Feynman integrals and graphs

In this lecture we will consider the problem of evaluating the integral

$$\int_{\mathbb{R}^n} e^{-B(x,x)/2+S(x)}dx.$$

Here $B(x,x)$ is a positive definite symmetric bilinear form on $\mathbb{R}^n$ and

$$S(x) = \sum_{m\geq 3} g_m S_m(x^{\otimes m})/m!.$$

$S_m \in S^m V^*$. In other words, S_m is a homogeneous polynomial function on V of degree m. Of course, such an integral can diverge but we understand it in the formal sense, in other words, it takes values in the ring $\mathbb{R}[[g_1, \ldots, g_n, \ldots]]$.

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It turns out that the answer is formulated combinatorially in terms of graphs. This relationship between asymptotic expansions of integrals and combinatorics of graphs underlies all further links between field theory and the theory of operads.

We start by reminding the notion of a graph. Our description here will be somewhat informal, in order to avoid overburdening the reader with the precise details. A more complete description is contained in [4].

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1. *A finite set, also denoted by Γ, consisting of the half-edges of Γ.*
2. *A partition $V(\Gamma)$ of Γ. The elements of $V(\Gamma)$ are called the vertices of Γ.*
3. *A partition $E(\Gamma)$ of Γ into sets having cardinality equal to two. The elements of $E(\Gamma)$ are called the edges of Γ.*

We say that a vertex $v \in V$ has valency n if v has cardinality n. The elements of v are called the *incident half-edges* of v. We will consider only those graphs whose internal vertices have valency ≥ 3.
There is an obvious notion of isomorphism for graphs. Two graphs Γ and Γ' are said to be isomorphic if there is a bijection $\Gamma \to \Gamma'$ which preserves the structures described by items (1–3) of Definition 1.

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There is an obvious notion of isomorphism for graphs. Two graphs Γ and Γ' are said to be isomorphic if there is a bijection $\Gamma \to \Gamma'$ which preserves the structures described by items (1–3) of Definition 1. We will now temporarily leave graphs and become interested in computing certain type of integrals. Among those the simplest are as follows.

Example

Consider the integral

$$I = \int_{-\infty}^{\infty} e^{-x^2/2} x^{2m} dx.$$

Integrating by parts, using induction and the well-known identity $\int_{\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$ we obtain

$$I = \sqrt{2\pi}(2m-1)(2m-3)\ldots 1 = \sqrt{2\pi}\frac{(2m)!}{2^m m!}.$$

Furthermore, it is clear that $I = \int_{-\infty}^{\infty} e^{-x^2/2} x^{2m+1} dx = 0$ for any m as the integrand is an odd function.

Note that $\frac{(2m)!}{2^m m!}$ is the number of splittings of the set $1, 2, \ldots, 2m$ into pairs. The set of such splittings is acted upon by the permutation group S_{2m} and the stabilizer of an element is isomorphic to the semidirect product of S_m and $(\mathbb{Z}/2)^m$.

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Proposition

Let $f_1, \dots, f_N$ be linear functions on V. Denote by $\langle f_1, \dots, f_N \rangle_0$ the ratio

$$\frac{\int_{\mathbb{R}^n} f_1(x) \dots f_N(x) e^{-B(x,x)/2} dx}{\int_{\mathbb{R}^n} e^{-B(x,x)/2} dx}$$

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Then $\langle f_1, \dots, f_N\rangle_0 = 0$ if N is odd and

$$\langle f_1, \dots, f_N\rangle_0 = \sum B^{-1}(f_{i_1}, f_{i_2}), \dots, B^{-1}(f_{i_{N-1}}, f_{i_N}) \tag{0.1}$$

where the summation is extended over all partitions of the set $1, \dots, N$ into pairs $(i_1, i_2), \dots, (i_{N-1}, i_N)$.

Proof.

It is clear that the integrand is an odd function for N odd and therefore the integral vanishes in this case. Let N be even. By linear change of variables we could reduce B to a diagonal form $B = x_1^2 + \ldots + x_n^2$; then $B^{-1}(x_i, x_j) = \delta_{ij}$. Note that since both sides of (0.1) are symmetric multilinear functions on in the variables f_i it is sufficient to check it for the case when $f_1 = \ldots = f_N = f$. In this case the right hand side of (0.1) is equal to $\frac{(2m)!}{2^m m!} B^{-1}(f, f)^N$. Furthermore, by a linear change of variables preserving the quadratic form B and taking f to one of the coordinate functions x_i (up to a factor) one can reduce the left hand side of (0.1) to a one dimensional integral and thus to Example 2. $\square$

Now consider a more general situation, i.e. the ratio of integrals

$$\langle f_1, \dots f_N \rangle := \frac{\int_{\mathbb{R}^n} f_1 \dots f_N e^{-B(x,x)/2+S(x)} dx}{\int_{\mathbb{R}^n} e^{-B(x,x)/2} dx}.$$

The expression $\langle f_1, \dots f_N \rangle$ takes values in $\mathbb{R}[[g_1, \dots, g_n, \dots]]$; one may or may not be able to set the parameters g_i to be equal to actual numbers. For example, the integral $\int_{-\infty}^{\infty} e^{-x^2/2-gx^4}$ actually converges for any number g whereas $\int_{-\infty}^{\infty} e^{-x^2/2+gx^4}$ only makes sense as a formal power series in g.

Remark

The expression $\langle f_1, \dots f_N \rangle$ is known in quantum field theory as an expectation value of the observable $f_1 \dots f_N$.

Let $\mathbf{n} = (n_3, n_4, \ldots)$ be any sequence of nonnegative integers which is eventually zero. Denote by $G(N, \mathbf{n})$ the set of isomorphism classes of graphs which have N 1-valent vertices labeled by the numbers $1, \ldots, N$ and n_i unlabeled i-valent vertices. The labeled vertices are called external, the unlabeled ones internal. Then for any graph $\Gamma \in G(N, \mathbf{n})$ we construct a multilinear function F_Γ which associates to $f_1, \ldots, f_N$ a number $F_\Gamma(f_1 \ldots, f_N)$ as follows. We attach to any 1-valent vertex of Γ labeled by i the vector f_i; to any m-valent vertex the tensor S_m. We then take the tensor product of these tensors and take contractions along edges using the form B^{-1}. The we have the following result.

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Theorem

$$\langle f_1, \dots f_N \rangle = \sum_{\mathbf{n}} \Big(\prod_i g_i^{n_i}\Big) \sum_{\Gamma \in G(N, \mathbf{n})} |Aut(\Gamma)|^{-1} F_\Gamma(f_1 \dots, f_N) \tag{0.2}$$

where $Aut(\Gamma)$ is the (finite) group of automorphisms of Γ which fix the external vertices.

Proof.

