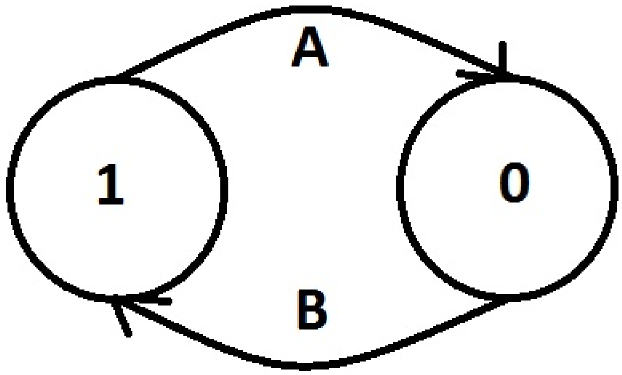
Random Processes
Solved Problems

Johar M. Ashfaque

Example 1. Consider the two-state, continuous-time Markov process with transition rate diagram



for some positive constants A and B. The generator matrix is given by

$$Q = \begin{pmatrix} -A & A \\ B & -B \end{pmatrix}.$$

Solve the forward Kolmogorov equation for a given initial distribution $\pi(0)$.
Solution. The equation for $\pi_1(t)$ is

$$\frac{\partial \pi_1(t)}{\partial t} = -A\pi_1(t) + B\pi_2(t); \quad \pi_1(0) \text{ given.}$$

But $\pi_1(t) = 1 - \pi_2(t)$ so

$$\frac{\partial \pi_1(t)}{\partial t} = -(A+B)\pi_1(t) + B; \quad \pi_1(0) \text{ given.}$$

By differentiation we check that this equation has the solution

$$\pi_t(t) = \pi_1(0)e^{-(A+B)t} + \frac{B}{A+B}(1 - e^{-(A+B)t})$$

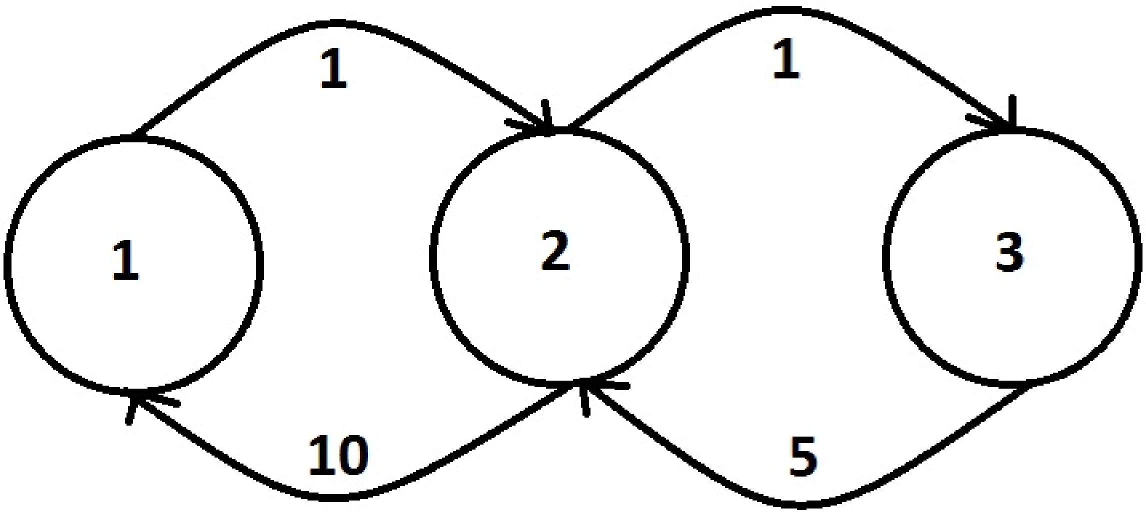
so that

$$\pi(t) = \pi(0)e^{-(A+B)t} + \left(\frac{B}{A+B}, \frac{A}{A+B}\right)(1 - e^{-(A+B)t}).$$

For any initial distribution $\pi(0)$

$$\lim_{t\to\infty} \pi(t) = \left(\frac{B}{A+B}, \frac{A}{A+B}\right).$$

The rate of convergence is exponential, with rate parameter $A + B$ and the limiting distribution is the unique probability distribution satisfying $\pi Q = 0$.
Example 2. Let $(X_t : t \geq 0)$ be a time-homogeneous Markov process with the transition rate diagram



1. Write down the rate matrix Q.
2. Find the equilibrium probability distribution pi.
3. Let $\tau = \min\{t \geq 0 : X_t = 3\}$ and let $a_i = E[\tau | X_0 = i]$ for $1 \leq i \leq 3$. Clearly, $a_3 = 0$. Derive equations for a_1 and a_2 by considering the possible values of $X_t(h)$ for small values of $h > 0$ and taking the limit as $h \to 0$. Solve the equations to find a_1 and a_2.

Solution. 1.

$$Q = \begin{pmatrix} -1 & 1 & 0 \\ 10 & -11 & 1 \\ 0 & 5 & -5 \end{pmatrix}.$$

2.

$$\pi = \left(\frac{50}{56}, \frac{5}{56}, \frac{1}{56} \right).$$

3. Consider X_h to get

$$a_1 = h + (1-h)a_1 + ha_2 + \mathrm{o}(h)$$

$$a_2 = h + 10a_1 + (1-11h)a_2 + \mathrm{o}(h)$$

Letting $h \to 0$, gives

$$1 - a_1 + a_2 = 0$$

and

$$1 + 10a_1 - 11a_2 = 0$$

and therefore

$$a_1 = 12, \qquad a_2 = 11.$$