1.1 Linear Waves

For simplicity, assume only one space dimension, and that a typical field variable is $u(x,t)$. Linear waves can then be represented by the Fourier component,

$$u = A\exp\left(ikx - i\omega t\right), \tag{1}$$

where k is the *wavenumber*, ω is the wave *frequency* and A is the wave amplitude, which may also be a function of k. The full solution is obtained by a superposition of such components. Equation (1) is a **kinematic** expression, valid for all physical systems which support waves. The **dynamics** of the system will lead to the **dispersion relation**

$$\omega = \omega(k), \tag{2}$$

whose precise form is determined by the physical system under consideration. For instance, for water waves,

$$\omega^2 = gk\tanh\left(kh\right), \tag{3}$$

where h is the still water depth, and g is gravity. Here there are two branches of the dispersion relation.

1.2 Linear Waves

An important feature of linear waves is that the dispersion relation captures the full system in Fourier space. That is, if the physical system takes the schematic form

$$D(i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x})\, u = 0\,. \quad \text{then} \quad D(\omega, k) = 0\,, \tag{4}$$

whose solutions are the branches $\omega = \omega(k)$. For stable waves, ω is real-valued for all real-valued k.

There are two important velocities,

$$\text{Phase velocity :} \quad c = \frac{\omega}{k}\,, \quad \text{and} \quad \text{Group velocity :} \quad c_g = \frac{d\omega}{dk}\,. \tag{5}$$

For a dispersive wave system, they are different. The phase of the wave (e.g. a wave crest) propagates with velocity c, but the wave energy propagates with the velocity c_g. The wave energy E for each Fourier component is typically given by an expression of the form $E = F(k)|A|^2$. For instance, for water waves $E = g|A|^2/2$ where A is the surface elevation above the still-water depth.

1.3 Group velocity

For linear waves, we can use Fourier superposition to represent the solution of an initial-value problem in the form,

$$u(x,t) = \int_{-\infty}^{\infty} F(k) \exp\left(i(kx - \omega t)\right) dk\,, \quad \omega = \omega(k)\,. \tag{6}$$

Here $F(k)$ is the Fourier transform of $f(x) = u(x,0)$. In general, when the dispersion relation has several branches, there is similar expression for each branch, and the full solution is the sum over each branch. Each Fourier component evolves independently with frequency ω related to the wavenumber k through the dispersion relation (2).
An important distinction now emerges between **dispersive** systems and **non-dispersive** systems. A non-dispersive system is one for which $c = c_0$ is a constant, **independent of** k, so that the dispersion relation reduces to

$$\omega = c_0 k \quad \text{so that} \quad u_t + c_0 u_x = 0\,. \tag{7}$$

in this case also $c_g = c_0$. The solution of (7) is

$$u = f(x - c_0 t) \quad \text{where} \quad f(x) = u(x,0)\,, \tag{8}$$

which can also be deduced from (6).

1.4 Group velocity

A **dispersive** system is one for which $d^2\omega/dk^2 \neq 0$ and hence c, c_g both depend on k with $c \neq c_g$, except possibly at certain exceptional values of k. In this case the representation (6)

$$u(x,t) = \int_{-\infty}^{\infty} F(k) \exp\left(i(kx - \omega(k)t)\right) dk\,,$$

is not in general susceptible to an explicit evaluation. Instead for large times, as $t \to \infty$, the integral can be evaluated asymptotically using the method of **stationary phase** which leads to the condition that the dominant contribution at the location x is given by

$$\frac{x}{t} = \frac{d\omega}{dk} = c_g(k)\,. \tag{9}$$

Thus, the Fourier component with wavenumber k propagates with the **group velocity**. Since this varies as k varies, the initial value **disperses**.

1.5 Wave packet

To obtain a wave packet, we suppose that the initial conditions are such that $F(k)$ has a dominant component centred at $k = k_0$. The dispersion relation (2) is then approximated by

$$\omega = \omega_0 + c_g(k - k_0) + \delta(k - k_0)^2 \,, \quad \text{where} \quad \delta = \frac{1}{2}\frac{d^2\omega}{dk^2} \,. \tag{10}$$

Here both c_g and δ are evaluated at $k = k_0$. The expression (6) becomes

$$u(x,t) \approx A(x,t) \exp\left(i\{k_0 x - \omega_0 t\}\right) , \tag{11}$$

$$\text{where} \quad A(x,t) = \int_{-\infty}^{\infty} F(k_0 + \tau) \exp\left(i\{\tau(x - c_g t)\} - i\delta\tau^2 t\right) d\tau \,, \tag{12}$$

where the variable of integration has been changed from k to $\tau = k - k_0$. Here the sinusoidal factor $\exp\left(i\{k_0 x - \omega_0 t\}\right)$ is a carrier wave with a phase velocity ω_0/k_0, while the (complex) amplitude $A(x,t)$ describes the wave packet. Since the term proportional to τ^2 in the exponent in (12) is a small correction term, it can be seen that to leading order, the amplitude A propagates with the group velocity c_g, while the small correction term gives a dispersive correction term proportional to $t^{-1/2}$.

1.6 Wave packet

$$A(x,t) = \int_{-\infty}^{\infty} F(k_0+\tau)\exp\left(i\{\tau(x-c_g t)\} - i\delta\tau^2 t\right) d\tau\,,$$

Indeed, it can be shown that

$$A(x,t) \approx F(k_0)\left(\frac{2\pi}{|\delta|t}\right)^{\frac{1}{2}}\exp\left(\frac{i\pi}{4}\mathrm{sign}\delta\right), \qquad \text{for} \quad x = c_g(k_0)t\,. \tag{13}$$

This same result can be obtained directly from (6) by using the method of stationary phase, valid here in the limit when $t \to \infty$ (see, for instance, Lighthill (1978) or Whitham (1974)). In this case, it is not necessary to assume also that $F(k)$ is centred at k_0, and so wave packets are the generic long-time outcome of initial-value problems.

1.7 Kinematic wave theory

Next note that the group velocity appears naturally in the kinematic theory of waves (see Lighthill (1978) or Whitham (1974)). Thus, let the wave field be defined asymptotically by

$$u(x,t) \sim A(x,t)\exp\left(i\theta(x,t)\right), \tag{14}$$

where $A(x,t)$ is the (complex) wave amplitude, and $\theta(x,t)$ is the phase, which is assumed to be rapidly-varying compared to the amplitude. Then it is natural to define the local wave frequency and wavenumber by

$$\omega = -\frac{\partial\theta}{\partial t}, \qquad k = \frac{\partial\theta}{\partial x}. \tag{15}$$

Note that the expression (11) has the required form (14). Then cross-differentiation leads to the kinematic equation for the conservation of waves,

$$\frac{\partial k}{\partial t} + \frac{\partial\omega}{\partial x} = 0. \tag{16}$$

1.8 Kinematic wave theory

Next it can be shown that to leading order in an asymptotic theory, the dispersion relation (2)

$$\omega = \omega(k)\,,$$

holds for the frequency and wavenumber defined by (15). This is substituted into equation (16)

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0\,,$$

to yield

$$\frac{\partial k}{\partial t} + c_g \frac{\partial k}{\partial x} = 0\,, \tag{17}$$

with a similar equation for the frequency. Thus, both the wavenumber and frequency propagate with the group velocity, a fact which can also be seen in (11, 12). Equation (17) is itself a simple wave equation, which can be readily integrated by the method of characteristics, that is

$$k = \text{constant} \quad \text{on} \quad \frac{dx}{dt} = c_g(k)\,.$$

Note that the group velocity c_g is a function of k, so that (17) is a nonlinear equation.

1.9 Several spatial variables

Next, suppose that the physical system contains several spatial variables, represented by the vector $\mathbf{x}$, for example $\mathbf{x} = (x, y)$. Then the phase variable $(kx - \omega t)$ in (1) is replaced by $(\mathbf{k} \cdot \mathbf{x} - \omega t)$, where $\mathbf{k}$ is the vector wavenumber, for example $\mathbf{k} = (k, l)$ and so $\mathbf{k} \cdot \mathbf{x} = kx + ly$. Then (1) is replaced by

$$u = A \exp\left(i\mathbf{k} \cdot \mathbf{x} - i\omega t\right), \tag{18}$$

The dispersion relation (2) is replaced by

$$\omega = \omega(\mathbf{k}). \tag{19}$$

The phase velocity and group velocity are now given by, replacing (5),

$$\text{Phase velocity :} \quad \mathbf{c} = \frac{\omega}{\kappa}\frac{\mathbf{k}}{\kappa}, \quad \text{and} \quad \text{Group velocity :} \quad \mathbf{c_g} = \nabla_{\mathbf{k}}\omega. \tag{20}$$

Thus the group velocity can differ from the phase velocity in both magnitude and direction. A striking example of the latter arises for internal waves, whose group velocity is perpendicular to the phase velocity (see Lighthill (1978)).

1.10 Energy propagation

Because the group velocity is the velocity of the wave packet as a whole, it is no surprise to find that it can also be identified with energy propagation. Indeed, it can be shown that in most linearised physical systems, an equation of the following form can be derived,

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla \cdot (\mathbf{c}_g \mathcal{A}) = 0\,, \tag{21}$$

where $\mathcal{A}$ is the **wave action density**, and is proportional to the square of the wave amplitude $|A|^2$, with the factor being a function of the wavenumber $\mathbf{k}$. Typically, in an inertial frame of reference, the wave action is the wave energy density divided by the frequency.

Introduction: Nonlinear effects

Next, consider a **dispersive** system. In the linearized theory, each Fourier component with wavenumber k propagates with its own group velocity, and so the system disperses. Then the effects of nonlinearity typically arise in three scenarios.

(1) **Long waves**: Here $k \to 0$. Because the dispersion relation can be made to satisfy the antisymmetry condition $\omega(k) = -\omega(-k)$ (ensuring real-valued solutions), it follows that when also $\omega(0) = 0$, we have that $\omega = c_0 k + O(k^3)$, and so $c_g = c_0 + O(k^2)$, with weak dispersion.
(2) **Wave packets**: Here it is assumed that the wave energy is concentrated around a finite wavenumber k_0 say. Consequently, there is only weak dispersion, and approximately the wave group propagates with a constant group velocity $c_{g0} = c_g(k = k_0)$.
(3) **Resonant Wave interactions**: Due to nonlinearity, two linear waves with wavenumbers $k_{1,2}$ say, will interact to form another wave with wavenumber $k_0 = k_1 + k_2$. If the corresponding frequencies are resonant, that is $\omega_0 \approx \omega_1 + \omega_2$ $(\omega_i = \omega(k = k_i))$, then there can be a strong effect.

2.1 Korteweg-de Vries (KdV) Equation

Here we consider the long-wave regime, where $k \to 0$, and assume that we can use the approximate dispersion relation

$$\omega = c_0 k - \beta k^3 \,, \tag{24}$$

with an error of $O(k^5)$. This translates to an evolution equation

$$u_t + c_0 u_x + \beta u_{xxx} = 0 \,, \tag{25}$$

where we recall that $-i\omega = \partial/\partial t, ik = \partial/\partial x$ for each Fourier component. The dominant term is $u_t + c_0 u_x \approx 0$, showing that the wave propagates with speed c_0 unchanged, except for the effect of the weak dispersion due to the term u_{xxx}. This small effect needs to be balanced by nonlinearity, and in many physical systems this has the quadratic form $\mu u u_x$, for some constant coefficient μ. Thus the model equation takes the form

$$u_t + c_0 u_x + \mu u u_x + \beta u_{xxx} = 0 \,. \tag{26}$$

This is the famous **Korteweg-de Vries (KdV)** equation, first derived in the water-wave context in 1895, and subsequently found to hold in many physical systems. Note that for $\beta = 0$ this reduces to the Hopf equation (22), and so can be regarded as a regularization of that due to linear dispersion.

2.2 KdV Equation: Solitons

The KdV equation is, in the reference frame moving with speed c_0 (transform $x \to x - c_0 t$),

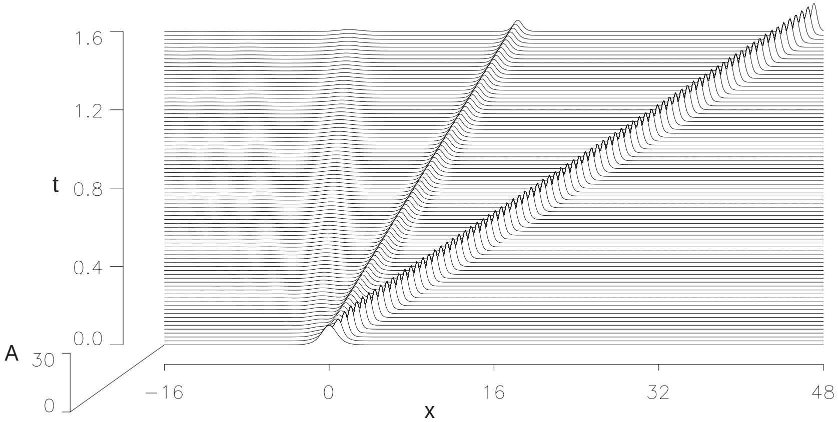
$$u_t + \mu u u_x + \beta u_{xxx} = 0\,. \tag{27}$$

This is an **integrable** equation,a result first established in the 1960's by Kruskal and collaborators. Its principal solutions are **solitons**. A single soliton is the **solitary wave**, an isolated and steadily-propagating pulse, given by

$$u = a\,\mathrm{sech}^2(\gamma(x - Vt))\,, \quad V = \frac{\mu a}{3} = 4\beta\gamma^2\,. \tag{28}$$

This is a one-parameter family of solutions, parametrized by the amplitude a say. The speed V is proportional to the amplitude a and is positive (negative) as $\beta > (<)0$, and is also proportional to the square of the wavenumber γ; thus large waves are thinner and travel faster. They are waves of elevation (depression) when $\mu\beta > (<)0$. Integrability means that the general initial-value problem for a localized initial condition can be solved through the **Inverse Scattering Transform** (IST), with the generic outcome of a finite number of solitons propagating in the positive x-direction, and some dispersing radiation, propagating in the negative x-direction (when $\mu\beta > 0$).

2.3 KdV Equation: Solitons



The generation of three solitons from a localized initial condition for the KdV equation

$$A_t + 6AA_x + A_{xxx} = 0\,.$$

2.4 Nonlinear Schrödinger Equation

Here we assume that the solution is a **narrow-band wave packet**, where the wave energy in Fourier space is concentrated around a dominant wavenumber k_0. The dispersion relation $\omega = \omega(k)$ can then be approximated for $k \approx k_0$ by

$$\omega - \omega_0 = c_{g0}\,(k - k_0) + \delta(k - k_0)^2\,, \tag{29}$$

where $\omega_0 = \omega(k_0), c_g = c_g(k_0)$ and $\delta = c_{gk}(k_0)/2$, and we recall that $c_g(k) = d\omega/dk$, so that $c_{gk} = \omega_{kk}$. This translates to an evolution equation for the wave amplitude

$$i(A_t + c_{g0}A_x) + \delta A_{xx} = 0\,, \quad u = A\exp\,(ik_0x - i\omega_0 t) + \text{c. c.}\,. \tag{30}$$

Here c. c. denotes the comples conjugate. The **envelope** function $A(x,t)$ is slowly-varying with respect to the carrier phase $k_0x - \omega_0 t$. The dominant term is $A_t + c_{g0}A_x \approx 0$, showing that the wave envelope propagates with the group velocity c_{g0}, modified by the effect of weak dispersion due to the term A_{xx}. The result is well-known in quantum mechanics as the Schrödinger equation.
The small dispersion effect needs to be balanced by nonlinearity, and in many physical systems this has the typical cubic form $\nu|A|^2A$, for some constant coefficient ν.

2.5 Nonlinear Schrödinger Equation

Thus the model evolution equation for the wave envelope is the **nonlinear Schrödinger equation** (NLS), expressed here in the reference frame moving with speed c_{g0} (transform $x \to x - c_{g0}t$),

$$iA_t + \nu|A|^2A + \delta A_{xx} = 0\,. \tag{31}$$

Like the KdV equation it is a valid model for many physical systems, including notably water waves and nonlinear optics, a result first realized in the late 1960's. Remarkably, like the KdV equation, it as an **integrable** equation through the IST, first established by Zakharov and collaborators in 1972. It also has soliton solutions, and the single soliton or solitary wave solution is

$$A = a\,\mathrm{sech}(\gamma(x - Vt))\,\exp\left(iKx - i\Omega t\right), \tag{32}$$

$$\nu a^2 = 2\delta\gamma^2\,, \quad \Omega = \delta(K^2 - \gamma^2)\,, \quad V = 2\delta K\,. \tag{33}$$

This solution exists only when $\delta\nu > 0$, the so-called focussing case. It forms a two-parameter family, the parameters being the amplitude a and "chirp" wavenumber K; however, K amounts to a perturbation of the carrier wavenumber k to $k + K, |K| << |k|$, and so can be removed by a gauge transformation.

2.6 Higher Space Dimensions

In two space dimensions the wavenumber becomes a vector $\mathbf{k} = (k, l)$ and the dispersion relation is then

$$\omega = \omega(\mathbf{k}) = \omega(k, l)\,, \tag{34}$$

where the wave phase is now $\mathbf{k} \cdot \mathbf{x} - \omega t = kx + ly - \omega t$. The phase velocity is $\mathbf{c} = \omega \mathbf{k}/\kappa^2$, where $\kappa = |\mathbf{k}|$. The group velocity becomes the vector

$$\mathbf{c}_g = \nabla_{\mathbf{k}} \cdot \omega = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}\right). \tag{35}$$

In general the group velocity and the phase velocity differ in both **magnitude** and **direction**. For water waves the dispersion relation is

$$\omega^2 = g\kappa \tanh \kappa h\,. \tag{36}$$

This is an example of an **isotropic** medium, as the wave frequency depends only on the wavenumber magnitude, and not its direction. In this case the group velocity is parallel to the wavenumber $\mathbf{k}$, and hence parallel to the phase velocity, with a magnitude $c_g = |\mathbf{c}_g| = d\omega/d\kappa$.

2.7 Kadomtsev-Petviashvili Equation

The KP equation is the two-dimensional extension of the KdV equation for isotropic systems, and is given by, in the reference frame moving with the linear long-wave speed c_0 in the x-direction,

$$(u_t + \mu u u_x + \beta u_{xxx})_x + \frac{c_0}{2} u_{yy} = 0 \,. \tag{37}$$

This equation assumes that there is weak diffraction in the y-direction, that is $\partial/\partial y << \partial/\partial x$. The linear terms can be deduced from the linear dispersion relation $\omega = \omega(\kappa), \kappa = (k^2 + l^2)^{1/2}$, where it assumed that $l^2 << k^2$. Thus in the long-wave limit, since $\kappa \approx k + l^2/2k$,

$$\omega \approx c_0 k - \beta k^3 + \frac{c_0 l^2}{2k} \cdots \,.$$

Recalling that $-i\omega \sim \partial/\partial t, ik \sim \partial/\partial x, il \sim \partial/\partial y$, we se that (37) follows. When $c_0\beta > 0$ holds in (37), this is the KPII equation, and it can be shown that then the solitary wave (28) is stable to transverse disturbances. This is the case for water waves. On the other hand if $\beta c_0 < 0$ holds, this is the KPI equation for which the solitary wave is unstable; instead this equation supports "lump" solitons. Like the KdV equation, both KPI and KPII are integrable equations.

2.8 Benney-Roskes Equation

For systems with an isotropic dispersion relation, the two-dimensional extension of the NLS equation is, in the reference frame moving with the x-component c_{g0} of the group velocity in the x-direction

$$iA_t + \nu|A|^2 A + \delta A_{xx} + \delta_1 A_{yy} + QA = 0\,. \tag{38}$$

Here Q is a wave-induced mean flow expression, which satisfies a forced long-wave equation. The precise form depends on the particular physical system being considered. For water waves, where $c_0^2 = gh$, it is

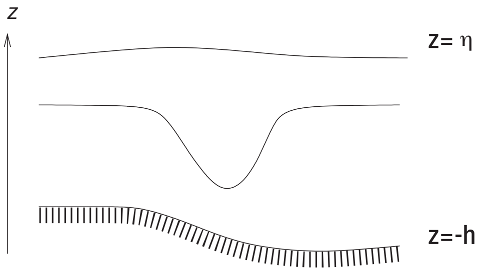
$$(1 - \frac{c_{g0}^2}{c_0^2})Q_{xx} + Q_{yy} + \nu_1 |A|^2_{yy} = 0\,. \tag{39}$$

The resulting system (38, 39) is the Benney-Roskes equations, also known as the Davey-Stewartson equations. The linear terms in (38) can be found by expanding the dispersion relation as in the one-dimensional case (29), so that for $k \approx k_0, l \approx 0$,

$$\omega - \omega_0 = c_{g0}\,(k - k_0) + \delta(k - k_0)^2 + \delta_1 l^2\,,$$

where, as before $\delta = \omega_{kk}(k_0, 0)/2 = c_{gk}(k_0, 0)/2$ and $\delta_1 = \omega_{ll}(k_0, 0) = c_{g0}/2k_0$. For water waves $\delta < 0, \delta_1 > 0$ and $c_{g0} < c_0$, so that (38) is hyperbolic, but (39) is elliptic.

In the basic state the fluid has **density** $\rho_0(z)$, a corresponding pressure $p_0(z)$ such that $p_{0z} = -g\rho_0$ describes the basic hydrostatic equilibrium, and a horizontal shear flow $u_0(z)$ in the $x-$direction. We shall consider only the case when the bottom is flat, that is h is a constant. Extensions to variable depth are possible and would lead to a variable-coefficient KdV equation.



