Complex manifolds

Definition

Let M be a $2m$-dimensional topological manifold. A coordinate atlas $\{(U, \phi_U : U \to \mathbb{C}^m)\}$ is called *holomorphic* if the transition functions $\phi_U \circ \phi_V^{-1}$ are holomorphic functions between subsets of $\mathbb{C}^m$; in this case the coordinate charts ϕ_U are called *local holomorphic coordinates*. The manifold M is called *complex* if it admits a holomorphic atlas.
Two holomorphic atlases are called *equivalent* if their union is a holomorphic atlas. An equivalence class of holomorphic atlases on M is called a *complex structure*.

Remark: Obviously, a complex manifold of dimension m is a smooth (real) manifold of dimension $2m$. We will denote the underlying real manifold by $M_{\mathbb{R}}$.

Example

Complex projective space $\mathbb{P}^m = \mathbb{C}P^m$, the set of (complex) lines in $\mathbb{C}^{m+1}$, i.e. the set of equivalence classes of the relation

$$(z_0, \dots, z_m) \sim (\alpha z_0, \dots, \alpha z_m), \quad \forall \alpha \in \mathbb{C}^* (= \mathbb{C} \setminus \{0\})$$

on $\mathbb{C}^{m+1} \setminus \{0\}$. In other words $\mathbb{C}P^m = (\mathbb{C}^{m+1} \setminus \{0\}) / \sim$.
We denote the equivalence class of $(z_0, \dots, z_m)$ by $[z_0, \dots, z_m]$ and call $(z_0, \dots, z_m)$ *homogeneous coordinates*.
The complex charts are defined as for real projective space $\mathbb{R}P^m$:

$$U_j = \{[z_0, \dots, z_m] : z_j \neq 0\}, \quad j = 0, \dots, m$$

$$\phi_j : U_j \to \mathbb{C}^m, \qquad \phi_j([z_0, \dots, z_m]) = \left(\frac{z_0}{z_j}, \dots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \dots, \frac{z_m}{z_j} \right).$$

Example

1. *Complex Grassmanian* $\mathrm{Gr}_p(\mathbb{C}^m)$; this is the set of all p-dimensional vector subspaces of $\mathbb{C}^m$. Note $\mathrm{Gr}_1(\mathbb{C}^{m+1}) = \mathbb{C}P^m$.
2. The *torus* $T^2 := \mathbb{R}^2/\mathbb{Z}^2$ is a complex manifold of dimension 1.
3. *Level sets of submersions* $f : \mathbb{C}^{m+1} \to \mathbb{C}$. If f is holomorphic and its differential df does not vanish at any point of $f^{-1}(c)$, then $f^{-1}(c)$ is a complex submanifold.For example *Fermat hypersurfaces*:

$$\left\{(z_0, \dots, z_m) \in \mathbb{C}^m : \sum_{j=0}^{m} z_j^{d_j} = 1\right\}, \quad d_0, \dots, d_m \in \mathbb{N}.$$

4. Similarly, *homogeneous* polynomials f on $\mathbb{C}^{m+1}$ give complex submanifolds of $\mathbb{C}P^m$.
5. *Complex Lie groups*: $GL(n,\mathbb{C})$, $O(n,\mathbb{C})$, etc.; all open subsets of the space of all $n \times n$ matrices: $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$.

Some linear algebra

Definition

Let $V_\mathbb{R}$ be a real vector space. A *complex structure* on $V_\mathbb{R}$ is a linear map $J : V_\mathbb{R} \longrightarrow V_\mathbb{R}$ satisfying $J^2 = -\mathrm{Id}$

Let V be any m-dimensional complex vector space and let $V_\mathbb{R}$ be the underlying $2m$-dimensional vector space. Then the $\mathbb{R}$-linear map defined by $Jv = \mathrm{i}v$ is an almost complex structure.

Conversely, given any $2m$-dimensional real vector space $V_\mathbb{R}$ with a complex structure J, let $V_\mathbb{C} := \mathbb{C} \otimes V_\mathbb{R}$ be the associated $2m$-dimensional complex vector space. The complex structure extends to a $\mathbb{C}$-linear map $J : V_\mathbb{C} \to V_\mathbb{C}$. Let $V^{1,0}$ and $V^{0,1}$ denote the eigenspaces of J with eigenvalues i and $-\mathrm{i}$. There is a canonical isomorphism $V_\mathbb{R} \to V^{1,0}$, $X \mapsto X - \mathrm{i}JX$ with inverse $Z \mapsto (Z + \bar{Z})/2$. This endows $V_\mathbb{R}$ with the structure of a complex vector space.

Remark: vector spaces that carry almost complex structures are necessarily even-dimensional!

Almost complex manifolds

Definition

An *almost complex structure* on a smooth real manifold M is an endomorphism J of the tangent bundle such that $J^2 = -1$ (i.e. J is a family of complex structures J_x on $T_x M$ that depend smoothly on x). The pair (M, J) is called an *almost complex manifold* in this case.

Remark: Obviously, an almost complex manifold has even dimension, but not every even-dimensional smooth manifold admits an almost complex structure, e.g. S^4 does not [Borel & Serre, 1951] showed that the only spheres admitting an almost complex structure are S^2 (=$\mathbb{C}P^1$) and S^6.

Example

Any complex manifold carries a canonical almost complex structure, defined as follows. Let $(z^1, \ldots, z^m) : U \to \mathbb{C}^m$ be a holomorphic chart on a complex manifold and let $(x^1, y^1, \ldots, x^m, y^m)$ be real coordinates such that $z^j = x^j + \mathrm{i}y^j$ for $j = 1, \ldots, m$. We define

$$J(\partial/\partial x^j) = \partial/\partial y^j, \quad J(\partial/\partial y^j) = -\partial/\partial x^j \quad \forall j = 1, \ldots, m.$$

We will show later that this definition is independent of the choice of holomorphic coordinates.

Remark: It is not true however that every almost complex structure is obtained from a complex structure. S^6 admits an almost complex structure, but it is still an open problem whether it can also be made into a complex manifold.

The holomorphic tangent bundle

Definition

Let (M, J) be an almost complex manifold. The *complexified tangent bundle* is $T_{\mathbb{C}}M := \mathbb{C} \otimes_{\mathbb{R}} TM$. The *holomorphic* (*antiholomorphic*) tangent bundle of M is the eigenbundle $T^{1,0}M$ ($T^{0,1}M$) of J in $T_{\mathbb{C}}M$ with eigenvalue i ($-\mathrm{i}$).
The *holomorphic cotangent bundle* (*anti-holomorphic cotangent bundle*) is the subbundle $\Lambda^{1,0}M$ ($\Lambda^{0,1}M$) of $\Lambda^1_{\mathbb{C}}M$ consisting of all cotangent vectors u such that $u(X) = 0$ $\forall X \in T^{0,1}M$ ($\forall X \in T^{1,0}M$).
Finally, $\Lambda^{p,q}M := \Lambda^p(\Lambda^{1,0}M) \wedge \Lambda^q(\Lambda^{0,1}M)$. Sections of $\Lambda^{p,q}M$ are called (p, q)*-forms*, and the space of all such forms is denoted by $\Omega^{p,q}M$.

There are canonical splittings:

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M, \quad \Lambda^r_{\mathbb{C}}M = \bigoplus_{p+q=r} \Lambda^{p,q}M.$$

Example

On a complex manifold, in terms of local holomorphic coordinates z^j near a point $p \in M$,

$$T^{1,0}_p M = \mathrm{span}\left\{\frac{\partial}{\partial z_j}\right\}, \quad T^{0,1}_p M = \mathrm{span}\left\{\frac{\partial}{\partial \bar{z}_j}\right\}$$

If u^j are another set of holomorphic coordinates then

$$\frac{\partial}{\partial u^j} = \frac{\partial z^k}{\partial u^j}\frac{\partial}{\partial z^k},$$

because $\partial \bar{z}^k / \partial u^j = 0$. A similar statement holds for $\partial / \partial \bar{u}^j$. It follows that J is independent of the choice of complex coordinates, since the splitting $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ determines J.

A section of $T^{1,0}M$ can be written $X = X^j \partial / \partial z^j$. It follows that if both X and Y are sections of $T^{1,0}M$, then so is their Lie bracket $[X, Y]$.

The Newlander-Nirenberg Theorem

Theorem (Newlander-Nirenberg)

Let (M, J) be an almost complex manifold. The almost complex structure J comes from a complex structure if and only if

$$X, Y \in \Gamma(T^{1,0}M) \implies [X, Y] \in \Gamma(T^{1,0}M).$$

Remark: An equivalent condition is the vanishing of the *Nijenhuis tensor*:

$$N(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y].$$

As for the proof, we have just have seen the ”only if” part.
The “if” part is very hard. See Kobayashi & Nomizu for a proof under an additional assumption that M and J are real-analytic, and Hörmander’s “Introduction to Complex Analysis in Several Variables” for a proof in full generality.

The dual picture

Example

In local holomorphic coordinates z^j on a complex manifold M, (p, q)-forms u may be written as follows:

$$u = \frac{1}{p!q!} u_{j_1 \ldots j_p \bar{k}_1 \ldots \bar{k}_q}(z, \bar{z}) \, dz^{j_1} \wedge \ldots \wedge dz^{j_p} \wedge d\bar{z}^{k_1} \wedge \ldots \wedge d\bar{z}^{k_q}$$

(The coefficients $u_{j_1 \ldots j_p \bar{k}_1 \ldots \bar{k}_j}$ can be chosen anti-symmetric in the j’s and k’s.) A direct calculation shows that

$$u \in \Omega^{p,q}M \Rightarrow du \in \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M.$$

This property characterises complex manifolds…

Theorem

Let (M,J) be an almost complex manifold of dimension $2n$. The following conditions are equivalent:

(i) *J is a complex structure.*

(ii) $d\Omega^{1,0}M \subset \Omega^{2,0}M \oplus \Omega^{1,1}M$.

(iii) $d\Omega^{p,q}M \subset \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M$ *for all* $0 \leq p,q \leq m$.

Proof.

The only non-trivial bit is (ii) $\Longrightarrow$ (i). Use the following elementary formula for the exterior derivative of a 1-form:

$$2d\omega(Z,W) = Z\big(\omega(W)\big) - W\big(\omega(Z)\big) - \omega([Z,W]).$$

$\square$

Holomorphic maps

Definition

A smooth map $f : M_1 \to M_2$ between two complex manifolds is called *holomorphic* if $\psi_V \circ f \circ \phi_U^{-1}$ is a holomorphic map between open subsets in $\mathbb{C}^n$, for any charts (U,ϕ_U) in M_1 and (V,ψ_V) in M_2. A holomorphic bijection $f : M_1 \longrightarrow M_2$ with holomorphic inverse is called a *biholomorphism*

Definition

A smooth map $f : (M_1,J_1) \longrightarrow (M_2,J_2)$ between two almost complex manifolds is called *J-holomorphic* if the differential of f commutes with the almost complex structures, i.e. $f_* \circ J_1 = J_2 \circ f_*$ as maps from T_pM_1 to $T_{f(p)}M_1$.

It is left as an exercise to show that every holomorphic map is J-holomorphic.

Summary 1

• An *almost complex manifold* is a pair (M,J), where M is a smooth real manifold and $J : TM \to TM$ has $J^2 = -Id$. $T^{1,0}M$, $T^{0,1}M$ denote the $\pm\mathrm{i}$-eigenspaces of J in $T^{\mathbb{C}}M$. $T^{1,0}M = \{X - iJX : X \in TM\}$.
• If M is a *complex manifold* with local coordinates $(z_1,\dots,z_n)$, then $T^{1,0}M$ is spanned by $\partial/\partial z_1,\dots,\partial/\partial z_n$; $T^{1,0}M$ is called the $(1,0)$ or *holomorphic tangent bundle*.
• A complex manifold is always an almost complex manifold in a natural way. Conversely, an almost complex manifold (M,J) is a complex manifold (i.e. complex coordinates exist) iff $[\Gamma(T^{0,1}M),\Gamma(T^{0,1}M)] \subset \Gamma(T^{0,1}M)$ (Newlander–Nirenberg Theorem).
• We can similarly decompose $T^*M \otimes \mathbb{C}$ into $T^{1,0}M$ and $T^{0,1}M$, and the whole exterior algebra into a direct sum: $\Lambda M \otimes \mathbb{C} = \bigoplus \Lambda^{p,q}M$.
• On a complex manifold, $\Lambda^{p,q}M$ is spanned by

$$dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.$$

• Sections of $\Lambda^{p,q}M$ are called (p,q)*-forms*. (M,J) is a complex manifold iff $d\Omega^{1,0}M \subset \Omega^{2,0}M \oplus \Omega^{1,1}M$ equivalently, $d\Omega^{p,q}M \subset \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M$ for all $0 \le p,q \le m$.

The Dolbeault operator

Recall that, on a complex manifold, $\mathrm{d}(\Omega^{p,q}M) \subset \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M$. We may therefore write $\mathrm{d} = \partial + \bar\partial$, where $\partial : \Omega^{p,q}M \to \Omega^{p+1,q}M$ and $\bar\partial : \Omega^{p,q}M \to \Omega^{p,q+1}M$.