The proof follows from Wick's lemma. Note that (0.2) is a special case of (0.1) for $\mathbf{n} = 0$. We can regard every i-valent vertex of Γ as a collection of i 1-valent vertices sitting close to each other. Every such graph with k vertices corresponds to a summand in $(S)^k/k!$ in the expansion of e^S. Furthermore, each graph $\Gamma \in G(N, \mathbf{n})$ determines precisely $|Aut(\Gamma)|^{-1} \prod i!^{n_i} \prod n_i!$ different pairings of these 1-valent vertices. This finishes the proof. $\square$

Remark

The function F_Γ is called the Feynman amplitude of the graph Γ. Particularly important special case is when Γ has no external vertices (in which case it is called a vacuum graph). Then F_Γ can be paired with any symmetric tensor S giving rise to a number. Namely, the m-valent part of S is associated with m-valent vertices of Γ. We then take the tensor product of these tensors over all vertices of Γ and contract along the edges using the inverse of the given scalar product. These inverses are called propagators in physics. The resulting number is the Feynman amplitude of Γ. Note that the Feynman amplitude is defined unambiguously because S is a symmetric tensor and because the bilinear form B is symmetric. Therefore it does not matter in which order we multiply and contract our tensors. If S and B were arbitrary, we would have to specify the ordering on the half-edges and edges of Γ in order for the Feynman amplitude to be well-defined. If S is cyclically symmetric, then it could be paired with so-called ribbon graphs. Further generalization is possible when V is a super-vector (or $\mathbb{Z}/2$)-graded vector space. Then the integral would also have to be understood in the super-sense. For the relevant discussion see [3].

Here's another version of Feynman's theorem; it is in this form that it is used most frequently by physicists. Below $b(\Gamma)$ stands for the number of edges minus the number of vertices of a graphs Γ. Note that $b(\Gamma)$ is the first Betti number of Γ. Consider the following quantity

$$\langle f_1 \dots f_N \rangle_h := \frac{\int_{\mathbb{R}^n} f_1 \dots f_N e^{-B(x,x)/2 - \frac{1}{h} S(x)} dx}{\int_{\mathbb{R}^n} e^{-B(x,x)/2} dx}.$$

Proposition

$$\langle f_1, \dots f_N \rangle_h = \sum_{\mathbf{n}} (\prod_i) g_i^{n_i}) \sum_{\Gamma \in G(N, \mathbf{n}} \frac{h^{b(\Gamma)}}{|Aut(\Gamma)|} F_\Gamma (f_1 \dots, f_N) \qquad (0.3)$$

where $Aut(\Gamma)$ is the (finite) group of automorphisms of Γ which fix the external vertices.

The last remark we need to make is that the right hand sides of the formulas (0.1) and (0.2) make sense without the assumption that the form B is positive definite. The following argument, allows one to make sense (formally) of the left hand sides as well. We need to make sense of integrals of the form

$$\int_{\mathbb{R}^n} f_1 \dots f_N e^{-B(x,x)/2} dx \tag{0.4}$$

where B is a nondegenerate symmetric bilinear form and f_i's are linear functions on $\mathbb{R}^n$. Suppose now that B is *not* positive definite.

A linear change of variables allows one to consider only the case when the quadratic function $B(x,x)$ has the form $\sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2$. We will introduce the function $g(t)$ as

$$g(t) := \int_{\mathbb{R}^n} f_1 \dots f_N e^{-\frac{1}{2}\left[\sum_{i=1}^{k}(x_i)^2 + \sum_{i=k+1}^{n}(tx_i)^2\right]} dx.$$

Then $g(t)$ is well-defined for nonzero real t and we can analytically continue g for arbitrary nonzero $t \in \mathbb{C}$; thus our integral (0.4) equals $g(i)$. For example, $\int_{-\infty}^{\infty} e^{\frac{1}{2}x^2} dx$ will be equal to $-i\sqrt{2\pi}$.

Remark

The formal manipulation described above is known in physics as the Wick rotation. It is easy to check that $\int_{\mathbb{R}^n} f_1 \dots f_N e^{-\frac{1}{2}B(x,x)} dx$ *as defined above obeys the standard rules of integration (i.e. the change of variables formula and integration by parts still hold) although the integral itself exists only formally. Theorem 5 will continue to hold.*

The geometric meaning of the Wick rotation is as follows. Consider $f_1, \ldots, f_N, S$ and B as functions defined on $\mathbb{C}^n$. Choose a real slice in $\mathbb{C}^n$, i.e. a real subspace V such that $V \otimes_{\mathbb{R}} V \cong \mathbb{C}$ and such that the function B is a sum of squares on V. Then perform integration over V. The resulting power series does not depend on the choice of a real slice (by the uniqueness of the analytic continuation. Note that we the integral in both the numerator and denominator of $\langle f_1 \ldots f_N \rangle$ could be complex but the ratio is always real. This is a small example of a situation often encountered in physics – the intermediate calculations might make only formal sense (like $\infty - \infty = 0$) but the end result is correct nonetheless.

Quantum field theory is a vast and complicated subject whose prerequisites are classical field theory, including special relativity, and quantum mechanics. Our modest goal in this lecture is to describe the formal logical structure of quantum mechanics and quantum field theory from the point of view of Feynman integrals.

Classical Mechanics.

One starts with classical mechanics which studies the motion of a particle (or a system of particles) subject to some force field. The position of our system corresponds to a point in a certain manifold – the *configuration space*. For example, the configuration space of system of n free point particles is $\mathbb{R}^{3n}$.

Classical Mechanics.

One starts with classical mechanics which studies the motion of a particle (or a system of particles) subject to some force field. The position of our system corresponds to a point in a certain manifold – the *configuration space*. For example, the configuration space of system of n free point particles is $\mathbb{R}^{3n}$. We are interested in the evolution of the state in the configuration space. Let $x(t)$ be the corresponding trajectory with $x(0)$ and $x(1)$ be its initial and final points. One introduces an *action functional* $S[x] = \int_0^1 L(x, \dot{x})dt$; here L is a function depending on x and $\dot{x}$; it is called the *Lagrangian* of the system. Sometimes L is also allowed to depend on the higher derivatives of x. The Lagrangian determines the dynamics of the system. For example, for a particle of mass m in a potential field with potential energy U the Lagrangian has the form $L = \frac{m\dot{x}^2}{2} - U(x)$.

The (classical) trajectory of the system is the minimum of the functional S, it is the so-called *least action principle*. This trajectory is, therefore, obtained from the equation $\delta S = 0$ which leads to the well-known Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}L - \frac{\partial L}{\partial x} = 0.$$

For example $L = \frac{m\dot{x}^2}{2} - U(x)$ leads to the Newton law $m\ddot{x} = -U'(x)$.

Classical field theory.

The situation here is similar. The 'position' of a field is a point in a certain infinite-dimensional space, typically a space of functions or sections of a vector bundle. The evolution of our system is a trajectory in this infinite-dimensional configuration space. These trajectories are usually functions of several space variables and one time variable. One introduces a Lagrangian on this space of fields; as above, it should be local, i.e. it should depend on the fields and their partial derivatives. The resulting Euler-Lagrange equations will then describe the dynamics of these classical fields.

