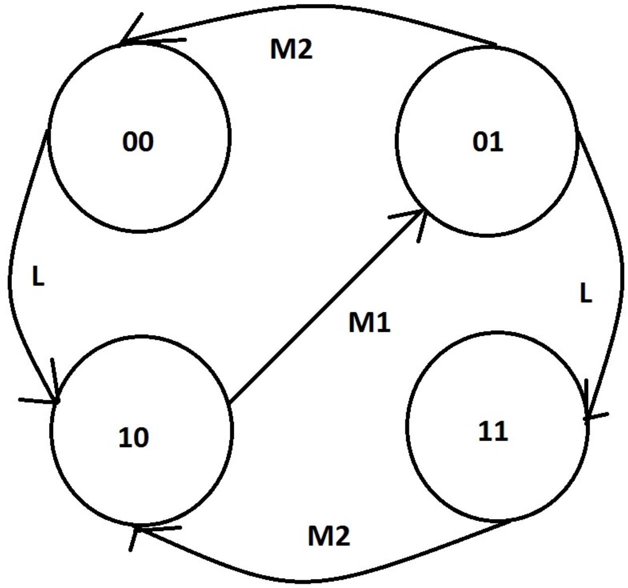
$$hM2 + o(h)$$

independently of the state of the stage one. Finally, the probability of two or more arrival, transfer, or departure events during $[t, t+h)$ is $o(h)$.

1. What is an appropriate state-space for this model?
2. Sketch a transition rate diagram.
3. Write down the Q matrix.
4. Derive the throughput, assuming that $L = M1 = M2 = 1$.

Solution.

$$\mathcal{S} = \{00, 01, 10, 11\}.$$



$$Q = \begin{pmatrix} -L & 0 & L & 0 \\ M2 & -M2-L & 0 & L \\ 0 & M1 & -M1 & 0 \\ 0 & 0 & M2 & -M2 \end{pmatrix}.$$

- $\eta = (\pi_{00} + \pi_{01})L = (\pi_{01} + \pi_{11})M2 = \pi_{10}M1$. If $L = M1 = M2 = 1$, then

  $$\pi = (.2, .2, .4, .2)$$

  and

  $$\eta = .4$$