Chapter 1

Mathematical modelling of fluids

Contents

1.1 Introduction

Mechanics is a field of science that encompasses statics, kinematics and dynamics. Statics is the study of forces and toques when there is no motion. Kinematics is the study of motion regardless of its causes. Dynamics is the study of forces and torques where there is motion. Fluid dynamics is thus the study of the motion of liquids, gases and plasmas (e.g. water, air, interstellar plasma). While it is a relatively old subject (dating back to the 18th century with Newton, Euler and Lagrange for example), it is still a very active research area. Here are some currently very active research areas:

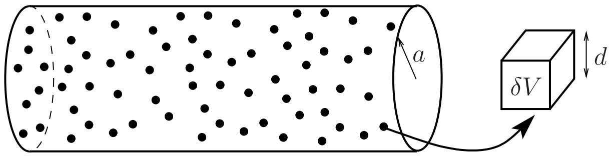
- biological fluids – blood and air flows, swimming organisms, drug delivery
- aerodynamics and hydrodynamics – aeroplanes, ships
- industrial fluids – casting, injection molding, mixing
- environmental fluids – pollution, water and wind power
- geophysical fluids – earth's core, atmosphere and ocean, weather
- astrophysical fluids – galaxies, stars, interstellar medium

Fundamentally all these fluids obey the same physical laws of motion, however, they differ widely in the lengthscale they display and in some of their physical properties (density, viscosity). As a consequence, fluids can show different types of motion depending upon what effects are dominant.

1.2 Continuum hypothesis

(Ref.: Paterson, §III.1)

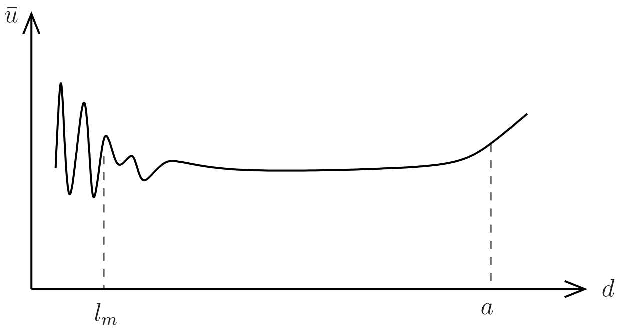
In fluid dynamics, we do not attempt to calculate the motion of individual molecules — there are far too many of them, (a cubic centimetre of water contains of the order of 10^{23} molecules of typical size $l_m \simeq 1\mathring{A} = 10^{-10}$ m) and their individual motion is dominated by high frequency fluctuations caused by collisions with neighbouring molecules. Instead, we represent the average motion of a "blob" of fluid called a *fluid particle* of length d and volume δV.



We choose d so that:

- $d \gg l_m$ (molecular scale) – δV contains many molecules and the fluctuations due to individual motions are averaged out.

- $d \ll a$ (macroscopic scale) – δV is approximately a point in space.



Given this choice ($l_m \ll d \ll a$), the average velocity $\bar{u}$ is a smooth function of the variables and independent of d.

Continuum hypothesis: Molecular details can be smoothed out by assigning to the velocity at a point P the average velocity in a fluid element δV centred in P.

We can thus define the velocity field $\mathbf{u}(\mathbf{x}, t)$ as a smooth function of time and position, i.e., $\mathbf{u}$ is differentiable and integrable. Note that shock waves break this assumption.

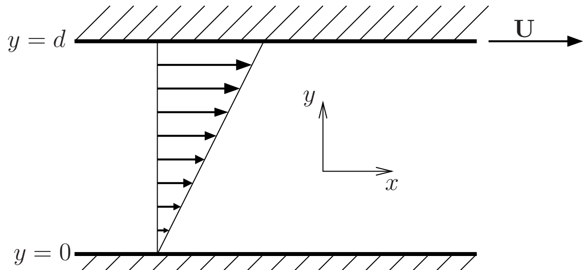
Similarly, $\rho(\mathbf{x}, t) = \dfrac{\text{mass in } \delta V}{\delta V}$ is the local density of mass.

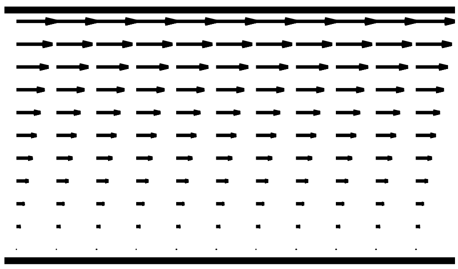
1.3 Velocity field

The fluid velocity is defined, within the continuum hypothesis, as the *vector field* $\mathbf{u}(\mathbf{x}, t)$, function of space and time.

Example 1.1
Shear flow: consider the flow between two parallel plates when one is moved relative to the other with a constant velocity U.

$$\mathbf{u} = \begin{pmatrix} \frac{Uy}{d} \\ 0 \\ 0 \end{pmatrix} = \frac{Uy}{d}\hat{\boldsymbol{\imath}}.$$



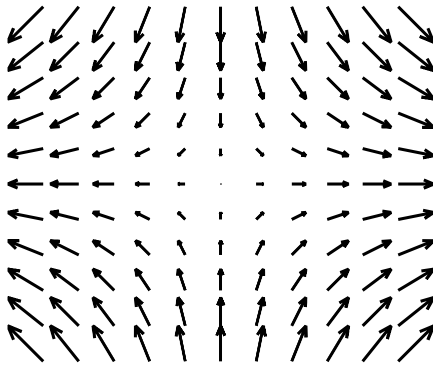


The direction of the flow is indicated by an arrow and its magnitude by the arrow length.

Stagnation-point flow: consider a flow where the velocity cancels at the origin.

$$\mathbf{u} = \begin{pmatrix} Ex \\ -Ey \\ 0 \end{pmatrix} = Ex\hat{\boldsymbol{\imath}} - Ey\hat{\boldsymbol{\jmath}}.$$

The point $\mathbf{x} = 0$ where $\mathbf{u} = 0$ is called a *stagnation point.*



1.3.1 Particle paths

(Ref.: Paterson, §III.2)

One method for visualising fluid motion is to follow the motion of a "tracer" particle in the flow.
Let a particle be released at time t_0 and at position $\mathbf{x}_0$ within the velocity field. Since the particle moves with the fluid velocity:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}, t) \quad \text{such that} \quad \mathbf{x} = \mathbf{x}_0 \text{ at } t = t_0. \tag{1.1}$$

Example 1.2 (Stagnation point flow)

$$\mathbf{u}(\mathbf{x}, t) = \begin{pmatrix} Ex \\ -Ey \\ 0 \end{pmatrix} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = Ex, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -Ey, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = 0$$

$$\Rightarrow x(t) = x_0\mathrm{e}^{Et}, \quad y(t) = y_0\mathrm{e}^{-Et}, \quad z(t) = z_0 \quad \text{if } \mathbf{x} = \mathbf{x}_0 = (x_0, y_0, z_0) \text{ at } t = t_0.$$

Note that particles at the stagnation point $x_0 = y_0 = z_0 = 0$ do not move since $\mathbf{u} = 0$.

The time variable, t, can be eliminated to show that particle paths are hyperbolae of equation $y = \dfrac{x_0 y_0}{x}$.

1.3.2 Streamlines

A streamline is a line everywhere tangent to the local fluid velocity at time t. If the line is parametrised by a parameter s ("distance" along the streamline), then:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{u}(\mathbf{x}(s), t), \tag{1.2}$$

or, equivalently:

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} = \frac{\mathrm{d}z}{w} \quad (= \mathrm{d}s), \tag{1.3}$$

since $\mathbf{u}(\mathbf{x}, t)$ is not explicitly function of s. If the flow is steady ($\partial \mathbf{u}/\partial t = 0$), then the streamlines are the same as the particle paths. Note that the converse is not necessarily true.

1.4 Time derivatives

The time derivative $\partial \mathbf{u}/\partial t$ measures the rate of change of velocity at the fixed position $\mathbf{x}$. This is referred to as the *Eulerian* time-derivative. However, this does not give the acceleration of a fluid particle at this point, since the particle is moving through this point along its particle path. Instead we require the *convective* derivative (also called Lagrangian derivative and material derivative) $\mathrm{D}\mathbf{u}(\mathbf{x}, t)/\mathrm{D}t$, which is the rate of change of $\mathbf{u}$ along the particle path $\mathbf{x}(t)$, i.e., moving with the fluid.

Using the chain rule:

$$\begin{aligned}
\frac{\mathrm{D}f}{\mathrm{D}t} &\equiv \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t), t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\
\Rightarrow \frac{\mathrm{D}f}{\mathrm{D}t} &= \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} \\
\Rightarrow \frac{\mathrm{D}f}{\mathrm{D}t} &= \frac{\partial f}{\partial t} + (\mathbf{u}\cdot\nabla) f. \qquad (1.4)
\end{aligned}$$

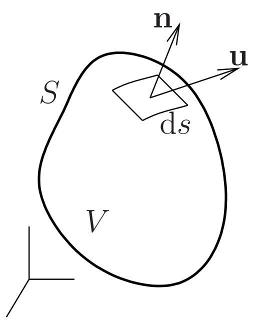
Hence the acceleration of a fluid particle is:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}. \tag{1.5}$$

1.5 Conservation of mass

In any situation, the mass of a fluid must be conserved. For a continuous material, this principle is expressed in the form of the *continuity equation*.

Consider a volume V, fixed in space, with surface S and outward normal $\mathbf{n}$.



The total mass in V is:

$$M_V = \int_V \rho \, \mathrm{d}V,$$

where ρ is the density of mass (mass per unit volume).

M_V can only change if mass is carried inside or outside the volume by the fluid.

The mass flowing through the surface per unit time (i.e. the mass flux) is:

$$\frac{\mathrm{d}M_V}{\mathrm{d}t} = -\int_S \rho \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S,$$

and therefore:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V = \int_V \frac{\partial\rho}{\partial t}\,\mathrm{d}V = -\int_S \rho\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S, \quad \text{since } V \text{ is fixed.}$$

Applying the divergence theorem:

$$\int_V \frac{\partial\rho}{\partial t}\,\mathrm{d}V = -\int_V \nabla\cdot(\rho\mathbf{u})\,\mathrm{d}V \Rightarrow \int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right]\mathrm{d}V = 0.$$

Since V is arbitrary, this equation must hold for all volume V. Thus, the *continuity equation*:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{1.6}$$

holds at all points in the fluid. Expanding the divergence as $\nabla\cdot(\rho\mathbf{u}) = \rho\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\rho$, we obtain the Lagrangian form of the continuity equation:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{u} = 0. \tag{1.7}$$

The density of a fluid particle moving with the fluid only changes if there is an expansion or a contraction of the flow.

1.6 Incompressibility

In an *incompressible* fluid, the density of each fluid particle remains constant and the continuity equation (1.7) reduces to:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \Leftrightarrow \rho\nabla\cdot\mathbf{u} = 0.$$

So, for an incompressible flow:

$$\nabla\cdot\mathbf{u} = 0. \tag{1.8}$$

This places a restriction on the form of the fluid velocity:

$$\mathbf{u} = \nabla\times\mathbf{\Psi}, \tag{1.9}$$

for some vector field $\mathbf{\Psi}$. Since $\nabla\cdot(\nabla\times\mathbf{\Psi}) = 0$ for any vector field $\mathbf{\Psi}$, this automatically satisfies the continuity equation.

1.6.1 Two dimensional flows

Let us consider the flow:

$$\mathbf{u} = \begin{pmatrix} u(x,y) \\ v(x,y) \\ 0 \end{pmatrix} = u(x,y)\,\hat{\boldsymbol{\imath}} + v(x,y)\,\hat{\boldsymbol{\jmath}}.$$

We can introduce $\mathbf{\Psi} = \psi(x,y)\hat{\boldsymbol{k}}$, where $\psi(x,y)$ is a scalar function such that:

$$u = \frac{\partial\psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial\psi}{\partial x}. \tag{1.10}$$

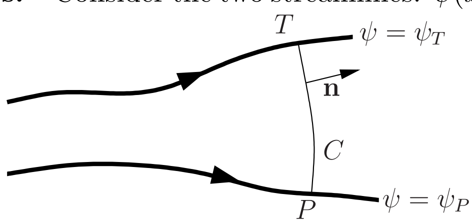
The function $\psi(x,y)$ is called the streamfunction and has a number of important properties.

Streamlines: the gradient of the streamfunction is orthogonal to the velocity field:

$$\mathbf{u} \cdot \nabla\psi = u\frac{\partial\psi}{\partial x} + v\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} = 0.$$

A streamline is a line on which the streamfunction is constant. Thus, on a streamline: $\mathrm{d}\psi = 0 \Rightarrow \dfrac{\partial\psi}{\partial x}\mathrm{d}x + \dfrac{\partial\psi}{\partial y}\mathrm{d}y = -v\mathrm{d}x + u\mathrm{d}y = 0$. So, $\mathbf{u} \times \mathbf{dl} = 0$, i.e., the streamline element $\mathbf{dl} = (\mathrm{d}x, \mathrm{d}y)$ is parallel to $\mathbf{u}$.

Flux between streamlines: Consider the two streamlines: $\psi(x, y) = \psi_P$ and $\psi(x, y) = \psi_T$.

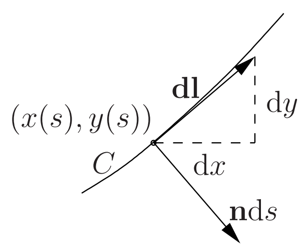


The fluid flow or *flux* through $C : \{x(s), y(s)\}$, an arbitrary curve connecting P and T, is:

$$Q = \int_P^T \mathbf{u} \cdot \mathbf{n}\,\mathrm{d}s. \tag{1.11}$$

Let $\mathbf{dl} = \mathrm{d}x\hat{\boldsymbol{\imath}} + \mathrm{d}y\hat{\boldsymbol{\jmath}} = \mathrm{d}s\left(\dfrac{\mathrm{d}x}{\mathrm{d}s}\hat{\boldsymbol{\imath}} + \dfrac{\mathrm{d}y}{\mathrm{d}s}\hat{\boldsymbol{\jmath}}\right)$ be an infinitesimal displacement along the curve C. We define the infinitesimal vector normal to $\mathbf{dl}$:

$$\mathbf{n}\,\mathrm{d}s = \mathrm{d}y\hat{\boldsymbol{\imath}} - \mathrm{d}x\hat{\boldsymbol{\jmath}} = \left(\frac{\mathrm{d}y}{\mathrm{d}s}\hat{\boldsymbol{\imath}} - \frac{\mathrm{d}x}{\mathrm{d}s}\hat{\boldsymbol{\jmath}}\right)\mathrm{d}s.$$



$$\text{So,}\quad Q = \int_P^T \left(\frac{\partial\psi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial\psi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s}\right)\mathrm{d}s = \int_P^T \frac{\mathrm{d}\psi}{\mathrm{d}s}\mathrm{d}s = \int_{\psi_P}^{\psi_T} \mathrm{d}\psi = \psi_T - \psi_P.$$

Hence, the flux between two streamlines is equal to the streamfunction difference between the two streamlines. Consequently, the flow is faster when the streamlines are close together.

Following from the definition of the streamfunction: $\|\mathbf{u}\| = \|\nabla\psi\|$, which shows that the speed of the flow increases with the gradient of the streamfunction.

Example 1.3
A bath-plug vortex can be defined as $\mathbf{u} = \begin{pmatrix} \dfrac{y}{x^2+y^2} \\ \dfrac{-x}{x^2+y^2} \end{pmatrix}$. It is incompressible since:

$$\nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} + \frac{(-2y)(-x)}{(x^2+y^2)^2} = 0.$$

So, $\dfrac{\partial \psi}{\partial y} = u = \dfrac{y}{x^2+y^2} = \dfrac{1}{2}\dfrac{2y}{x^2+y^2} \Rightarrow \psi(x,y) = \dfrac{1}{2}\ln\left(x^2+y^2\right) + \alpha(x)$.
Then $v = -\dfrac{\partial \psi}{\partial x} = -\dfrac{x}{x^2+y^2} + \dfrac{\mathrm{d}\alpha}{\mathrm{d}x} \Rightarrow \dfrac{\mathrm{d}\alpha}{\mathrm{d}x} = 0$. So, α is constant and:

$$\psi(x,y) = \frac{1}{2}\ln\left(x^2+y^2\right) \quad \text{(choosing } \alpha = 0\text{)}.$$

This flow is easier to visualise if we use polar coordinates (r,θ) in the (x,y)-plane, so that the streamfunction becomes $\psi = \dfrac{1}{2}\ln\left(x^2+y^2\right) = \ln r$. The streamfunction is independent of θ which shows that the streamlines are circles about the origin.
In polar coordinates, the velocity field is:

$$\mathbf{u} = \nabla\times\left(\psi(r,\theta)\hat{\mathbf{e}}_{\boldsymbol{z}}\right) \;\Rightarrow u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r} \tag{1.12}$$

$$\Rightarrow u_r = 0, \quad u_\theta = -\frac{1}{r}. \tag{1.13}$$

1.6.2 Axisymmetric flows

We consider:

$$\mathbf{u} = u(r,z)\,\hat{\mathbf{e}}_{\boldsymbol{r}} + w(r,z)\,\hat{\mathbf{e}}_{\boldsymbol{z}}.$$

Examples include flows in a circular pipe or past a sphere. In this case $\boldsymbol{\Psi}$ is in the $\hat{\mathbf{e}}_{\boldsymbol{\theta}}$ direction and we can define:

$$\boldsymbol{\Psi} = \frac{1}{r}\Psi(r,z)\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}.$$

where $\Psi(r,z)$ is the *Stokes streamfunction* (using Ψ to distinguish from planar streamfunction ψ). Note the prefactor $\frac{1}{r}$ in the definition.
The fluid velocity is given by:

$$\mathbf{u} = \nabla\times\left(\frac{1}{r}\Psi(r,z)\,\hat{\mathbf{e}}_{\boldsymbol{\theta}}\right), \tag{1.14}$$

so, in cylindrical polar coordinates:

$$w(r,z) = \frac{1}{r}\frac{\partial \Psi}{\partial r} \quad \text{and} \quad u(r,z) = -\frac{1}{r}\frac{\partial \Psi}{\partial z}. \tag{1.15}$$

Stokes streamfunctions have properties analogous to planar streamfunctions.

i. <u>Ψ is constant on streamlines</u>

$$\begin{aligned}\mathbf{u}\cdot\nabla\Psi = u\frac{\partial \Psi}{\partial r} + w\frac{\partial \Psi}{\partial z} &= \frac{1}{r}\left(ru\frac{\partial \Psi}{\partial r} + rw\frac{\partial \Psi}{\partial z}\right)\\ &= \frac{1}{r}\left(-\frac{\partial \Psi}{\partial z}\frac{\partial \Psi}{\partial r} + \frac{\partial \Psi}{\partial r}\frac{\partial \Psi}{\partial z}\right)\\ &= 0.\end{aligned}$$

Thus, the gradient of Ψ is orthogonal to the velocity field. Moreover:

$$\begin{aligned} d\Psi = 0 &\Rightarrow \frac{\partial \Psi}{\partial r}dr + \frac{\partial \Psi}{\partial z}dz = 0\\ &\Rightarrow rwdr - rudz = 0\\ &\Rightarrow wdr - udz = 0\\ &\Rightarrow \mathbf{dl}\times\mathbf{u} = 0,\end{aligned}$$

implying that Ψ is constant in the direction of the flow.

For axisymmetric flows it is useful to think of *streamtubes*: surface of revolution spanned by all the streamlines through a circle about the axis of symmetry.

ii. Relation between volume flux and streamtubes
The volume flux, or fluid flow, between two streamtubes with $\Psi = \Psi_i$ and $\Psi = \Psi_o$ is:

$$Q = \int_S \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S = 2\pi(\Psi_o - \Psi_i). \tag{1.16}$$

Proof

$$\begin{aligned} Q = \int_S \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S &= \int_S \nabla \times \left(\frac{1}{r} \Psi \, \hat{\mathbf{e}}_{\boldsymbol{\theta}} \right) \cdot \mathbf{n} \, \mathrm{d}S, \quad \text{(definition of } \Psi) \\ &= \oint_{C_o} \frac{1}{r} \Psi \, \hat{\mathbf{e}}_{\boldsymbol{\theta}} \cdot \mathrm{d}\mathbf{l} + \oint_{C_i} \frac{1}{r} \Psi \, \hat{\mathbf{e}}_{\boldsymbol{\theta}} \cdot \mathrm{d}\mathbf{l}, \quad \text{(Stokes theorem)} \\ &= \Psi_o \oint_{C_o} \frac{1}{r} \hat{\mathbf{e}}_{\boldsymbol{\theta}} \cdot \mathrm{d}\mathbf{l} + \Psi_i \oint_{C_i} \frac{1}{r} \hat{\mathbf{e}}_{\boldsymbol{\theta}} \cdot \mathrm{d}\mathbf{l}, \quad (\Psi \equiv \Psi_{\{o,i\}} \text{ onto } C_{\{o,i\}}) \\ &= \Psi_o \int_0^{2\pi} \mathrm{d}\theta + \Psi_i \int_{2\pi}^{0} \mathrm{d}\theta = 2\pi(\Psi_o - \Psi_i) \quad (\text{Note, } \mathrm{d}\mathbf{l} = \mathrm{d}r \, \hat{\mathbf{e}}_{\boldsymbol{r}} + r\mathrm{d}\theta \, \hat{\mathbf{e}}_{\boldsymbol{\theta}}). \end{aligned}$$

□

Example 1.4
For a uniform flow parallel to the axis, $u = 0$ and $w = U$,

$$\frac{1}{r}\frac{\partial \Psi}{\partial r} = U \quad \text{and} \quad \frac{\partial \Psi}{\partial z} = 0 \Rightarrow \Psi(r) = \frac{1}{2} U r^2.$$

(We choose the integration constant such that $\Psi = 0$ on the axis, at $r = 0$).

Now consider a streamtube of radius a.

The volume flux

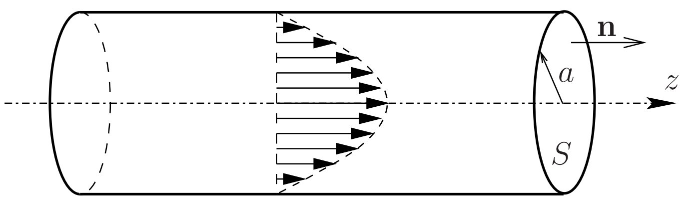
$$Q = \int_S \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S = \int_S \mathbf{u} \cdot \hat{\mathbf{e}}_{\boldsymbol{z}} \, \mathrm{d}S = \int_S w \, \mathrm{d}S = U \int_S \mathrm{d}S = \pi U a^2.$$

Also,

$$2\pi(\Psi_o - \Psi_i) = 2\pi(\Psi(a) - \Psi(0)) = 2\pi \left(\frac{1}{2} U a^2 - 0 \right) = \pi U a^2 \quad \text{as required.}$$

Example 1.5
Consider a flow in a long pipe of radius a:

$$u = 0, \quad w = \frac{U}{a^2}(a^2 - r^2) \quad \text{with} \quad \begin{cases} w = 0 \text{ on } r = a, \\ w = U \text{ on } r = 0. \end{cases}$$



$$\frac{\partial\Psi}{\partial z} = -ru = 0 \Rightarrow \Psi \equiv \Psi(r),$$
$$\text{and } \frac{\mathrm{d}\Psi}{\mathrm{d}r} = rw = \frac{U}{a^2}(a^2r - r^3) \Rightarrow \Psi(r) = \frac{U}{a^2}\left(\frac{a^2r^2}{2} - \frac{r^4}{4}\right) + C.$$

Hence,

$$\Psi(r) = \frac{Ur^2}{4a^2}(2a^2 - r^2) \qquad \text{(choose } C \text{ such that } \Psi(0) = 0).$$

So, $\Psi(0) = 0$ and $\Psi(a) = Ua^2/4$, and the volume flux

$$Q = \int_S \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = 2\pi\left(\Psi(a) - \Psi(0)\right) = \frac{\pi}{2}Ua^2.$$

Indeed,

$$\begin{aligned}\int_S \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S &= \int_0^{2\pi}\int_0^a wr\,\mathrm{d}r\mathrm{d}\theta = \frac{2\pi U}{a^2}\int_0^a (a^2r - r^3)\,\mathrm{d}r, \quad (\text{since } \mathrm{d}S = r\mathrm{d}\theta\mathrm{d}r)\\ &= \frac{2\pi U}{a^2}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_0^a = \frac{\pi U a^2}{2}, \quad \text{as required.}\end{aligned}$$

1.7 Viscosity

Let us return to the shear-flow between two parallel plates when one is moved relative to the other, with a constant velocity U: $\mathbf{u} = Uy/d\hat{\boldsymbol{\imath}}$. The motion of the upper plate requires a force, $\mathbf{F}$, that is proportional to its surface area, A. Since this force results from the rate at which the fluid is being deformed, it should be proportional to U/d so that:

$$F = \mu\frac{AU}{d}.$$

Here, μ is a constant that depends only on the properties of the fluid and is called *dynamic viscosity*. It is convenient to define two new quantities: the *stress*, $\sigma = F/A$, and the *shear rate*, $\dot{\gamma} = U/d$, so that the relation becomes:

$$\sigma = \mu\dot{\gamma}. \tag{1.17}$$

Fluids that obey equation (1.17) are referred to as *Newtonian fluids*. In practice, most fluids, including air, water and even sticky fluids like golden syrup, obey this relationship to high degree of accuracy within a wide range of viscosities. Dynamic viscosities have dimension $ML^{-1}T^{-1}$ and their S.I. unit is the Pa·s (Pascal second).
The dynamic viscosity of air is of the order of 10^{-5} Pa·s, that of water is approximately 10^{-3} Pa·s, that of golden syrup around 10^2 Pa·s, while magma in the Earth's interior has dynamic viscosities of around 10^{22} Pa·s. Note that some fluids, such as those containing polymers, do not obey this law. These fluids are out of the scope of the present course.

1.8 Cartesian tensors

The pressure, p, is a scalar quantity and can be represented by a function equal to its value at each point in space.

