The definition of a metric

Definition – Metric

A metric on a set X is a function d that assigns a real number to each pair of elements of X in such a way that the following properties hold.

1. Non-negativity: $d(x, y) \geq 0$ with equality if and only if $x = y$.
2. Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. Triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

- A metric space is a pair (X, d), where X is a set and d is a metric defined on X. The metric is often regarded as a distance function.
- The usual metric on $\mathbb{R}$ is the one given by $d(x, y) = |x - y|$.
- A metric can be used to define limits and continuity of functions. In fact, the ε-δ definition for functions on $\mathbb{R}$ can be easily adjusted so that it applies to functions on an arbitrary metric space.

Examples of metrics in $\mathbb{R}^k$

- The usual metric in $\mathbb{R}^k$ is the Euclidean metric d_2 defined by

$$d_2(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^{k} |x_i - y_i|^2 \right]^{1/2}.$$

- The metric d_1 is defined using the formula

$$d_1(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{k} |x_i - y_i|.$$

- One may define a metric d_p for each $p \geq 1$ by setting

$$d_p(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^{k} |x_i - y_i|^p \right]^{1/p}.$$

- Finally, there is a metric d_∞ which is defined by

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max_{1 \leq i \leq k} |x_i - y_i|.$$

Examples of other metrics

- The discrete metric on a nonempty set X is defined by letting

$$d(x,y) = \left\{ \begin{array}{ll} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{array} \right\}.$$

- Let $C[a,b]$ denote the set of all continuous functions $f\colon [a,b] \to \mathbb{R}$. A metric on $C[a,b]$ is then given by the formula

$$d_1(f,g) = \int_a^b |f(x) - g(x)| \, dx.$$

- Another metric on $C[a,b]$ is given by the formula

$$d_\infty(f,g) = \sup_{a \leq x \leq b} |f(x) - g(x)|.$$

Here, the supremum could also be replaced by a maximum.

Technical inequalities

Theorem 1.1 – Technical inequalities

Suppose that $x, y \geq 0$ and let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be arbitrary vectors in $\mathbb{R}^k$.

1. **Young's inequality**: If $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then
$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$
2. **Hölder's inequality**: If $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then
$$\sum_{i=1}^{k} |a_i| \cdot |b_i| \leq \left[\sum_{i=1}^{k} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{k} |b_i|^q\right]^{1/q}.$$
3. **Minkowski's inequality**: If $p > 1$, then
$$d_p(\boldsymbol{a}, \boldsymbol{b}) \leq d_p(\boldsymbol{a}, \boldsymbol{c}) + d_p(\boldsymbol{c}, \boldsymbol{b}).$$

Open balls

Definition – Open ball

Suppose (X, d) is a metric space and let $x \in X$ be an arbitrary point. The open ball with centre x and radius $r > 0$ is defined as

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

- The open balls in $\mathbb{R}$ are the open intervals $B(x, r) = (x - r, x + r)$. The open interval (a, b) has centre $(b + a)/2$ and radius $(b - a)/2$.
- If the metric on X is discrete, then $B(x, 1) = \{x\}$ for all $x \in X$.
- The open ball $B(0, 1)$ in $X = [0, 2]$ is given by $B(0, 1) = [0, 1)$.

Definition – Bounded

Let (X, d) be a metric space and $A \subset X$. We say that A is bounded, if there exist a point $x \in X$ and some $r > 0$ such that $A \subset B(x, r)$.

Open sets

Definition – Open set

Given a metric space (X, d), we say that a subset $U \subset X$ is open in X if, for each point $x \in U$ there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. In other words, each $x \in U$ is the centre of an open ball that lies in U.

Theorem 1.2 – Main facts about open sets

1. If X is a metric space, then both $\varnothing$ and X are open in X.
2. Arbitrary unions of open sets are open.
3. Finite intersections of open sets are open.
4. Every open ball is an open set.
5. A set is open if and only if it is a union of open balls.

- Infinite intersections of open sets are not necessarily open.
- If the metric on X is discrete, then every subset of X is open in X.

Convergence of sequences

Definition – Convergence

Let (X, d) be a metric space. We say that a sequence $\{x_n\}$ of points of X converges to the point $x \in X$ if, given any $\varepsilon > 0$ there exists an integer N such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.

