Random Vectors and Minimum Mean Squared Error Estimation Solved Problems

Johar M. Ashfaque

Example 1. Let Θ be uniformly distributed on the interval $[0, \pi]$. Suppose $Y = \cos\Theta$ is to be estimated of the form $a + b\Theta$. What numerical values of a and b minimize the mean square error?
Solution.

$$\hat{E}[Y|\Theta] = E[Y] + \frac{Cov(\Theta, Y)}{Var\Theta}(\Theta - E[\Theta])$$

where

$$E[Y] = \frac{1}{\pi}\int_0^\pi \cos\theta d\theta = 0$$

$$E[\Theta] = \frac{\pi}{2}$$

$$Var(\Theta) = \frac{\pi^2}{12}$$

$$E[\Theta Y] = \int_0^\pi \frac{\theta\cos\theta}{\pi}d\theta = -\frac{2}{\pi}$$

and

$$Cov(\Theta, Y) = E[\Theta Y] - E[\Theta]E[Y] = -\frac{2}{\pi}.$$

Hence

$$\hat{E}[Y|\Theta] = -\frac{24}{\pi^3}\left(\Theta - \frac{\pi}{2}\right).$$

Therefore the optimal choice is

$$a = \frac{12}{\pi^2}$$

and

$$b = -\frac{24}{\pi^3}.$$

Example 2. For what real values of a and b is the following matrix the covariance matrix of some real-valued random vector?

$$K = \begin{pmatrix} 2 & 1 & b \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix}.$$

Solution. Set $a = 1$ to make K symmetric. Choose b so that the determinants of the following seven matrices are non-negative

$$\begin{pmatrix} 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & b \\ b & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & b \\ 1 & 1 & 0 \\ b & 0 & 1 \end{pmatrix}.$$

The fifth matrix has determinant $2 - b^2$ and

$$\det(K) = 2 - 1 - b^2 = 1 - b^2.$$

Hence K is a valid covariance matrix if and only if $a = 1$ and $-1 \leq b \leq 1$.

Example 3. Let X and Y be jointly Gaussian random variables with mean zero and covariance matrix

$$Cov \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 18 \end{pmatrix}.$$

Express the answers in terms fo Φ defined by

$$\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds.$$

1. Find $P\{|X-1| \geq 2\}$.
2. What is the conditional density of X given that $Y = 3$?
3. Find $P\{|X - E[X|Y] \geq 1\}$.

Solution. 1.

$$P\{|X-1| \geq 2\} = P\{X \leq 1 \text{ or } X \geq 3\} = P\{\frac{X}{2} \leq -\frac{1}{2}\} + P\{\frac{X}{2} \geq \frac{3}{2}\} = \Phi(-\frac{1}{2}) + 1 - \Phi(\frac{3}{2}).$$

2. Given $Y = 3$, the conditional density of X is Gaussian

$$E[X] + \frac{Cov(X,Y)}{Var(Y)}(3 - E[Y]) = 1$$

and

$$Var(X) - \frac{Cov(X,Y)^2}{Var(Y)} = 4 - \frac{6^2}{18} = 2.$$

3. The estimation error $X - E[X|Y]$ is Gaussian, has mean zero and variance 2 and is independent of Y. Thus the probability is

$$\Phi(-\frac{1}{\sqrt{2}}) + 1 - \Phi(\frac{1}{\sqrt{2}})$$

which can also be wrriten as

$$2\Phi(-\frac{1}{\sqrt{2}}).$$

Example 4. Let X and Y be square integrable random variables and let $Z = E[X|Y]$ so Z is the MMSE estimator of X given Y. Show that the LMSSE estimator of X given Y is also the LMSSE estimator of Z given Y.

Solution. To show that $\hat{E}[X|Y]$ is the LMMSE estimator of $E[X|Y]$, it suffices by the orthogonality principle to note that $\hat{E}[X|Y]$ is linear in $(1, Y)$ and to prove that

$$E[X|Y] - \hat{E}[X|Y]$$

is orthogonal to 1 and to Y. However

$$E[X|Y] - \hat{E}[X|Y]$$

can be written as the difference of two random variables

$$X - E[X|Y]$$

and

$$X - \hat{E}[X|Y]$$

which are each orthogonal to 1 and to Y. Thus

$$E[X|Y] - \hat{E}[X|Y]$$

is also orthogonal to 1 and Y.

Example 5. Let X and Y be jointly continuous random variables with the pdf

$$f_{XY}(x,y) = \begin{cases} x+y & 0 \le x, y \le 1 \\ 0 & \text{else} \end{cases}.$$

Find $E[X|Y]$ and $\hat{E}[X|Y]$.
Solution. To find $E[X|Y]$ we first identify $f_Y(y)$ and $f_{X|Y})(x|y)$:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \begin{cases} \frac{1}{2}+y & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}.$$

Therefore $f_{X|Y})(x|y)$ is defined only for $0 \le y \le$ and for such y is given by

$$f_{X|Y})(x|y) = \begin{cases} \frac{x+y}{\frac{1}{2}+y} & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}.$$

So for $0 \le y \le 1$

$$E[X|Y=y] = \int_0^1 x f_{X|Y})(x|y)dx = \frac{2+3y}{3+6y}.$$

To find $\hat{E}[X|Y]$ use $E[X] = E[Y] = \frac{7}{12}$, $Var(Y) = \frac{11}{144}$ and

$$Cov(X,Y) = -\frac{1}{144}$$

so that

$$\hat{E}[X|Y] = \frac{7}{12} - \frac{1}{11}\left(Y - \frac{7}{12}\right).$$