The fluid velocity, $\mathbf{u}$, is a vector field and can be represented by its coefficients (u_1, u_2, u_3) with respect to a set of Cartesian axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ as a column vector:

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

This can be represented more compactly as u_i where $i = 1, 2, 3$.
The pressure gradient ∇p is another vector quantity and is given by:

$$\nabla p = \begin{pmatrix} \frac{\partial p}{\partial x_1} \\ \\ \frac{\partial p}{\partial x_2} \\ \\ \frac{\partial p}{\partial x_3} \end{pmatrix},$$

or more compactly in suffix notation as $\partial p/\partial x_j$ where $j = 1, 2, 3$.
The scalar product of these two vectors $\mathbf{u} \cdot \nabla p$ is given by:

$$\mathbf{u} \cdot \nabla p = u_1 \frac{\partial p}{\partial x_1} + u_2 \frac{\partial p}{\partial x_2} + u_3 \frac{\partial p}{\partial x_3},$$

and in suffix notation:

$$\mathbf{u} \cdot \nabla p = u_i \frac{\partial p}{\partial x_i}.$$

Here, we are using the Einstein convention: repeated suffixes denote summations.
The basic rules of suffix notation are:

i. A suffix that appears once is called a *free* index. The number of free indices denote the type of quantity in question. A scalar quantity has no free index, a vector has one and an nth rank tensor has n. Terms that are added or equated must have the same free indices.

ii. If a suffix appears twice it is called a *dummy* index. Since we sum over dummy indices, the number of pairs of dummy indices does not affect the type of the quantity being described. It is also possible to change the index name without affecting the result. However, it is important not to use a letter already in use as a free index.

As we have already seen, taking the gradient of a scalar produces a vector quantity and so taking the gradient of vector produces a quantity with two associated dimensions, called a second rank tensor. Since there are three components of velocity and three coordinate directions, the velocity gradient $\nabla \mathbf{u}$ has 9 components. It can be represented in the form of a matrix as:

$$\nabla \mathbf{u} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix},$$

but it is much more convenient to write this in suffix notation as:

$$\frac{\partial u_i}{\partial x_j},$$

where $(i,j) = (1,2,3)^2$. In terms of the matrix representation, i denotes the row and j the column of the entry.
Note that the Kronecker delta δ_{ij} is another example of a second rank tensor:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \tag{1.18}$$

which, in matrix representation, is the identity matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1.8.1 Scalar and vector products

We have already seen that calculating the scalar product of two vectors comes down to summing over a pair of indices:

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i.$$

This operation is equivalent to the action of the Kronecker delta on the two vectors **a** and **b** since

$$\mathbf{a} \cdot \mathbf{b} = a_i \delta_{ij} b_j = a_i b_i \qquad \text{since } \delta_{ij} b_j = b_i.$$

The vector product can also be represented in Einstein notation by introducing the alternating tensor ϵ_{ijk}:

$$\epsilon_{ijk} = \begin{cases} 1 & ijk = \text{even, i.e., 123, 231 or 312} \\ -1 & ijk = \text{odd, i.e., 132, 213 or 321} \\ 0 & i = j,\ j = k \text{ or } k = i \end{cases}, \tag{1.19}$$

which is a third rank tensor. Since ϵ_{ijk} has 3 free indices, the resulting quantity $\epsilon_{ijk} a_j b_k$ is a vector with index i:

$$c_i = \epsilon_{ijk} a_j b_k$$

, which has the following components:

$$c_1 = a_2 b_3 - a_3 b_2, \qquad c_2 = a_3 b_1 - a_1 b_3, \qquad c_3 = a_1 b_2 - a_2 b_1,$$

and so represents the product of the vectors **a** and **b**.
We can extend these products to tensors. For example, $\mathbf{a} \cdot \mathbf{A} = a_i A_{ij}$ is a vector formed from the scalar product of the vector **a** with the first index of the tensor **A**. Note that, by convention, the dot signifies scalar product of the two neighbouring indices. This product may be performed using the matrix notation by writing **a** as a row vector and then multiplying it by the matrix **A**,

$$\mathbf{a} \cdot \mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

Similarly, $\mathbf{A} \cdot \mathbf{a} = A_{ij} a_j$ may be performed in matrix notation as

$$\mathbf{A} \cdot \mathbf{a} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Note that these two scalar products give different results unless $\mathbf{A}$ is symmetric (i.e., $A_{ij} = A_{ji}$). For example, if we use $K_{ij} = \dfrac{\partial u_i}{\partial x_j}$ to denote the velocity gradient, then:

$$[\mathbf{K} \cdot \mathbf{u}]_j = K_{ji} u_i = u_i K_{ji} = u_i \frac{\partial u_j}{\partial x_i} = [\mathbf{u} \cdot \nabla \mathbf{u}]_j,$$

whereas:

$$[\mathbf{u} \cdot \mathbf{K}]_j = u_i K_{ij} = u_i \frac{\partial u_i}{\partial x_j} = \frac{1}{2} \frac{\partial}{\partial x_j} (u_i u_i) = \frac{1}{2} [\nabla u^2]_j.$$

We can also form products between two indices on the same tensor. For example

$$\delta_{ij} A_{ij} = A_{ii} = A_{11} + A_{22} + A_{33} = \mathrm{Tr}\mathbf{A},$$

the trace of matrix $\mathbf{A}$, which is a scalar quantity.
The scalar product of two second rank tensors $\mathbf{A}$ and $\mathbf{B}$ is another second rank tensor $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$ where

$$C_{ij} = A_{ik} B_{kj}.$$

This is equivalent to matrix multiplication. We can also form the double dot product $\mathbf{A} : \mathbf{B}$, which is the scalar formed by contracting i with j.

$$\mathbf{A} : \mathbf{B} = \delta_{ij} A_{ik} B_{kj} = A_{ik} B_{ki},$$

which is equal to the trace of $\mathbf{C}$.
We can also apply cross-products between components of a tensor. For example:

$$c_i = \epsilon_{ijk} A_{jk},$$

is a vector with components:

$$c_1 = A_{23} - A_{32}, \qquad c_2 = A_{31} - A_{13}, \qquad c_3 = A_{12} - A_{21}.$$

Finally, we have the triple product rule:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

which results from the following relationship between ϵ_{ijk} and δ_{ij}:

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}. \tag{1.20}$$

One way to remember this rule is: second-with-second $\times$ third-with-third - alternative pairings.

1.8.2 ∇, $\nabla \cdot$ and $\nabla \times$

We have already seen that we can write the gradient of a scalar and of a vector as

$$\frac{\partial p}{\partial x_j} \quad \text{and} \quad \frac{\partial u_i}{\partial x_j} \quad \text{respectively.}$$

Taking the gradient increases the rank of a tensor by one, from scalar to vector, vector to second rank tensor, etc.
The divergence is obtained by taking the dot product between nabla and one of the indices of the tensor. For a vector $\mathbf{u}$:

$$\nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i}.$$

Note that this is simply the product between the gradient operator, δ_{ij} and the vector $\mathbf{u}$. Similarly, we can define the divergence of a tensor A_{ij} as:

$$\frac{\partial}{\partial x_i} A_{ij}.$$

This is a vector quantity. Note that the summation can be over either of the two indices, so we can obtain a second vector using the second index:

$$\frac{\partial}{\partial x_j} A_{ij}.$$

By convention, the notation $\nabla \cdot \mathbf{A}$ is taken to mean summation over the first index (the one closest to the dot). The potential for ambiguities in this formulation means that it is better to stick to suffix notation when dealing with tensors.
Finally we can obtain the curl of a vector or tensor by the action of ϵ_{ijk} on the gradient:

$$[\nabla \times \mathbf{u}]_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}.$$

Example 1.6
Let us consider the conservation equation:

$$\frac{\partial c}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

where c is scalar quantity and $\mathbf{j} = c\mathbf{u}$ is the vector flux of c. We write the divergence term in Cartesian coordinates:

$$\nabla \cdot \mathbf{j} = \frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{\partial j_3}{\partial x_3},$$

so, in suffix notation, we have:

$$\nabla \cdot \mathbf{j} = \frac{\partial j_k}{\partial x_k},$$

where $k = 1, 2, 3$. The flux $\mathbf{j}$ is the product of the scalar c with the vector $\mathbf{u}$, so:

$$\mathbf{j} = (j_1, j_2, j_3) = (cu_1, cu_2, cu_3),$$

which is written in index notation as:

$$j_k = cu_k.$$

Now substituting into the conservation equation we obtain,

$$\frac{\partial c}{\partial t} + \nabla \cdot \mathbf{j} = \frac{\partial c}{\partial t} + \frac{\partial j_k}{\partial x_k} = \frac{\partial c}{\partial t} + \frac{\partial}{\partial x_k}(cu_k) = 0.$$

Finally we can apply the product rule to the differential to give:

$$\frac{\partial}{\partial x_k}(cu_k) = u_k \frac{\partial c}{\partial x_k} + c \frac{\partial u_k}{\partial x_k},$$

so that the equation becomes:

$$\frac{\partial c}{\partial t} + u_k \frac{\partial c}{\partial x_k} + c \frac{\partial u_k}{\partial x_k} = 0. \tag{1.21}$$

In the usual (Gibbs) vector notation, this writes:

$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c + c \nabla \cdot \mathbf{u} = 0. \tag{1.22}$$

Example 1.7
Now suppose the quantity c is replaced by a vector $\mathbf{v}$. The equivalent conservation law would be of the form:

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

where the flux $\mathbf{J}$ is a second rank tensor. By replacing c with v_i in equation (1.21), we obtain:

$$\frac{\partial v_i}{\partial t} + u_k \frac{\partial v_i}{\partial x_k} + v_i \frac{\partial u_k}{\partial x_k} = 0, \tag{1.23}$$

which is clear and unambiguous. Written out in full, this represents the equations:

$$\begin{aligned}
\frac{\partial v_1}{\partial t} + \left(u_1 \frac{\partial v_1}{\partial x_1} + u_2 \frac{\partial v_1}{\partial x_2} + u_3 \frac{\partial v_1}{\partial x_3} \right) + v_1 \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) = 0, \\
\frac{\partial v_2}{\partial t} + \left(u_1 \frac{\partial v_2}{\partial x_1} + u_2 \frac{\partial v_2}{\partial x_2} + u_3 \frac{\partial v_2}{\partial x_3} \right) + v_2 \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) = 0, \\
\frac{\partial v_3}{\partial t} + \left(u_1 \frac{\partial v_3}{\partial x_1} + u_2 \frac{\partial v_3}{\partial x_2} + u_3 \frac{\partial v_3}{\partial x_3} \right) + v_3 \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) = 0.
\end{aligned}$$

In this case, $J_{ki} = u_k v_i$ and $[\nabla \cdot \mathbf{J}]_i = \frac{\partial}{\partial x_k}(u_k v_i)$.

1.9 Strain-rate and vorticity tensors

Let us now examine the velocity gradient $\partial u_i / \partial x_j$. For an incompressible flow, $\nabla \cdot \mathbf{u} = 0$ and so this tensor has zero trace. There are still 8 remaining components. A useful simplification is to decompose the velocity gradient into the sum of a symmetric and an antisymmetric tensor:

$$\frac{\partial u_i}{\partial x_j} = E_{ij} + \Omega_{ij}, \quad \text{where} \quad E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{and} \quad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right). \tag{1.24}$$

It is easily verified that $E_{ij} = E_{ji}$ and $\Omega_{ij} = -\Omega_{ji}$.
The symmetric tensor, $\mathbf{E}$, is called the *strain-rate tensor* and the antisymmetric tensor, $\boldsymbol{\Omega}$, is called the *vorticity tensor*.
Recall that the vorticity $\omega = \nabla \times \mathbf{u}$. In suffix notation:

$$\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}. \tag{1.25}$$

Multiplying this equation by ϵ_{ilm} and using the triple product rule, we obtain:

$$\epsilon_{ilm}\omega_i = \epsilon_{ijk}\epsilon_{ilm} \frac{\partial u_k}{\partial x_j} = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \frac{\partial u_k}{\partial x_j} = \frac{\partial u_m}{\partial x_l} - \frac{\partial u_l}{\partial x_m} = 2\Omega_{ml},$$

so that:

$$\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k. \tag{1.26}$$

This result is clear if we write $\boldsymbol{\Omega}$ in matrix notation:

$$\boldsymbol{\Omega} = \frac{1}{2} \begin{pmatrix} 0 & \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} & \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial u_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} & 0 & \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial u_2} \\ \frac{\partial u_3}{\partial x_1} - \frac{\partial u_2}{\partial u_3} & \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Example 1.8
Consider the simple shear flow:

$$\mathbf{u} = \begin{pmatrix} \dot{\gamma} y \\ 0 \\ 0 \end{pmatrix},$$

using (x, y, z) rather than (x_1, x_2, x_3) for Cartesian coordinates. The gradient of velocity writes:

$$\nabla \mathbf{u} = \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

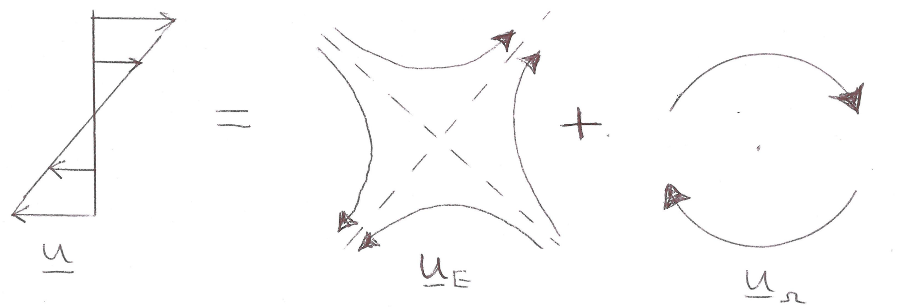
Therefore, the strain-rate and vorticity tensors for this flow are:

$$\mathbf{E} = \frac{1}{2} \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\Omega} = \frac{1}{2} \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ -\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us now reconstrust the linear flows corresponding to the symmetric and antisymmetric parts of the velocity gradient tensor, $\mathbf{u_E} = \mathbf{E} \cdot \mathbf{x}$ and $\mathbf{u_\Omega} = \boldsymbol{\Omega} \cdot \mathbf{x}$:

$$\mathbf{u_E} = \begin{pmatrix} \frac{\dot{\gamma}}{2} y \\ \\ \frac{\dot{\gamma}}{2} x \\ \\ 0 \end{pmatrix}, \qquad \mathbf{u_\Omega} = \begin{pmatrix} \frac{\dot{\gamma}}{2} y \\ \\ -\frac{\dot{\gamma}}{2} x \\ \\ 0 \end{pmatrix}.$$

The streamlines of $\mathbf{u_E}$ are given by $x^2 - y^2 = \text{cst}$, while the streamlines of $\mathbf{u_\Omega}$ are circles $x^2 + y^2 = \text{cst}$.



Thus $\mathbf{u_E}$ is a hyperbolic flow with extension along the line $y = x$ (and contraction along $y = -x$), while $\mathbf{u_\Omega}$ is a clockwise rotation.

Furthermore, since $\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k$, it follows that:

$$u_{\Omega i} = \Omega_{ij} x_j = -\frac{1}{2}\epsilon_{ijk}\omega_k x_j = \frac{1}{2}\epsilon_{ikj}\omega_k x_j = \frac{1}{2}[\omega \times \mathbf{x}]_i,$$

i.e., $\mathbf{u_\Omega}$ is a solid body rotation at an angular velocity of $\omega/2$. It is the strain-rate $\mathbf{E}$ that produces a deformation of the fluid.

1.10 Polar tensors

So far we have only discussed tensors with respect to fixed Cartesian coordinates, however, in many problems it is often more convenient to work in polar coordinates. Tensor calculus in polar coordinates tends to be more complicated because of the rotation of the base vectors. For example in cylindrical polar coordinates (r, θ, z) the gradient of the velocity (u_r, u_θ, u_z) is given by

$$\begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{1}{r}\frac{\partial u_z}{\partial \theta} & \frac{\partial u_z}{\partial z} \end{pmatrix}.$$

Note that when we take the trace we recover the formula for $\nabla \cdot \mathbf{u}$ in cylindrical polar coordinates

$$\nabla \cdot \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}.$$

As this is not a course in tensor calculus we shall not attempt to derive these formulae. Instead we will refer to a formula sheet when using cylindrical or spherical polar coordinates.

Chapter 2

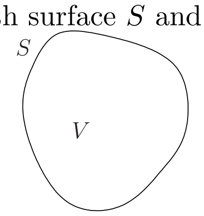
The Navier–Stokes equation

Contents

So far, we have considered mass conservation and the rate of deformation of the fluid. Now, we shall consider the forces acting on the fluid and how they relate to the velocity gradient.

2.1 Fluid momentum transport

Consider a fixed volume of fluid V with surface S and outward normal $\mathbf{n}$.



We consider the changes to the total fluid momentum contained within V. Applying Newton's law of motion to this volume we have:

$$\begin{pmatrix}\text{rate of increase}\\ \text{of momentum}\\ \text{in } V\end{pmatrix} = \begin{pmatrix}\text{net inward flow}\\ \text{of momentum}\\ \text{through } S\end{pmatrix} + \begin{pmatrix}\text{net force}\\ \text{acting}\\ \text{on } V\end{pmatrix} \tag{2.1}$$

The momentum density in the fluid is given by $\rho\mathbf{u}$ and so the first contribution to equation (2.1) is:

$$\begin{pmatrix}\text{rate of increase}\\ \text{of momentum}\\ \text{in } V\end{pmatrix} = \frac{d}{dt}\left(\int_V \rho\mathbf{u}dV\right) = \int_V \frac{\partial}{\partial t}\left(\rho\mathbf{u}\right)dV,$$

The net flow of momentum through the surface S is given by:

$$\begin{pmatrix}\text{net inward flow}\\ \text{of momentum}\\ \text{through surface } S\end{pmatrix} = -\int_S \rho\mathbf{u}\mathbf{u}\cdot\mathbf{n}dS,$$

which in suffix notation is written as:

$$-\int_S \rho u_i u_j n_j dS.$$

Now, to convert this to a volume integral, we apply the divergence theorem by replacing n_j with $\partial/\partial x_j$ so that:

$$-\int_S \rho u_i u_j n_j dS = -\int_V \frac{\partial}{\partial x_j}(\rho u_i u_j)\, dV.$$

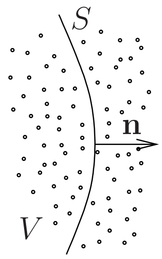
The forces acting on the fluid can be divided into two groups:

i. **Body forces:** These are external forces acting on the fluid, such as gravity or electromagnetic forces (which are important in astrophysical fluids and also for flows within the Earth's core), however we shall only consider gravity which exerts a force:

$$\int_V \rho \mathbf{g} dV.$$

ii. **Molecular forces:** These are short-range due to interactions between fluid molecules on either side of the surface S, which exert a force:

$$\int_S \mathbf{f} dS.$$



Since we wish to use the divergence theorem to transform this into a volume integral we define the *stress tensor*, $\boldsymbol{\tau}$, such that:

$$f_i = n_j \tau_{ji},$$

where $\mathbf{n}$ is the normal to the surface. With this definition, the molecular force acting on the fluid is given by:

$$\int_S \tau_{ji} n_j dS = \int_V \frac{\partial \tau_{ji}}{\partial x_j} dV.$$

Putting all these contributions back into equation (2.1), we obtain:

$$\int_V \left[\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) - \rho g_i - \frac{\partial \tau_{ji}}{\partial x_j}\right] dV = 0. \tag{2.2}$$

Since V is arbitrary, we obtain the momentum equation:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) - \rho g_i - \frac{\partial \tau_{ji}}{\partial x_j} = 0.$$

Note that the first two terms correspond to the conservation equation for the vector $\rho \mathbf{u}$. We can simplify this equation by noting from mass conservation (equation (1.6)) that:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0,$$

so that:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho \frac{Du_i}{Dt}.$$

Hence, the equation for fluid momentum transport is:

$$\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} = \rho g_i + \frac{\partial \tau_{ji}}{\partial x_j}, \tag{2.3}$$

or written in the usual vector notation:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \rho \mathbf{g} + \nabla \cdot \boldsymbol{\tau}. \tag{2.4}$$

Like the equation for mass conservation, this equation applies to all continuous materials. However, this is not sufficient to predict the motion of the fluid since we need an additional equation relating the molecular forces represented by the stress tensor $\boldsymbol{\tau}$ to the fluid motion.

2.2 Constitutive equations

The equation defining the stress tensor $\boldsymbol{\tau}$ is called the constitutive equation.

2.2.1 Ideal fluid

The simplest constitutive equation for a fluid is that of an ideal or inviscid fluid for which the only surface force is the pressure. For an incompressible fluid, pressure arises from the resistance to changes in volume and acts along the direction of the normal $\mathbf{n}$:

$$\mathbf{f} = -P\mathbf{n}.$$

Hence, for an ideal fluid: $n_j \tau_{ji} = -P n_i$, yielding $\tau_{ji} = -P\delta_{ij}$. Substituting into the momentum transport equation, we obtain the Euler equation:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \rho \mathbf{g} - \nabla P. \tag{2.5}$$

2.2.2 Newtonian fluid

In most situations, additional forces come into play. In the example case of a shear flow, we observed a force that takes the following form per unit area: $\mathbf{f} = \mu\dot{\gamma}\mathbf{e}_x$, where we have defined the shear-rate $\dot{\gamma} = \partial u_x/\partial y$. In addition to the pressure, there is therefore a contribution to the stress caused by the shearing motion:

$$\tau_{ij} = -P\delta_{ij} + \sigma_{ij}, \tag{2.6}$$

where σ_{ij} is the viscous stress. The tensor $\boldsymbol{\sigma}$ is proportional to the velocity gradient. However, as shown in chapter 1, only the symmetric part of the velocity gradient involves the deformation of the fluid. Considering an isotropic fluid, the stress must be of the form:

$$\sigma_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \tag{2.7}$$

where μ is the dynamic viscosity.
In the shear flow case, the only non-zero component of the velocity gradient is $\partial u_1/\partial x_2 = \dot{\gamma}$, so that $\sigma_{21} = \sigma_{12} = \mu\dot{\gamma}$. Notice that the stress tensor is symmetric.

Example: Let us consider the flow which occurs when we stretch out a cylinder of fluid: $w = \dot{\epsilon}z$.
Mass conservation requires that the volume of fluid in the column remains fixed:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

which can be achieved by setting $u = -\frac{1}{2}\dot{\epsilon}x$ and $v = -\frac{1}{2}\dot{\epsilon}y$. The velocity gradient tensor $\nabla\mathbf{u}$ is therefore given by:

$$\nabla\mathbf{u} = \begin{pmatrix} -\frac{1}{2}\dot{\epsilon} & 0 & 0 \\ 0 & -\frac{1}{2}\dot{\epsilon} & 0 \\ 0 & 0 & \dot{\epsilon} \end{pmatrix}.$$

Hence, the viscous stress has components, $\sigma_{xx} = \sigma_{yy} = -\mu\dot{\epsilon}$ and $\sigma_{zz} = 2\mu\dot{\epsilon}$.

2.3 The Navier–Stokes equation

For a Newtonian fluid:

$$\frac{\partial \tau_{ji}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[-P\delta_{ji} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] = -\frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j x_j} + \mu\frac{\partial^2 u_j}{\partial x_j x_i},$$

However, since $\nabla \cdot \mathbf{u} = 0$, the last term is zero and we obtain the governing equations for an incompressible Newtonian fluid:

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla P + \mu\nabla^2\mathbf{u} + \rho\mathbf{g}, \tag{2.8}$$

often solved together with the incompressibility constraint: $\nabla \cdot \mathbf{u} = 0$.

2.4 Hydrostatic and dynamic pressure

If there is no flow, the Navier–Stokes equation reduces to a balance between gravity and pressure:

$$-\nabla P + \rho\mathbf{g} = 0.$$

The resulting pressure solution:

$$P_H = P_0 + \rho\mathbf{g}\cdot\mathbf{x},$$

is referred to as *hydrostatic pressure*. Although gravity is responsible for driving some flows such as rivers or gravity waves, in many cases it is simply balanced by the hydrostatic pressure. As a consequence, it is often useful to subtract off the hydrostatic pressure by writing the pressure in the form:

$$P = P_H + p \tag{2.9}$$

where p is referred to as the *dynamic pressure*. This reduces the Navier–Stokes equation to:

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu\nabla^2\mathbf{u}. \tag{2.10}$$

2.5 Boundary conditions

In addition to the equation for the stress tensor, we need to know what boundary conditions to apply. In general, both the velocity and the forces must be continuous at a fluid boundary, however, the nature of the boundary impacts the way these laws are expressed.

2.5.1 Solid boundaries

Where a fluid is in contact with a solid surface moving at velocity $\mathbf{U}$, there is friction between the solid surface and the fluid: the velocity in the fluid $\mathbf{u}$ must be equal to the velocity of the solid surface:

$$\mathbf{u} = \mathbf{U},$$

on the boundary. As well as matching the velocity, we also have a boundary condition on the stress. By definition, the surface force density applied by the boundary on the fluid is equal to

$$\mathbf{f} = \mathbf{n} \cdot \boldsymbol{\tau}, \tag{2.11}$$

where $\mathbf{n}$ is the outward pointing normal to the surface. By Newton's third law, the fluid imposes an equal and opposite force density on the solid boundary.

2.5.2 Free surfaces

If the fluid is in contact with air (or a fluid of much lower viscosity), the only force exerted by the air on the fluid results from the atmospheric pressure P_{atm}. In the absence of surface forces, the force applied by the air to the fluid is $-P_{\text{atm}}\mathbf{n}$. The force balance implies:

$$\mathbf{n} \cdot \boldsymbol{\tau} = -P\mathbf{n} + \mathbf{n} \cdot \boldsymbol{\sigma} = -P_{\text{atm}}\mathbf{n}. \tag{2.12}$$

Consequently, $\boldsymbol{\sigma}$ must be orthogonal to $\mathbf{n}$: there is no force parallel to the surface. A free surface cannot support shear.
We still require one additional boundary condition. Let the position of the surface be given by $f(\mathbf{x}, t)$=0. Since all points on the surface must remain on the surface:

$$\frac{Df}{Dt} = 0.$$

In particular, if the surface remains fixed in time, we have:

$$\mathbf{u} \cdot \nabla f = 0,$$

where $\nabla f = \mathbf{n}$ is the normal to the surface. It follows:

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{0}.$$

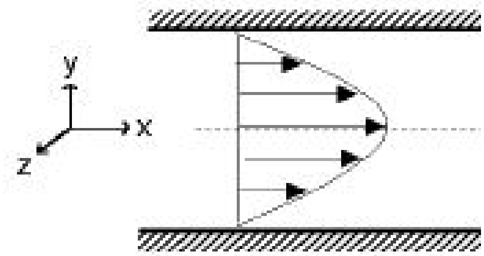
2.5.3 Boundary between immiscible fluids

At a boundary between two fluids of different viscosities, both the velocity $\mathbf{u}$ and force density $\mathbf{n} \cdot \boldsymbol{\tau}$ must be continuous.
Furthermore, if the surface between the fluids remains fixed in time then $\mathbf{u} \cdot \mathbf{n} = 0$. Consequently, the condition that $\mathbf{n} \cdot \boldsymbol{\tau}$ is continuous reduces to both P and $\mu\mathbf{n}\cdot\nabla\mathbf{u}$ being continuous.

2.6 One dimensional flow examples

2.6.1 Plane Poiseuille flow

Let us consider the steady flow of fluid along a channel driven by a pressure gradient. We define Cartesian coordinates with x along the channel in the direction of flow and y across the channel, with boundaries at $y = \pm h$.