One of the simplest examples is given by the relativistic free scalar field the Lagrangian of which has the form

$$L = 1/2(\partial_t\phi\partial_t\phi - \partial_{x_1}\phi\partial_{x_1}\phi - \partial_{x_2}\phi\partial_{x_2}\phi - \partial_{x_3}\phi\partial_{x_3}\phi - m^2\phi^2).$$

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Using standard formalism of summation over repeated indices it could be rewritten as $L = 1/2(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$. The corresponding equation of motion is the so-called Klein-Gordon equation:

$$\Box\phi + m^2\phi = 0,$$

where $\Box\phi := \frac{\partial^2}{\partial t^2}\phi - \frac{\partial^2}{\partial x_1^2}\phi + \frac{\partial^2}{\partial x_2^2}\phi + \frac{\partial^2}{\partial x_3^2}\phi$.

Quantum theory.

In quantum mechanics the *space of states* of a system is a Hilbert space V, usually taken to be the space of L_2-functions on the configuration space. An *observable* is then a self-adjoint operator on V.
The dynamics of a quantum system is described by a self-adjoint operator H called a *Hamiltonian* which may or may not depend on the time t. The equation of motion is the *Schrödinger equation*

$$\frac{d}{dt}\psi(t) = -\frac{i}{h} H\psi(t).$$

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The operator $e^{-\frac{i}{h}H}$ is called the *evolution operator*; is is unitary since H is self-adjoint.

Quantization

The problem of *quantization* is: given a classical system (described by a certain Lagrangian on a configuration space) construct the corresponding quantum system, i.e. specify a self-adjoint operator H (or the corresponding unitary operator U) on the space of L_2-functions on the configuration space.

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The following result gives a 'solution' to the quantization problem via *Feynman path integral*.

Theorem

The integral kernel of the operator U is given by the following formula

$$U(x_1, x_2) = \int e^{\frac{i}{\hbar} S} Dx(t), \tag{0.1}$$

where S is the classical action functional

One has to make a few comments concerning the above statement. First of all, its claim is that the operator U acts on a function f as $U(f) = \int U(x, x_1) f(x_1) dx_1$. Furthermore, the integral (0.1) is taken over the space of all paths starting at x_1 and ending at x_2.

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Next, we don't really expect any actual 'proof' of the above theorem. It could be justified by showing that the result agrees with the one obtained from the so-called 'canonical' quantization which we do not discuss. Really, the above result is best regarded as an axiom.

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The foregoing was the Feynman integral approach to quantum mechanics; it is widely regarded as satisfactory.
Unfortunately, quantum mechanics does not incorporate in a consistent way relativity theory. The point is that if we consider a system of *interacting* particles we have to allow the possibility of creation and annihilation of new particles.
Thus, even if we initially start with a single particle its quantum ‘trajectory’ is in fact not a curve: it could split into several curves, some of which could later merge etc. In other words, we have a *graph*, not a curve. In order to be consistent with the Feynman integral formulation of quantum mechanics we would have to integrate not over paths but over graphs. Should we then consider all possible (infinitely many) configurations of graphs? The introduction of a measure on this space also poses very serious problems.

This leads to another point of view – so-called *secondary quantization* according to which a particle should be considered to generate a field theory in it own right. This way our 'configuration space' is infinite-dimensional from the start.

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Further, the corresponding Lagrangian will have a quadratic part (corresponding to non-interacting particles) and the higher-degree part (which corresponds to interactions). The corresponding classical field could then be quantized according to the Feynman recipe. It has to be mentioned that this interpretation of a quantum particle as a classical field goes by the name 'wave-particle duality'.

Perturbation expansion of Feynman integrals.

Let us consider an interacting massive scalar field theory determined by the action functional

$$S[\phi] = \int_{\mathbb{R}^{n+1}} \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2\phi^2 - U(\phi))d^{n+1}x.$$

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Here ϕ is a function defined on $\mathbb{R}^{n+1}$ and $U(\phi)$ is a polynomial function: $U(\phi) = U_3\phi^3 + U_4\phi^4 + \ldots + U_k\phi^k$. The term $U(\phi)$ is called the interaction term.

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$$S_0[\phi] = \int_{\mathbb{R}^{n+1}} \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2\phi^2)d^{n+1}x.$$

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We need to compute integrals of the form

$$\langle f_1 \ldots f_l \rangle := \frac{\int f_1 \ldots f_l e^{\frac{i}{\hbar} S} D\phi}{\int e^{\frac{i}{\hbar} S_0} D\phi},$$

the so-called l-point function of our theory.

To be able to apply the formalism of Lecture 1 we perform the Wick rotation $x_0 = it$ to convert the Feynman integral to the Euclidean form

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Here

$$S_{0Eu}[\phi] = \int_{\mathbb{R}^{n+1}} \frac{1}{2}((\partial_t \phi)^2 + \dots + (\partial_{x_n} \phi)^2 + m^2 \phi^2) d^{n+1}x;$$

$$S_{Eu}[\phi] = \int_{\mathbb{R}^{n+1}} \frac{1}{2}((\partial_t \phi)^2 + \dots + (\partial_{x_n} \phi)^2 - m^2 \phi^2 - U(\phi)) d^{n+1}x.$$

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and so $S_{0Eu}[\phi] = \langle A\phi, \phi \rangle$ where the operator A is the linear operator

$$A\phi = (\Delta\phi - m^2)/2$$

and

$$\langle \phi, \psi \rangle = \int_{\mathbb{R}^{n+1}} \phi\psi d^{n+1}x.$$

To find the inverse to the bilinear form $-(A\phi, \phi)$ we need to invert the operator $-A$. Its inverse will be the integral operator with kernel $G(x-y)$, the fundamental solution of $-A$, i.e. the solution of the differential equation

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It is easy to check that for $n = 0$ (quantum mechanics) $G = \frac{2e^{m|x|}}{m}$. Note that for $n > 0$ (QFT case) the Green function G has a singularity at 0; logarithmic for $n = 1$ and polynomial for $n > 1$. It is because of these singularities that the quantum field theory is vastly more complicated than quantum mechanics.

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- Let $G_\Gamma(f_1, \dots\ f_l, x_1, \dots)$ be the product of all these functions.
- Finally set

$$F_\Gamma(f_1, \dots, f_l) = \prod_j U_{v(j)} \int G_\Gamma(f_1, \dots, f_l, x_1, \dots) dx_1 dx_2 \dots,$$

where $v(j)$ is the valency of the jth vertex.

The correlation function $\langle f_1 \dots f_l \rangle$ if the equal to

$$\sum_{\Gamma} \frac{h^{b(\Gamma)}}{|Aut(\Gamma)} F_{\Gamma}(f_1, \dots, f_l).$$

Here the summation is extended over all graphs Γ whose internal vertices have valencies ≥ 3 and $b(\Gamma)$ is the number of edges minus the number of vertices of Γ.

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Remark

Note that for a QFT case ($n > 0$) the Green functions have singularities at 0 which makes the amplitude of graphs with loops formally undefined. In good cases the amplitudes still make sense as distributions. In worse (typical) cases one has to deal with divergences which gives rise to *renormalization*. This difficulty also presents itself when one attempts to canonically quantize a classical field theory, i.e. introduce a suitable Hilbert space, Hamiltonian operators etc.