3.2: Euler equations

In standard notation, the equations of motion relative to this basic state are

$$\begin{aligned}
\rho_0(u_t + u_0 u_x + w u_{0z}) + p_x = -\rho_0(u u_x + w u_z) \\
-\rho(u_t + u_0 u_x + w u_{0z} + u u_x + w u_z)\,, && (40) \\
p_z + g\rho = -(\rho_0 + \rho)(w_t + u_0 w_x + u w_x + w w_z)\,, && (41) \\
g(\rho_t + u_0 \rho_x) - \rho_0 N^2 w = -g(u\rho_x + w\rho_z)\,, && (42) \\
u_x + w_z = 0\,. && (43)
\end{aligned}$$

Here $(u_0 + u, w)$ are the velocity components in the (x, z) directions, $\rho_0 + \rho$ is the density, $p_0 + p$ is the pressure and t is time. $N(z)$ is the **buoyancy frequency**, defined by

$$\rho_0 N^2 = -g\rho_{0z}\,. \tag{44}$$

The boundary conditions are

$$\begin{aligned}
w = 0 \quad \text{at} \quad z = -h\,, && (45) \\
p_0 + p = 0\,, \quad \text{at} \quad z = \eta\,, && (46) \\
w = \eta_t + u_0\eta_x + u\eta_x \quad \text{at} \quad z = \eta\,. && (47)
\end{aligned}$$

Then to describe internal solitary waves we seek solutions whose horizontal length scales are much greater than h, and whose time scales are much greater than N^{-1}. We shall also assume that the waves have small amplitude. Then the dominant balance is obtained by equating to zero the terms on the left-hand side of (40 - 43), together with the linearization of the free surface boundary conditions. We then obtain the set of equations describing linear long wave theory. To proceed it is useful to use the **vertical particle displacement** ζ as the primary dependent variable. It is defined by

$$\zeta_t + u_0\zeta_x + u\zeta_x + w\zeta_z = w\,. \tag{48}$$

Note that it then follows that the perturbation density field is given by $\rho = \rho_0(z - \zeta) - \rho_0(z) \approx \rho_0 N^2\zeta$ as $\zeta \to 0$, where we have assumed that as $x \to -\infty$, the density field relaxes to its basic state. The isopycnal surfaces (i.e. $\rho_0 + \rho =$ constant) are then given by $z = z_0 + \zeta$ where z_0 is the level as $x \to -\infty$. In terms of ζ, the kinematic boundary condition (47) becomes simply $\zeta = \eta$ at $z = \eta$.

3.4: Linear long waves

Linear long wave theory is now obtained by omitting the right-hand side of equations (40 - 43), and simultaneously linearizing boundary conditions (46,47). Solutions are sought in the form

$$\zeta = A(x - ct)\phi(z)\,, \tag{49}$$

while the remaining dependent variables are then given by

$$u = A[(c - u_0)\phi]_z\,, \quad p = A\rho_0(c - u_0)^2\phi_z\,, \quad \rho = A\rho_0 N^2\phi\,. \tag{50}$$

Here c is the linear long wave speed, and the modal functions $\phi(z)$ are defined by the boundary-value problem,

$$\{\rho_0(c - u_0)^2\phi_z\}_z + \rho_0 N^2\phi = 0\,, \quad \text{for} \quad -h < z < 0\,, \tag{51}$$

$$\phi = 0 \quad \text{at} \quad z = -h\,, \quad (c - u_0)^2\phi_z = g\phi \quad \text{at} \quad z = 0\,, \tag{52}$$

Equation (51) is the well-known **Taylor-Goldstein** equation, here in the long-wave limit.

3.5: Linear long-wave modes

Typically, the boundary-value problem (51, 52) defines an infinite sequence of **linear long-wave modes**, $\phi_n^{\pm}(z)$, $n = 0, 1, 2, \ldots,$ with corresponding speeds $c_n^{\pm}$. Here, the superscript "$\pm$" indicates waves with $c_n^+ > u_M = \max u_0(z)$ and $c_n^- < u_M = \min u_0(z)$ respectively. We shall confine our attention to these regular modes, and consider only stable shear flows. Nevertheless, we note that there may also exist singular modes with $u_m < c < u_M$ for which an analogous theory can be developed. Note that it is useful to let $n = 0$ denote the **surface gravity waves** for which c scales with $\sqrt{gh}$, and then $n = 1, 2, 3, \ldots$ denotes the **internal gravity waves** for which c scales with Nh. In general, the boundary-value problem (51, 52) is readily solved numerically. Typically, the surface mode ϕ_0 has no extrema in the interior of the fluid and takes its maximum value at the surface $z = 0$, while the internal modes $\phi_n^{\pm}(z)$, $n = 1, 2, 3, \ldots,$ have $n - 1$ extremal points in the interior of the fluid, and vanish near $z = 0$ (and, of course, also at $z = -h$).

3.6: Linear long-wave modes

It can now be shown that, within the context of linear long wave theory, any localised initial disturbance will evolve into a set of outwardly propagating modes, each given by an expression of the form (49). Indeed, it can be shown that the solution of the linearised long wave equations is given asymptotically by

$$\zeta \sim \sum_{n=0}^{\infty} A_n^{\pm}(x - c_n^{\pm} t)\phi_n^{\pm}(z) \,, \quad \text{as} \quad t \to \infty \,. \tag{53}$$

Here the amplitudes $A_n^{\pm}(x)$ are determined from the initial conditions, Assuming thats the speeds $c_n^{\pm}$ of each mode are sufficiently distinct, it is sufficient for **large times to consider just a single mode**. Henceforth, we shall omit the indices and assume that the mode has speed c, amplitude A and modal function $\phi(z)$. Then, as time increases, the hitherto neglected **nonlinear terms** begin to have an effect, and cause wave steepening. However, this is opposed by the terms representing **linear wave dispersion**, also neglected in the linear long wave theory. A balance between these two effects emerges as time increases and the outcome is the KdV equation for the wave amplitude.

3.7: Asymptotic expansion

The formal derivation of the evolution equation requires the introduction of **two small parameters**, α and ϵ, respectively characterising the wave amplitude and dispersion. The **KdV balance** requires $\alpha = \epsilon^2$, with a corresponding **timescale** of ϵ^{-3}. The asymptotic analysis required is well understood, so we shall give only a brief outline here. We introduce the scaled variables

$$T = \epsilon\alpha t\,, \quad X = \epsilon(x - ct)\,, \tag{54}$$

and then put

$$\zeta = \alpha A(X,T)\phi(z) + \alpha^2\zeta_2 + \ldots\,, \tag{55}$$

with similar expressions analgous to (50) for the other dependent variables. At leading order, we get the linear long wave theory for the modal function $\phi(z)$ and the speed c, defined by the modal equations (51, 52). Since the modal equations are homogeneous, we are free to impose a normalization condition on $\phi(z)$. A commonly used condition is that $\phi(z_m) = 1$ where $|\phi(z)|$ achieves a maximum value at $z = z_m$. In this case the amplitude αA is uniquely defined as the amplitude of ζ (to $0(\alpha)$) at the depth z_m.

Then, at the next order, we obtain the equation for ζ_2,

$$\{\rho_0(c-u_0)^2\zeta_{2Xz}\}_z + \rho_0 N^2\zeta_{2X} = M_2\,, \quad \text{for} \quad -h < z < 0\,, \tag{56}$$

$$\zeta_{2X} = 0 \text{ at } z = -h\,, \quad \rho_0(c-u_0)^2\zeta_{2Xz} - \rho_0 g\zeta_{2X} = N_2 \text{ at } z = 0\,. \tag{57}$$

Here the **inhomogeneous terms** M_2, N_2 are known in terms of $A(X,T)$ and $\phi(z)$, and are given by

$$M_2 = 2\{\rho_0(c-u_0)\phi_z\}_z A_T + 3\{\rho_0(c-u_0)^2\phi_z^2\}_z AA_X \\ -\rho_0(c-u_0)^2\phi A_{XXX}\,, \tag{58}$$

$$N_2 = 2\{\rho_0(c-u_0)\phi_z\}A_T + 3\{\rho_0(c-u_0)^2\phi_z^2\}AA_X\,. \tag{59}$$

The left-hand side of equations (56, 57) are identical to the equations defining the modal function, that is 51, 52), and hence can be solved only if a certain compatibility condition is satisfied. In general the compatibility condition is that the inhomogenous terms in (56, 57) should be orthogonal to the solutions of the adjoint of the modal equations (51, 52). Here, rather than following this general procedure, we obtain the compatibility condition by a direct construction of ζ_2.

3.9: Compatibility condition

Thus a formal solution of (56) which satisfies the first boundary condition in (57) is

$$\zeta_{2X} = A_{2X}\phi + \phi \int_{-h}^{z} \frac{M_2\psi}{W}\,dz - \psi \int_{-h}^{z} \frac{M_2\phi}{W}\,dz\,, \tag{60}$$

$$\text{where} \qquad W = \rho_0(c-u_0)^2\{\phi_z\psi - \psi_z\phi\}\,. \tag{61}$$

Here $\psi(z)$ is a solution of the modal equation (51) which is linearly independent of $\phi(z)$, and so, in particular $\psi(-h) \neq 0$. Here W is the Wronskian and is a constant independent of z. Indeed, the expression (61) can be used to obtain ψ explicitly in terms of ϕ. Next, we insist that the expression (60) for ζ_{2X} should satisfy the second boundary condition in (57). The result is the compatibility condition

$$\int_{-h}^{0} M_2\phi\,dz = [N_2\phi]_{z=0}\,. \tag{62}$$

Note that the amplitude A_2 is left undetermined at this stage.

3.10 Korteweg-deVries equation

Substituting the expressions (58, 59) into (62) we obtain the required evolution equation for A, namely the **KdV equation**

$$A_T + \mu A A_X + \lambda A_{XXX} = 0\,. \tag{63}$$

Taking into account the scaling (54) this is just (26), where here the coefficients μ and λ are given by

$$I\mu = 3\int_{-h}^{0} \rho_0 (c - u_0)^2 \phi_z^3 \, dz\,, \tag{64}$$

$$I\lambda = \int_{-h}^{0} \rho_0 (c - u_0)^2 \phi^2 \, dz\,, \tag{65}$$

$$\text{where} \qquad I = 2\int_{-h}^{0} \rho_0 (c - u_0) \phi_z^2 \, dz\,. \tag{66}$$

The KdV equation (63) is to be solved with the initial condition $A(X, T = 0) = A_0(X)$ where $A_0(X)$ is determined from the linear long wave theory, and is in essence the projection of the original initial conditions onto the relevant linear long wave mode. As mentioned before, localized initial conditions lead to the generation of a finite number of solitary waves, or **internal solitons**.

3.11 Korteweg-de Vries equation

$$I\mu = 3\int_{-h}^{0} \rho_0 (c-u_0)^2 \phi_z^3 \, dz \,, \tag{67}$$

$$I\lambda = \int_{-h}^{0} \rho_0 (c-u_0)^2 \phi^2 \, dz \,, \tag{68}$$

$$\text{where} \qquad I = 2\int_{-h}^{0} \rho_0 (c-u_0) \phi_z^2 \, dz \,. \tag{69}$$

Confining attention to waves propagating to the right, so that $c > u_M = \max u_0(z)$, we see that I and λ are always positive. For the surface mode, $\phi_z > 0$ and $\phi(0) = 1$ so we see that $\mu > 0$. Further, recalling that for the internal modes the modal functions are normalised so that $\phi(z_m) = 1$ where z_m is an extremal point, then it is readily shown that for the usual situation of a near-surface pycnocline, μ is negative for the first internal mode. However, in general μ can take either sign, and in some special situations may even be zero. Explicit evaluation of the coefficients μ and λ requires knowledge of the modal function, and hence they are usually evaluated numerically.

3.12 Water waves

To illustrate the procedure, consider first the case of **water waves**. We put the density $\rho =$ constant so that then $N^2 = 0$ (44). Then

$$\phi = \frac{z+h}{h} \quad \text{for} \quad -h < z < 0\,, \quad c = (gh)^{1/2}\,. \tag{70}$$

$$\text{and so} \quad \mu = \frac{3c}{2h}\,, \quad \lambda = \frac{ch^2}{6}\,. \tag{71}$$

Thus the KdV equation for water waves is, in the original variables,

$$\zeta_t + c\zeta_x + \frac{3c}{2h}\zeta\zeta_x + \frac{ch^2}{6}\zeta_{xxx} = 0\,. \tag{72}$$

Note that here $z_m = 0$ so that $A = \zeta$, the free surface displacement, to leading order. For zero surface tension, this is the equation derived by Korteweg and de Vries in 1895 (and first by Boussinesq in 1870's).

3.13 Interfacial waves

Similarly, for **interfacial waves**, let the density be a constant ρ_1 in an upper layer of height h_1 and $\rho_2 > \rho_1$ in the lower layer of height $h_2 = h - h_1$. That is

$$\rho_0(z) = \rho_1 H(z + h_1) + \rho_2 H(-z - h_1)\,, \quad \rho_0 N^2 = g(\rho_2 - \rho_1)\delta(z + h_1)\,.$$

Here $H(z)$ is the Heaviside function and $\delta(z)$ is the Dirac δ-function. For simplicity, we shall assume that $\rho_1 \approx \rho_2$, the usual situation in the ocean, and also then the upper boundary condition for $\phi(z)$ becomes just $\phi(0) \approx 0$. Then we find that

$$\phi = \frac{z + h}{h_2} \text{ for } -h < z < h_1\,, \quad \phi = -\frac{z}{h_1} \text{ for } -h_1 < z < 0\,, \tag{73}$$

$$c^2 = \frac{g(\rho_2 - \rho_1)}{\rho_2}\frac{h_1 h_2}{h_2 + h_2}\,, \quad \mu = \frac{3c(h_1 - h_2)}{h_1 h_2}\,, \quad \lambda = \frac{c h_1 h_2}{6}\,. \tag{74}$$

Note that the nonlinear coefficient μ for these interfacial waves is negative (positive) when $h_1 < (>) h_2$ (that is, the when interface is closer (further) to the free surface than to the bottom), The case when $h_1 \approx h_2$ leads to the necessity to use an extended KdV equation which contains a cubic nonlinear term.

3.14 Higher-order KdV equations

Proceeding to the next highest order will yield an equation set analogous to (56, 57) for ζ_3, whose compatibility condition then determines an evolution equation for the second-order amplitude A_2. We shall not give details here, but note that using the transformation $A + \alpha A_2 \to A$, and then combining the KdV equation (63) with the evolution equation for A_2 will lead to a **higher-order KdV** equation

$$A_T + \mu A A_X + \lambda A_{XXX}$$
$$+\alpha\{\lambda_1 A_{XXXXX} + \sigma A^2 A_X + \mu_1 A A_{XXX} + \mu_2 A_X A_{XX}\} = 0 . \qquad (75)$$

Explicit expressions for the coefficients are known. Note that to be **Hamiltonian**, $\mu_2 = 2\mu_1$. However, this equation is not unique, as the near-identity transformation $A \to A + \alpha(aA^2 + bA_{XX})$ asymptotically reproduces the same equation but with altered coefficients,

$$(\lambda_1, \sigma, \mu_1, \mu_2) \to (\lambda_1, \sigma - a\mu, \mu_1, \mu_2 - 6a\lambda + 2b\mu) .$$

Further when $\mu \neq 0, \lambda \neq 0$, the enhanced transformation,

$$A \to A + \alpha(aA^2 + bA_{XX} + a' A_X \int^X A\, dX + b' X A_T) ,$$

can asymptotically reduce (75) to the KdV equation.

3.15 Extended KdV equation

A particularly impotant special case arises when the coefficient μ (63) is close to zero. Then the cubic nonlinear term in the higher-order KdV equation is the most important, and the KdV equation (63) is replaced by the **extended KdV** (or Gardner) equation,

$$A_T + \mu A A_X + \alpha\sigma A^2 A_X + \lambda A_{XXX} = 0\,. \tag{76}$$

For $\mu \approx 0$, a rescaling is needed and the optimal choice is to assume that μ is $0(\epsilon)$, and then replace A with A/ϵ. In effect the amplitude parameter is ϵ in place of ϵ^2. This is an integrable equation, in canonical form

$$A_t \, + \, 6AA_x \, + \, 6\beta A^2 A_x \, + \, A_{xxx} \, = \, 0\,. \tag{77}$$

Like the KdV equation, the Gardner equation is integrable by the inverse scattering transform. Here the coefficient β can be either positive or negative, and the structure of the solutions depends crucially on which sign is appropriate.

3.16. Extended KdV Equation

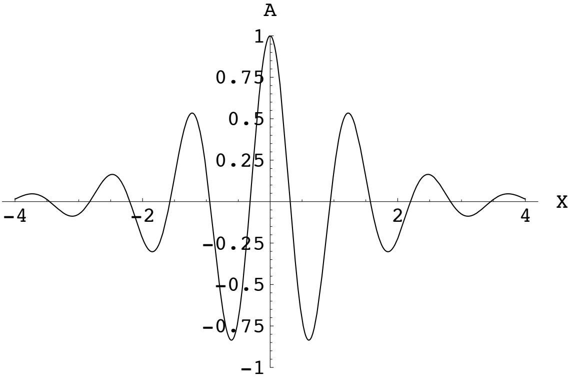
The solitary wave solutions are given by

$$A = \frac{a}{b + (1-b)\cosh^2 \gamma(x - Vt)}\,, \tag{78}$$

$$\text{where} \quad V = a(2+\beta a) = 4\gamma^2\,, \quad b = \frac{-\beta a}{(2+\beta a)}\,. \tag{79}$$

There are two cases to consider. If $\beta < 0$, then there is a single family of solutions such that $0 < b < 1$ and $a > 0$. As b increases from 0 to 1, the amplitude a increases from 0 to a maximum of $-1/\beta$ while the speed V also increases from 0 to a maximum of $-1/\beta$. In the limiting case when $b \to 1$ the solution (78) describes the so-called "thick" solitary wave, which has a flat crest of amplitude $a_m = -1/\beta$. The figure shows solitary wave solutions (78) of the extended KdV equation; upper panel for $\beta < 0$; lower panel for $\beta > 0$.

4.1 Wave packets



Plot of a wave packet, in scaled coordinates, for a case when the ratio of the carrier wavenumber to the envelope wavenumber is 5.

4.2 Wave packets

First, we recall from the first lecture, that the linearized system supports sinusoidal waves, described by the kinematic expression (1), that is

$$u = A \exp\left(ikx - i\omega t\right), \tag{80}$$

Here u is one of the relevant physical flow variables, k is the wavenumber, ω is the wave frequency, while A is the constant complex-valued amplitude. The dynamics of the system will lead to the **dispersion relation** (2), that is

$$\omega = \omega(k), \tag{81}$$

defining the frequency as a function of wavenumber. The phase velocity $c = \omega/k$ is likewise a function of wavenumber. Here we consider only stable waves for which ω is real-valued for all real wavenumbers k. For instance, for gravity-capillary waves the dispersion relation is

$$\frac{c^2}{gh} = \frac{(1 + Bq^2)}{q} \tanh q, \qquad q = kh, \tag{82}$$

Here $B = \Sigma/gh^2$ is the Bond number, where $\rho\Sigma$ is the coefficient of surface tension (ρ is the water density), and has a value of 74 $dynes/cm$ at $20^o C$.

4.3 Narrow-band wave packet

Next, as described in the first lecture, we can use Fourier superposition to obtain the more general expression (6), that is

$$u(x,t) = \int_{-\infty}^{\infty} F(k)\exp\left(ikx - i\omega(k)t\right)dk\,, \tag{83}$$

where $F(k)$ is the Fourier transform of $u(x,0)$. To obtain a wave packet, we suppose that $F(k)$ has a dominant component centred at $k = k_0$. The dispersion relation is then approximated by (10), that is

$$\omega = \omega_0 + c_g(k - k_0) + \delta(k - k_0)^2, \quad c_g = \frac{d\omega}{dk}\,, \quad \delta = \frac{1}{2}\frac{d^2\omega}{dk^2}\,. \tag{84}$$

Here, both c_g, δ are evaluated at $k = k_0$. The expression (83) then becomes

$$u(x,t) \approx A(x,t)\exp\left(i(k_0x - \omega_0 t)\right), \tag{85}$$

$$\text{where} \quad A(x,t) = \int_{-\infty}^{\infty} F(k_0 + \tau)\exp\left(i(\tau(x - c_g t)) - i\delta\tau^2 t\right)d\tau\,, \tag{86}$$

and the variable of integration has been changed from k to $\tau = k - k_0$.

4.3 Narrow-band wave packet

$$u(x,t) \approx A(x,t)\exp\left(i(k_0 x - \omega_0 t)\right),$$

$$A(x,t) = \int_{-\infty}^{\infty} F(k_0+\tau)\exp\left(i(\tau(x-c_g t)) - i\delta\tau^2 t\right)d\tau\,.$$

Here the sinusoidal factor $\exp\left(i(k_0 x - \omega_0 t)\right)$ is a **carrier** wave with a phase velocity ω_0/k_0, while the (complex) amplitude $A(x,t)$ describes the **envelope** thus forming a wave packet.

It is important to note here that within this narrow-band approximation, the envelope amplitude A satisfies the **linear Schrödinger equation**

$$i(A_t + c_g A_x) + \delta A_{xx} = 0\,. \tag{87}$$

The first two term (in the brackets) are the dominant terms, and we conclude that to leading order the wave envelope propagates with the **group velocity** $c_g = d\omega/dk$.

4.4 Weakly nonlinear waves

The theory of linearised waves is valid when the initial conditions are such that the waves have sufficiently small amplitudes. However, after a sufficiently long time (or if the initial conditions describe waves of moderate or large amplitudes), the effects of the nonlinear terms in the physical system need to be taken into account. As discussed previously, there are two principal cases when weak nonlinearity needs to be taken into account, namely, long waves and wave packets. The first case has been discussed in the third lecture and it is the second case which is of interest here.