The fluid velocity, $\mathbf{u}$, satisfies no-slip boundary conditions at the walls: $\mathbf{u} = 0$ at $y = \pm h$. We can use the symmetries of the configuration to reduce the flow:

- $\mathbf{u} \cdot \hat{\mathbf{y}} = 0$: no wall-normal flow
- $\mathbf{u} \cdot \hat{\mathbf{z}} = 0$: unidirectional velocity in the direction of the pressure gradient
- $\partial_z \equiv 0$: invariance in the spanwise direction

The first two hypotheses lead to the velocity having only one non-vanishing component while the third one indicates it only varies in the streamwise and wall-normal directions: $\mathbf{u} = u_x(x, y)\hat{\mathbf{x}}$.
We recall the writing of the incompressibility constraint in Cartesian coordinates:

$$\partial_x u_x + \partial_y u_y + \partial_z u_z = 0, \tag{2.13}$$

and of the Navier–Stokes equation:

$$\rho \left[\partial_t u_x + u_x \partial_x u_x + u_y \partial_y u_x + u_z \partial_z u_x\right] = -\partial_x p + \mu \left[\partial_x^2 u_x + \partial_y^2 u_x + \partial_z^2 u_x\right], \tag{2.14}$$
$$\rho \left[\partial_t u_y + u_x \partial_x u_y + u_y \partial_y u_y + u_z \partial_z u_y\right] = -\partial_y p + \mu \left[\partial_x^2 u_y + \partial_y^2 u_y + \partial_z^2 u_y\right], \tag{2.15}$$
$$\rho \left[\partial_t u_z + u_x \partial_x u_z + u_y \partial_y u_z + u_z \partial_z u_z\right] = -\partial_z p + \mu \left[\partial_x^2 u_z + \partial_y^2 u_z + \partial_z^2 u_z\right]. \tag{2.16}$$

The incompressibility constraint (2.13) yields:

$$\partial_x u_x = 0, \tag{2.17}$$

which, together with the starting hypotheses provides:

$$\mathbf{u} = u_x(y)\hat{\mathbf{x}}. \tag{2.18}$$

The Navier–Stokes equation in the wall-normal and spanwise directions (2.15) and (2.16) give:

$$\partial_y p = \partial_z p = 0, \tag{2.19}$$

thus:

$$p = p(x). \tag{2.20}$$

Eventually, the Navier–Stokes equation in the streamwise direction (2.14) gives:

$$0 = -\partial_x p + \mu \partial_y^2 u_x. \tag{2.21}$$

The solution reads:

$$u_x = \frac{\partial_x p}{2\mu} y^2 + k_1 y + k_2, \tag{2.22}$$

where k_1 and k_2 are solved for using the boundary conditions:

$$\frac{\partial_x p}{2\mu} h^2 + k_1 h + k_2 = 0, \tag{2.23}$$
$$\frac{\partial_x p}{2\mu} h^2 - k_1 h + k_2 = 0, \tag{2.24}$$

yielding:

$$k_1 = 0, \qquad k_2 = -\frac{\partial_x p\, h^2}{2\mu}. \tag{2.25}$$

The trivial laminar flow in a channel, also called *plane Poiseuille flow*, then reads:

$$u_x = \frac{G\,h^2}{2\mu}\left(1 - \frac{y^2}{h^2}\right), \tag{2.26}$$

with $G = -\partial_x p$. Hence, the velocity has a parabolic profile, with a shear-stress at the wall given by: $\sigma_{yx} = \mu du_x/dy = \mp Gh$. The streamfunction is given by:

$$\frac{\partial \psi}{\partial y} = u_x = \frac{G}{2\mu}\left(h^2 - y^2\right),$$

so that:

$$\psi = \frac{Gy}{6\mu}\left(3h^2 - y^2\right).$$

The volume flow per unit depth is given by $\psi(h) - \psi(-h) = 2Gh^3/3\mu$.

2.6.2 Hagen–Poiseuille flow

The equivalent axisymmetric problem where a fluid flows along a cylindrical pipe is often referred to as Hagen–Poiseuille or Poiseuille flow in the name of the scientists who first derived and measured this flow.
We consider a pipe of radius a and use cylindrical polar coordinates based on the axis of the cylinder so that the fluid velocity is of the form: $\mathbf{u} = w(r)\hat{\mathbf{e}}_{\boldsymbol{z}}$. As was the case for channel flow, this flow automatically satisfies the incompressibility condition and, since we look for a steady flow: $D\mathbf{u}/Dt = 0$. The components of the Navier–Stokes equation reduce to:

$$\begin{aligned}
0 &= -\frac{\partial p}{\partial r}, \\
0 &= -\frac{1}{r}\frac{\partial p}{\partial \theta}, \\
0 &= -\frac{\partial p}{\partial z} + \mu \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right).
\end{aligned}$$

The pressure is a function of z alone and is of the form $p(z) = p_0 - Gz$ where G is the magnitude of the pressure gradient. We obtain:

$$\frac{d}{dr}\left(r\frac{dw}{dr}\right) = -\frac{Gr}{\mu}.$$

Integrating with respect to r yields:

$$\frac{dw}{dr} = \frac{A}{r} - \frac{Gr}{2\mu},$$

for some constant A. The velocity is smooth everywhere, so $A = 0$. Finally, integrating again and applying the boundary condition $w(a) = 0$, we obtain:

$$w(r) = \frac{G}{4\mu}\left(a^2 - r^2\right).$$

Hence, using the result from example 1.5, the volume flow through the pipe is equal to:

$$Q = \frac{G\pi a^4}{8\mu},$$

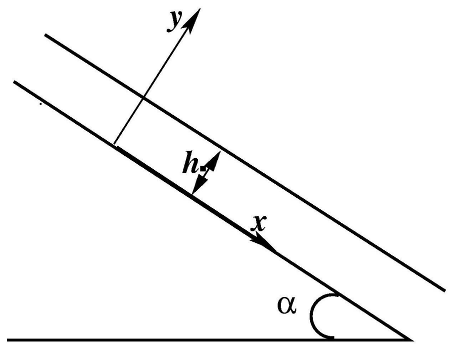
and so the pressure difference required to pump a fluid of viscosity μ at a volume flow rate Q along a pipe of radius a and length L is given by:

$$\Delta p = \frac{8\mu QL}{\pi a^4}.$$

This solution gives a good approximation provided that the pipe is long and straight and that the fluid is sufficiently viscous. It also provides a good method for measuring the viscosity of a fluid.

2.6.3 Flow down an inclined plane

A plane inclined at an angle α to the horizontal is coated with a layer of fluid of thickness h.
Let us define the Cartesian coordinates: x is directed down the slope and y perpendicular to the slope. Using the symmetries of the configuration to simplify the flow, we assume that the fluid velocity is of the form $\mathbf{u} = (u(y), 0, 0)$.



Since gravity is driving the flow, we consider the full pressure and include the body force due to gravity. We look for a steady flow, so $D\mathbf{u}/Dt = 0$. The Navier–Stokes equation reduces to:

$$\nabla P = \rho\mathbf{g} + \mu\nabla^2\mathbf{u}.$$

The gravitational acceleration $\mathbf{g}$ is given by:

$$\mathbf{g} = (g\sin\alpha, -g\cos\alpha, 0).$$

Hence, the x and y components of the momentum equation reduce to:

$$\frac{\partial P}{\partial x} = \mu\frac{d^2u}{dy^2} + \rho g\sin\alpha, \tag{2.27}$$

$$\frac{\partial P}{\partial y} = -\rho g\cos\alpha. \tag{2.28}$$

At the free surface $y = h$, the boundary condition reads: $\mathbf{n}\cdot\boldsymbol{\tau} = -P_{\mathrm{atm}}\mathbf{n}$. In this case, $\mathbf{n} = (0, 1)$, which implies that:

$$\tau_{yy} = -P = -P_{\mathrm{atm}}, \quad \tau_{yx} = \sigma_{yx} = \mu\frac{du}{dy} = 0.$$

From equation (2.28) the pressure, P, is given by:

$$P = P_{\mathrm{atm}} + \rho g\cos\alpha(h - y).$$

so that $\partial P/\partial x = 0$. Therefore, upon integrating equation (2.27), we get:

$$u(y) = -\frac{\rho g\sin\alpha}{2\mu}y^2 + Ay + B,$$

where A and B are integration constants determined by the boundary conditions: $u = 0$ at $y = 0$ and $du/dy = 0$ at $y = h$. Eventually, we obtain:

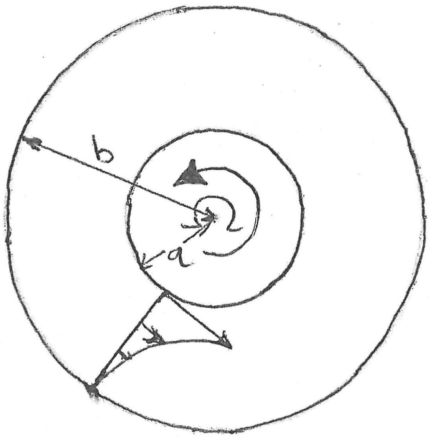
$$u(y) = \frac{\rho g\sin\alpha}{2\mu}y(2h - y).$$

Hence, the flow profile is again parabolic and corresponds to the flow in the bottom half of the channel.

2.6.4 Taylor–Couette flow

So far, we have only considered flows in which the fluid particles move in straight lines at constant speed so that the acceleration is equal to zero. Let us now consider a flow with curved streamlines so that $D\mathbf{u}/Dt \neq 0$.

Consider a fluid flow between two concentric cylinders of radii a and b respectively, where the inner cylinder is rotating at an angular velocity Ω. We define the cylindrical polar coordinates about the axis of the cylinders so that $\mathbf{u} = (0, v(r), 0)$ with boundary conditions $v(a) = a\Omega$ and $v(b) = 0$.



Again, this form of the fluid velocity automatically satisfies $\nabla \cdot \mathbf{u} = 0$. However, although the flow is steady, $D\mathbf{u}/Dt \neq 0$. Indeed, substituting the form of the fluid velocity, we find:

$$\mathbf{u} \cdot \nabla \mathbf{u} = \left(-\frac{v^2}{r}, 0, 0 \right).$$

Substituting into the Navier–Stokes equation, we obtain:

$$-\frac{\rho v^2}{r} = -\frac{\partial p}{\partial r}, \tag{2.29}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr} \right) - \frac{v}{r^2} \right], \tag{2.30}$$

$$0 = -\frac{\partial p}{\partial z}. \tag{2.31}$$

From equation (2.30), we see that $\partial p/\partial \theta$ is independent of θ. However, since p is periodic in θ, $p(\theta + 2\pi) = p(\theta)$ and so $\partial p/\partial \theta = 0$. Equation (2.30) reduces to:

$$\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} = 0.$$

This is a Cauchy equation with general solution:

$$v = \frac{A}{r} + Br.$$

Applying the boundary conditions, we obtain:

$$v(r) = \frac{\Omega a^2}{b^2 - a^2}\left(\frac{b^2}{r} - r \right). \tag{2.32}$$

We can find the pressure by integrating equation (2.29):

$$p(r) = p(a) + \rho \int_a^r \frac{v^2(r')}{r'} dr'.$$

Since $v^2/r > 0$, the pressure increases with the distance to the center. This is the reason why the free-surface dips near a rotating rod.

In order to calculate the surface forces we need to obtain the velocity gradient, which is given in cylindrical polar coordinates by:

$$\nabla \mathbf{u} = \begin{pmatrix} \dfrac{\partial u}{\partial r} & \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial r} & \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} + \dfrac{u}{r} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial r} & \dfrac{1}{r}\dfrac{\partial w}{\partial \theta} & \dfrac{\partial w}{\partial z} \end{pmatrix}.$$

Hence, in this flow:

$$\sigma_{r\theta} = \mu \left(\frac{dv}{dr} - \frac{v}{r} \right) = -\frac{2\mu \Omega b^2}{(b^2 - a^2)}.$$

The torque required to rotate the inner cylinder is given by:

$$\mathbf{T} = \int \mathbf{r} \times \mathbf{f} dS,$$

where $\mathbf{r}$ is the radial vector and:

$$\mathbf{f} = -\hat{\mathbf{r}} \cdot \boldsymbol{\tau},$$

since $\mathbf{n} = -\hat{\mathbf{r}}$. Thus, the magnitude of the torque T applied to the inner cylinder is given by:

$$T = -\int_S r\tau_{r\theta} dS = -L \int_0^{2\pi} a\tau_{r\theta} a d\theta = -2\pi a^2 L \sigma_{r\theta} = \frac{4\pi \mu L \Omega a^2 b^2}{(b^2 - a^2)}, \tag{2.33}$$

where L is the length of the Taylor–Couette cell. This experiment provides a practical method for measuring viscosity.

Chapter 3

The Reynolds number

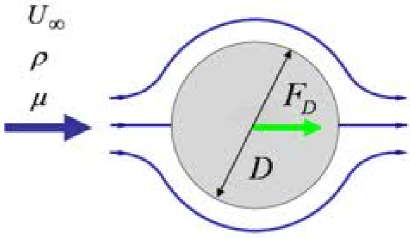
Contents

In the previous chapter, we found analytic solutions of the full Navier–Stokes equation. Such examples are rare and, in general, fluid flows can only be investigated through a combination of experiments, numerical simulations and approximate analytical solutions. Before embarking on any of these, it is useful to reduce the number of parameters as much as possible.

3.1 Dynamic similarity

A good starting point is to consider under what conditions are two flows "dynamically equivalent", by which we mean that they have the same flow pattern even though the scales and fluid properties may be different.

Let us consider the flow pattern generated by an obstacle (e.g. a sphere) of size D in a uniform flow of speed U in a fluid of density ρ and viscosity μ.



This problem has four dimensional parameters, D, U, ρ and μ. We can use these parameters to define a new system of units based upon independent units for mass (M), length (L) and time (T). Note that $[D] = L$, $[U] = L \cdot T^{-1}$, $[\rho] = M \cdot L^{-3}$ and $[\mu] = M \cdot L^{-1} \cdot T^{-1}$. It is thus logical to choose:

$$L = D, \quad \text{and } T = \frac{D}{U}.$$

Lastly, since both ρ and μ involve mass, we can choose either:

$$M = \rho D^3 \quad \text{or } M = \frac{\mu D^2}{U}.$$

As a consequence, there is a single independent dimensionless group that can formed from the combination of D, U, ρ and μ:

$$\mathrm{Re} = \frac{\rho U D}{\mu}. \tag{3.1}$$

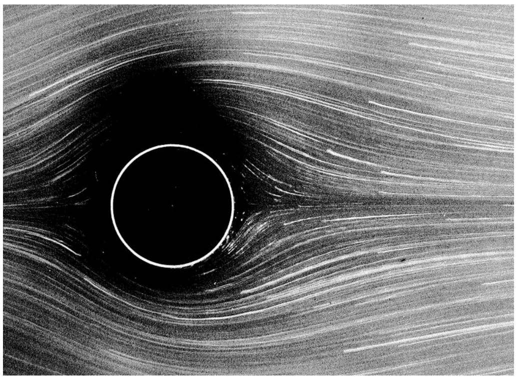
This number is called the *Reynolds number*. It indicates the balance between inertia and viscous forces. Flows with the same value of the Reynolds number (and dimensionless geometry) but different values of the dimensional parameters D, U, ρ and μ display the same flow pattern and are thus dynamically similar. By selecting fluid properties and flow rates appropriately, we can make smaller or larger scale models that give the same flow pattern.

3.2 Flow past a cylinder

To illustrate how the Reynolds number can be used to characterise a flow, let us consider the flow past a cylinder. Recall that for an inviscid fluid the potential flow pattern is fore-aft symmetric and produces zero drag, but that this flow is not seen in practice.

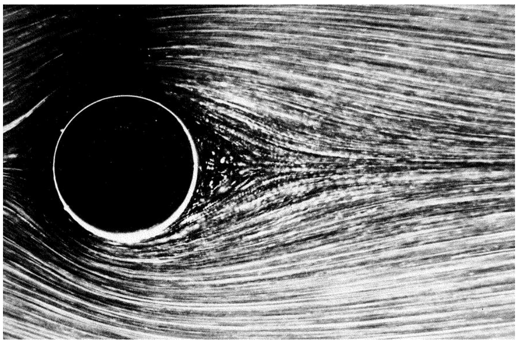
- $Re < 1$

  For small values of the Reynolds number the flow is nearly fore-aft symmetric. However, this flow pattern is distinct from the potential flow solution as it satisfies $\mathbf{u} = \mathbf{0}$ on the cylinder surface.

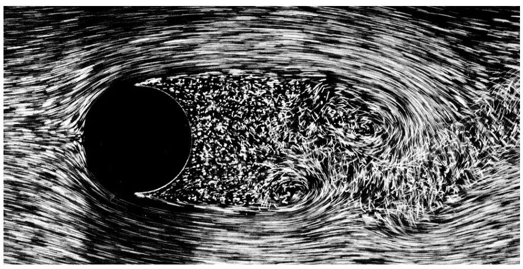
- $1 < Re < 46$

  As the Reynolds number increases, the flow loses its fore-aft symmetry and two recirculating eddies appear on the downstream side of the cylinder. These cells grow in size as the Reynolds number increases. Although the flow is no longer fore-aft symmetric it remains steady.

- $46 < Re$

  Above Reynolds numbers of around 46, the flow is no longer steady. The eddies behind the cylinder become unsteady and are shed alternately from the two sides, forming a double line of eddies known as a von Kármán vortex street.

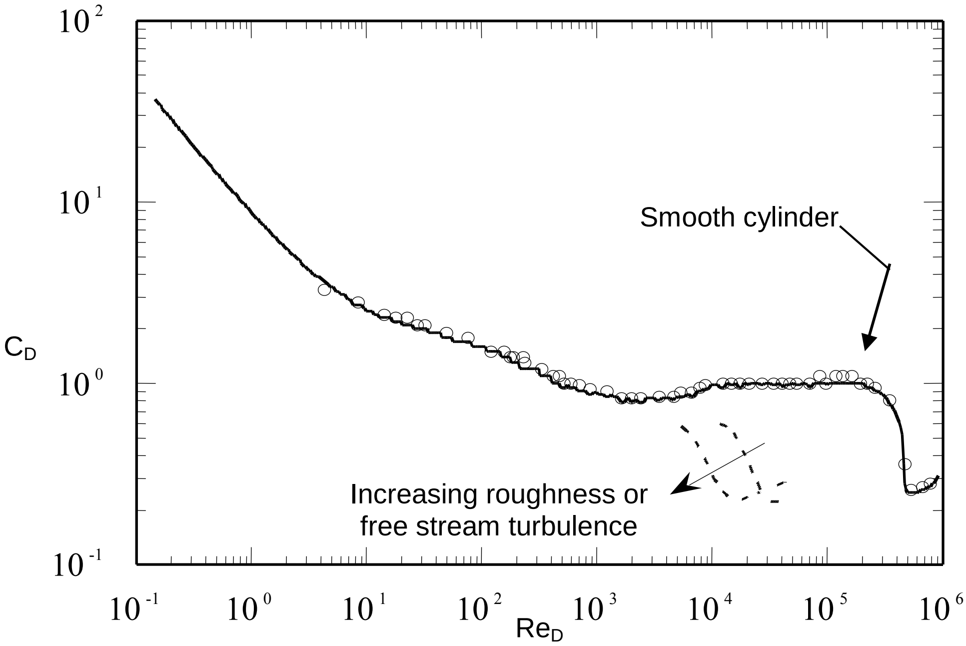
  As the Reynolds number increases further the flow in this wake region behind the cylinder becomes chaotic.

Simulations at $Re = 25$, $Re = 50$, $Re = 100$ and $Re = 220$:
https://www.youtube.com/watch?v=8WtEuw0GLg0.

As well as looking at the flow pattern, we can also measure the drag force on the cylinder. Forces have units of $M \cdot L \cdot T^{-2}$ so, if we use ρ to define the unit of mass, we can write the drag force in the form:

$$F = \frac{1}{2}\rho U^2 A C_D(\mathrm{R}e),$$

where A is the cross-sectional area and C_D is a dimensionless number that is a function of the object shape and the Reynolds number. Note that the factor $\frac{1}{2}$ is introduced by convention. The graph below shows how the drag coefficient on a cylinder varies with the Reynolds number.



For low Reynolds numbers, the drag coefficient decreases roughly as $1/\mathrm{R}e$ and levels out to an $O(1)$ value for Reynolds numbers above 200. A sharp drop occurs at Reynolds numbers between 100,000 and 1,000,000.
The fact that the drag coefficient remains approximately constant over a wide range of Reynolds numbers makes it useful for defining how the shape of an object affects the drag force. For example, cars typically have a drag coefficient in the range 0.25 to 0.5. The boxy shapes, such as Range Rovers tend to be at the high end, whereas the best energy efficient designs have drag coefficients around 0.25.

3.3 The Reynolds number and the Navier–Stokes equation

The Reynolds number arises naturally from consideration of the terms in the Navier–Stokes equation (2.10):

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu \nabla^2 \mathbf{u}.$$

For a steady flow past an obstacle, the size of the left-hand side can be estimated as:

$$|\rho \mathbf{u} \cdot \nabla \mathbf{u}| \sim \frac{\rho U^2}{D},$$

while that of the viscous term as:

$$|\mu \nabla^2 \mathbf{u}| \sim \frac{\mu U}{D^2}.$$

Hence, if we take the ratio of these two terms, we get:

$$\frac{|\rho \mathbf{u} \cdot \nabla \mathbf{u}|}{|\mu \nabla^2 \mathbf{u}|} \sim \frac{\rho U^2}{D} \times \frac{D^2}{\mu U} = \frac{\rho U D}{\mu} = Re.$$

The Reynolds number can be thought of as the ratio of the relative sizes of the terms governing fluid inertia and viscosity.
If the Reynolds number is large then $|\rho \mathrm{D}\mathbf{u}/\mathrm{D}t| \gg |\mu \nabla^2 \mathbf{u}|$. The pressure gradient balances $\rho \mathbf{u} \cdot \nabla \mathbf{u}$ and the pressure differences over the obstacle are of size ρU^2. The drag force is thus roughly of the size of $\rho U^2 A$, so that the drag coefficient is of order unity.
Conversely, if the Reynolds number is small then $|\mu \nabla^2 \mathbf{u}| \gg |\rho \mathrm{D}\mathbf{u}/\mathrm{D}t|$ and the pressure gradient balances $|\mu \nabla^2 \mathbf{u}|$. This gives a pressure difference of the size of $\mu U/D$ and hence the magnitude of the force arising from viscous drag scales as:

$$\mu \frac{U}{D} A = \rho U^2 A \left(\frac{\mu}{\rho U D} \right) = \frac{\rho U^2 A}{Re},$$

so that the drag coefficient scales as $1/\mathrm{R}e$, as was found for the case of the cylinder.
We can also obtain the Reynolds number from the Navier–Stokes equation by non-dimensionalising it, i.e. by choosing units based upon the natural length and time scales. We substitute:

$$\mathbf{u} = U\mathbf{u}^*, \qquad \mathbf{x} = D\mathbf{x}^*, \qquad t = \frac{D}{U} t^*,$$

where $\mathbf{u}^*$ and $\mathbf{x}^*$ are now dimensionless vector quantities, and choose $\mu U/D$ as the unit for pressure:

$$p = \mu \frac{U}{D} p^*.$$

The Navier–Stokes equation becomes:

$$\frac{\rho U^2}{D} \frac{\mathrm{D}\mathbf{u}^*}{\mathrm{D}t^*} = -\mu \frac{U}{D^2} \nabla^* p^* + \mu \frac{U}{D^2} \nabla^{*2} \mathbf{u}^*,$$

and, dividing by $\mu U/D^2$, we have:

$$\mathrm{Re} \frac{\mathrm{D}\mathbf{u}^*}{\mathrm{D}t^*} = -\nabla^* p^* + \nabla^{*2} \mathbf{u}^*.$$

Conservation of mass remains:

$$\nabla^* \cdot \mathbf{u}^* = 0,$$

and so the only parameter in the governing equations is the Reynolds number.

3.4 Flow at low and high Reynolds numbers

A small (large) Reynolds number suggests that the inertia (viscosity) terms are small compared to the other terms in the Navier–Stokes equation and might be neglected. We do however need to be careful as the correct scales for U and D are not always obvious. The small Reynolds number case describes slow viscous flows where we can neglect $\rho \mathrm{D}\mathbf{u}/\mathrm{D}t$. The resulting equations:

$$-\nabla p + \mu \nabla^2 \mathbf{u} = 0, \qquad \nabla \cdot \mathbf{u} = 0, \tag{3.2}$$

are called *Stokes equations* and the corresponding solutions *Stokes flows*. As the Stokes equations do not contain $\mathrm{D}\mathbf{u}/\mathrm{D}t$, they are linear and not directly dependent on time. They

are considerably easier to solve than the full Navier–Stokes equations. Indeed, the exact solutions given in the previous chapter are in fact solutions of the Stokes equations.
The opposite limit of high Reynolds number flows is more complicated. Excluding the viscous term from the Navier–Stokes equation reduces it to the Euler equation for an ideal fluid:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p.$$

However, in doing this, we removed the term with the highest spatial derivative, $\nabla^2\mathbf{u}$, which is mathematically dangerous since it means that we cannot impose the full no-slip boundary condition. Therefore there must be a layer of fluid, called a *boundary layer* near the surface where the shear-rates are sufficiently high that viscosity cannot be neglected. In many cases however, these layers are sufficiently thin that we can neglect them and in these cases the Euler equation (and hence the Bernoulli equation) gives a good approximation to the flow. In other flows, such as flow past a cylinder, these boundary layers can grow in size and affect large regions of the flow.

