Preliminaries: Vectors, Tensors, and Coordinate Transformations

Definition 1.1 Manifold

A manifold M is a set of points which can locally be mapped into $\mathbb{R}^N$ for some N = 0, 1, 2, …. The number N will be called the dimension of the manifold. 1.

1. The mapping must be continuous, with continuous inverse (i.e. a homeomorphism).
2. The mapping must be one-to-one.
3. If two mappings overlap, one set of coordinates must be a differentiable function of the other set.

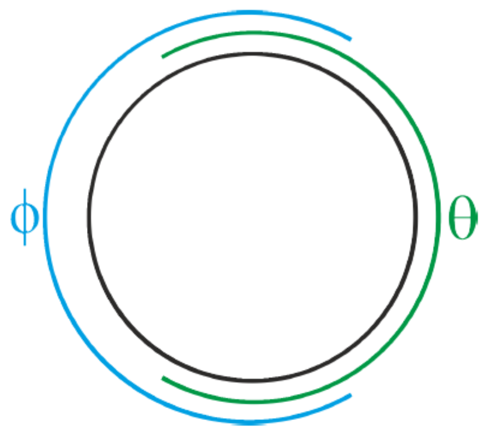
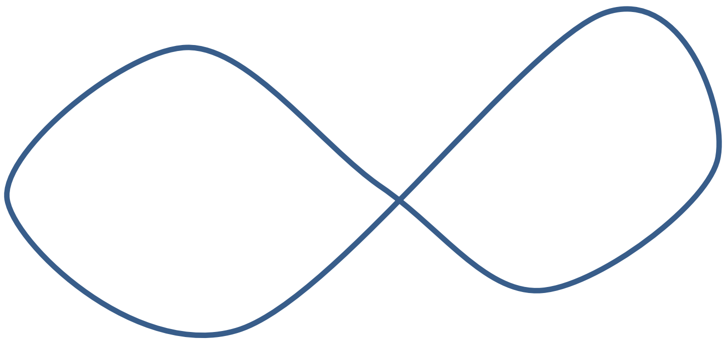


Figure 1.2: Two coordinate regions on the circle S^1. Here θ is only defined on the right two thirds of the circle, while ϕ is only defined on the left two thirds.

Not a manifold



Two-torus

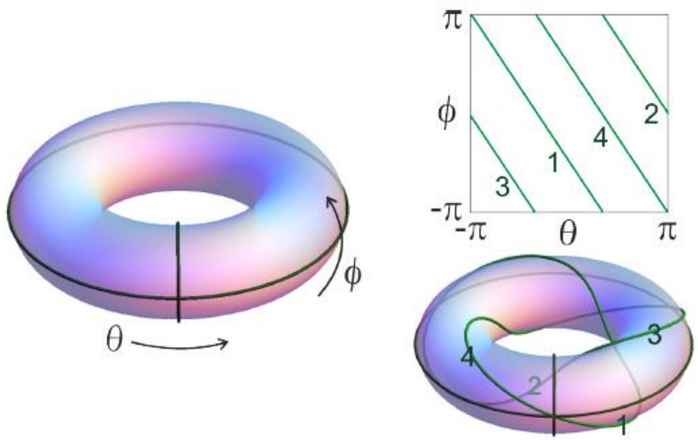
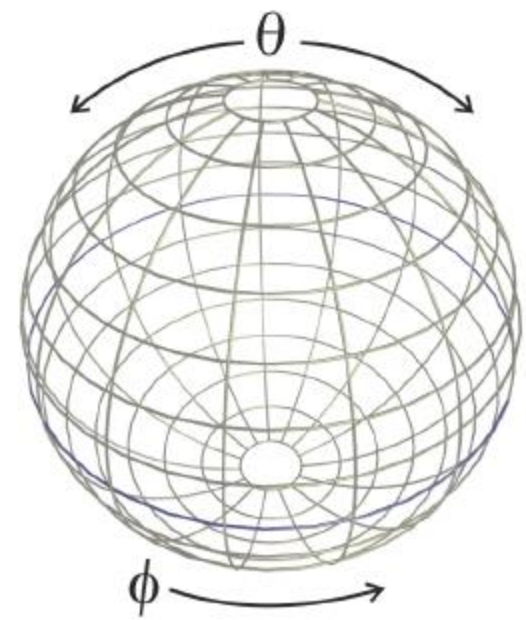


Figure 1.5: *Left Panel:* The 2-Torus, T^2 is the surface of a doughnut where θ represents the toroidal or axial angle while ϕ represents the poloidal or meridional angle. *Right Panel:* By making two perpendicular cuts one along, e.g. $\phi = \pi$ and the other along $\theta = \pi$, we may open the torus and lay it flat so that it can be seen as a 2-dimensional sheet. A trefoil knot which goes 2 times around θ and 3 times around ϕ is shown.

Sphere



Coordinate system labels

We will label coordinate systems primarily by either unprimed symbols (X) and primed symbols (X'), for example

$$\begin{array}{lcl} \textit{Rest Frame} & \mathsf{X} = & (\mathsf{X}^1, \ldots, \mathsf{X}^N) \\ \textit{Moving Frame} & \mathsf{X}' = & \left(\mathsf{X}'^1, \ldots, \mathsf{X}'^N\right) \end{array} \tag{1.24}$$

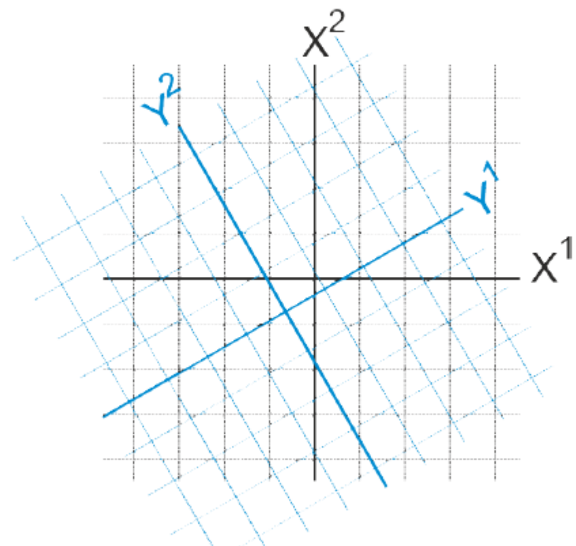
or by capital letters, for example

$$\begin{array}{lcl} \textit{Euclidean Coordinates} & \mathsf{E} = & (\mathsf{E}^1, \ldots, \mathsf{E}^N) \\ \textit{Spherical Coordinates} & \mathsf{S} = & (\mathsf{S}^1, \ldots, \mathsf{S}^N) \end{array} \tag{1.25}$$

Recall that X^1 is the first coordinate in a reference frame X, e.g. in three dimensional cartesian space $\mathsf{X}^1 = x$, $\mathsf{X}^2 = y$, and $\mathsf{X}^3 = z$.

Suppose point P has coordinates $\mathsf{X} = (\mathsf{X}^1, \mathsf{X}^2, \ldots, \mathsf{X}^N)$ in one coordinate system, and $\mathsf{Y} = (\mathsf{Y}^1, \mathsf{Y}^2, \ldots, \mathsf{Y}^N)$ in another. Then the coordinates $(\mathsf{Y}^1, \mathsf{Y}^2, \ldots, \mathsf{Y}^N)$ must be smooth (differentiable) functions of $(\mathsf{X}^1, \mathsf{X}^2, \ldots, \mathsf{X}^N)$. We can write

$$\begin{aligned} \mathsf{Y}^1 &= \mathsf{Y}^1(\mathsf{X}^1, \mathsf{X}^2, \ldots, \mathsf{X}^N) \\ \mathsf{Y}^2 &= \mathsf{Y}^2(\mathsf{X}^1, \mathsf{X}^2, \ldots, \mathsf{X}^N) \\ &\vdots \\ \mathsf{Y}^n &= \mathsf{Y}^n(\mathsf{X}^1, \mathsf{X}^2, \ldots, \mathsf{X}^N) \end{aligned}$$



1.4 Things that Live on Manifolds

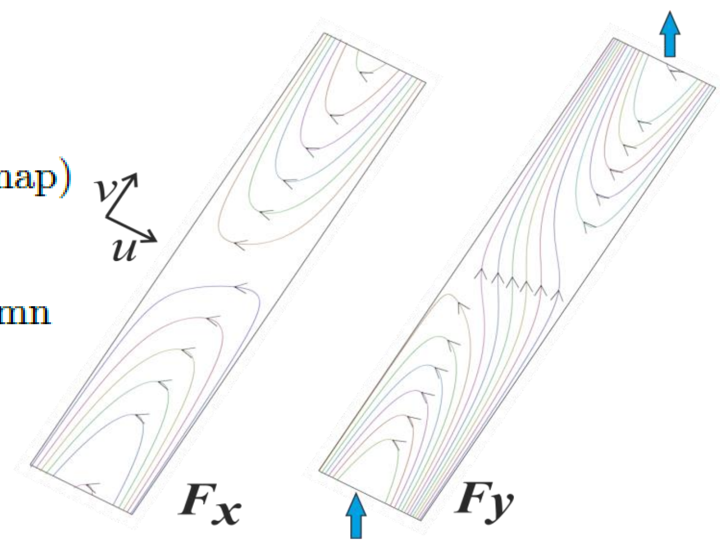
As manifolds are n-dimensional spaces, a number of objects such as fields, lines, and points may reside on or in them. In this section we describe the nature of these objects.

1.4.1 Scalar fields

Definition 1.2 *Scalar fields:* Scalar fields are functions which assign values (numbers) to points on a given manifold. More formally, a scalar field is a function f which maps a manifold M to the set of real numbers: $f : M \to \mathbb{R}$.