Lemma

$$\partial^2 = 0, \quad ,\bar\partial^2 = 0, \quad \partial\bar\partial + \bar\partial\partial = 0.$$

Proof.

$0 = d^2 = (\partial + \bar\partial)^2 = \partial^2 + \bar\partial^2 + (\partial\bar\partial + \bar\partial\partial)$ and the three operators in the last term take values in different subbundles of $\Lambda^*M \otimes \mathbb{C}$. □

Definition

The operator $\bar\partial : \Omega^{p,q}M \to \Omega^{p,q+1}M$ is called the *Dolbeault operator*. A p-form ω of type $(p,0)$ is called *holomorphic* if $\bar\partial\omega = 0$.

Dolbeault cohomology

$\bar{\partial}$ defines *Dolbeault cohomology groups* of a complex manifold, analogous to de Rham cohomology:

$$Z_{\bar{\partial}}^{p,q}(M) := \{\omega \in \Omega^{p,q}(M) : \bar{\partial}\omega = 0\} \qquad \text{— the } \bar{\partial}\text{-closed forms.}$$
$$H_{\bar{\partial}}^{p,q}(M) := Z_{\bar{\partial}}^{p,q}(M)/\bar{\partial}\Omega^{p,q-1}(M).$$

Warning: Dolbeault cohomology is not a topological invariant: it depends on the complex structure.
A holomorphic map $f : M \to N$ between complex manifolds induces a map $f^* : H_{\bar{\partial}}^{p,q}(N) \to H_{\bar{\partial}}^{p,q}(M)$.

Lemma (Dolbeault Lemma)

For B a ball in $\mathbb{C}^n$, $H_{\bar{\partial}}^{p,q}(B) = 0$ for $q > 0$.

For a proof, see Griffiths and Harris, p. 25.
This lemma implies that $H_{\bar{\partial}}^{p,q}(\mathbb{C}^m) = 0$ for $q > 1$. Note however that $H_{\bar{\partial}}^{p,0}(\mathbb{C}^m)$ has infinite dimension!

Example ($\mathbb{CP}^1$)

We claim that $H_{\bar{\partial}}^{1,0}(\mathbb{CP}^1) \cong 0$ and that $H_{\bar{\partial}}^{0,1}(\mathbb{CP}^1) \cong 0$.
Consider first $H_{\bar{\partial}}^{1,0}$. Let w_0 and w_1 be the coordinates on the two patches U_0, U_1, so that $w_1 = 1/w_0$. Any $(1,0)$-form ϕ can be written $\phi = \phi_0(w_0)\,\mathrm{d}w_0$ on U_0, and $\phi = \phi_1(w_1)\,\mathrm{d}w_0$ on U_1. Since $\mathrm{d}w_1 = -\mathrm{d}w_0/w_0^2$, $\phi_0 = -\phi_1/w_0^2$ on $U_0 \cap U_1$. This implies that $\phi_0 \to 0$ as $w_0 \to \infty$, and that ϕ_0 is a bounded function on $\mathbb{C}$. $\bar{\partial}\phi = 0$ implies that ϕ_0 and ϕ_1 are holomorphic functions. Therefore by Liouville's theorem, ϕ_0 is constant. Since $\phi_0 \to 0$ at ∞, $\phi_0 = 0$. Hence $\phi = 0$.
Now consider $H_{\bar{\partial}}^{0,1}$. Let ψ be any (0,1)-form. By the Dolbeault lemma, there are functions $f_i : U_i \to \mathbb{C}$ such that $\psi = \bar{\partial}f_i$ on U_i. On $U_0 \cap U_1$, $f_1 - f_0$ is an anti-holomorphic function, so write

$$f_1 - f_0 = \sum_{i=-\infty}^{\infty} a_i w_0^i.$$

Then $g = f_0 + \sum_{i=1}^{\infty} a_i w_0^i = f_1 - \sum_{i=0}^{\infty} a_{-i} w_1^i$ is a function on $\mathbb{CP}^m$ such that $\bar{\partial}g = \psi$.

First example of a complex vector bundle

Example

Let M be a smooth n-dimensional manifold and let V be a complex k-dimensional vector space. The *trivial vector bundle* over M of rank k is the pair (E,π), where $E = M \times V$ and $\pi : E \to M$ is the projection.

Roughly speaking, a vector bundle is something that looks like this example locally...

Complex vector bundles

Definition

Let M be a smooth n-dimensional manifold. A smooth *complex vector bundle* of rank k on M consists of a $2k+n$-dimensional manifold E and a surjective submersion $\pi : E \to M$ such that

- for each $x \in M$, $E_x := \pi^{-1}(x)$ has the structure of a complex k-dimensional vector space;
- for each $x \in M$ there exists an open set $U \ni x$ and a diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k$ such that

  $$pr_U \circ \phi = \pi \quad \text{(where } pr_U : U \times \mathbb{C}^k \to U \text{ is the projection)}$$

  and $\phi_U|_{E_y}$ is a vector space isomorphism for each $y \in U$.

The maps ϕ_U are called *trivialisations* of E over U. The vector spaces E_x are called the *fibres* of E.

Example (The tangent bundle)

If M is any smooth manifold, the complexified tangent bundle $T_{\mathbb{C}}M$ is an example of a complex vector bundle. Given local coordinates $U \subset M$, $x : U \to \mathbb{R}^n$, a local trivialisation is given by

$$\phi_U : (p, X) \to \left(p, (X^1, X^2, \ldots, X^n)\right), \quad \text{where } p \in U,\ X = X^i \frac{\partial}{\partial x^i} \in T_pM.$$

If M is an almost complex manifold, the holomorphic and anti-holomorphic tangent bundles $T^{1,0}M$ and $T^{0,1}M$ are also examples of complex vector bundles. In fact, they are "sub-bundles" of $T_{\mathbb{C}}M$...

Definition

Let (E, π_E) be a complex vector bundle over a smooth Riemannian manifold M. A sub-bundle of E is a vector bundle of the form (F, π_F), where $F \subset E$ is a submanifold, $\pi_F = \pi_E|_F$, and for each $p \in M$, F_p is a linear subspace of E_p.

Remark: We often omit π and simply write a vector bundle as $E \to M$.

Transition functions

Given any pair ϕ_U, ϕ_V of trivialisations, their *transition function* is the smooth map $g_{UV} : U \cap V \to GL(k, \mathbb{C})$ defined by

$$\phi_U \circ \phi_V^{-1}(x, v) = (x, g_{UV}(x)v) \quad \forall x \in U \cap V, v \in \mathbb{C}^k.$$

They satisfy

$$g_{UV}(x)g_{VU}(x) = I, \quad g_{UV}(x)g_{VW}(x)g_{WU}(x) = I \quad (x \in U \cap V).$$

Conversely, given an open cover $\mathcal{U} = \{U_\alpha\}$ of M and C^∞-maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{C})$ satisfying these identities, there is a unique complex vector bundle $E \to M$ with transition functions $\{g_{\alpha\beta}\}$.

New vector bundles from old

If E, F are two vector bundles with transition functions g_{UV}, h_{UV}, one can form (using vector bundle operations):

- The *dual bundle* E^*, with transition functions $g_{UV}(x)^T$.
- The *direct sum* $E \oplus F$, with transition functions

$$\begin{pmatrix} g_{UV}(x) & 0 \\ 0 & h_{UV}(x) \end{pmatrix} \in GL(\mathbb{C}^k \oplus \mathbb{C}^l)$$

- The *tensor product* $E \otimes F$, with transition functions

$$g_{UV}(x) \otimes h_{UV}(x) \in GL(\mathbb{C}^k \otimes \mathbb{C}^l)$$

- The *determinant bundle* $\Lambda^k E$ (where k is the rank of E), with transition functions $\det g_{UV} \in \mathbb{C}^*$.
- The bundle $\mathrm{Hom}(E, F) := F \otimes E^*$.
- If F is a sub-bundle of E there is a *quotient bundle* E/F. The transition functions for F, E and E/F are of the form

$$h_{UV}, \quad g_{UV} = \begin{pmatrix} h_{UV} & k_{UV} \\ 0 & j_{UV} \end{pmatrix}, \quad \text{and } j_{UV}$$

Vector bundles and maps

Definition

A *homomorphism* between vector bundles E and F on M is given by a smooth map $f : E \to F$, such that for each $p \in M$ the restrictions $f_p := f|_{E_x} : E_x \to F_x$ are linear. We have the obvious notions of $\ker(f)$ — a subbundle of E, and $\mathrm{Im}(f)$ — a subbundle of F. Also, f is called an *isomorphism* if each f_p is an isomorphism.
A vector bundle E on M is called *trivial* if E is isomorphic to the product bundle $M \times \mathbb{C}^k$.

Definition

Given a C^∞-map $f : M \to N$ and a vector bundle $E \xrightarrow{\pi} N$, we define the *pullback bundle* f^*E on M by

$$f^*E = \{(x, e) \in M \times E : \ f(x) = \pi(e)\}; \text{ so } (f^*E)_x = E_{f(x)}.$$

Sections

Definition

A *section* of a vector bundle $E \longrightarrow M$ is a smooth map $s : M \longrightarrow E$ such that $\pi \circ s = id_M$. The space of sections is denoted by $\Gamma(E)$.

For example, a vector field is a section of the tangent bundle.

Definition

A *local frame* for a vector bundle $E \longrightarrow M$ consists of an open subset $U \subset M$ and a set $\{e_1, e_2, \ldots, e_k\}$ of sections of the vector bundle $\pi^{-1}(U)$ such that for each $p \in U$, $\{e_1(p), e_2(p), \ldots, e_k(p)\}$ are a basis for E_p.

Given a local frame, a section s can be represented by functions $s^i : U \longrightarrow \mathbb{C}$ with $i = 1, \ldots, k$, such that $s(p) = \sum_i s^i(p) e_i(p)$.
Exercise: Show that a local frame determines a local trivialisation, and conversely.

Holomorphic vector bundles

Definition

Let M be a complex m-dimensional manifold. A *holomorphic vector bundle* of rank k on M consists of a $k+m$-dimensional complex manifold E and a surjective submersion $\pi : E \longrightarrow M$ such that

- for each $x \in M$, $E_x := \pi^{-1}(x)$ has the structure of a complex k-dimensional vector space;
- for each $x \in M$ there exists an open set $U \ni x$ and a biholomorphism $\phi_U : \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^k$ such that

$$pr_U \circ \phi = \pi \quad (\text{where } pr_U : U \times \mathbb{C}^k \longrightarrow U \text{ is the projection})$$

  and $\phi_U|_{E_y}$ is a vector space isomorphism for each $y \in U$.

A complex vector bundle is holomorphic if and only if its transition functions $g_{UV} : U \cap V \longrightarrow GL(k, \mathbb{C})$ are holomorphic functions.

Example (Tangent and related bundles)

Examples include $T^{1,0}M$ and $\Lambda^{p,0}M$. However, $T^{0,1}M$ and $\Lambda^{p,q}M$ with $q > 0$ are NOT holomorphic vector bundles. The top exterior power $\Lambda^{m,0}M$ is called the *canonical bundle* over M.

Example (The tautological bundle on $\mathbb{C}P^n$)

The tautological line bundle $\pi : L \to \mathbb{C}P^n$ is defined by setting $L_A = A$, where A is a line in $\mathbb{C}^{n+1}$ defining a point of $\mathbb{P}^n = \mathbb{C}P^n$.
To show that this bundle is holomorphic, it suffices to show that its transition functions are holomorphic.
Recall that we covered $\mathbb{C}P^n$ wth $n+1$ open subsets $U_j = \{[z_0, \dots, z_n] : z_j \neq 0\}$. A trivialisation of L over U_j is given by

$$\phi_j : \pi^{-1}U \ni \big(\ell, (z_0, \dots, z_n)\big) \mapsto \big(\ell, z_j\big) \in U_j \times \mathbb{C},$$

hence $\phi_i\phi_j^{-1}([z], \lambda) = ([z], z_i z_j^{-1}\lambda)$, so the transition functions are $g_{ij}([z]) = z_i z_j^{-1}$. Since the transition functions are holomorphic, L is a holomorphic line bundle.

Everything that we defined for complex vector bundles carries over to the holomorphic setting, with "smooth" replaced by "holomorphic". Briefly:

- A *holomorphic subbundle* F of a holomorphic vector bundle E must be a complex submanifold of E.
- A *homomorphism* or *isomorphism* f between two holomorphic vector bundles F and E must be a holomorphic map.
- A *holomorphic section* of a holomorphic vector bundle $E \to M$ is a holomorphic function $s : M \to E$.
- The pull-back of a a holomorphic bundle $E \to N$ by $f : M \to N$ is holomorphic as long as f is holomorphic.
- One can define duals, direct sums, quotients, and tensor products of holomorphic bundles as for complex bundles.

Remark: a holomorphic bundle may be non-trivial as a holomorphic bundle and trivial as a complex vector bundle. Holomorphic triviality is a stronger condition than triviality!