As described above, the linear theory predicts that a localised initial state will typically evolve into wave packets, with a dominant wavenumber k and corresponding frequency $\omega(k)$ given by (81), within which each wave phase propagates with the phase speed c, but whose envelope propagates with the group velocity c_g. Note that we will henceforth replace k_0, ω_0 of the previous discussion with k, ω for notational convenience. After a long time, the packet tends to disperse around the dominant wavenumber, which tendency is opposed by **cumulative nonlinear effects**.

4.5 Nonlinear Schrödinger (NLS) equation

In order to describe this situation, we replace the linear expression (13) with

$$u = \epsilon A(X,T)\exp(i\theta) + \text{c. c.} + \cdots, \tag{88}$$

$$\text{where} \quad \theta = kx - \omega(k)t\,,\; X = \epsilon(x - c_g t)\,,\; T = \epsilon^2 t\,. \tag{89}$$

Here $\epsilon << 1$ is a small parameter measuring the wave amplitude and we have scaled the linear dispersive effects to balance the leading order nonlinear effects, while the omitted terms are $O(\epsilon^2)$; the notation c. c. denotes the complex conjugate. The outcome for these uni-directional waves is described by the **nonlinear Schrödinger (NLS) equation**

$$iA_T + \delta A_{XX} + \mu|A|^2 A = 0\,. \tag{90}$$

Here the coefficient δ of the linear dispersive term is defined by (84) and we note that the linear part of (90) agrees with (87) when we take account of the rescaling (89). The coefficient μ of the nonlinear term needs to be found separately for each specific physical system. The NLS equation is integrable in both the focussing ($\delta\mu > 0$) and the defocussing ($\delta\mu < 0$) cases.

4.6 Derivation for water waves

In order to indicate how this is derivation proceeds, we shall give a brief description for the case of water waves. In this case the coefficient μ is given by

$$\mu = -\frac{k^2\omega}{4\sigma^4}(9\sigma^4 - 10\sigma^2 + 9) + \frac{gk\omega}{2\sigma^2(gh - c_g^2)}(2\sigma(3 - \sigma^2) + 3q(1 - \sigma^2)^2)\,, \tag{91}$$

$$\text{where} \quad \sigma = \tanh q\,, q = kh\,.$$

Note that the first term is always negative and the second term is always positive. In deep water ($q \to \infty$) the second term vanishes, and the coefficient $\mu \to -2\omega k^2 < 0$. In general $\mu < 0 (> 0)$ according as $q > q_c (q < q_c)$, where $q_c = 1.36$.

4.7 Derivation for gravity-capillary waves

The full Euler equations for an inviscid, incompressible fluid in irrotational flow can be reduced to the solution of Laplace's equation for a velocity potential $\phi = \phi(x, z, t)$,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \,, \quad -h < z < \zeta \,. \tag{92}$$

Here $z = \zeta(x, t)$ denotes the location of the free surface, and we will restrict our attention to the two-dimensional case when there is no dependence on the horizontal transverse variable y. This is to be solved with boundary conditions at the rigid bottom $z = -h$ and at the free surface,

$$\phi_z = 0 \quad \text{at} \quad z = -h \,, \tag{93}$$

$$\zeta_t + u\zeta_x = w \,, \quad \text{at} \quad z = \zeta \,, \tag{94}$$

$$\phi_t + g\zeta + \frac{1}{2}|\mathbf{u}|^2 = \Sigma \frac{\zeta_{xx}}{(1 + \zeta_x^2)^{3/2}} \,, \quad \text{at} \quad z = \zeta \,. \tag{95}$$

Here $\mathbf{u} = (u, w) = (\phi_x, \phi_z)$ is the velocity field, and we recall that $\rho\Sigma$ is the coefficient of surface tension (ρ is the water density), included here as it leads to some important issues.

4.8 Derivation for gravity-capillary waves

To derive the NLS equation (90), we now seek an asymptotic expansion of the form (88),

$$\zeta = \epsilon A(X, T) \exp(i\theta) + \text{c. c.} + \epsilon^2 \zeta^{(2)} + \epsilon^3 \zeta^{(3)} + \cdots , \tag{96}$$

where the phase variable θ and the slow variables X, T are defined by (89). There are analogous expansions for u, w and ϕ. At the leading order we obtain the dispersion relation (82). It is convenient to write this in the form,

$$D(\omega, k) \equiv \omega^2 - gk(1 + Bq^2) \tanh q = 0 , \tag{97}$$

where we recall that $q = kh$. At the next order, we find that the envelope A moves with group velocity, a result already anticipated in the scaling, see (54).

4.9 Derivation for gravity-capillary waves

We then seek expressions for the **second harmonic** component $\zeta^{(2)}$ in the form

$$\zeta^{(2)} = \zeta_2^{(2)}(X, T) \exp(2i\theta) + \text{c.c} + \zeta_0^{(2)}(X, T) \,. \tag{98}$$

To leading order the second harmonic $\zeta_2^{(2)}$ is governed by an equation of the form

$$D(2\omega, 2k)\zeta_2^{(2)} = \nu_2 A^2 \,. \tag{99}$$

where ν_2 is a complex-valued constant which depends on q, B. Then provided that there is no **second harmonic resonance**, that is $D(2\omega, 2k) \neq 0$ (and so $k \neq k_S$ where $D(2\omega(k_S), 2k_S) = 0$), it follows that to leading order,

$$\zeta_2^{(2)} = \frac{\nu_2 A^2}{D(2\omega, 2k)} \,. \tag{100}$$

4.10 Derivation for gravity-capillary waves

The **mean term** $\zeta_0^{(2)}$ is real-valued and satisfies an equation of the form

$$\frac{\partial^2 \zeta_0^{(2)}}{\partial t^2} - gh\,\frac{\partial^2 \zeta_0^{(2)}}{\partial x^2} = \nu_0\,\epsilon^2 |A|^2_{XX}\,, \tag{101}$$

where ν_0 is a real-valued constant which depends on q, B. The simplest method to obtain this equation is to average the full Euler equations over the phase θ; the left-hand side of (101) is easily recognised as the long-wave operator, as should be expected. Indeed, taking the limit $\omega, k \to 0$ in (97) gives $D(\omega, k) \approx \omega^2 - ghk^2$ which is the Fourier transform of the left-hand side of (101). Then, taking account of the fact that $\zeta_0^{(2)}$ depends on the slow variables X, T, it is readily seen that this equation can be solved at leading order by a term of the form $\zeta_0^{(2)} = \nu_0 |A|^2/(c_g^2 - gh)$. Note that we must exclude any possible **long-short wave resonance** when $q = q_L$ or $c_g^2 = gh$.

4.11 Wave resonances

It is pertinent to note here that the possible occurrence of a second harmonic resonance, or a long-short wave resonance, at certain wavenumbers is indicative of the possible presence of **resonant wave triads**. That is the wave with a wavenumber-frequency pair (k, ω) may be in resonance with the pairs (k_1, ω_1) and (k_2, ω_2), that is, whenever

$$k_1 + k_2 = k, \;\; \omega_1 + \omega_2 = \omega \, . \tag{102}$$

Such triads exist for capillary-gravity waves when $0 < B < 1/3$, but **not for water waves ($B = 0$)**, or for $B > 1/3$. Second harmonic resonance occurs for $k_1 = k_2, \omega_1 = \omega_2$, and long-wave resonance when $k_2 \approx 0$. The presence of a resonant triad will allow a modulation of the primary wave on a time-scale of ϵ^{-1}, which is an order of magnitude faster than that being considered here. Thus, in this circumstance, for the validity of the NLS model, it is necessary to suppose *a priori* that the wavenumber spectrum is sufficiently narrow-banded to exclude any resonant triads.

At the third order in ϵ, the term of interest in $\zeta^{(3)}$ is that proportional to $\exp(i\theta)$, say $\zeta_1^{(3)} \exp i\theta$. At leading order we get an equation of the form

$$D(\omega, k)\zeta_1^{(3)} = 2\omega(iA_T + \delta A_{XX}) + i\mu_2\zeta_2^{(2)}A^* + i\mu_0\zeta_0^{(2)}A + i\mu_1 A^2 A^* . \tag{103}$$

Here μ_0, μ_1, μ_2 are constants depending on q, B and A^* is the complex conjugate of A. The origin of the linear terms on the right-hand side of (103) has already been discussed, see (87). The first two nonlinear terms arise due to the quadratic interaction of the second harmonic and the mean with the primary harmonic, while the third nonlinear term is due to the cubic self-interaction of the primary harmonic. SInce the dispersion relation (97) holds the left-hand side of (103) is zero, and so the NLS equation (90) is obtained with

$$2\omega\mu = \frac{\mu_2\nu_2}{D(2\omega, 2k)} + \frac{\mu_0\nu_0}{c_g^2 - gh} + \mu_1 . \tag{104}$$

The NLS equation (90) is a canonical model for weakly nonlinear wave packets in many physical systems, including a variety of fluid flows, and very prominently in **nonlinear optics**. Like the KdV equation, the NLS equation is integrable with an associated inverse scattering transform, a result first shown by Zakharov and Shabat in 1972. There are two cases, the so-called **focussing** NLS equation when $\delta\mu > 0$ and the **defocussing** NLS equation when $\delta\mu < 0$. For water waves without surface tension $2\delta = \partial c_g/\partial k < 0$, and so we have the focussing (defocussing) NLS equation according as $\mu < 0 (> 0)$, that is, as above from (91), as $q (= kh) > 1.36 (q < 1.36)$.

4.14 Envelope solitons

The focussing NLS equation has solitary wave solutions, **bright solitons**, given by

$$A = a\,\mathrm{sech}(\gamma X)\exp(-i\sigma T)\,, \text{ where } \mu a^2 = 2\delta\gamma^2\,,\ \sigma = -\frac{1}{2}\mu a^2\,. \quad (105)$$

Note that this one-parameter family can be extended to a two-parameter family through the gauge transformation

$$X \to X + VT\,, \quad A \to A\exp(iPX + i\delta P^2 T)\,, \quad \text{where} \quad V = 2\delta P\,. \quad (106)$$

Of course, this gauge transformation amounts to a small shift $k \to k + \epsilon P$ in the carrier wavenumber, with consequent small shifts in the carrier frequency and group velocity. On the other hand the defocussing NLS equations has no such solitary wave solutions which decay to zero at infinity; instead it has solitary waves riding on a non-zero background, **dark solitons**.

4.15 Modulational instability

A key property of the NLS equation is that plane waves are **modulationally unstable (stable)** in the focussing (defocussing) case. That is, the NLS equation has the exact plane wave solution,

$$A = A_0 \exp(i\mu|A_0|^2 t)\,, \tag{107}$$

which is then perturbed with a small-amplitude modulation proportional to $\exp(iKx + \Omega t)$. It is readily found that the growth rate Ω is given by

$$\Omega^2 = K^2(2\delta\mu|A_0|^2 - \delta^2 K^2)\,. \tag{108}$$

Thus in the focussing NLS case when $\delta\mu > 0$ there is a positive growth rate for modulation wavenumbers K such that $K^2 < 2\mu|A_0|^2/\delta|$. On the other hand Ω is pure imaginary for all K in the defocussing case when $\delta\mu < 0$. The maximum growth rate occurs for $K = K_M = (\mu/\delta)^{1/2}|A_0|$ and the instability is due to the generation of side bands with wavenumbers $k \pm K_M$. As the instability grows the full NLS equation (90) is needed to describe the long-time outcome of the collapse of the uniform plane wave into several soliton wave packets, each described by (105).

4.16 Modulational instability

The implication for **water waves** (that is, the Bond number $B = 0$) is that plane Stokes waves in **deep water ($q = kh > 1.36$) are unstable**. This remarkable result was first discovered by Benjamin and Feir in 1967 using a different theoretical approach and in some experiments, again by Zakharov in 1968 from the NLS equation, and has since been confirmed in several experiments.

Resonant Wave Interactions: Introduction

Let $\omega = \omega(\mathbf{k})$ be a linear dispersion relation of some system of nonlinear dispersive PDEs. We assume that solutions of the linearised equation can be obtained by a superposition of plane waves (Fourier harmonics):

$$A \exp[i(\mathbf{kx} - \omega t)],$$

where ω is a wave frequency, $\mathbf{k}$ is a wavevector (or wavenumber).

To admit 3-wave interactions
(i) the system must admit a resonant triad, i.e. 3 linear waves satisfying

$$\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3} = 0, \quad \omega_1 + \omega_2 + \omega_3 = 0$$

(ii) the system should have quadratic nonlinearities

Resonant quartets satisfying

$$\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3} + \mathbf{k_4} = 0, \quad \omega_1 + \omega_2 + \omega_3 + \omega_4 = 0$$

may arise at the next order.
Resonant wave interactions play an important role in nonlinear optics, plasma physics, fluid mechanics, etc.

5.1. Three-wave interactions

Let us consider a class of scalar PDEs of the form

$$L(\frac{\partial}{\partial t};\frac{\partial}{\partial x})\phi=\sum_i (M^{(i)}\phi)(N^{(i)}\phi)+\sum_i (P^{(i)}\phi)(Q^{(i)}\phi)(R^{(i)}\phi)+\ldots, \quad (1)$$

where L, M, N, P, Q, R are scalar linear differential operators in $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$. We assume that equations are already expanded in the vicinity of a stable equilibrium, and $\phi=0$ is a solution of (1). We then seek a solution in the form

$$\phi=\varepsilon\phi^{(1)}+\varepsilon^2\phi^{(2)}+\ldots, \quad (2)$$

where

$$\frac{\partial}{\partial t}\to\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial T_1}+\ldots.; \quad \frac{\partial}{\partial x}\to\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial X_1}+\ldots;$$
$$T_1=\varepsilon t, \quad X_1=\varepsilon x.$$

Then, the operator L can be written as an expansion in ε,

$$L(\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial T_1}+\ldots;\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial X_1}+\ldots)\to L(\frac{\partial}{\partial t};\frac{\partial}{\partial x})+\varepsilon(L_1\frac{\partial}{\partial X_1}+L_2\frac{\partial}{\partial T_1})+\ldots \quad (3)$$

5.1. Three-wave interactions

Subsituting (2) and (3) into (1) and collecting equal powers of ε, we obtain

$$O(\varepsilon): \quad L\phi^{(1)} = 0, \tag{4}$$

$$O(\varepsilon^2): \quad L\phi^{(2)} = -(L_1 \frac{\partial}{\partial X_1} + L_2 \frac{\partial}{\partial T_1})\phi^{(1)} + \sum_i (M^{(i)}\phi^{(1)})(N^{(i)}\phi^{(1)}) \tag{5}$$

Equation (4) is the linearisation of equation (1). Let's take its solution to be

$$\phi^{(1)} = \sum_{n=1}^{3} A_n(T_1, X_1)\exp(i\theta_n) + c.c.,$$

where $\theta_n = k_n x - \omega_n t, \quad n = 1, 2, 3,$ and

$$k_1 + k_2 + k_3 = 0, \quad \omega_1 + \omega_2 + \omega_3 = 0$$

is assumed to be a single resonant triad. (The quadratic nonlinearities in (5) give rise to various harmonics. The equality

$$\theta_1 + \theta_2 + \theta_3 = 0$$

means that three of them will be resonant.)

5.1. Three-wave interactions

Let us denote

$$L(\frac{\partial}{\partial t};\frac{\partial}{\partial x})\exp(i\theta_n) = D(\omega_n, k_n)\exp(i\theta_n)$$

Here, $D(\omega, k) = 0$ is the dispersion relation. Then, in $O(\varepsilon^2)$ we obtain

$$\begin{aligned}
L\phi^{(2)} &= -i\sum_{n=1}^{3}\left(D_{\omega_n}\frac{\partial A}{\partial T_1} - D_{k_n}\frac{\partial A}{\partial X_1}\right)\exp(i\theta_n)\\
&+ (m_1^* n_2^* + m_2^* n_1^*)A_1^* A_2^* \exp(-i(\theta_1+\theta_2))\\
&+ (m_2^* n_3^* + m_3^* n_2^*)A_2^* A_3^* \exp(-i(\theta_2+\theta_3))\\
&+ (m_3^* n_1^* + m_1^* n_3^*)A_3^* A_1^* \exp(-i(\theta_3+\theta_1))\\
&+ c.c. + \text{other quadratic and higher-order terms}
\end{aligned}$$

Here, $D_{\omega_n} = \frac{\partial D}{\partial \omega}|_{k=k_n,\omega=\omega_n}$; $\quad D_{k_n} = \frac{\partial D}{\partial k}|_{k=k_n,\omega=\omega_n}$. Note, that

$$D_k + \frac{d\omega}{dk}D_\omega = 0.$$

5.1. Three-wave interactions

In solving for $\phi^{(2)}$ we run into a problem: the right hand side "resonates" with the solution of the homogeneous equation, i.e. the particular solution for $\phi^{(2)}$ will contain fast growing terms (so-called "secular terms"), making our asymptotic expansion invalid for $t > \varepsilon^{-1}$. To avoid secular growth we require

$$\begin{aligned}
\left(\frac{\partial}{\partial T_1} + c_1 \frac{\partial}{\partial X_1}\right) A_1 &= i\mu_1 A_2^* A_3^*, \\
\left(\frac{\partial}{\partial T_1} + c_2 \frac{\partial}{\partial X_1}\right) A_2 &= i\mu_2 A_3^* A_1^*, \\
\left(\frac{\partial}{\partial T_1} + c_3 \frac{\partial}{\partial X_1}\right) A_3 &= i\mu_3 A_1^* A_2^*
\end{aligned} \tag{6}$$

and obtain the three-wave resonance (or triad) equations. If the original system is conservative, then μ_i are real. The equations are integrable by the IST (Zakharov and Manakov 1973, 1976; Kaup 1976).

See Appendix 1 for the alternative derivation for Lagrangian systems.

5.2. Four-wave interactions

Three-wave interactions appear as the first nonlinear corrections to a linear theory if secular terms arise at $O(\varepsilon^2)$. However, if no secular terms arise at this order, then these equations need not be satisfied, and resonant quartets may arise instead at $O(\varepsilon^3)$. An important example is given by water waves for which the dispersion relation $\omega = \pm\sqrt{g|\mathbf{k}|}$ does not allow resonant triads. Also, four-wave resonances are crucial for equations whose lowest order nonlinearity is cubic. The resonance conditions are

$$\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3} + \mathbf{k_4} = 0, \quad \omega_1 + \omega_2 + \omega_3 + \omega_4 = 0.$$

Introducing the slow variables $X_2 = \varepsilon^2 x$ and $T_2 = \varepsilon^2 t$, one may obtain

$$\left(\frac{\partial}{\partial T_2} + c_m \frac{\partial}{\partial X_2}\right) A_m = \sum_{p=1}^{4} \mu_{mp} A_m A_p A_p^* + \sum_{q,r,s \neq m} \nu_{mqrs} A_q^* A_r^* A_s^*$$

(Benney and Newell 1967).

See Appendix 2 for the derivation of the equations for Lagrangian systems.

5.3. A graphical method for finding the resonant triads

The graphical method described below was suggested by Ziman 1960, Ball 1964. First, pick any point A on one of the branches of the dispersion curve. Then reproduce all branches of the dispersion relation, with the origin translated from O to A (dashed lines). Each intersection of a dashed and a solid curve (e.g., the point B) represents a second wave that can participate with A in a resonant triad. From B, draw a vector parallel and equal to $\underline{AO}$. By construction, this vector ends on a dispersion curve, at C. Therefore, the points A, B and C all lie on a dispersion curve, and $\underline{OA} + \underline{OC} = \underline{OB}$, so that $\omega_A + \omega_C + (-\omega_B) = 0$ and $k_A + k_C + (-k_B) = 0$.

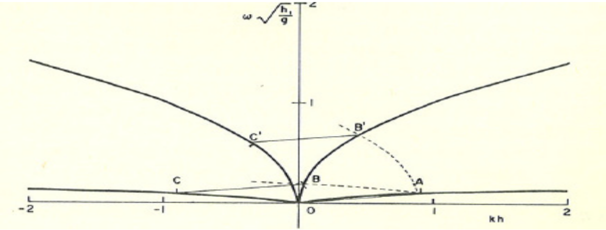


Figure: from Ablowitz & Segur 1981, p. 308 (dispersion relation for surface and internal waves in infinitely deep 2-layer fluid).

5.4. Solution of triad equations with no spatial variation (or for travelling waves)

Let us suppose that the spatial variation of the amplitudes can be neglected. The scaling

$$A_1 = -\frac{ia_1}{\sqrt{|\mu_2\mu_3|}}, \; A_2 = -\frac{ia_2}{\sqrt{|\mu_1\mu_3|}}, \; A_3 = -\frac{ia_3}{\sqrt{|\mu_1\mu_2|}}$$

yields

$$\dot{a}_1 = \sigma_1 a_2^* a_3^*, \quad \dot{a}_2 = \sigma_2 a_3^* a_1^*, \quad \dot{a}_3 = \sigma_3 a_1^* a_2^*, \tag{7}$$

where $\sigma_n = sign\ \mu_n$. The same equations can be obtained for the travelling waves

$$a_n = a_n(X_1 - vT_1), \quad n = 1, 2, 3,$$

rescaling $\mu_n \to \frac{\mu_n}{c_n - v}$ (e.g., $\sigma_n = sign\ \frac{\mu_n}{c_n - v}$).