Examples of scalar fields:

- M = Surface of the Earth (approximately S^2)
  $f(P)$ = Temperature at point P (as in a weather map)
- M = Support column $(B^2 \times I^1)$
  $f(P)$ = Load at a point P, inside the support column



1.4.2 Curves

The scalar fields defined above map the manifold M to the Real line $\mathbb{R}$. Suppose we now do the reverse. Assign to the real number λ a point $\gamma(\lambda)$ on M. Stringing these points together provides a curve on M, as shown in figure 1.7.

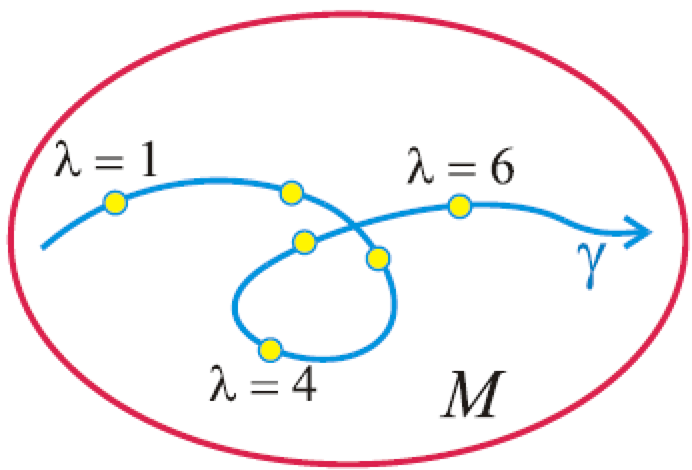


Figure 1.7: A curve $\gamma : M \to \mathbb{R}$

Vectors

Consider the curve $\gamma : \mathbb{R} \to M$ (or $\gamma : I^1 \to M$, or $S^1 \to M$). On an N-dimensional manifold, the curve provides a set of N coordinate functions of λ, i.e.

$$\gamma(\lambda) = \left(\mathsf{X}^1(\lambda), \ldots, \mathsf{X}^N(\lambda)\right) . \tag{1.29}$$

These n functions have derivatives which show how fast they change with respect to λ. Taken together, they show the direction the curve is travelling.

Definition 1.4 *Tangent Vectors* The tangent vector to the curve γ is given by

$$\overline{\mathbf{V}}(\lambda) = \begin{pmatrix} \mathrm{d}\mathsf{X}^1/\mathrm{d}\lambda \\ \vdots \\ \mathrm{d}\mathsf{X}^N/\mathrm{d}\lambda \end{pmatrix} \tag{1.30}$$

Note that the set of coordinates of a point - e.g. $(\mathsf{X}^1, \ldots, \mathsf{X}^N)$ is *not* a vector!

Gradients

Definition 1.5 *Gradients* Given a function $f : M \to \mathbb{R}$, the gradient is the set of partial derivatives

$$\nabla f = \left(\frac{\partial f}{\partial X^1}, \frac{\partial f}{\partial X^2}, \ldots, \frac{\partial f}{\partial X^N} \right). \tag{1.31}$$

Gradients and Vectors have different Coordinate transformations!

(except for Cartesian coordinates)

Notation

Components of a gradient

$$\partial_a f \equiv \frac{\partial f}{\partial \mathsf{X}^a}$$

Components of a vector

$$V_{\mathsf{L}}^a = \frac{\mathrm{d}\mathsf{L}^a}{\mathrm{d}\lambda}$$

Definition 1.6 *Einstein Summation* Given two objects, one indexed with superscripts (i.e. a vector) $A = (A^1, \ldots, A^N)$, and one with subscripts (i.e. a gradient) $B = (B_1, \ldots, B_N)$, we define

$$B_c A^c \equiv \sum_{c=1}^{N} B_c A^c \tag{1.35}$$

Cartesian Coordinates: Here we will sometimes be lazy and allow both indices to be upper or lower in Einstein summation.

1.5.2 How to transform vectors

Consider a curve $\gamma(\lambda)$ with X coordinates $(\mathsf{X}^1(\lambda), \ldots, \mathsf{X}^n(\lambda))$. The tangent vector $\overline{\mathbf{V}}_\mathsf{X}$ will have components $\begin{pmatrix} \mathrm{d}\mathsf{X}^1/\mathrm{d}\lambda \\ \vdots \\ \mathrm{d}\mathsf{X}^N/\mathrm{d}\lambda \end{pmatrix}$ In another coordinate system $\mathsf{Y} = (\mathsf{Y}^1, \ldots, \mathsf{Y}^n)$, the same tangent vector has components $\overline{\mathbf{V}}_\mathsf{Y} = \begin{pmatrix} \mathrm{d}\mathsf{Y}^1/\mathrm{d}\lambda \\ \vdots \\ \mathrm{d}\mathsf{Y}^N/\mathrm{d}\lambda \end{pmatrix}$.

Given $\overline{\mathbf{V}}_\mathsf{X}$, how do we find $\overline{\mathbf{V}}_\mathsf{Y}$? All we need is the chain rule. The first component of the tangent vector in Y is $V^1_\mathsf{Y} = \mathrm{d}\mathsf{Y}^1/\mathrm{d}\lambda$:

$$V^1_\mathsf{Y} = \frac{\mathrm{d}\mathsf{Y}^1}{\mathrm{d}\lambda} = \sum_{a=1}^{N} \frac{\partial \mathsf{Y}^1}{\partial \mathsf{X}^a} \frac{\mathrm{d}\mathsf{X}^a}{\mathrm{d}\lambda}. \tag{1.36}$$

Using the Einstein summation convention and noting that $\mathrm{d}\mathsf{X}^a/\mathrm{d}\lambda = V^a_\mathsf{X}$, we can rewrite this formula in the simpler form

$$V^1_\mathsf{Y} = \frac{\partial \mathsf{Y}^1}{\partial \mathsf{X}^a} V^a_\mathsf{X}. \tag{1.37}$$

The formula for an arbitrary component (b say) of $\overline{\mathbf{V}}$ is

$$V^b_\mathsf{Y} = \frac{\partial \mathsf{Y}^b}{\partial \mathsf{X}^a} V^a_\mathsf{X}. \tag{1.38}$$

Using primed and unprimed coordinates (instead of X and Y) the formula becomes

$$V'^b = \frac{\partial \mathsf{X}'^b}{\partial \mathsf{X}^a} V^a. \tag{1.39}$$

1.5.3 How to transform gradients

To transform gradients from one coordinate system to another, we again use the chain rule. A single coordinate $(\nabla_{\mathsf{Y}} f)_1$ of the gradient in the new coordinate system is

$$(\nabla_{\mathsf{Y}} f)_1 = \partial_{\mathsf{Y}\,1} f \frac{\partial f}{\partial \mathsf{Y}^1} = \sum_{c=1}^{N} \frac{\partial f}{\partial \mathsf{X}^c} \frac{\partial \mathsf{X}^c}{\partial \mathsf{Y}^1}. \tag{1.40}$$

Again using Einstein notation, this can be written

$$\partial_{\mathsf{Y}\,1} f = \frac{\partial \mathsf{X}^c}{\partial \mathsf{Y}^1} \partial_{\mathsf{X}\,c} f. \tag{1.41}$$

An arbitrary component (b say) of ∇f can be expressed as

$$\partial_{\mathsf{Y}\,b} f = \frac{\partial \mathsf{X}^c}{\partial \mathsf{Y}^b}\, \partial_{\mathsf{X}\,c} f \tag{1.42}$$

or using primed and unprimed coordinates (instead of X and Y):

$$\partial'_b f = \frac{\partial \mathsf{X}^c}{\partial \mathsf{X}'^b}\, \partial_c f \tag{1.43}$$

Many objects in physics transform like gradients (for example the electromagnetic vector potential). These objects will be called *forms*. Some textbooks call these objects *covariant* vectors.

1.5.4 Inverse nature of gradient and vector transformations

Note that the transformation matrices for vectors and gradient in equations (1.38) and (1.42), i.e. $\partial \mathsf{Y}^b/\partial \mathsf{X}^a$ and $\partial \mathsf{X}^c/\partial \mathsf{Y}^b$, are *inverses*. We can prove this by examining the product of the two transformation matrices. Let Δ be the product of these two matrices. Then

$$\begin{aligned} \Delta^c{}_a &= \sum_{b=1}^{N} \frac{\partial \mathsf{Y}^b}{\partial \mathsf{X}^a} \frac{\partial \mathsf{X}^c}{\partial \mathsf{Y}^b}, && (1.44) \\ &= \sum_{b=1}^{N} \frac{\partial \mathsf{X}^c}{\partial \mathsf{Y}^b} \frac{\partial \mathsf{Y}^b}{\partial \mathsf{X}^a}, && (1.45) \end{aligned}$$

which by the chain rule is just

$$\Delta^c{}_a = \frac{\partial \mathsf{X}^c}{\partial \mathsf{X}^a}. \tag{1.46}$$

Here

$$\frac{\partial \mathsf{X}^c}{\partial \mathsf{X}^a} = \begin{cases} 1 & c = a \\ 0 & c \neq a \end{cases} \tag{1.47}$$

is just the Kronecker delta, or identity matrix

$$\Delta^c{}_a = \frac{\partial \mathsf{X}^c}{\partial \mathsf{Y}^b} \frac{\partial \mathsf{Y}^b}{\partial \mathsf{X}^a} = \delta^c{}_a = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & & \end{pmatrix}. \tag{1.48}$$