Summary 2

• A *(complex) vector bundle* over a smooth manifold M is a smooth map $\pi : E \to M$, from a smooth manifold E with fibres $E_x = \pi^{-1}(x)$ ($x \in M$) which are vector spaces of some finite dimension k, such that π is *locally trivial*, i.e., there exists an open cover $\mathcal{U}$ of M such that, for every $U \in \mathcal{U}$, we have a diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k$ and these diffeomorphisms satisfy $\phi_U \circ \phi_V^{-1}(x, v) = (x, g_{UV}(x)v)$, where $g_{UV} : U \cap V \to GL(k, \mathbb{C})$ is a smooth map.
• E is a *holomorphic* vector bundle if M is a complex manifold and g_{UV} are holomorphic functions for all $U, V \in \mathcal{U}$; this implies that E is a complex manifold.
• A *section* of a vector bundle $E \to M$ is a smooth map $s : M \to E$ such that $s(x) \in E_x$ for all $x \in M$ (like a vector field). The *space of sections* is denoted by $\Gamma(E)$.
• A *frame* for E over U is a choice of a vector space basis $(s_1(x), \ldots, s_k(x))$ for E_x which varies smoothly with $x \in U$ (i.e., each $s_i : U \to E$ is a smooth section.
• If E is a holomorphic vector bundle, we can speak of *holomorphic sections* and *frames*.

Connections

Connections are a way to differentiate sections.

Notation: given a complex vector bundle $E \to M$, $\Lambda^r E := \Lambda^r M \otimes E$ and $\Omega^r E := \Gamma(\Lambda^r E)$ (and similarly for $\Lambda^{p,q} E$ and $\Omega^{p,q} E$).

Definition

Let $E \to M$ be a complex vector bundle over a smooth manifold M. A *connection* on E is a $\mathbb{C}$-linear map $D : \Gamma(E) \to \Omega^1 E$ satisfying the Leibniz rule:

$$D(fs) = (df) \otimes s + f(Ds) \quad \forall f \in C^\infty(M),\ s \in \Gamma(E).$$

Let D be a connection on E and let X be a vector field on M. The *covariant derivative with respect to X* is the operator $D_X : \Gamma(E) \to \Gamma(E)$ defined by

$$D_X s = Ds(X) \quad \forall s \in \Gamma(E).$$

Connection matrix

Let us choose a local frame $e = (e_1, \ldots, e_k)$ for E over U. Then there is a matrix of 1-forms $\theta = (\theta_{ij})$ such that

$$De_i = \sum_j \theta_{ij} e_j \qquad (i = 1, \ldots, k)$$

θ is called the *connection matrix* (or *gauge potential*) with respect to the frame e.
For example $\Gamma^k_{ij} dx^i$ is the connection matrix for the *Levi-Civita connection* in the frame $\partial/\partial x^i$.
Since $D(fs) = df \otimes s + f \cdot Ds$, the data e and θ determine D.

The connection matrix depends on the choice of e: if $(e'_1, \ldots, e'_k)$ is another frame with $e'(x) = g(x)e(x)$ $(g(x) \in GL(k, \mathbb{C}))$, then

$$De' = dg\, e + g\, De = dg\, e + g\,\theta\, e = dg\, g^{-1} e' + g\,\theta\, g^{-1} e'.$$

So the connection matrix w.r.t. e' is

$$\theta' = g\theta g^{-1} + dg\, g^{-1}.$$

The matrix-valued function g is sometimes called a *gauge transformation*.

Example

Consider the trivial bundle $\mathbb{C}^2 \times \mathbb{R}^3$ of rank 2 over $\mathbb{R}^3$.
Its standard (global) frame is $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
Set $v = \partial/\partial x_1$ and

$$\theta = \begin{pmatrix} dx_1 & dx_2 \\ dx_2 & dx_3 \end{pmatrix}.$$

Then $De_1 = \theta_{11} e_1 + \theta_{12} e_2 = \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix}$ and $D_v e_1 = e_1$. Similarly, $D_v e_2 = 0$. Therefore, for a general section $s = \begin{pmatrix} s_1(x_1, x_2, x_3) \\ s_2(x_1, x_2, x_3) \end{pmatrix}$,

$$\begin{aligned} D_v(s) &= D_v(s_1 e_1 + s_2 e_2) \\ &= ds_1(v) e_1 + s_1 D_v e_1 + ds_2(v) e_2 + s_2 D_v e_2 \\ &= \frac{\partial s_1}{\partial x_1} e_1 + s_1 e_1 + \frac{\partial s_2}{\partial x_1} + 0 = \begin{pmatrix} \frac{\partial s_1}{\partial x_1} + s_1 \\ \frac{\partial s_2}{\partial x_1} \end{pmatrix}. \end{aligned}$$

Curvature

Any connection $D : \Gamma(E) \to \Omega^1 E$ can be extended to an operator $D : \Omega^p E \to \Omega^{p+1} E$ by imposing the Leibniz rule:

$$D(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^p \omega \wedge D\sigma, \ \ \forall \omega \in \Omega^p M, \sigma \in \Gamma(E).$$

Lemma

The operator $D^2 : \Gamma(E) \to \Omega^2 E$ is tensorial, i.e. $D^2(fs) = fD^2 s$ $\forall f \in C^\infty(M)$, $s \in \Gamma(E)$.

Proof.

$D^2(fs) = D(\mathrm{d}f \otimes s + fDs) = \mathrm{d}^2 f - \mathrm{d}f \otimes Ds + \mathrm{d}f \otimes Ds + fD^2 s = fD^2 s.$ □

Definition

The *curvature* of a connection D is the section $R^D \in \Omega^2 \operatorname{Hom}(E, E)$ such that $R^D(s) = D^2 s \ \forall s \in \Gamma(E)$.

Curvature matrix

Given a local frame $e = (e_1, \ldots, e_k)$, the curvature R^D is represented by a matrix-valued 2-form (Θ_{ij}) such that $R^D(s_i e_i) = s_i \Theta_{ij} e_j$ for all local sections $s = s_i e_i$.One can calculate Θ in terms of θ:

$$\begin{aligned} D^2 s_i e_i &= D(\mathrm{d}s_i e_i + s_i \theta_{ij} e_j) \\ &= -\mathrm{d}s_i \wedge \theta_{ij} s_j + \mathrm{d}s_i \wedge \theta_{ij} e_j + s_i \mathrm{d}\theta_{ij} e_j - s_i \theta) ij \wedge \theta_{jk} e_k \\ &= s_i (\mathrm{d}\theta_{ij} - \theta_{ik} \wedge \theta_{kj}) e_j. \end{aligned}$$

Hence $\Theta_{ij} = \mathrm{d}\theta_{ij} - \theta_{ik} \wedge \theta_{kj}$, or, more succinctly,

$$\Theta = \mathrm{d}\theta - \theta \wedge \theta.$$

Exercise: Show that if $e'(x) = g(x)e(x)$ is another frame then $\Theta' = g\Theta g^{-1}$.
Exercise: Prove the *Bianchi identity*: $\mathrm{d}\Theta + \theta \wedge \Theta - \Theta \wedge \theta = 0$.

Holomorphic vector bundles

Definition

Let M be a complex m-dimensional manifold. A *holomorphic vector bundle* of rank k on M consists of a $k+m$-dimensional complex manifold E and a **holomorphic** surjective submersion $\pi : E \longrightarrow M$ such that

- for each $x \in M$, $E_x := \pi^{-1}(x)$ has the structure of a complex k-dimensional vector space;
- for each $x \in M$ there exists an open set $U \ni x$ and a **biholomorphism** $\phi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{C}^k$ such that

$$pr_U \circ \phi = \pi \quad \text{(where } pr_U : U \times \mathbb{C}^k \longrightarrow U \text{ is the projection)}$$

  and $\phi_U|_{E_y}$ is a vector space isomorphism for each $y \in U$.

A complex vector bundle is holomorphic if and only if its transition functions $g_{UV} : U \cap V \rightarrow GL(k, \mathbb{C})$ are holomorphic functions.

Example (Tangent and related bundles)

Examples include $T^{1,0}M$ and $\Lambda^{p,0}M$. However, $T^{0,1}M$ and $\Lambda^{p,q}M$ with $q > 0$ are NOT holomorphic vector bundles (**Exercise:** why not?). The top exterior power $K := \Lambda^{m,0}M$ is called the *canonical bundle* over M.

Example (The tautological bundle on $\mathbb{C}P^n$)

The tautological line bundle $\pi : L \rightarrow \mathbb{C}P^n$ is defined by setting $L_A = A$, where A is a line in $\mathbb{C}^{n+1}$ defining a point of $\mathbb{P}^n = \mathbb{C}P^n$.
To show that this bundle is holomorphic, it suffices to show that its transition functions are holomorphic.
Recall that we covered $\mathbb{C}P^n$ with $n+1$ open subsets $U_j = \{[z_0, \ldots, z_n] : z_j \neq 0\}$. A trivialisation of L over U_j is given by

$$\phi_j : \pi^{-1}U \ni \big([z_0, \ldots, z_n], \lambda(z_0, \ldots, z_n)\big) \mapsto \big([z_0, \ldots, z_n], \lambda z_j\big) \in U_j \times \mathbb{C}.$$

Hence $\phi_i \phi_j^{-1}([z], p) = ([z], z_i z_j^{-1} p)$, so the transition functions are $g_{ij}([z]) = z_i z_j^{-1}$. Since the transition functions are holomorphic, L is a holomorphic line bundle.

Everything that we defined for complex vector bundles carries over to the holomorphic setting, with “smooth” replaced by “holomorphic”. Briefly:

- A *holomorphic subbundle* F of a holomorphic vector bundle E must be a complex submanifold of E.
- A *homomorphism* or *isomorphism* f between two holomorphic vector bundles F and E must be a holomorphic map.
- A *holomorphic section* of a holomorphic vector bundle $E \to M$ is a holomorphic function $s : M \to E$.
- The pull-back of a a holomorphic bundle $E \to N$ by $f : M \to N$ is holomorphic as long as f is holomorphic.
- One can define duals, direct sums, quotients, and tensor products of holomorphic bundles as for complex bundles.

Remark: a holomorphic bundle may be non-trivial as a holomorphic bundle and trivial as a complex vector bundle. Holomorphic triviality is a stronger condition than triviality!

The $\bar\partial$ operator

Lemma

*Let $E \to M$ be a **holomorphic** vector bundle over a complex manifold. There is a unique operator $\bar\partial : \Omega^{p,q}E \to \Omega^{p,q+1}E$ such that $\bar\partial s = 0$ for any holomorphic section $s \in \Gamma(E)$ and which obeys the Leibniz rule:*

$$\bar\partial(\omega \wedge s) = \bar\partial\omega \wedge s + (-1)^{p+q}\omega \wedge \bar\partial s \quad \forall \omega \in \Omega^{p,q}M,\ s \in \Omega^{r,s}E.$$

Proof.

Let e_i be a local holomorphic frame for E, and write a section ω as $\omega = \omega_i e_i$ with ω_i locally-defined (p, q)-forms. The properties of $\bar\partial$ stated in the lemma imply that

$$\bar\partial\omega = \bar\partial\omega_i \wedge e_i.$$

It is straightforward to check that this is independent of the choice of holomorphic frame e. □

Remark: Note that $\bar\partial^2 = 0$.

Summary 3

• A *connection* on a complex vector bundle $E \to M$ is an operator $D : \Gamma(E) \to \Omega^1(E)$ which satisfies the Leibniz rule $D(fs) = (df) \otimes s + f \cdot Ds$ for $f \in C^\infty(M)$, $s \in \Gamma(E)$.
• In a local frame $e = (e_j)$ for E, $De_i = \sum \theta_{ij} e_j$. The matrix θ of 1-forms is called the *connection matrix (w.r.t. the frame e)*.
• The *curvature* of a connection is the section $R^D \in \Omega^2 \operatorname{Hom}(E, E)$ such that $D^2 = R^D$. In a local frame it has matrix $\Theta = d\theta - \theta \wedge \theta$.
• A *holomorphic vector bundle* is a vector bundle whose projection and trivialisation maps are holomorphic.

Holomorphic and complex vector bundles

The next theorem answers the question: when can a complex vector bundle be made into a holomorphic vector bundle?

Theorem

Let $E \to M$ be a complex vector bundle over a complex manifold and let $\bar\partial : \Omega^{p,q}E \to \Omega^{p,q+1}E$ be an operator that satisfies the Leibniz rule. Then $\bar\partial$ is induced from the structure of a holomorphic vector bundle on E if and only if $\bar\partial^2 = 0$.

This theorem can be proved using the Newlander-Nirenberg theorem.

Definition

A connection on a holomorphic vector bundle is said to be *compatible* with the holomorphic structure if, for every $s \in \Omega^{p,q}E$, the component of Ds in $\Omega^{p,q+1}E$ equals $\bar\partial s$.

The theorem implies that a complex vector bundle with a connection D admits a compatible holomorphic structure iff the component of R^D in $\Omega^{0,2} \operatorname{Hom}(E, E)$ vanishes.