- When a sequence $\{x_n\}$ converges to a point x, we say that x is the limit of the sequence and we write $x_n \to x$ as $n \to \infty$ or simply
$$\lim_{n\to\infty} x_n = x.$$
- A sequence $\boldsymbol{x}_n = (x_{n1}, x_{n2}, \ldots, x_{nk})$ of points in $\mathbb{R}^k$ converges if and only if each of the components x_{ni} converges in $\mathbb{R}$.

Theorem 1.3 – Limits are unique

The limit of a sequence in a metric space is unique. In other words, no sequence may converge to two different limits.

Closed sets

Definition – Closed set

Suppose (X, d) is a metric space and let $A \subset X$. We say that A is closed in X, if its complement $X - A$ is open in X.

Theorem 1.4 – Main facts about closed sets

1. If a subset $A \subset X$ is closed in X, then every sequence of points of A that converges must converge to a point of A.
2. Both $\varnothing$ and X are closed in X.
3. Finite unions of closed sets are closed.
4. Arbitrary intersections of closed sets are closed.

- The last two statements can be established using De Morgan's laws

$$X - \bigcup_i U_i = \bigcap_i (X - U_i), \qquad X - \bigcap_i U_i = \bigcup_i (X - U_i).$$

Continuity in metric spaces

Definition – Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is continuous at $x \in X$ if, given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \quad \Longrightarrow \quad d_Y(f(x), f(y)) < \varepsilon.$$

We also say that f is continuous, if f is continuous at all points.

- One may express the above definition in terms of open balls as

$$y \in B(x, \delta) \quad \Longrightarrow \quad f(y) \in B(f(x), \varepsilon).$$

- If $f\colon X \to Y$ is a constant function, then f is continuous.
- Every function $f\colon X \to Y$ is continuous, if d_X is discrete.

Theorems involving continuity

Theorem 1.5 – Composition of continuous functions

Suppose $f: X \to Y$ and $g: Y \to Z$ are continuous functions between metric spaces. Then the composition $g \circ f: X \to Z$ is continuous.

Theorem 1.6 – Continuity and sequences

Suppose $f: X \to Y$ is a continuous function between metric spaces and let $\{x_n\}$ be a sequence of points of X which converges to $x \in X$. Then the sequence $\{f(x_n)\}$ must converge to $f(x)$.

Theorem 1.7 – Continuity and open sets

A function $f: X \to Y$ between metric spaces is continuous if and only if $f^{-1}(U)$ is open in X for each set U which is open in Y.

Lipschitz continuity

Definition – Lipschitz continuous

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is Lipschitz continuous, if there is a constant $L \geq 0$ such that

$$d_Y(f(x), f(y)) \leq L \cdot d_X(x, y) \quad \text{for all } x, y \in X.$$

Theorem 1.8 – Main facts about Lipschitz continuity

1. Every Lipschitz continuous function is continuous.
2. If a function $f\colon [a, b] \to \mathbb{R}$ is differentiable and its derivative is bounded, then f is Lipschitz continuous on $[a, b]$.

- The function $f(x) = x^2$ is Lipschitz continuous on $[0, 1]$.
- The function $f(x) = \sqrt{x}$ is not Lipschitz continuous on $[0, 1]$.

Convergence of functions

Definition – Pointwise and uniform convergence

Let $\{f_n(x)\}$ be a sequence of functions $f_n \colon X \to \mathbb{R}$, where X is a metric space. We say that $f_n(x)$ converges pointwise to $f(x)$ if, given any $\varepsilon > 0$ there exists an integer N such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \geq N.$$

We also say that f_n converges to f uniformly on X if, given any $\varepsilon > 0$ there exists an integer N such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \geq N \text{ and all } x \in X.$$

- For pointwise convergence, one gets to choose N depending on x.
- For uniform convergence, the same choice of N should work for all x.
- If a sequence converges uniformly, then it also converges pointwise.

Pointwise and uniform convergence

Theorem 1.9 – Pointwise and uniform convergence

1. To say that $f_n(x) \to f(x)$ pointwise is to say that

$$|f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$

2. To say that $f_n \to f$ uniformly on X is to say that

$$\sup_{x \in X} |f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$

Theorem 1.10 – Uniform limit of continuous functions

The uniform limit of continuous functions is continuous: if each f_n is continuous and $f_n \to f$ uniformly on X, then f is continuous on X.

- The pointwise limit of continuous functions need not be continuous. For instance, x^n converges to 0 if $0 \leq x < 1$ and to 1 if $x = 1$.