Example: Polar and Cartesian Systems

Lets examine how we transform to and from polar coordinates in the plane. The two coordinate systems are defined as

$$\mathsf{C} = (\mathsf{C}^1, \mathsf{C}^2) = (x, y) \qquad \text{Cartesian,} \tag{1.57}$$

$$\mathsf{P} = (\mathsf{P}^1, \mathsf{P}^2) = (r, \phi) \qquad \text{polar,} \tag{1.58}$$

where the transformation from polar to Cartesians is defined as

$$\mathsf{C}^1 = x = r\cos\phi = \mathsf{P}^1 \cos \mathsf{P}^2, \tag{1.59}$$

$$\mathsf{C}^2 = y = r\sin\phi = \mathsf{P}^1 \sin \mathsf{P}^2. \tag{1.60}$$

Alternatively from Cartesian to polar coordinates we have

$$\mathsf{P}^1 = r = \sqrt{x^2 + y^2} = \sqrt{(\mathsf{C}^1)^2 + (\mathsf{C}^2)^2}, \tag{1.61}$$

$$\mathsf{P}^2 = \phi = \arctan\frac{y}{x} = \arctan\frac{\mathsf{C}^2}{\mathsf{C}^1}. \tag{1.62}$$

The transformation matrix (Jacobian), from polar coordinates to Cartesians is:

$$\frac{\partial \mathsf{C}^a}{\partial \mathsf{P}^b} = \begin{pmatrix} \partial \mathsf{C}^1/\partial \mathsf{P}^1 & \partial \mathsf{C}^1/\partial \mathsf{P}^2 \\ \partial \mathsf{C}^2/\partial \mathsf{P}^1 & \partial \mathsf{C}^2/\partial \mathsf{P}^2 \end{pmatrix} \tag{1.63}$$

$$= \begin{pmatrix} \partial x/\partial r & \partial x/\partial \phi \\ \partial y/\partial r & \partial y/\partial \phi \end{pmatrix} \tag{1.64}$$

$$= \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix} \tag{1.65}$$

while the Jacobian for the transformation from Cartesian to polar coordinates is

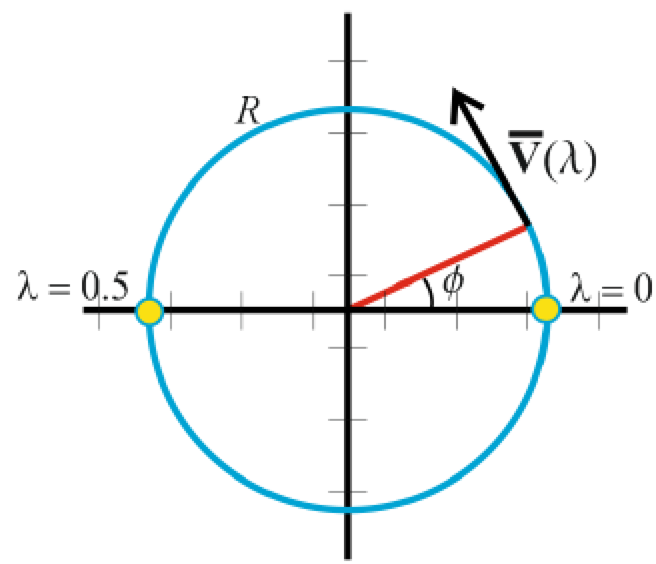
$$\frac{\partial \mathsf{P}^b}{\partial \mathsf{C}^d} = \begin{pmatrix} \partial \mathsf{P}^1/\partial \mathsf{C}^1 & \partial \mathsf{P}^1/\partial \mathsf{C}^2 \\ \partial \mathsf{P}^2/\partial \mathsf{C}^1 & \partial \mathsf{P}^2/\partial \mathsf{C}^2 \end{pmatrix} \tag{1.66}$$

$$= \begin{pmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial \phi/\partial x & \partial \phi/\partial y \end{pmatrix} \tag{1.67}$$

$$= \begin{pmatrix} x/\sqrt{x^2+y^2} & y/\sqrt{x^2+y^2} \\ -y/(x^2+y^2) & x/(x^2+y^2), \end{pmatrix}$$

or using r and ϕ

$$\frac{\partial \mathsf{P}^b}{\partial \mathsf{C}^d} = \begin{pmatrix} \cos\phi & \sin\phi \\ \frac{-\sin\phi}{r} & \frac{\cos\phi}{r} \end{pmatrix}.$$



Note that the two matrices 1.65 and 1.69 *are* inverses.

$$\begin{aligned}\gamma(\lambda) &= \left(\mathsf{C}^1, \mathsf{C}^2\right) \\ &= (x(\lambda), y(\lambda))\end{aligned}$$

while in polar coordinates

$$\begin{aligned}\gamma(\lambda) &= \left(\mathsf{P}^1, \mathsf{P}^2\right) && (1.73)\\ &= (r(\lambda), \phi(\lambda)) && (1.74)\\ &= (R, 2\pi\lambda)\,. && (1.75)\end{aligned}$$

We may now compute the tangent vector separately for each coordinate system.

In Cartesian coordinates, the tangent vector is

$$\begin{aligned}V^a_{\mathsf{C}}(\lambda) &= \frac{\mathrm{d}\mathsf{C}^a}{\mathrm{d}\lambda} && (1.76)\\ &= \begin{pmatrix}\mathrm{d}x/\mathrm{d}\lambda \\ \mathrm{d}y/\mathrm{d}\lambda\end{pmatrix} && (1.77)\\ &= \begin{pmatrix}-2\pi R\sin 2\pi\lambda \\ 2\pi R\cos 2\pi\lambda\end{pmatrix}, && (1.78)\end{aligned}$$

while in polar coordinates

$$\begin{aligned}V^a_{\mathsf{P}}(\lambda) &= \frac{\mathrm{d}\mathsf{P}^a}{\mathrm{d}\lambda} && (1.79)\\ &= \begin{pmatrix}\mathrm{d}r/\mathrm{d}\lambda \\ \mathrm{d}\phi/\mathrm{d}\lambda\end{pmatrix} && (1.80)\\ &= \begin{pmatrix}0 \\ 2\pi\end{pmatrix}. && (1.81)\end{aligned}$$

Let's check that we recover the expression for the tangent vector in Cartesian coordinates equation (1.78) by transforming the expression in polar coordinates equation (1.81) according to the transformation law, equation (1.65):

$$V^a_{\mathsf{C}}(\lambda) = \frac{\partial \mathsf{C}^a}{\partial \mathsf{P}^b} V^b_{\mathsf{P}} \qquad (\text{at } r = R, \phi = 2\pi\lambda) \tag{1.82}$$

$$= \begin{pmatrix} \cos 2\pi\lambda & -r\sin 2\pi\lambda \\ \sin 2\pi\lambda & r\cos 2\pi\lambda \end{pmatrix} \begin{pmatrix} 0 \\ 2\pi \end{pmatrix} \tag{1.83}$$

$$= \begin{pmatrix} -2\pi R\sin 2\pi\lambda \\ 2\pi R\cos 2\pi\lambda \end{pmatrix} . \tag{1.84}$$

It works!

On a manifold, we can combine a vector, $\overline{\mathbf{V}}$, and a gradient, ∇f, naturally by using Einstein summation:

$$\begin{aligned} \overline{\mathbf{V}} \cdot \nabla f &= \sum_{a=1}^{N} V^a \, \partial_a f \\ &= V^a \, \partial_a f. \end{aligned} \tag{1.85}$$

This sum gives a number called the *directional derivative.*

1.6.2 Invariance

The directional derivative is the same in all coordinate systems and is thus said to be invariant under coordinate transformations. This invariance arises from the inverse nature of the transformation of forms and vectors. Consider the transform from $\mathsf{X} \to \mathsf{X}'$:

$$V^a = \frac{\partial \mathsf{X}^a}{\partial \mathsf{X}'^b} V'^b \tag{1.86}$$

$$\partial_a f = \frac{\partial \mathsf{X}'^c}{\partial \mathsf{X}^a} \partial'_c f \tag{1.87}$$

$$\Rightarrow \overline{\mathbf{V}} \cdot \nabla f = V^a \partial_a f = \left(\frac{\partial \mathsf{X}^a}{\partial \mathsf{X}'^b} \frac{\partial \mathsf{X}'^c}{\partial \mathsf{X}^a} \right) V'^b \partial'_c f \tag{1.88}$$

$$= (\delta^c{}_b) V'^b \partial'_c f \tag{1.89}$$

$$= V'^c \partial'_c f \tag{1.90}$$

Thus

$$\overline{\mathbf{V}} \cdot \nabla f = V^a \partial_a f = V'^c \partial'_c f. \tag{1.91}$$

The directional derivative has the same form and gives the same number in any coordinate system.

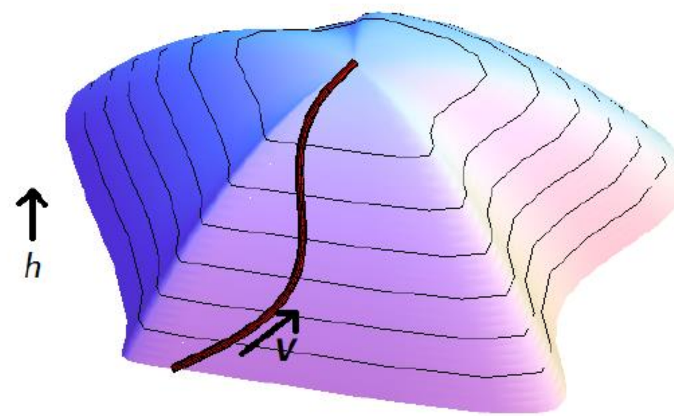
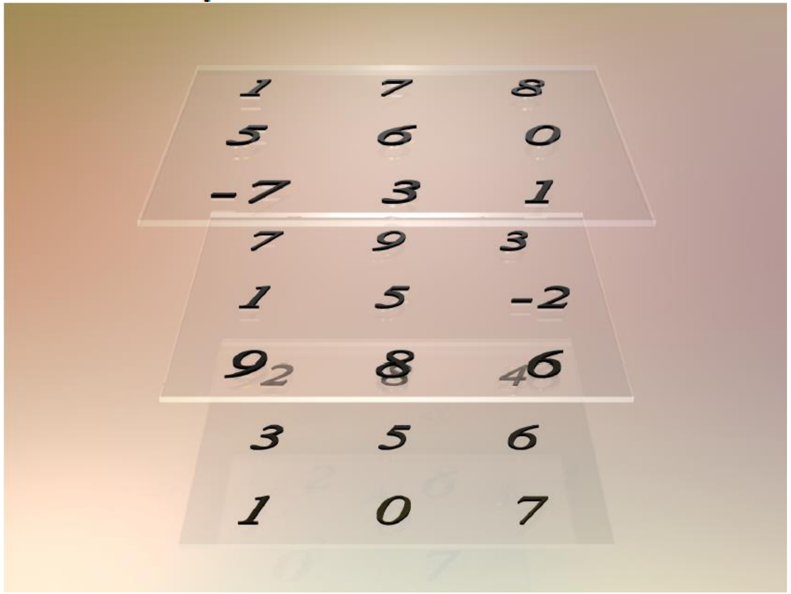


Figure 1.11: Suppose $f = h =$ height of a point on a mountain. The contours of h are shown. If V gives the speed a person climbs the mountain, then $\mathrm{d}h/\mathrm{d}t = V \cdot \nabla h$ will give the rate at which the person increases her altitude.