Definition

Let $E \longrightarrow M$ be a complex vector bundle over a smooth manifold. A *Hermitian metric* on E is a pairing $\langle \cdot, \cdot \rangle : \Gamma(E) \times \Gamma(E) \longrightarrow \Gamma(E)$ such that, for all $s, t, u \in \Gamma(E)$, all $f, g \in C^\infty(M)$, and all $x \in M$:

1. $\langle t, s \rangle = \overline{\langle s, t \rangle}$
2. $\langle fs + tg, u \rangle = f\langle s, u \rangle + g\langle t, u \rangle$
3. $\langle s, s \rangle \geq 0$, and $\langle s, s \rangle(x) = 0 \Leftrightarrow s(x) = 0$

The pair $(E, \langle \cdot, \cdot \rangle)$ is called a *Hermitian vector bundle* in this case.

Remark: It follows that each fibre E_x carries a Hermitian metric which depends smoothly on x.

Definition

A connection D on a hermitian vector bundle (E, h) is called *hermitian* or *compatible* with the hermitian structure if

$$\mathrm{d}\langle s, t \rangle = \langle Ds, t \rangle + \langle s, Dt \rangle \quad \forall s, t \in \Gamma(E).$$

Exercise: Show that in a frame e such that $\langle e_i, e_j \rangle = \delta_{ij}$ the connection matrix for a hermitian connection is anti-hermitian.

Theorem

If $E \longrightarrow M$ is a Hermitian holomorphic vector bundle over a complex manifold, then there is unique connection D (called the Chern connection) compatible with both the metric and the complex structure.

Proof.

Let $(e_1, \ldots, e_k)$ be a holomorphic frame for E and put $h_{ij} = \langle e_i, e_j \rangle$. If D is compatible with the complex structure, then De_i is of type $(1,0)$, and, as D is compatible with the metric:

$$\mathrm{d}h_{ij} = \langle De_i, e_j \rangle + \langle e_i, De_j \rangle = \sum_p \theta_{ip} h_{pj} + \sum_p \bar{\theta}_{jp} h_{ip}.$$

The first term is of type $(1,0)$ and the second of type $(0,1)$, so:

$$\partial h_{ij} = \sum_p \theta_{ip} h_{pj}, \quad \bar{\partial} h_{ij} = \sum_p \overline{\theta_{jp}} h_{ip}.$$

Therefore $\partial h = \theta h$ and $\bar{\partial} h = h \bar{\theta}^T$. The unique solution to the first equation is $\theta = \partial h \, h^{-1}$; this solves the second because $\bar{h} = h^T$. □

Curvature of the Chern connection in a holomorphic frame

Recall that the curvature matrix of a connection D with respect to any frame $e = (e_1, \dots, e_n)$ is given by

$$\Theta = d\theta - \theta \wedge \theta,$$

where θ is the connection matrix w.r.t. e.
Recall that, for the Chern connection, $\theta = \partial h h^{-1}$ in a holomorphic frame e (where $h_{ij} = \langle e_i, e_j \rangle$). So...

$$d\theta = (\partial + \bar\partial)\theta = \bar\partial\theta + \partial(\partial h h^{-1}) = \bar\partial\theta - \partial h \wedge \partial(h^{-1}) = \bar\partial\theta + \partial h h^{-1} \wedge \partial h h^{-1}.$$

Hence the curvature matrix of the Chern connection w.r.t. a holomorphic frame is given by

$$\Theta = \bar\partial\theta$$

.This calculation shows that the curvature is a $(1,1)$-form.
In the case of a line bundle, if $h = \langle e_1, e_1 \rangle$, we have

$$\theta = \partial \log h, \quad \Theta = \bar\partial\partial \log h.$$

Example (Curvature of the tautological bundle of $\mathbb{C}P^m$)

The tautological bundle L is a subbundle of the trivial bundle $\mathbb{P}^m \times \mathbb{C}^{m+1}$:

$$L := \{(\ell, z) \in \mathbb{P}^n \times \mathbb{C}^{m+1} \,:\, z \in \ell \subset \mathbb{C}^{m+1}\}.$$

The standard Hermitian metric on $\mathbb{C}^{m+1}$ defines a Hermitian metric on $\mathbb{P}^m \times \mathbb{C}^{m+1}$ and hence on L. We will calculate the curvature of the associated Chern connection on L. A holomorphic frame over U_0 is given by:

$$e : [z_0, \dots, z_m] \to \big([z_0, \dots, z_m], (1, z_1/z_0, \dots, z_m/z_0)\big).$$

$h := \langle e, e \rangle = 1 + \sum_{j=1}^m |w_j|^2$, where $w_j = z_j/z_0$ are local coordinates. So

$$\Theta = \bar\partial\partial \log h = -\frac{\sum_{j=1}^n dw^j \wedge d\bar w^j}{1 + \sum_{j=1}^n |w_j|^2} + \frac{\sum_{i,j=1}^n \bar w_i \, dw_i \wedge w_j \, d\bar w_j}{(1 + \sum_{j=1}^n |w_j|^2)^2}.$$

Hermitian manifolds

Definition

A *Hermitian manifold* (M, g) consists of a complex manifold M and a Riemannian metric g on M for which the almost complex structure J is an isometry, i.e.:

$$g(JX, JY) = g(X, Y)\ \forall X, Y \in \Gamma(TM).$$

Setting $\omega(X, Y) = g(JX, Y)$ defines a $(1, 1)$-form ω called the *fundamental form* of (M, g).

The holomorphic tangent bundle of a Hermitian manifold carries a canonical positive definite Hermitian inner product (or *Hermitian metric*), defined by

$$\langle X, Y\rangle = 2g(\bar{X}, Y) \quad \forall X, Y \in \Gamma(T^{1,0}M).$$

Conversely, given a complex manifold M and a positive definite Hermitian inner product $\langle\cdot,\cdot\rangle$ on $T^{1,0}M$, setting

$$g(X, Y) = \langle X - \mathrm{i}JX, Y - \mathrm{i}JY\rangle \forall X, Y \in \Gamma(TM)$$

makes (M, g) a Hermitian manifold.

Hermitian manifolds in local coordinates

Given local holomorphic coordinates z^i, let $h_{ij} = h(\partial/\partial z^i, \partial/\partial z^j)$. Then

$$g = \frac{1}{2} h_{ij}(\mathrm{d}z^i \mathrm{d}\bar{z}^j + \mathrm{d}\bar{z}^j \mathrm{d}z^i), \quad \omega = \frac{\mathrm{i}}{2} h_{ij}\, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j.$$

In particular, $h = g \pm \mathrm{i}\omega$, where $h = h_{ij}\, \mathrm{d}\bar{z}^i \wedge \mathrm{d}z^j$ is the section of $\Lambda^{0,1}M \otimes \Lambda^{1,0}M$ that defines $\langle\cdot,\cdot\rangle$.

Example (Connection and curvature in dimension one)

Let M be a *Riemann surface*, i.e., a 1-dimensional complex manifold. Let $z = x + \mathrm{i}y$ be a local coordinate and $\partial/\partial z$ a local holomorphic frame, then a Hermitian metric on $T^{1,0}M$ is written as $h\,dz \otimes d\bar{z}$, for a local function $h > 0$. The connection matrix of the Chern connection is $\partial h\,h^{-1} = \dfrac{\partial \log h}{\partial z}\mathrm{d}z$, and the curvature matrix is

$$\Theta = \bar{\partial}\partial \log h = \frac{\partial^2 \log h}{\partial \bar{z}\partial z}\mathrm{d}\bar{z} \wedge \mathrm{d}z = \left(-\frac{1}{4}\Delta \log h\right)\mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

Now, the fundamental form on M is $\omega = \dfrac{\mathrm{i}}{2}h\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$ and hence

$$\Theta = -\mathrm{i}K\omega,$$

where $K = (-\Delta \log h)/2h$ is the usual Gauss curvature of a surface.

Sign of the curvature

Definition

The curvature $R_E \in \Gamma\left(\Lambda^{1,1}M \otimes Hom(E,E)\right)$ of the Chern connection of a Hermitian holomorphic vector bundle $E \longrightarrow M$ is called *positive at* $x \in M$ if the Hermitian matrix $R_E(x)(v,\bar{v}) \in Hom(E_x,E_x)$ is positive definite $\forall 0 \neq v \in T_x^{1,0}M$. We write $R_E(x) > 0$ if this is the case. If $R_E(x) > 0$ for every $x \in M$ the curvature is said to be *positive* and we write $R_E > 0$. *Negative*, *non-negative*, and *positive* curvature are defined similarly and written $R_E < 0$, $R_E \geq 0$, $R_E \leq 0$.

In a local frame, R_E is positive at x if the matrix $\Theta(x)(v,\bar{v})$ is positive $\forall 0 \neq v \in T^{1,0}M$.

Example

1. The curvature of the tautological bundle $J_{\mathbb{P}^n}$ is negative.
2. The curvature the holomorphic tangent bundle of a Riemann surface is positive if and only if the Gauss curvature is positive.

Sub-bundles

Let us compute the curvature R_F of the Chern connection of a holomorphic subbundle $F \subset E$ (with the induced Hermitian metric). Let $N = F^\perp$. This is a C^∞ complex subbundle of E.
Choose a local frame for E consisting of a **holomorphic** frame for F and **any** frame for N. Let θ_E, θ_F be corresponding the matrices for the Chern connections on E and F.
Write

$$\theta_E = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).$$

Since θ_E is compatible with the metric, $B = -C^\dagger$.
Since $F \subset E$ is holomorphic and θ is compatible with the holomorphic structure, A and C are a matrices of $(1,0)$-forms.
A defines a holomorphic Hermitian connection on A, so $A = \theta_F$. So

$$\Theta_E = d\theta_E - \theta_E \wedge \theta_E = \begin{pmatrix} d\theta_F - \theta_F \wedge \theta_F + B \wedge B^\dagger & * \\ * & * \end{pmatrix}.$$

Therefore $R_F - R_E|_F \leq 0$: *curvature decreases in holomorphic subbundles.*

Example

If M is a complex submanifold of $\mathbb{C}^n$ and $F = T^{1,0}M \subset T^{1,0}\mathbb{C}^n\big|_M$ the curvature of $T^{1,0}M$ is non-positive. In particular, if M is a Riemann surface then its *Gauss* curvature $K \leq 0$.

A similar calculation for the quotient bundle E/F shows that

$$R_{E/F} - R_E|_F \geq 0,$$

i.e. the curvature *increases* in holomorphic *quotient* bundles.

Example

Consider a holomorphic vector bundle $E \to M$ which is "spanned by its sections", i.e. there exist holomorphic sections $s_1, \dots, s_k \in \Gamma(E)$ (with $k \geq \operatorname{rank}(E)$) such that $s_1(x), \dots, s_k(x)$ span E_x $\forall x \in M$. Then

$$M \times \mathbb{C}^k \to E, \quad (x, \lambda) \mapsto \sum_{j=1}^{k} \lambda_j s_j(x)$$

is surjective. Thus E is a quotient bundle of a trivial bundle and $R_Q \geq 0$ with respect to the obvious metric.

Summary 4

• A complex vector bundle is holomorphic iff it admits an operator $\bar{\partial} : \Omega^{p,q}(E) \longrightarrow \Omega^{p,q+1}(E)$ which obeys the Leibniz rule with $\bar{\partial}^2 = 0$.
• A *Hermitian metric* $\langle \cdot, \cdot \rangle$ on a complex vector bundle E is a smoothly varying Hermitian inner product on each fibre $E_x \ \ (x \in M)$.
• The *Chern connection* on a Hermitian holomorphic bundle is the unique connection compatible with both structures. In a local *holomorphic* frame its connection and curvature matrices are given by $\theta = \partial h h^{-1}$ and $\Theta = \bar{\partial}\theta$.
• A Hermitian manifold is a complex Riemannian manifold such that $g(JX, JY) = g(X, Y)$.
• The curvature of the Chern connection can be positive *positive* or *negative* and

1. curvature decreases in holomorphic subbundles;
2. curvature increases in holomorphic quotient bundles.

The Ricci form

Definition

Let $E \longrightarrow M$ be a complex vector bundle over a smooth manifold, let D be a connection on E and let $R^D \in \Omega^2 \,\mathrm{Hom}(E, E)$ be its curvature. The *Ricci form of D* is the 2-form $\mathrm{tr}\, R^D \in \Lambda^2 M$.

Lemma

The Ricci form defines a cohomology class $\left[\mathrm{tr}\, R^D\right] \in H^2_{\mathrm{DR}}(M)$ that does not depend on D.

Proof.

Choose a local frame and let $\Theta = \mathrm{d}\theta - \theta \wedge \theta$ be the matrix of 2-forms representing R^D. Since $\theta \wedge \theta$ is traceless the Ricci form is equal $\mathrm{tr}\,\Theta = \mathrm{tr}\,\mathrm{d}\theta = \mathrm{d}\,\mathrm{tr}\,\theta$. Therefore the Ricci form is locally exact, hence closed.
Now let D' be another connection. Then $A = D - D'$ is a well-defined section of $\Lambda^1 M \otimes \mathrm{Hom}(E, E)$. So $\mathrm{tr}\, R^D - \mathrm{tr}\, R^{D'} = \mathrm{d}\,\mathrm{tr}\, A$ is exact. $\square$

The first Chern class

Definition

The cohomology class $c_1(E) = \frac{\mathrm{i}}{2\pi}\left[\operatorname{tr} R^D\right] \in H^2_{\mathrm{DR}}(M)$ is called *the first Chern class* of E.

