The Grassmannian

To turn the Grassmannian into a variety, we need a coordinate system for subspaces.

For projective space, a homogeneous coordinate-tuple $[Z_0, \dots, Z_n]$ represents an equivalence class of points in $\mathbb{A}^{n+1}$, namely all points on the same line through the origin.
This equivalence can be seen as coming from a **group action**. The multiplicative group K^* acts on $\mathbb{A}^{n+1} \setminus \{0\}$ by scalar multiplication and each point of $\mathbb{P}^n$ corresponds to an **orbit** of this action, in other words, $\mathbb{P}^n$ is the **quotient space** $(\mathbb{A}^{n+1} \setminus \{0\})/K^*$.

We can try the same for the Grassmannian: A k-dimensional subspace of K^n is spanned by k vectors. So we look at the space of all k-tuples of linearly independent vectors, which we think of as the rows of $k \times n$-matrices.
The group $\mathrm{GL}_k(K)$ acts on this space by multiplication from the left:

$$\begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,k} \\ \vdots & \ddots & \vdots \\ \lambda_{k,1} & \cdots & \lambda_{k,k} \end{pmatrix} \cdot \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} \end{pmatrix}$$

and two $k \times n$-matrices have the same row span if and only if they are in the same orbit under this group action. So we can identify $G(k,n)$ with the quotient space

$$\mathrm{Mat}^{(k)}_{k\times n}(K)/\mathrm{GL}_k(K).$$

where $\mathrm{Mat}^{(k)}$ is the set of matrices of rank k.

The Grassmannian

Looking further at the group action

$$\begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,k} \\ \vdots & \ddots & \vdots \\ \lambda_{k,1} & \cdots & \lambda_{k,k} \end{pmatrix} \cdot \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} \end{pmatrix},$$

we see that if the first $k \times k$-minor of the matrix on the right is non-zero, the orbit contains a unique element of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & b_{1,1} & b_{1,2} & \cdots & b_{1,n-k} \\ 0 & 1 & \cdots & 0 & b_{2,1} & b_{2,2} & \cdots & b_{2,n-k} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & b_{k,1} & b_{k,2} & \cdots & b_{k,n-k} \end{pmatrix}.$$

Conversely, we obtain a matrix of rank k for any $k \times (n-k)$-matrix B on the right. In other words, the row spans of matrices of this form are in bijection with an affine space $\mathbb{A}^{k(n-k)}$.

But this involved a choice coming from the assumption that the *first* $k \times k$-minor is non-zero. In general, we have to permute columns first. So we see in this way that the Grassmannian $G(n,k)$ is covered by $\binom{n}{k}$ copies of affine spaces $\mathbb{A}^{k(n-k)}$. (Note the analogy with projective space!)

In particular, whatever the Grassmannian is as a variety, it must be of dimension $k(n-k)$.

The Grassmann algebra

While the above description of the Grassmannian in terms of matrices works fine for understanding it as a set, it is not very convenient for the goal of finding an embedding of the Grassmannian into projective space. Instead, it is better to employ some multilinear algebra.

The **Grassmann algebra** or **exterior algebra** is the algebra of antisymmetric tensors.

Let V be a vector space of finite dimension n. The **tensor algebra** is the non-commutative algebra $T(V) = \bigoplus_{k\geqslant 0} V^{\otimes k}$, where $V^{\otimes k}$ is the k-th tensor power of V, spanned by all tensors $v_1 \otimes \cdots \otimes v_k$ with $v_1, \ldots, v_k \in V$. The product in $T(V)$ is given by the tensor product, i.e. it the map $V^{\otimes k} \times V^{\otimes \ell} \to V^{\otimes k+\ell}$, defined as the bilinear extension of $(v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_\ell) \mapsto v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_\ell$.

The **exterior algebra** $\bigwedge V$ is the residue class ring of $T(V)$ modulo the ideal generated by all tensors of the form $v \otimes v$ for $v \in V$. The residue class of a basis tensor $v_1 \otimes \ldots \otimes v_k$ is denoted

$$v_1 \wedge \cdots \wedge v_k.$$

We call the elements of $\bigwedge V$ **multivectors**. The exterior algebra inherits the grading from the tensor algebra, i.e. it has a decomposition $\bigwedge V = \bigoplus \bigwedge^k V$, where $\bigwedge^k V$ is spanned by all multivectors of the form $v_1 \wedge \cdots \wedge v_k$ for $v_1, \ldots, v_k \in V$. In particular, $\bigwedge^1 V = V$ and $\bigwedge^0 V = K$.