Tensors

Rank	*Name*	*Example*
0	Scalars	functions of position $f(X^1, X^2)$
1	Vectors and forms	gradient $\nabla f = (\partial_1 f, \partial_2 f)$
2	Second rank tensors	$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
3	Third rank tensors	(see figure 2)
$\vdots$	$\vdots$	$\vdots$



2nd order and higher rank tensors

Second rank tensors can be viewed as square matrices, albeit with special properties when transforming between coordinate systems. We will consider transformations between unprimed and primed coordinates.

1. First consider a tensor T^{ab} with two *superscripts*. This object will transform like a product of two vectors:

$$T'^{ab} = \frac{\partial \mathsf{X}'^{a}}{\partial \mathsf{X}^{c}} \frac{\partial \mathsf{X}^{b'}}{\partial \mathsf{X}^{d}} T^{cd}. \tag{2.5}$$

2. Let $\mathcal{M}$ be a *mixed* tensor with components $M^{a}{}_{b}$. A tensor with one superscript and one subscript transforms as a vector on the superscript and as a form on the subscript. Then when transforming from one coordinate system to another,

$$M'^{a}{}_{b} = \frac{\partial \mathsf{X}'^{a}}{\partial \mathsf{X}^{c}} \frac{\partial \mathsf{X}^{d}}{\partial \mathsf{X}'^{b}} M^{c}{}_{d} \tag{2.6}$$

3. Finally, consider a tensor N_{ab} with two *subscripts*. This object will transform like a product of two forms:

$$N'_{ab} = \frac{\partial \mathsf{X}^{c}}{\partial \mathsf{X}'^{a}} \frac{\partial \mathsf{X}^{d}}{\partial \mathsf{X}'^{b}} N_{cd}. \tag{2.7}$$

Definition 2.4 *Type* $\begin{pmatrix} p \\ q \end{pmatrix}$ *tensors*

A type $\begin{pmatrix} p \\ q \end{pmatrix}$ tensor has p upper indices and q lower indices. The product with q vectors and p forms (summing over all indices) returns a scalar.

For example, if $T^{ab}{}_c{}^d{}_{ef}{}^g$ is a type $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ tensor (7th rank) then

$$\mu = T^{ab}{}_c{}^d{}_{ef}{}^g A_a B_b C^c D_d E^e F^f G_g \tag{2.10}$$

is a scalar where $\underline{\mathbf{A}}, \underline{\mathbf{B}}, \underline{\mathbf{D}}, \underline{\mathbf{G}}$ are forms, and $\overline{\mathbf{C}}, \overline{\mathbf{E}}, \overline{\mathbf{F}}$ are vectors.

Tensor Arithmetic

1. **Tensor Addition:** Tensors can be simply added together.

$$R^a{}_b{}^c + S^a{}_b{}^c = T^a{}_b{}^c$$

Note: All terms have the *same* indices!

2. **Tensor Composition:** A second rank tensor can be composed by combining one form and one vector. Suppose we have a vector $\overline{\mathbf{V}}$ and form $\underline{\mathbf{W}}$, with values (in some coordinate system) $V^1 = 1, V^2 = 2, W_1 = 20, W_2 = 30$. We can then create a tensor by writing

$$M^a{}_b = V^a W_b = \begin{pmatrix} 20 & 30 \\ 40 & 60 \end{pmatrix} \tag{2.19}$$

Higher order tensors can also be composed to form tensors of even higher orders!

3. **Contraction with another Tensor:** Tensors can be summed over one or more repeated indices to give a new tensor with fewer indices. For example, with the above values for $\overline{\mathbf{V}}$ and $\underline{\mathbf{W}}$, we can create a scalar

$$\mu = V^a W_a \quad (= 80 \text{ in the above example}) \tag{2.20}$$

Two second rank tensors need to be contracted twice to produce a scalar: for example

$$\lambda = M^a{}_b M^b{}_a \tag{2.21}$$

or, combining fully contra-variant and fully covariant second rank tensors,

$$\lambda = T^{ab} N_{ab} \tag{2.22}$$

will be a scalar.

4. **Contraction within a Tensor:** Mixed Tensors can self contract across a single sub and superscript:

$$P^{ab}{}_{bc} = Q^a{}_c \tag{2.23}$$

$$M^a{}_a = 80 = \mu \quad \text{in the above example} \tag{2.24}$$

For 2nd rank mixed tensors, the self contraction $Tr(\mathcal{M}) = M^a{}_a$, is called the trace of $\mathcal{M}$.

5. **Symmetrizing and anti-Symmetrizing (for 2nd rank tensors):** Tensors can obey certain symmetries. Given a tensor with components T^{ab}, we define

$$S^{ab} = \frac{1}{2}\left(T^{ab} + T^{ba}\right) \tag{2.25}$$

$$\text{and} \quad A^{ab} = \frac{1}{2}\left(T^{ab} - T^{ba}\right) \tag{2.26}$$

These tensors then exhibit the following properties:

$$(1) \quad S^{ab} + A^{ab} = T^{ab} \tag{2.27}$$

$$(2) \quad S^{ab} = S^{ba} \qquad \text{(Symmetric)} \tag{2.28}$$

$$(3) \quad A^{ab} = -A^{ba} \qquad \text{(Antisymmetric)} \tag{2.29}$$

Double Contractions

Theorem (very useful!) Let S^{ab} be a symmetric tensor, and A_{ab} an anti-symmetric tensor. Then their double contraction vanishes:

$$A_{ab}S^{ab} = 0. \tag{2.34}$$

Proof. To evaluate the double contraction of A_{ab} and S^{ab}, note that we can exchange the dummy labels a and b (as they are being summed over, it does not matter what each one is called; hence the term "dummy index"). Thus

$$\mu \equiv A_{ab}S^{ab} = A_{ba}S^{ba}. \tag{2.35}$$

Now use the symmetry and anti-symmetry of S^{ab} and A_{ab}:

$$\mu = A_{ba}S^{ba} = (-A_{ab})(+S^{ab}) = -\mu. \tag{2.36}$$

This can only be true if the double contraction $\mu \equiv A_{ab}S^{ab}$ is zero.

Tensor Densities

2.4.2 Tensor densities

As second rank tensors (in a given coordinate system) look like matrices, we can ask whether the determinant of a matrix is also a tensor. As the determinant is a single number, it would have to be a scalar. In fact, determinants are not scalars, but transform in a particular manner.

2.4.3 The Levi–Civita symbol

The Levi–Civita symbol[1] is completely antisymmetric. Its usefulness will become clear later, when we will use it in the manipulation of tensors, for example in calculating cross products. It can be written in two dimensions as

$$\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.38}$$

and in 3-dimensions as

$$\epsilon^{1ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{2.39}$$

$$\epsilon^{2ab} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{2.40}$$

$$\epsilon^{3ab} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.41}$$

The Metric: Definition

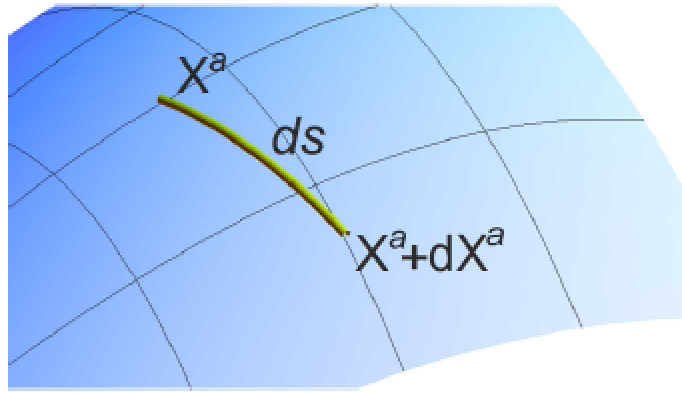


Figure 2.2: The distance ds between two points X^a and $\mathsf{X}^a + \mathrm{d}\mathsf{X}^a$.