The first Chern class is a topological invariant.

It is also integral, i.e. $\int_\Sigma c_1(E) \in \mathbb{Z}$ for any real 2-dimensional submanifold $\Sigma \subset M$. **Note added:** The first Chern number of a trivial

vector bundle is zero, because a trivial vector bundle admits a connection with zero curvature. The first Chern number of a line bundle (i.e. a vector bundle of rank 1) is zero if and **only if** it is trivial.

Example (Tautological bundle on $\mathbb{P}^1$)

. Recall that, for the Chern connection induced from $\mathbb{P}^1 \times \mathbb{C}^2$, we computed the curvature matrix in the chart U_0 with holomorphic coordinate w as

$$\Theta = \frac{1}{\left(1+|w|^2\right)^2} d\bar{w} \wedge dw.$$

Now, $H^2_{\mathrm{DR}}(\mathbb{P}^1)$ is identified with $\mathbb{C}$ via integration: $\omega \mapsto \int_{\mathbb{P}^1} \omega$. Thus

$$c_1(L) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \frac{1}{\left(1+|w|^2\right)^2} d\bar{w} \wedge dw = \frac{-1}{\pi} \int_{[0,\infty)\times[0,2\pi]} \frac{r}{\left(1+r^2\right)^2} dr \wedge d\theta,$$

where $w = re^{\mathrm{i}\theta}$. Hence

$$c_1(L) = \frac{-1}{\pi} \int_0^{+\infty} dr \int_0^{2\pi} \frac{r}{\left(1+r^2\right)^2} d\theta = -2 \int_0^{+\infty} \frac{r}{\left(1+r^2\right)^2} dr = -1.$$

In fact, over any $\mathbb{P}^n$, $c_1(L) = -1$. (This follows from the above because the restriction of the canonical bundle on $\mathbb{P}^n$ to $\mathbb{P}^1$ equals the canonical bundle on $\mathbb{P}^1$).

First Chern numbers of tensor products etc

Exercise: Let M be a compact complex manifold Let E and F be complex vector bundles over M of ranks m and n, respectively. Then:

(i) $c_1(\Lambda^m E) = c_1(E)$,

(ii) $c_1(E \oplus F) = c_1(E) + c_1(F)$,

(iii) $c_1(E \otimes F) = n c_1(E) + m c_1(F)$, note order of n, m,

(iv) $c_1(E^*) = -c_1(E)$,

(v) $c_1(f^* E) = f^* c_1(E)$.

Chern number of a complex manifold

Definition

Let M be a complex manifold of dimension n. The *first Chern class of* M is $c_1(M) := c_1\left(T^{1,0}M\right) = c_1\left(\Lambda^n T^{1,0}M\right) = -c_1(K_M)$.

Example

$c_1(\mathbb{P}^n) = c_1(K^*) = c_1\left((L^*)^{\otimes(n+1)}\right) = (n+1)c_1(L^*) = n+1$. In particular, $c_1(\mathbb{P}^1) = 2$.

For $\mathbb{P}^1$, this is just the Gauss–Bonnet theorem; in general, for any Hermitian metric on a compact surface S,

$$c_1(S) = \frac{1}{2\pi}\int_S K\omega = \chi(S), \text{ the } \textit{Euler characteristic of } S.$$

Examples of manifolds with $c_1(M) = 0$

Observe that $c_1(M) = 0$ if the canonical bundle K_M is trivial, i.e. there exists a non-vanishing holomorphic $(n,0)$-form on M where $n = \dim_{\mathbb{C}} M$.

- $\mathbb{C}^n$. Other boring examples include any M with $H^2(M) = 0$.
- Quotients of $\mathbb{C}^n$ by finite groups of biholomorphisms, e.g. by lattices: abelian varieties (complex tori).
- The quadric $Q = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : \ z_1^2 + z_2^2 + z_3^2 = 1\}$. This is a complexification of S^2 and so $H^2(Q) \neq 0$. The following holomorphic 2-form is non-vanishing on Q and trivialises K_Q:

$$z_1 dz_2 \wedge dz_3 + z_2 dz_3 \wedge dz_1 + z_3 dz_1 \wedge dz_2.$$

- The famous K3-surface (one of them, anyway): $S = \{[z_0, z_1, z_2, z_3] \in \mathbb{P}^3 : \ z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$.

First Chern class of a hypersurface

Let $\iota : V \hookrightarrow M$ be a complex hypersurface in a complex n-manifold M.
Let $\Lambda^{1,0} M|_V = \iota^* \Lambda^{1,0} M$ denote the *restriction* of $\Lambda^{1,0} M$ to V.
Let $N^* \subset \Lambda^{1,0} M|_V$ denote the bundle of forms ϕ such that $\phi(X) = 0$ $\forall X \in T^{1,0} V$. Then

$$\Lambda^{1,0} V \cong \Lambda^{1,0} M|_V \,/\, N^*.$$

It follows that

$$\Lambda^{n,0} M|_V \cong \Lambda^{n-1,0} V \otimes N^*,$$

and hence that

$$-c_1(M) = -c_1(V) + c_1(N^*).$$

Let f, f' be local holomorphic defining functions for V over open sets $U, U' \subset M$.
Then $\iota^* \mathrm{d}f$, $\iota^* \mathrm{d}f'$ are local holomorphic frames for V. On $U \cap U'$:

$$\iota^* \mathrm{d}f = \iota^* \mathrm{d}\left(\frac{f}{f'} f'\right) = \iota^* \left(\frac{f}{f'} \mathrm{d}f' + f' \mathrm{d}\frac{f}{f'}\right) = \left(\frac{f}{f'} \circ \iota\right) \iota^* \mathrm{d}f'.$$

Hence N^* has transition function f/f'.

Hypersurfaces in $\mathbb{P}^n$

A homogeneous polynomial $P(z_0, \dots, z_n)$ of degree d defines a hypersurface $V \subset \mathbb{P}^n$.
The local definining equations for V on $U_j = \{z_j \neq 0\}$ are $P_j := P/z_j^d$.
So N^* has transition functions $(z_i/z_j)^d$.
The bundle with transition function (z_i/z_j) is the tautological bundle L, so $N^* \cong (L^*)^d$ and $c_1(N^*) = -d$.
Recall that $c_1(\mathbb{P}^n) = n+1$. Hence

$$c_1(V) = c_1(\mathbb{P}^n) + c_1(N^*) = n+1-d.$$

Thus the 'K3 surface' $\{[z_0, z_1, z_2, z_3] \in \mathbb{P}^3 \,:\, z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$ has $c_1 = 0$.

Motivating Kähler manifolds

Let M be a Hermitian manifold (i.e. a complex manifold with a compatible Riemannian metric g).
The real and holomorphic tangent bundles are isomorphic:

$$TM \cong T^{1,0}M, \quad X \mapsto X - \mathrm{i}JX.$$

Moreover, $T^{1,0}M$ is a hermitian holomorphic vector bundle, with hermitian metric

$$\langle X - \mathrm{i}JX, Y - \mathrm{i}JY \rangle = g(X, Y) \quad \forall X, Y \in \Gamma(TM).$$

There are thus two natural connections on TM:

1. The Levi-Civita connection (g-compatible and torsion-free)
2. The Chern connection (compatible with Hermitian and complex structures).

When are these connections the same?

Lemma

Let ∇ be a connection on the tangent bundle of a Hermitian manifold M. Then ∇ can be identified with a connection D on $T^{1,0}M$ if and only if $\nabla J = 0$. Moreover,

1. *D is compatible with the holomorphic structure if and only if the torsion of ∇ is a section of $\Lambda^{2,0} \otimes T^{1,0}M \oplus \Lambda^{0,2} \otimes T^{0,1}M$.*
2. *D is compatible with the Hermitian structure if and only if ∇ is compatible with g.*

Note that

$$\begin{aligned}
(\nabla J)(X) &:= \nabla(JX) - J\nabla X = 0 \quad \forall X \in \Gamma(TM) \\
(\nabla g)(X,Y) &:= \mathrm{d}(g(X,Y)) - g(\nabla X, Y) - g(X, \nabla Y) \quad \forall X, Y \in \Gamma(TM) \\
D\langle W, Z\rangle &:= \mathrm{d}\langle W, Z\rangle - \langle DW, Z\rangle - \langle W, DZ\rangle \quad \forall W, Z \in \Gamma(T^{1,0}M).
\end{aligned}$$

Thus the second condition may be restated $\nabla g = 0 \Leftrightarrow D\langle \cdot, \cdot \rangle = 0$.
The lemma implies that equality of the Chern and Levi-Civita connections is equivalent to J being parallel w.r.t. the Levi-Civita connection, and to the Chern connection being torsion free.

Proof.

Choose local holomorphic coordinates z^j and work with the frame $\partial/\partial z^j, \partial/\partial \bar{z}^j$ for $T_{\mathbb{C}}M$. A connection on $T^{1,0}M$ with connection matrix θ_j^k yields a real connection on TM with matrix:

$$\begin{pmatrix} \theta & 0 \\ 0 & \bar{\theta} \end{pmatrix}.$$

This clearly makes $J = \mathrm{diag}(\mathrm{i}, \ldots, \mathrm{i}, -\mathrm{i}, \ldots, -\mathrm{i})$ parallel. Conversely, any connection that makes J parallel must be of this form.
The torsion of this connection is given by

$$T = (\theta^l_{jk}\, \mathrm{d}z^j \wedge \mathrm{d}z^k + \theta^l_{\bar{j}k}\, \mathrm{d}\bar{z}^j \wedge \mathrm{d}z^k) \otimes \frac{\partial}{\partial z^l} + \text{c.c.},$$

where $\theta^l_k = \theta^l_{jk}\, \mathrm{d}z^j + \theta^l_{\bar{j}k}\, \mathrm{d}\bar{z}^j$. The connection on $T^{1,0}M$ is holomorphic if and only if θ^k_j are (1,0)-forms, and this is clearly equivalent to the condition stated for the torsion.
The last part follows from the fact that $(\langle, \cdot, \rangle, J)$ determine g and (g, J) determine $\langle, \cdot, \rangle$. □

Theorem

Let (M, g) be a Hermitian manifold and let $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ be its fundamental form. The following conditions are equivalent:

(i) *J is parallel for the Levi–Civita connection ∇.*

(ii) *The Chern connection D has zero torsion.*

(iii) *The Levi–Civita and the Chern connections coincide.*

(iv) *The fundamental form ω of g is closed, $\mathrm{d}\omega = 0$.*

(v) *For each point $p \in M$, there exists a smooth real-valued function f in a neighbourhood of p, such that $\omega = \mathrm{i}\partial\bar{\partial}f$.*

(vi) *For each point $p \in M$, there exist complex coordinates w centred at p (called holomorphic normal coordinates), such that $g(w) = 1 + O(|w|^2)$.*

Proof strategy: We have show $(i) \Leftrightarrow (iii) \Leftrightarrow (ii)$; now show $(i) \implies (iv) \implies (v) \implies (vi) \implies (ii)$.

$(i) \implies (iv)$

If J and g are both parallel w.r.t. the Levi–Civita connection, then so is ω (since $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$).
To show that $\mathrm{d}\omega = 0$ we use the following lemma:

Lemma

Let M be a manifold, let $\phi \in \Omega^p M$ and let ∇ be a torsion-free connection on TM. Then

$$\mathrm{d}\phi = \nabla \wedge \phi.$$

The notation "$\nabla \wedge \phi$" instructs you to first evaluate $\nabla\phi$, which is a section of $T^*M \otimes \Lambda^p M$, and then replace $\otimes$ with $\wedge$. Thus for $\omega \in \Lambda^2 M$, the lemma says

$$\mathrm{d}\omega(X, Y, Z) = \nabla_X\omega(Y, Z) - \nabla_Y\omega(X, Z) + \nabla_Z\omega(X, Y).$$

Clearly, $\nabla\omega = 0$ implies $\mathrm{d}\omega = 0$.
The lemma can be proved by writing the left and right hand sides in a coordinate frame $\mathrm{d}x^j$, and recalling that vanishing torsion means that the connection matrix $\Gamma^k_j = \Gamma^k_{ij}\,\mathrm{d}x^i$ satisfies $\Gamma^k_{ij} = \Gamma^k_{ji}$.

$(iv) \implies (v)$

Lemma ($\partial\bar\partial$ lemma)

Let M be a complex manifold such that $H^{0,1}(M) = 0$ and let ϕ be an exact real (1,1)-form on M. Then there exists a real function $f : M \to \mathbb{R}$ such that $\phi = i\partial\bar\partial f$.

Thus if $U \subset M$ is diffeomorphic to a ball then ω is exact over U, and $\omega = i\partial\bar\partial f$ because $H^{0,1}(U) = 0$ (by the Dolbeault lemma).

Proof.