Cauchy sequences

Definition – Cauchy sequence

Let (X, d) be a metric space. A sequence $\{x_n\}$ of points of X is called Cauchy if, given any $\varepsilon > 0$ there exists an integer N such that

$$d(x_m, x_n) < \varepsilon \quad \text{for all } m, n \geq N.$$

Theorem 1.11 – Convergent implies Cauchy

In a metric space, every convergent sequence is a Cauchy sequence.

Theorem 1.12 – Cauchy implies bounded

In a metric space, every Cauchy sequence is bounded.

- A Cauchy sequence does not have to be convergent. For instance, the sequence $x_n = 1/n$ is Cauchy but not convergent in $X = (0, 2)$.

Completeness of $\mathbb{R}$

Definition – Complete metric space

A metric space (X, d) is called complete if every Cauchy sequence of points of X actually converges to a point of X.

Theorem 1.13 – Cauchy sequence with convergent subsequence

Suppose (X, d) is a metric space and let $\{x_n\}$ be a Cauchy sequence in X that has a convergent subsequence. Then $\{x_n\}$ converges itself.

Theorem 1.14 – Completeness of $\mathbb{R}$

1. Every sequence in $\mathbb{R}$ which is monotonic and bounded converges.
2. **Bolzano-Weierstrass theorem**: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.
3. The set $\mathbb{R}$ of all real numbers is a complete metric space.

Completeness

Theorem 1.15 – Examples of complete metric spaces

1. The space $\mathbb{R}^k$ is complete with respect to its usual metric.
2. The space $C[a, b]$ is complete with respect to the d_∞ metric.

- The space $\mathbb{R}^k$ is complete with respect to any d_p metric. One can prove this fact by noting that $d_\infty(\boldsymbol{x}, \boldsymbol{y}) \leq d_p(\boldsymbol{x}, \boldsymbol{y}) \leq k^{1/p} d_\infty(\boldsymbol{x}, \boldsymbol{y})$.
- The space $C[a, b]$ is not complete with respect to the d_1 metric. One can find Cauchy sequences that converge to a discontinuous function.
- The set $A = \{1/n : n \in \mathbb{N}\}$ is not complete. It contains a sequence which converges in $\mathbb{R}$, but this sequence does not converge in A.

Theorem 1.16 – Subsets of a complete metric space

Suppose (X, d) is a complete metric space and let $A \subset X$. Then A is complete if and only if A is closed in X.

Banach's fixed point theorem

Definition – Contraction

Let (X,d) be a metric space. We say that a function $f\colon X \to X$ is a contraction, if there exists a constant $0 \leq \alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha \cdot d(x,y) \quad \text{for all } x, y \in X.$$

Theorem 1.17 – Banach's fixed point theorem

If $f\colon X \to X$ is a contraction on a complete metric space X, then f has a unique fixed point, namely a unique point x with $f(x) = x$.

- Every contraction is Lipschitz continuous, hence also continuous.
- Consider the function $f\colon (0,1) \to (0,1)$ defined by $f(x) = x/2$. This is easily seen to be a contraction, but it has no fixed point on $(0,1)$. Thus, one does need X to be complete for the theorem to hold.

Application in differential equations

Theorem 1.18 – Existence and uniqueness of solutions

Consider an initial value problem of the form

$$y'(t) = f(t, y(t)), \qquad y(0) = y_0.$$

If f is continuous in t and Lipschitz continuous in y, then there exists a unique solution $y(t)$ which is defined on $[0, \varepsilon]$ for some $\varepsilon > 0$.

- To say that $y(t)$ is a solution is to say that $y(t)$ is a fixed point of

$$\mathcal{A}(y(t)) = y_0 + \int_0^t f(s, y(s))\, ds.$$

- In general, solutions of differential equations need not be defined for all times. For instance, $y(t) = 1/(1-t)$ is the unique solution of

$$y'(t) = y(t)^2, \qquad y(0) = 1.$$

  This solution is defined at time $t = 0$ but not at time $t = 1$.

Completion of a metric space

Theorem 1.19 – Completion of a metric space

Given a metric space (X, d), there exist a metric space (X', d') and a distance preserving map $f\colon X \to X'$ such that X' is complete.

- A distance preserving map is called an isometry, while X' is called a completion of X. It is easy to check that every distance preserving map is injective. Thus, one can always regard X as a subset of X'.
- The proof of this theorem is somewhat long, but the general idea is to define a relation on the set of Cauchy sequences in X by letting

$$\{x_n\} \sim \{y_n\} \quad \Longleftrightarrow \quad \lim_{n\to\infty} d(x_n, y_n) = 0.$$

This turns out to be an equivalence relation and the completion X' is the set of all equivalence classes with metric d' defined by

$$d'([x_n], [y_n]) = \lim_{n\to\infty} d(x_n, y_n).$$

Topological space

Definition – Topology

A topology T on a set X is a collection of subsets of X such that

1. The topology T contains both the empty set $\varnothing$ and X.
2. Every union of elements of T belongs to T.
3. Every finite intersection of elements of T belongs to T.