Definition 2.6 *Metric* Given two nearby points, $(\mathsf{X}^1, \ldots, \mathsf{X}^N)$ and $(X^1 + \mathrm{d}\mathsf{X}^1, \ldots, X^N + \mathrm{d}\mathsf{X}^N)$, on a manifold M, a distance ds can be defined by introducing a new object, the *metric tensor* g_{ab}. The distance satisfies

$$\mathrm{d}s^2 = g_{11}\,\mathrm{d}\mathsf{X}^1\mathrm{d}\mathsf{X}^1 + g_{12}\,\mathrm{d}\mathsf{X}^1\mathrm{d}\mathsf{X}^2 + \ldots + g_{NN}\,\mathrm{d}\mathsf{X}^N\mathrm{d}\mathsf{X}^N \tag{2.49}$$

or

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}\mathsf{X}^a\mathrm{d}\mathsf{X}^b. \tag{2.50}$$

The metric is symmetric

We will refer to $\mathrm{d}s^2$ as the *metric line element*. Note that $\mathrm{d}s^2$ means $(\mathrm{d}s)^2$, not $\mathrm{d}(s^2)$. Let us decompose g_{ab} into symmetric and anti-symmetric parts:

$$g_{ab} = S_{ab} + A_{ab} \tag{2.51}$$
$$\mathrm{d}s^2 = (S_{ab} + A_{ab}) \left(\mathrm{d}\mathsf{X}^a \mathrm{d}\mathsf{X}^b\right) \tag{2.52}$$

where $S_{ab} = S_{ba}$ and $A_{ab} = -A_{ba}$. Consider the anti-symmetric contributions to $\mathrm{d}s^2$, $A_{ab}\mathrm{d}\mathsf{X}^a\mathrm{d}\mathsf{X}^b$. This is the double contraction of an anti-symmetric tensor A_{ab} with a symmetric tensor $\mathrm{d}\mathsf{X}^a\mathrm{d}\mathsf{X}^b$, so by equation (2.34)

$$A_{ab}\mathrm{d}\mathsf{X}^a\mathrm{d}\mathsf{X}^b = 0. \tag{2.53}$$

As A_{ab} does not contribute to the line element, it is superfluous. Thus we may simply get rid of it, defining g_{ab} to be symmetric:

$$g_{ab} = g_{ba} \tag{2.54}$$

Euclidean metrics

2.7.1 Two dimensional Euclidean space

The two-dimensional Euclidean Plane is designated as E^2. Often a Cartesian coordinate system is used in E^2. In this case $\mathrm{d}x = \mathrm{d}\mathsf{C}^1$ and $\mathrm{d}y = \mathrm{d}\mathsf{C}^2$. The differential distance or line element

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}\mathsf{C}^i\mathrm{d}\mathsf{C}^j \tag{2.57}$$

becomes

$$\mathrm{d}s^2 = g_{C ij}\mathrm{d}x\mathrm{d}y \tag{2.58}$$

or, when multiplying out

$$\mathrm{d}s^2 = g_{C11}\,\mathrm{d}x^2 + g_{C12}\,\mathrm{d}x\mathrm{d}y + g_{C21}\,\mathrm{d}y\mathrm{d}x + g_{C22}\,\mathrm{d}y^2. \tag{2.59}$$

Since the Pythagorean theorem states that the square of the distance between two points is $\mathrm{d}x^2 + \mathrm{d}y^2$, by inspection we can identify $g_{C11} = g_{C22} = 1$ while the other terms involving the off diagonal elements of g_{Cij} must vanish, implying $g_{C12} = g_{C21} = 0$. We may therefore write g_{Cij} as

$$g_{Cij} = \delta_{ij} \qquad (i, j = 1, 2) \tag{2.60}$$

or in matrix form:

$$g_{Cab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.61}$$

Polar Coordinates

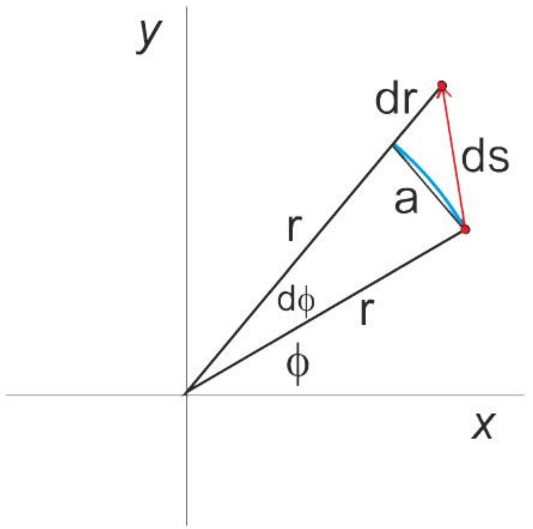


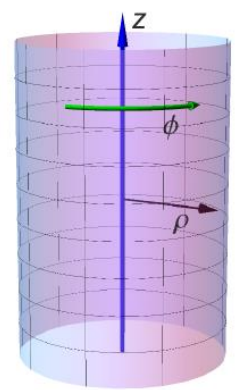
Figure 2.3: The length ds in polar coordinates

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\phi^2$$

or, in terms of the metric tensor $\mathrm{d}s^2 = g_{ab}\,\mathrm{dP}^a\,\mathrm{dP}^b$,

$$g_{Pab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Cylindrical Coordinates



$$(\mathsf{X}^1, \mathsf{X}^2, \mathsf{X}^3) = (\rho, \phi, z) \tag{2.72}$$

In this case a small step in the radial (ρ) and height (z) directions is given simply by $ds_\rho = \mathrm{d}\rho$, and $ds_z = \mathrm{d}z$. A step in the azimuthal direction is given by $ds_\phi = \rho \mathrm{d}\phi$. The metric line element is then given by

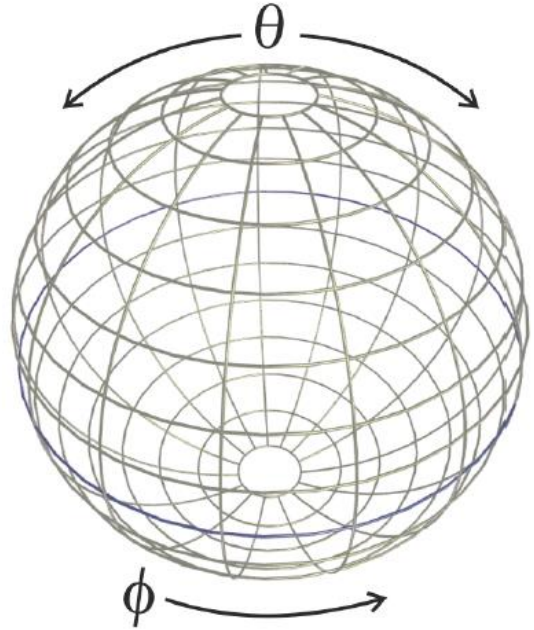
$$\mathrm{d}s^2 = ds_\rho^2 + ds_\phi^2 + ds_z^2 \tag{2.73}$$

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi^2 + dz^2 \tag{2.74}$$

which corresponds to the metric

$$g_{Cyl\ ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.75}$$

Spherical Coordinates



$$
\begin{aligned}
\mathrm{d}s^2 &= ds_r^2 + ds_\theta^2 + ds_\phi^2 \\
\mathrm{d}s^2 &= \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2 \\
g_{Pab} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}.
\end{aligned}
$$

Scalar Products and Magnitudes for Vectors

If we have a Riemannian manifold defined by a unique metric we can define the scalar product by writing

$$\overline{\mathbf{U}} \cdot \overline{\mathbf{V}} = g_{ab} U^a V^b \tag{2.84}$$

which is invariant in all coordinate systems, because the covariant tensor g is dual to two vectors. The magnitude of $\overline{\mathbf{V}}$ is then

$$|\overline{\mathbf{V}}| = \sqrt{\overline{\mathbf{V}} \cdot \overline{\mathbf{V}}} = \sqrt{g_{ab} V^a V^b} \tag{2.85}$$

In order to define the scalar product between two forms, we will employ the inverse of the metric. Let g^{ab} be a second rank tensor which gives the identity matrix when contracted with g_{bc}:

$$g^{ab} g_{bc} = \delta^a{}_c. \tag{2.86}$$

For two forms $\underline{A}, \underline{B}$ we then define

$$\underline{A} \cdot \underline{B} = g^{ab} A_a B_b \tag{2.87}$$

The magnitude of the form $\underline{A}$ is

$$|\underline{A}| = \sqrt{\underline{A} \cdot \underline{A}} = \sqrt{g^{ab} A_a A_b} \tag{2.88}$$

Raising and Lowering Indices

Note that $\overline{\mathbf{U}} \cdot \overline{\mathbf{V}} = g_{ab} U^a V^b$ is a scalar. We can write this scalar as

$$\overline{\mathbf{U}} \cdot \overline{\mathbf{V}} = (g_{ab} U^a)\, V^b \tag{2.89}$$

The term $g_{ab} U^a$ is 'dual' to a vector – It takes one vector, $\overline{\mathbf{V}}$ with components V^b, and returns a scalar number. Therefore, given a vector $\overline{\mathbf{U}}$, we can define a form $\underline{\mathbf{U}}$ where

$$U_b = g_{ab} U^a. \tag{2.90}$$

Similarly given a form $\underline{W}$ we can define a vector

$$W_a = g_{ab} W^b \tag{2.91}$$

The metric very usefully allows us to *raise* or *lower* the index. By contraction with the metric, we can turn vectors into forms and vice versa.