Chapter 4

Slow viscous flow

Contents

4.1 Stokes flow

For flows where the Reynolds number:

$$\mathrm{Re} = \frac{\rho U D}{\mu} \ll 1,$$

we can neglect the term $\rho \mathrm{D}\mathbf{u}/\mathrm{D}t$ in the Navier–Stokes equation so that the equations governing the fluid flow become the Stokes equations:

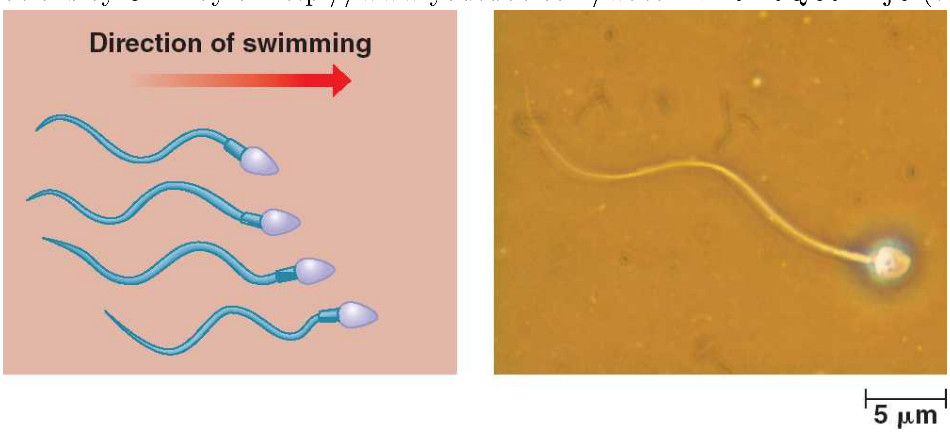
$$\mu \nabla^2 \mathbf{u} = \nabla p, \tag{4.1}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{4.2}$$

In the case of gravity driven flows, equation (4.1) is replaced by:

$$\mu \nabla^2 \mathbf{u} = \nabla P - \rho \mathbf{g}. \tag{4.3}$$

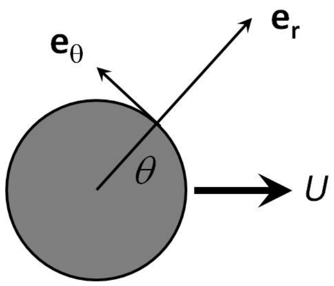
Unlike the Navier–Stokes equation, the Stokes equations are linear, which means that we can construct solutions using the principle of linear superposition. Furthermore, since there are no time derivatives the solutions are *time reversible* under the reflection $(\mathbf{u}, t) \to -(\mathbf{u}, t)$ (see demonstrations by G.I. Taylor http://www.youtube.com/watch?v=51-6QCJTAjU (t=13:13)).



This is bad news for small water-living creatures as it leads to the *scallop theorem*: if the swimming motion of a micro-organism is time reversible, it produces no net forward motion. Consequently, micro-organisms use wave motions along their flagella for propulsion, rather than the side-to-side motion employed by fish.

4.2 Translating sphere

Let us consider the Stokes flow generated by a sphere of radius a moving at speed U. Using spherical polar coordinates centred on the sphere, the velocity of the sphere is given by $(U\cos\theta, -U\sin\theta, 0)$, where θ represents the angle between the first vector of the coordinate frame and the direction of motion.



The potential flow solution for a sphere of radius a moving at speed U is given in spherical polar coordinates by:

$$\mathbf{u} = \nabla\phi = \left(U\frac{a^3}{r^3}\cos\theta, -U\frac{a^3}{2r^3}\sin\theta, 0\right).$$

However, although this satisfies the boundary condition for the r component of velocity, u, it does not satisfy the boundary condition for v. This solution also had the unrealistic property that the drag force on the sphere was identically zero: the *D'Alembert paradox*.
The flow is two-dimensional, so we may assume that the velocity is of the form:

$$\mathbf{u} = (u(r,\theta), v(r,\theta), 0)\,,$$

so that $\nabla\cdot\mathbf{u} = 0$ becomes:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v\sin\theta\right) = 0, \tag{4.4}$$

while equation (4.1) reduces (or expands) to:

$$\frac{\mu}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) - 2u - 2\frac{\partial v}{\partial\theta} - 2v\cot\theta\right] - \frac{\partial p}{\partial r} = 0, \tag{4.5}$$

$$\frac{\mu}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + 2\frac{\partial u}{\partial\theta} - \frac{v}{\sin^2\theta}\right] - \frac{1}{r}\frac{\partial p}{\partial\theta} = 0. \tag{4.6}$$

Given the boundary conditions, it is natural to seek a solution for the velocity of the form:

$$u = f(r)\cos\theta, \qquad v = g(r)\sin\theta,$$

where $f(r)$ and $g(r)$ satisfy $f(a) = U$, $g(a) = -U$ and f and g both tend to zero as $r\to\infty$. Substituting into equation (4.4) gives:

$$f + \frac{r}{2}\frac{df}{dr} + g = 0. \tag{4.7}$$

Careful inspection of equation (4.5) shows that all the terms coming from the fluid velocity are proportional to $\cos\theta$, which suggests that the pressure is of the form:

$$p(r,\theta) = p_0 + h(r)\cos\theta,$$

where p_0 is a constant. Substituting into equations (4.5) and (4.6), we obtain:

$$\frac{\mu}{r^2}\left[\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - 4f - 4g\right] - \frac{dh}{dr} = 0, \tag{4.8}$$

$$\frac{\mu}{r}\left[\frac{d}{dr}\left(r^2\frac{dg}{dr}\right) - 2g - 2f\right] + h = 0. \tag{4.9}$$

Thus, we have three coupled linear ODE for f, g and h. Eliminating h by differentiating equation (4.9), adding equation (4.8) and substituting for g from equation (4.7), we obtain:

$$\frac{d^4f}{dr^4} + \frac{8}{r}\frac{d^3f}{dr^3} + \frac{8}{r^2}\frac{d^2f}{dr^2} - \frac{8}{r^3}\frac{df}{dr} = 0. \tag{4.10}$$

This is a fourth order Cauchy equation and has solutions of the form r^m where m are the roots of:

$$m^4 + 2m^3 - 5m^2 - 6m = 0,$$

that is, $m = -3, -1, 0$ and 2. The general solution for f is of the form:

$$f = Ar^2 + B + \frac{C}{r} + \frac{D}{r^3}. \tag{4.11}$$

The boundary conditions as $r \to \infty$ implies that we must have $A = B = 0$, hence f is of the form:

$$f = \frac{C}{r} + \frac{D}{r^3}.$$

Substituting into equation (4.7):

$$g = -f - \frac{r}{2}\frac{df}{dr} = -\frac{C}{2r} + \frac{D}{2r^3}.$$

Hence, applying the wall boundary conditions ($f = U$, $g = -U$ at $r = a$), we obtain:

$$C = \frac{3aU}{2}, \qquad D = -\frac{Ua^3}{2},$$

that is:

$$u = \left(\frac{3a}{2r} - \frac{a^3}{2r^3}\right)U\cos\theta, \tag{4.12}$$

$$v = -\left(\frac{3a}{4r} + \frac{a^3}{4r^3}\right)U\sin\theta. \tag{4.13}$$

$$\tag{4.14}$$

Finally, from equation (4.9), we obtain $h = 3\mu Ua/2r^2$, giving:

$$p = p_0 + \frac{3\mu Ua}{2r^2}\cos\theta. \tag{4.15}$$

From this solution, we can calculate the drag force acting on the sphere:

$$\mathbf{F} = -\int_S \mathbf{f}\,dS = -\int_S \mathbf{n}\cdot\boldsymbol{\tau}\,dS.$$

Note that the sign is negative because this is the force applied by the fluid onto the boundary. This force is directed along the axis, so we need only consider the component in the direction $(\cos\theta, -\sin\theta, 0)$. The normal to the surface is $-\mathbf{e}_r$, so:

$$F_z = \int_S (\tau_{rr}\cos\theta - \tau_{r\theta}\sin\theta)dS.$$

To find τ_{rr} and $\tau_{r\theta}$, we need the components of the strain-rate tensor E_{rr} and $E_{r\theta}$, which, in spherical polar coordinates, are given by:

$$E_{rr} = \frac{\partial u}{\partial r}, \qquad E_{r\theta} = \frac{1}{2}\left(r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \theta}\right).$$

Hence, for $r = a$:

$$E_{rr} = 0, \qquad E_{r\theta} = \frac{3U}{4a}\sin\theta,$$

and therefore:

$$\tau_{rr}\cos\theta - \tau_{r\theta}\sin\theta = -p_0\cos\theta - \frac{3\mu U}{2a}\cos^2\theta - \frac{3\mu U}{2a}\sin^2\theta = -p_0\cos\theta - \frac{3\mu U}{2a}.$$

The term $p_0\cos\theta$ integrates to zero over the surface of the sphere, leaving:

$$F_z = -\frac{3\mu U}{2a}S,$$

where $S = 4\pi a^2$ is the surface area of the sphere. We obtain:

$$F_z = -6\pi\mu a U, \tag{4.16}$$

which is the *Stokes drag force* on a sphere, a classical result in fluid dynamics.
If the sphere is falling due to gravity, then the associated force is:

$$F_z = -\frac{4\pi a^3}{3}\Delta\rho g,$$

where $\Delta\rho$ is the difference in density between the material of the sphere and the surrounding fluid, giving a fall speed for the sphere equal to:

$$U = \frac{2\Delta\rho g a^2}{9\mu}.$$

It is tempting to try to perform the same calculation in two dimensions for a moving cylinder, however, the solution fails because, the equivalent general solution to equation (4.11) is:

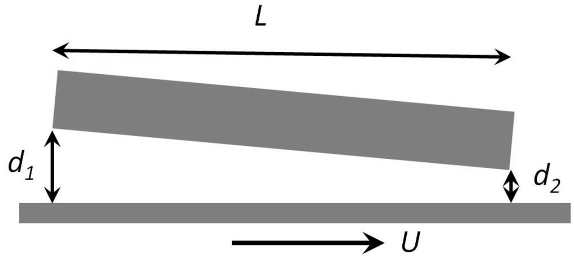
$$f = Ar^2 + B + C\log(r) + \frac{D}{r^2}.$$

and it is not possible to apply the boundary conditions at $r = a$ and $r \to \infty$ simultaneously. This result is known as *Stokes paradox* and was addressed by Oseen.

4.3 Lubrication flows

In many practical applications, the fluid flows in thin films where the boundaries are nearly parallel. These flows are "nearly" unidirectional and we can exploit this property to find approximate solutions.

4.3.1 The slider bearing flow



Consider a slider bearing where the fluid flows in the gap between the surface $y = 0$ moving at velocity $(U, 0, 0)$ and a fixed block at $y = h(x)$ of length L, where the two ends are at equal pressure p_0. We consider the case where the gap between the block and the moving surface varies slowly with x, so that $|dh/dx| \ll 1$. Specifically, we wish to consider the case:

$$h(x) = d_1 + \frac{(d_2 - d_1)}{L}x,$$

where $|h'| = |d_2 - d_1|/L \ll 1$. Since this is a two dimensional problem, we shall neglect z so that the velocity:

$$\mathbf{u} = (u(x, y), v(x, y)),$$

satisfies the following equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.17}$$

$$\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial p}{\partial x}, \tag{4.18}$$

$$\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = \frac{\partial p}{\partial y}, \tag{4.19}$$

together with the boundary conditions: $u(x, 0) = U$, $u(x, h(x)) = 0$ and $v(x, 0) = v(x, h(x)) = 0$.

Constant gap

For a constant gap, $h' = 0$, we would set the x derivatives of the velocity to zero, so that:

$$\begin{aligned} \frac{dv}{dy} &= 0, \\ \mu\frac{d^2u}{dy^2} &= \frac{\partial p}{\partial x}, \\ \mu\frac{d^2v}{dy^2} &= \frac{\partial p}{\partial y}. \end{aligned}$$

From the first of these equations, we deduce that $v(y) = 0$. The third equation gives $\partial p/\partial y = 0$ which implies that p is only a function of x:

$$\mu\frac{d^2u}{dy^2} = \frac{dp}{dx} = -G.$$

Hence, the pressure is given by

$$p = p_0 - Gx,$$

but, $p(L) = p_0$ implies $G = 0$ and hence:

$$\frac{d^2u}{dy^2} = 0.$$

The boundary conditions yields: $u = U(h - y)/h$.

Variable gap

Let us now consider $h' \ll 1$ and, for simplicity, rescale the equations (4.17) to (4.19) by choosing appropriate scales for variation in x and y. The natural choice for y is the average gap, $\bar{h} = \frac{1}{2}(d_1 + d_2)$, so we shall define:

$$y = \bar{h}y^*.$$

However, x derivatives only arise from changes in h so the appropriate scale is $\bar{h}/h'$:

$$x = \frac{\bar{h}}{h'}x^*.$$

We can additionally write the velocity and pressure in the form:

$$u(x, y) = Uu^*(x^*, y^*), \quad v(x, y) = Uv^*(x^*, y^*), \quad p(x, y) = \frac{\mu U}{\bar{h}}p^*(x^*, y^*).$$

Substituting these scalings into equations (4.17)–(4.19), we obtain:

$$h'\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{4.20}$$

$$h'^2\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} = h'\frac{\partial p^*}{\partial x^*}, \tag{4.21}$$

$$h'^2\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}} = \frac{\partial p^*}{\partial y^*}. \tag{4.22}$$

The boundary conditions on u^* and v^* become $u^*(x^*, 0) = 1$, $u^*(x^*, h^*(x^*)) = 0$ and $v^*(x^*, 0) = v^*(x^*, h^*(x^*)) = 0$, where $h(x) = \bar{h}h^*(x^*)$. Noting that $|h'| \ll 1$, we seek solutions in the form:

$$u^* = u_0^* + h'u_1^* + \ldots.$$

Substituting in equation (4.20), we find:

$$\frac{\partial v^*}{\partial y^*} = -h'\frac{\partial u_0^*}{\partial x^*} + \ldots.$$

We thus write:

$$v^* = h'v_1^* + h'^2v_2^* + \ldots. \tag{4.23}$$

To find the size of the leading order pressure term, we need to look at equations (4.21) and (4.22). From equation (4.21), we have:

$$h'\frac{\partial p^*}{\partial x^*} = \frac{\partial^2 u_0^*}{\partial y^{*2}} + \ldots,$$

and from equation (4.22):

$$\frac{\partial p^*}{\partial y^*} = h'\frac{\partial^2 v_1^*}{\partial y^{*2}} + \ldots.$$

The pressure p^* can be as large as $1/h'$ and thus takes the form:

$$p^* = \frac{1}{h'}p^*_{-1} + \dots. \tag{4.24}$$

Let us now consider the leading order terms in h' in each of the equations:

$$\frac{\partial u_0^*}{\partial x^*} + \frac{\partial v_1^*}{\partial y^*} = 0, \tag{4.25}$$

$$\frac{\partial^2 u_0^*}{\partial y^{*2}} = \frac{\partial p^*_{-1}}{\partial x^*}, \tag{4.26}$$

$$\frac{\partial p^*_{-1}}{\partial y^*} = 0. \tag{4.27}$$

The last two equations are analogous to those we found for $h' = 0$ the only change is that u_0^* is function of both x^* and y^*. Since the leading order gives $\partial p^*_{-1}/\partial y^* = 0$, we can integrate equation (4.26) to obtain:

$$u_0^* = \frac{dp^*_{-1}}{dx^*}\frac{y^{*2}}{2} + A(x^*)y^* + B(x^*),$$

and applying the boundary conditions, we obtain:

$$u_0^* = \frac{1}{2}\frac{dp^*_{-1}}{dx^*}y^*(y^* - h^*) + 1 - \frac{y^*}{h^*}.$$

To find v_1^*, we integrate equation (4.25):

$$\begin{aligned} v_1^* &= -\int_0^{y^*} \frac{\partial u_0^*}{\partial x^*} dy^* \\ &= \frac{1}{12}\frac{d^2 p^*_{-1}}{dx^{*2}} y^{*2}(3h^* - 2y^*) + \frac{1}{2}\frac{dh^*}{dx^*}\left(\frac{1}{2}\frac{dp^*_{-1}}{dx^*} - \frac{1}{h^{*2}}\right) y^{*2}. \end{aligned}$$

Hence, applying the boundary condition ($v^* = 0$ at $y^* = h^*$), we obtain:

$$\frac{1}{12}\frac{d^2 p^*_{-1}}{dx^{*2}} h^{*3} + \frac{1}{4}\frac{dp^*_{-1}}{dx^*} h^{*2}\frac{dh^*}{dx^*} = \frac{1}{2}\frac{dh^*}{dx^*},$$

which can be rewritten as:

$$\frac{d}{dx^*}\left(h^{*3}\frac{dp^*_{-1}}{dx^*}\right) = 6\frac{dh^*}{dx^*}.$$

This equation is called the *Reynolds equation*. Upon integrating, we obtain:

$$\frac{dp^*_{-1}}{dx^*} = 6\left(\frac{1}{h^{*2}} + \frac{A^*}{h^{*3}}\right),$$

for some constant A^*.
Now, let us go back to the original unstarred variables. Recall that:

$$p = \frac{\mu U}{\bar{h}}\left(\frac{1}{h'}p^*_{-1} + \dots\right),$$

so the leading order reads:

$$\frac{dp}{dx} \approx \frac{6\mu U}{h^3}(h + A), \tag{4.28}$$

where $A = \bar{h}A^*$.
For the specific case of the slider bearing $h(x) = d_1 + (d_2 - d_1)/Lx$:

$$\begin{aligned} p(L) &= p_0 - \frac{3\mu UL}{d_2 - d_1}\left[\frac{2}{h} + \frac{A}{h^2}\right]_0^L \\ &= p_0 + \frac{3\mu UL}{d_2^2 d_1^2}\left[2d_1 d_2 + A(d_1 + d_2)\right], \end{aligned}$$

and hence, to get $p_0 = p(L)$, it follows that $A = -2d_1d_2/(d_1 + d_2)$, and:

$$\frac{dp}{dx} = \frac{6\mu U(d_2 - d_1)}{h^3}\left(\frac{x}{L} - \frac{d_1}{d_1 + d_2}\right).$$

To leading order, the fluid velocity reads:

$$u(x, y) = \frac{3U(d_2 - d_1)}{h^3}\left(\frac{x}{L} - \frac{d_1}{d_1 + d_2}\right) y(y - h) + U\left(1 - \frac{y}{h}\right).$$

Note that if $d_1 > d_2$ then $p(x) > p_0$ within the bearing and so there is a positive upward force on the bearing, whereas if $d_2 > d_1$ the force is negative.

4.3.2 Alternative method for the slider bearing flow

In the above, we used a formal expansion to find the dominant terms in the governing equations by scaling and non-dimensionalising the equations. However, with sufficient experience it is possible to recognise the dominant terms in each equation without needing to rescale everything. Let us return to the original dimensional equations:

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\ \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) &= \frac{\partial p}{\partial x}, \\ \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) &= \frac{\partial p}{\partial y}, \end{aligned}$$

and seek out the largest terms in each equation by estimating the size of each term. Let us denote U the size of u, V the size of v and P the size of p. We also need estimates for the size of derivatives. Since variations in y occur over the gap h, let us denote $\partial\ /\partial y$ as being of size $1/h$, and since $h' = dh/dx$, we can estimate x derivatives as being of size h'/h. Let us now look at the terms in equation (4.17):

$$\begin{array}{ccc} \dfrac{\partial u}{\partial x} & + & \dfrac{\partial v}{\partial y} = 0, \\ \dfrac{Uh'}{h} & & \dfrac{V}{h} \end{array}$$

Since these two terms have to balance to yield a two-dimensional flow, we can deduce:

$$V \sim Uh'. \tag{4.29}$$

Notice that this is equivalent to the scaling we deduced in equation (4.23). Equation (4.18) has three terms,

$$\begin{array}{ccccc} \mu\left(\dfrac{\partial^2 u}{\partial x^2}\right. & + & \left.\dfrac{\partial^2 u}{\partial y^2}\right) & = & \dfrac{\partial p}{\partial x}. \\ \mu\dfrac{Uh'^2}{h^2} & & \mu\dfrac{U}{h^2} & & \dfrac{Ph'}{h} \end{array}$$

Since $|h'| \ll 1$, the first term is small compared with the second term, so we can deduce the scale for pressure:

$$P \sim \mu \frac{U}{hh'}, \tag{4.30}$$

the same scaling found in equation (4.24). Finally, equation (4.19) also has three terms:

$$\begin{array}{ccccc} \mu\left(\dfrac{\partial^2 v}{\partial x^2}\right. & + & \left.\dfrac{\partial^2 v}{\partial y^2}\right) & = & \dfrac{\partial p}{\partial y}. \\ \mu\dfrac{Vh'^2}{h^2} & & \mu\dfrac{V}{h^2} & & \dfrac{P}{h} \end{array}$$

We find that the scale of $\partial p/\partial y$ is larger than the terms on the left-hand side by a factor $1/h'^2$. Note that equation (4.19) yields a smaller scale for P.
By neglecting all but the dominant terms in equations (4.17)–(4.19), we obtain:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.31}$$

$$\mu\frac{\partial^2 u}{\partial y^2} \simeq \frac{\partial p}{\partial x}, \tag{4.32}$$

$$\frac{\partial p}{\partial y} \simeq 0. \tag{4.33}$$

Equation (4.33) shows that the pressure is approximately independent of y and so, integrating equation (4.32) and applying the boundary conditions at $y = 0$ and $y = h(x)$, we get:

$$u = \frac{1}{2\mu}\frac{dp}{dx}y(y-h) + U - \frac{Uy}{h},$$

so that:

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{1}{2\mu}\frac{d^2p}{dx^2}y(y-h) + \left(\frac{1}{2\mu}\frac{dp}{dx} - \frac{U}{h^2}\right)\frac{dh}{dx}y.$$

Upon integration and application of the boundary condition ($v = 0$ at $y = 0$), we get:

$$v = \frac{1}{12\mu}\frac{d^2p}{dx^2}y^2(3h-2y) + \frac{1}{2}\frac{dh}{dx}\left(\frac{1}{2\mu}\frac{dp}{dx} - \frac{U}{h^2}\right)y^2.$$

The boundary condition ($v = 0$ on $y = h$) yields the Reynolds equation:

$$\frac{h^3}{12\mu}\frac{d^2p}{dx^2} + \frac{h^2}{4\mu}\frac{dh}{dx}\frac{dp}{dx} = \frac{U}{2}\frac{dh}{dx}.$$

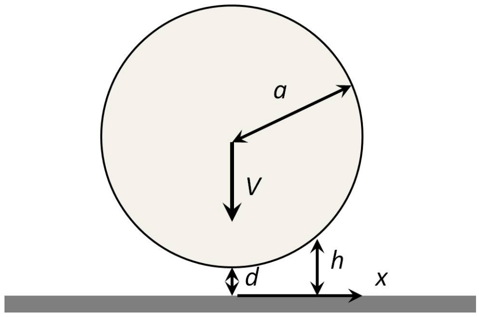
Integrating once returns:

$$\frac{dp}{dx} = \frac{6\mu U}{h^3}(h+A),$$

as found in equation (4.28) above.

4.3.3 Cylinder approaching a wall

Consider the motion of a cylinder towards a wall in the limit where the minimum gap d is small compared to the radius of the cylinder a.



We use Cartesian coordinates with the origin located at the position on the wall nearest to the cylinder with the y axis directed normal to the wall towards the cylinder and x directed in the plane of the wall perpendicular to the cylinder axis. The velocity takes the form:

$$\mathbf{u} = (u(x,y), v(x,y)).$$

In these coordinates, the position of the cylinder surface is given by:

$$h(x) = d + a - \sqrt{a^2 - x^2}.$$

However, we are interested in the flow in the region $|x| \ll a$, so:

$$h(x) \approx d + \frac{x^2}{2a}.$$

Note that $dh/dx = x/a$ is small provided that $|x| \ll a$. Let us choose the minimum gap d as the scale for y. The scale for x is less obvious. Let us write:

$$h(x) = d\left(1 + \frac{x^2}{2ad}\right).$$

We can see that variations in h are felt for $x \sim \sqrt{ad}$, so we shall rescale x and y as follows:

$$x = \sqrt{ad}x^* = \frac{d}{\epsilon}x^*, \quad y = dy^*,$$

where $\epsilon = \sqrt{d/a} \ll 1$. With this scaling, $h(x) = dh^*(x^*)$, where:

$$h^*(x^*) = 1 + \frac{x^{*2}}{2}. \tag{4.34}$$

The governing equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.35}$$

$$\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial p}{\partial x}, \tag{4.36}$$

$$\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = \frac{\partial p}{\partial y}, \tag{4.37}$$

with boundary conditions: $u(x,0) = v(x,0) = 0$, $u(x,h(x)) = 0$ and $v(x,h(x)) = -V$. The velocity v being of size V, we can expand:

$$v(x,y) = V\left(v_0(x^*,y^*) + \epsilon v_1(x^*,y^*) + \dots\right),$$

and therefore from conservation of mass:

$$\frac{\partial u}{\partial x} \sim \frac{V}{d},$$

so that u must be of size $\epsilon^{-1}V$:

$$u(x,y) = V\left(\epsilon^{-1}u_{-1}(x^*,y^*) + \dots\right).$$

Thus, the flow mainly goes in the x direction. Turning to equation (4.36), we have:

$$\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 u}{\partial x^2} = \mu\frac{V}{d^2\epsilon}\frac{\partial^2 u^*_{-1}}{\partial y^{*2}} + \dots,$$

and hence:

$$p \sim \frac{\mu V}{d\epsilon^2}.$$

Therefore, the pressure can be expanded in the following fashion:

$$p(x,y) = \frac{\mu V}{d}\left(\epsilon^{-2}p^*_{-2}(x^*,y^*) + \dots\right).$$

Substituting into equations (4.35) to (4.37) and keeping only the leading order terms in ϵ, we have:

$$\frac{\partial u^*_{-1}}{\partial x^*} + \frac{\partial v^*_0}{\partial y^*} = 0, \tag{4.38}$$

$$\frac{\partial^2 u^*_{-1}}{\partial y^{*2}} = \frac{\partial p^*_{-2}}{\partial x^*}, \tag{4.39}$$

$$\frac{\partial p^*_{-2}}{\partial y^*} = 0, \tag{4.40}$$

with boundary conditions $u^*_{-1}(x^*,0) = u^*_{-1}(x^*,h^*) = 0$, $v^*_0(x^*,0) = 0$ and $v^*_0(x^*,h^*) = -1$. As before, $\partial p^*_{-2}/\partial y^* = 0$, so that the solution of equation (4.39), satisfying the boundary conditions, is given by:

$$u^*_{-1} = \frac{1}{2}\frac{dp^*_{-2}}{dx^*}y^*(y^*-h^*). \tag{4.41}$$

From equation (4.38), we get:

$$v^*_0 = \frac{1}{12}\frac{d^2p^*_{-2}}{dx^{*2}}y^{*2}(3h^*-2y^*) + \frac{1}{4}\frac{dp^*_{-2}}{dx^*}\frac{dh^*}{dx^*}y^{*2}.$$

Applying the boundary condition at $y^* = h^*$, we obtain:

$$-1 = \frac{h^{*3}}{12}\frac{d^2p^*_{-2}}{dx^{*2}} + \frac{h^{*2}}{4}\frac{dp^*_{-2}}{dx^*}\frac{dh^*}{dx^*}.$$

The associated *Reynolds equation* is:

$$\frac{d}{dx^*}\left(h^{*3}\frac{dp^*_{-2}}{dx^*}\right) = -12. \tag{4.42}$$

Integrating once, we find:

$$\frac{dp^*_{-2}}{dx^*} = -12\frac{x^*}{h^{*3}} + \frac{A^*}{h^{*3}}.$$

Since this flow is symmetric about $x = 0$, $dp^*_{-2}/dx^* = 0$ at $x^* = 0$ and so $A^* = 0$. It follows:

$$u^*_{-1} = -\frac{6x^*}{h^{*3}} y^*(y^* - h^*).$$

Recall from equation (4.34) that $dh^*/dx^* = x^*$ which yields:

$$p^*_{-2}(x^*) = p^*_\infty + \frac{6}{h^{*2}},$$

where p^*_∞ is the dimensionless pressure at infinity.
Having found the pressure, we can now calculate the force that the cylinder exerts on the fluid. This force is oriented in the y direction and its amplitude given by:

$$F_y = L \int_{-a}^{a} f_y dx,$$

where L is the length of the cylinder and $\mathbf{f} = \mathbf{n} \cdot \boldsymbol{\tau}$. Here:

$$\mathbf{n} = \frac{1}{\sqrt{1 + h'^2}} \Big(-h', 1 \Big),$$

where $h' = \partial h/\partial_x$. For $|x| \ll a$, $\mathbf{n} \approx (-h', 1)$ and hence:

$$f_y = -p - \mu \frac{dh}{dx} \left(\frac{du}{dy} + \frac{dv}{dx} \right) + 2\mu \frac{dv}{dy}.$$

Since the pressure is of size $\epsilon^{-2}\mu V/d$, it constitutes the dominant contribution to f_y:

$$F_y = -L \int_{-a}^{a} (p(x) - p_\infty) dx.$$

Changing variables to $x^* = \epsilon d^{-1} x = \epsilon^{-1} a^{-1} x$, we get:

$$\begin{aligned} F_y &= -\frac{\epsilon^{-2}\mu V L}{d} \int_{-\epsilon^{-1}}^{\epsilon^{-1}} \left(p^*_{-2}(x^*) - p^*_\infty \right) d\epsilon^{-1} dx^* \\ &= -\epsilon^{-3} 6\mu V L \int_{-\epsilon^{-1}}^{\epsilon^{-1}} \frac{1}{(1 + \frac{x^{*2}}{2})^2} dx^*. \end{aligned}$$

To perform this integral, we substitute $x^* = \sqrt{2} \tan u$ and obtain:

$$\begin{aligned} F_y &= -\epsilon^{-3} 6\sqrt{2} \mu V L \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 u du \\ &= -\frac{9\sqrt{2}}{4} \pi \mu V L \left(\frac{a}{d} \right)^{3/2}. \end{aligned}$$

If the cylinder is falling towards the wall under a constant force $F = -Mg$:

$$V = \frac{4Mg d^{3/2}}{9\sqrt{2}\pi\mu L a^{3/2}} = -\frac{d}{dt} d,$$

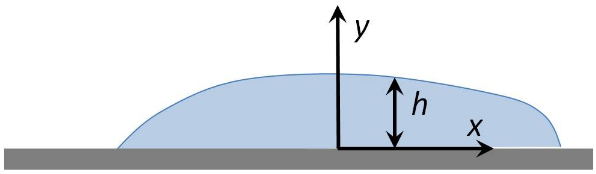
and therefore:

$$\frac{1}{\sqrt{d(t)}} = \frac{1}{\sqrt{d_0}} + \frac{2Mg}{9\sqrt{2}\pi\mu L a^{3/2}} t,$$

where d_0 is the gap at $t = 0$. The cylinder will not contact the wall in finite time.