- A topological space (X, T) consists of a set X and a topology T.
- Every metric space (X, d) is a topological space. In fact, one may define a topology to consist of all sets which are open in X. This particular topology is said to be induced by the metric.
- The elements of a topology are often called open. This terminology may be somewhat confusing, but it is quite standard. To say that a set U is open in a topological space (X, T) is to say that $U \in T$.

Examples of topological spaces

- The discrete topology on a set X is defined as the topology which consists of all possible subsets of X.
- The indiscrete topology on a set X is defined as the topology which consists of the subsets $\varnothing$ and X only.
- Every metric space (X, d) has a topology which is induced by its metric. It consists of all subsets of X which are open in X.

Definition – Metrisable space

A topological space (X, T) is called metrisable, if there exists a metric on X such that the topology T is induced by this metric.

- The discrete topology on X is metrisable and it is actually induced by the discrete metric. On the other hand, the indiscrete topology on X is not metrisable, if X has two or more elements.

Convergence of sequences

Definition – Convergence

Let (X, T) be a topological space. A sequence $\{x_n\}$ of points of X is said to converge to the point $x \in X$ if, given any open set U that contains x, there exists an integer N such that $x_n \in U$ for all $n \geq N$.

- When a sequence $\{x_n\}$ converges to a point x, we say that x is the limit of the sequence and we write $x_n \to x$ as $n \to \infty$ or simply

$$\lim_{n\to\infty} x_n = x.$$

- When X is a metric space, this new definition of convergence agrees with the definition of convergence in metric spaces.

Theorem 2.1 – Limits are not necessarily unique

Suppose that X has the indiscrete topology and let $x \in X$. Then the constant sequence $x_n = x$ converges to y for every $y \in X$.

Closed sets

Definition – Closed set

Suppose (X, T) is a topological space and let $A \subset X$. We say that A is closed in X, if its complement $X - A$ is open in X.

Theorem 2.2 – Main facts about closed sets

1. If a subset $A \subset X$ is closed in X, then every sequence of points of A that converges must converge to a point of A.
2. Both $\varnothing$ and X are closed in X.
3. Finite unions of closed sets are closed.
4. Arbitrary intersections of closed sets are closed.

- We have already established these statements for metric spaces and our proofs apply almost verbatim in the case of topological spaces.

Closure of a set

Definition – Closure

Suppose (X, T) is a topological space and let $A \subset X$. The closure $\overline{A}$ of A is defined as the smallest closed set that contains A. It is thus the intersection of all closed sets that contain A.

- The interval $A = [0, 1)$ has closure $\overline{A} = [0, 1]$.
- The interval $A = (0, 1)$ has closure $\overline{A} = [0, 1]$.

Theorem 2.3 – Main facts about the closure

1. One has $A \subset \overline{A}$ for any set A.
2. If $A \subset B$, then $\overline{A} \subset \overline{B}$ as well.
3. The set A is closed if and only if $\overline{A} = A$.
4. The closure of $\overline{A}$ is itself, namely $\overline{\overline{A}} = \overline{A}$.

Interior of a set

Definition – Interior

Suppose (X, T) is a topological space and let $A \subset X$. The interior A° of A is defined as the largest open set contained in A. It is thus the union of all open sets contained in A.

- The interval $A = [0, 1]$ has interior $A^\circ = (0, 1)$.
- The interval $A = [0, 1)$ has interior $A^\circ = (0, 1)$.

Theorem 2.4 – Main facts about the interior

1. One has $A^\circ \subset A$ for any set A.
2. If $A \subset B$, then $A^\circ \subset B^\circ$ as well.
3. The set A is open if and only if $A^\circ = A$.
4. The interior of A° is itself, namely $(A^\circ)^\circ = A^\circ$.

Boundary of a set

Definition – Boundary

Suppose (X, T) is a topological space and let $A \subset X$. The boundary of A is defined as the set $\partial A = \overline{A} \cap \overline{X - A}$.

Definition – Neighbourhood

Suppose (X, T) is a topological space and let $x \in X$ be an arbitrary point. A neighbourhood of x is simply an open set that contains x.