What is the significance of QFT for pure mathematics? The general philosophy is that given a mathematical object (say, a smooth or holomorphic manifold) one associates a classical field theory with it and then quantizes this field theory. The obtained expectation values of natural observables of the theory (i.e. functions on the space of fields or corresponding quantum operators) will then be invariants of the original geometric object.
We now list a few examples of QFT particularly important in mathematics.

Chern-Simons theory

Chern-Simons theory on a 3-dimensional manifold M is one of the simplest field theories to formulate. Our space of fields will be connections in an U_n-bundle over M and the action functional will have the form:

$$S_{cs}(A) = \int_M Tr(AdA + 2/3A^3).$$

Yang-Mills theory

Of great interest is also Yang-Mills theory. Let M be a smooth manifold with a Riemannian metric g. Since g determines an identification between the tangent and the cotangent bundles of M it gives a way to identify covariant tensors with contravariant ones. In particular, we can pair any two-forms ω, ψ: in coordinates we have $\omega = \omega_{ij} dx^i dx^j ; \psi = \psi_{ij} dx^i dx^j$ and $g = g_{ij} dx^i dx^j$, then

$$\langle \omega, \psi \rangle = \omega_{ij} g^{ik} g^{jl} \omega_{kl}.$$

Furthermore, using the volume form dV (which is determined by g) we can define a global scalar product:

$$(\omega, \psi) = \int_M \langle \omega, \psi \rangle dV.$$

Let our space of fields be again the space of connections on a U_n-bundle over M, for a connection A denote by F_A its curvature form, it is indeed a two-form on A with values in U_n. The Yang-Mills functional has the form:

$$S_{YM}[A] = (F_A, F_A).$$

This functional for $n = 1$ and M being the four-dimensional spacetime describes electrodynamics (in vacuum). The equations of motion are the classical Maxwell equations. The quantum theory is known as *quantum electrodynamics*; it was shown to agree with experiments with extreme precision despite relying on such (mathematically suspicious) manipulations as renormalization.
The corresponding field theory for general 4-manifolds is related to to Donaldson-Witten invariants of 4-manifolds.

Supersymmetric Σ-models

A particularly interesting example, related to many topics of current interest is the so-called $N = 2$ supersymmetric Σ-model where the space of fields is (very roughly) the space of maps from a Riemann surface to a fixed spacetime manifold. One then *twists* this theory to obtain another one, whose correlators do not depend on the metric in the target manifold (in fact there are two such twists: A and B). This is the theory of topological strings. We cannot even briefly touch this subject but mention that this theory is a source of *mirror symmetry* and is, to a great extent, the motivation for much of the developments covered in our course. A comprehensive introduction into these ideas is the book [3].

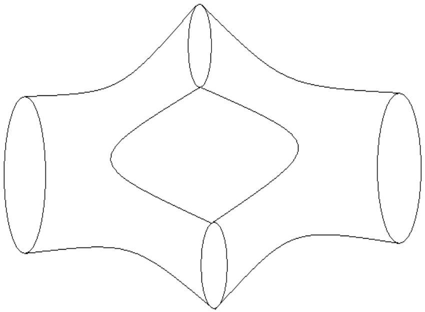
Overview

This lecture is devoted to reviewing string theory as an instance of QFT. A comprehensive modern course in string theory is [2] while [4] is a very readable introduction not assuming almost any background. A mathematical introduction to string theory is given in [1].

The formal structure of string theory resembles that of quantum mechanics except that particles are regarded as linear, rather than point objects. Most of string theory is done in *first* quantization which might account for some of its deficiencies. We will not even attempt to describe the approaches to second quantization of strings or *string field theory* (cf. [3]) although that was one of the first examples of operadic structures appearing in physics.

In string theoretic approach a particle is described by a circle (closed string) or an interval (open string). A string propagates through spacetime sweeping a two-dimensional surface – its *worldsheet*. Various elementary particles observed in nature (photon, electron etc.) correspond to various excited states of a string which are analogous to quantum mechanical states of a harmonic oscillator. Among these states one finds the *graviton* state, which corresponds to the particle generating the gravitational field. In this sense string theory predicts gravity which is considered by some as evidence for its validity. This picture is in marked contrast with the QFT picture where elementary particles corresponds to irreducible representations of the Lorentz group.

What makes string theory an attractive alternative to QFT is that it is free from the so-called *ultraviolet divergences*, which are due to the singularities of the Green functions. The related circumstance is that the interaction of strings is described by the topology of the worldsheet. One string could split into two which could then merge again etc.



The analogue of the path integral for strings will be the integral over all two-dimensional surfaces. What makes this much more feasible than the analogous problem for interacting point-particles is that there are much fewer topological configurations of two-dimensional surfaces than those of graphs; recall that any surface is classified topologically by its *genus*. This, in conjunction with a vast group of symmetries of string theory, makes most of the divergences disappear.

In this lecture we consider very briefly the theory of quantum bosonic string and how one come naturally to the critical dimension 26. One should note that most physicists consider the bosonic string to be unrealistic since it does not contain *fermionic states*, i.e. those states which are antisymmetric with respect to the permutation of two identical particles. Since there are known elementary particles that are fermions, the bosonic string cannot accurately describe nature. Another objection against the bosonic string is the presence of *tachyon states* corresponding to particles traveling faster than light. These difficulties are overcome in *superstring* theory which is considerably more involved than the bosonic string theory, yet shares many essential features with the latter.

Classical theory

We start with a description of the Dirichlet action functional of which the string action will be a special case. Let Σ and M be manifolds of dimensions d and n and with non-degenerate metrics h and g respectively. Our space of *fields* will be the space of smooth maps $\phi : \Sigma \to M$. Consider the following action functional:

$$S[\phi, g] = \int_\Sigma |d\phi|^2 \sqrt{|\det h|} d^d x.$$

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$$S[\phi, g] = \int_\Sigma |d\phi|^2 \sqrt{|\det h|} d^d x.$$

Let us explain this notation. Let V and W be two vector spaces with nondegenerate bilinear forms h and g. Then the space of all linear maps $V \to W$ also has a bilinear form: for two linear maps f and ϕ we define $\langle f, \phi \rangle$ as

$$\langle f, \phi \rangle = Tr(f^* \circ \phi),$$

where $f^* : W \to V$ is the adjoint map determined by the forms g and h.

In coordinates: let $(a^i_{i'})$ and $(b^j_{j'})$ be the matrices of f and ϕ respectively. We would like to multiply these matrices and take a trace. This is, of course, impossible, but we can use the bilinear forms h and g to lower and raise indices appropriately so that the multiplication becomes possible. We have:

$$\langle f, \phi \rangle = h^{j'i'} a^i_{i'} b^j_{j'} g_{ij}.$$

It follows that we can define the norm of a map f as $|f| := \sqrt{\langle f, f \rangle}$.