4.3.4 Free surface flows

Let us now consider the gravitational spreading of a blob of viscous fluid on a surface.



For simplicity, we shall consider a two-dimensional blob whose height is given by $y = h(x,t)$. In order to use lubrication theory, we shall assume that $|dh/dx \ll 1|$ so that the normal to the surface is approximately $(-h', 1)$. At leading order, the boundary conditions at $y = h$ are:

$$-P + 2\mu\frac{\partial v}{\partial y} = -P_{atm}, \qquad \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0,$$

together with $u = v = 0$ at $y = 0$. The kinematic boundary condition at the free surface requires that:

$$v = \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}. \tag{4.43}$$

The Stokes equations read:

$$\begin{aligned}\nabla P &= \rho\mathbf{g} + \mu\nabla^2\mathbf{u},\\ \nabla\cdot\mathbf{u} &= 0,\end{aligned}$$

and reduce to:

$$\frac{\partial P}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{4.44}$$

$$\frac{\partial P}{\partial y} = -\rho g + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{4.45}$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \tag{4.46}$$

Rather than scale these equations, let us look for the dominant terms. We shall use U and V to denote the size of the velocity components u and v respectively, and take $1/h$ as the size of $\partial\ /\partial y$, and since $h' = dh/dx$ we can estimate x derivatives as being of size h'/h.
Since this flow is driven by gravity, P and ρg must balance so that $P \sim \rho gh$. Equation (4.44) gives:

$$\begin{array}{ccccc}\dfrac{\partial P}{\partial x} & = \mu & \Bigg(\dfrac{\partial^2 u}{\partial x^2} & + & \dfrac{\partial^2 u}{\partial y^2}\Bigg). \\[2ex] h'\rho g & & \dfrac{\mu U}{h^2}h'^2 & & \dfrac{\mu U}{h^2}\end{array}$$

Hence, we can estimate that:

$$U \sim \frac{\rho g h^2}{\mu}h',$$

and neglect the x derivatives of u so that:

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{1}{\mu}\frac{\partial P}{\partial x}.$$

Furthermore, from equation (4.46):

$$V \sim h'U \sim \frac{\rho g h^2}{\mu} h'^2.$$

Hence, the sizes of the terms in equation (4.45) are:

$$\begin{array}{ccccccc} \dfrac{\partial P}{\partial y} & = -\rho g & + & \mu \Bigg(\dfrac{\partial^2 v}{\partial x^2} & + & \dfrac{\partial^2 v}{\partial y^2} \Bigg). \\ \rho g & \rho g & & h'^4 \rho g & & h'^2 \rho g \end{array}$$

Equation (4.45) reduces to:

$$\frac{\partial P}{\partial y} = -\rho g,$$

so that

$$P = -\rho g y + A(x).$$

Furthermore, since $|dv/dy| \sim h'^2(\rho g h)/\mu$, we can also neglect this term in the boundary condition. As $P = P_{atm}$ at $y = h$:

$$P = P_{atm} + \rho g(h - y), \tag{4.47}$$

and therefore $\partial P/\partial x = \rho g \partial h/\partial x$. It follows:

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{\rho g}{\mu} \frac{\partial h}{\partial x}.$$

Again since $|dv/dx| \sim h'^3(\rho g h)/\mu$, we can neglect its contribution to the boundary conditions and the boundary conditions on u simplify to $u = 0$ on $y = 0$ and $du/dy \approx 0$ on $y = h$. The leading order solution for $|h'| \ll 1$ is:

$$u = -\frac{\rho g}{2\mu} \frac{\partial h}{\partial x} y(2h - y). \tag{4.48}$$

From equation (4.46):

$$v = \frac{\rho g}{6\mu} \left[\frac{\partial^2 h}{\partial x^2} y^2 (3h - y) + 3 \left(\frac{\partial h}{\partial x} \right)^2 y^2 \right],$$

and so, at $y = h$:

$$v = \frac{\rho g}{6\mu} \left[2 \frac{\partial^2 h}{\partial x^2} h^3 + 3 \left(\frac{\partial h}{\partial x} \right)^2 h^2 \right].$$

Substituting into the kinematic boundary condition gives:

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu} \left[\frac{\partial^2 h}{\partial x^2} h^3 + 3 \left(\frac{\partial h}{\partial x} \right)^2 h^2 \right] = \frac{\rho g}{3\mu} \frac{\partial}{\partial x} \left(h^3 \frac{\partial h}{\partial x} \right). \tag{4.49}$$

This is a non-linear diffusion equation for $h(x, t)$.

4.4 Hele-Shaw flow

Let us consider now the case of the general flow in the gap between two parallel plates separated by a distance h.

At the boundaries $z = 0$ and $z = h$, we impose no slip boundary conditions: $\mathbf{u} = 0$. We assume that the variations of $\mathbf{u}$ in the (x, y) plane are slow so that $\nabla^2 \mathbf{u} \approx \partial^2 \mathbf{u}/\partial z^2$.
The Stokes equations reduce to:

$$\frac{\partial p}{\partial x} \approx \mu \frac{\partial^2 u}{\partial z^2},$$
$$\frac{\partial p}{\partial y} \approx \mu \frac{\partial^2 v}{\partial z^2},$$
$$\frac{\partial p}{\partial z} \approx 0,$$

and the mass conservation equation reads:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

The velocity in the (x, y) plane is given by:

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x} z(h - z),$$
$$v = -\frac{1}{2\mu}\frac{\partial p}{\partial y} z(h - z).$$

Integrating over z, we obtain the average velocity in the (x, y) plane:

$$\bar{u} = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{1}{h}\int_0^h z(h - z)dz = -\frac{h^2}{12\mu}\frac{\partial p}{\partial x},$$
$$\bar{v} = -\frac{1}{2\mu}\frac{\partial p}{\partial y}\frac{1}{h}\int_0^h z(h - z)dz = -\frac{h^2}{12\mu}\frac{\partial p}{\partial y},$$

which we can write as:

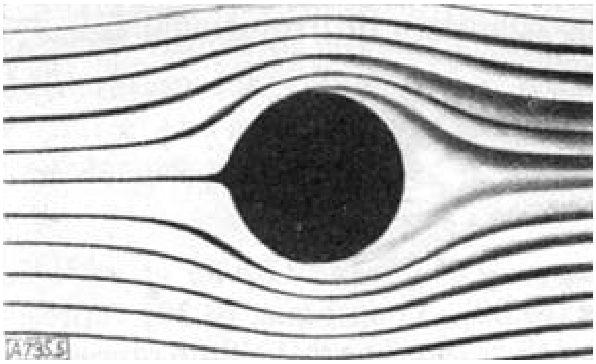
$$\bar{\mathbf{u}} = -\frac{h^2}{12\mu}\nabla_H p, \tag{4.50}$$

where ∇_H is the restriction of the gradient operator to the (x, y) plane.
Hence, the two-dimensional flow in the (x, y) plane is a potential flow with the velocity potential $\Phi = -h^2/12\mu p$. Furthermore, integrating the continuity equation between 0 and h, we obtain:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0,$$

and so this flow is incompressible and hence:

$$\nabla_H \cdot \bar{\mathbf{u}} = 0. \tag{4.51}$$



Chapter 5

Vorticity diffusion and boundary layers

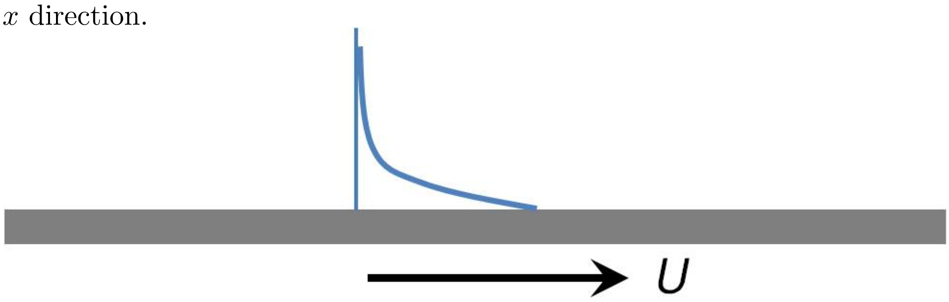
Contents

In this chapter we will turn our attention to high Reynolds number flows. As we noted earlier, it is mathematically dangerous to ignore the viscous terms in the Navier–Stokes equation because, by removing the highest spatial derivative, we are no longer able to satisfy all the boundary conditions at a solid boundary.

5.1 Start-up flows

5.1.1 Flow near an impulsively moved boundary

To illustrate what happens near a boundary, let us consider the flow above a solid wall at $y = 0$. Initially, the fluid is at rest. At time $t = 0$, the boundary starts to move with velocity U in the x direction.



In this simple flow, we can assume that:

$$\mathbf{u} = (u(y,t), 0, 0).$$

Also, since this flow is driven by the motion of the boundary and not an external pressure gradient, we can assume that $\partial p/\partial x = 0$.
The Navier–Stokes equation:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu \nabla^2 \mathbf{u},$$

reduces to:

$$\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2}, \tag{5.1}$$

and is accompanied with the boundary conditions: $u = U$ on $y = 0$ and $U \to 0$ as $y \to \infty$. We also impose the initial condition: $u = 0$ at $t = 0$.
The velocity $u(x,t)$ thus satisfies the diffusion equation with diffusivity $\nu = \mu/\rho$, where ν is the kinematic viscosity. This problem is equivalent to that of finding the temperature distribution in a semi-infinite bar when one end is suddenly heated to a constant temperature.
We seek a *similarity solution*:

$$u(y,t) = f(\eta), \text{ where } \eta = yt^a,$$

for some constant a. Using the chain rule:

$$\frac{\partial}{\partial y} = t^a\frac{d}{d\eta},$$
$$\frac{\partial}{\partial t} = ayt^{a-1}\frac{d}{d\eta},$$

so that equation (5.1) becomes:

$$ayt^{a-1}\frac{df}{d\eta} = \nu t^{2a}\frac{d^2 f}{d\eta^2},$$

and therefore:

$$\frac{d^2 f}{d\eta^2} - \frac{ayt^{-a-1}}{\nu}\frac{df}{d\eta} = 0.$$

For the similarity solution to exist, this equation must only contain y and t in the combination $\eta = yt^a$ and therefore $-a-1 = a$. We get: $a = -\frac{1}{2}$. Solutions thus exists for the similarity variable $\eta = y/\sqrt{t}$ and satisfy:

$$\frac{d^2 f}{d\eta^2} + \frac{\eta}{2\nu}\frac{df}{d\eta} = 0.$$

Substituting $v = df/d\eta$ we have:

$$\frac{dv}{d\eta} = -\frac{\eta}{2\nu}v,$$

which has general solution:

$$v = \frac{df}{d\eta} = A\exp\left(-\frac{\eta^2}{4\nu}\right).$$

Integrating again, we obtain:

$$f = A\int_0^\eta \exp\left(-\frac{\eta^2}{4\nu}\right)d\eta + B.$$

The above integral can be expressed in terms of the error function:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-x^2\right)dx.$$

Substituting $x = \eta/2\sqrt{\nu}$, we have:

$$f = A\sqrt{\nu\pi}\text{erf}\left(\frac{\eta}{2\sqrt{\nu}}\right) + B.$$

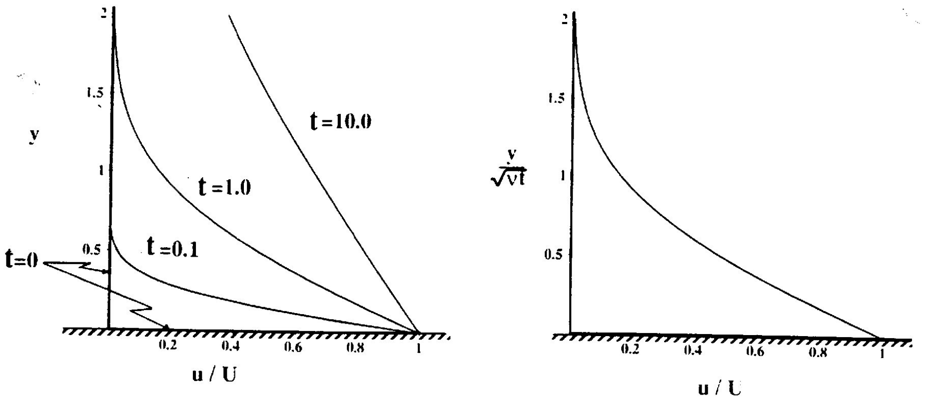
In terms of the original variables, this solution reads:

$$u(y,t) = A\sqrt{\nu\pi}\mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right) + B.$$

The boundary conditions on $y = 0$ imposes $B = U$ and, since $\mathrm{erf}(x) \to 1$ as $x \to \infty$, the other boundary condition requires $A\sqrt{\nu\pi} = -U$. We eventually get:

$$u(y,t) = U\left[1 - \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right]. \tag{5.2}$$

Finally, $u(y,0) = 0$ holds for all $y > 0$, so all the boundary conditions are satisfied.
The velocity $u(y,t)$ will be approximately zero wherever $y/2\sqrt{\nu t}$ is large. In addition, for a fixed value of y, the velocity will remain less than $0.01U$ until a time t such that $y \approx 4\sqrt{\nu t}$. Hence, at time t, the fluid is only moving within a narrow region of thickness $4\sqrt{\nu t}$. This narrow region is called the *viscous boundary layer*. Note that the boundary layer thickness is independent of U.



As an example, let us consider water for which $\nu \approx 10^{-6}\mathrm{m}^2 \cdot \mathrm{s}^{-1}$. After one second, the boundary layer thickness is around 4mm. After 100 seconds, it is still only 4cm in size. For lower viscosity fluids, the effects of the boundary are felt in a narrower region next to the boundary.

5.1.2 Start-up of shear flow

Let us now modify the previous problem by considering the start-up of a shear flow between two parallel plates located at $y = 0$ and $y = h$. Once again, we begin to move the lower plate with velocity U at $t = 0$. The problem is the same as that above except that the boundary condition at infinity is replaced by one at $y = h$. The velocity now satisfies:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{5.3}$$

together with the boundary conditions: $u(0,t) = U$ and $u(h,t) = 0$, and the initial condition $u(y,0) = 0$.
First, we observe that the steady solution $u_s = U\,(1 - y/h)$ satisfies the equation and the boundary conditions. We then write:

$$u(y,t) = u_s + v(y,t),$$

and seek a separable solution of the form:

$$v(y,t) = T(t)Y(y).$$

This gives:

$$YT' = \nu TY'',$$

so that:

$$\frac{Y''}{Y} = \frac{1}{\nu}\frac{T'}{T} = k,$$

where k is the constant of integration. Since u_s takes care of the moving boundary, we want to find solutions satisfying $Y(0) = Y(h) = 0$. We thus choose solutions of the form:

$$Y(y) = \sin\left(\frac{n\pi y}{h}\right),$$

so that:

$$\frac{Y''}{Y} = -\frac{n^2\pi^2}{h^2}.$$

It follows:

$$\frac{T'}{T} = -\frac{\nu n^2\pi^2}{h^2},$$

and so we have separable solutions of the form:

$$v_n = \exp\left(-\frac{\nu n^2\pi^2}{h^2}t\right)\sin\left(\frac{n\pi y}{h}\right).$$

The general solution for u satisfying the boundary conditions is:

$$u(y,t) = U\left(1 - \frac{y}{h}\right) + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{\nu n^2\pi^2}{h^2}t\right)\sin\left(\frac{n\pi y}{h}\right).$$

The initial condition at $t = 0$ requires:

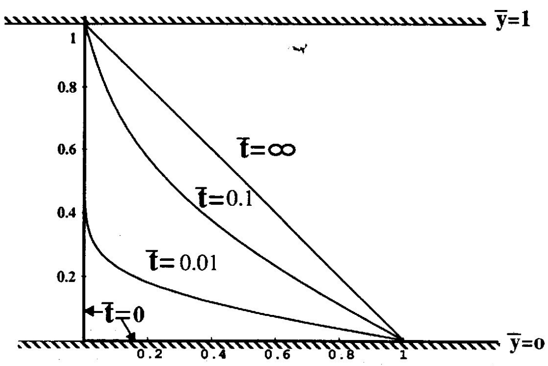
$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{h}\right) = -U\left(1 - \frac{y}{h}\right),$$

for $0 < y < h$. We can determine the a_n using Fourier series properties:

$$a_n = \frac{2U}{h}\int_0^h \left(\frac{y}{h} - 1\right)\sin\left(\frac{n\pi y}{h}\right)dy = -\frac{2U}{n\pi},$$

Hence, the solution is:

$$u(y,t) = U\left(1 - \frac{y}{h}\right) - \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{\nu n^2\pi^2}{h^2}t\right)\sin\left(\frac{n\pi y}{h}\right). \tag{5.4}$$



This flow resembles that of the unbounded plate until the boundary layer grows to the width of the channel. The solution then approaches the steady state u_s. Note that the slowest decaying exponential in the sum corresponds to $n = 1$. As a result, the flow reaches u_s on a time of order $h^2/(\nu\pi^2)$. For water in a 1cm channel, this time is about 10s and scales inversely with ν so that in a fluid of lower viscosity it becomes longer.

5.2 Vorticity dynamics

A more fundamental way of viewing these flows is to consider the vorticity of the flow: $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. Starting from the Navier–Stokes equation and dividing by ρ, we have:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}.$$

By using the vector identity:

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \cdot \nabla \mathbf{u},$$

we can rewrite this as:

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\nabla \left(\frac{p}{\rho} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} \right) + \nu \nabla^2 \mathbf{u}.$$

Taking the curl of this equation yields:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = \nu \nabla^2 \boldsymbol{\omega}.$$

Using the vector identity:

$$\nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = \mathbf{u}\nabla \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \nabla \mathbf{u} - \boldsymbol{\omega}\nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \boldsymbol{\omega},$$

noting that $\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \times \mathbf{u}) = 0$ and considering the fluid incompressible, we have:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = \frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}. \tag{5.5}$$

The left-hand side of this equation represents the rate of change of vorticity of a fluid particle. The term $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$ corresponds to the vorticity enhancement by velocity gradients parallel to the direction of $\boldsymbol{\omega}$. It represents *vortex stretching*. Finally, $\nu \nabla^2 \mathbf{u}$ is the diffusion of vorticity due to viscosity.

For the special case of a two-dimensional planar flow:

$$\mathbf{u} = (u(x, y), v(x, y), 0),$$

the vorticity reads:

$$\boldsymbol{\omega} = (0, 0, \omega), \text{ where } \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

In this case, the vorticity equation reduces to:

$$\frac{D\omega}{Dt} = \nu \nabla^2 \omega. \tag{5.6}$$

Hence, in the limit $\nu \to 0$, the vorticity of a fluid particle remains constant. It is simply advected with the flow. When viscosity is considered, the vorticity obeys a diffusion equation (in the frame of the moving fluid).

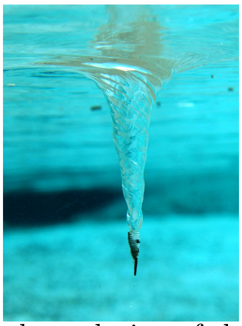
5.2.1 Impulsively moving boundary revisited

Let us now return to the case of the flow above a boundary that is set in motion at time $t = 0$. Initially, the vorticity is zero everywhere, except at $y = 0$ where the fluid velocity jumps from U to 0. At time t, the velocity is given by equation (5.2). The vorticity ω reads:

$$\omega = -\frac{\partial u}{\partial y} = \frac{U}{\sqrt{\pi \nu t}} \exp\left(-\frac{y^2}{4\nu t} \right).$$

This is a Gaussian distribution of standard deviation $\sqrt{2\nu t}$. Hence, as times increases, the vorticity gradually spreads away from the boundary over a distance of order $\sqrt{2\nu t}$.

5.2.2 Decay of a line vortex



Let us consider the decay of a line vortex in which the fluid velocity is given in plane polar coordinates (r, θ) by:

$$\mathbf{u} = \frac{\Gamma}{2\pi r}\mathbf{e}_\theta.$$

This is an idealised version of the "bath plug" vortex.

If we assume that this flow remains axisymmetric, then we can seek a solution of the form:

$$\mathbf{u} = v(r,t)\mathbf{e}_\theta.$$

The θ component of the Navier–Stokes equation becomes:

$$\rho\frac{\partial v}{\partial t} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{v}{r^2}\right].$$

Since the pressure is periodic, $\partial p/\partial\theta = 0$, and thus:

$$\frac{\partial v}{\partial t} = \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right]. \tag{5.7}$$

At $t = 0$, $v(r,0) = \Gamma/2\pi r$, so we seek a solution of the form:

$$v(r,t) = \frac{\Gamma}{2\pi r}f(\eta),$$

where $\eta = r/\sqrt{t}$, so that:

$$\begin{aligned}
\frac{\partial v}{\partial t} &= -\frac{\Gamma}{2\pi}\frac{1}{2t^{3/2}}\frac{df}{d\eta},\\
\frac{\partial v}{\partial r} &= \frac{\Gamma}{2\pi r^2}\left(\eta\frac{df}{d\eta} - f\right),\\
\frac{\partial^2 v}{\partial r^2} &= \frac{\Gamma}{2\pi r^3}\left(\eta^2\frac{d^2 f}{d\eta^2} - 2\eta\frac{df}{d\eta} + 2f\right).
\end{aligned}$$

Substituting into equation (5.7), we have:

$$-\frac{\Gamma}{2\pi r^3}\frac{\eta^3}{2}\frac{df}{d\eta} = \nu\frac{\Gamma}{2\pi r^3}\left(\eta^2\frac{d^2 f}{d\eta^2} - 2\eta\frac{df}{d\eta} + 2f + \eta\frac{df}{d\eta} - f - f\right),$$

which simplifies to:

$$-\frac{df}{d\eta} = 2\nu\left(\frac{1}{\eta}\frac{d^2 f}{d\eta^2} - \frac{1}{\eta^2}\frac{df}{d\eta}\right) = 2\nu\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df}{d\eta}\right).$$

Integrating both sides yields:

$$-f = \frac{2\nu}{\eta}\frac{df}{d\eta} + A.$$

We can thus write:

$$\frac{df}{d\eta} + \frac{\eta}{2\nu}f = -A\frac{\eta}{2\nu}.$$

And the solution is:

$$f = -A + B\exp\left(-\frac{\eta^2}{4\nu}\right),$$

giving:

$$v(r,t) = \frac{\Gamma}{2\pi r}\left[-A + B\exp\left(-\frac{r^2}{4\nu t}\right)\right].$$

Taking the limit $t \to 0$:

$$v(r,0) \to -A\frac{\Gamma}{2\pi r},$$

yielding $A = -1$. However, for the solution to be bounded at $r = 0$, we need $B = A = -1$. The velocity solution then reads:

$$v(r,t) = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right]. \tag{5.8}$$

We find that the viscosity only affects a region of size of order $2\sqrt{\nu t}$.

i. For $r \gg 2\sqrt{\nu t}$, the solution remains close to the initial irrotational flow with:

$$v \approx \frac{\Gamma}{2\pi r}.$$

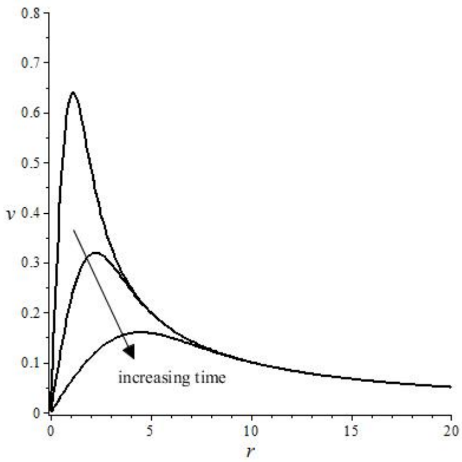
ii. For $r \ll 2\sqrt{\nu t}$, we can expand the exponential as:

$$v \approx \frac{\Gamma}{2\pi r}\left(1 - 1 + \frac{r^2}{4\nu t} + \dots\right),$$

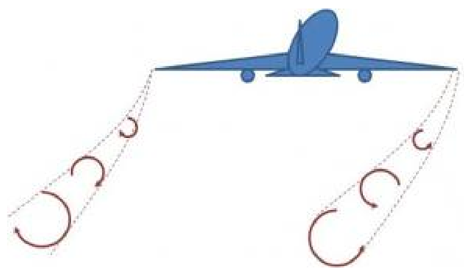
so that:

$$v \approx \frac{\Gamma r}{8\pi\nu t}.$$

This corresponds to solid body rotation at an angular velocity of $\Gamma/8\pi\nu t$.