Write $\phi = d\psi$ for an exact 1-form ψ. Write $\psi = \chi + \chi'$, with $\chi \in \Omega^{1,0}M$ and $\chi' \in \Omega^{0,1}M$. Since ψ is real, $\chi' = \bar\chi$.
Decomposing to types gives $\phi = d\psi = \partial\chi + (\bar\partial\chi + \partial\bar\chi) + \bar\partial\bar\chi$.
Since $\phi \in \Omega^{1,1}M$, $\partial\chi = 0 = \bar\partial\bar\chi$.
Since $H^{0,1}(M) = 0$ $\exists u : M \to \mathbb{C}$ such that $\bar\chi = \bar\partial u$.
Since $\partial\bar\partial + \bar\partial\partial = 0$, $\phi = \bar\partial\partial\bar u + \partial\bar\partial u = \partial\bar\partial(u - \bar u) = i\partial\bar\partial \mathrm{Im}(u)$. $\square$

$(v) \implies (vi)$

Write $g = \frac{1}{2}h_{jk}(dz^j d\bar z^k + d\bar z^k dz^j)$ and $\omega = \frac{i}{2}h_{jk}\, dz^j \wedge d\bar z^k$ in local coordinates near a point $z = 0$, with $h_{kj} = \overline{h_{jk}}$. By linear change of coordinates we can arrange that $h_{jk}(0) = \delta_{jk}$. Then

$$h_{jk} = \delta_{jk} + z^l a_{ljk} + \bar z^l \bar a_{lkj} + O(|z|^2).$$

Since $\omega = i\partial\bar\partial f$ for some function f, $a_{ljk} = 2\partial^3 f/\partial z^l \partial z^j \partial\bar z^k$, and hence $a_{ljk} = a_{jlk}$. Let

$$w^j = z^j + \tfrac{1}{2}a_{lkj}z^k z^l, \quad \text{so that} \quad dw^j = dz^j + a_{klj}z^k dz^l.$$

Then

$$\begin{aligned}
\tfrac{i}{2}dw^j \wedge d\bar w^j &= \tfrac{i}{2}(dz^j \wedge d\bar z^j + a_{klj}z^k dz^l \wedge d\bar z^j + dz^j \wedge \bar a_{klj}\bar z^k d\bar z^l) + O(|z|^2) \\
&= \omega + O(|z|^2) \\
&= \omega + O(|w|^2).
\end{aligned}$$

$(vi) \implies (ii)$

Let $p \in M$, and choose coordinates z near, such that $g = \frac{1}{2} h_{jk}(dz^j \, d\bar{z}^k + d\bar{z}^k \, dz^j)$ with

$$h_{jk} = \delta_{jk} + O(|z|^2).$$

Since the derivatives of h_{jk} vanish at $z = 0$, the connection matrix $\theta = \partial h h^{-1}$ of the Chern connection vanishes at $z = 0$.
Therefore the Chern connection has vanishing torsion at p.
Since the choice of p was arbitrary, the Chern connection is torsion-free.

This completes the proof.

Kähler manifolds

Definition

A Hermitian manifold satisfying any one of the six equivalent conditions in the preceding theorem is called a *Kähler manifold*, its metric is called a *Kähler metric*, and its fundamental form ω is called the *Kähler form*. A local function f such that $\omega = i\partial\bar{\partial} f$ is called a *Kähler potential*.

Example ($\mathbb{C}^m$)

$$g = \frac{1}{2} \sum_j dz^j \, d\bar{z}^j + d\bar{z}^j \, dz^j, \quad \omega = \frac{i}{2} \sum_j dz^j \wedge d\bar{z}^j.$$

This is Kähler, by (vi). A Kähler potential is given by

$$f = \frac{1}{2} |z|^2.$$

Example (Fubini–Study metric on $\mathbb{P}^m$)

Choose standard coordinates $w^j = z^j/z^0$ over U_0, with $j = 1, \ldots, m$. Let

$$f_0 = \ln\left(1 + \sum_{j=1}^{m} |w^j|^2\right), \quad \omega = \mathrm{i}\partial\bar{\partial} f_0.$$

We claim that this ω extends to the whole of $\mathbb{P}^m$. To show this, define f_j analogously on U_j. Then on $U_j \cap U_0$,

$$f_j - f_0 = \ln\left(\sum_{l=0}^{m} \frac{|z^l|^2}{|z^j|^2}\right) - \ln\left(\sum_{l=0}^{m} \frac{|z^l|^2}{|z^0|^2}\right) = \ln\frac{z^0}{z^j} + \ln\frac{\bar{z}^0}{\bar{z}^j}.$$

Therefore $\mathrm{i}\partial\bar{\partial} f_j - \mathrm{i}\partial\bar{\partial} f_0 = 0$, and we may write $\omega = \mathrm{i}\partial\bar{\partial} f_j$ on U_j.
By direct calculation, $\mathrm{i}\partial\bar{\partial} f_0 = \frac{\mathrm{i}}{2} h_{jk}\, \mathrm{d}w^j \wedge \mathrm{d}\bar{w}^k$, where h is the matrix

$$h = \frac{1}{1+|w|^2}\left(Id_k - \frac{ww^\dagger}{1+|w|^2}\right) > \frac{1}{1+|w|^2}\left(Id_k - \frac{ww^\dagger}{|w|^2}\right) \geq 0$$

Therefore $g = \frac{1}{2} h_{jk}(\mathrm{d}w^j\, \mathrm{d}\bar{w}^j + \mathrm{d}\bar{w}^j\, \mathrm{d}w^j)$ is positive definite, so $(\mathbb{P}^m, g)$ is Kähler.

New Kähler metrics from old

Remark: It can be shown that the Fubini–Study metric on $\mathbb{P}^m$ is invariant under the natural action of $\mathrm{U}(m+1)$.

Proposition

A complex submanifold of a Kähler manifold, equipped with the induced metric, is Kähler.

Proof.

Let $\iota : N \to M$ be the inclusion map. Since N is a complex submanifold, $\iota_* \circ J_N = J_M \circ \iota_*$. Therefore the fundamental form $\omega_N = g_N(J_N\cdot, \cdot)$ equals the pull-back $\iota^*\omega_M$ of the fundamental form of M. Since ω_M is closed, so too is ω_N. □

Proposition

The product of two Kähler manifolds is Kähler.

The proof is an **exercise**!

The Ricci form

Proposition

Let (M,g) be a Kähler manifold, let J be its almost complex structure and let

$$Ric(X,Y) := \mathrm{tr}(V \mapsto R(V,X)Y)$$

be its Ricci tensor. be its Ricci tensor. Then

$$Ric(JX,JY) = Ric(X,Y) \quad \forall X,Y \in TM$$

Definition

The *Ricci form* of a Kähler manifold is the (1,1)-form

$$\rho(X,Y) = Ric(JX,Y) \quad \forall X,Y \in TM.$$

ρ is a 2-form because
$\rho(Y,X) = Ric(JY,X) = Ric(X,JY) = Ric(JX,J^2Y) = -\rho(X,Y)$.
It is of type (1,1) because
$\rho(JX,JY) = Ric(J^2X,JY) = Ric(J^3X,J^2Y) = \rho(X,Y)$.

Proof.

Since the Chern connection makes J parallel ($\nabla J = 0$),

$$R(X,Y)JZ = JR(X,Y)Z \quad \forall X,Y,Z \in TM.$$

Since the Chern connection has curvature of type (1,1),

$$R(JX,JY)Z = R(X,Y)Z \quad \forall X,Y,Z \in TM.$$

Therefore

$$\begin{aligned}
Ric(JX,JY) &= \mathrm{tr}(V \mapsto R(V,JX)JY) \\
&= \mathrm{tr}(V \mapsto JR(JV,J^2X)Y) \quad \text{identies above} \\
&= \mathrm{tr}(V \mapsto J^2R(V,J^2X)Y) \quad \text{cyclicity of trace} \\
&= Ric(X,Y)
\end{aligned}$$

$\square$

Properties of the Ricci form

Theorem

Let ρ be the Ricci form of a Kähler manifold (M, g). Then

1. $\mathrm{i}\rho$ *is equal to the curvature of the Chern connection of the canonical bundle.*
2. ρ *is closed:* $\mathrm{d}\rho = 0$.
3. *In local coordinates where* $g = \frac{1}{2}h_{jk}(\mathrm{d}z^j\,\mathrm{d}\bar{z}^k + \mathrm{d}\bar{z}^k\,\mathrm{d}z^j)$, $\rho = -\mathrm{i}\partial\bar{\partial}\ln\det h$.
4. $[\rho/2\pi] = c_1(M)$ *in* $H^2(M)$.

$\mathrm{i}\rho$ = curvature of K

By J-compatibility of ∇ and Bianchi I:

$$JR(X,Y)V = -JR(Y,V)X - JR(V,X)Y = R(V,Y)JX - R(V,X)JY.$$

Therefore

$$\mathrm{tr}(V \mapsto JR(X,Y)V) = Ric(Y,JX) - Ric(X,JY) = 2\rho(X,Y).$$

We evaluate this trace over TM as a trace over $T_{\mathbb{C}}M$.
With respect to the splitting $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$,

$$J = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \text{ and } R = \begin{pmatrix} \Theta & 0 \\ 0 & \bar{\Theta} \end{pmatrix},$$

where Θ is the curvature of the Chern connection on $T^{1,0}M$. Hence

$$\rho(X,Y) = \frac{\mathrm{i}}{2}\mathrm{tr}(\Theta(X,Y)) - \frac{\mathrm{i}}{2}\mathrm{tr}(\bar{\Theta}(X,Y)).$$

The Chern connection θ on $T^{1,0}M$ induces a holomorphic Hermitian connection $-\mathrm{tr}\,\theta$ on $K = \det(\Lambda^{1,0}M)$. By uniqueness, this is the Chern connection of K. It's curvature is $\Theta_K = -\mathrm{tr}\,\Theta$. Since the connection is Hermitian, $\bar{\Theta}_K = -\Theta_K$. So $\mathrm{i}\rho = \Theta_K$.

Everything else follows

The remaining parts of the theorem follow from the identity $i\rho = \Theta_K$:

- ρ is closed by the (second) Bianchi identity: $d\Theta_K = 0$.
- If $g = \frac{1}{2}h_{jk}(dz^j\, d\bar{z}^k + d\bar{z}^k\, dz^j)$ then

$$\langle dz^1 \wedge \ldots \wedge dz^k, dz^1 \wedge \ldots \wedge dz^k \rangle = \det(h)^{-1}.$$

  Therefore $\rho = -i\Theta_K = -i\bar{\partial}\partial \ln(\det(h)^{-1}) = -i\partial\bar{\partial} \ln \det h$.
- $c_1(M) = -c_1(K) = -[\frac{i}{2\pi}\Theta_K] = [\frac{1}{2\pi}\rho]$.

Hodge star and Laplacian

Definition

Let (M, g) be a compact oriented n-dimensional **Riemannian** manifold and let Vol_g be a volume form. The *Hodge star operator* is the unique operator $* : \Omega^p M \to \Omega^{n-p} M$ such that

$$u \wedge *v = g(u, v)\mathrm{Vol}_g \quad \forall u, v \in \Omega^p M.$$

The L^2 inner product on $\Omega^p M$ is

$$\langle u, v \rangle_{L^2} = \int_M g(\bar{u}, v)\mathrm{Vol}_g = \int_M \bar{u} \wedge *v.$$

Remark: if E_i is an orthonormal frame for TM and e^i is the dual frame for T^*M then the metric g on $\Omega^p M$ is defined such that the forms $e^{i_1} \wedge \ldots \wedge e^{i_p}$, $i_1 < \ldots < i_p$ are orthonormal.
Exercise: show that $*^2 = (-1)^{p(n-p)}$.

Definition

Let (M,g) be as above. The codifferential is the map

$$\mathrm{d}^* : \Omega^p M \to \Omega^{p-1} M, \quad \mathrm{d}^* = (-1)^{pn+1} * \mathrm{d} *.$$

The *Hodge Laplacian* (or Laplace-Beltrami operator) is

$$\triangle_g : \Omega^p M \to \Omega^p M, \quad \triangle_g := \mathrm{d}\mathrm{d}^* + \mathrm{d}^* \mathrm{d}.$$

Lemma

The codifferential is the L^2 adjoint of the exterior derivative, and the Hodge-Laplacian is self-adjoint.