Orthogonal and non-orthogonal coordinates

For orthogonal coordinates, the metric is diagonal. Often the diagonal element g_{ii} is called

$$h_i = \sqrt{g_{ii}} \qquad \text{(no summation)}$$

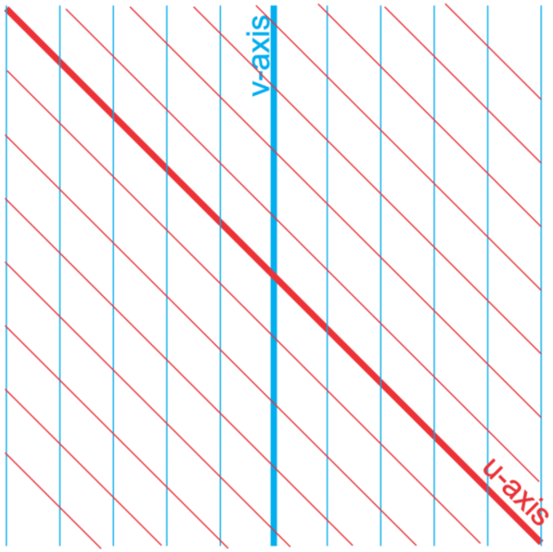
Example: Spherical coordinates

$$\begin{aligned} h_1 &= h_r = 1 \\ h_2 &= h_\theta = r \\ h_3 &= h_\phi = r \sin\theta \end{aligned}$$

Orthogonal and non-orthogonal coordinates

1. Draw the coordinate axes (what *is* a coordinate axis?)

$$
\begin{aligned}
u &= x \\
v &= x + y
\end{aligned}
$$



Here the inverse transformations are $x = u$ and $y = v - u$. Thus

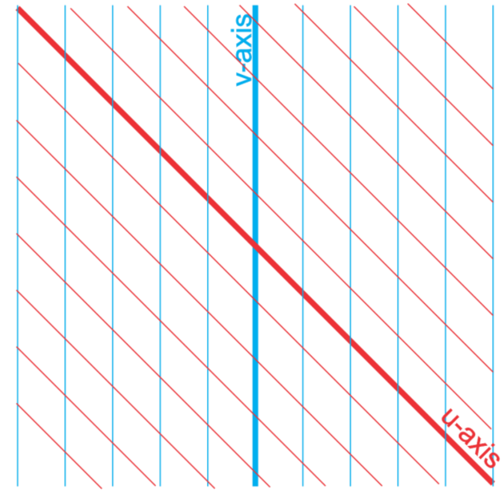
$$ds^2 = dx^2 + dy^2 = du^2 + (d(v-u))^2 = 2du^2 - 2du\ dv + dv^2. \tag{3}$$

The corresponding metric tensor is

$$g_{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}. \tag{4}$$

$$
\begin{aligned}
u &= x \\
v &= x + y
\end{aligned}
$$

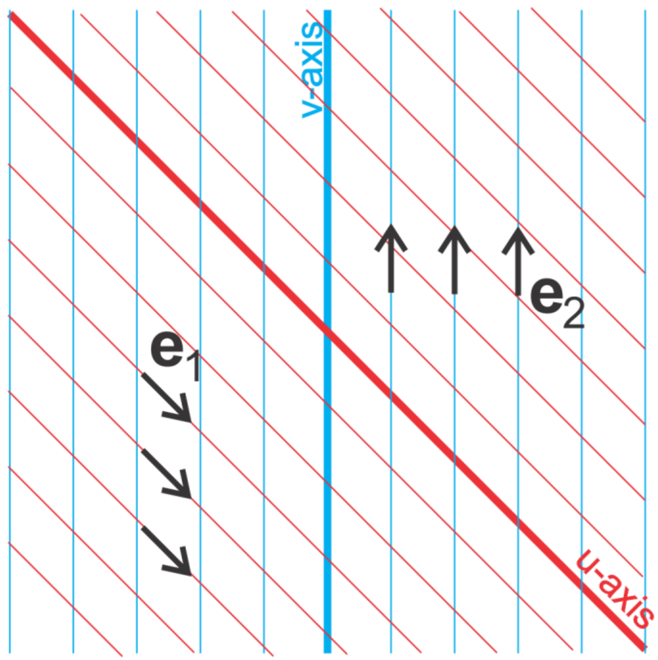
Basis Vectors



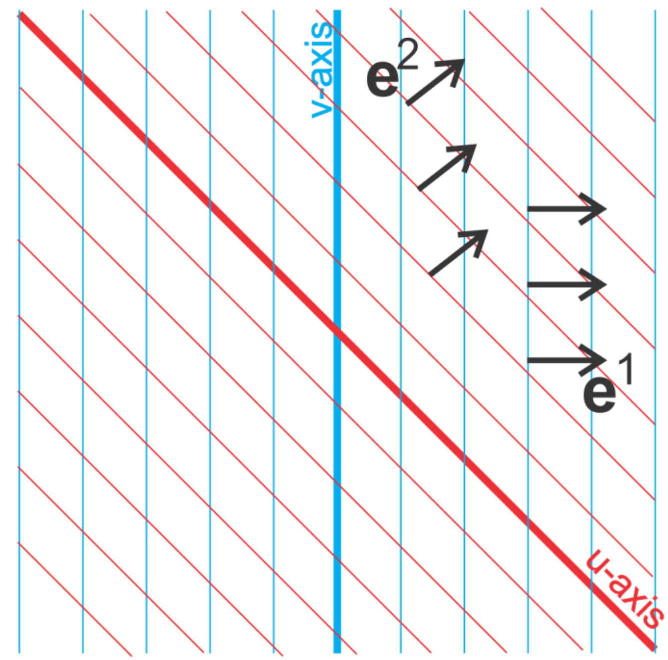
What are the basis vectors?

There are two reasonable choices:

Covariant basis vectors



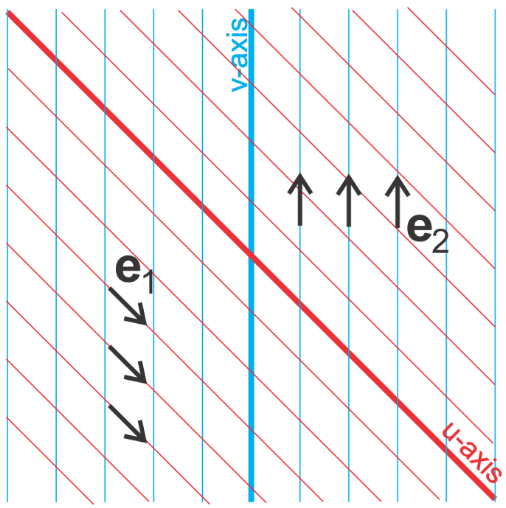
Contravariant basis vectors



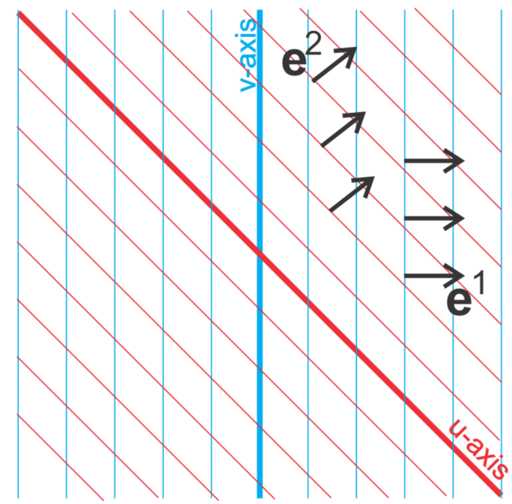
$$
\begin{aligned}
u &= x \\
v &= x + y
\end{aligned}
$$

Duality

Covariant basis vectors



Contravariant basis vectors



$$
\begin{aligned}
\mathbf{e_1} &= (1, -1) \\
\mathbf{e_2} &= (0, 1)
\end{aligned}
$$

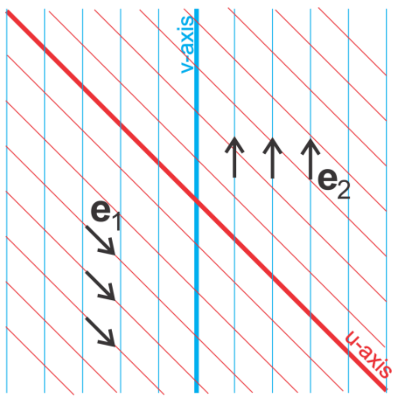
$$\mathbf{e^i} \cdot \mathbf{e_j} = \delta^i_j$$

$$
\begin{aligned}
\mathbf{e^1} &= (1, 0) \\
\mathbf{e^2} &= (1, 1)
\end{aligned}
$$

Duality

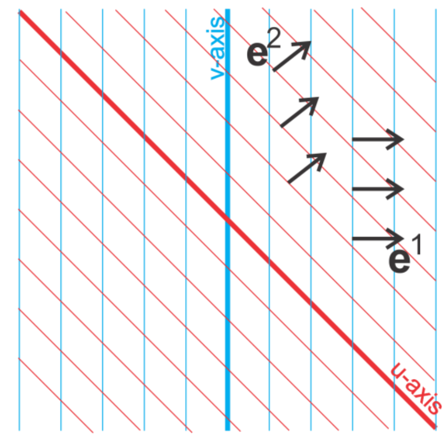
$$u = x$$
$$v = x + y$$

Covariant basis vectors



$$\mathbf{e_1} = (1, -1)$$
$$\mathbf{e_2} = (0, 1)$$

Contravariant basis vectors



$$\mathbf{e^1} = (1, 0)$$
$$\mathbf{e^2} = (1, 1)$$

$$\mathbf{e^i} \cdot \mathbf{e_j} = \delta^i_j$$

$$\mathbf{e_1} = \frac{\partial x^i}{\partial u}$$

$$\mathbf{e_2} = \frac{\partial x^i}{\partial v}$$

$$\mathbf{e^1} = \nabla u = \frac{\partial u}{\partial x^i}$$

$$\mathbf{e^2} = \nabla v = \frac{\partial v}{\partial x^i}$$

Relation to metric:

$$g_{ij} = \mathbf{e_i} \cdot \mathbf{e_j}$$

Inverse metric tensor:

$$g^{ij} = \mathbf{e^i} \cdot \mathbf{e^j}$$

For orthogonal coordinates, we will sometimes combine the orthogonal elements with the basis vectors:

Orthogonal elements:

$$\mathbf{h_i} = h_i \mathbf{e_i} = \sqrt{g_{ii}} \mathbf{e_i}$$

Cartesian Coordinates

basis $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} \equiv \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$