The intensity of the vortex decreases with time as the “core” spreads out radially. This process is quite slow for low viscosity fluids, which is why vortex lines persist for a large distance/long time behind an aircraft.



5.3 High Reynolds number flows

In the above examples, the flow is two dimensional, so $\mathbf{u} \cdot \nabla\omega = 0$ and the vorticity equation (5.5) reduces to the diffusion equation. By balancing the terms on either side of this equation, we can obtain an estimate for the thickness δ of the boundary layer. If Ω is the approximate magnitude of the vorticity then:

$$|\nu\nabla^2\omega| \sim \nu\frac{\Omega}{\delta^2},$$

while:

$$\left|\frac{\partial\omega}{\partial t}\right| \sim \frac{\Omega}{t}.$$

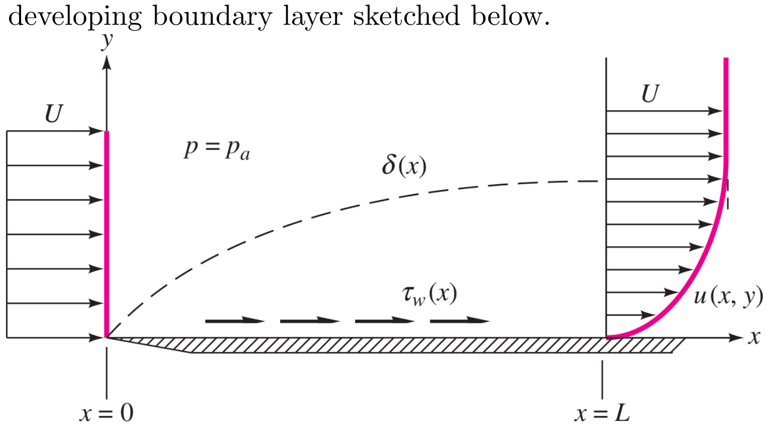
Combining both expressions, we get:

$$\frac{\Omega}{t} \sim \nu\frac{\Omega}{\delta^2},$$

giving $\delta^2 \sim \nu t$. The boundary layer thickness is $O(\sqrt{\nu t})$, as we have seen in the previous examples.

5.3.1 Blasius boundary layer

We consider the developing boundary layer sketched below.



Such a two-dimensional stationary flow is governed by the following Navier–Stokes equations:

$$\rho\left(u\partial_x u + v\partial_y u\right) = -\partial_x p + \mu\left(\partial_x^2 u + \partial_y^2 u\right), \tag{5.9}$$

$$\rho\left(u\partial_x v + v\partial_y v\right) = -\partial_y p + \mu\left(\partial_x^2 v + \partial_y^2 v\right), \tag{5.10}$$

where ρ is the fluid density, u is the streamwise (x-) velocity, v the wall-normal (y-) velocity, p is the pressure and μ is the fluid dynamic viscosity. The flow is incompressible, so we pose:

$$\partial_x u + \partial_y v = 0. \tag{5.11}$$

Lastly, the boundary conditions are:

$$u = v = 0 \qquad \text{at } y = 0, \tag{5.12}$$

$$(u, v) \to (U, 0) \qquad \text{at } y \to \infty. \tag{5.13}$$

Asymptotically reduced equations

Dynamics occur on a(n arbitrary) length scale L in the streamwise direction and δ, the boundary layer thickness, in the wall-normal direction. The spatial derivatives then follow the scalings:

$$\partial_x \sim \frac{1}{L}, \qquad \partial_y \sim \frac{1}{\delta}. \tag{5.14}$$

In addition, these length scales are not comparable: $L \gg \delta$. We can thus introduce a small parameter $\epsilon \ll 1$ such that:

$$\frac{\delta}{L} = \epsilon. \tag{5.15}$$

The streamwise velocity u is of the same order of magnitude as the velocity infinitely far away from the plate U but the order of magnitude of the wall-normal velocity v is yet to be determined. To that end, we assume that both terms in the incompressibility constraint are of the same order of magnitude:

$$\frac{U}{L} \sim \frac{v}{\delta}, \tag{5.16}$$

which gives:

$$v \sim \frac{U\delta}{L} \tag{5.17}$$

$$\Rightarrow \qquad v \sim \epsilon\, U, \tag{5.18}$$

implying that the wall-normal velocity is smaller than the streamwise one.
We can now rescale the wall-normal quantities according to the streamwise quantities. We define the rescaled quantities by:

$$x^* = \frac{x}{L}, \tag{5.19}$$

$$y^* = \frac{y}{\delta} = \frac{y}{\epsilon\, L}, \tag{5.20}$$

$$u^* = \frac{u}{U}, \tag{5.21}$$

$$v^* = \frac{v}{\epsilon\, U}, \tag{5.22}$$

$$p^* = \frac{p}{\rho U^2}, \tag{5.23}$$

where the pressure is nondimensionalised in such a way that it remains of the same order of magnitude as the other terms.
Remark: in the case of an incompressible flow, the pressure can be thought of as a mathematical function whose role is to ensure incompressibility.
Using the dimensionless variables, the incompressibility constraint reads:

$$\frac{U}{L}\partial_{x^*} u^* + \frac{\epsilon U}{\epsilon L}\partial_{y^*} v^* = 0, \tag{5.24}$$

yielding:

$$\partial_{x^*} u^* + \partial_{y^*} v^* = 0. \tag{5.25}$$

Similarly, the Navier–Stokes equation becomes:

$$\rho\left(\tfrac{U^2}{L}u^*\partial_{x^*}u^* + \tfrac{U^2}{L}v^*\partial_{y^*}u^*\right) = -\frac{\rho U^2}{L}\partial_{x^*}p^* + \mu\left(\frac{U}{L^2}\partial^2_{x^*}u^* + \frac{U}{\epsilon^2 L^2}\partial^2_{y^*}u^*\right), \tag{5.26}$$

$$\rho\left(\tfrac{\epsilon U^2}{L}u^*\partial_{x^*}v^* + \tfrac{\epsilon U^2}{L}v^*\partial_{y^*}v^*\right) = -\frac{\rho U^2}{\epsilon L}\partial_{y^*}p^* + \mu\left(\frac{\epsilon U}{L^2}\partial^2_{x^*}v^* + \frac{U}{\epsilon L^2}\partial^2_{y^*}v^*\right), \tag{5.27}$$

and simplifies into:

$$u^*\partial_{x^*}u^* + v^*\partial_{y^*}u^* = -\partial_{x^*}p^* + \frac{1}{Re_L}\partial^2_{x^*}u^* + \frac{1}{\epsilon^2 Re_L}\partial^2_{y^*}u^*, \tag{5.28}$$

$$u^*\partial_{x^*}v^* + v^*\partial_{y^*}v^* = -\frac{1}{\epsilon^2}\partial_{y^*}p^* + \frac{1}{Re_L}\partial^2_{x^*}v^* + \frac{1}{\epsilon^2 Re_L}\partial^2_{y^*}v^*, \tag{5.29}$$

where we have introduced the Reynolds number $Re_L = \rho UL/\mu$.
We are interested in $Re_L \gg 1$. As $\epsilon \ll 1$, the term $\partial^2_{x^*} u^*/Re_L$ is the smallest term in equation (5.28) and $\partial^2_{x^*} v^*/Re_L$ is the smallest term in equation (5.29). We drop them.
To keep a balance between advection (left-hand-side) and diffusion (right-hand-side), and therefore retain sensible physics, we impose $\epsilon^2 Re_L = 1$. As a result, the small quantity we have introduced is now related to the Reynolds number:

$$\epsilon = \frac{\delta}{L} = Re_L^{-1/2}. \tag{5.30}$$

Consequently, the leading order of system (5.28), (5.29) is:

$$u^*\partial_{x^*} u^* + v^*\partial_{y^*} u^* = -\partial_{x^*} p^* + \partial^2_{y^*} u^*, \tag{5.31}$$
$$\partial_{y^*} p^* = 0, \tag{5.32}$$

where equation (5.32) implies that the pressure does not vary across the boundary layer, but only along it.
We can then express the pressure at any point in the critical layer by applying the Bernoulli equation on a streamline away from the boundary layer:

$$p + \frac{\rho u^2}{2} = \text{cst}, \tag{5.33}$$

which gives in dimensionless form:

$$p^* + \frac{{u^*}^2}{2} = \text{cst}. \tag{5.34}$$

Outside the boundary layer, the velocity is $u^* = 1$, so we have:

$$p^* + \frac{1}{2} = \text{cst}, \tag{5.35}$$
$$\Rightarrow \qquad \partial_{x^*} p^* = 0, \tag{5.36}$$

a relation valid for all y^*.
The resulting set of reduced equations for the boundary layer writes:

$$\partial_{x^*} u^* + \partial_{y^*} v^* = 0, \tag{5.37}$$
$$u^*\partial_{x^*} u^* + v^*\partial_{y^*} u^* = \partial^2_{y^*} u^*, \tag{5.38}$$

and is to be solved together with the following boundary conditions:

$$u^* = v^* = 0 \qquad \text{at } y^* = 0, \tag{5.39}$$
$$u^* \to 1 \qquad \text{at } y^* \to \infty. \tag{5.40}$$

A second boundary condition for v is not necessary as v is only derived once with respect to y in the above equations.

The Blasius equation

While studying the dimensional boundary layer equations:

$$\partial_x u + \partial_y v = 0, \tag{5.41}$$
$$u\partial_x u + v\partial_y u = \nu \partial^2_y u, \tag{5.42}$$

where $\nu = \mu/\rho$, Blasius conjectured that the boundary layer is self-similar, i.e., that at any given point along the boundary layer, the velocity profile is the same to a stretching factor on the spatial dimension. He wrote:

$$\eta = y\left(\frac{U}{\nu x}\right)^{1/2}, \qquad \frac{u(x,y)}{U} = f'(\eta), \tag{5.43}$$

where $f(\eta)$ is a dimensionless quantity and $f'(\eta)$ denotes its derivative with respect to the dimensionless wall-normal coordinate η.
One of the virtues of this rescaling is that the wall-normal direction is rescaled by a quantity proportional to the laminar boundary layer thickness. In other words, the laminar boundary layer is mapped onto a rectangle.
As the flow is incompressible and two-dimensional, we introduce a streamfunction ψ such that:

$$u = \partial_y\psi, \qquad v = -\partial_x\psi. \tag{5.44}$$

The incompressibility constraint is automatically verified:

$$\partial_x u + \partial_y v = \partial_x(\partial_y\psi) + \partial_y(-\partial_x\psi) = 0. \tag{5.45}$$

Using the streamfunction, equation (5.42) reduces down to:

$$\partial_y\psi\,\partial_x\partial_y\psi - \partial_x\psi\,\partial_y^2\psi = \nu\partial_y^3\psi. \tag{5.46}$$

Equation (5.46) has yet to be written in terms of Blasius's variables (5.43). To do so, we note that the definition of the streamfunction (5.44) implies:

$$Uf'(\eta) = \partial_\eta\psi\partial_y\eta \tag{5.47}$$

$$\Rightarrow \quad \psi = U\int_0^\eta \frac{1}{\partial_y\eta}f'(\eta)d\eta, \tag{5.48}$$

providing the new definitions:

$$\psi = U\gamma(x)f(\eta), \qquad \gamma(x) = (\nu x/U)^{1/2}. \tag{5.49}$$

With these new variables, the spatial derivatives of ψ become:

$$\begin{aligned}
\partial_x\psi \quad &= U\left(\partial_x\gamma f + \gamma\partial_\eta f\partial_x\eta\right) && (5.50)\\
&= U\left(\gamma' f - \gamma f'\frac{y\gamma'}{\gamma^2}\right) && (5.51)\\
&= U\gamma' f - Uf'\frac{y\gamma'}{\gamma}, && (5.52)
\end{aligned}$$

and:

$$\begin{aligned}
\partial_y\psi \quad &= U\gamma\partial_\eta f\partial_y\eta && (5.53)\\
&= U\gamma f'\frac{1}{\gamma} && (5.54)\\
&= Uf', && (5.55)
\end{aligned}$$

where $\gamma' = \partial_x\gamma$ and $f' = \partial_\eta f$.

The terms of equation (5.46) therefore become:

$$\partial_y\psi\,\partial_x\partial_y\psi = \left(Uf'\right)\,\partial_x\left(Uf'\right) \tag{5.56}$$

$$= \left(Uf'\right)\,Uf''\left(-\frac{y\gamma'}{\gamma^2}\right) \tag{5.57}$$

$$= -U^2\frac{y\gamma'}{\gamma^2}f'f'', \tag{5.58}$$

$$\partial_x\psi\,\partial_y^2\psi = \left(U\gamma' f - Uf'\frac{y\gamma'}{\gamma}\right)\,\partial_y\left(Uf'\right) \tag{5.59}$$

$$= \left(U\gamma' f - Uf'\frac{y\gamma'}{\gamma}\right)\,U\frac{1}{\gamma}f'' \tag{5.60}$$

$$= U^2\frac{\gamma'}{\gamma}ff'' - U^2\frac{y\gamma'}{\gamma^2}f'f'', \tag{5.61}$$

$$\nu\partial_y^3\psi = \nu\partial_y^2\left(Uf'\right) \tag{5.62}$$

$$= \nu U\partial_y\left(\frac{1}{\gamma}f''\right) \tag{5.63}$$

$$= \nu U\frac{1}{\gamma^2}f'''. \tag{5.64}$$

In the end, equation (5.46) simplifies into:

$$-U^2\frac{y\gamma'}{\gamma^2}f'f'' - \left(U^2\frac{\gamma'}{\gamma}ff'' - U^2\frac{y\gamma'}{\gamma^2}f'f''\right) = \nu U\frac{1}{\gamma^2}f''', \tag{5.65}$$

$$\nu U\frac{1}{\gamma^2}f''' + U^2\frac{\gamma'}{\gamma}ff'' = 0, \tag{5.66}$$

$$f''' + \frac{U\gamma'\gamma}{\nu}ff'' = 0. \tag{5.67}$$

The variable γ is easily obtained from definition (5.49) and yields $U\gamma'\gamma/\nu = 1/2$. The equation Blasius obtained for the dimensionless quantity f is then:

$$f''' + \frac{1}{2}ff'' = 0. \tag{5.68}$$

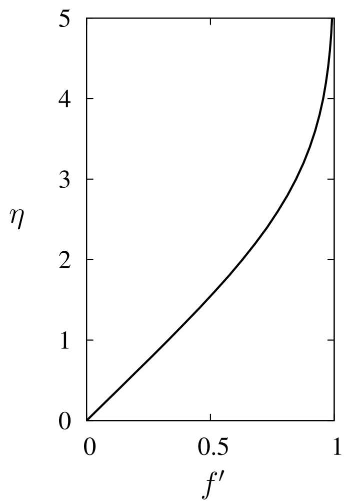
This equation is complemented with the boundary conditions (5.39) and (5.40) that now write:

$$f' = f = 0, \qquad \eta = 0, \tag{5.69}$$

$$f' \to 1, \qquad \eta \to \infty. \tag{5.70}$$

Solution

Equation (5.68) is generally solved numerically, as below.



This laminar boundary layer solution is self-similar: the same profile holds at any given position x along the boundary layer, the only change being a stretching coefficient in the wall-normal direction as η is a linear function of y and depends on x.
The boundary layer equation (5.68) and this solution are valid for $x \in [0; L_{max}]$ and $\eta \in \mathcal{R}^+$, where L_{max} represents the point at which a change in dynamics occurs that violates one of the hypotheses made. This point can arise due to a transition where the boundary layer becomes turbulent. There, wall-normal velocities become of the same order as streamwise velocities because of the creation and advection of eddies and the whole analysis carried out here breaks down.
Since $u/U \to 1$ as $\eta \to \infty$, we can define the boundary layer thickness as the region in which $u/U \leq 0.99$, or in other words $f' \leq 0.99$. The data from the figure above provides a boundary layer thickness of $\eta \approx 5.0$. Therefore:

$$\delta \left(\frac{U}{\nu x}\right)^{1/2} \approx 5.0, \tag{5.71}$$

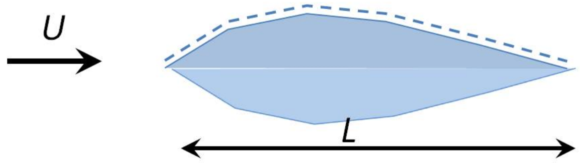
$$\Rightarrow \quad \delta \approx 5.0 \left(\frac{\nu x}{U}\right)^{1/2}, \tag{5.72}$$

$$\Rightarrow \quad \frac{\delta}{x} \approx 5.0 \left(\frac{\nu}{U x}\right)^{1/2}, \tag{5.73}$$

$$\Rightarrow \quad \frac{\delta}{x} \approx 5.0 Re_x^{-1/2}. \tag{5.74}$$

5.3.2 Boundary layer on a solid boundary

Now, let us consider a steady flow of magnitude U past a streamlined body of length L.



The advection term, $\mathbf{u} \cdot \nabla\omega$, balances diffusion:

$$\mathbf{u} \cdot \nabla\omega = \nu \nabla^2 \omega.$$

We expect the vorticity to vary over the length of the body, so that:

$$|\mathbf{u} \cdot \nabla\omega| \sim \frac{U\Omega}{L}.$$

Balancing this with the diffusion term, we have:

$$\frac{U\Omega}{L} \sim \nu \frac{\Omega}{\delta^2},$$

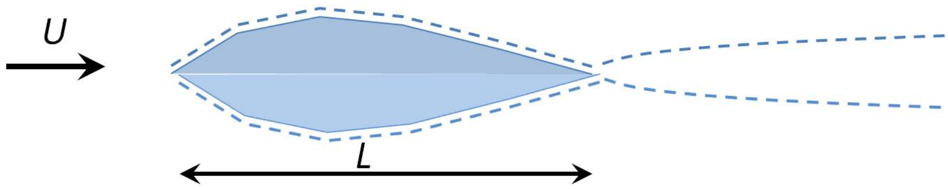
and hence:

$$\delta^2 = \frac{\nu L}{U} = L^2 \left(\frac{\nu}{UL} \right).$$

Thus, the vorticity is confined to a boundary layer of thickness of the order of $L\mathrm{Re}^{-1/2}$, where $\mathrm{Re} = UL/\nu$ is the Reynolds number. Provided the Reynolds number is large, the boundary layer thickness is small compared with the size of the body.

5.3.3 Wake behind a streamlined body

So far, we have argued that there exists a thin boundary layer on the surface of the body in which the vorticity is concentrated. The fluid in this boundary layer is advected around the surface of the body until the point where the flow separates behind it. This creates a region of vorticity behind the body which is referred to as the wake.



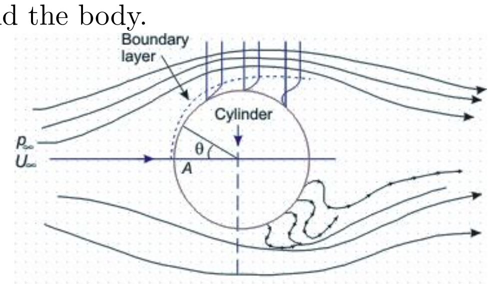
We can estimate the structure of the wake from the form of the flow above an impulsively started plate. In the frame of the fluid particle, the thickness of the wake grows as $\delta = \sqrt{\nu t}$ while the fluid is being carried away from a fixed point on the boundary at a speed U. Hence the thickness of the wake at a distance x behind the fixed point is approximately:

$$\delta = \sqrt{\frac{\nu x}{U}}.$$

It is parabolic, however, by the time the wake has grown to the size of the body, it has diffused so much that it is no longer detectable.

5.3.4 Separation from a bluff body

The above arguments are based on the case when the body has a streamlined shape so that its surface can be approximated by a flat plate. However, in the case of a bluff body such as a sphere or a cylinder, the boundary layer detaches from the body before it reaches its end. The reason that this happens is the presence of an opposing pressure gradient in the boundary layer. In the case of flow past a cylinder, the potential flow solution has high pressure at the front and back of cylinder, but low pressure at the top and bottom. Thus the pressure gradient changes sign at the top of cylinder. Provided $u > 0$ away from the boundary layer, $\partial p/\partial x > 0$ past the top of the cylinder. This drives the inner flow in the opposite direction to the flow outside the boundary layer and lifts the boundary layer away from the surface, causing a large wake behind the body.

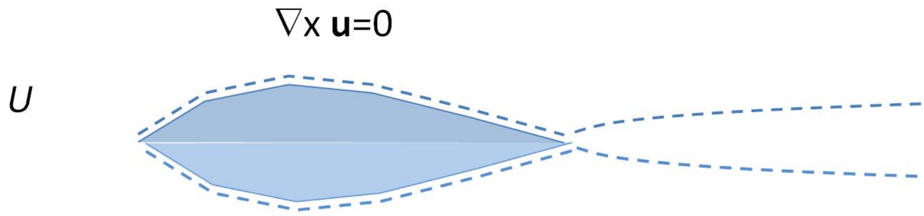


Chapter 6

Irrotational flow — Complex potential and aerofoils

Contents

In the previous chapter, we discussed high Reynolds numbers flow past a streamlined object. In these flows, vorticity is only generated at the object surface and is therefore confined to a thin boundary layer around the surface of the object and to a narrow wake behind it.



Consequently outside these boundary layers the flow has no vorticity and is described as being irrotational.

6.1 Velocity potential

If $\nabla \times \mathbf{u} = 0$, we can write:

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \qquad w = \frac{\partial \phi}{\partial z}.$$

Note that we can add any arbitrary function of t to ϕ.

6.1.1 Incompressible flow

If the fluid is also incompressible, then:

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0,$$

so that ϕ satisfies the Laplace equation.

For example,

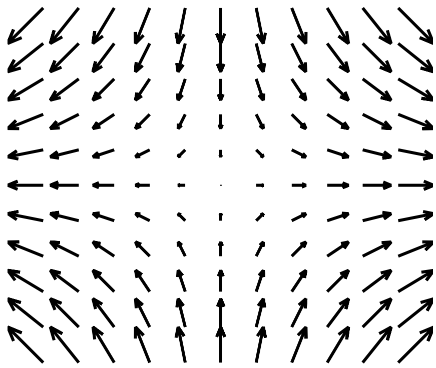
$$\phi = \frac{1}{2} E(x^2 - y^2),$$

satisfies

$$\nabla^2 \phi = 0.$$

and corresponds to the stagnation point flow:

$$\mathbf{u} = \nabla \phi = \begin{pmatrix} Ex \\ -Ey \\ 0 \end{pmatrix} = Ex\hat{\boldsymbol{\imath}} - Ey\hat{\boldsymbol{\jmath}}.$$



In cases where the domain is not infinite, we need boundary conditions. Since we are not including viscous boundary layers, we cannot fully impose the condition $\mathbf{u} = \mathbf{U}$ on a rigid boundary. However, since the volume of the boundary layers is small, we can assume mass is conserved:

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n},$$

where $\mathbf{n}$ is the normal to the surface. The boundary condition on the velocity potential is:

$$\mathbf{n} \cdot \nabla \phi = \frac{\partial \phi}{\partial \mathbf{n}} = \mathbf{U} \cdot \mathbf{n}.$$

This is a *Neumann boundary condition*: it involves the first spatial derivative only. The solution of the Laplace equation is unique up to an arbitrary additive constant. There is a unique solution to the fluid velocity satisfying these boundary conditions.

6.1.2 The Bernoulli equation

Since we neglect the effects of viscosity, the Navier–Stokes equation reduces to the Euler equation (2.5):

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \rho \mathbf{g} - \nabla P.$$

Using the vector identity:

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2} \nabla (\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \cdot \nabla \mathbf{u},$$

we can rewrite this as:

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\nabla \left(\frac{P}{\rho} + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - \mathbf{g} \cdot \mathbf{x} \right).$$

For a potential flow ($\boldsymbol{\omega} = 0$) and since $\mathbf{u} = \nabla \phi$, we obtain:

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - \mathbf{g} \cdot \mathbf{x} \right) = 0,$$

yielding:

$$\frac{\partial \phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} - \mathbf{g}\cdot\mathbf{x} = F(t), \tag{6.1}$$

or, using the dynamic pressure $p = P - \rho\mathbf{g}\cdot\mathbf{x}$:

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} = F(t).$$

As a result, for a steady irrotational flow, $p + \frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}$ is a constant. The pressure is lower in the regions of higher flow speed.

6.2 Planar potential flows

If the flow is confined to the (x, y) plane:

$$\mathbf{u} = (u(x,y), v(x,y), 0)\,,$$

and we can express the fluid velocity for an incompressible flow in terms of the streamfunction ψ:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}.$$

If, additionally, the flow is irrotational, we have:

$$\begin{aligned}\omega = 0 &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\ &= -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2},\end{aligned}$$

and therefore the streamfunction ψ satisfies the Laplace equation.
As a result, for planar incompressible irrotational flows, the velocity potential and the streamfunction are related by:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \tag{6.2}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{6.3}$$

The right-hand pair of equalities in the above equations are known as the *Cauchy–Riemann equations* and have a connection with complex variable theory as we shall see in the next section.
One property of the Cauchy–Riemann equations is that this relation between ϕ and ψ is sufficient to show that both functions satisfy the Laplace equation. Summing the x derivative of equation (6.2) with the y derivative of equation (6.3), we get:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0.$$

We can show in a similar way that:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

6.3 The complex derivative

Let f be a complex function of z where $z = x + iy$. The derivative of f with respect to z is defined as:

$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}.$$

Note that this definition requires that the limit is the same for all infinitesimal increments δz which can be in any direction in the complex plane. As a consequence, differentiability of a complex function imposes restrictions on its real and imaginary parts.