Coming back to our manifold situation we see that for $\phi : \Sigma \to M$ the expression $|d\phi|^2$ is a well-defined function on Σ which we can integrate against the canonical measure on Σ determined (up to a sign) by the metric h; specifically it is given by the formula $\sqrt{|\det h|}d^n x$. In coordinates we have:

$$S_\Sigma(\phi) = \int_\Sigma \sqrt{|\det h|} h^{\alpha\beta} \partial_\alpha \phi^\mu \partial_\beta \phi^\nu g_{\mu\nu} dx^d. \tag{0.1}$$

Note the following symmetries of the Dirichlet action:

- Diffeomorphisms in the source space:

$$\phi'^{\mu}(x') = \phi^{\mu}(x);$$

$$\frac{\partial x'^{c}}{\partial x'^{a}} \frac{\partial x'^{d}}{\partial x'^{b}} h_{cd}(x') = h_{ab}(x).$$

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- Invariance with respect to diffeomorphisms in the target space preserving the form g:

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- Two-dimensional Weyl invariance. If $d = 2$ then S does not change under a conformal scaling of the metric h:

$$\phi'(x) = \phi(x);$$

$$h'_{ab}(x) = e^{c(x)}h_{ab}(x)$$

  for an arbitrary function $c(x)$.

We now set $d = 2$ and assume that h is a Lorentzian metric in which case $|\det h| = -\det h$. In that case the action (0.1) is called the Polyakov action for the bosonic string. Its diffeomorphism invariance implies that the internal motion and geometry of the string has *no physical meaning*. The invariance with respect to the symmetries in the target spaces translates into the usual Poincaré invariance whereas the Weyl invariance crucially leads one to considering *moduli spaces of Riemann surfaces*.

Let us now vary the functional S_Σ with respect to h. First of all, note observe the following formula for the derivative of a determinant:

$$(\det A)' = \det A \operatorname{Tr}(A' A^{-1}),$$

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We have:

$$\begin{aligned}
\delta_h S &= \int_\Sigma [\frac{1}{2}(-\det h)^{-1/2}(\det h) \operatorname{Tr}(\delta h h^{-1}) \operatorname{Tr}(h^{-1}\gamma) - \sqrt{-\det h} \operatorname{Tr}(h^{-1}\delta h h^{-1}\gamma)] dx \\
&= \int_\Sigma \sqrt{-\det h}[\frac{1}{2} Tr(\delta h h^{-1}) \operatorname{Tr}(h^{-1}\gamma) - \operatorname{Tr}(h^{-1}\delta h h^{-1}\gamma)] dx \\
&= \int_\Sigma \sqrt{-\det h} \operatorname{Tr} \delta(h^{-1}) \left(\gamma - \frac{1}{2} h \operatorname{Tr}(h^{-1}\gamma)\right)
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Here we used the identity $\delta(h^{-1}) = -h^{-1}\delta h h^{-1}$.

It follows that for a critical metric (the solution of the equation $\delta_h S = 0$) we have

$$\gamma - \frac{1}{2} h \operatorname{Tr}(h^{-1}\gamma) = 0.$$

In other words, the critical metric h is proportional to the induced metric γ. This implies

$$\operatorname{Tr}(h^{-1}\gamma) = 2 \frac{\sqrt{-\det \gamma}}{\sqrt{-\det h}}.$$

Plugging this into the Polyakov action we get

$$S = 2 \int_\Sigma \sqrt{-\gamma} d^d x.$$

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The latter action (up to a factor) is called the *Nambu-Goto action*. It is, therefore equivalent (classically) to the Polyakov action. Its geometric meaning is simply twice the area of the worldsheet.

Let us now consider the case of a *free* string, i.e. when the surface Σ is either a cylinder $S^1 \times \mathbb{R}$ (closed string) or $I \times \mathbb{R}$ (open string). One also imposes suitable boundary conditions in the case of open or closed strings. We will not discuss these boundary conditions.

We will fix the standard flat metric $dt^2 - dx^2$ on $\mathbb{R} \times S^1$ and the standard Lorentzian metric $g = diag(-1, 1)$. We take M to be the flat Minkowski space. In that case the string Lagrangian will have the form

$$L = \partial_t \phi^\mu \partial_t \phi^\mu - \partial_x \phi^\mu \partial_x \phi^\mu.$$

The corresponding Euler-Lagrange equations are:

$$(\partial_t^2 - \partial_x^2)\phi^\mu = 0$$

In other words, each field ϕ^μ satisfies the Klein-Gordon massless equation. It is not hard to solve these equations; let us introduce the *light-cone coordinates*:

$$\sigma^+ = t + x, \sigma^- = t - x.$$

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These equations can be rewritten as

$$\partial_+\partial_-\phi^\mu = 0$$

whose general solutions are

$$\phi^\mu = \phi_L^\mu(\sigma^+) + \phi_R^\mu(\sigma^-).$$

Quantization.

We now discuss quantization of interacting strings and see how moduli spaces of Riemann surfaces arise in this context. Our treatment of this important subject will be very sketchy since the complete treatment relies on many deep facts of both physics and algebraic geometry which we cannot discuss here.

Quantization.

We now discuss quantization of interacting strings and see how moduli spaces of Riemann surfaces arise in this context. Our treatment of this important subject will be very sketchy since the complete treatment relies on many deep facts of both physics and algebraic geometry which we cannot discuss here. First of all, any complex structure on a 2-dimensional surface gives rise to a Riemannian metric; indeed one takes locally the metric $dzd\bar{z}$ and then glues those with a partition of unity. Conversely, any Riemannian metric gives rise to a complex structure. It is clear that if two Riemannian metrics are conformally equivalent (i.e. differ at each point by a factor) then the corresponding complex structures are also equivalent; indeed, an easy calculation shows that the metrics $dzd\bar{z}$ and $dz'd\bar{z}'$ are conformally equivalent if and only if the coordinate change $z' = z'(z)$ is holomorphic.

It follows that the moduli space of complex structures on a 2-dimensional surface Σ is isomorphic to the space of all conformal classes of metrics on S modulo the group *Diff* of all diffeomorphisms of S.

It follows that the moduli space of complex structures on a 2-dimensional surface Σ is isomorphic to the space of all conformal classes of metrics on S modulo the group *Diff* of all diffeomorphisms of S. The Polyakov path integral has the form:

$$\int e^{-S[\phi,g]} D\phi Dg.$$

Here the integral is taken over all Riemannian metrics g on a surface Σ and over all fields ϕ. We have taken the Euclidean version of the Feynman integral, the Minkowskian version could be reduced to it via the Wick rotation.
This integral is not quite right since it contains an enormous overcounting since the configurations ϕ, g and ϕ', g' which are Diff$\times$Weyl equivalent represent the same physical configuration. Thus, we need to take precisely one element from each equivalence class; this is called *gauge-fixing*. The conclusion is that we are left with integrating over the moduli space of Riemann surfaces.

One final remark: this is still not quite right: we should have checked that the Feynman ‘measure’ on the space of metrics, not just the action functional is invariant with respect to Diff$\times$Weyl. If this is not so it indicates at the presence of a ‘quantum anomaly’. It turns out that the anomaly is indeed present unless the dimension of the target space is 26.