The qualitative behaviour of (7) can be inferred from the Manley-Rowe relations

$$\frac{d}{dt}(\sigma_1|a_1|^2 - \sigma_2|a_2|^2) = \frac{d}{dt}(\sigma_2|a_2|^2 - \sigma_3|a_3|^2) = \frac{d}{dt}(\sigma_3|a_3|^2 - \sigma_1|a_1|^2) = 0 \tag{8}$$

5.4. Solution of triad equations with no spatial variation (or for travelling waves)

Motivating example ('parametric resonance' instability):

Consider the case $a_1 \approx const, \quad a_2, a_3 \ll a_1.$

Linearised ('pump wave') approximation has the form:

$$\dot{a}_1 = 0, \quad \dot{a}_2 = \sigma_2 a_1^* a_3^*, \quad \dot{a}_3 = \sigma_3 a_1^* a_2^*,$$

implying $\ddot{a}_k = \sigma_2\sigma_3|a_1|^2 a_k \quad (k = 2, 3).$

If $\sigma_2\sigma_3 = -1$, then a_2 and a_3 are periodic, with frequency $|a_1|$.

If $\sigma_2\sigma_3 = 1$, then a_2 and a_3 are exponential, with growth and decay rates $\pm|a_1|$.

5.4. Solution of triad equations with no spatial variation (or for travelling waves)

There are 2 cases to consider:

(i) one of the interaction coefficients has a different sign from the other two, e.g.

$$-\sigma_1 = \sigma_2 = \sigma_3 = 1$$

(ii) the three interaction coefficients have the same sign

$$\sigma_1 = \sigma_2 = \sigma_3 = 1$$

(other sign combinations reduce to that by changes of signs of a_n).

Case (i) is the most common. The Manley-Rowe relations (8) then indicate that the three amplitudes are bounded by the maximum of the initial amplitudes. The three waves exchange energy periodically.

In case (ii) the Manley-Rowe relations impose no bounds on the wave amplitudes. The amplitudes can grow simultaneously and so rapidly that they develop a singularity in a finite time. This is called explosive resonant interaction. It is shown to take place in fluid mechanics if the equilibrium configuration consists of a stably stratified fluid with shear (Craik and Adam 1979, Vanneste 1995).

5.4. Solution of triad equations with no spatial variation (or for travelling waves)

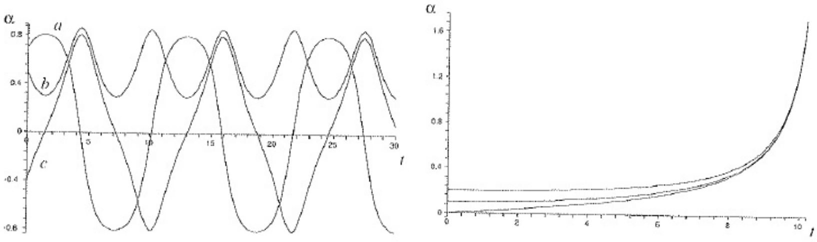


Figure: from Grimshaw (ed.) 2005, p. 79 (some typical solutions of the triad equations with no spatial variation: left panel has been obtained in case (i), right panel has been obtained in case (ii)).

5.4. Solution of triad equations with no spatial variation (or for travelling waves)

The triad equations (7) admit the qualitative phase plane analysis, and can be solved in closed form in terms of elliptic functions, which generally play a prominant role in nonlinear wave theory. To do that, we write $a_i = |a_i| e^{i\phi_i}$ and derive from (7)

$$\frac{d|a_1|}{dt} = \sigma_1 |a_2||a_3| \cos\theta, \quad \frac{d|a_2|}{dt} = \sigma_2 |a_3||a_1| \cos\theta,$$
$$\frac{d|a_3|}{dt} = \sigma_3 |a_1||a_2| \cos\theta, \tag{9}$$

where $\theta = \phi_1 + \phi_2 + \phi_3$, as well as

$$\frac{d\phi_1}{dt} = -\frac{\sigma_1 |a_2||a_3|}{|a_1|} \sin\theta, \quad \frac{d\phi_2}{dt} = -\frac{\sigma_2 |a_3||a_1|}{|a_2|} \sin\theta,$$
$$\frac{d\phi_3}{dt} = -\frac{\sigma_3 |a_1||a_2|}{|a_3|} \sin\theta. \tag{10}$$

Equations (10) imply $\frac{d\theta}{dt} \cos\theta + \frac{d}{dt}[\ln|a_1 a_2 a_3|] \sin\theta = 0,$ yielding the integral of motion $W = |a_1 a_2 a_3| \sin\theta = const.$

5.4. Solution of triad equations with no spatial variation (or for travelling waves)

Using this integral and integrals following from the Manley-Rowe relations, we obtain

$$\begin{aligned}\frac{d|a_1|^2}{dt} &= 2\sigma_1|a_1a_2a_3|\cos\theta = 2\sigma_1\sqrt{|a_1a_2a_3|^2(1-\sin^2\theta)} \\ &= 2\sigma_1\sqrt{|a_1|^2\sigma_2(\sigma_1|a_1|^2-C_3)\sigma_3(\sigma_1|a_1|^2-C_2)-W^2}, \quad (11)\end{aligned}$$

where $C_3 = \sigma_1|a_1|^2 - \sigma_2|a_2|^2 = const$, $C_2 = \sigma_1|a_1|^2 - \sigma_3|a_3|^2 = const$, and similar equations for $|a_2|^2$ and $|a_3|^2$.
This equation yields

$$\frac{1}{2}\dot{Z}^2 + U(Z) = 0,$$

where

$$U(Z) = 2[W^2 - \sigma_1\sigma_2\sigma_3 Z(Z-C_2)(Z-C_3)],$$

allowing for the phase-plane analysis.

5.4. Solution of triad equations with no spatial variation (or for travelling waves)

The equation can be solved in elliptic functions. For example, when $\sigma_1\sigma_2\sigma_3 = -1$, it can be rewritten as $\frac{dy}{d\tau} = \sqrt{(y_3 - y)(y - y_2)(y - y_1)}$, where $y_1 < y_2 < y < y_3$, by the appropriate rescaling of the variables, and solved in elliptic functions as

$$y = y_3 - (y_3 - y_2)\, sn^2\, \left(\frac{\sqrt{y_3 - y_1}}{2}(\tau - \tau_0), m\right), \quad m = \frac{y_3 - y_2}{y_3 - y_1}.$$

After that, the phases can be reconstructed by quadratures.

5.5. Remarks about elliptic functions

By the end of the 18th century it was realised that elliptic integrals $\int \frac{dx}{\sqrt{P(x)}}$, where $P(x)$ is a cubic or quartic polynomial, could not be expressed in terms of elementary functions (or their inverses). The elliptic functions have been introduced in 19th century as inversions of the elliptic integrals. There are two standard forms, known as Jacobi elliptic functions and Weierstrass elliptic functions. Jacobi elliptic functions are used to solve differential equations of the form

$$\frac{d^2x}{dt^2} = Ax^3 + Bx^2 + Cx + D.$$

Weierstrass elliptic functions are used to solve

$$\frac{d^2x}{dt^2} = Ax^2 + Bx + C.$$

The three main Jacobi functions are denoted as $sn\ (u, m)$, $cn\ (u, m)$ and $dn\ (u, m)$, where m is called the elliptic modulus. These functions arise from the inversion of the elliptic integral of the first kind,

$$u = F(\phi, m) = \int_0^{\phi} \frac{dt}{\sqrt{1 - m \sin^2 t}}, \quad 0 < m < 1.$$

5.5. Remarks about elliptic functions

The Jacobi amplitude is introduced as $\phi = F^{-1}(u, m) = am\ (u, m)$.
Then, the Jacobi functions are introduced as

$$
\begin{aligned}
sn\ (u, m) &= \sin\phi = \sin[am\ (u, m)], \\
cn\ (u, m) &= \cos\phi = \cos[am\ (u, m)], \\
dn\ (u, m) &= \sqrt{1 - m\sin^2\phi} = \sqrt{1 - m\sin^2[am\ (u, m)]}
\end{aligned}
$$

They satisfy the differential equations:

$$
\begin{aligned}
sn\ (t, m) : \quad & \frac{d^2x}{dt^2} = -(1 + m)x + 2mx^3, \\
cn\ (t, m) : \quad & \frac{d^2x}{dt^2} = -(1 - 2m)x - 2mx^3, \\
dn\ (t, m) : \quad & \frac{d^2x}{dt^2} = (2 - m)x - 2x^3,
\end{aligned}
$$

and have the following important limits:

$$
\begin{aligned}
sn\ (u, 0) &= \sin u, & sn\ (u, 1) &= tanh\ u, \\
cn\ (u, 0) &= \cos u, & cn\ (u, 1) &= sech\ u, \\
dn\ (u, 0) &= 1, & dn\ (u, 1) &= sech\ u.
\end{aligned}
$$

5.5. Remarks about elliptic functions

Most famous Weierstrass elliptic function is the $\wp$-function, which satisfies the following equation:

$$[\wp'(z)]^2 = 4\wp^3 - g_2\wp - g_3. \tag{12}$$

While trigonometric functions $\sin z, \cos z, \tan z$ satisfy $f(z+2\pi) = f(z)$, elliptic functions can be characterized as the doubly periodic functions $f(z+2\omega_1) = f(z), \quad f(z+2\omega_2) = f(z)$ with pole singularities. (Here, ω_1 and ω_2 are complex periods.) The simplest elliptic functions have two poles in the *fundamental period parallelogram* $[0, 2\omega_1, 2\omega_2, 2\omega_1 + 2\omega_2]$: the *Jacobi elliptic functions* have two simple poles of opposite residue, and the *Weierstrass elliptic function* has a single double pole with zero residue at each point of the period lattice.

There exist formulae relating Jacobi and Weierstrass functions.

The main Jacobi functions are implemented in *Mathematica* as JacobiSN[z,m], JacobiCN[z,m], JacobiDN[z,m]. The Weierstrass $\wp$-functiion is implemented as WeierstrassP[z, g2, g3].

5.6. Second harmonic generation

Second harmonic generation can be viewed as a special case of a resonant triad in which $\omega_3 = \omega_1, \quad \omega_2 = -2\omega_1.$ We may identify A_3 with A_1 to obtain

$$\left(\frac{\partial}{\partial T_1} + c_1 \frac{\partial}{\partial X_1}\right) A_1 = i\mu_1 A_1^* A_2^*,$$
$$\left(\frac{\partial}{\partial T_1} + c_2 \frac{\partial}{\partial X_1}\right) A_2 = i\mu_2 (A_1^*)^2 \qquad (13)$$

Equations can be rewritten in the form

$$\frac{\partial A_1}{\partial \xi} = A_1^* A_2^*, \quad \frac{\partial A_2}{\partial \tau} = (A_1^*)^2.$$

The second harmonic A_2 is generated by the fundamental A_1, even if it was not present initially.

5.6. Second harmonic generation

Experimental demonstration of second harmonic generation by Franken, Hill, Peters and Weinreich in 1961 (the beginning of nonlinear optics). They focused a ruby laser beam ($0.6943\mu m$) on the front surface of a quartz crystal, and detected radiation emitted from the back surface at twice the frequency of the incoming beam, i.e. blue light ($0.347\mu m$).

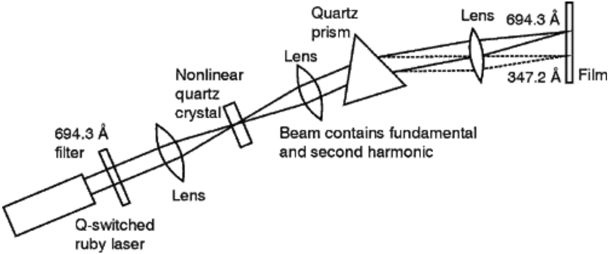


Figure: “Second Harmonic Generation” at http://images.google.co.uk/

5.7. Long-short wave resonance

Long-short wave resonance can be viewed as a special case of a resonant triad in which $k_1 = k_2 + k_3,\ \omega_1 = \omega_2 + \omega_3$, where k_1 and k_2 are close, i.e. $k_1 = k + \varepsilon\kappa,\ k_2 = k - \varepsilon\kappa,\ k_3 = 2\varepsilon\kappa.$ Then, to leading order the condition for frequencies yields

$$2\varepsilon\kappa \frac{d\omega}{dk} = \omega_3 \quad \text{or} \quad \frac{d\omega}{dk} = \frac{\omega_3}{k_3},$$

i.e. the group velocity of the short wave (wavenumber k) equals the phase velocity of the long wave (wavenumber $2\varepsilon\kappa$). Not every dispersion relation admits this type of resonance. However, systems with several branches of the dispersion curve often admit it, and there are other important cases, e.g., interaction between gravity and capillary waves in the context of water waves (Benney 1977). The possibility of a long-short wave resonance in plasma physics was shown by Zakharov (1972). Typically, equations can be written in the form:

$$iA_\tau + \beta A_{\xi\xi} = \gamma A n, \quad n_\tau = -\Delta\{AA^*\}_\xi.$$

Appendix 1. Three-wave interactions: alternative derivation for Lagrangian systems

For systems that can be derived from a variational principle,

$$\delta \int \int L(\mathbf{u}, \dot{\mathbf{u}}, \nabla \mathbf{u}, ...) d\mathbf{x} dt = 0,$$

the interaction equations can be derived by introducing the expansion (solution of the linearized problem)

$$\mathbf{u} = \varepsilon \sum_{n=1}^{3} A_n \hat{\mathbf{u}}_n \exp[i(\mathbf{k_n x} - \omega_n t)] + c.c.$$

into the Lagrangian density $L(\mathbf{u}, \dot{\mathbf{u}}, \nabla \mathbf{u}, ...)$, performing the integration over space, and taking the variations of the resulting functional with respect to the amplitudes. For simplicity, we assume that $A_n = A_n(T_1)$ (no spatial variation). Typically, one obtains a variational principle of the form

$$\delta \int (\varepsilon^2 L^{(2)} + \varepsilon^3 L^{(3)} + ...) dT_1 = 0,$$

Three-wave interactions: alternative derivation for Lagrangian systems

where

$$L^{(2)} = \sum_{n=1}^{3} D(k_n,\omega_n) A_n A_n^* + c.c.,$$

$$L^{(3)} = \sum_{n=1}^{3} iD_{\omega_n} \frac{dA_n}{dT_1} A_n^* + \frac{1}{3} \sum_{n\neq m\neq l} \mu_{nml}\; A_n^* A_m^* A_l^* + c.c.$$

with μ_{nml} constants symmetric in (n,m,l). The variations with respect to the amplitudes A_n (or A_n^*) give at leading order the dispersion relation:

$$D(k_n,\omega_n) = 0$$

At next order, the variations with respect to A_n^* give the interaction equations

$$D_{\omega_n} \frac{dA_n}{dT_1} = \frac{i}{2} \sum_{m\neq l(\neq n)} \mu_{nml}\; A_m^* A_l^*$$

Appendix 2. Four-wave interactions: derivation for Lagrangian systems

To obtain the four-wave resonance equations for systems that can be derived from a variational principle,

$$\delta \int \int L(\mathbf{u}, \dot{\mathbf{u}}, \nabla \mathbf{u}, ...) d\mathbf{x} dt = 0,$$

one can substitute the expansion (solution of the linearized problem)

$$\mathbf{u} = \varepsilon \sum_{n=1}^{4} A_n \hat{\mathbf{u}}_n \exp[i(\mathbf{k_n x} - \omega_n t)] + c.c.$$

into the Lagrangian density $L(\mathbf{u}, \dot{\mathbf{u}}, \nabla \mathbf{u}, ...)$, perform the integration over space, and take the variations of the resulting functional with respect to the wave amplitudes. In this case, we assume that $A_n = A_n(T_2)$ where $T_2 = \varepsilon^2 t$. Typically, in the absence of resonant triads, one obtains a variational principle of the form

$$\delta \int (\varepsilon^2 L^{(2)} + \varepsilon^4 L^{(4)} + ...) dT_2 = 0,$$

Four-wave interactions: derivation for Lagrangian systems

where

$$L^{(2)} = \sum_{n=1}^{4} D(k_n, \omega_n) A_n A_n^* + c.c.,$$

$$L^{(4)} = i\ [\sum_{n=1}^{4} D_{\omega_n} \frac{dA_n}{dT_2} A_n^* + \sum_{n,m} \alpha_{nm}\ A_n A_n^* A_m A_m^*$$

$$+ \sum_{n \neq m \neq l \neq p} \beta_{nmlp}\ A_n^* A_m^* A_l^* A_p^* + c.c.]$$

Variations with respect to the amplitudes A_n (or A_n^*) give at leading order the dispersion relation, while at next order variations with respect to A_n^* give the interaction equations.

6.1 Fermi-Pasta-Ulam (FPU) problem

The FPU problem is considered to be the starting point of the modern nonlinear wave theory. After the war, in Los Alamos, Fermi, Pasta and Ulam (1955) numerically studied the dynamics of an anharmonic chain of particles. The model consisted of identical equidistant (by a distance a) particles connected to their nearest neighbours by weakly nonlinear springs. Let u_n denote the displacement of the n-th particle from the equilibrium. Then,

$$m\ddot{u}_n = f(u_{n+1} - u_n) - f(u_n - u_{n-1}), \quad n = 1, ..., N,$$

where $f(\Delta u)$ was given by $f(\Delta u) = k\Delta u + \alpha(\Delta u)^2$. The ends of the chain were fixed: $u_0 = u_{N+1} = 0$. Thus, the dynamics of the chain was described by the system

$$\begin{aligned} & m\ddot{u}_n = k(u_{n+1} - 2u_n + u_{n-1}) + \alpha[(u_{n+1} - u_n)^2 - (u_n - u_{n-1})^2], \quad (1) \\ & \text{for} \quad n = 1, ..., N, \\ & u_0 = u_{N+1} = 0 \end{aligned}$$

(typically, N was equal to 64).

A general solution of the linearised system ($\alpha = 0$) is given by a superposition of normal modes;

$$u_n^k(t) = A_k \sin\left(\frac{\pi kna}{N+1}\right) \cos\left(\omega_k t + \delta_k\right), \quad k = 1, ..., N,$$

$$\omega_k = 2\sqrt{\frac{k}{m}} \sin\frac{\pi ka}{2(N+1)}.$$

There is no energy transfer between the modes in the linear approximation. In the nonlinear chain ($\alpha \neq 0$), modes become coupled. It was expected that if all the initial energy was put into a single mode (or a few first modes), the nonlinear coupling would yield equal distribution of the energy among the normal modes. However, the numerical results were surprising: for example, if the energy was initially in the mode of lowest frequency, it returned almost entirely to that mode after interaction with a few other low frequency modes (FPU recurrence).

6.2 Continuum approximation: Boussinesq equation

This strange result has motivated Zabusky and Kruskal (1965) to consider the FPU problem in the so-called continuum approximation. One assumes that

$$u_n(t) = u(x_n, t) = u(na, t), \quad u_{n\pm1} = u(x_n \pm a, t),$$

and the displacement field u varies slowly justifying the Taylor expansion

$$\begin{aligned} u_{n\pm1}(t) &\approx u(x_n,t) \pm a u'(x_n,t) + \frac{1}{2} a^2 u''(x_n,t) \\ &\pm \frac{1}{6} a^3 u'''(x_n,t) + \frac{1}{24} a^4 u''''(x_n,t) + \ldots \end{aligned} \tag{2}$$

Substituting (2) into (1) and dropping the label n yields equation:

$$u_{tt} - c^2 u_{xx} = \varepsilon c^2 (u_x u_{xx} + \delta^2 u_{xxxx}), \tag{3}$$

where $c^2 = \frac{ka^2}{m}, \varepsilon = \frac{2\alpha a}{k}, \delta^2 = \frac{a^2}{12\varepsilon}$. Here, the leading order nonlinear and dispersive contributions are balanced at the same order of ε. This is the Boussinesq equation. It describes waves, which can propagate both to the right, and to the left (the two-way long-wave equation).

6.3 Further reduction: KdV equation

A further reduction to the KdV equation is possible if we look for an asymptotic miltiple-scale expansion of the solution of (3) of the form

$$u = f(\xi, T) + \varepsilon u^{(1)}(x,t) + \ldots, \quad \text{where} \quad \xi = x - ct, T = \varepsilon t,$$

whereupon (3) gives us

$$u^{(1)}_{tt} - c^2 u^{(1)}_{xx} = 2cf_{\xi T} + c^2 f_\xi f_{\xi\xi} + c^2 \delta^2 f_{\xi\xi\xi\xi} + \ldots$$

The function $u^{(1)}$ will grow linearly in $\eta = x + ct$, unless

$$2cf_{\xi T} + c^2 f_\xi f_{\xi\xi} + c^2 \delta^2 f_{\xi\xi\xi\xi} = 0.$$

By setting $q = \frac{f_\xi}{6}, \tau = \frac{cT}{2}$, it reduces to the canonical form of the KdV equation

$$q_\tau + 6qq_\xi + \delta^2 q_{\xi\xi\xi} = 0. \tag{4}$$

Kruskal and Zabusky numerically studied the dynamics of the KdV equation with sinusoidal initial conditions (for small δ^2, periodic boundary conditions), and discovered that the appearing solitons interact with each other elastically. They have called the waves solitons because of the analogy with particles.

6.3 Further reduction: KdV equation

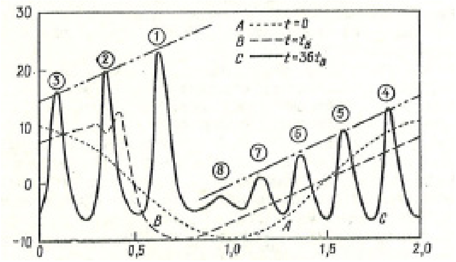


Figure: Zabusky and Kruskal's numerical solution, 1965

Kruskal and Zabusky explained the FPU recurrence as follows. The initial condition generates a family of solitons, moving with different speeds. Since the system studied was of finite length, solitons eventually reassembled in the (x, t) plane and approximately recreated the initial condition $u(x, 0)$.