$\boldsymbol{x} = x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k} = x_i\boldsymbol{e}_i$

Cylindrical Coordinates

basis $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z\}$

$$\boldsymbol{e}_r = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}, \quad \boldsymbol{e}_\theta = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}.$$

$$\frac{\partial \boldsymbol{e}_r}{\partial\theta} = \boldsymbol{e}_\theta, \quad \frac{\partial \boldsymbol{e}_\theta}{\partial\theta} = -\boldsymbol{e}_r.$$

Spherical Coordinates

basis $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi\}$

$$\boldsymbol{e}_r = \sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k},$$

$$\boldsymbol{e}_\theta = \cos\theta\cos\phi\,\boldsymbol{i} + \cos\theta\sin\phi\,\boldsymbol{j} - \sin\theta\,\boldsymbol{k},$$

$$\boldsymbol{e}_\phi = -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j},$$

$$\frac{\partial \boldsymbol{e}_r}{\partial\theta} = \boldsymbol{e}_\theta, \quad \frac{\partial \boldsymbol{e}_r}{\partial\phi} = \sin\theta\,\boldsymbol{e}_\phi, \quad \frac{\partial \boldsymbol{e}_\theta}{\partial\theta} = -\boldsymbol{e}_r,$$

$$\frac{\partial \boldsymbol{e}_\theta}{\partial\phi} = \cos\theta\,\boldsymbol{e}_\phi, \quad \frac{\partial \boldsymbol{e}_\phi}{\partial\theta} = 0, \quad \frac{\partial \boldsymbol{e}_\phi}{\partial\phi} = -\sin\theta\,\boldsymbol{e}_r - \cos\theta\,\boldsymbol{e}_\theta.$$

For all coordinate systems, we have

$$d\boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial s_i} ds_i = \boldsymbol{h}_i ds_i,$$

where for rectangular coordinates,

$$s_1 = x_i, \quad s_2 = x_2, \quad s_3 = x_3, \quad \boldsymbol{h}_1 = \boldsymbol{i}, \quad \boldsymbol{h}_2 = \boldsymbol{j}, \quad \boldsymbol{g}_3 = \boldsymbol{k};$$

for cylindrical polar coordinates,

$$s_1 = r, \quad s_2 = \theta, \quad s_3 = z, \quad \boldsymbol{h}_1 = \boldsymbol{e}_r, \quad \boldsymbol{h}_2 = r\boldsymbol{e}_\theta, \quad \boldsymbol{h}_3 = \boldsymbol{e}_z;$$

for spherical polar coordinates,

$$s_1 = r, \quad s_2 = \theta, \quad s_3 = \phi, \quad \boldsymbol{h}_1 = \boldsymbol{e}_r, \quad \boldsymbol{h}_2 = r\boldsymbol{e}_\theta, \quad \boldsymbol{h}_3 = r\sin\theta\,\boldsymbol{e}_\phi.$$

We also define $\boldsymbol{h}^{(1)}, \boldsymbol{h}^{(2)}, \boldsymbol{h}^{(3)}$ such that

$$\boldsymbol{h}^{(i)} \cdot \boldsymbol{h}_j = \delta_{ij}.$$

When $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$ are orthogonal to each other,

$$\boldsymbol{h}^{(i)} \text{ is parallel to } \boldsymbol{h}_i, \quad \text{and} \quad |\boldsymbol{h}^{(i)}| = \frac{1}{|\boldsymbol{h}_i|}.$$

Additional Notes from Yibin Fu, U. Keele

Gradient, divergence and curl

Definition

Given that the scalar field $p(\boldsymbol{x})$ is differentiable, the *gradient* of p, denoted by $\operatorname{grad} p$, is defined by

$$dp = (\operatorname{grad} p) \cdot d\boldsymbol{x}. \tag{1.2}$$

Since

$$dp = \frac{\partial p}{\partial s_i} ds_i,$$

we have

$$(\operatorname{grad} p) \cdot d\boldsymbol{x} = \frac{\partial p}{\partial s_i} \boldsymbol{h}^{(i)} \cdot \boldsymbol{h}_j ds_j = \frac{\partial p}{\partial s_i} \boldsymbol{h}^{(i)} \cdot d\boldsymbol{x}.$$

Thus,

$$\boxed{\operatorname{grad} p = \frac{\partial p}{\partial s_i} \boldsymbol{h}^{(i)}} \tag{1.3}$$

Example 2.12: In rectangular coordinates, we have

$$\operatorname{grad} p = \frac{\partial p}{\partial x_1} \boldsymbol{i} + \frac{\partial p}{\partial x_2} \boldsymbol{j} + \frac{\partial p}{\partial x_3} \boldsymbol{k}.$$

Example 2.13: In cylindrical polar coordinates, we have

$$s_1 = r, \;\; s_2 = \theta, \;\; s_3 = z, \;\; \boldsymbol{h}^{(1)} = \boldsymbol{e}_r, \;\; \boldsymbol{h}^{(2)} = \frac{1}{r}\boldsymbol{e}_\theta, \;\; \boldsymbol{h}^{(3)} = \boldsymbol{e}_z;$$

and thus,

$$\operatorname{grad} p = \frac{\partial p}{\partial s_i}\,\boldsymbol{h}^{(i)} = \frac{\partial p}{\partial r}\,\boldsymbol{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\,\boldsymbol{e}_\theta + \frac{\partial p}{\partial z}\,\boldsymbol{e}_z\,.$$

In spherical polar coordinates, we have

$$s_1 = r, \;\; s_2 = \theta, \;\; s_3 = \phi, \;\; \boldsymbol{h}^{(1)} = \boldsymbol{e}_r, \;\; \boldsymbol{h}^{(2)} = \frac{1}{r}\boldsymbol{e}_\theta, \;\; \boldsymbol{h}^{(3)} = \frac{1}{r\sin\theta}\boldsymbol{e}_\phi,$$

and so

$$\operatorname{grad} p = \frac{\partial p}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\boldsymbol{e}_\phi\,.$$

Definition

Given that the vector field $\boldsymbol{u}(\boldsymbol{x})$ is differentiable, the *gradient* and *divergence* of $\boldsymbol{u}$, denoted by $\operatorname{grad}\boldsymbol{u}$ and $\operatorname{div}\boldsymbol{u}$, respectively, are defined by

$$d\boldsymbol{u} = \operatorname{grad}\boldsymbol{u}[d\boldsymbol{x}], \quad \operatorname{div}\boldsymbol{u} = \operatorname{tr}(\operatorname{grad}\boldsymbol{u}). \tag{1.4}$$

Since

$$d\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial s_i} ds_i,$$

we have

$$\operatorname{grad}\boldsymbol{u}[d\boldsymbol{x}] = \frac{\partial \boldsymbol{u}}{\partial s_i} ds_i = \left(\frac{\partial \boldsymbol{u}}{\partial s_i} \otimes \boldsymbol{h}^{(i)}\right) \boldsymbol{h}_j ds_j = \left(\frac{\partial \boldsymbol{u}}{\partial s_i} \otimes \boldsymbol{h}^{(i)}\right) d\boldsymbol{x}.$$

Thus,

$$\boxed{\operatorname{grad}\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial s_i} \otimes \boldsymbol{h}^{(i)}, \quad \operatorname{div}\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial s_i} \cdot \boldsymbol{h}^{(i)}}. \tag{1.5}$$

Example 2.14: In rectangular coordinates, we have

$$\operatorname{grad}\boldsymbol{u} = \frac{\partial u_i}{\partial x_j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \quad \operatorname{div}\boldsymbol{u} = u_{i,i}.$$

Example 2.15: If $\boldsymbol{u} = u(r,\theta,z)\boldsymbol{e}_r + v(r,\theta,z)\boldsymbol{e}_\theta + w(r,\theta,z)\boldsymbol{e}_z$ in terms of cylindrical polar coordinates, then

$$\operatorname{grad}\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial s_i} \otimes \boldsymbol{h}^{(i)} = \frac{\partial \boldsymbol{u}}{\partial r} \otimes \boldsymbol{e}_r + \frac{\partial \boldsymbol{u}}{\partial \theta} \otimes \frac{1}{r}\boldsymbol{e}_\theta + \frac{\partial \boldsymbol{u}}{\partial z} \otimes \boldsymbol{e}_z$$

$$= \frac{\partial u}{\partial r}\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \frac{\partial v}{\partial r}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_r + \frac{\partial w}{\partial r}\boldsymbol{e}_z \otimes \boldsymbol{e}_r$$

$$+ \left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta + \left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right)\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \frac{1}{r}\frac{\partial w}{\partial \theta}\boldsymbol{e}_z \otimes \boldsymbol{e}_\theta$$

$$+ \frac{\partial u}{\partial z}\boldsymbol{e}_r \otimes \boldsymbol{e}_z + \frac{\partial v}{\partial z}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_z + \frac{\partial w}{\partial z}\boldsymbol{e}_z \otimes \boldsymbol{e}_z,$$

or, in matrix form,

$$\operatorname{grad}\boldsymbol{u} = \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial r} & \frac{1}{r}\frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial z} \end{bmatrix}, \quad \operatorname{div}\boldsymbol{u} = \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} + \frac{u}{r}.$$