Let us recall the description of a quantum particle. Let **H** be its space of states and H the Hamiltonian operator. Then the propagation of the state is given by the evolution operator $e^{-\frac{i}{\hbar}H}$. This is the $0+1$-dimensional field theory. Here 0 refers to the dimension of the particle itself and 1 is the additional time dimension.

To make this theory *topological* we require that the evolution operator depend only on the topology of the interval. That is the same as asking that it do not depend on t. This, in turn is equivalent to H being identically zero. Thus, topological quantum mechanics is completely trivial and uninteresting.

We now move to the dimension $1 + 1$; a suitable generalization also exists in higher dimensions. The situation is now much more interesting because there are many topologically inequivalent two-manifolds. Below when we say '2-dimensional surface' we shall always mean 'an oriented 2-dimensional surface', all homeomorphisms will be tacitly assumed to preserve the orientation. Let us consider the following category $\mathscr{C}$.

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- The objects of $\mathscr{C}$ are finite sets, thought of as disjoint unions of circles S^1.

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- The composition $f \circ g$ of two morphisms is defined by glueing the outgoing boundary components of g to the incoming components of f.

Note that the unit morphism corresponds to the cylinder connecting two circles. The category $\mathscr{C}$ has a monoidal structure, cf. *MacLane*: the product of two objects or morphisms is their disjoint union. Without going into a protracted discussion of monoidal categories we simply say that a monoidal structure is the abstraction of the notion of a cartesian product on sets or of a tensor product of vector spaces in that it is a bilinear operation together with a suitable coherent associativity isomorphism. Our monoidal structure is also *symmetric*, or commutative.

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Definition

A (closed) topological field theory (closed TFT for short) is a monoidal functor F from $\mathscr{C}$ to the category of vector spaces over $\mathbb{C}$.

Note that the requirement that F be monoidal simply means that F takes disjoint unions of circles and the corresponding cobordisms into tensor products of vector spaces and their linear maps. It turns out that a closed TFT is equivalent to to a purely algebraic collection of data called a commutative *Frobenius algebra*.

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Definition

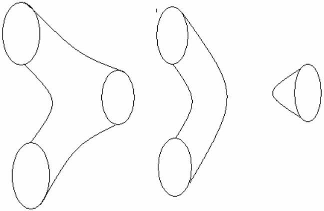
A Frobenius algebra is a (unital) associative algebra A possessing a non-degenerate symmetric scalar product $\langle , \rangle$ which is invariant in the sense that

$$\langle ab, c\rangle = \langle a, bc\rangle$$

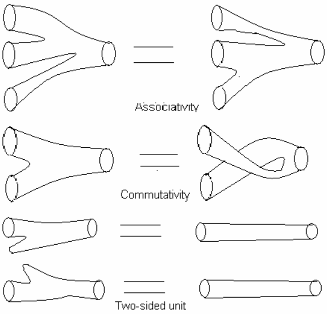
for any $a, b, c \in A$.

A unital Frobenius algebra has a trace $\mathrm{Tr}(a) := \langle a, 1\rangle$; it is clear that there is a 1-1 correspondence between invariant scalar products and traces. An example of a Frobenius algebra is given by a group algebra of a finite group, it will be commutative if the group is commutative. The corresponding trace is the usual augmentation in the group algebra.

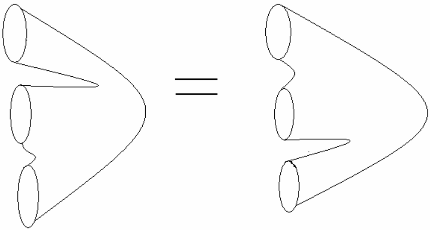
Suppose that one has a closed TFT F. We construct a Frobenius algebra A according to the following recipe. The underlying space of A will be $F(S^1)$. The multiplication map $A \otimes A \to A$, the unit map $\mathbb{C} \to A$ and the scalar product $A \otimes A \to \mathbb{C}$ are obtained by applying F to the following cobordisms where the incoming boundaries are positioned on the left and the outgoing ones on the right:



The following pictures show that the constructed multiplication is associative, commutative and has a two-sided unit; the first of these pictures is known as a pair of pants



The following picture proves the invariance condition:



Theorem

The above construction gives a $1-1$ *correspondence between isomorphism classes of commutative Frobenius algebras and isomorphism classes of closed TFT's.*

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Proof.

To construct a closed TFT out of a Frobenius algebra note that any 2-dimensional surface with boundary could be sewn from pairs of pants. To finish the proof one needs to show that the resulting functor *does not depend* on the choice of the pants decomposition of a surface. Informally speaking, that means that there are no further relations in a closed TFT besides associativity, commutativity and the invariance condition. This is done in [3] using Morse theory; another proof (of a differently phrased but equivalent result) is given in [2]. □

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Consider the category $O\mathscr{C}$ whose objects are unions of intervals I and whose morphisms are 2-dimensional surfaces with boundary components. We require that a set of intervals – open boundaries – are embedded in the union of all boundaries. The complement of the open boundaries are *free* boundaries; the latter can be either circles or intervals. The open boundaries are parametrized and partitioned into *incoming* and outgoing open boundaries.

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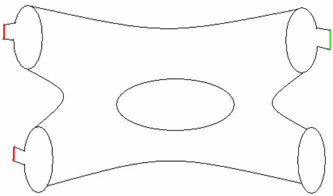
The composition is defined by glueing at the open boundary intervals. Clearly the unit morphism between two intervals is represented by a rectangle connecting them. The following picture illustrates this definition.

Here the incoming open boundaries are painted red whereas the outgoing open boundaries are green. The free boundaries are not colored.



Note that the category $O\mathscr{C}$ is monoidal with disjoint union determining the monoidal structure.

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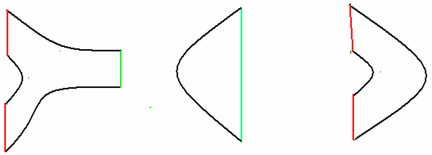
Definition

An open TFT is a monoidal functor from $O\mathscr{C}$ to the category of vector spaces.

Let us now construct a (generally noncommutative) Frobenius algebra A from an open TFT F. The underlying space of A will be the result of applying F to I. The multiplication map, the unit map and the scalar product are obtained by applying F to the following pictures.



It is straightforward to prove the associativity, the unit axiom and the invariance property. Note that that the product is not necessarily commutative. This is because there is no orientation-preserving homeomorphism of a disc which fixes one point on its boundary and switches two points.