Theorem 2.5 – Characterisation of closure/interior/boundary

Suppose (X, T) is a topological space and let $A \subset X$.

1. $x \in \overline{A} \iff$ every neighbourhood of x intersects A.
2. $x \in A^\circ \iff$ some neighbourhood of x lies within A.
3. $x \in \partial A \iff$ every neighbourhood of x intersects A and $X - A$.

Interior, closure and boundary: examples

Theorem 2.6 – Interior, closure and boundary

One has $A^\circ \cap \partial A = \varnothing$ and also $A^\circ \cup \partial A = \overline{A}$ for any set A.

Set	**Interior**	**Closure**	**Boundary**
$\{1\}$	$\varnothing$	$\{1\}$	$\{1\}$
$[0,1)$	$(0,1)$	$[0,1]$	$\{0,1\}$
$(0,1)\cup(1,2)$	$(0,1)\cup(1,2)$	$[0,2]$	$\{0,1,2\}$
$[0,1]\cup\{2\}$	$(0,1)$	$[0,1]\cup\{2\}$	$\{0,1,2\}$
$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{Q}$	$\varnothing$	$\mathbb{R}$	$\mathbb{R}$
$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\varnothing$

Limit points

Definition – Limit point

Let (X,T) be a topological space and let $A \subset X$. We say that x is a limit point of A if every neighbourhood of x intersects A at a point other than x.

Theorem 2.7 – Limit points and closure

Let (X,T) be a topological space and let $A \subset X$. If A' is the set of all limit points of A, then the closure of A is $\overline{A} = A \cup A'$.

- Intuitively, limit points of A are limits of sequences of points of A.
- The set $A = \{1/n : n \in \mathbb{N}\}$ has only one limit point, namely $x = 0$.
- Every point of $A = (0,1)$ is a limit point of A, while $A' = [0,1]$.
- A set is closed if and only if it contains its limit points.

Continuity in topological spaces

Definition – Continuity

A function $f\colon X \to Y$ between topological spaces is called continuous if $f^{-1}(U)$ is open in X for each set U which is open in Y.

Theorem 2.8 – Composition of continuous functions

Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions between topological spaces. Then the composition $g \circ f\colon X \to Z$ is continuous.

Theorem 2.9 – Continuity and sequences

Let $f\colon X \to Y$ be a continuous function between topological spaces and let $\{x_n\}$ be a sequence of points of X which converges to $x \in X$. Then the sequence $\{f(x_n)\}$ must converge to $f(x)$.

Subspace topology

Definition – Subspace topology

Let (X, T) be a topological space and let $A \subset X$. Then the set

$$T' = \{U \cap A : U \in T\}$$

forms a topology on A which is known as the subspace topology.

Theorem 2.10 – Inclusion maps are continuous

Let (X, T) be a topological space and let $A \subset X$. Then the inclusion map $i \colon A \to X$ which is defined by $i(x) = x$ is continuous.

Theorem 2.11 – Restriction maps are continuous

Let $f \colon X \to Y$ be a continuous function between topological spaces and let $A \subset X$. Then the restriction map $g \colon A \to Y$ which is defined by $g(x) = f(x)$ is continuous. This map is often denoted by $g = f|_A$.

Product topology

Definition – Product topology

Given two topological spaces (X, T) and (Y, T'), we define the product topology on $X \times Y$ as the collection of all unions $\bigcup_i U_i \times V_i$, where each U_i is open in X and each V_i is open in Y.

Theorem 2.12 – Projection maps are continuous

Let (X, T) and (Y, T') be topological spaces. If $X \times Y$ is equipped with the product topology, then the projection map $p_1 \colon X \times Y \to X$ defined by $p_1(x, y) = x$ is continuous. Moreover, the same is true for the projection map $p_2 \colon X \times Y \to Y$ defined by $p_2(x, y) = y$.

Theorem 2.13 – Continuous map into a product space

Let X, Y, Z be topological spaces. Then a function $f \colon Z \to X \times Y$ is continuous if and only if its components $p_1 \circ f$, $p_2 \circ f$ are continuous.

Hausdorff spaces

Definition – Hausdorff space

We say that a topological space (X, T) is Hausdorff if any two distinct points of X have neighbourhoods which do not intersect.

- If a space X has the discrete topology, then X is Hausdorff.
- If a space X has the indiscrete topology and it contains two or more elements, then X is not Hausdorff.