The Grassmann algebra

The algebra $\bigwedge V$ has the following properties for all $\omega, \eta, \vartheta \in \bigwedge V$, $\alpha \in K$.

(1) $\omega \wedge (\eta \wedge \vartheta) = (\omega \wedge \eta) \wedge \vartheta$ (Associativity)

(2) $\omega \wedge (\eta + \vartheta) = \omega \wedge \eta + \omega \wedge \vartheta$, $(\omega + \eta) \wedge \vartheta = \omega \wedge \vartheta + \eta \wedge \vartheta$ (Bilinearity)

(3) $\alpha(\omega \wedge \eta) = (\alpha\omega) \wedge \eta = \omega \wedge (\alpha\eta)$

(4) $0 \wedge \omega = \omega \wedge 0 = 0$

Futhermore, for all $v \in V = \bigwedge^1 V$, we have

(5) $v \wedge v = 0$. (Antisymmetry)

From $(v+w)\wedge(v+w) = 0$, it follows that $v \wedge w = -v \wedge w$ (which is equivalent to (5) if $\mathrm{char}(K) \neq 2$) and by induction $v_1 \wedge \cdots \wedge v_k = \mathrm{sgn}(\sigma)(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)})$ for all permutations $\sigma \in S_k$.

Now let $v_1, \ldots, v_n$ be a basis of V. Then we can use bilinearity to expand every multivector in $\bigwedge V$ in terms of this basis. Explicitly, we obtain

$$\left(\sum a_{i,1} v_i\right) \wedge \cdots \wedge \left(\sum a_{i,k} v_i\right) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \begin{vmatrix} a_{i_1,1} & \cdots & a_{i_1,k} \\ \vdots & & \vdots \\ a_{i_k,1} & \cdots & a_{i_k,k} \end{vmatrix} v_{i_1} \wedge \cdots \wedge v_{i_k}$$

for $k \leqslant n$. In particular, we see that every multivector in $\bigwedge^n V$ is a multiple of $v_1 \wedge \cdots \wedge v_n$, with the coefficient of a multivector of the form $w_1 \wedge \cdots \wedge w_n$ given by the determinant of the coefficient matrix of $w_1, \ldots, w_n$ in terms of the basis $v_1, \ldots, v_n$.

Note that, because of (5), we never need repeated basis elements. In particular, we find

$$\dim \bigwedge^k V = \binom{n}{k}$$

for all $k \leqslant n$ and $\bigwedge^k V = 0$ for all $k > n$.

The Plücker embedding

We now use the Grassmann algebra to realize the Grassmannian as a projective variety.

Let W be a k-dimensional subspace of V with basis $v_1, \ldots, v_k$. The multivector $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V$ is determined by W up to a scalar, by what we just saw: If we pick a different basis, the corresponding multivector in $\bigwedge^k V$ is obtained by multiplying with the determinant of the base change. So we have a well-defined map

$$\psi : G(k, V) \to \mathbb{P}(\textstyle\bigwedge^k V).$$

The image of ψ is the set of **totally decomposable multivectors** of $\bigwedge^k V$. (While general multivectors in $\bigwedge^k V$ are sums of totally decomposable ones.)

The map ψ is injective. To see this, let

$$L_\omega = \{v \in V : \omega \wedge v = 0\}$$

for any $\omega \in \bigwedge^k V$. This is a linear subspace of V. For $\omega = v_1 \wedge \cdots \wedge v_k$ as above, we find $L_\omega = W$ (see also the lemma on the next slide). So $\omega \mapsto L_\omega$ is the inverse of ψ (on its image).

In conclusion, we identified the Grassmannian $G(k, V)$ with the set of totally decomposable multivectors in $\mathbb{P}(\bigwedge^k V)$. This is called the **Plücker embedding** of $G(k, V)$.

It remains to show that the totally decomposable multivectors form a closed subset of $\mathbb{P}(\bigwedge^k V)$ and to find the equations that describe it.