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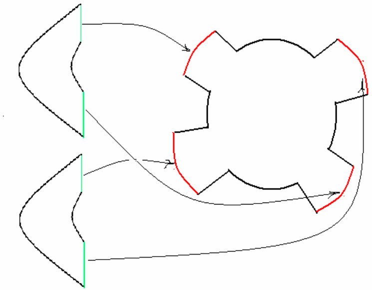
Theorem

The above construction gives a $1-1$ *correspondence between isomorphism classes of Frobenius algebras and isomorphism classes of open TFT's.*

To prove this theorem one has, first of all, decompose any 2-dimensional surface with boundary into pairs of 'flat pants', i.e. discs with three intervals embedded in the boundary circle. It is clear that we could obtain a disc with any number of free boundaries (this would correspond to taking the iterated product in the corresponding Frobenius algebra. Further, glueing those free boundary intervals (which corresponds to composing with the disc with two open boundaries) one can build any 2-surface with any number of free boundaries. For example, glueing four boundary intervals of a disc as indicated in the picture below, one obtains a torus with one free boundary:



Again, one has to prove that no other relations besides associativity and invariance are present; for this see [4] or [2].

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Note that one can also meaningfully consider an *open-closed theory* which combines both open and closed glueing. Restricting to its closed (open) sector will give a closed (open) TFT. The corresponding algebraic structure is a pair of two Frobenius algebras, a map between them and a compatibility condition known as 'Cardy condition'. These issues are treated in detail in [4].

Finally we mention another generalization of the notion of TFT; the functor on the cobordism category can take values in the category of graded, $\mathbb{Z}/2$-graded (or super-) vector spaces or the corresponding categories of complexes. The above results readily generalize; the relevant algebraic structures are (super-) graded or differential graded Frobenius algebras, commutative or not.

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An example of a graded Frobenius algebra is given by a cohomology ring of a manifold; the invariant scalar product being given by the Poincaré duality form.

Another example is given by the Dolbeault algebra of a Calabi-Yau manifold. The Dolbeault algebra of any complex manifold M has the form

$$\bigoplus_i \Omega^{0,i},$$

where $\Omega^{(0,i)}$ is the space of $(0,i)$ differential forms, i.e. forms which could be locally written as $f^{l_1 \ldots l_i} d\bar{z}_{l_1} \wedge \ldots d\bar{z}_{l_i}$ where $f^{l_1 \ldots l_i}$ is a holomorphic function. It is in fact a complex with respect to the $\bar{\partial}$-differential.

A Calabi-Yau manifold possesses a top-dimensional holomorphic form ω; wedging with this form following by integration over the fundamental cycle of M determines a trace on the Dolbeault algebra making it into a kind of Frobenius algebra (albeit infinite dimensional). The homotopy category of differential graded modules over this 'Frobenius' algebra is equivalent to the derived category of coherent analytic sheaves on M according to a recent result of J. Block [1]. This leads to an algebraic approach to the construction of Gromov-Witten invariants on Calabi-Yau manifolds.

Recall the observation that we made in Lecture 4: a topological field theory is nothing but a Frobenius algebra. This observation is extremely fruitful and we will try to generalize and build on it.

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Recall the definition of the category $\mathcal{C}$: its objects are disjoint of circles and the objects are topological cobordisms. This is a category of sets; we can turn it into a linear category by taking linear spans of the sets of morphisms. Thus, we obtain a category whose sets of morphisms are vector spaces and compositions are linear maps. We denote the category thus obtained by $l\mathcal{C}$. The passage from $\mathcal{C}$ to $l\mathcal{C}$ is quite general and is similar to the passing from a group to its group algebra.

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We then consider *linear* functors from $l\mathcal{C}$ to the category of vector spaces (or graded vector spaces or complexes of vector spaces). Here by linear functors we mean those functors that map spaces of morphisms in $l\mathcal{C}$ *linearly* into spaces of morphisms of vector spaces. It is clear that such functors are in 1-1 correspondence with all functors from $\mathcal{C}$ to vector spaces; these are thus topological field theories.

We are going to define a certain ‘derived’ version of TFT’s of various flavors. To this end consider the *topological* category $\mathcal{C}onf$ whose objects are again the disjoint unions of parametrized circles but whose morphisms are Riemann surfaces, i.e. 2-dimensional surfaces with a choice of a complex structure (equivalently, a choice of a conformal class of a Riemannian metric). Two morphisms are regarded to be equal if the corresponding Riemann surfaces are biholomorphically equivalent.

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In terms of metrics this can be phrased as follows: each conformal class of a metric contains a unique representative of constant curvature -1; two such are then considered equivalent if there exists a diffeomorphism of the surface taking one to another. [This is reminiscent of our discussion of the Diff$\times$Weyl invariance of the Polyakov action.]

As before, the category $\mathcal{C}onf$ is symmetric monoidal. We have the following result.

Proposition

The category $\pi_0\mathcal{C}onf$ whose objects are the same as those of $\mathcal{C}$ and whose morphisms are π_0 of the spaces of morphisms of $\mathcal{C}$ is equivalent to the category $\mathcal{C}$.

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It is well-known that the moduli spaces of Riemann surfaces of a fixed genus are connected. Therefore the set of connected components of these moduli spaces is labeled by the genus. Two Riemann surfaces are homeomorphic if and only if they have the same genus. $\square$

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The analogous results hold for open and open-closed analogues of the category $\mathcal{C}onf$. One needs to use the fact that moduli spaces of Riemann surfaces with boundaries having the same genus and the same number of boundary components is connected and that these two numbers form a complete topological invariant of a 2-dimensional surface.

We would like to consider the monoidal functors from $\mathcal{C}onf$ to vector spaces. This leads to the notion of *conformal field theory* (CFT). More precisely, the notion of a CFT should also include the *complex-analytic structure* on the moduli space of Riemann surfaces. A result of Huang [2] states that the notion of a CFT is more or less equivalent to the notion of a *vertex operator algebra*.

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Definition

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As before, this definition could be modified in several ways. Firstly, we can consider *open* TCFT's or, more generally, open-closed TCFT's. Secondly, we can consider the graded, $\mathbb{Z}/2$-graded vector spaces or complexes of vector spaces. The image of S^1 under TCFT is called the *state space*, if it has grading then it is usually called in physics literature *ghost number* and the differential on it (if present) is called the *BRST operator*.

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One could consider other versions of field theories related to the category $\mathcal{C}onf$ and its open or open-closed analogues. Namely, instead of considering a dg category one could take its *homology* which will result in a graded category and consider monoidal functors from t into vector spaces. It is natural to call such functors *cohomological field theories* but this term has already been reserved for a slightly different notion.

Note that the moduli spaces we are considering are *non-compact*. Indeed, imagine the holomorphic sphere $\mathbb{C}P^1$ embedded into $\mathbb{C}P^2$ as the locus of the equation $xy = \epsilon z$ where x, y, z are the homogeneous coordinates in $\mathbb{C}P^2$ and $\epsilon \neq 0$. When ϵ approaches zero our sphere degenerates into a singular surface that is topologically a wedge of two spheres. There is a natural compactification of the moduli space $\mathscr{M}_g$ of smooth Riemann surfaces of genus g obtained by adding surfaces with simple double points. This compactification $\overline{\mathscr{M}}_g$ is called the *Deligne-Mumford compactification* and it is known to be a smooth orbifold, in particular, it has Poincaré duality in its rational cohomology. There is also the corresponding notion $\overline{\mathscr{M}}_{g,n}$ for Riemann surfaces with n marked points.