Proof.

$$\begin{aligned}\langle \mathrm{d}u, v\rangle_{L^2} &= \int_M \mathrm{d}\bar{u} \wedge *v = (-1)^{p+1} \int_M \bar{u} \wedge \mathrm{d}*v \\ &= (-1)^{pn+p(1-p)+1} \int_M \bar{u} \wedge *(*\mathrm{d}*v) = \langle u, \mathrm{d}^* v\rangle_{L^2}.\end{aligned}$$

Furthermore $(\mathrm{d}^*\mathrm{d})^* = \mathrm{d}^*(\mathrm{d}^*)^* = \mathrm{d}^*\mathrm{d}$ and similarly $(\mathrm{d}\mathrm{d}^*)^* = \mathrm{d}\mathrm{d}^*$. □

Example ($\mathbb{R}^3$ with $g = \sum_i \mathrm{d}x^i \, \mathrm{d}x^i$)

If $f : \mathbb{R}^3 \to \mathbb{R}$ is a 0-form then

$$\begin{aligned}
\mathrm{d}f &= \frac{\partial f}{\partial x^1}\mathrm{d}x^1 + \frac{\partial f}{\partial x^2}\mathrm{d}x^2 + \frac{\partial f}{\partial x^3}\mathrm{d}x^3 \\
*\mathrm{d}f &= \frac{\partial f}{\partial x^1}\mathrm{d}x^2 \wedge \mathrm{d}x^3 - \frac{\partial f}{\partial x^2}\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \frac{\partial f}{\partial x^3}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
\mathrm{d}*\mathrm{d}f &= \frac{\partial^2 f}{(\partial x^1)^2}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \frac{\partial^2 f}{(\partial x^2)^2}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&\quad + \frac{\partial^2 f}{(\partial x^3)^2}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
\mathrm{d}\mathrm{d}f &= \frac{\partial^2 f}{(\partial x^1)^2} + \frac{\partial^2 f}{(\partial x^2)^2} + \frac{\partial^2 f}{(\partial x^3)^2}
\end{aligned}$$

and $\mathrm{d}^* f = 0$, so

$$\triangle_g f = \mathrm{d}^*\mathrm{d}f = -*\mathrm{d}*\mathrm{d}f = -\sum_i \frac{\partial^2 f}{(\partial x^i)^2}.$$

Hodge Laplacian as a gradient

Lemma

The Hodge Laplacian is the gradient of the energy functional

$$E_g : \Omega^p M \to \mathbb{R}, E_g : \phi \mapsto \frac{1}{2}\left(\langle \mathrm{d}\phi, \mathrm{d}\phi\rangle_{L^2} + \langle \mathrm{d}^*\phi, \mathrm{d}^*\phi\rangle_{L^2}\right),$$

i.e. if ϕ_t is a family of p-forms smoothly parametrised by $t \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} E_g[\phi_t]\Big|_{t=0} = \mathrm{Re}\,\langle \dot\phi_0, \triangle_g \phi_0\rangle_{L^2}.$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E_g[\phi_t] &= \frac{1}{2}\big(\langle \mathrm{d}\dot\phi_t, \mathrm{d}\phi_t\rangle_{L^2} + \langle \mathrm{d}\phi_t, \mathrm{d}\dot\phi_t\rangle_{L^2} \\
&\quad + \langle \mathrm{d}^*\dot\phi_t, \mathrm{d}^*\phi_t\rangle_{L^2} + \langle \mathrm{d}^*\phi_t, \mathrm{d}^*\dot\phi_t\rangle_{L^2}\big) \\
&= \mathrm{Re}\,\left(\langle \mathrm{d}\dot\phi_t, \mathrm{d}\phi_t\rangle_{L^2} + \langle \mathrm{d}^*\dot\phi_t, \mathrm{d}^*\phi_t\rangle_{L^2}\right) \\
&= \mathrm{Re}\,\left(\langle \dot\phi_t, \mathrm{d}^*\mathrm{d}\phi_t\rangle_{L^2} + \langle \dot\phi_t, \mathrm{d}\mathrm{d}^*\phi_t\rangle_{L^2}\right) \\
&= \mathrm{Re}\,\langle \dot\phi_t, \triangle_g \phi_t\rangle_{L^2}.
\end{aligned}$$

□

Dolbeault Laplacians

Let M be a compact m-complex-dimensional Hermitian manifold.
Exercise: Show that $*(\Omega^{p,q}M) = \Omega^{m-q,m-p}M$.
Exercise: Show that the operators

$$\begin{aligned}
\partial^* : \Omega^{p,q}M \to \Omega^{p-1,q}M, \quad \partial^* &= -*\bar\partial*\\
\bar\partial^* : \Omega^{p,q}M \to \Omega^{p,q-1}M, \quad \bar\partial^* &= -*\partial*
\end{aligned}$$

are the L^2-adjoints of ∂ and $\bar\partial$.

Definition

The *Hodge Laplacians* are the operators $\triangle_\partial, \triangle_{\bar\partial} : \Omega^{p,q}M \to \Omega^{p.q}M$, $\triangle_\partial = \partial\partial^* + \partial^*\partial$, $\triangle_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$.

Exercise: Show that the Dolbeault Laplacians are the gradients of

$$E_\partial[\phi] = \frac{1}{2}\left(\langle \partial\phi, \partial\phi\rangle + \langle \partial^*\phi, \partial^*\phi\rangle\right), \quad E_{\bar\partial}[\phi] = \frac{1}{2}\left(\langle \bar\partial\phi, \bar\partial\phi\rangle + \langle \bar\partial^*\phi, \bar\partial^*\phi\rangle\right).$$

Laplacians on Kähler manifolds

Theorem

Let (M, g) be a Kähler manifold. Then $\triangle_g = 2\triangle_\partial = 2\triangle_{\bar\partial}$.

Proof.

Let $p \in M$ and choose normal coordinates $z^j = x^j + \mathrm{i}y^j$ near p such that $g_{jk} = \delta_{jk} + O(|z|^2)$. Let ϕ be any form.
Then $\triangle_g \phi = \triangle_\delta \phi + O(|z|)$.
In particular, $\triangle_g \phi(0) = \triangle_\delta \phi(0)$.
Similarly, the Dolbeault Laplacians agree with their Euclidean versions at $z = 0$.
By direct calculation, $\triangle_g = 2\triangle_\partial = 2\triangle_{\bar\partial}$ holds for the Euclidean metric.
Therefore $\triangle_g \phi = 2\triangle_\partial \phi = 2\triangle_{\bar\partial} \phi$ at p.
Since p was arbitrary, $\triangle_g \phi = 2\triangle_\partial \phi = 2\triangle_{\bar\partial} \phi$ everywhere. □

Cohomology of compact Kähler manifolds I

Proposition

If (M, g) is an m-complex-dimensional compact Kähler manifold and $0 \leq q \leq m$ then $H^{2q}_{DR}(M) \neq 0$.

Hence the complex manifold $S^1 \times S^{2m-1}$ admits no Kähler metric!

Proof.

Let ω be the Kähler form. Then ω is closed, and hence so is ω^q.
Suppose that $\omega^q = \mathrm{d}\psi$ for some $\psi \in \Omega^{q-1}M$. By Stokes' theorem:

$$\int_M \omega^m = \int_M \mathrm{d}\psi \wedge \omega^{m-q} = \int_M \mathrm{d}(\psi \wedge \omega^{m-q}) = 0.$$

By direct calculation in an orthonormal basis, $\omega^m = m!\,\mathrm{Vol}_g$, so

$$\int_M \omega^m = m!\,\mathrm{Vol}(M) \neq 0,$$

a contradiction. Thus ω^q is not exact, and $0 \neq [\omega^q] \in H^{2q}_{DR}(M)$. □

Cohomology of compact Kähler manifolds II

Proposition

Let (M,g) be an m-complex-dimensional compact Kähler manifold and $0 \leq q \leq m$. The inclusion $\Omega^{q,0}M \hookrightarrow \Omega^q M$ induces an injective map $H^{q,0}_{\bar\partial}(M) \hookrightarrow H^q_{DR}(M)$.

Proof.

If $\eta \in \Omega^{q,0}M$ and $\bar\partial\eta = 0$ we call η a *holomorphic* $(q,0)$*-form*. Assume:

Lemma

A non-zero holomorphic $(q,0)$-form is never exact.

Choose any non-zero element of $H^{q,0}_{\bar\partial}(M)$; this is represented by a unique holomorphic $(q,0)$-form η. Then $d\eta = (\partial + \bar\partial)\eta = \partial\eta$ is an exact holomorphic $(q+1,0)$-form, so $d\eta = 0$ by the lemma. Therefore η defines a class $[\eta] \in H^{2q}_{DR}(M)$. This class is not trivial, because η is not exact, by the lemma. □

Proof of lemma.

Let η be a non-zero holomorphic $(q,0)$-form, and suppose for contradiction that $\eta = d\psi$ for some $\psi \in \Omega^{q-1}M$. Then since $d\bar\eta = 0$ and $d\omega = 0$,

$$\int_M \eta \wedge \bar\eta \wedge \omega^{m-q} = \int_M d(\psi \wedge \bar\eta \wedge \omega^{m-q}) = 0$$

by Stokes' theorem. On the other hand,

$$\int_M \eta \wedge \bar\eta \wedge \omega^{m-q} = C_{m,q} \int_M \eta \wedge *\eta = C_{m,q} \langle \eta, \eta \rangle_{L^2} \neq 0,$$

(where $C_{m,q} = (-\mathrm{i})^{q^2} 2^q (m-q)!$), a contradiction. Therefore η is not exact. □

The proposition that we just proved is a special case of the Hodge theorem, which will be proved in a few slides' time...

Harmonic forms

Definition

A p-form ϕ on a Riemannian manifold is called *co-closed* if $d^*\phi = 0$ and *harmonic* if $\triangle_g \phi = 0$. The space of all harmonic p-forms is denoted $\mathcal{H}^p(M)$.

Lemma

A differential form on a compact Riemannian manifold is harmonic if and only if it is both closed and co-closed.

Proof.

The "if" direction is obvious, so suppose that ϕ is a harmonic p-form. Then

$$0 = \langle \triangle_g \phi, \phi \rangle_{L^2} = \langle (dd^* + d^* d)\phi, \phi \rangle_{L^2} = \langle d\phi, d\phi \rangle_{L^2} + \langle d^*\phi, d^*\phi \rangle_{L^2}.$$

The right hand side is zero if and only if $d\phi = 0$ and $d^*\phi = 0$. $\square$

The Hodge–de Rham theorem

Theorem (Hodge–de Rham)

Let (M, g) be a compact Riemannian manifold. Then there are L^2-orthogonal decompositions,

$$\Omega^p M = \mathcal{H}^p(M) \oplus d(\Omega^{p-1} M) \oplus d^*(\Omega^{p+1} M).$$

Corollary (Hodge isomorphism)

On a compact Riemannian manifold (M, g) the natural map $\mathcal{H}^p(M) \to H^p_{DR}(M)$, $\phi \mapsto [\phi]$ is an isomorphism.

Proof of corollary.

Let $\phi \in \Omega^p(M)$ and write $\phi = \phi_H + d\psi + d^*\chi$ with $\phi_H \in \mathcal{H}^p(M)$. If $d^*\chi = 0$ then ϕ is closed. Conversely, if ϕ is closed then

$$0 = \langle d\phi, \chi \rangle_{L^2} = \langle dd^*\chi, \chi \rangle = \langle d^*\chi, d^*\chi \rangle$$

and $d^*\chi = 0$. So $\ker d = \mathcal{H}^p(M) \oplus d(\Omega^{p-1} M)$. Therefore $H^p_{DR}(M) = \ker d / \operatorname{im} d \cong \mathcal{H}^p(M)$. $\square$

Corollary (Poincaré duality)

Let M be a compact n-dimensional manifold. Then $H^k_{DR}(M) \cong H^{n-k}_{DR}(M)$.

Proof.

Choose any Riemannian metric on M. It suffices to show that the Hodge star induces a linear map $\mathcal{H}^k_{DR}(M) \to \mathcal{H}^{n-k}_{DR}(M)$, i.e. to show that if ϕ is harmonic then $*\phi$ is harmonic. This is left as an **exercise**. The linear map must then be an isomorphism, because $*^2 = (-1)^{k(n-k)}$. □

Sketch proof of Hodge–de Rham theorem.

Want to show: $\Omega^p M = \mathcal{H}^p(M) \oplus \mathrm{d}(\Omega^{p-1}M) \oplus \mathrm{d}^*(\Omega^{p+1}M)$.
Easy to show that the three factors are orthogonal:
$\langle \mathrm{d}\psi, \mathrm{d}^*\phi\rangle_{L^2} = \langle \mathrm{d}^2\psi, \phi\rangle_{L^2} = 0$ shows that $\mathrm{d}(\Omega^{p-1}M) \perp \mathrm{d}^*(\Omega^{p+1}M)$;
if $\omega \in \mathcal{H}^p(M)$ then $\mathrm{d}\omega = 0$, so $\langle \omega, \mathrm{d}^*\psi\rangle = \langle \mathrm{d}\omega, \psi\rangle = 0$ and $\mathcal{H}^p(M) \perp \mathrm{d}^*(\Omega^{p+1}M)$; ...
Next show the three factors span $\Omega^p M$ – this is hard! Things would be easy if $\Omega^p(M)$ were finite-dimensional: in that case, since $\triangle_g$ is self-adjoint, it would restrict to an invertible operator $A : \mathcal{H}^p(M)^\perp \to \mathcal{H}^p(M)^\perp$. For any $\psi \in \mathcal{H}^p(M)^\perp$ we could write

$$\psi = AA^{-1}\psi = \triangle_g(A^{-1}\psi) = \mathrm{d}(\mathrm{d}^*A^{-1}\psi) + \mathrm{d}^*(\mathrm{d}A^{-1}\psi),$$

and conclude that $\mathcal{H}^p(M)^\perp = \mathrm{d}(\Omega^{p-1}M) \oplus \mathrm{d}^*(\Omega^{p+1}M)$.
Since $\Omega^p(M)$ is NOT finite-dimensional life is harder. You need to work with a completion of $\Omega^p(M)$ with respect to a Lebesgue or Sobolev norm and appeal to linear analysis to show that A is invertible (usually by a circuitous route). You also need to show that things in the kernel of $\triangle_g$ really are smooth solutions, rather than weak solutions. □

Dolbeault decomposition theorem

Theorem (Dolbeault decomposition theorem)

Let (M,g) be a compact Hermitian manifold, and let $\mathcal{H}^{p,q} = \ker(\triangle_{\bar\partial}) \subset \Omega^{p,q}M$. Then there are L^2-orthogonal decompositions,

$$\Omega^{p,q}M = \mathcal{H}^{p,q}(M) \oplus \bar\partial(\Omega^{p,q-1}M) \oplus \bar\partial^*(\Omega^{p,q+1}M).$$

Corollary

Dolbeault isomorphism theorem On a compact Hermitian manifold (M,g) the natural map $\mathcal{H}^{p,q}(M) \to H^{p,q}_{\bar\partial}(M)$, $\phi \mapsto [\phi]$ is an isomorphism.