Theorem 2.14 – Main facts about Hausdorff spaces

1. Every metric space is Hausdorff.
2. Every subset of a Hausdorff space is Hausdorff.
3. Every finite subset of a Hausdorff space is closed.
4. The product of two Hausdorff spaces is Hausdorff.
5. A convergent sequence in a Hausdorff space has a unique limit.

Connected spaces, part 1

Definition – Connected

Two sets A, B form a partition $A|B$ of a topological space (X, T), if they are nonempty, open and disjoint with $A \cup B = X$. We say that the space X is connected, if it has no such partition $A|B$.

Theorem 2.15 – Some facts about connected spaces

1. To say that X is connected is to say that the only subsets of X which are both open and closed in X are the subsets $\varnothing, X$.
2. The continuous image of a connected space is connected: if X is connected and $f\colon X \to Y$ is continuous, then $f(X)$ is connected.
3. A subset of $\mathbb{R}$ is connected if and only if it is an interval.
4. If a connected space A is a subset of X and the sets U, V form a partition of X, then A must lie entirely within either U or V.

Connected spaces, part 2

Theorem 2.16 – Some more facts about connected spaces

1. If A is a connected subset of X, then $\overline{A}$ is connected as well.
2. Consider a collection of connected sets U_i that have a point in common. Then the union of these sets is connected as well.
3. The product of two connected spaces is connected.

Definition – Connected component

Let (X, T) be a topological space. The connected component of a point $x \in X$ is the largest connected subset of X that contains x.

Theorem 2.17 – Connected components are closed

Let (X, T) be a topological space. Then X is the disjoint union of its connected components and each connected component is closed in X.

Compact spaces, part 1

Definition – Compactness

Let (X, T) be a topological space and let $A \subset X$. An open cover of A is a collection of open sets whose union contains A. An open subcover is a subcollection which still forms an open cover. We say that A is compact if every open cover of A has a finite subcover.

- The intervals $(-n, n)$ with $n \in \mathbb{N}$ form an open cover of $\mathbb{R}$, but this cover has no finite subcover, so $\mathbb{R}$ is not compact.
- Suppose $\{x_n\}$ is a sequence that converges to the point x. Then the set $A = \{x, x_1, x_2, x_3, \ldots\}$ is easily seen to be compact.

Theorem 2.18 – Compactness and convergence

Suppose that X is a compact metric space. Then every sequence in X has a convergent subsequence.

Theorem 2.19 – Main facts about compact spaces

1. A compact subset of a Hausdorff space is closed.
2. A closed subset of a compact space is compact.
3. The interval $[a, b]$ is compact for all real numbers $a < b$.
4. The continuous image of a compact space is compact: if X is compact and $f\colon X \to Y$ is continuous, then $f(X)$ is compact.
5. If X is compact and $f\colon X \to \mathbb{R}$ is continuous, then f is bounded.
6. If X is compact and $f\colon X \to \mathbb{R}$ is continuous, then there exist points $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in X$.
7. The product of two compact spaces is compact.

Theorem 2.20 – Heine-Borel theorem

A subset of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.

Homeomorphisms

Definition – Homeomorphism

A function $f\colon X \to Y$ between topological spaces is a homeomorphism if f is bijective, continuous and its inverse f^{-1} is continuous. When such a function exists, we say that X and Y are homeomorphic.

Theorem 2.21 – Main facts about homeomorphisms

1. Consider two homeomorphic topological spaces. If one of them is connected or compact or Hausdorff, then so is the other.
2. Suppose $f\colon X \to Y$ is bijective and continuous. If X is compact and Y is Hausdorff, then f is a homeomorphism.

- Every open interval (a, b) is homeomorphic to $\mathbb{R}$. Thus, a complete space can be homeomorphic with a space which is not complete.
- There is no closed interval $[a, b]$ that is homeomorphic to $\mathbb{R}$ because the former space is compact and the latter space is not.

Uniform continuity in metric spaces

Definition – Uniformly continuous

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is uniformly continuous if, given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \quad \Longrightarrow \quad d_Y(f(x), f(y)) < \varepsilon \qquad \text{for all } x, y \in X.$$

Theorem 2.22 – Main facts about uniform continuity

1. Every Lipschitz continuous function is uniformly continuous.
2. Every uniformly continuous function is continuous.
3. When X is compact, a function $f\colon X \to Y$ is continuous on X if and only if it is uniformly continuous on X.

- $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$ but not Lipschitz.
- $f(x) = 1/x$ is continuous on $(0, \infty)$ but not uniformly continuous.