6.4 KdV equation: cnoidal waves and solitons

Considering the travelling waves $u = u(\xi)$, $\xi = x - ct$ of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

we obtain an ordinary differential equation $u''' + 6uu' - cu' = 0$. Integrating once with respect to ξ gives $u'' + 3u^2 - cu = A, \quad A = const.$ Integrating again after multiplication by u' gives

$$\frac{1}{2}(u')^2 = -u^3 + \frac{c}{2}u^2 + Au + B, \quad B = const. \tag{5}$$

Imposing the boundary conditions, $u, u', u'' \to 0$ as $\xi \to \pm\infty$ which describe the solitary waves, we obtain $A = B = 0$ and

$$(u')^2 = u^2(c - 2u).$$

Separating the variables, $\int \frac{du}{u\sqrt{c-2u}} = \pm \int d\xi$, and integrating, we obtain the solitary wave solution:

$$u = \frac{1}{2}c \; sech^2 \left\{\frac{\sqrt{c}}{2}(x - ct - x_0)\right\}, \quad x_0 = const, \quad c \geq 0.$$

6.4 KdV equation: cnoidal waves and solitons

The qualitative nature of travelling wave solutions for arbitrary A and B can be determined by the phase plane analysis.

Periodic solutions of the KdV equation are called cnoidal waves. See Appendix 1 for the description of the cnoidal waves in elliptic functions.

The solitary wave solution can be rewritten in the form

$$u = 2\frac{\partial^2}{\partial x^2}\ln[1 + e^{-\sqrt{c}(x-ct-x_0)}],$$

suggesting the transformation

$$u(t,x) = 2\frac{\partial^2}{\partial x^2}\ln\tau(t,x)$$

(Cole-Hopf transformation), which is used to reduce the KdV equation to a bilinear equation for τ,

$$\tau\tau_{xxxx} - 4\tau_x\tau_{xxx} + 3\tau_{xx}^2 + \tau\tau_{xt} - \tau_x\tau_t = 0. \qquad (6)$$

6.4 KdV equation: cnoidal waves and solitons

Looking for solutions of of this equation of the form

$$\tau = 1 + \sum_{n=1}^{\infty} \varepsilon^n \tau^{(n)},$$

where $\tau^{(1)} = e^{\theta_1} + e^{\theta_2}, \ \theta_i = k_i x - k_i^3 t + \delta_i$, one can obtain an exact two-soliton solution:

$$u = 2\frac{\partial^2}{\partial x^2} \ln \tau, \quad \text{where} \quad \tau = 1 + e^{\theta_1} + e^{\theta_2} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{\theta_1+\theta_2}.$$

This is know as Hirota's method (1971). For the KdV equation and other integrable equations, it can be extended to describe the N-soliton solutions for $N \geq 3$. (Integrability of the KdV equation by the IST was shown by Gardner, Green, Kruskal and Miura in 1967, 1974. We will return to the IST and solitons for the KdV later.)

See Appendix 2 for the discussion of similar solutions for the Boussinesq equation

$$u_{tt} - u_{xx} = u_x u_{xx} + u_{xxxx}.$$

6.5 Remark: Boussinesq systems for water waves

Boussinesq originally derived a system of two first-order (in time) equations for weakly nonlinear surface waves in shallow water (1881). A system describes waves which can travel both to the left and to the right. There are several asymptotically equivalent versions of this system, which are all called Boussinesq systems, e.g.

$$\eta_t + u_x + (u\eta)_x = 0,$$
$$u_t + \eta_x + uu_x - u_{xxt} = 0;$$

or

$$\eta_t + u_x + (u\eta)_x = 0,$$
$$u_t + \eta_x + uu_x + \eta_{xtt} = 0.$$

Here, u is the horizontal velocity, η is the free surface elevation. Boussinesq then reduced a system of first-order equations to one second-order equation, using further approximations.

See Appendix 3 for the derivation and discussion of the Boussinesq-type equation for long longitudinal waves in a solid waveguide.

6.6 Frenkel-Kontorova (FK) model

The original FK model (1938) was proposed to describe dislocations in metals. Consider the situation when an additional semi-infinite plane of atoms is inserted into a perfect crystal lattice. The atoms of the "interface layer" are treated as a one-dimensional chain subjected to an external periodic potential produced by the surrounding atoms.

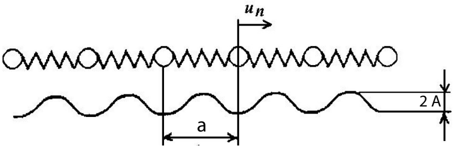


Figure: Frenkel-Kontorova model.

The Lagrangian of the system is given by

$$\mathcal{L} = T - \Pi = \sum_n \frac{m\dot{u}_n^2}{2} - \sum_n \frac{\alpha}{2}(u_{n+1} - u_n)^2 - \sum_n A\left(1 - \cos\frac{2\pi u_n}{a}\right)$$

Here, m is the particle mass, u_n is the displacement of the nth particle, α is the elastic constant, A is the amplitude of the on-site periodic potential, a is the period of the chain.

6.7 FK model and sine-Gordon (SG) equation

Introducing the dimensionless variables,

$$u_n \to \frac{2\pi u_n}{a}, \qquad t \to \frac{2\pi}{a}\sqrt{\frac{A}{m}}\ t, \qquad \alpha \to \alpha\frac{(a/2\pi)^2}{A},$$

the equations of motion $\left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}_n}\right) - \frac{\partial \mathcal{L}}{\partial u_n} = 0\right)$ can be written as

$$\ddot{u}_n - \alpha(u_{n+1} - 2u_n + u_{n-1}) = -\sin u_n$$

Assuming that there exists a function $u(x,t)$ such that $u(na,t) = u_n(t)$ and using

$$u((n\pm 1)a,t) = u(na,t) \pm a u_x(na,t) + \frac{1}{2}a^2 u_{xx}(na,t) + O(a^3)$$

we obtain the sine-Gordon (SG) equation: $u_{tt} - \alpha a^2 u_{xx} = -\sin u$.
Scaling $x \to \frac{x}{a\sqrt{\alpha}}$, the SG equation can be rewritten in the canonical form:

$$u_{tt} - u_{xx} = -\sin u \tag{7}$$

The solutions for travelling waves admit the phase plane analysis, and can be given in terms of elliptic functions. See Appendix 4 for an overview of other applications of the sine-Gordon equation.

6.8 Sine-Gordon equation: Bäcklund Transformations

Solitary waves of the SG equation are called kinks and breathers. We will obtain these solutions here using Bäcklund transformations.

Bäcklund transformations (BT) for the SG equation were devised in 1880s in differential geometry and are attributed to Bianchi and Bäcklund. They have the form:

$$\begin{aligned}\frac{1}{2}(\phi+\tilde{\phi})_\xi &= \alpha \sin\frac{\phi-\tilde{\phi}}{2},\\ \frac{1}{2}(\phi-\tilde{\phi})_\eta &= \frac{1}{\alpha}\sin\frac{\phi+\tilde{\phi}}{2},\end{aligned} \tag{8}$$

where both ϕ and $\tilde{\phi}$ are solutions of the SG equation (1), and can be viewed as a transformation of the SG equation into itself. BT allow one to construct hierarchies of solutions, starting from some simple known solutions.

Are there any other Klein-Gordon equations $\phi_{\xi\eta} = F(\phi)$, admitting Bäcklund transformation of a similar form? (see Appendix 5.)

6.9 Bäcklund Transformations: Kink and Antikink

Given a solution ϕ_0 of the SG equation, BT (8) allows one to find a 2-parameter family of solutions. Consider the trivial solution $\phi = 0$. Substituting it into (8), we obtain

$$\tilde{\phi}_\xi = -2\alpha \sin\frac{\tilde{\phi}}{2}, \quad \tilde{\phi}_\eta = -\frac{2}{\alpha}\sin\frac{\tilde{\phi}}{2}.$$

Integrating,

$$2\alpha\xi = -\int^{\tilde{\phi}} \frac{d\tilde{\phi}}{\sin\frac{\tilde{\phi}}{2}} = -2\ln\left(\tan\frac{\tilde{\phi}}{4}\right) + p(\eta),$$

$$\frac{2}{\alpha}\eta = -\int^{\tilde{\phi}} \frac{d\tilde{\phi}}{\sin\frac{\tilde{\phi}}{2}} = -2\ln\left(\tan\frac{\tilde{\phi}}{4}\right) + q(\xi).$$

Therefore, $\tan\frac{\tilde{\phi}}{4} = \exp\left(a\xi + \frac{1}{a}\eta + \delta\right)$, where $a = -\alpha, \delta = const$.
Thus, the trivial solution $\phi = 0$ has generated a new solution:

$$\phi_1 = \tilde{\phi} = 4\tan^{-1}\exp(a\xi + \frac{1}{a}\eta + \delta). \tag{9}$$

6.9 Bäcklund Transformations: Kink and Antikink

Reverting to t and x:

$$\phi_1 = \tilde{\phi} = 4\tan^{-1}\exp\left[\sigma\frac{x - vt}{\sqrt{1 - v^2}} + \delta\right],$$

where $v = \frac{1-a^2}{1+a^2}$, $\sigma = \pm 1$ (we can assume $\delta = 0$, up to shifts in x, t).

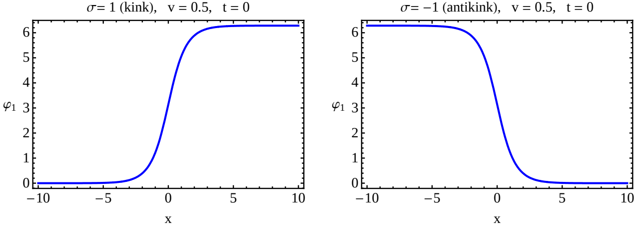


Figure: Kink and antikink for $v = 0.5$ at $t = 0$.

Remarks. Note that derivatives of these solutions are hump-shaped, like a solitary wave.

Also note that $2\pi n$, where n is an integer can be added to any solution of the SG equation due to invariance of the equation with respect to this transformation. (See Appendix 6.)

6.10 Addition theorem

Now we can again use the BT (8), so that ϕ_1 generates a new solution ϕ_2. To do this, put $\phi = \phi_1$ into the BT (8), and integrate to find a new solution. This new solution can then also be used to find another solution, etc. However, the direct integration is not easy, and the actual way to use the BTs for finding new solutions is based on the fact that BTs commute, which is shown on the diagram below.

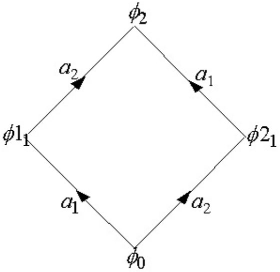


Figure: http://homepages.tversu.ru/∼s000154/collision/main.html

Remark. To be precise, BTs can be *made* commutative by adjusting the integration constants (e.g., McLaughlin and Scott 1973, Seeger et al. 1953).

6.10 Addition theorem

Let us take a sequence of pairs $(\phi_0, \phi 1_1), (\phi 1_1, \phi_2), (\phi_0, \phi 2_1), (\phi 2_1, \phi_2)$.
Then, we have the following BTs (follow the diagram):

$$\frac{1}{2}(\phi_0 + \phi 1_1)_\xi = \alpha_1 \sin\frac{\phi_0 - \phi 1_1}{2}, \tag{10}$$

$$\frac{1}{2}(\phi_0 - \phi 1_1)_\eta = \frac{1}{\alpha_1} \sin\frac{\phi_0 + \phi 1_1}{2}; \tag{11}$$

$$\frac{1}{2}(\phi 1_1 + \phi_2)_\xi = \alpha_2 \sin\frac{\phi 1_1 - \phi_2}{2}, \tag{12}$$

$$\frac{1}{2}(\phi 1_1 - \phi_2)_\eta = \frac{1}{\alpha_2} \sin\frac{\phi 1_1 + \phi_2}{2}; \tag{13}$$

$$\frac{1}{2}(\phi_0 + \phi 2_1)_\xi = \alpha_2 \sin\frac{\phi_0 - \phi 2_1}{2}, \tag{14}$$

$$\frac{1}{2}(\phi_0 - \phi 2_1)_\eta = \frac{1}{\alpha_2} \sin\frac{\phi_0 + \phi 2_1}{2}; \tag{15}$$

$$\frac{1}{2}(\phi 2_1 + \phi_2)_\xi = \alpha_1 \sin\frac{\phi 2_1 - \phi_2}{2}, \tag{16}$$

$$\frac{1}{2}(\phi 2_1 - \phi_2)_\eta = \frac{1}{\alpha_1} \sin\frac{\phi 2_1 + \phi_2}{2}. \tag{17}$$

6.10 Addition theorem

Taking (10) - (12) - (14) + (16) and using the trigonometric formula $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$, one can obtain

$$\alpha_1 \sin\frac{\phi_0 - \phi 1_1 + \phi 2_1 - \phi_2}{4} = \alpha_2 \sin\frac{\phi_0 + \phi 1_1 - \phi 2_1 - \phi_2}{4},$$

which yields, by using $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$ with $\alpha = \phi_0 - \phi_2$ and $\beta = \phi 2_1 - \phi 1_1$,

$$\tan\frac{\phi_0 - \phi_2}{4} = \frac{\alpha_2 + \alpha_1}{\alpha_2 - \alpha_1}\tan\frac{\phi 2_1 - \phi 1_1}{4}. \tag{18}$$

The same formula follows from the second group of equations, (11), (13), (15) and (17).

The relation (18) is called the addition theorem for the Bäcklund transformations. This relation allows one to obtain new solution ϕ_2 algebraically, from the known solutions $\phi_0, \phi 1_1, \phi 2_1$, without any integration. Repeatedly applying this theorem, one can obtain successively new solutions by means of purely algebraic manipulations.

6.11 Two-wave interactions (kink-kink, kink-antikink, etc.)

Let $\phi_0 = 0$. Then $\phi 1_1$ and $\phi 2_1$ are (solutions are parametrized by $a_i = -\alpha_i$):

$$\phi i_1 = 4\tan^{-1}\exp\theta_i \quad (+2\pi n_i), \quad \theta_i = \left[\frac{a_i^2+1}{2a_i}(x - \frac{1-a_i^2}{1+a_i^2}\ t) + \delta_i\right]$$

Using the addition theorem (18) and the formula $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$, we find the two-soliton solution:

$$\phi_2 = 4\tan^{-1}\left[\frac{a_1+a_2}{a_1-a_2}\cdot\frac{e^{\theta_2}-e^{\theta_1}}{1+e^{\theta_1+\theta_2}}\right]. \qquad (19)$$

Choosing different values of parameters a_1 and a_2, one can obtain solutions describing various interaction scenarios (kink-kink, kink-antikink, etc.)

6.11 Two-wave interactions (kink-kink, kink-antikink, etc.)

Example 1. The symmetric kink-kink solution (head-on collision, same velocities) is obtained when $a_2 = -1/a_1,\ a_1 > 0$ and $\delta_1 = \delta_2 = 0$. It can be written as

$$\phi_2 = 4\tan^{-1}\left[\frac{v\ \sinh\ \frac{x}{\sqrt{1-v^2}}}{\cosh\ \frac{vt}{\sqrt{1-v^2}}}\right], \quad \text{where} \quad v = \frac{1-a_1^2}{1+a_1^2}. \tag{20}$$

Considering the asymptotics of (19) as $t \to -\infty$ (and then $x \to -\infty$ and $x \to +\infty$) and as $t \to +\infty$ (and then $x \to -\infty$ and $x \to +\infty$), one can find the phase shift to be $2\sqrt{v^2-1}\log v$ (the only "evidence" of interaction).

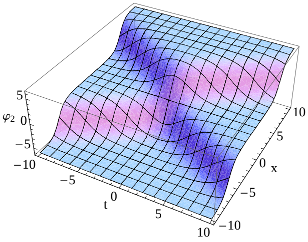


Figure: Head-on collision of two kinks for $v = 0.6$.

6.11 Two-wave interactions (kink-kink, kink-antikink, etc.)

Example 2. The symmetric kink-antikink solution (head-on collision, same velocities) is obtained when $a_2 = 1/a_1,\ a_1 < 0$ and $\delta_1 = \delta_2 = 0$. It can be written as

$$\phi_2 = 4\tan^{-1}\left[\frac{\sinh\frac{vt}{\sqrt{1-v^2}}}{v\,\cosh\frac{x}{\sqrt{1-v^2}}}\right], \quad \text{where} \quad v = \frac{1-a_1^2}{1+a_1^2}. \qquad (21)$$

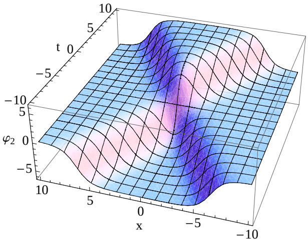


Figure: Head-on collision of kink and antikink for $v = 0.6$.

6.12 Breather

The symmetric kink-antikink solution (21) takes an interesting form if the velocity parameter v is allowed to be imaginary. Setting

$$v = \frac{i\omega}{\sqrt{1-\omega^2}}, \quad \omega < 1,$$

one obtains the breather

$$\phi_2 = 4\tan^{-1}\left[\frac{\sqrt{1-\omega^2}}{\omega}\cdot\frac{\sin\omega t}{\cosh\sqrt{1-\omega^2}\,x}\right], \tag{22}$$

which is a localized, but oscillating in time solution.

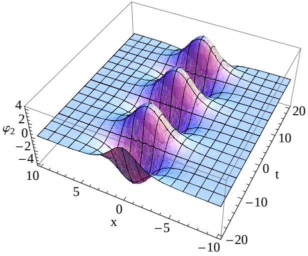


Figure: A stationary breather for $\omega = 0.5$.

6.12 Breather

The solution (22) describes a stationary breather. Using the Lorentz symmetry of the SG equation:

$$x \to \frac{x - ct}{\sqrt{1 - c^2}}, \quad t \to \frac{t - cx}{\sqrt{1 - c^2}},$$

one can obtain a moving breather:

$$\phi_2 = 4\tan^{-1}\left[\frac{\sqrt{1-\omega^2}}{\omega} \cdot \frac{\sin[\frac{\omega(t-cx)}{\sqrt{1-c^2}}]}{\cosh\,[\frac{\sqrt{1-\omega^2}(x-ct)}{\sqrt{1-c^2}}]}\right].$$

Here, an envelope velocity c is equal to the reciprocal of its carrier velocty c^{-1}.

6.13 Soliton "ladder"

The procedure, described above, can be continued, resulting in the so-called soliton "ladder", shown below:

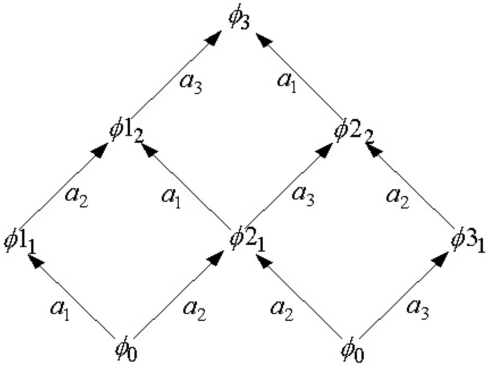


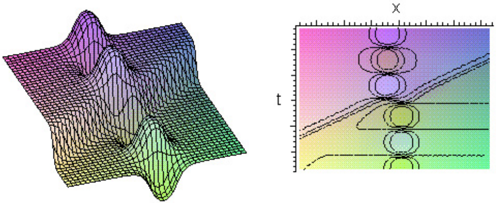
Figure: http://homepages.tversu.ru/∼s000154/collision/main.html

6.14 Soliton interactions

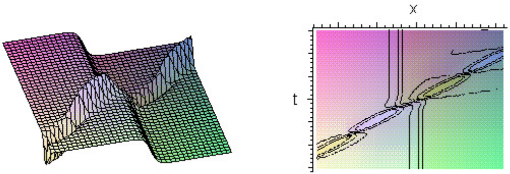
- Interactions of any solitons are accompanied by phase shifts only (elastic interactions).
- Interactions have a two-particle nature: when several solitons collide, a shift of any soliton involved into the interaction is equal to the sum of the shifts caused by its independent interactions with other solitons.

6.14 Soliton interactions

Standing Breather Moving Kink collision



Standing Kink and Moving Breather collision



Visit: http://homepages.tversu.ru/~s000154/collision/main.html

to see some movies, showing kink-kink, kink-antikink, kink-breather, etc. interactions.

Appendix 1. KdV equation: cnoidal waves and solitons

Periodic solutions can be given in terms of elliptic functions. Indeed, (5) yields

$$\begin{aligned} u' &= \pm\sqrt{-2u^3 + cu^2 + 2Au + 2B} \\ &= \pm\sqrt{2(u_3 - u)(u - u_2)(u - u_1)} = \pm\sqrt{P(u)}, \end{aligned} \tag{23}$$

where u_i are the three roots of the cubic polynomial. If c, A and B are such that the three roots are real $(u_1 < u_2 < u_3)$, then $P(u) \geq 0$ for $u_2 < u < u_3$. Integration of (23) yields $\xi - \xi_0 = \pm\int_u^{u_3} \frac{dy}{\sqrt{P(y)}}$, where $\xi_0 = const$. The standard substitution $y = u_3 - (u_3 - u_2)\sin^2\theta$ leads to

$$\xi - \xi_0 = \pm\sqrt{\frac{2}{u_3 - u_1}}\int_0^{\phi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}, \quad m = \frac{u_3 - u_2}{u_3 - u_1},$$

where $u = u_3 - (u_3 - u_2)\sin^2\phi = u_2 + (u_3 - u_2)\cos^2\phi$, and gives us the periodic solution (cnoidal wave):

$$u(x - ct) = u_2 + (u_3 - u_2)\ cn^2\ [\sqrt{\frac{u_3 - u_1}{2}}(x - ct - \xi_0); m].$$

KdV equation: cnoidal waves and solitons

The period of this solution in ξ is given by

$$\lambda = \frac{2K(m)}{\sqrt{(u_3 - u_1)/2}}, \quad \text{where} \quad K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}$$

is the complete elliptic integral of the first kind.
Note, that the so-called trigonometric limit $(m \to 0)$ recovers the harmonic waves, and the solitonic limit $(m \to 1)$ yields the solitary wave solution.