Both proofs are similar to their Riemannian counterparts.

Corollary (Serre duality)

On a compact complex m-dimensional manifold M, $H^{p,q}(M) \cong H^{m-p,m-q}(M)$.

Proof: Choose a Hermitian metric and show that $\psi \mapsto *\bar\psi$ is an isomorphism $\mathcal{H}^{p,q}(M) \to \mathcal{H}^{m-p,m-q}(M)$.

Theorem (Hodge)

Let (M,g) be a compact Kähler manifold. Then

$$H^k_{DR}(M) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(M), \quad \text{and} \quad H^{q,p}_{\bar\partial}(M) \cong \overline{H^{p,q}_{\bar\partial}(M)}.$$

Proof.

Since $\triangle_g = 2\triangle_{\bar\partial}$, $\mathcal{H}^k(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M)$.
Since $\triangle_{\bar\partial} = \triangle_\partial$, $\psi \mapsto \bar\psi$ is an isomorphism $\mathcal{H}^{p,q}(M) \to \mathcal{H}^{q,p}(M)$.
The results then follow by the Hodge and Dolbeault isomophism theorems. □

Theorem

Let ϕ_t be a continuous family of isometries of a compact Kähler manifold parametrised by $t \in \mathbb{R}$ such that $\phi_{s+t} = \phi_s \circ \phi_t$. Then ϕ_t is holomorphic for all $t \in \mathbb{R}$.

The theorem is **false** on non-compact Kähler manifolds. E.g. SO(4) is a path-connected group of isometries of $\mathbb{C}^2$, but only its subgroup U(2) acts holomorphically.
The theorem is also false for actions of discrete groups. E.g. the antipodal map $S^2 \to S^2$, $\mathbf{x} \mapsto -\mathbf{x}$ (where $\mathbf{x} \in \mathbb{R}^3$ satisfies $\mathbf{x}.\mathbf{x} = 1$) is not holomorphic with respect to the complex structure on $\mathbb{P}^1 \simeq S^2$ (**Exercise**).
Remark: By setting $X = \dot{\phi}_t|_{t=0}$, we learn that Killing vectors X on compact Kähler manifolds commute with J: $[X, JY] = J[X, Y]$ for all vector fields Y.

Proof.

We show that $\phi_t^*\omega = \omega$. The result follows, since g and ω determine J.
Since $*\omega = \frac{1}{(m-1)!}\omega^{m-1}$ and $d\omega = 0$, $d^*\omega = -*d*\omega = 0$. Hence ω is harmonic.
Since ϕ_t is homotopic to the identity map, $\phi_t^*\omega$ and ω define the same cohomology class in $H^2_{DR}(M)$.
Since ϕ_t is an isometry, $\triangle_g \phi_t^*\omega = \phi_t^*\triangle_g\omega = 0$ and $\phi_t^*\omega$ is also harmonic.
Therefore $\phi_t^*\omega - \omega$ is a harmonic form homologous to zero.
By the Hodge–de Rham theorem, it must vanish.
So $\phi_t^*\omega = \omega$. $\square$

Sectional curvature

Let (M,g) be a Riemannian manifold and let $\pi \subset T_xM$ be a plane. Recall that the *sectional curvature* of π is the Gauss curvature of a surface passing through x tangent to π.
Given $X, Y \in \pi$, the sectional curvature of π is equal to

$$K_x(X,Y) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2},$$

where $R(U,V,X,Y) = g(U, R(X,Y)V)$.

Definition

Let (M,g) be a *Hermitian* manifold. The *holomorphic sectional curvatures* are the sectional curvatures of tangent planes to one-complex-dimensional submanifolds.

A plane π is tangent to a complex submanifold if and only if it is fixed by J. So the holomorphic sectional curvatures are the quantities

$$K_x(X,JX) = \frac{R(X,JX,X,JX)}{g(X,X)^2}, \quad X \in T_xM.$$

Holomorphic sectional curvature determines curvature

Recall that

Theorem

Let (M,g) be a Riemannian manifold. Then the sectional curvatures of M determine the curvature tensor.

Kähler manifolds are even better:

Theorem

Let (M,g) be a Kähler manifold. Then the holomorphic sectional curvatures of M determine the curvature tensor.

Both proofs are similar to the proof that a quadratic form B on a vector space V is determined by the quantities $B(v,v)$, $v \in V$. In that case, one extracts $B(u,v)$ as the coefficient of $2t$ in

$$B(u+tv, u+tv) = B(u,u) + 2tB(u,v) + t^2B(v,v).$$

We will prove only the second of these two theorems.

Proof: Holomorphic sectional curvature determines curvature.

We show that the holomorphic sectional curvatures determine the sectional curvatures, and appeal to the previous theorem.
The coefficient of t^2 in $R(X+tY, JX+tJY, X+tY, JX+tJY)$ is

$$R(X,JY,X,JY)+R(X,JX,Y,JY)+R(X,JY,Y,JX)+ \\ R(Y,JX,Y,JX)+R(Y,JY,X,JX)+R(Y,JX,X,JY).$$

Using symmetry properties of R and the fact that $R(U,V,JX,JY)=R(U,V,X,Y)=R(JU,JV,X,Y)$, this equals

$$2R(X,JY,X,JY)+4R(X,JX,Y,JY).$$

The first term here looks like a sectional curvature. We make the second look like a sectional curvature by using the Bianchi identity:

$$\begin{aligned} R(X,JX,Y,JY) &= -R(X,Y,JY,JX)-R(X,JY,JX,Y) \\ &= R(X,Y,X,Y)+R(X,JY,X,JY). \end{aligned}$$

□

Proof (cont'd).

So the coefficient of t^2 in $R(X+tY, JX+tJY, X+tY, JX+tJY)$ is

$$6R(X,JY,X,JY)+4R(X,Y,X,Y).$$

Similarly, the coefficient of t^2 in $R(X+tJY, JX-tY, X+tJY, JX-tY)$ is

$$6R(X,Y,X,Y)+4R(X,JY,X,JY).$$

$R(X,Y,X,Y)$ is equal to a linear combination of these two quantities. □

If you haven't seen the proof that sectional curvatures determine the curvature tensor, it is similar in spirit to this one.

Constant holomorphic sectional curvature

Definition

A Kähler manifold (M,g) is said to have constant holomorphic sectional curvature if $K_x(X,JX)$ does not depend on the point $x \in M$ or on the plane $\pi \subset T_xM$ spanned by X, JX.

Example

$\mathbb{C}^m$ has constant sectional curvature equal to zero.

Example ($\mathbb{P}^m$ has positive constant sectional curvature)

Rather than a direct calculation, we give a conceptual proof. Let

$$F_{1,2} = \{\text{linear subspaces } \ell, m \subset \mathbb{C}^{m+1} : \dim \ell = 1, \dim m = 2, \ell \subset m\}.$$

Pairs $(\ell, m) \in F_{1,2}$ are called *flags*.
There is a bijection

$$\begin{aligned} F_{1,2} &\rightarrow \{(x,\pi) : x \in \mathbb{P}^m, \pi \subset T_xM \text{ is a } \mathbb{C}\text{-linear subspace}\} \\ (\ell, m) &\mapsto \left([z_0 : \ldots : z_m], \frac{\mathrm{d}}{\mathrm{d}t}[z_0 + tv_0 : \ldots : z_m + tv_m]\Big|_{t=0} \right), \end{aligned}$$

where $z, v \in \mathbb{C}^{m+1}$ are such that $\ell = \mathrm{span}\{z\}$, $m = \mathrm{span}\{z, v\}$.
$\mathrm{U}(m+1)$ acts transitively on $\mathbb{P}^m$ and on $F_{1,2}$, i.e. any flag can be mapped to any other by an element of $\mathrm{U}(m+1)$.
The metric and complex structure on $\mathbb{P}^m$ are invariant under this action (see second exercise sheet).
Therefore the holomorphic section curvature is a function on $F_{1,2}$ which is invariant under a transitive group action, hence constant.

Example (Complex hyperbolic space)

$$\begin{aligned}\mathbb{C}H^m &= \{w \in \mathbb{C}^m : |w|^2 < 1\} \\ \omega &= \mathrm{i}\partial\bar{\partial}\ln(1-|w|^2)\}.\end{aligned}$$

This is a Kähler manifold with constant *negative* sectional curvature.

Theorem

The only complete simply connected Kähler manifolds with constant sectional curvature are $\mathbb{P}^m$, $\mathbb{C}^m$ and $\mathbb{C}H^m$.

The Calabi conjecture

Let (M,g) be a compact Kähler manifold.
Recall that $c_1(M) := c_1(T^{1,0}M) = \frac{\mathrm{i}}{2\pi}\left[\operatorname{tr} R^D\right] \in H^2_{DR}(M)$, where R^D is the curvature of any connection D on $T^{1,0}M$.
If D is the Chern connection R^D is type (1,1), so $c_1(M) \in H^2_{DR}(M) \cap H^{1,1}_{\bar{\partial}}(M)$.
Recall that $[\rho] = 2\pi c_1(M)$.
Can every form ϕ such that $[\phi] = c_1(M)$ be realised as the Ricci form of some Kähler metric?

Theorem (Calabi-Yau)

Let (M,g) be a compact Kähler manifold with Kähler form ω and let ϕ be a real (1,1)-form representing $c_1(M)$. Then there exists a unique Kähler metric on M whose Kähler form $\tilde{\omega}$ satisfies $[\tilde{\omega}] = [\omega]$ and whose Ricci form $\tilde{\rho}$ satisfies $\tilde{\rho} = \phi$.

Conjectured by Calabi, proved by Yau in 1977.

Kähler-Einstein manifolds

Recall that a Riemannian manifold is called *Einstein* if $Ric = \lambda g$ for some constant $\lambda \in \mathbb{R}$.
A complex manifold is called *Kähler-Einstein* if it admits a metric which is both Kähler and Einstein.

Does a given Kähler manifold admit a Kähler-Einstein metric?

Note that $\rho = \lambda\omega$ on a Kähler-Einstein manifold, so $\lambda[\omega] = 2\pi c_1(M)$.
Since $\omega > 0$, one requires $\lambda = 0$, > 0 or < 0 according to whether $c_1(M)$ is positive, negative, or zero. W.l.o.g., $\lambda = 0, \pm 1$.

Existence of Kähler-Einstein metrics

If $c_1(M) = 0$, there is a Kähler-Einstein metric with $\rho = 0$, by the Calabi-Yau theorem.

Theorem (Aubin–Yau, 1978)

Let M be a compact Kähler manifold with $c_1(M) < 0$. Then there exists a unique Kähler metric whose fundamental form $\tilde{\omega}$ and Ricci form $\tilde{\rho}$ satisfy $\tilde{\rho} = -\tilde{\omega}$.

It is not true that every Kähler manifold with $c_1(M) > 0$ admits a Kähler-Einstein metric.
Berman shows in arXiv:1205.6214 that a manifold with $c_1(M) > 0$ must be "K-(poly)stable" (a similar result was obtained by Tian in 1997 with more restrictive assumptions).
Chen, Donaldson Sun, and, independently, Tian, showed in late 2012 that any K-(poly)stable Kähler manifold with $c_1(M) > 0$ admits a Kähler-Einstein metric.

Constant scalar curvature Kähler metrics

A *constant scalar curvature Kähler metric* is a Kähler metric whose scalar curvature $S := \operatorname{tr} Ric$ is constant.
Kähler-Einstein metrics obviously have constant scalar curvature. In fact, more is true on compact Kähler manifolds:

$$S = \text{constant and } [\rho] = \lambda[\omega] \quad \Leftrightarrow \quad g \text{ is Kähler-Einstein.}$$

Stoppa has shown (2009) that constant scalar curvature Kähler manifolds are necessarily K-stable. The proof of the converse (i.e. existence of constant scalar curvature Kähler metrics on K-stable manifolds) remains an important open problem.