If $f(z) = g(x, y) + ih(x, y)$ is complex differentiable, where g and h are both real functions then, taking $\delta z = \delta x$, we obtain:

$$\frac{df}{dz} = \frac{\partial g}{\partial x} + i\frac{\partial h}{\partial x}. \tag{6.4}$$

If, instead, we take the increment in the imaginary direction $\delta z = i\delta y$, then:

$$\frac{df}{dz} = \frac{1}{i}\frac{\partial g}{\partial y} + \frac{\partial h}{\partial y} = \frac{\partial h}{\partial y} - i\frac{\partial g}{\partial y}. \tag{6.5}$$

Comparing the real and imaginary terms in equations (6.4) and (6.5), we obtain the Cauchy–Riemann equations:

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y}, \tag{6.6}$$

$$\frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}, \tag{6.7}$$

from which we deduce that both g and h satisfy the two-dimensional Laplace equation.

Hence the real and imaginary parts of any complex differentiable function $f(z)$ are solutions of the Laplace equation.

6.4 The complex potential

This powerful result allows us to use complex differentiable functions to derive two-dimensional potential flows. We define the complex potential as:

$$w(z) = \phi(x, y) + i\psi(x, y), \tag{6.8}$$

since the Cauchy–Riemann equations guarantee that ϕ and ψ will have the appropriate properties.

The fluid velocity is obtained from equation (6.4):

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv, \tag{6.9}$$

or alternatively:

$$u + iv = \overline{\frac{dw}{dz}},$$

where the overbar indicates complex conjugation.

Furthermore:

$$|\mathbf{u}| = \sqrt{u^2 + v^2} = \left(\frac{dw}{dz}\overline{\frac{dw}{dz}}\right)^{1/2} = \left|\frac{dw}{dz}\right|, \tag{6.10}$$

and the velocity direction, α, can be expressed as:

$$\alpha = \arg\left(\overline{\frac{dw}{dz}}\right) = -\arg\left(\frac{dw}{dz}\right). \tag{6.11}$$

6.4.1 Examples of complex potentials

i. **Uniform flow:** If $u = U_0 \cos\alpha$, $v = U_0 \sin\alpha$, then:

$$\frac{dw}{dz} = U_0 \cos\alpha - iU_0 \sin\alpha = U_0 e^{-i\alpha},$$

and so the corresponding complex potential is:

$$w = U_0 e^{-i\alpha} z.$$

ii. **Saddle point flow:** Consider:

$$w = Az^2.$$

If A is real, then:

$$w = A(x+iy)^2 = A(x^2 - y^2) + 2iAxy = \phi(x,y) + i\psi(x,y).$$

Differentiating, we have:

$$u - iv = 2Az = 2A(x+iy).$$

iii. **Source/Sink:** The logarithm of a complex number $z = re^{i\theta}$ is given by:

$$\log(z) = \log(re^{i\theta}) = \log(r) + i\theta. \tag{6.12}$$

For a real number A:

$$w = A\log(z) = A\log(r) + iA\theta.$$

Thus, $\phi = A\log(r)$ where $r = \sqrt{x^2+y^2}$. This flow is most easily recognised in polar coordinates,

$$u_r = \frac{\partial \phi}{\partial r} = \frac{A}{r}.$$

It represents a 2D source (sink) flow out of (into) the origin for $A > 0$ ($A < 0$).

The strength of the source is defined from the flow rate through a circle centered around the origin:

$$q = \int_0^{2\pi} u_r a d\theta = 2\pi A,$$

and the complex potential for a source of strength q at the origin is given by:

$$w(z) = \frac{q}{2\pi}\log(z).$$

iv. **Vortex:** If A is imaginary, $iA = k$, then $\phi = k\theta$ which represents the vortex:

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{k}{r}.$$

The circulation about the origin is:

$$\Gamma = \int_0^{2\pi} v_\theta a d\theta = 2\pi k,$$

so that the complex potential for a vortex of circulation Γ at the origin is given by:

$$w(z) = -\frac{i\Gamma}{2\pi}\log(z).$$

More generally, the complex potential for a vortex at $z = z_0$ is given by:

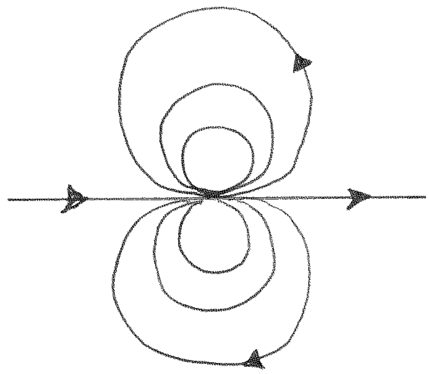
$$w(z) = -\frac{i\Gamma}{2\pi}\log(z - z_0).$$

v. **Dipole:** Consider the complex potential

$$w = -\frac{m}{z}.$$

We find the velocity by differentiating:

$$\begin{aligned} u - iv = \frac{dw}{dz} = \frac{m}{z^2} = \frac{m\bar{z}^2}{|z^4|} \\ &= \frac{m}{(x^2+y^2)^2}\,(x-iy)^2 \\ &= \frac{m}{(x^2+y^2)^2}\left(x^2 - y^2 - i2xy\right), \end{aligned}$$



so that:

$$\begin{aligned} u &= \frac{m(x^2-y^2)}{(x^2+y^2)^2}, \\ v &= \frac{2mxy}{(x^2+y^2)^2}, \end{aligned}$$

which is a dipole flow.

We can also obtain other flows through combinations of these complex potentials.

6.5 Imposing boundary conditions

As we noted above, we can only impose the continuity of the normal component of velocity for potential flows. In order to describe a fixed boundary, it is sufficient to impose that the streamfunction is constant along the boundary. We thus need to ensure that the imaginary part of the complex potential is constant on the boundary.

6.5.1 Method of images

For simple boundaries, this can be achieved using the method images. For example, the complex potential due to a line vortex at $(0, d)$ is given by:

$$w(z) = -\frac{i\Gamma}{2\pi}\log(z - id).$$

Let us try to find the flow in $y > 0$ when there is a wall at $y = 0$. We need to impose that the real axis $y = 0$ is a streamline. We can do this by placing an "image" vortex at the point $y = -id$ of opposite strength. This will cancel out the y component of the velocity on $y = 0$. The complex potential for this image system is given by:

$$w(z) = -\frac{i\Gamma}{2\pi}\log(z-id) + \frac{i\Gamma}{2\pi}\log(z+id) = -\frac{i\Gamma}{2\pi}\log\left(\frac{z-id}{z+id}\right).$$

As a consequence, the streamlines are given by:

$$\left|\frac{z-id}{z+id}\right| = \text{cst}.$$

In particular, on the real axis $z \pm id = x \pm id$,

$$|z - id| = |z + id| = \sqrt{d^2 + x^2},$$

and so $\psi = \Gamma/2\pi \log 1 = 0$.
We can achieve a generalisation of this principle for any complex potential $f(z)$ by considering the function:

$$w(z) = f(z) + \overline{f(\overline{z})},$$

for which w is a function of z (and not another combination of x and y).
Remark: Taking the complex conjugate of a function is done by replacing each term by its complex conjugate (in other words we replace i by $-i$). For example, if:

$$f(z) = i \log(z + b),$$

then

$$\overline{f(z)} = -i \log(\overline{z} + \overline{b}).$$

In this case, $\overline{f(z)}$ is a function of $\overline{z}$ and not z. However $\overline{f(\overline{z})}$ is a function of z:

$$\overline{f(\overline{z})} = -i \log(z + \overline{b}).$$

Returning to the general case, on the real axis, $z = x$,

$$w(x) = f(x) + \overline{f(x)},$$

which is real (from the definition of the complex conjugate) and so $\psi = 0$. In the case of the line vortex above:

$$f(z) = -\frac{i\Gamma}{2\pi} \log(z - id),$$

and so:

$$\overline{f(\overline{z})} = \frac{i\Gamma}{2\pi} \log(\overline{\overline{z} - id}) = \frac{i\Gamma}{2\pi} \log(z + id).$$

6.5.2 Milne-Thompson circle theorem

There exists a similar construction for the case when the boundary is the circle $|z| = a$:

$$w(z) = f(z) + \overline{f\left(\frac{a^2}{\overline{z}}\right)}.$$

On the circle $z = ae^{i\theta}$, we have:

$$w(ae^{i\theta}) = f(ae^{i\theta}) + \overline{f(ae^{i\theta})},$$

and so $\psi = 0$.
Flow past a cylinder. We can use the Milne-Thompson circle theorem to find the complex potential for a flow past a cylinder. We start with the uniform flow $f(z) = U_0 z e^{-i\alpha}$. Applying the circle theorem:

$$w(z) = U_0 z e^{-i\alpha} + \frac{\overline{U_0 a^2 e^{-i\alpha}}}{\overline{z}} = U_0 z e^{-i\alpha} + U_0 e^{i\alpha} \frac{a^2}{z}.$$

We can now add a line vortex at $z = 0$ so that the circle $|z| = a$ remains a streamline:

$$w(z) = U_0 z e^{-i\alpha} + U_0 e^{i\alpha} \frac{a^2}{z} - \frac{i\Gamma}{2\pi} \log z. \tag{6.13}$$

6.5.3 Conformal mapping

Another way to find flow solutions for particular boundary conditions is to use a transformation that maps the boundary onto that of a different problem for which we know the solution.

Flow round a corner. Let us start with the simple complex potential:

$$w(z) = U_0 z,$$

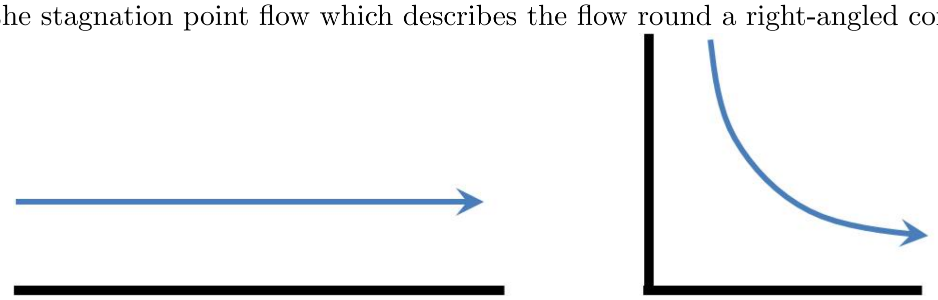
which represents the uniform flow in the positive x direction in the upper half-plane ($y > 0$). This flow admits a straight boundary at $y = 0$. Let us map this onto the quadrant $x > 0, y > 0$ by taking:

$$Z = z^{1/2}.$$

This maps the point $z = re^{i\theta}$ onto $Z = r^{1/2}e^{i\theta/2}$, mapping the negative x-axis $\theta = \pi$ onto the positive y-axis $\theta = \pi/2$. The corresponding complex potential is given by:

$$W(Z) = U_0 Z^2.$$

This is the stagnation point flow which describes the flow round a right-angled corner.

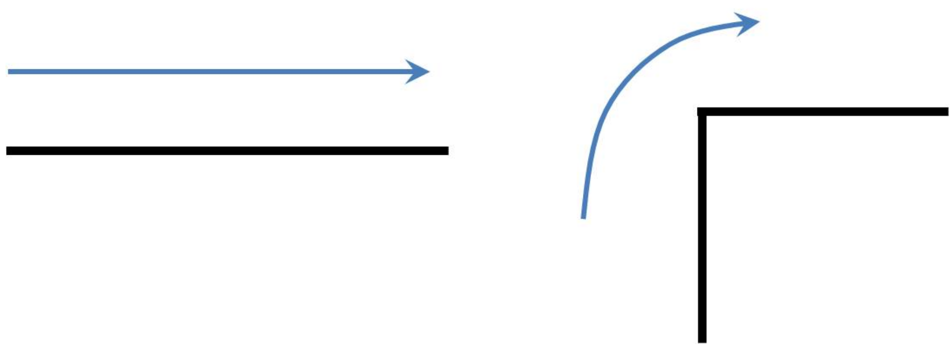


We can generalise this to a flow around a corner of angle α by taking $Z = z^{\alpha/\pi}$ so that $z = Z^{\pi/\alpha}$ and:

$$W(Z) = U_0 Z^{\pi/\alpha}.$$

For example, $\alpha = 3\pi/2$ models the flow around the outside of a right-angled corner:

$$W(Z) = U_0 Z^{2/3}.$$



6.5.4 The Joukowski transformation

Another useful transformation is the Joukowski transformation:

$$Z = z + \frac{c^2}{z}. \tag{6.14}$$

Let us consider the effect of this transformation on the circle $|z| = a$:

$$Z(ae^{i\theta}) = a\left(e^{i\theta} + \frac{c^2}{a^2}e^{-i\theta}\right) = a\left[\left(1 + \frac{c^2}{a^2}\right)\cos\theta + i\left(1 - \frac{c^2}{a^2}\right)\sin\theta\right].$$

Hence, for $0 \leq c \leq a$, it maps the $|z| = a$ circle onto the ellipse with semi-axes $(a + c^2/a)$ and $(a - c^2/a)$.
The inverse mapping is given by:

$$z = \frac{Z}{2} + \left(\frac{Z^2}{4} - c^2\right)^{1/2},$$

where we have taken the positive square root so that z maps onto Z when $c = 0$.
Using the solution for the flow past a cylinder with circulation Γ in equation (6.13), we can obtain the flow past an ellipse at an angle α to the major axis and with circulation Γ:

$$W(Z) = U_0\left[\frac{Z}{2} + \left(\frac{Z^2}{4} - c^2\right)^{1/2}\right]e^{-i\alpha} + \frac{U_0 a^2 e^{i\alpha}}{\frac{Z}{2} + \left(\frac{Z^2}{4} - c^2\right)^{1/2}}$$
$$- \frac{i\Gamma}{2\pi}\log\left[\frac{Z}{2} + \left(\frac{Z^2}{4} - c^2\right)^{1/2}\right].$$

Note that if we take $c = a$, the ellipse collapses onto a flat plate along the X axis running from $X = -2a$ to $X = 2a$.

6.5.5 Flow past a finite flat plate

Rather than working with the rather messy expression above, we can determine the fluid velocity using the conformal mapping and the complex potential in the original z-space. Since $w(z) = W(Z)$:

$$\frac{dw}{dz} = \frac{dW}{dZ}\frac{dZ}{dz}.$$

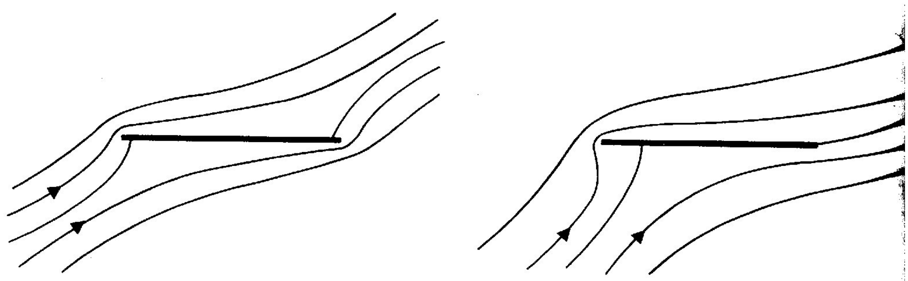
The velocity components u and v are given by:

$$U - iV = \frac{dW}{dZ} = \frac{dw}{dz}\left(\frac{dZ}{dz}\right)^{-1}, \tag{6.15}$$

and so, for the flow past a flat plate, we have:

$$U - iV = \frac{U_0 z^2 e^{-i\alpha} - U_0 e^{i\alpha}a^2 - \frac{i\Gamma z}{2\pi}}{z^2 - a^2}. \tag{6.16}$$

Note that the denominator vanishes at $z = \pm a$, indicating that the fluid velocity is infinite at the edges of the plate unless the numerator also vanishes there.



(a) $\Gamma=0$ (b) $\Gamma=-4\pi Ua\sin\alpha$

The position of the points of flow separation is controlled by the circulation Γ. Consequently, we can choose Γ so that the separation occurs at the trailing edge. This is referred to as the *Kutta condition*: we want to cancel the numerator in equation (6.16) at $z = a$ (trailing edge):

$$U_0 e^{-i\alpha} - U_0 e^{i\alpha} - \frac{i\Gamma}{2\pi a} = 0,$$

yielding: $\Gamma = -4\pi U_0 a \sin\alpha$.

6.5.6 The Joukowski aerofoil

While the flat-plate provides a rough approximation to a thin aerofoil (such as in a paper aeroplane), real aerofoils have a rounded front and sharp trailing edge. We can generate a symmetric aerofoil of this kind by moving the circle centre along the x-axis to $z = -b$ before applying the Joukowski transformation. The result is an aerofoil whose shape is given by:

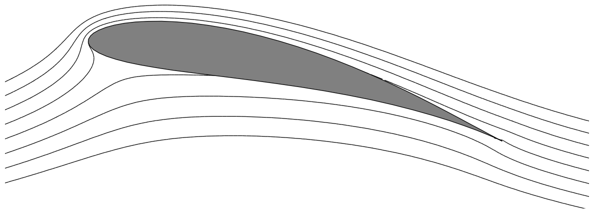
$$Z = -b + (a+b)e^{i\theta} + \frac{a^2}{-b + (a+b)e^{i\theta}}.$$

The corresponding fluid velocity is given by:

$$U - iV = \frac{\left(U_0 e^{-i\alpha} - U_0 e^{i\alpha}\dfrac{a+b^2}{(z+b)^2} - \dfrac{i\Gamma}{2\pi(z+b)}\right)}{\left(1 - \dfrac{a^2}{z^2}\right)},$$

and the Kutta condition gives $\Gamma = -4\pi U_0 (a+b)\sin\alpha$.

Better still is a cambered aerofoil, which is formed by moving the centre of the circle to $z = be^{i\beta}$ (see http://s6.aeromech.usyd.edu.au/aerodynamics/jflow2.php).

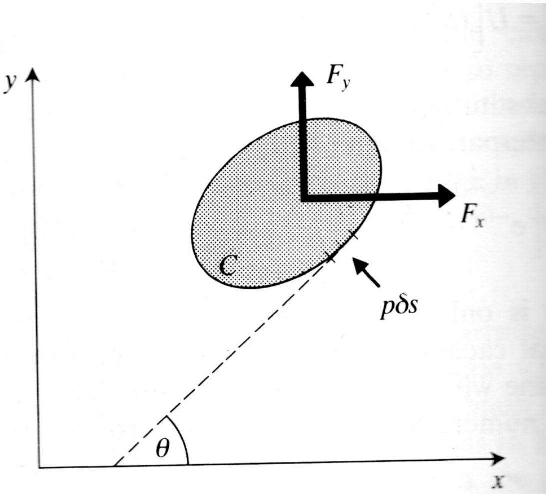


6.6 Forces on streamlined bodies

Our remaining task is to calculate the force exerted by the flow on the aerofoil. Since we ignore viscous effects, this force comes from the fluid pressure and is given by:

$$\mathbf{F} = -\int_S p\mathbf{n}ds, \qquad (6.17)$$

where $\mathbf{n}$ is the normal to the surface.



Consider an element of the surface δS that is at an angle θ to the x-axis. The outward normal to this surface element is $(\sin\theta, -\cos\theta)$. The contribution from this element to the force is:

$$\delta \mathbf{F} = -p(\sin\theta, -\cos\theta)\delta s.$$

For a steady flow, the pressure is given by the Bernoulli equation (6.1):

$$p = E - \frac{\rho}{2}(u^2 + v^2),$$

for some constant E. The contribution from E integrates to zero around a closed loop, so:

$$\mathbf{F} = \frac{\rho}{2}\int_S (u^2 + v^2)(\sin\theta, -\cos\theta)ds.$$

Let us write the force in complex form:

$$F_x - iF_y = \frac{\rho}{2}\oint_C \left|\frac{dw}{dz}\right|^2 (\sin\theta + i\cos\theta)ds,$$

where C is the contour in the complex plane corresponding to the surface S of the body, and we have used equation (6.10) to write the speed as $|dw/dz|$. Furthermore:

$$(\sin\theta + i\cos\theta)ds = i(\cos\theta - i\sin\theta)ds = i\overline{dz},$$

so:

$$F_x - iF_y = \frac{i\rho}{2}\oint_C \frac{dw}{dz}\overline{\left(\frac{dw}{dz}\right)}\overline{dz}.$$

However,

$$\overline{\left(\frac{dw}{dz}\right)}\overline{dz} = \overline{dw},$$

and since C is a streamline, dw is real:

$$\overline{dw} = dw = \frac{dw}{dz}dz,$$

yielding:

$$F_x - iF_y = \frac{i\rho}{2}\oint_C \left(\frac{dw}{dz}\right)^2 dz, \tag{6.18}$$

which is known as the *Blasius theorem*.

6.6.1 Forces on cylinder

Applying the Blasius theorem to the flow past a cylinder, we have:

$$F_x - iF_y = \frac{i\rho}{2} \oint_C \left(U_0 e^{-i\alpha} - U_0 e^{i\alpha} \frac{a^2}{z^2} - \frac{i\Gamma}{2\pi z} \right)^2 dz,$$

where C is the circle $|z| = a$. To evaluate this integral, let us consider:

$$\oint_{|z|=a} z^{-n} dz,$$

where n is an integer. Substituting $z = ae^{i\theta}$, we have:

$$\begin{aligned}
\oint_{|z|=a} z^{-n} dz &= \int_0^{2\pi} a^{1-n} i e^{i(1-n)\theta} d\theta \\
&= \begin{cases} \frac{a^{1-n}}{1-n} \left[e^{i(1-n)\theta} \right]_0^{2\pi} = 0 & n \neq 1 \\ i \left[\theta \right]_0^{2\pi} = 2\pi i & n = 1 \end{cases} .
\end{aligned}$$

This is a special case of the Cauchy residue theorem and it can be shown more generally that the result holds for any closed curve around the origin. The only non-zero contribution to the integral thus comes from the coefficient of z^{-1}:

$$F_x - iF_y = \frac{i\rho}{2} 2\pi i \left(-\frac{iU_0 e^{-i\alpha}\Gamma}{\pi} \right) = i\rho U_0 \Gamma e^{-i\alpha} = \rho U_0 \Gamma (\sin\alpha + i\cos\alpha).$$

Hence: $F_x = \rho U_0 \Gamma \sin\alpha$ and $F_y = -\rho U_0 \Gamma \cos\alpha$. Let us consider $\alpha = 0$. In the case $\Gamma = 0$, there is no force acting on the cylinder. For $\Gamma \neq 0$, the only force component is a lift force directed perpendicular to the direction of flow: $F_y = -\rho U_0 \Gamma$. This is called the *Magnus effect* and is responsible for aerodynamic forces acting on balls in sports (e.g. football, tennis, table tennis).

6.6.2 Forces on an elliptical body

Let us now consider the flow around an ellipse. One way to calculate it to consider the velocity potential $W(Z)$. It is however easier to use the conformal transformation to perform the integral around the cylinder. From equation (6.18):

$$F_x - iF_y = \frac{i\rho}{2} \oint_C \left(\frac{dW}{dZ} \right)^2 dZ,$$

where C is the curve around the surface of the ellipse. We can transform this surface back to a circle of radius a using the Joukowski transformation:

$$Z(z) = z + \frac{c^2}{z}.$$

Noting from equation (6.15) that $dW/dZ = (dw/dz) \cdot (dZ/dz)^{-1}$:

$$\begin{aligned}
F_x - iF_y &= \frac{i\rho}{2} \oint_{|z|=a} \left(\frac{dw}{dz} \right)^2 \left(\frac{dZ}{dz} \right)^{-1} dz \\
&= \frac{i\rho}{2} \oint_{|z|=a} \left(U_0 e^{-i\alpha} - U_0 e^{i\alpha} \frac{a^2}{z^2} - \frac{i\Gamma}{2\pi z} \right)^2 \left(1 - \frac{c^2}{z^2} \right)^{-1} dz.
\end{aligned}$$

Since $c < a$, we can use the binomial expansion:

$$\left(1 - \frac{c^2}{z^2}\right)^{-1} = 1 + \frac{c^2}{z^2} + \ldots,$$

and the property:

$$\oint_{|z|=a} z^{-n} dz = \begin{cases} 2\pi i & n = 1 \\ 0 & n \neq 1 \end{cases},$$

to evaluate the integral by identifying the coefficient of z^{-1}. We obtain:

$$F_x - iF_y = i\rho U_0 \Gamma e^{-i\alpha}.$$

Note that this result is independent of c and is identical to the case of the cylinder. The case of the flat-plate requires a little more care because the integrand is singular at $z^2 = a^2$. We can get around this problem by performing the integral on a larger circle (by exploiting the Cauchy theorem for contour integration, since the integrand does not contain any singularities in the flow domain).

6.6.3 The Kutta–Joukowski lift theorem

The result we have just obtained for an ellipse applies in general to any shaped body. The proof requires using a few results from complex analysis, but it is worth it.
We choose the origin $z = 0$ to lie inside the body and apply the Blasius theorem around the surface contour of the body C_b:

$$F_x - iF_y = \frac{i\rho}{2} \oint_{C_b} \left(\frac{dw}{dz}\right)^2 dz.$$

We now take a second contour C_R, which is a large circle around the origin of radius R where R is much larger than the body we are considering.
A key result from complex analysis is that for a complex differentiable function $f(z)$ that has no singularities inside a closed curve C:

$$\oint_C f(z) dz = 0.$$

In this case, $f(z) = (dw/dz)^2$ has no singularities outside the body. Taking C to be the curve composed of $C_R \cup -C_b$, the Blasius integral can be changed into an integral around C_R:

$$F_x - iF_y = \frac{i\rho}{2} \oint_{C_R} \left(\frac{dw}{dz}\right)^2 dz.$$

Far away from the object, we can expand dw/dz as a power series in z:

$$\frac{dw}{dz} = U - \frac{i\Gamma}{2\pi z} + \ldots,$$

where U is the fluid velocity at large distance and Γ is the circulation around C_R. Note that Γ must be equal to the circulation around C_b since the flow is irrotational. Using the result we used for the cylinder, we find:

$$F_x - iF_y = \frac{i\rho}{2} 2\pi i \left(-\frac{iU\Gamma}{\pi}\right) = i\rho U\Gamma. \tag{6.19}$$

This is the *Kutta–Joukowski theorem.* The force is in the direction iU and so is perpendicular to the flow direction. In particular, if U is in the positive x-direction then the force is equal to $F_y = -\rho U\Gamma$. Since this force is perpendicular to the direction of flow it is referred to as a *lift force.*

6.6.4 Lift on an aerofoil

Using the Kutta condition, the lift force on the flat plate is given by:

$$F_x - iF_y = i\rho U\Gamma = i\rho U_0 e^{-i\alpha}(-4\pi U_0 a \sin\alpha) = -4\rho U_0^2 \pi a \sin\alpha\,(\sin\alpha + i\cos\alpha)\,.$$

Since the length of the aerofoil is $L = 4a$, the lift force per unit length on a wing reads:

$$F_L = F_y \cos\alpha - F_x \sin\alpha = \rho U_0^2 L\pi \sin\alpha. \tag{6.20}$$

Generating lift on a symmetric aerofoil requires a finite angle of attack. Lift then increases with the angle α. However, if this angle becomes too large, the flow separates and generates a catastrophic loss of lift and increase in drag. This leads to stalling.