Appendix 2: Boussinesq equation

We consider the Boussinesq equation

$$u_{tt} - u_{xx} = u_x u_{xx} + u_{xxxx}.$$

Looking for the travelling waves $u = u(x - ct)$, we obtain

$$(c^2 - 1)u'' - u'u'' - u'''' = 0,$$

and, integrating, $(c^2 - 1)u' - \frac{1}{2}(u')^2 - u''' = A$. Multiplying by u'', integrating, and introducing $e = \frac{1}{6}u'$, we obtain an equation which can be integrated in elliptic functions.

Boussinesq equation: cnoidal waves and solitons

The solitary wave is a "solitonic limit" of the cnoidal wave, and can be written in the form

$$e = 2\frac{\partial^2}{\partial x^2}\ln\tau,\ \ \tau = 1 + e^{kx+\epsilon k\sqrt{1+k^2}t+\delta}, \epsilon = \pm 1.$$

Hirota's method can be used to obtain a two-soliton solution (Hirota, 1973) to the equation $e_{tt} - e_{xx} = 3(e^2)_{xx} + e_{xxxx}$ for the function $e = \frac{1}{6}u_x$ in the form:

$$e = 2\frac{\partial^2}{\partial x^2}\ln\tau, \quad \tau = 1 + e^{\theta_1} + e^{\theta_2} + Ae^{\theta_1+\theta_2},$$

$$\text{where} \quad \theta_i = k_i x + \epsilon_i k_i\sqrt{1+k_i^2}t + \delta_i,\ \epsilon_i = \pm 1,$$

$$A = \frac{3(k_1-k_2)^2 + (\epsilon_1\sqrt{1+k_1^2} - \epsilon_2\sqrt{1+k_2^2})^2}{3(k_1+k_2)^2 + (\epsilon_1\sqrt{1+k_1^2} - \epsilon_2\sqrt{1+k_2^2})^2},$$

and it can be extended to N-soliton solutions for $N \geq 3$ (equation is integrable by the IST, as shown by Zakharov in 1974).

Appendix 3: Long longitudinal waves in an elastic rod

A Boussinesq-type equation with two kinds of dispersive terms (so-called Doubly Dispersive Equation (DDE)) has been derived for long longitudinal waves in an elastic rod (Samsonov, 1984).
Consider a rod of circular cross section of radius a.

Figure: Elastic rod of circular cross section.

In the framework of nonlinear dynamic elasticity, on can consider the variational problem for the action functional S written as

$$S = \int_{t_0}^{t_1} \int_{\Omega} \mathcal{L}(\mathbf{U}, \mathbf{U}_t, \mathbf{U}_x, \dots x, t)\; d\Omega\; dt, \tag{24}$$

where $\mathcal{L}(\mathbf{U}, \mathbf{U}_t, \mathbf{U}_x \dots x, t)$ is the Lagrangian density per unit volume, t is time, Ω is a space domain occupied by the rod, $\mathbf{U} = \{u, v, w\}$ is the displacement vector in the Lagrangian coordinates (x, y, z).

Long longitudinal waves in an elastic rod

The Lagrangian density $\mathcal{L}$ is the difference of the kinetic energy density T and the density Π of potential energy, as follows:

$$\mathcal{L} = T - \Pi = \rho(\partial \mathbf{U}/\partial t)^2/2 - \rho\Pi(I_k), \tag{25}$$

where ρ is the density, and $I_k = I_k(\mathbf{C})$ are the invariants of Cauchy-Green's deformation tensor $\mathbf{C} = [\nabla\mathbf{U} + (\nabla\mathbf{U})^T + \nabla\mathbf{U}\cdot(\nabla\mathbf{U})^T]/2$:

$$I_1 = tr\ \mathbf{C};\ \ I_2 = (1/2)[(tr\ \mathbf{C})^2 - (tr\ \mathbf{C}^2)];\ \ I_3 = \det\mathbf{C}.$$

For isotropic compressible nonlinearly elastic materials which can be described by so-called Murnaghan's 5-constant model:

$$\Pi = (\lambda + 2\mu)I_1^2/2 - 2\mu I_2 + (l + 2m)I_1^3/3 - 2m I_1 I_2 + n I_3 + \ldots$$

where λ and μ are Lame's coefficients, and l, m and n are Murnaghan's moduli.

To simplify the problem, one can use the planar cross section hypothesis and the approximate (Love's) relations for the transverse displacements via the linear longitudinal strain component as follows:

$$u \approx u(x,t),\ \ v \approx -y\nu u_x,\ \ w \approx -z\nu u_x, \tag{26}$$

where $\nu = \frac{\lambda}{2(\lambda+\mu)}$ is Poisson's ratio.

Long longitudinal waves in an elastic rod

Assuming the scaling $\varepsilon \sim \frac{Wave\ amplitude}{Wave\ length} \sim \left(\frac{Bar\ width}{Wave\ length}\right)^2$, one can obtain the approximate expressions for the invariants valid for the small amplitude long longitudinal elastic waves. The approximate kinetic energy density per unit volume is written as

$$T = \frac{1}{2}\rho\left({u_t}^2 + {v_t}^2 + {w_t}^2\right) = \frac{\rho}{2}\left[{u_t}^2 + (y^2+z^2)\,\nu^2\,{u_{xt}}^2\right] + \ldots$$

The approximate potential energy density per unit volume may be written as follows:

$$\Pi = \frac{1}{2}\left[E{u_x}^2 + \frac{\beta}{3}{u_x}^3 + \mu\nu^2(y^2+z^2){u_{xx}}^2\right] + \ldots,$$

where the nonlinearity coefficient β depends on Murnaghan's moduli l, m, n, Young's modulus $E = \mu\frac{3\lambda+2\mu}{\lambda+\mu}$, and Poisson's ratio ν.

Long longitudinal waves in an elastic rod: the DDE

Then, from (24) and (25) one obtains the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}_\sigma}{\partial u} - \left[\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}_\sigma}{\partial u_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}_\sigma}{\partial u_x}\right)\right] + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}_\sigma}{\partial u_{xx}}\right) + \frac{\partial^2}{\partial x \partial t}\left(\frac{\partial \mathcal{L}_\sigma}{\partial u_{xt}}\right) + \dots = 0, \tag{27}$$

where $\mathcal{L}_\sigma = \int_\sigma \mathcal{L} d\sigma$ is the Lagrangian density per unit length ($\sigma = \pi a^2$ is the cross section area). Equation (27) yields the so-calle DDE (doubly dispersive equation) for long nonlinear longitudinal displacement waves in a bar of circular cross section in a form (Samsonov 1984):

$$u_{tt} - c^2 u_{xx} = \frac{\beta}{\rho} u_x u_{xx} + \frac{\nu^2 a^2}{2}(u_{tt} - c_1^2 u_{xx})_{xx}, \tag{28}$$

where $c = \sqrt{E/\rho}$, and $c_1 = \sqrt{\mu/\rho} = c/\sqrt{2(1+\nu)}$. This is equation of Boussinesq type with two kinds of dispersive terms.
Differentiating (28) with respect to x, we obtain the following equation for the linear strain component $e \equiv u_x$:

$$e_{tt} - c^2 e_{xx} = \frac{\beta}{2\rho}(e^2)_{xx} + \frac{\nu^2 a^2}{2}\left(e_{tt} - c_1^2 e_{xx}\right)_{xx}. \tag{29}$$

Long longitudinal waves in an elastic rod: the DDE

The one-parameter family of exact solitary wave solutions of (29) can be written in dimensional physical variables in the form

$$e = e_0 \, \mathrm{sech}^2 \frac{1}{\Lambda} (x - st) \,, \tag{30}$$

where

$$e_0 = \frac{3\rho(s^2 - c^2)}{\beta}, \quad \Lambda^2 = \frac{\nu^2 a^2}{1+\nu} \left[1 + \frac{(1+2\nu)s^2}{s^2 - c^2} \right].$$

If the value of s is close to c, say, $s = c(1 + \varepsilon^2 s_1 + \ldots)$, then (30) can be approximately written as

$$e \approx \varepsilon^2 \tilde{e} = \varepsilon^2 \frac{6\rho c^2 s_1}{\beta} \, \mathrm{sech}^2 \sqrt{s_1} \frac{\xi - s_1 c T}{\sqrt{\frac{\nu^2 a^2}{2} \frac{1+2\nu}{1+\nu}}},$$

$$\text{where} \quad \xi = \varepsilon(x - ct), \; T = \varepsilon^3 t, \; (\text{and } \varepsilon^2 s_1 \approx \frac{s-c}{c}),$$

which satisfies the KdV equation, derivable from (29)

$$\tilde{e}_T + \frac{\beta}{2\rho c} \tilde{e} \tilde{e}_\xi + \frac{\nu^2 a^2 c}{8} \frac{1+2\nu}{1+\nu} \tilde{e}_{\xi\xi\xi} = 0. \tag{31}$$

Appendix 4: Sine-Gordon equation; other applications

Historically, the SG equation has first appeared in the differential geometry of surfaces of a constant negative Gaussian curvature (Bour 1862, also Bianchi, Enneper). The study in the context of differential geometry has revealed the possibility to generate new solutions of the SG equation from the known solutions (Bianchi 1879, Bäcklund 1882). Seeger and Kochendörfer (1950-1951) have used Bäcklund transformations to find kink-antikink and breather solutions. In 1962 Perring and Skyrme introduced the SG equation as a one-dimensional model of the scalar field theory modelling a classical particle. Soon after that, the SG equation appeared as an equation describing the so-called long Josephson junctions in the theory of weak superconductivity (Josephson 1965). In this context, the kink solution describes a fluxon, a quantum of magnetic field. The mechanical analog of the FK chain as the chain of coupled pendulums was introduced by Scott in 1969. The FK model and its continuum approximation continue to find numerous other applications and generalizations in solid state physics and biology (e.g., in the surface reconstruction phenomena, the proton conductivity of hydrogen-bonded chains, the dynamics of DNA, etc.)

Appendix 5: Bäcklund Transformations and integrability

Exercise

Are there any other Klein-Gordon equations

$$\phi_{\xi\eta} = F(\phi), \tag{32}$$

admitting Bäcklund transformations of the form:

$$\frac{1}{2}(\phi + \tilde{\phi})_\xi = f(\frac{\phi - \tilde{\phi}}{2}), \tag{33}$$

$$\frac{1}{2}(\phi - \tilde{\phi})_\eta = g(\frac{\phi + \tilde{\phi}}{2}) \ ? \tag{34}$$

Bäcklund Transformations and integrability

Differentiating (33) with respect to η and (34) with respect to ξ, we obtain

$$\frac{1}{2}(\phi+\tilde{\phi})_{\xi\eta} = f'(\frac{\phi-\tilde{\phi}}{2})g(\frac{\phi+\tilde{\phi}}{2}), \tag{35}$$

$$\frac{1}{2}(\phi-\tilde{\phi})_{\xi\eta} = g'(\frac{\phi+\tilde{\phi}}{2})f(\frac{\phi-\tilde{\phi}}{2}), \tag{36}$$

which implies

$$\phi_{\xi\eta} = F(\phi) = f'(\frac{\phi-\tilde{\phi}}{2})g(\frac{\phi+\tilde{\phi}}{2}) + g'(\frac{\phi+\tilde{\phi}}{2})f(\frac{\phi-\tilde{\phi}}{2}), \tag{37}$$

$$\tilde{\phi}_{\xi\eta} = F(\tilde{\phi}) = f'(\frac{\phi-\tilde{\phi}}{2})g(\frac{\phi+\tilde{\phi}}{2}) - g'(\frac{\phi+\tilde{\phi}}{2})f(\frac{\phi-\tilde{\phi}}{2}). \tag{38}$$

Introducing functions $v = (\phi+\tilde{\phi})/2$ and $w = (\phi-\tilde{\phi})/2$, we rewrite the previous equations as

$$F(v+w) = g(v)f'(w) + g'(v)f(w), \tag{39}$$

$$F(v-w) = g(v)f'(w) - g'(v)f(w). \tag{40}$$

Bäcklund Transformations and integrability

Differentiating either of these equations with respect to v and then w, and either adding or subtracting the results, we obtain

$$\frac{g''(v)}{g(v)} = \frac{f''(w)}{f(w)} = \lambda = const.$$

Thus,

$$g'' = \lambda g, \quad f'' = \lambda f,$$

and λ can be assumed to be $-1, 0, 1$. For $\lambda = -1$ we find

$$g(v) = \beta \sin v, \quad f(w) = \alpha \sin w. \tag{41}$$

Substituting (41) into (39) and (40), we obtain $F(\phi) = \sin\phi$ for $\beta = \alpha^{-1}$, giving us the BT for the sine-Gordon equation. The choice of $\lambda = 1$ yields the sinh-Gordon equation, $\lambda = 0$ - the linear Klein-Gordon equation. Thus, the SG equation is a very special equation (existence of the BT for it is related to the integrability of this equation by the IST, which was establishded by Takhtajan and Faddeev, and AKNS 1974).

Appendix 6: Symmetries of the SG equation

The following continuous and discrete symmetries are admitted by the SG equation:

$$t \to t + t_0, \quad x \to x, \quad \phi \to \phi \quad \text{(shift in t)},$$
$$t \to t, \quad x \to x + x_0, \quad \phi \to \phi \quad \text{(shift in x)},$$
$$t \to t, \quad x \to x, \quad \phi \to \phi + 2\pi n, \quad \text{n is any integer},$$
$$\text{(discrete shifts in } \phi\text{)},$$

$$t \to -t, \quad x \to x, \quad \phi \to \phi \quad \text{(reflection in } t\text{)},$$
$$t \to t, \quad x \to -x, \quad \phi \to \phi \quad \text{(reflection in } x\text{)},$$
$$t \to t, \quad x \to x, \quad \phi \to -\phi \quad \text{(reflection in } \phi\text{)},$$

$$t \to \frac{t - cx}{\sqrt{1 - c^2}}, \quad x \to \frac{x - ct}{\sqrt{1 - c^2}}, \quad \phi \to \phi,$$
$$\text{(Lorenz transformation)}.$$

For example, if ϕ is a solution, then $-\phi$ is a solution as well, etc.

7.1 Brief historical overview

1834 John Scott Russel observes "the great wave of translation" (single isolated hump of permanent form) propagating along the Edinburgh - Glasgow Canal:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water...it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed..." (from 1844 'Report on Waves')

1871-1877 Boussinesq obtains the analytical solution

$$u(x,t) = a\,\mathrm{sech}^2[\beta(x - ct)].$$

1895 Korteweg and de Vries derive the equation (KdV)

$$u_t - 6uu_x + u_{xxx} = 0.$$

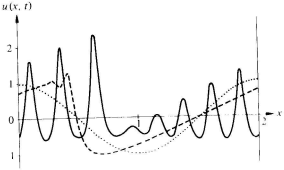
7.1 Brief historical overview

1955 Fermi, Pasta, Ulam (FPU) problem (see Lecture 6).

1965 Zabusky and Kruskal consider the FPU problem in the continuum limit (KdV), introduce the term "soliton":

$$u_t + uu_x + \delta^2 u_{xxx} = 0\,, \qquad \delta = 0.022 \tag{1}$$

with initial conditions $u(x,0) = \cos \pi x$, $0 \leq x \leq 2$, periodic boundary conditions.



- generation of solitary waves
- elastic interactions
- recurrence

7.1 Brief historical overview

1967-1974 Gardner, Green, Kruskal, Miura (GGKM) discover the infinite number of conservation laws and the Inverse scattering transform.

1968 Lax reformulates the method in the operator form, which was important for the generalisations of the method.

1972 Zakharov and Shabat develop the method for the Nonlinear Shrödinger (NLS) equation.

1972-1974 several other equations are shown to be integrable (Zakharov, Manakov, ...)

1974 Ablowitz, Kaup, Newell, Segur (AKNS) develop a general scheme, which shows that the method can be applied to many other equations.

Further developments (finite-gap solutions, 2D, etc.)

7.1 Brief historical overview

It was known that Burgers' equation

$$u_t + uu_x - u_{xx} = 0,$$

combining effects of nonlinearity and diffusion may be reduced to the linear heat equation

$$\phi_t - \phi_{xx} = 0$$

by the transformation $u = -2\frac{\phi_x}{\phi}$ (Cole 1951, Hopf 1950).
It is believed (see Whitham) that the analogy of the KdV equation with Burger's equation has motivated GGKM to consider $u \sim \frac{\phi_{xx}}{\phi}$ and take it further to

$$u = \frac{\phi_{xx}}{\phi} + \lambda.$$

Rewritten as

$$\phi_{xx} + (\lambda - u)\phi = 0 \tag{2}$$

this was the well-known time-independent Schrödinger equation from quantum mechanics (the Sturm-Liouville equation with potential u and eigenvalue λ).

7.1 Brief historical overview

Moreover, it turned out that the KdV equation (7) can be viewed as a compatibility condition for two linear differential equations for the **same** auxiliary function $\phi(x, t; \lambda)$:

$$\mathbf{L}\phi \equiv \left(-\partial_{xx}^2 + u\right)\phi = \lambda\phi\,, \tag{3}$$

$$\begin{aligned} \phi_t = \mathbf{A}\phi &\equiv \left(-4\partial_{xxx}^3 + 6u\partial_x + 3u_x + C\right)\phi & (4)\\ &= (-u_x + C)\phi + (4\lambda + 2u)\phi_x\,. & (5) \end{aligned}$$

Here λ is a spectral parameter (which can generally depend on time) and $C(\lambda, t)$ is determined by the normalisation of ϕ. Equation (3) with appropriate boundary conditions constitutes the spectral problem and Eq. (4) the evolution problem. Direct calculation shows that the compatibility condition $(\phi_{xx})_t = (\phi_t)_{xx}$ yields the KdV equation (7) for $u(x, t)$ provided

$$\lambda_t = 0\,, \tag{6}$$

that is the KdV evolution is isospectral i.e. it preserves the spectrum λ of the operator **L** in (3).

The operators **L** and **A** in (3), (4) are often referred to as the Lax pair.

Remarks

- Equation (3): $-\partial_{xx}^2 + u(x)\phi = \lambda\phi$ is the quantum-mechanical Schrödinger equation describing the wave function ϕ of a quantum particle with energy $-\lambda$ in the potential $u(x,t)$. Thus, one can take advantage of the well-developed direct and inverse scattering problems for the Schrödinger operator $\mathbf{L}$. The representation of the KdV equation as a compatibility condition of the two linear equations, and the direct and inverse scattering problems for one of them form the basis of the Inverse Scattering Transform.
- The KdV equation (7) can be represented in an operator form as $\mathbf{L}_t = \mathbf{LA} - \mathbf{AL} \equiv [\mathbf{L}, \mathbf{A}]$. This (Lax) operator representation provides a route for constructing the KdV hierarchy by appropriate choice of the operator $\mathbf{A}$. Indeed, given the $\mathbf{L}$-operator (3) the $\mathbf{A}$-operator in the Lax pair is determined up to an operator commuting with $\mathbf{L}$, which makes it possible to construct an infinite number of equations associated with the same spectral problem but having different evolution problems. Such 'higher' KdV equations play important role in constructing nonlinear multiperiodic (multiphase) solutions to the original KdV equation (7).

7.2 Initial value problem (Cauchy problem)

We consider the KdV equation in its canonical dimensionless form

$$u_t - 6uu_x + u_{xxx} = 0\,. \tag{7}$$

We shall be interested in solving the KdV equation (7) in the class of functions decaying sufficiently fast together with their first derivatives far from the origin. With this aim in view, we consider initial data

$$u(x,0) = u_0(x)\,, \qquad u_0(x) \to 0\,, \quad u_0'(x) \to 0 \quad \text{as} \quad |x| \to \infty\,. \tag{8}$$

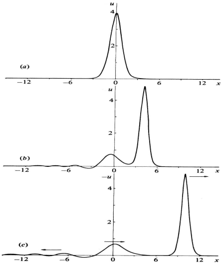


Figure: Decay of a localised initial profile into solitons and radiation (showing $-u$, from Drazin and Johnson).

7.3 Direct Scattering Problem

We consider the KdV equation in the class of functions sufficiently rapidly decaying as $|x| \to \infty$. (One can impose the Faddeev's condition

$$\int_{-\infty}^{+\infty} (1 + |x|)|u(x)|dx < \infty \,, \tag{9}$$

which ensures applicability of the scattering analysis in the sequel, but this is not the necessary condition.)
Now we turn to the Schrödinger equation (3)

$$-\partial_{xx}^2 \phi + u\phi = \lambda\phi.$$

The **spectrum** of the Schrödinger equation (3) is the set of values λ for which there is a bounded solution $\phi(x)$ for all x.

Direct Scattering Problem: For a given potential $u(x)$, the problem is to find the **spectrum** of the linear operator $\mathbf{L} = -\partial_{xx}^2 + u$, and to construct the corresponding functions $\phi(x; \lambda)$.