The spaces $\overline{\mathscr{M}}_{g,n}$ form a category $DM\mathcal{C}$. Its objects are disjoint unions of points (thought of as infinitesimal circles) and its morphisms are the Deligne-Mumford spaces of surfaces whose marked points are partitioned into two classes – incoming and outgoing. The composition is simply the glueing of surfaces at marked points. The monoidal structure is given by the disjoint unions of points and surfaces. Denote the *homology* of this category by $hoDM\mathcal{C}$.
Then the *cohomological field theory* in the terminology of Kontsevich-Manin is a monoidal functor $hoDM\mathcal{C}$ to the category of vector spaces. We will return to this notion in the later lectures, for now we'll just say that they are related to many topics of much curent interest such as *Frobenius manifolds* and *mirror symmetry*. These topics are the subject of [3].

The categories $\mathcal{C}, \mathcal{C}onf, DM\mathcal{C}$ etc. are examples of *PROP's*. A PRO is a symmetric monoidal category whose objects are identified with the set of natural numbers and the tensor product on them is given by addition.

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A PROP is a PRO together with a right action of the permutation group S_m and a left action of S_n on the set $Mor(m, n)$ compatible with the monoidal structure and the compositions of morphisms. A good discussion of PROP's is found in [1]. One of the simplest examples of PROP's is an *endomorphism PROP* for which $Mor(m, n) = \mathrm{Hom}(V^{\otimes m}, V^{\otimes n})$ where V is a vector space. Another example is the space of the PROP of graphs with a flow. Such graphs have a collection of external edges partitioned into two subsets – inputs and outputs. Furthermore each edge is oriented in such a way that each path along the edges compatible with the given orientation will end at one of the output edges. The composition is defined by grafting the input edges to the output edges.

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An *algebra* over a PROP P is a morphism of RPOP's from P into a suitable endomorphism PROP which is the same as a monoidal functor from P to vector spaces.

At this point we change the viewpoint slightly and will consider *operads* rather then PROP's. To be sure, PROPs are more general than operads and certain structures (e.g. bialgebras) cannot be described by operads. However, for the purposes of treating such objects as TCFT's operads are adequate and their advantage is that they are considerably smaller than PROP's. Informally speaking, an operad retains only part of the information encoded in a PROP: about the morphisms with only one output. Consequently, the composition is only partially defined. Here's the definition of an operad in vector spaces; a similar definition makes sense in any symmetric monoidal category.

Definition

An operad $\mathcal{O}$ is a collection of vector spaces $\mathcal{O}(n)$, $n \geq 0$ supplied with actions of permutation groups S_n and a collection of composition morphisms for $1 \leq i \leq n$:

$$\mathcal{O}(n) \otimes \mathcal{O}(n') \to \mathcal{O}(n + n' - 1)$$

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- *Equivariance: compositions are equivariant with respect to the actions of the permutation groups.*
- *Associativity: for each $1 \leq j \leq a, b, c$, $f \in \mathcal{O}(a) h \in \mathcal{O}(c)$*

$$(f \circ_j g) \circ_i h = \begin{cases} (f \circ_i h) \circ_{j+c-1} g, 1 \leq i \leq j \\ f \circ_j (g \circ_{i-j+1} h), j \leq i < b + j \\ (f \circ_{i-b+1} h) \circ g, j + b \leq i \leq a + b - 1 \end{cases} .$$

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- *Unitality. There exists $e \in \mathcal{O}(1)$ such that*

$$f \circ_i e = e \text{ and } e \circ_1 g = g.$$

It is useful to visualize this definition thinking of the operad of *trees*. A tree is an oriented graph without loops, such that each vertex has no more than one outgoing edge and at least on incoming edge. The edges abutting only one vertex are called *external*. There is a unique outgoing external edge called the *root*, the rest of the external edges are called the *leaves*.
Set $Trees(n)$ to be the set of trees with n leaves. This will be an operad of *sets*; to pass from it to an operad of vector spaces on simply takes the linear span of everything in sight.

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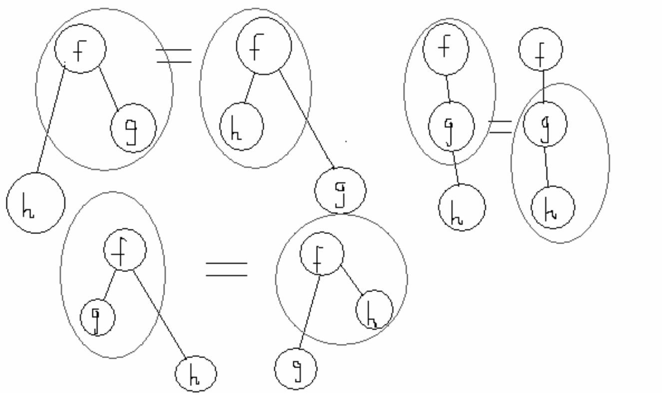

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Remark

In many ways operads behave like associative algebras. In fact, they are generalizations of associative algebras – their definition implies that the component $\mathcal{O}(1)$ of an operad is an associative algebra.
Recall that an associative algebra could be defined as a monoid in the symmetric monoidal category of vector space: this is a vector space V together with a structure map $V \otimes V \to V$ satisfying appropriate axioms. Similarly we can consider the category of Σ-modules whose objects are collections $\{O(n)\}, n \geq 0$. There is a certain non-symmetric 'tensor product' on the category of Σ-modules and operads could be defined as monoids with respect to this monoidal structure. We will not use this point of view.

Example

We have seen one example of an operad – the operad of trees. Closely related to it are various operads constructed from moduli spaces of Riemann surfaces. Take, for example, the Deligne-Mumford operad whose nth space is the $\overline{\mathbb{M}}_{0,n+1}$, the compactified moduli space of Riemann surfaces with $n+1$ marked points. One views the first n marked point as inputs and the remaining on as the output; the composition maps are simply glueing at marked points. One can also consider uncompactified versions of this operad with either closed or open glueings. These are *topological* operads; to obtain a linear operad one takes its singular or cellular complex or its homology. Another example is the *endomorphism operad* $\mathcal{E}(V)(n) := \mathrm{Hom}(V^{\otimes n}, V)$ where V is a vector space (or a graded vector space etc.

There is a functor from PROP's to operads forgetting part of the structure. Namely, if $\mathcal{P}$ is a PROP then the associated operad is $\mathcal{O}(n) := \mathcal{P}(n, 1)$. For example, we can speak about the *endomorphism operad* of a vector space $\mathcal{E}(V)(n) := \mathrm{Hom}(V^{\otimes n}, V)$. Conversely, one can write down a 'free' PROP generated by a given operad; these functors are adjoint.

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Definition

Let $\mathcal{O}$ be an operad. An algebra over $\mathcal{O}$ is a map of operads $\mathcal{O} \to \mathcal{E}(V)$ where V is a vector space (graded vector space etc.)

Note the similarity of between the notions of an algebra over an operad and over a PROP. Indeed, an algebra over a PROP, freely generated by an operad $\mathcal{O}$ is the same as an $\mathcal{O}$-algebra which follows from the adjointness of the corresponding functors.