6.6.5 Torque on a streamlined body

The torque $\mathbf{T}$ about the origin due to a force $\mathbf{F}$ acting at $\mathbf{x}$ is given by $\mathbf{T} = \mathbf{x} \times \mathbf{F}$. Hence, for a planar flow, the torque about the origin due to the force (dF_x, dF_y) acting at (x, y) is:

$$dT = x dF_y - y dF_x.$$

This can be written in complex variable notation:

$$dT = \Re\{iz(dF_x - idF_y)\}.$$

where $\Re\{\dots\}$ denotes the real part. Using the Blasius formula for the force on the surface of a rigid body (equation (6.18)), we get:

$$dF_x - idF_y = \frac{i\rho}{2}\left(\frac{dw}{dz}\right)^2 dz,$$

and the torque on the body is:

$$T = -\frac{\rho}{2}\Re\left\{\oint_C z\left(\frac{dw}{dz}\right)^2 dz\right\}.$$

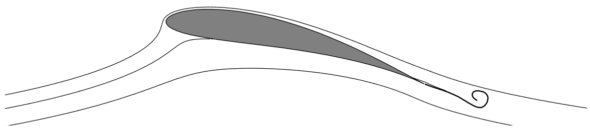
For the flow past a cylinder where the complex potential is given by equation (6.13), we have:

$$\begin{aligned}
T &= -\frac{\rho}{2}\Re\left\{\oint_C z\left(U_0 e^{-i\alpha} - U_0 e^{i\alpha}\frac{a^2}{z^2} - \frac{i\Gamma}{2\pi z}\right)^2 dz\right\} \\
&= -\frac{\rho}{2}\Re\left\{\oint_C z\left(U_0^2 e^{-2i\alpha} - \frac{i\Gamma U_0 e^{-i\alpha}}{\pi z} - \frac{2U_0 a^2}{z^2} - \frac{\Gamma^2}{4\pi^2 z^2} + \dots\right) dz\right\} \\
&= -\frac{\rho}{2}\Re\left\{2\pi i\left(-2U_0 a^2 - \frac{\Gamma^2}{4\pi^2}\right)\right\} \\
&= 0.
\end{aligned}$$

Consequently, there is no aerodynamic torque on a cylinder.

6.7 Origin of circulation around a wing

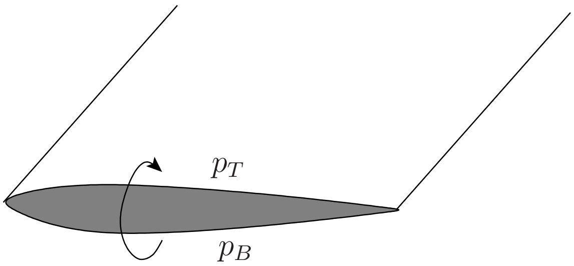
When the plane is stationary on the runway, there is no circulation around the wings. In the absence of viscosity the flow remains vortex free. However, when the plane starts to move, small viscous effects in the boundary layer allow the aerofoil to shed a vortex off the trailing edge.



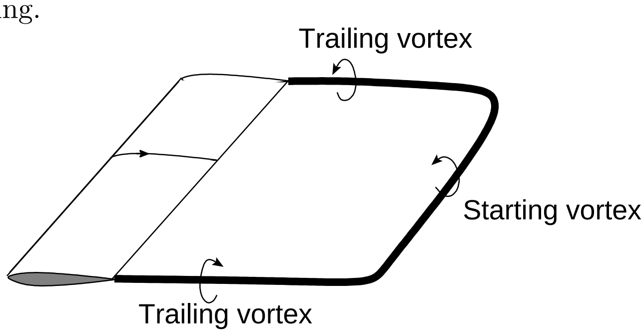
This vortex, called the starting vortex, remains behind on the runway. Its circulation is equal and opposite to the circulation around the wing.

6.8 Three-dimensional aerofoils

No wing is infinitely long. Special care needs to be taken with wingtips.



Since the pressure is different on the upper and lower surfaces of the wing, there is a pressure gradient driving the flow around the edge of the wing. This leads to the creation of a vortex at the edge of the wing.

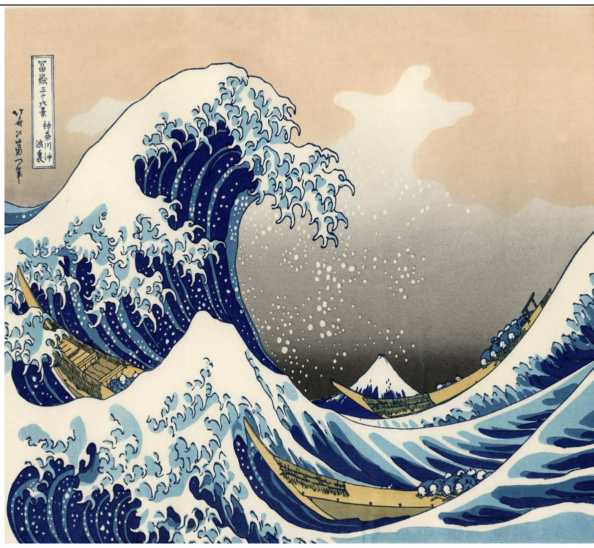


These trailing vortices are part of a single vortex tube formed by the wings and comprising the trailing vortices and the starting vortex.

Chapter 7

Water Waves

Contents



In this final chapter we will look at how waves travel in fluids. One of the most commonly observed waves are surface waves on water. These vary in scale from tsunamis or tidal waves on oceans to small ripples on ponds or bowls of water. There can also be internal waves within fluids, such as atmospheric lee waves.

Real waves are very complex and so in order to analyse them, we will have to make a number of approximations.

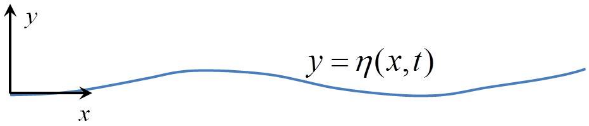
7.1 Surface Gravity Waves on Deep Water

Let us investigate two dimensional water waves where the water surface is given by:

$$y = \eta(x, t)$$

and the fluid velocity for $y < \eta(x, t)$ is:

$$\mathbf{u} = (u(x, y, t), v(x, y, t), 0).$$



We shall assume that the flow is irrotational, so that $\nabla \times \mathbf{u} = 0$ which is a reasonable assumption in water waves. We define a velocity potential $\phi(x, y, t)$, such that

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}.$$

Mass conservation yields:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{7.1}$$

We shall also assume that the water is very deep so that $u = \frac{\partial \phi}{\partial x} \to 0$ and $v = \frac{\partial \phi}{\partial y} \to 0$ as $y \to -\infty$.

7.1.1 Boundary Conditions at the Free Surface

We also require boundary conditions at the free surface $y = \eta(x, t)$. The first of these comes from the Bernoulli equation (6.1):

$$\frac{\partial \phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} - \mathbf{g} \cdot \mathbf{x} = F(t).$$

At the free surface $y = \eta(x, t)$, the pressure is equal to P_{atm} and:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2\right) + g\eta = F(t) - \frac{P_{\text{atm}}}{\rho}.$$

Since adding an arbitrary function of t to the velocity potential does not change the fluid velocity, we define ϕ so that the right-hand-side is zero:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2\right) + g\eta = 0. \tag{7.2}$$

Lastly, the kinematic boundary condition at $y = \eta(x, t)$ reads:

$$\frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v. \tag{7.3}$$

7.1.2 Linearised Boundary Conditions

The boundary conditions (7.2) and (7.3) are rather nasty nonlinear differential equations involving ϕ and η so we need to simplify the problem further. Let us assume that the wave amplitude η is small compared to the wavelength. This allows us to:

i. Impose the boundary conditions at $y = 0$.

ii. Neglect quadratic and higher order terms in both η and ϕ.

Under this approximation, equations (7.2) and (7.3) reduce to,

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \qquad \text{at } y = 0, \tag{7.4}$$

and

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial y} = 0 \qquad \text{at } y = 0. \tag{7.5}$$

7.1.3 Harmonic Waves

Let us now look for solutions of equation (7.1) that satisfy boundary conditions (7.4),(7.5) and $\partial\phi/\partial y = 0$ as $y \to -\infty$. We seek separable solutions of the form $\phi = X(x)Y(y)T(t)$. Substituting into equation (7.1), we have:

$$X''YT = -XY''T,$$

so that:

$$\frac{Y''}{Y} = -\frac{X''}{X} = k^2,$$

with k a constant. We require $Y' \to 0$ as $y \to \infty$, which is satisfied by:

$$Y = \exp(ky),$$

and $k > 0$. Additionally, we get:

$$X = A\exp(ikx) + B\exp(-ikx).$$

We can eliminate η in equations (7.4) and (7.5) by taking the time derivative of equation (7.4) and substituting into equation (7.5) to get:

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0.$$

Using the above forms for ϕ and Y, we get:

$$\frac{d^2T}{dt^2} = -gkT.$$

The solution for T reads:

$$T = C\exp(i\omega t) + D\exp(-i\omega t),$$

where the frequency ω satisfies $\omega^2 = gk$. The solution for the velocity potential takes the form:

$$\phi(x, y, t) = \exp(ky)\left[A\exp(ikx) + B\exp(-ikx)\right]\left[C\exp(-i\omega t) + D\exp(i\omega t)\right],$$

with $\eta(x, t)$ given by

$$\eta(x, t) = \frac{i\omega}{g}\left[A\exp(ikx) + B\exp(-ikx)\right]\left[C\exp(-i\omega t) - D\exp(i\omega t)\right],$$

which we can rewrite in the form:

$$\eta(x, t) = a\cos(kx - \omega t + \alpha) + b\cos(kx + \omega t - \beta). \tag{7.6}$$

for some constants a, b, α and β. The solutions are in the form of plane waves moving to the left and right, where the angular frequency ω and wavelength k are related by:

$$\omega^2 = gk.$$

This last equation is called the *dispersion relation*.

7.1.4 Fluid Motion

In order to examine the motion of the fluid, let us consider the case of a wave moving to the right and set $\alpha = 0$ so that $\eta(x,t) = a\cos(kx - \omega t)$. To find the corresponding velocity potential, we use equation (7.4):

$$\frac{\partial \phi}{\partial t} = -g\eta = -ga\cos(kx - \omega t) \qquad \text{at } y = 0.$$

The velocity potential is:

$$\phi = \frac{ag}{\omega}\exp(ky)\sin(kx - \omega t). \tag{7.7}$$

Therefore, the fluid velocity is:

$$u = \frac{\partial \phi}{\partial x} = a\omega\exp(ky)\cos(kx - \omega t), \tag{7.8}$$

$$v = \frac{\partial \phi}{\partial y} = a\omega\exp(ky)\sin(kx - \omega t). \tag{7.9}$$

Since the velocity is small and periodic in time we can assume that the fluid particles do not move far from their original positions and hence the position $(x(t), y(t))$ of a particle initially located at (x_0, y_0) satisfies:

$$\frac{dx}{dt} = a\omega\exp(ky_0)\cos(kx_0 - \omega t),$$
$$\frac{dy}{dt} = a\omega\exp(ky_0)\sin(kx_0 - \omega t).$$

and hence:

$$x - x_0 = -a\exp(ky_0)\left(\sin(kx_0 - \omega t) - \sin(kx_0)\right),$$
$$y - y_0 = a\exp(ky_0)\left(\cos(kx_0 - \omega t) - \cos(kx_0)\right).$$

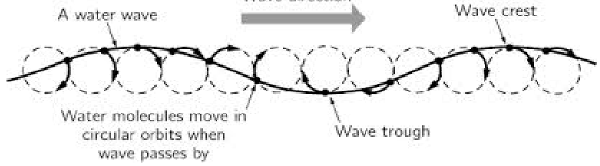
The particle path thus satisfies:

$$(x - x_c)^2 + (y - y_c)^2 = a^2\exp(2ky_0)$$

where

$$x_c = x_0 + a\exp(ky_0)\sin(kx_0),$$
$$y_c = y_0 - a\exp(ky_0)\cos(kx_0).$$

The particle paths are circles of radius $a\exp(ky_0)$. Thus, although the wave propagates in the x-direction, the fluid itself does not propagate, but moves in small circles.



7.2 Wave Speed

Calculating how fast the waves travel turns out to be more complicated than it first appears.

7.2.1 Phase Velocity

We can determine the speed at which the wave crests move. For a wave of the form:

$$\eta(x,t) = a\cos(kx - \omega t) = a\cos\left[k\left(x - \frac{\omega}{k}t\right)\right],$$

the angle within the cosine function remains constant along the trajectory:

$$x = x_0 + \frac{\omega}{k}t.$$

We define the *phase speed* or *phase velocity* as:

$$c_p = \frac{\omega}{k}. \tag{7.10}$$

for the case of gravity waves on deep water, we have:

$$c_p = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}, \tag{7.11}$$

where $\lambda = 2\pi/k$ is the wavelength. In particular the crests of long waves travel faster than shorter waves. For example a wave of length 10cm has a wave speed of 0.4 m·s^{-1} whereas a wave of 1 km in length travels at 40 m·s^{-1}. Consequently, unlike for acoustic waves, waves of different lengths travel at different speeds and will disperse. These waves are called *dispersive waves.*

7.2.2 Energy Flow

Let us now determine the energy associated with the wave motion. The potential energy at position x is purely gravitational:

$$V(x) = \int_0^\eta \rho g y dy = \frac{\rho g \eta^2}{2}. \tag{7.12}$$

For the case $\eta = a\cos(kx - \omega t)$, we have:

$$V(x,t) = \frac{\rho g a^2}{2}\cos^2(kx - \omega t),$$

which, averaged over a cycle yields:

$$\overline{V} = \frac{\rho g a^2}{4}.$$

To find the kinetic energy per unit length, we use the solution for the fluid velocity (equations (7.8) and (7.9)):

$$\overline{T}(x,t) = \int_{-\infty}^0 \left(\frac{\rho}{2}\right)(u^2 + v^2)dy = \frac{\rho a^2 g^2 k^2}{2\omega^2}\int_{-\infty}^0 \exp(2ky)dy = \frac{\rho a^2 g^2 k}{4\omega^2} = \frac{\rho a^2 g}{4},$$

using the dispersion relation $gk = \omega^2$. Hence adding this to the potential energy, the average energy per unit length associated with the wave is:

$$\overline{E} = \overline{V} + \overline{T} = \frac{\rho g a^2}{2}.$$

This energy is transported by the fluid pressure. The rate of work of the fluid pressure at x is given by

$$\frac{dW}{dt} = \int_{-\infty}^0 (P - P_H)u dy$$

where P_H is the hydrostatic pressure. Using the Bernoulli equation, we find $P - P_H = -\rho\frac{\partial\phi}{\partial t}$, so:

$$\frac{dW}{dt} = -\rho\int_{-\infty}^{0}\frac{\partial\phi}{\partial t}udy = \frac{a^2g^2k}{\omega}\cos^2(kx-\omega t)\int_{-\infty}^{0}\exp(2ky)dy = \frac{a^2g\omega}{2k}\cos^2(kx-\omega t).$$

Averaged over a cycle, we obtain:

$$\overline{\frac{dW}{dt}} = \frac{a^2g\omega}{4k}.$$

The speed of energy transport is:

$$\frac{1}{\overline{\overline{E}}}\overline{\frac{dW}{dt}} = \frac{\omega}{2k},$$

which is equal to half the phase speed. This unexpected result demonstrates that the wave energy is not transported at the same rate.

7.2.3 The Group Velocity

To see how the energy can move at a different velocity to the wave crests let us consider the evolution of a wave packet containing waves of frequency close to k_0. At $t = 0$, we can define η from its Fourier transform:

$$\eta(x,0) = f(x) = \int_{-\infty}^{\infty}F(k)\exp(ikx)dk \tag{7.13}$$

but where $F(k)$ is only non-zero for values of k close to k_0. At time t, the surface is given by:

$$\eta(x,t) = \int_{-\infty}^{\infty}F(k)\exp(ikx - i\omega(k)t)dk$$

where $\omega(k)$ is the corresponding frequency. Now since $F(k)$ is non-zero only for $k \approx k_0$, we can approximate $\omega(k)$ by its Taylor series:

$$\omega(k) \approx \omega(k_0) + \frac{d\omega}{dk}(k-k_0),$$

and hence:

$$\eta(x,t) \approx \int_{-\infty}^{\infty}F(k)\exp\left[ikx - i\omega(k_0)t - i\frac{d\omega}{dk}(k-k_0)t\right]dk,$$

giving:

$$\eta(x,t) \approx \exp\left[i(k_0x - \omega(k_0))t\right]\int_{-\infty}^{\infty}F(k)\exp\left[i(k-k_0)\left(x - \frac{d\omega}{dk}t\right)\right]dk,$$

which indicates a wave with wavenumber k_0 moving at wavespeed $\omega(k_0)$ within an envelope moving at speed:

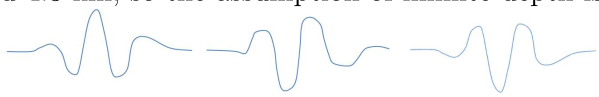
$$c_g = \frac{d\omega}{dk}, \tag{7.14}$$

where c_g is called the *group velocity*.
For gravity waves, $\omega = \sqrt{gk}$, and therefore:

$$c_g = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{c_p}{2}.$$

Note that for non-dispersive waves, ω/k is constant and $c_g = c_p$.

Although the crests of gravity waves travel at c_p, the disturbance itself travels at half this speed. You can see this if you drop a stone in a pond. The disturbance travels radially outwards at the group velocity but individual crests appear at the back of this disturbance, travel to the front and then disappear again. Consequently while the phase speed for a 1km wave is 40 m·s^{-1} the disturbance only travels at 20m·s^{-1}. However ocean waves can still travel quickly. Tsunamis or tidal waves associated with earthquakes can have wavelengths in excess of 100 km and so in the deep ocean can travel at speeds of around 200 m·s^{-1}. Note that the average ocean is around 4.3 km, so the assumption of infinite depth is no longer valid.



7.3 Gravity Waves on Shallower Water

Let us now consider the case where there is an ocean bed at $y = -H$. The boundary conditions at the free surface, $y = 0$, remain given by equations (7.4) and (7.5):

$$\frac{\partial \phi}{\partial t} + g\eta = 0,$$
$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial y} = 0.$$

The only difference is that the boundary condition $v = 0$ must be imposed at $y = -H$ rather than at $y \to \infty$. This problem is set as an exercise on the example sheet. The solution for ϕ corresponding to the surface wave disturbance:

$$\eta = a\cos(kx - \omega t), \tag{7.15}$$

is:

$$\phi = \frac{ga}{\omega}\frac{\cosh(k(y+H))}{\cosh kH}\sin(kx - \omega t), \tag{7.16}$$

with the dispersion relation given by

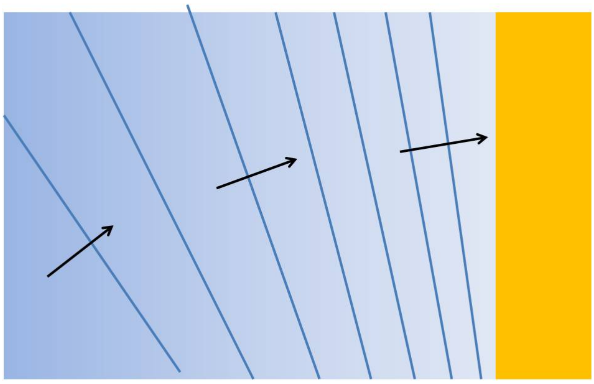
$$\omega^2 = gk\tanh kH. \tag{7.17}$$

Hence, the phase speed is:

$$c_p = \left[\frac{g\tanh kH}{k}\right]^{1/2}. \tag{7.18}$$

In the limit $kH \to \infty$, $\tanh kH \to 1$ and we recover the deep water case where $c_p = \sqrt{g/k}$. However, in the opposite limit when $kH \ll 1$, $\tanh kH \simeq kH$ yielding $c_p \to \sqrt{gH}$. Thus, in shallow water the phase speed is independent of k. The waves are non-dispersive and the group velocity is equal to the phase velocity.

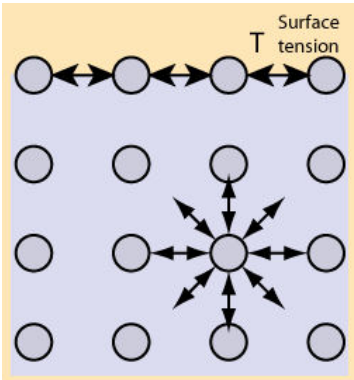
One consequence of this result is that waves slow down as the sea gets shallower. This causes the waves to become shorter and steeper. In the deep ocean a Tsunami wave is only around a metre in height with a wavelengths of around 100km and is hard to detect. However, as it reaches the coast, its wavelength reduces and the height increases around 30 fold. This is also the reason why waves always appear to come in parallel to the shore-line.



7.4 Capillary Waves

So far we have considered waves in which the force trying to restore equilibrium is due to gravity. While this is the dominant force for waves on open water, for short waves there is an alternative restoring force due to surface tension.

7.4.1 Surface Tension



In a liquid, there are short-range attractive forces between molecules. For molecules away from the boundaries there are equal numbers of molecules in all directions and so these forces cancel each other out. However, for molecules at the surface there is an imbalance in the force due to the greater affinity that the molecules have for themselves as opposed to molecules of the neighbouring fluid which results in a tension in the surface that resists expansion of the surface area.

The surface tension between air and water is 0.072 N$\cdot$m^{-1}. This is sufficient to prevent a paper-clip from sinking and allows small insects called pond skaters to "skate" over the surface of ponds.

If the surface is flat, then the tension forces on opposite sides cancel, however, if the surface is curved, then surface tension produces a net surface force in the normal direction. As a consequence the surface force is not continuous (as assumed in equation (2.12)), but has a jump given by:

$$\mathbf{n} \cdot \boldsymbol{\tau} = -P\mathbf{n} + \mathbf{n} \cdot \boldsymbol{\sigma} = -P_{\text{atm}}\mathbf{n} - \gamma(\nabla \cdot \mathbf{n})\mathbf{n}, \tag{7.19}$$

where γ is the surface tension and $\nabla \cdot \mathbf{n}$ is the surface curvature.

For an inviscid fluid, $\boldsymbol{\sigma}$ is assumed to be negligible and we get:

$$P = P_{\text{atm}} + \gamma(\nabla \cdot \mathbf{n}).$$

If the surface is given by $f(x,y) = y - \eta(x,t) = 0$:

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{1+(\frac{\partial \eta}{\partial x})^2}}\left(-\frac{\partial \eta}{\partial x}, 1\right),$$

and hence the curvature is:

$$\nabla \cdot \mathbf{n} = \frac{\partial}{\partial x}\left(-\frac{\frac{\partial \eta}{\partial x}}{\sqrt{1+(\frac{\partial \eta}{\partial x})^2}}\right).$$

For linear waves, $|\frac{\partial \eta}{\partial x}| \ll 1$, and:

$$\nabla \cdot \mathbf{n} \simeq -\frac{\partial^2 \eta}{\partial x^2}.$$

Thus, the pressure at the surface $y = \eta(x,t)$ is:

$$P = P_{\text{atm}} - \gamma \frac{\partial^2 \eta}{\partial x^2}.$$

7.4.2 Short waves on deep water

Applying the Bernoulli equation at the surface (and neglecting the term (u^2+v^2)), we get:

$$\frac{\partial \phi}{\partial t} + \frac{P_{\text{atm}}}{\rho} - \frac{\gamma}{\rho}\frac{\partial^2 \eta}{\partial x^2} + g\eta = E(t),$$

and so by choosing the origin of ϕ suitably the Bernoulli equation at $y = 0$ becomes:

$$\frac{\partial \phi}{\partial t} - \frac{\gamma}{\rho}\frac{\partial^2 \eta}{\partial x^2} + g\eta = 0. \tag{7.20}$$

The kinematic boundary condition remains given by equation (7.5):

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial y} = 0$$

and the velocity potential ϕ satisfies:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

with $\frac{\partial \phi}{\partial y} \to 0$ as $y \to -\infty$.
We again look for a solution of the form $\eta(x,t) = a\cos(kx - \omega t)$, so that the boundary conditions at $y = 0$ become:

$$\frac{\partial \phi}{\partial t} = -a\cos(kx-\omega t)\left(\frac{\gamma k^2}{\rho} + g\right),$$
$$\frac{\partial \phi}{\partial y} = a\omega \sin(kx - \omega t).$$

A suitable solution of Laplace's equation is:

$$\phi = B\exp(ky)\sin(kx - \omega t),$$

where

$$-B\omega\cos(kx-\omega t) = -a\cos(kx-\omega t)\left(\frac{\gamma k^2}{\rho}+g\right),$$
$$Bk\sin(kx-\omega t) = a\omega\sin(kx-\omega t).$$

The dispersion relation becomes:

$$\omega^2 = gk + \frac{\gamma k^3}{\rho}. \tag{7.21}$$

The phase speed is given by:

$$c_p = \frac{\omega}{k} = \left(\frac{g}{k}+\frac{\gamma k}{\rho}\right)^{1/2},$$

and the group velocity by:

$$c_g = \frac{g+3\gamma k^2/\rho}{2\left(gh+\gamma k^3/\rho\right)^{1/2}}.$$

For small k, the phase speed decreases with k for $k < k_c$, where

$$k_c = \sqrt{\frac{\rho g}{\gamma}}.$$

This limit corresponds to gravity waves. However, above this critical wave number, the phase speed increases with k and surface tension becomes the dominant mechanism. Waves in this regime are referred to as *capillary waves*. The wavelength corresponding to k_c is:

$$\lambda_c = 2\pi\sqrt{\frac{\gamma}{\rho g}}$$

is around 1.7 cm in water.
Thus water waves of lengths shorter than around 1 cm are driven by surface tension rather than gravity, and in the limit $k \gg k_c$,

$$c_p = \sqrt{\frac{\gamma k}{\rho}}, \qquad c_g = \frac{3}{2}\sqrt{\frac{\gamma k}{\rho}},$$

indicating that the group velocity is faster than the phase velocity. Hence crests appear at the front of the disturbance and travel backwards relative to the wave packet.

7.5 Nonlinear waves

All the cases in this section have been based on the assumption that the amplitude is small compared with wavelength so that $|\frac{\partial\eta}{\partial x}| \ll 1$. However, this is not always the case in practice. In particular waves approaching a beach become nonlinear and overturn causing them to break. Recently it has also been recognised that nonlinear waves can occur in the deep ocean where they appear as dangerous high amplitude steep waves, known as rogue waves. There are also special nonlinear waves called solitary waves that can travel long distances without changing shape, as famously discovered by John Scott Russell on the Glasgow-Edinburgh Canal.