7.3 Direct Scattering Problem: discrete spectrum

Since $u \to 0$ as $x \to \pm\infty$, $\phi'' \sim -\lambda\phi$ as $x \to \pm\infty$, and ϕ is asymptotically a linear combination of $e^{\pm i\sqrt{\lambda}x}$, implying that ϕ decays exponentially at infinity if $\lambda < 0$ and oscillates at infinity if $\lambda > 0$. The spectrum is divided into 2 parts, respectively.

Discrete (or point) spectrum:

If $\lambda < 0$, we let $k = \sqrt{-\lambda} > 0$, and choose

$$\phi(x) \sim \alpha e^{kx} \quad \text{as} \quad x \to -\infty$$

to have a bounded solution at negative infinity. Then, as $x \to +\infty$, generally

$$\phi(x) \sim \beta e^{kx} + \gamma e^{-kx},$$

where the constants β and γ depend of α, k and u. However, for some values of λ (called discrete eigenvalues of the Schrödinger operator) $\beta = 0$ and $\phi(x)$ may be bounded also as $x \to +\infty$, implying that

$$\phi \to 0 \quad \text{as} \quad x \to \pm\infty.$$

These functions are called eigenfunctions (bound states).

7.3 Direct Scattering Problem: discrete spectrum

It is easy to see that if $u > 0$, then there are no discrete eigenvalues. Indeed, $\lambda\phi = (-\frac{d^2}{dx^2} + u)\phi$, implying that

$$\lambda = \frac{< \phi, (-\frac{d^2}{dx^2} + u)\phi >}{< \phi, \phi >} = \frac{< \phi', \phi' > + < u\phi, \phi >}{< \phi, \phi >}.$$

Thus, if $u(x) > 0$, then $\lambda > 0$, and there are no discrete eigenvalues. Therefore, to have discrete eigenvalues $u(x)$ has to take negative values for some x.

It is known that there is a finite number of discrete eigenvalues λ_n belonging to the square-integrable eigenfunctions ϕ_n (i.e. such functions that $\int_{-\infty}^{\infty} \phi_n^2 dx < \infty$):

$$\min(u) < \lambda_1 < \lambda_2 < \cdots < \lambda_p < 0.$$

There are two standard choices for the normalisation of ϕ_n:

- $\phi_n \sim e^{-k_n x}$ as $x \to \infty$ (the Jost solutions),
- $\int_{-\infty}^{\infty} \phi_n^2 dx = 1$. (Then $\phi_n \sim c_n e^{-k_n x}$ as $x \to +\infty$.) We choose this normalisation.

7.3 Direct Scattering Problem: continuous spectrum

Continuous Spectrum:

If $\lambda > 0$ then ϕ oscillates as $x \to \pm\infty$. It is known that solutions are bounded for all $\lambda > 0$, but ϕ is not square-integrable. For the continuous spectrum we write $\sqrt{\lambda} = k$ and define the solution by scattering states:

$$\phi \sim e^{-ikx} + b(k)e^{ikx} \quad \text{as} \quad x \to +\infty\,, \tag{10}$$

$$\phi \sim a(k)e^{-ikx} \quad \text{as} \quad x \to -\infty\,. \tag{11}$$

This solution of the Schrödinger Eq. (3) describes scattering *from the right* of the incident wave e^{-ikx} on the potential $u(x)$. Then $b(k)$ represents a reflection coefficient and $a(k)$ a transmission coefficient. The functions $b(k)$ and $a(k)$ are not independent and the scattering data can be characterized by a single function $b(k)$. In particular, $|a|^2 + |b|^2 = 1$ (conservation of energy in the scattering). To prove that, one needs to consider the Wronskian of ϕ and ϕ^*:

$$W(\phi, \phi^*) = \phi\phi^{*'} - \phi^*\phi'.$$

Firstly, $\frac{d}{dx}W(\phi, \phi^*) = \phi\phi^{*''} - \phi^*\phi'' = 0$. Then, since $W(\phi, \phi^*) = const$, it can be calculated at $+\infty$ and $-\infty$, yielding $|a|^2 + |b|^2 = 1$.

7.3 Direct Scattering Problem: time-evolution

Further, when

$$u = \frac{\phi_{xx}}{\phi} + \lambda$$

is substituted into the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0,$$

the outcome may be written in the form

$$\phi^2 \frac{d\lambda}{dt} + \frac{\partial}{\partial x}(\phi_x Q - \phi Q_x) = 0, \tag{12}$$

where

$$Q = \phi_t + \phi_{xxx} - 3(u + \lambda)\phi_x$$

$$= \phi_t + u_x\phi - 2(u + 2\lambda)\phi_x.$$

This is applicable to any eigenfunction and corresponding eigenvalue.

7.3 Direct Scattering Problem: time-evolution of discrete eigenvalues

Evolution of spectral data:

Let us first consider the bound states (discrete spectrum). Let $\lambda = -k_n^2$ and $\phi = \phi_n$. Integrating (12), we obtain

$$-(k_n^2)_t \int_{-\infty}^{\infty} \phi_n^2 dx = [\phi_{n_x} Q_n - \phi_n Q_{n_x}]_{-\infty}^{\infty},$$

where $Q_n = Q(k_n, \phi_n)$. Since

$$\int_{-\infty}^{\infty} \phi_n^2 dx = 1,$$

and $\phi_n, Q_n \to 0$ as $x \to \pm\infty$, the previous relation implies

$$(k_n^2)_t = 0 \quad \text{or} \quad k_n = const,$$

i.e. each discrete eigenvalue $-k_n^2$ is a constant of motion.

7.3 Direct Scattering Problem: time-evolution of normalisation constants

Further, $\phi_{n_x} Q_n - \phi_n Q_{n_x} = g_n(t) = 0$ (evaluating at infinity), and thus, $\frac{Q_n}{\phi_n} = h_n(t)$, implying $Q_n\phi_n = h_n\phi_n^2$ or

$$\frac{1}{2}(\phi_n^2)_t + (u\phi_n^2 - 2\phi_{n_x}^2 + 4k_n^2\phi_n^2)_x = h_n\phi_n^2.$$

Integrating, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{\infty}\phi_n^2 dx\right) = h_n\int_{-\infty}^{\infty}\phi_n^2 dx$$

implying that $h_n = 0$ (since $\int_{-\infty}^{\infty}\phi_n^2 dx = 1$).
Therefore,

$$Q_n = \phi_{n_t} + u_x\phi_n - 2(u - 2k_n^2)\phi_{n_x} = 0$$

- a time-evolution equation for $\phi_n(x;t)$.

Since $u, u_x \to 0$ and $\phi_n(x,t) \sim c_n(t)e^{-k_n x}$ as $x \to +\infty$, then considering the time-evolution equation at infinity we obtain

$$\frac{dc_n}{dt} - 4k_n^3 c_n = 0 \quad \text{or} \quad c_n(t) = c_n(0)e^{4k_n^3 t},$$

where $c_n(0)$ are the normalisation constants determined at $t = 0$.

7.3 Direct Scattering Problem: time-evolution of transmission and reflection coefficients

For the continuos spectrum, $\lambda = k^2 (> 0)$. We choose $k = const \quad (\frac{d\lambda}{dt} = 0)$, then since

$$\phi^2 \frac{d\lambda}{dt} + \frac{d}{dx}(\phi_x Q - \phi Q_x) = 0,$$

we obtain $\phi_x Q - \phi Q_x = g(t,k)$. For the continuous eigenfunction

$$\phi(x;t,k) \sim a(k;t) e^{-ikx} \quad \text{as} \quad x \to -\infty,$$

then

$$Q \sim (\frac{da}{dt} + 4ik^3 a) e^{-ikx} \quad \text{as} \quad x \to -\infty,$$

and $\phi_x Q - \phi Q_x \to 0$ as $x \to -\infty$. Thus, $g(t,k) = 0$ and $\frac{Q}{\phi} = h(t;k)$ or $Q = h\phi$. As $x \to -\infty$ this implies

$$\frac{da}{dt} + 4ik^3 a = ha.$$

7.3 Direct Scattering Problem: time-evolution of transmission and reflection coefficients

As $x \to +\infty$

$$Q(x,t;k) \sim \frac{db}{dt} e^{ikx} + 4ik^3(e^{-ikx} - be^{ikx}),$$

then

$$\frac{db}{dt} e^{ikx} + 4ik^3(e^{-ikx} - be^{ikx}) = h(e^{-ikx} + be^{ikx}).$$

From above,

$$\frac{db}{dt} - 4ik^3 b = hb, \quad h(t;k) = 4ik^3,$$

which implies $\frac{da}{dt} = 0$. Hence, we obtain the time evolution of the transmission and reflection coefficients

$$a(k;t) = a(k;0); \quad b(k;t) = b(k;0)e^{8ik^3 t}, \quad \text{for} \quad t \geq 0.$$

7.3 Direct scattering problem: summary

Summary I (direct scattering problem):

Scattering data is found by solving

$$\phi_{xx} + (\lambda - u_0(x))\phi = 0,$$

where $u_0(x) = u(x,0)$ is the initial condition for the KdV equation. It consists of

$$\{k_n, c_n(0), b(k;0)\}.$$

Summary II (time evolution of scattering data):

Time evolution of the scattering data (when $u(x,t)$ evolves according to the KdV equation) is given by

$$\{k_n = const; \quad c_n(t) = c_n(0)e^{4k_n^3 t}; \quad b(k;t) = b(k;0)e^{8ik^3 t}\}.$$

8.1 Inverse scattering problem: continuous spectrum only

Let us consider first potentials with no discrete spectrum. Then, $\lambda = k^2$ (continuous spectrum). Let

$$\phi_+ \sim e^{ikx} \quad \text{as} \quad x \to +\infty.$$

We look for the solution in the form

$$\phi_+(x;k) = e^{ikx} + \int_x^\infty K(x,z)e^{ikz}\,dz.$$

Differentiating twice,

$$\phi_{+xx} = e^{ikx}\left(-k^2 - \frac{dK(x,x)}{dx} - ikK(x,x) - K_x(x,x)\right) + \int_x^\infty K_{xx}(x,z)e^{ikz}\,dz.$$

Also, integrating by parts twice, and requiring $K, K_z \to 0$ as $z \to +\infty$,

$$\phi_+ = e^{ikx}\left(1 + \frac{iK(x,x)}{k} - \frac{K_z(x,x)}{k^2}\right) - \frac{1}{k^2}\int_x^\infty K_{zz}(x,z)e^{ikz}\,dz.$$

8.1 Inverse scattering problem: continuous spectrum only

Substituting the above into the Sturm-Liouville equation, we obtain

$$\begin{aligned} 0 &= \phi_{+xx} + (k^2 - u)\phi_+ \\ &= -e^{ikx}\left(u + 2\frac{dK(x,x)}{dx}\right) + \int_x^\infty (K_{xx} - K_{zz} - u(x)K)e^{ikz}dz, \end{aligned}$$

which is satisfied if

$$K_{xx} - K_{zz} - u(x)K = 0 \quad \text{for} \quad z > x,$$

and

$$u(x) = -2\frac{dK(x,x)}{dx} = -2\{K_x(x,x) + K_z(x,x)\}.$$

Also, we required that $K(x,z), K_z(x,z) \to 0$ as $z \to +\infty$.

This problem is known as a Goursat problem. It is known that its solution exists and is unique.

8.1 Inverse scattering problem: continuous spectrum only

Let us consider $\hat{\phi}$ - eigenfunction for the continuous spectrum such that

$$\hat{\phi}(x;k) \sim e^{-ikx} + b(k)e^{ikx} \quad \text{as} \quad x \to +\infty,$$
$$\hat{\phi}(x;k) \sim a(k)e^{-ikx} \quad \text{as} \quad x \to -\infty.$$

Since

$$\phi_+ \sim e^{ikx} \quad \text{as} \quad x \to +\infty,$$

we will obtain the correct behaviour as $x \to +\infty$ by writing

$$\hat{\phi} = \phi_+^* + b(k)\phi_+,$$

which gives us

$$\hat{\phi} = e^{-ikx} + b(k)e^{ikx} + \int_{-\infty}^{\infty} K(x,z)e^{-ikz}dz + b(k)\int_{-\infty}^{\infty} K(x,z)e^{ikz}dz,$$

where $K(x,z) = 0$ for $z < x$.

Then,

$$\int_{-\infty}^{\infty} K(x,z)e^{-ikz}dz = \hat{\phi} - e^{-ikx} - b(k)e^{ikx} - b(k)\int_{-\infty}^{\infty} K(x,z)e^{ikz}dz,$$

Inverting both sides (Fourier transform),

$$\begin{aligned} K(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \{&\hat{\phi}(x;k) - e^{-ikx} - b(k)e^{ikx} \\ &-b(k)\int_{-\infty}^{\infty} K(x,y)e^{iky}dy\}e^{ikz}dk. \end{aligned}$$

Let

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)e^{ikx}dk.$$

Then, the previous equation can be written in the form

$$K(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\hat{\phi} - e^{-ikx})e^{ikz}dk - F(x+z) - \int_{-\infty}^{\infty} K(x,y)F(y+z)dy.$$

8.1 Inverse scattering problem: Gelfand-Levitan-Marchenko equation

Choosing the appropriate integration contour in the complex k-plane and using Cauchy's theorem and Jordan's lemma, we obtain (noting that $z > x$, and $a \to 1, b \to 0$ as $|k| \to \infty$)

$$\int_{-\infty}^{\infty} (\hat{\phi} - e^{-ikx}) e^{ikz} dk = \int_{-\infty}^{\infty} (\hat{\phi} e^{ikx} - 1) e^{ik(z-x)} dk = 0.$$

Then,

$$K(x,z) + F(x+z) + \int_{x}^{\infty} K(x,y) F(y+z) dy = 0, \quad \text{for} \quad z > x$$

(Gelfand-Levitan-Marchenko (GLM) equation).

8.2 Inverse scattering problem: discrete and continuous spectrum

What changes if there is discrete spectrum?

The function

$$(\hat{\phi}e^{ikx} - 1)e^{ik(z-x)}$$

now has poles in the upper half-plane at

$$ik_1, \dots, ik_N.$$

Choosing the appropriate integration contour in the complex k-plane and using the Cauchy's residue theorem and Jordan's lemma, we obtain

$$\int_{-\infty}^{\infty} (\hat{\phi} - e^{-ikx})e^{ikz} dk = 2\pi i \sum_{n=1}^{N} \mathrm{res}\, (\hat{\phi}e^{ikz})\,|_{k=ik_n} \;. \tag{13}$$

We need to find the residues.

8.2 Inverse scattering problem: discrete and continuous spectrum

Consider

$$\phi_-(x;k) = e^{-ikx} + \int_{-\infty}^{x} L(x,z)e^{-ikz}dz.$$

From the behaviour as $x \to -\infty$ and $x \to +\infty$,

$$\hat{\phi} = a(k)\phi_- = \phi_+^* + b(k)\phi_+,$$

yielding

$$\phi_- = \frac{1}{a}\phi_+^* + \frac{b}{a}\phi_+.$$

Next, we observe that $\phi_+(x;ik_n) \sim e^{-k_n x}$ as $x \to +\infty$ and $\phi_-(x;ik_n) \sim e^{k_n x}$ as $x \to -\infty$ (Jost solutions for discrete eigenfunctions). Then,

$$\phi_n = c_n\phi_+(x;ik_n) = d_n\phi_-(x;ik_n),$$

where c_n is our (old) normalisation constant.

8.2 Inverse scattering problem: discrete and continuous spectrum

Next, we observe that the Wronskian $W(\phi_-, \phi_+) = const$ (check that $\frac{dW(\phi_-,\phi_+)}{dx} = 0$) and we can evaluate this constant as $x \to +\infty$, obtaining

$$W(\phi_-, \phi_+) = \frac{2ik}{a}.$$

Differentiating with respect to k,

$$\frac{dW(\phi_-, \phi_+)}{dk} = W(\phi_{-k}, \phi_+) + W(\phi_-, \phi_{+k}) = \frac{2i}{a} - 2ik\frac{a'}{a^2}. \qquad (14)$$

Also, differentiating $\phi_{xx} + (k^2 - u)\phi = 0$ we have

$$\phi_{xxk} + 2k\phi + (k^2 - u)\phi_k = 0,$$

while multiplying by ϕ_k we obtain

$$\phi_k\phi_{xx} + (k^2 - u)\phi\phi_k = 0.$$

Using the above,

$$\frac{dW(\phi_k, \phi)}{dx} = 2k\phi^2.$$

Let us choose $\phi = \phi_+, k = ik_n$ and integrate from $-\infty$ to ∞.

8.2 Inverse scattering problem: discrete and continuous spectrum

$$\lim_{x\to+\infty} W_n(\phi_{+k},\phi_+) - \lim_{x\to-\infty} W_n(\phi_{+k},\phi_+) = 2ik_n \int_{-\infty}^{\infty} \phi_+^2\, dx$$

$$= 2ik_n \int_{-\infty}^{\infty} \left(\frac{\phi_n}{c_n}\right)^2 dx = \frac{2ik_n}{c_n^2}, \qquad (15)$$

since $\int_{-\infty}^{\infty} \phi_n^2 dx = 1$. From (14),

$$\lim_{x\to-\infty} W_n(\phi_{-k},\phi_+) + \lim_{x\to-\infty} W_n(\phi_-,\phi_{+k}) = \lim_{x\to-\infty} [\frac{2i}{a} - 2ik\frac{a'}{a^2}].$$

Evaluating at $k = ik_n$, we obtain

$$\lim_{x\to-\infty} W_n(\phi_{-k},\frac{d_n}{c_n}\phi_-) + \lim_{x\to-\infty} W_n(\frac{c_n}{d_n}\phi_+,\phi_{+k}) = 2k_n\frac{a'}{a^2}. \qquad (16)$$

Now consider (15) - $\frac{d_n}{c_n}$ (16) to obtain

$$\lim_{x\to+\infty} W_n(\phi_{+k},\phi_+) - \frac{d_n^2}{c_n^2}\lim_{x\to-\infty} W_n(\phi_{-k},\phi_-) = 0 = \frac{2ik_n}{c_n^2} - \frac{2k_n d_n}{c_n}\frac{a'}{a^2}.$$

8.2 Inverse scattering problem: discrete and continuous spectrum

This yields $\frac{a'}{a^2} = \frac{i}{d_n c_n}$. Then,

$$\text{res } a(k)|_{k=ik_n} = -\frac{a^2}{a'}|_{k=ik_n} = id_n c_n,$$

and we can find the required residue:

$$\begin{aligned} \text{res } \left(\hat{\phi} e^{ikz}\right)|_{k=ik_n} &= \text{res } a(k)\phi_-(x;k)e^{ikz}|_{k=ik_n} \\ &= \text{res } a(k)\left(\frac{c_n}{d_n}\phi_+(x;k)\right)e^{ikz}|_{k=ik_n} \\ = \frac{c_n}{d_n}\phi_+(x;ik_n)e^{-k_n z} id_n c_n &= ic_n^2\phi_+(x;ik_n)e^{-k_n z}. \end{aligned}$$

Returning to the formula (13):

$$\int_{-\infty}^{\infty} (\hat{\phi} - e^{-ikx})e^{ikz} dk = -2\pi \sum_{n=1}^{N} c_n^2 e^{-k_n z}\{e^{-k_n x} + \int_x^{\infty} K(x,y)e^{-k_n y} dy\}.$$

8.2 Inverse scattering problem: Gelfand-Levitan-Marchenko equation

Therefore,

$$K(x,z)+F(x+z)+\int_x^\infty K(x,y)F(y+z)dy$$
$$=-\sum_{n=1}^{N} c_n^2 e^{-k_n z}\{e^{-k_n x}+\int_x^\infty K(x,y)e^{-k_n y}dy\}.$$

If we redefine F as

$$\tilde{F}(x)=F(x)+\sum_{n=1}^{N} c_n^2 e^{-k_n x}=\sum_{n=1}^{N} c_n^2 e^{-k_n x}+\frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)e^{ikx}dk,$$

then the equation takes the same form as before:

$$K(x,z)+\tilde{F}(x+z)+\int_x^\infty K(x,y)\tilde{F}(y+z)dy=0$$

(Gelfand-Levitan-Marchenko (GLM) equation).

Summary III (inverse scattering problem):

Given the scattering data $\{-k_n^2, c_n(t), b(k;t)\}$ for the equation

$$\phi_{xx} + (\lambda - u(x;t))\phi = 0$$

(we view t as a parameter), the potential $u(x;t)$ can be found as

$$u(x;t) = -2\frac{dK(x,x;t)}{dx},$$

where $K(x,y;t)$ is a solution of the Gelfand-Levitan-Marchenko equation

$$K(x,z;t) + F(x+z;t) + \int_x^\infty K(x,y;t)F(y+z;t)dy = 0$$

with

$$F(x;t) = \sum_{n=1}^{N} c_n^2(t)e^{-k_n x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k;t)e^{ikx}dk. \tag{17}$$

8.4 Integration of the KdV equation by the IST: summary

Thus, we have the scheme of integration of the KdV equation by the IST:

$$u(x,0) \mapsto S(0) \to S(t) \mapsto u(x,t)\,. \tag{18}$$

It is essential that at each step of this algorithm one has to solve a **linear problem**. One can notice that the described method of integration of the KdV equation is somewhat similar to the Fourier method for integration of linear partial differential equations with the role of the direct and inverse Fourier transform played by the direct and inverse scattering problems.