Example 2.16: If $\boldsymbol{u} = u(r, \theta, \phi)\boldsymbol{e}_r + v(r, \theta, \phi)\boldsymbol{e}_\theta + w(r, \theta, \phi)\boldsymbol{e}_\phi$ in terms of spherical polar coordinates, then

$$\operatorname{grad} \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial s_i} \otimes \boldsymbol{h}^{(i)} = \frac{\partial \boldsymbol{u}}{\partial r} \otimes \boldsymbol{e}_r + \frac{\partial \boldsymbol{u}}{\partial \theta} \otimes \frac{1}{r}\boldsymbol{e}_\theta + \frac{\partial \boldsymbol{u}}{\partial \phi} \otimes \boldsymbol{e}_\phi$$

$$= \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} & \frac{1}{r \sin\theta}\frac{\partial u}{\partial \phi} - \frac{w}{r} \\ \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & \frac{1}{r \sin\theta}\frac{\partial v}{\partial \phi} - \frac{w \cot\theta}{r} \\ \frac{\partial w}{\partial r} & \frac{1}{r}\frac{\partial w}{\partial \theta} & \frac{1}{r \sin\theta}\frac{\partial w}{\partial \phi} + \frac{u}{r} + \frac{v \cot\theta}{r} \end{bmatrix}$$

Thus,

$$\operatorname{div} \boldsymbol{u} = \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial w}{\partial \phi} + \frac{2u}{r} + \frac{v \cot\theta}{r}.$$

Example 2.17: Let p be a differentiable scalar field. Show that in terms of cylindrical and spherical polar coordinates, the Laplacian of p, denoted by $\bigtriangledown^2 p$ and defined by $\bigtriangledown^2 p = \mathrm{div}\,(\mathrm{grad}\, p)$, are respectively given by

$$\bigtriangledown^2 p = \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2},$$

and

$$\bigtriangledown^2 p = \frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial p}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p}{\partial \phi^2}.$$

Solution: Use

$$\mathrm{grad}\, p = \frac{\partial p}{\partial r}\,\boldsymbol{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\,\boldsymbol{e}_\theta + \frac{\partial p}{\partial z}\,\boldsymbol{e}_z.$$

$$\mathrm{div}\,\boldsymbol{u} = \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} + \frac{u}{r}$$

for cylindrical polar coordinates, and

$$\mathrm{grad}\, p = \frac{\partial p}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\boldsymbol{e}_\phi,$$

$$\mathrm{div}\,\boldsymbol{u} = \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi} + \frac{2u}{r} + \frac{v\cot\theta}{r},$$

for spherical polar coordinates.

Definition

The *curl* of $\boldsymbol{u}$, written $\operatorname{curl}\boldsymbol{u}$, is defined by

$$(\operatorname{curl}\boldsymbol{u})\cdot\boldsymbol{a} = \operatorname{div}(\boldsymbol{u}\wedge\boldsymbol{a}), \quad \forall\,\boldsymbol{a} \quad \Longrightarrow \quad \boxed{\operatorname{curl}\boldsymbol{u} = \boldsymbol{h}^{(j)}\wedge\frac{\partial\boldsymbol{u}}{\partial s_j}} \tag{1.6}$$

Example 2.18: If $\boldsymbol{u} = u(x_1,x_2,x_3)\boldsymbol{e}_1 + u_2(x_1,x_2,x_3)\boldsymbol{e}_2 + u_3(x_1,x_2,x_3)\boldsymbol{e}_3$ in terms of rectangular coordinates, then

$$\operatorname{curl}\boldsymbol{u} = (u_{3,2}-u_{2,3})\boldsymbol{e}_1 + (u_{1,3}-u_{3,1})\boldsymbol{e}_2 + (u_{2,1}-u_{1,2})\boldsymbol{e}_3.$$

Example 2.19: If $\boldsymbol{u} = u(r,\theta,z)\boldsymbol{e}_r + v(r,\theta,z)\boldsymbol{e}_\theta + w(r,\theta,z)\boldsymbol{e}_z$ in terms of cylindrical polar coordinates, then

$$\operatorname{curl}\boldsymbol{u} = \boldsymbol{e}_r\wedge\frac{\partial\boldsymbol{u}}{\partial r} + \frac{1}{r}\boldsymbol{e}_\theta\wedge\frac{\partial\boldsymbol{u}}{\partial\theta} + \boldsymbol{e}_z\wedge\frac{\partial\boldsymbol{u}}{\partial z}$$

$$= \left(\frac{1}{r}\frac{\partial w}{\partial\theta} - \frac{\partial v}{\partial z}\right)\boldsymbol{e}_r + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\boldsymbol{e}_\theta + \left(\frac{\partial v}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{v}{r}\right)\boldsymbol{e}_z.$$

Example 2.20: If $\boldsymbol{u} = u(r,\theta,\phi)\boldsymbol{e}_r + v(r,\theta,\phi)\boldsymbol{e}_\theta + w(r,\theta,\phi)\boldsymbol{e}_\phi$ in terms of spherical polar coordinates, then

$$\operatorname{curl}\boldsymbol{u} = \boldsymbol{e}_r \wedge \frac{\partial \boldsymbol{u}}{\partial r} + \frac{1}{r}\boldsymbol{e}_\theta \wedge \frac{\partial \boldsymbol{u}}{\partial \theta} + \frac{1}{r\sin\theta}\boldsymbol{e}_\phi \wedge \frac{\partial \boldsymbol{u}}{\partial \phi}$$

$$\begin{aligned} &= \left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi} + \frac{w\cot\theta}{r}\right)\boldsymbol{e}_r \\ &\quad + \left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} - \frac{\partial w}{\partial r} - \frac{w}{r}\right)\boldsymbol{e}_\theta \\ &\quad + \left(\frac{\partial v}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{v}{r}\right)\boldsymbol{e}_\phi. \end{aligned}$$

Definition

When the tensor field T is differentiable, its divergence, denoted by div T, is the vector field defined by $(\operatorname{div} T) \cdot \boldsymbol{a} = \operatorname{div}(T\boldsymbol{a}), \quad \forall\, \boldsymbol{a}$.

Using (1.4), we have

$$\operatorname{div}(T\boldsymbol{a}) = \boldsymbol{h}^{(i)} \cdot \frac{\partial (T\boldsymbol{a})}{\partial s_i} = \boldsymbol{h}^{(j)} \cdot \left(\frac{\partial T}{\partial s_j}\boldsymbol{a}\right) = \left(\boldsymbol{h}^{(j)} \cdot \frac{\partial T}{\partial s_j}\right) \cdot \boldsymbol{a}.$$

Thus,

$$\boxed{\operatorname{div} T = \boldsymbol{h}^{(j)} \cdot \frac{\partial T}{\partial s_j}} \tag{1.7}$$

Example 2.21: In rectangular coordinates, $T = T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, and

$$\operatorname{div} T = \boldsymbol{e}_j \cdot \frac{\partial}{\partial x_j}(T_{ik}\boldsymbol{e}_i \otimes \boldsymbol{e}_k) = \delta_{ij}\, T_{ik,j}\boldsymbol{e}_k = T_{jk,j}\boldsymbol{e}_k.$$

Example 2.22: Calculate div T when T is given by

$$T = T_{rr}\boldsymbol{e}_r \otimes \boldsymbol{e}_r + T_{r\theta}\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta + T_{rz}\boldsymbol{e}_r \otimes \boldsymbol{e}_z + T_{\theta r}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_r$$

$$+ T_{\theta\theta}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + T_{\theta z}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_z + T_{zr}\boldsymbol{e}_z \otimes \boldsymbol{e}_r + T_{z\theta}\boldsymbol{e}_z \otimes \boldsymbol{e}_\theta + T_{zz}\boldsymbol{e}_z \otimes \boldsymbol{e}_z$$

in terms of cylindrical polar coordinates.
Solution: From

$$\mathrm{div}\ T = \boldsymbol{h}^{(j)} \cdot \frac{\partial T}{\partial s_j} = \boldsymbol{e}_r \cdot \frac{\partial T}{\partial r} + \frac{1}{r}\boldsymbol{e}_\theta \cdot \frac{\partial T}{\partial \theta} + \boldsymbol{e}_z \cdot \frac{\partial T}{\partial z}$$

we find that the three components of div T are given by

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{\partial T_{zr}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r},$$
$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{z\theta}}{\partial z} + \frac{T_{\theta r} + T_{r\theta}}{r},$$
$$\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{rz}}{r},$$

respectively.

Example 2.23: Calculate div T when T is given by

$$T = T_{rr}\boldsymbol{e}_r \otimes \boldsymbol{e}_r + T_{r\theta}\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta + T_{r\phi}\boldsymbol{e}_r \otimes \boldsymbol{e}_\phi$$

$$+T_{\theta r}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_r + T_{\theta\theta}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + T_{\theta\phi}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_z$$

$$+T_{\phi r}\boldsymbol{e}_\phi \otimes \boldsymbol{e}_r + T_{z\phi\theta}\boldsymbol{e}_\phi \otimes \boldsymbol{e}_\theta + T_{\phi\phi}\boldsymbol{e}_\phi \otimes \boldsymbol{e}_\phi$$

in terms of spherical polar coordinates.
Solution: From

$$\operatorname{div} T = \boldsymbol{h}^{(j)} \cdot \frac{\partial T}{\partial s_j} = \boldsymbol{e}_r \cdot \frac{\partial T}{\partial r} + \frac{1}{r}\boldsymbol{e}_\theta \cdot \frac{\partial T}{\partial \theta} + \frac{1}{r\sin\theta}\boldsymbol{e}_\phi \cdot \frac{\partial T}{\partial \theta}$$

we find that the three components of div T are given by

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r}}{\partial \phi} + \frac{2T_{rr} - T_{\theta\theta} - T_{\phi\phi} + T_{\theta r}\cot\theta}{r},$$

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta}}{\partial \phi} + \frac{T_{\theta r} + 2T_{r\theta} + \cot\theta\, T_{\theta\theta} - \cot\theta\, T_{\phi\phi}}{r},$$

$$\frac{\partial T_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{2T_{r\phi} + T_{\phi r} + \cot\theta\, T_{\theta\phi} + \cot\theta\, T_{\phi\theta}}{r},$$

respectively.