The Plücker embedding

Lemma 3.1. *Let $\omega \in \bigwedge^k V$, $\omega \neq 0$. The space $L_\omega = \{v \in V : \omega \wedge v = 0\}$ has dimension at most k, with equality occuring if and only if ω is totally decomposable.*

Proof. Pick a basis $v_1, \ldots, v_s$ of L_ω and extend to a basis $v_1, \ldots, v_n$ of V. We express ω in this basis: For any choice of indices $I = \{i_1, \ldots, i_k\}$ with $1 \leqslant i_1 < \cdots < i_k \leqslant n$ let $\omega_I = v_{i_1} \wedge \cdots \wedge v_{i_k}$. Then ω can be written as

$$\omega = \sum_{I \subset \{1,\ldots,n\}, |I|=k} c_I \omega_I$$

for some scalars $c_I \in K$. For $j \in \{1, \ldots, n\}$, we find

$$\omega \wedge v_j = \sum c_I \omega_I \wedge v_j = \sum_{I : j \notin I} c_I \omega_I \wedge v_j.$$

Now for $j \leqslant s$, we have $v_j \in L_\omega$ and the equation $\omega \wedge v_j = 0$ shows $c_I = 0$ for all I with $j \notin I$. In other words, all I with $c_I \neq 0$ must contain $\{1, \ldots, s\}$. If $s > k$, there is no such I of length k, contradicting the fact that $\omega \neq 0$. If $s = k$, then there is exactly one such I, namely $I = \{1, \ldots, k\}$, hence ω is a multiple of $v_1 \wedge \cdots \wedge v_k$. Conversely, if ω is totally decomposable, say $\omega = w_1 \wedge \cdots \wedge w_k$, then $w_1, \ldots, w_k \in L_\omega$, hence $\dim L_\omega \geqslant k$. ■

The Plücker embedding

Lemma 3.1. *Let $\omega \in \bigwedge^k V$, $\omega \neq 0$. The space $L_\omega = \{v \in V : \omega \wedge v = 0\}$ has dimension at most k, with equality occuring if and only if ω is totally decomposable.*

This will be all we need: Fix $\omega \in \bigwedge^k V$, $\omega \neq 0$ and consider the map

$$\varphi(\omega) : \begin{cases} V & \to & \bigwedge^{k+1} V \\ v & \mapsto & \omega \wedge v \end{cases}.$$

By the lemma, we have $[\omega] \in G(k, V)$ if and only if the rank of $\varphi(\omega)$ is at most $n - k$.
The map $\bigwedge^k V \to \mathrm{Hom}(V, \bigwedge^{k+1} V)$ given by $\omega \mapsto \varphi(\omega)$ is linear. If we fix coordinates by fixing a basis of V, this means that the matrix $A(\omega)$ describing $\varphi(\omega)$ has linear entries, i.e. entries that are homogeneous of degree 1 in the coordinates. Therefore, $G(k, V)$ is defined by the vanishing of all $(n-k+1) \times (n-k+1)$-minors of this matrix. We have proved:

Theorem 3.2. *The Grassmannian $G(k, V)$ is a projective variety, embedded as a closed subset of $\mathbb{P}(\bigwedge^k V)$ under the Plücker embedding.* ∎

The Plücker embedding

Theorem 3.2. *The Grassmannian $G(k, V)$ is a projective variety, embedded as a closed subset of $\mathbb{P}(\bigwedge^k V)$ under the Plücker embedding.* ∎

Fix a basis of $v_1, \ldots, v_n$ of V and the corresponding basis $v_{i_1} \wedge \cdots \wedge v_{i_k}$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$ of $\bigwedge^k V \cong K^{\binom{n}{k}}$. If a subspace W is represented as the row span of a $k \times n$-matrix A, the formula

$$\left(\sum a_{i,1} v_i\right) \wedge \cdots \wedge \left(\sum a_{i,k} v_i\right) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \begin{vmatrix} a_{i_1,1} & \cdots & a_{i_1,k} \\ \vdots & & \vdots \\ a_{i_k,1} & \cdots & a_{i_k,k} \end{vmatrix} v_{i_1} \wedge \cdots \wedge v_{i_k},$$

which we saw earlier, shows what the Plücker embedding does in these coordinates: It maps the matrix A to the tuple of all $k \times k$-minors of A (of which there are $\binom{n}{k} = \dim(\bigwedge^k V)$).

