Basic Probability
Solved Problems

Johar M. Ashfaque

Example 1. Suppose there is an election with two candidates and six ballots turned in, such that four of the ballots are for the winning candidate and two of the ballots are for the other candidate. The ballots are opened and counted one at a time, in random order, with all orders equally likely. Find the probability that from the time the first ballot is counted until all the ballots are counted, the winning candidate has the majority of the ballots counted.
Solution. There are

$$\binom{6}{4} = 15$$

possibilities for the positions of the winning ballots and the event in question can be written as

$$\{110110, 110101, 111001, 111010, 111100\}$$

so the event has probability

$$\frac{5}{15} = \frac{1}{3}.$$

It can be shown in general that is k of the ballots are for the winning candidate and $n-k$ are for the losing candidate then the winning candidate has a strict majority throughout the counting with probability

$$\frac{2k-n}{n}.$$

This remains true even if the cyclic order of the ballots counted is fixed with only the identity of the first ballot counted being random and uniform over the n possibilities.4

Example 2.
1. Suppose that an event E is independent of itself. Show that either $P(E) = 0$ or $P(E) = 1$.
2. Events A and B and probabilities $P(A) = 0.3$ and $P(B) = 0.4$. What is $P(A \cup B)$ if A and B are independent? What is $P(A \cup B)$ if A and B are mutually exclusive?
3. Now suppose that $P(A) = 0.6$ and $P(B) = 0.8$. In this case, could the events A and B be independent? Could they be mutually exclusive?

Solution.
1. If E is an event independent of itself, then $P(E) = P(E \cap E) = P(E)P(E)$. This can happen if $P(E) = 0$. If $P(E) \neq 0$ then canceling the factor of $P(E)$ on either side yields $P(E) = 1$. In summary, either $P(E) = 0$ or $P(E) = 1$.
2. In general, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. On the one hand, if A and B are independent then

   $$P(A \cup B) = 0.3 + 0.4 - (0.3)(0.4) = 0.58.$$

   On the other hand, if A and B are mutually exclusive then

   $$P(A \cup B) = 0.3 + 0.4 = 0.7.$$

3. If $P(A) = 0.6$ and $P(B) = 0.8$ then the two events can be independent. However, if A and B were mutually exclusive then

   $$P(A) + P(B) = P(A \cup B) \leq 1$$

   so it would not be possible for A and B to be mutually exclusive if $P(A) = 0.6$ and $P(B) = 0.8$.

Example 3. At the end of the each day Professor Plum puts her glasses in her drawer with probability .90, leaves them on the table with probability .06, leaves them in her briefcase with probability .03 and she actually leaves them at the office with probability 0.01. The next morning she has no recollection of where she left the glasses. She looks for them, but each time she looks in a place the glasses are actually located, she misses finding them with probability 0.1, whether or not she already looked in the same place.

1. Given that Professor Plum did not find her glasses in her drawer after looking one time, what is the conditional probability that the glasses are on the table?
2. Given that she did not find the glasses after looking for them in the drawer and on the table once each, what is the conditional probability that they are in the briefcase?
3. Given that she failed to find the glasses after looking in the drawer twice, on the table twice, and in the briefcase once, what is the conditional probability that she left the glasses in the office?

Solution. Let D, T, B and O denote the events that the glasses are in the drawer, on the table, in the briefcase, or in the office respectively. These four events partition the probability space.

1. Let E denote the event that the glasses were not found in the first drawer search

$$P(T|E) = \frac{P(T \cap E)}{P(E)} = \frac{P(E|T)P(T)}{P(E|D)P(D) + P(E|D^c)P(D^c)} = \frac{(1)(0.06)}{(0.1)(0.9) + (1)(0.1)} = \frac{0.06}{0.19} \approx 0.315$$

where $P(D^c) = 1 - P(D) = 1 - 0.9 = 0.1$.

2. Let F denote the event that the glasses were not found after first drawer search and first table search

$$\begin{aligned} P(B|F) &= \frac{P(B \cap F)}{P(F)} \\ &= \frac{P(F|B)P(B)}{P(F|D)P(D) + P(F|T)P(T) + P(F|B)P(B) + P(F|O)P(O)} \\ &= \frac{(1)(0.03)}{(0.1)(0.9) + (0.1)(0.06) + (1)(0.03) + (1)(0.01)} \\ &\approx 0.22 \end{aligned}$$

3. Let G denote the event that the glasses were not found after the two drawer searches, two table searches and one briefcase search

$$\begin{aligned} P(O|G) &= \frac{P(O \cap G)}{P(G)} \\ &= \frac{P(G|O)P(O)}{P(G|D)P(D) + P(G|T)P(T) + P(G|B)P(B) + P(G|O)P(O)} \\ &= \frac{(1)(0.01)}{(0.1)^2(0.9) + (0.1)^2(0.06) + (0.1)(0.03) + (1)(0.01)} \\ &\approx 0.4225 \end{aligned}$$

Example 4. Suppose each corner of a cube is colored blue, independently of the other corners with some probability p. Let B denote the event that at least one face of the cube has all four corners colored blue.

1. Find the conditional probability of B given exactly five corners of the cube are colored blue.
2. Find $P(B)$, the unconditional probability of B.4

Solution. 1. There are 24 ways to color five corners of the cube so that at least one face has four clue corners. Since there are

$$\binom{8}{5} = 56$$

ways to select five out of eight corners

$$P(B|\text{exactly five corners colored blue}) = \frac{24}{56} = \frac{3}{7}.$$

2. By counting the number of ways that B can happen for different numbers of blue corners we find

$$P(B) = 6p^4(1-p)^4 + 24p^5(1-p)^3 + 24p^6(1-p)^2 + 8p^7(1-p) + p^8.$$

Example 5. Suppose X, Y and Z are random variables, each with mean zero and variance 20, such that $Cov(X, Y) = Cov(X, Z) = 10$ and $Cov(Y, Z) = 5$.

1. Find $Cov(X + Y, X - Y)$.
2. Find $Cov(3X + Z, 3X + Y)$.
3. Find $E[(X + Y)^2]$.

Solution. 1.

$$Cov(X+Y, X-Y) = Cov(X, X) - Cov(X, Y) + Cov(Y, X) - Cov(Y, Y) = Var(X) - Var(Y) = 0.$$

2.

$$Cov(3X+Z,3X+Y) = 9Var(X)+3Cov(X,Y)+3Cov(Z,X)+Cov(Z,Y) = 9\times20+3\times10+3\times10+5 = 245.$$

3. Since $E[X+Y]=0$

$$E[(X+Y)^2] = Var(X+Y) = Var(X) + 2Cov(X,Y) + Var(Y) = 20 + 2\times10 + 20 = 60.$$