The Plücker embedding of $G(k, n)$ as a space of matrices is given by the $k \times k$-minors.
The relations between these minors corresponding to the equations of $G(k, n)$ in $\mathbb{P}(\bigwedge^k V)$ are the **Plücker relations**.

Affine cover of the Grassmannian

We have seen how the Grassmannian $G(k,n)$ is covered by $\binom{n}{k}$ copies of $\mathbb{A}^{k(n-k)}$.
Let us see what that corresponds to under the Plücker embedding.

First, there is an abstract description:
Let Γ be any subspace of dimension $n-k$ of V, corresponding to a multivector $\eta \in \bigwedge^{n-k} V$.
The set

$$H_\Gamma = \left\{ W \in G(k,V) : \Gamma \cap W \neq \{0\} \right\}$$

is a hyperplane in $G(k,V)$. Namely, if $W = [\omega]$ for $\omega \in \bigwedge^k V$, then $\Gamma \cap W \neq \{0\}$ is equivalent to $\omega \wedge \eta = 0$. Since $\omega \wedge \eta$ is an element of $\bigwedge^n V$, which is one-dimensional, we can identify $\bigwedge^n V$ with K and thus interpret η as a linear form on $\bigwedge^k V$ given by $\omega \mapsto \omega \wedge \eta$. (Indeed, this amounts to a natural isomorphism $\bigwedge^{n-k} V \cong \bigwedge^k V^*$, up to scaling.)

Thus H_Γ is the hyperplane defined by η, so that $U_\Gamma = \mathbb{P}(\bigwedge^k V) \setminus H_\Gamma$ is an affine space. The intersection $G(k,V) \cap U_\Gamma$ thus corresponds to all k-dimensional subspaces of V that are complementary to Γ. Fix some k-dimensional subspace W_0 of V complementary to Γ. Then any other such subspace W can be viewed as the graph of a linear map $W_0 \to \Gamma$, and vice-versa. (Given W, the corresponding map is $w_0 \mapsto \gamma$, where $\gamma \in \Gamma$ is the unique element with $w_0 + \gamma \in W$. Conversely, given $\alpha\colon W_0 \to \Gamma$, let $W = \{w_0 + \alpha(w_0) : w_0 \in W_0\}$.) Since $W_0 \cong K^k$ and $\Gamma \cong K^{n-k}$, we find

$$G(k,V) \cap U_\Gamma \cong \mathrm{Hom}(W_0, \Gamma) \cong \mathrm{Mat}_{k\times(n-k)}(K) = \mathbb{A}^{k(n-k)}.$$

Affine cover of the Grassmannian

Now let $V = K^n$ and $\Gamma = \mathrm{span}(e_{k+1}, \ldots, e_n)$. Then any subspace W complementary to Γ has a unique basis given by the rows of a $k \times n$-matrix of the form

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & b_{1,1} & b_{1,2} & \cdots & b_{1,n-k} \\ 0 & 1 & \cdots & 0 & b_{2,1} & b_{2,2} & \cdots & b_{2,n-k} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & b_{k,1} & b_{k,2} & \cdots & b_{k,n-k} \end{pmatrix}.$$

This yields a bijection of $G(k,n) \cap U_\Gamma$ with $\mathbb{A}^{k(n-k)}$.

Under the Plücker embedding, we know that A is mapped to the tuple of all its $k \times k$-minors. But since the left part of A is the identity, the $k \times k$-minors of A are really just the minors of the matrix B of *any* size. Hence the Plücker embedding of $G(k,n) \cap U_\Gamma$ is given by all the minors of the matrix B.

Finally, since the affine parts $G(k,n) \cap U_\Gamma$ are irreducible open subsets of dimension $k(n-k)$ and have pairwise non-empty intersection, we conclude:

Corollary 3.3. *The Grassmannian $G(k,n)$ is an irreducible variety of dimension $k(n-k)$.* ∎

The Grassmannian $\mathbb{G}(1,3)$

The Grassmannian $\mathbb{G} = \mathbb{G}(1,3) = G(2,4)$ parametrizes lines in $\mathbb{P}^3$.
The Plücker embedding puts $\mathbb{G}$ into $\mathbb{P}(\bigwedge^2 K^4) \cong \mathbb{P}^5$. Writing $z_{ij} = v_i \wedge v_j$, $0 \leqslant i < j \leqslant 3$, the image is the quadratic hypersurface

$$\mathcal{V}(z_{01}z_{23} - z_{02}z_{13} + z_{03}z_{12})$$

called the **Plücker quadric**.

This and the following statements will be shown in the exercises.

Proposition 3.4. *For any point $p \in \mathbb{P}^3$ and plane $H \subset \mathbb{P}^3$ with $p \in H$, let $\Sigma_{p,H} \subset \mathbb{G}$ be the set of lines in $\mathbb{P}^3$ passing through p and lying in H. Under the Plücker embedding, $\Sigma_{p,H}$ is a line in $\mathbb{P}^5$. Conversely, every line in $\mathbb{G} \subset \mathbb{P}^5$ is of the form $\Sigma_{p,H}$ for some choice of p, H.*

Proposition 3.5. *For any point $p \in \mathbb{P}^3$, let $\Sigma_p \subset \mathbb{G}$ be the set of lines in $\mathbb{P}^3$ passing through p; for any plane $H \subset \mathbb{P}^3$, let $\Sigma_H \subset \mathbb{G}$ be the locus of lines lying in H. Under the Plücker embedding, both Σ_p and Σ_H are carried into planes in $\mathbb{P}^5$. Conversely, any plane $\Lambda \subset \mathbb{G} \subset \mathbb{P}^5$ is either of the form Σ_p for some point p or of the form Σ_H for some plane H.*

Proposition 3.6. *Let $\ell_1, \ell_2 \subset \mathbb{P}^3$ be skew lines (i.e. $\ell_1 \cap \ell_2 = \varnothing$). The set $Q \subset \mathbb{G}$ of lines in $\mathbb{P}^3$ meeting both is the intersection of $\mathbb{G}$ with a three-dimensional subspace of $\mathbb{P}^5$.*

Incidence Correspondences

Let $\mathbb{G}(k,n)$ be the Grassmannian of k-subspaces in $\mathbb{P}^n$ and put

$$\Sigma = \left\{(\Lambda, x) \in \mathbb{G}(k,n) \times \mathbb{P}^n : x \in \Lambda\right\}.$$

So Σ is the subvariety of $\mathbb{G}(k,n) \times \mathbb{P}^n$ whose fibre over a point $\Lambda \in \mathbb{G}(k,n)$ is just Λ itself as a subset of $\mathbb{P}^n$. To see that Σ is closed, it suffices to note that

$$\Sigma = \left\{(v_1 \wedge \cdots \wedge v_k, w) : v_1 \wedge \cdots \wedge v_k \wedge w = 0\right\}.$$

Proposition 3.7. *Let $\Phi \subset \mathbb{G}(k,n)$ be a closed subvariety. Then $\bigcup_{\Lambda \in \Phi} \Lambda$ is closed in $\mathbb{P}^n$.*

Proof. Let π_1, π_2 be the projection maps of Σ onto $\mathbb{G}(k,n)$ and $\mathbb{P}^n$. Then

$$\bigcup_{\Lambda \in \Phi} \Lambda = \pi_2(\pi^{-1}(\Phi)). \qquad \blacksquare$$

Proposition 3.8. *Let $X \subset \mathbb{P}^n$ be a projective variety. Then $\mathcal{C}_k(X) = \left\{\Lambda \in \mathbb{G}(k,n) : \Lambda \cap X \neq \varnothing\right\}$ is closed in $\mathbb{G}(k,n)$.*

Proof. We have

$$\mathcal{C}_k(X) = \pi_1(\pi_2^{-1}(X)). \qquad \blacksquare$$

The variety $\mathcal{C}_k(X)$ is called the **variety of incident subspaces**.

Proposition 3.9. *Let $X, Y \subset \mathbb{P}^n$ be two disjoint projective varieties. Let $J(X,Y)$ be the union of all lines $\overline{pq}$ with $p \in X$, $q \in Y$, called the **join of X and Y**. Then $J(X,Y)$ is closed in $\mathbb{P}^n$.*

Proof. The set $\mathcal{J}(X,Y) = \mathcal{C}_1(X) \cap \mathcal{C}_1(Y)$ is closed in the Grassmannian, hence $J(X,Y) = \bigcup_{\ell \in \mathcal{J}} \ell$ is closed in $\mathbb{P}^n$. $\blacksquare$