9.1 Reflectionless potentials and N-soliton solutions

Example: $N = 1$
Assuming $b(k) = 0$, $N = 1$ in (17) we obtain:
$F(x,t) = c(0)^2 \exp(-kx + 8k^3 t)$, where $c \equiv c_1$, $k \equiv k_1$. Then the solution of the GLM equation can be sought in the form $K(x,y;t) = M(x,t)\exp(-ky)$. After some simple algebra we get

$$M(x,t) = \frac{-2kc(0)^2 \exp(-kx + 8k^3 t)}{2k + c(0)^2 \exp(-2kx + 8k^3 t)} \,. \qquad (19)$$

As a result, we obtain

$$u = -2k^2 \mathrm{sech}^2(k(x - 4k^2 t - x_0)) \,, \qquad (20)$$

which is the solitary wave of the amplitude $a_s = 2k^2$ propagating to the right with the velocity $c_s = 4k^2$ and having the initial phase

$$x_0 = \frac{1}{2k} \ln \frac{c(0)^2}{2k} \,. \qquad (21)$$

9.1 Reflectionless potentials and N-soliton solutions

For arbitrary $N \in \mathbb{N}$ and $b(k) \equiv 0$ we have from (17)

$$F(x,t) = \sum_{n=1}^{N} c_n(t)^2 \exp(-k_n x)\,, \tag{22}$$

and therefore, seek the solution of the GLM equation in the form

$$K(x,y;t) = \sum_{n=1}^{N} M_n(x,t) \exp(-k_n y)\,. \tag{23}$$

One can obtain the general (Kay-Moses) representation for the reflectionless potential

$$u_N(x,t) = -2 \frac{\partial^2}{\partial x^2} \ln \det \mathrm{A}(x,t)\,. \tag{24}$$

Here A is the $N \times N$ matrix given by

$$A_{mn} = \delta_{mn} + \frac{c_n(0)^2}{k_n + k_m} e^{-(k_n+k_m)x + 8k_n^3 t}\,, \tag{25}$$

where δ_{km} is the Kronecker delta.

9.1 Reflectionless potentials and N-soliton solutions

Analysis of formulae (24), (25) shows that for $t \to \pm\infty$ the solution of the KdV equation corresponding to the reflectionless potential can be asymptotically represented as a **superposition of N single-soliton solutions** propagating to the right and ordered in space by their speeds (amplitudes):

$$u_N(x,t) \sim -\sum_{n=1}^{N} 2k_n^2 \mathrm{sech}^2[k_n(x - 4k_n^2 t - x_n^{\pm})] \quad \text{as} \quad t \to \pm\infty\,, \tag{26}$$

where the amplitudes of individual solitons are given by $a_n = 2k_n^2$ and the positions $\mp x_n$ of the n-th soliton as $t \to \mp\infty$ are given by

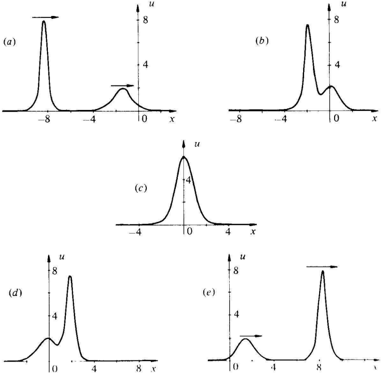
$$x_n^{\pm} = \frac{1}{2k_n} \ln \frac{c_n(0)^2}{2k_n} \pm \frac{1}{2k_n} \left\{ \sum_{m=1}^{n-1} \ln \left| \frac{k_n - k_m}{k_n + k_m} \right| - \sum_{m=n+1}^{N} \ln \left| \frac{k_n - k_m}{k_n + k_m} \right| \right\} . \tag{27}$$

Phase shift $x_n^{+} - x_n^{-}$ is the only result of interaction (Zakharov 1971, GGKM 1974).

One can infer from Eq. (26) that at $t \gg 1$, the tallest soliton with $n = N$ is at the front followed by the progressively shorter solitons behind, forming thus the triangle amplitude (velocity) distribution characteristic for noninteracting particles. At $t \to -\infty$ we get the reversed picture. The full solution (24), (25) thus describes the interaction (collision) of N solitons at finite times. For this reason it is called N-soliton solution.

The N-soliton solution is characterised by $2N$ parameters $k_1, \dots, k_N$, $c_1(0), \dots, c_N(0)$. Owing to isospectrality ($k_n = constant$), the solitons preserve their amplitudes (and velocities) in the interactions; the only change they undergo is the phase shift $\delta_n = x_n^+ - x_n^-$ due to collisions.

2-soliton solution: interaction of two solitons



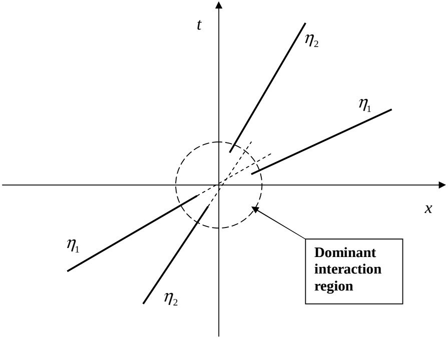
$$-u_2(x,t) \sim 2k_1^2 \mathrm{sech}^2[k_1(x - 4k_1^2 t \mp x_1)] + 2k_2^2 \mathrm{sech}^2[k_2(x - 4k_2^2 t \mp x_2)]$$

as $t \to \pm\infty$

9.1 Reflectionless potentials. Example: $N = 2$

For a two-soliton collision with $k_1 > k_2$ the phase shifts as $t \to +\infty$ are

$$\delta_1 = 2x_1 = \frac{1}{k_1} \ln\left(\frac{k_1 + k_2}{k_1 - k_2}\right) , \quad \delta_2 = 2x_2 = -\frac{1}{k_2} \ln\left(\frac{k_1 + k_2}{k_1 - k_2}\right) . \quad (28)$$



It follows from formula above that, as a result of the interaction, the taller soliton gets an additional shift forward by the distance δ_1 while the shorter soliton is shifted backwards by the distance $-\delta_2$.

9.2 N-soliton solutions: summary

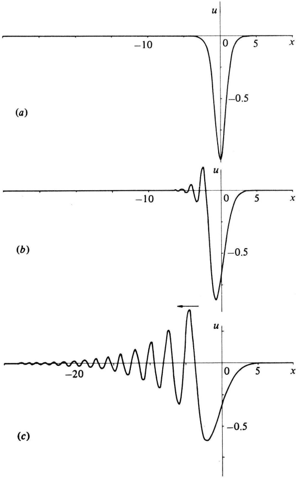
- The general phase-shift formula (27) and its 2-soliton reduction (28) are relevant only for sufficiently large times when individual solitons are separated enough for the asymptotic representation (26) to be applicable.
- One of the remarkable consequences of the formula (27) for the phase shifts is that the solitons in the N-soliton solution of the KdV equation interact only pairwise, i.e. the 'multi-particle' effects in the soliton interactions are absent.

9.3 Solitonsless potentials: nonlinear radiation

In contrast to the reflectionless potentials, characterised by the discrete spectrum, there are potentials characterised by purely continuous spectrum. In particular, this is the case for all positive potentials $u_0(x) \geq 0$. Now one has to deal with the second term alone in formula (17). In this case, the general expression for the solution similar to the N-soliton solution is not available.

An asymptotic analysis shows that, under the long-time evolution, such a potential transforms into the linear dispersive wave (the radiation) described by the linearised KdV equation but the detailed structure of this wave and its dependence on the initial data are complicated. However, the local qualitative behaviour is physically transparent: the linear radiation propagates to the left with the velocity close to the group velocity $c_g = -3k^2$ of a linear wavepacket and the lowest rate of the amplitude decay is consistent with the momentum conservation law in linear modulation theory.

Evolution of a “solitonless” profile



(Shows solution of $u_t + 6uu_x + u_{xxx} = 0$ for $u_0(x) < 0$.)

9.4 Evolution of an arbitrary decaying potential

Now we can qualitatively describe an asymptotic evolution of an arbitrary decaying potential satisfying the condition (9). The spectrum of such a potential generally contains both discrete and continuous components. The discrete component is responsible for the appearance in the asymptotic solution of the chain of solitons ordered by the amplitudes and moving to the right. At the same time, the continuous component contributes to the linear dispersive wave propagating to the left.
Thus, the long-time asymptotic outcome of the general KdV initial-value problem for decaying initial data can be represented in the form

$$u(x,t) \sim -\sum_{n=1}^{N} 2k_n^2 \mathrm{sech}^2(k_n(x-4k_n^2t-x_n)) + \text{linear radiation}\,, \qquad (29)$$

where the soliton amplitudes $a_n = 2k_n^2$ and the initial phases x_n, as well as the parameters of the radiation component, are determined from the scattering data for the initial potential. For a number of potentials the direct scattering problem can be solved explicitly (see example in Appendix 1). In many cases, one can use asymptotic estimates (see semi-classical asymptotics for 'large-scale' potentials in Appendix 2).

9.4 Arbitrary initial potential: comments

For a given initial potential $u_0(x)$:

- A simple **sufficient** condition for the appearance of at least one discrete eigenvalue in the spectrum (i.e. of a soliton in the solution) is

$$\int_{-\infty}^{+\infty} u_0(x)dx < 0\,. \tag{30}$$

- The upper bound for the number N of solitons in the solution can be estimated by the formula

$$N \leq 1 + \int_{-\infty}^{+\infty} |x||u_0(x)|dx\,. \tag{31}$$

9.3 Conservation laws (revisited)

The KdV equation $u_t - 6uu_x + u_{xxx} = 0$ can be represented in the form of a conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(-3u^2 + u_{xx}\right) = 0\,. \tag{32}$$

Indeed, equation (32) implies conservation of the integral 'mass',

$$\frac{d}{dt}\int_{-\infty}^{+\infty} u dx = 0 \tag{33}$$

provided the function $u(x,t)$ **vanishes** together with its spatial derivatives as $x \to \pm\infty$. (Remark: spatially **periodic** $u(x,t)$ is another possibility). Using simple algebra one can also obtain conservation equations for the 'momentum'

$$\frac{\partial}{\partial t}\frac{u^2}{2} + \frac{\partial}{\partial x}\left(uu_{xx} - \frac{1}{2}u_x^2 - 2u^3\right) = 0\,, \tag{34}$$

and for the 'energy'

$$\frac{\partial}{\partial t}\left(u^3 + \frac{1}{2}u_x^2\right) + \frac{\partial}{\partial x}\left(-\frac{9}{2}u^4 + 3u^2u_{xx} - 6uu_x^2 + u_x u_{xxx} - \frac{1}{2}u_{xx}^2\right) = 0\,. \tag{35}$$

9.3 Conservation laws

Miura transformation: $u = v^2 + v_x$. Substitution into the KdV equation gives

$$(2v + \frac{\partial}{\partial x})(v_t - 6v^2 v_x + v_{xxx}) = 0,$$

where $v_t - 6v^2 v_x + v_{xxx} = 0$ is called the modified KdV equation (mKdV).

Gardner transformation: $u = w + \epsilon w_x + \epsilon^2 w^2$, ϵ being an arbitrary parameter. This substitution yields

$$u_t - 6uu_x + u_{xxx} = (1 + 2\epsilon^2 w + \epsilon\partial_x)(w_t + (w_{xx} - 3w^2 - 2\epsilon^2 w^3)_x) = 0\,.$$

Now, if $w(x,t)$ satisfies the conservation law (Gardner),

$$w_t + (w_{xx} - 3w^2 - 2\epsilon^2 w^3)_x = 0\,, \tag{36}$$

then $u(x,t)$ is the solution of the KdV equation.

9.3 Conservation laws

We represent w in the form of an asymptotic expansion in powers of ϵ:

$$w = \sum_{n=0}^{\infty} \epsilon^n w_n \,.$$

Then, collecting equal powers of $\epsilon \ll 1$ we get subsequently:

$$w_0 = u \,, \qquad w_1 = -w_{0x} = -u_x \,, \qquad w_2 = -w_{1x} - w_0^2 = u_{xx} - u^2 \,, \qquad \ldots$$

Now, substituting this expansion $w = \sum_{n=0}^{\infty} \epsilon^n w_n$ into (36) we get an **infinite series** of the KdV conservation laws as coefficients at even powers of ϵ. Coefficients at odd powers are exact differentials ("trivial conservation laws"). The values

$$I_n = \int_{-\infty}^{\infty} w_{2n} dx \,, n = 0, 1, 2, \ldots$$

are called Kruskal integrals. In particular,

$$I_1 = \int u dx, I_2 = \int u^2/2 dx, I_3 = \int (u^3 + (u_x)^2/2) dx, \ldots$$

9.4 Hamiltonian structures

The KdV equation can be represented in the form

$$u_t = \frac{\partial}{\partial x}\frac{\delta F}{\delta u}, \tag{37}$$

where

$$F[u] = I_3 = \int_{-\infty}^{\infty} (u^3 + \frac{1}{2}u_x^2)dx \tag{38}$$

is the "energy" integral and the variational derivative for a functional $F[u] = \int_{x_1}^{x_2} f(x, u, u_x)dx$ is given by

$$\frac{\delta F}{\delta u} = \frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u_x}\right). \tag{39}$$

The representation (37) is a Hamiltonian form of the KdV equation with I_3 as the Hamiltonian (Gardner 1971).

See Appendix 3 for the discussion of a connection with the canonical Hamiltonian equations.

Equation (37) is supplied with the Poisson bracket

$$\{F,G\} = \int_{-\infty}^{\infty} \frac{\delta F}{\delta u} \frac{\partial}{\partial x} \left(\frac{\delta G}{\delta u} \right) dx \tag{40}$$

It can be shown that all Kruskal integrals I_k are in involution, i.e.

$$\{I_k, I_m\} = 0\,, \qquad \forall k, m\,. \tag{41}$$

Thus the KdV equation can be viewed as an infinite-dimensional integrable Hamiltonian system (by analogy with the Liouville integrability in the finite-dimensional case).

9.4 Hamiltonian structures: comments

The higher KdV flows can be defined as $u_{t_n} = \dfrac{\partial}{\partial x}\dfrac{\delta I_n}{\delta u}$, and they commute with each other (Zakharov, Faddeev 1971).
The KdV equation also has a second Hamiltonian structure (F. Magri 1978):

$$u_t = J_1 \frac{\delta I_3}{\delta u} = J_2 \frac{\delta I_2}{\delta u},$$

where $J_1 = \frac{\partial}{\partial x}$, $I_3 = \int\limits_{-\infty}^{\infty} (u^3 + \frac{1}{2}u_x^2)dx$ and $J_2 = -\frac{\partial^3}{\partial x^3} + 4u\frac{\partial}{\partial x} + 2u_x$, $I_2 = \frac{1}{2}\int_{-\infty}^{\infty} u^2 dx$, which allows one to define the recursion operator $R = J_2 J_1^{-1}$. Then, the KdV hierarchy can be obtained by repeatedly applying the recursion operator to the previous equations in the hierarchy:

$$\begin{aligned}
u_{t_0} &= u_x, \\
u_{t_1} &= Ru_x = 6uu_x - u_{xxx} = (3u^2 - u_{xx})_x, \\
u_{t_2} &= R(3u^2 - u_{xx})_x = (-\frac{\partial^3}{\partial x^3} + 4u\frac{\partial}{\partial x} + 2u_x)(3u^2 - u_{xx}) \\
&= 30u^2 u_x - 20u_x u_{xx} - 10uu_{xxx} + u_{xxxxx}, \quad \textit{etc.}
\end{aligned}$$

Appendix 1: *A sech*^{2} potential

Let us find the discrete spectrum of the Schrödinger equation

$$\Psi_{\chi\chi} + [\lambda - U(\chi)]\Psi = 0, \tag{42}$$

where, the potential is given by

$$U(\chi) = -A\ sech^2\ \frac{\chi}{l}.$$

The discrete eigenvalues for such a potential are:

$$\lambda = -k_n^2,\ k_n = \frac{1}{2l}\left[(1+4Al^2)^{1/2} - (2n-1)\right] > 0,\ n = 1, ..., N$$

(e.g., Landau and Lifshitz 1959).

$A\ sech^2$ potential

Key steps:

- From (42), ψ decays exponentially at infinity if $\lambda < 0$ and ψ oscillates sinusoidally at infinity if $\lambda > 0$ (for sufficiently rapidly decaying potential).
- If $U(\chi) > 0$ (i.e., $A < 0$), then the discrete spectrum is empty. Indeed, from (42),

$$\lambda = \frac{<\psi, (-\frac{d^2}{d\chi^2} + U(\chi))\psi>}{<\psi,\psi>} = \frac{<\psi',\psi'> + <U(\chi)\psi,\psi>}{<\psi,\psi>} > 0.$$

- Consider $A > 0$ $(U(\chi) < 0)$. Rescale χ to have $l = 1$.
- Substitution $T = \tanh x$ maps (42) to the associated Legendre equation

$$\frac{d}{dT}\left[(1-T^2)\frac{d\psi}{dT}\right] + \left[\nu(\nu+1) - \frac{\mu^2}{1-T^2}\right]\psi = 0, \qquad (43)$$

where $\nu = (-1+\sqrt{1+4A})/2,\ \mu = \sqrt{-\lambda}$.

$A\ sech^2$ potential

- Further substitution $\psi = (1 - T^2)^{\mu/2} w(u),\ u = (1 - T)/2$ maps (43) to a hypergeometric equation

$$u(1-u)w'' + [c - (1 + a + b)u]w' - abw = 0, \qquad (44)$$

with $a = \mu - \nu, b = \mu + \nu + 1, c = \mu + 1.$

- Require that ψ is finite at $T = 1$ (i.e. $u = 0$). Then, $w = {}_2F_1(a, b; c; u) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} u^n$, where ${}_2F_1[...]$ is the hypergeometric function, and $(a)_n = a(a + 1)(a + 2) \ldots (a + n - 1)$ is the Barnes symbol. This yields

$$\psi = (1 - T^2)^{\mu/2} {}_2F_1(a, b; c; \frac{1-T}{2}).$$

(Another linearly independent solution of (44) is not finite at $u = 0$.)

- Require that ψ is also finite at $T = -1$ (i.e. $u = 1$). Then, $\mu - \nu = -n$, where $n = 0, 1, 2, \ldots$. In this case the hypergeometric function is a polynomial of degree n, finite at $u = 1$ (e.g., Hochstadt 1986 or Abramowitz and Stegun 1972).

Appendix 2: Semi-classical asymptotics in the IST method

One of the important cases where some explicit analytic results of rather general form become available, occurs when the initial potential is a 'large-scale' function. Then, for positive $u_0(x)$ the Schrödinger operator (3) has a large number of bound states located close to each other so that the discrete spectrum can be characterized by a single continuous distribution function. In this case, an effective asymptotic description of the spectrum can be obtained with the use of the semi-classical Wentzel-Kramers-Brillouin (WKB) method.

We consider the KdV equation (7) with the large-scale positive initial data

$$u(x,0) = u_0(x/L) > 0\,, \qquad L \gg 1\,. \tag{45}$$

For simplicity, we assume that initial function (45) has a form of a single positive bump satisfying an additional condition

$$\int_{-\infty}^{\infty} {u_0}^{1/2} dx \gg 1\,, \tag{46}$$

An estimate following from Eq. (46) is $A^{1/2}L \gg 1$, where $A = \max(u_0)$.

Semi-classical asymptotics in the IST method

Assuming $A = \mathcal{O}(1)$ we introduce 'slow' variables $X = \epsilon x$, $T = \epsilon t$, where $\epsilon = 1/L \ll 1$ is a small parameter, into Eq. (7) to get the small-dispersion KdV equation:

$$u_T + 6uu_X + \epsilon^2 u_{XXX} = 0\,, \qquad \epsilon \ll 1 \tag{47}$$

with the initial condition

$$u(X,0) = u_0(X) \geq 0\,, \tag{48}$$

where

$$u_0(X) \ \text{ is } \mathrm{C}^1\,, \quad \text{and} \quad \int_{-\infty}^{\infty} u_0^{1/2} dX = \mathcal{O}(1)\,. \tag{49}$$

The associated Schrödinger equation (3) in the Lax pair assumes the form

$$-\epsilon^2 \phi_{XX} - u\phi = \lambda\phi\,, \qquad \epsilon \ll 1 \tag{50}$$

Semi-classical asymptotics in the IST method

The WKB analysis of the Schrödinger equation (50) yields that, for the potential $-u_0(X) \leq 0$ satisfying condition (49) the reflection coefficient is asymptotically zero,

$$\lim_{\epsilon \to 0} R(k) = 0\,, \tag{51}$$

while the eigenvalues $\lambda_n = -\eta_n^2$ ($n = 1, \ldots, N$ and $\eta_1 > \eta_2 > \cdots > \eta_N \geq 0$) are distributed in the range $-A \leq -\eta^2 \leq 0$ so that the density of the distribution of η_k's is given by the formula (Weyl's law)

$$\phi(\eta) = \frac{1}{\pi\epsilon} \int_{X^-(\eta)}^{X^+(\eta)} \frac{\eta}{\sqrt{u_0(X) - \eta^2}} dX\,, \tag{52}$$

so that $\phi(\eta)d\eta$ is the number of η_k's in the interval $(\eta; \eta + d\eta)$. Here the limits of integration $X^-(\eta) < X^+(\eta)$ are defined by $u_0(X^\pm) = \eta^2$.

Semi-classical asymptotics in the IST method

The total number of bound states N can be estimated as

$$N \sim \int_0^{A^{1/2}} \phi(\eta) d\eta = \frac{1}{\pi\epsilon} \int_{-\infty}^{+\infty} u_0^{1/2}(X) dX \gg 1 \,. \tag{53}$$

The inequality in (53) is equivalent to the condition (46) and clarifies its physical meaning. The norming constants $\beta_n(0)$ of the scattering data in the semi-classical limit are given by formulae

$$\beta_n = \exp\{\chi(\eta_n)/\epsilon\} \,, \tag{54}$$

where

$$\chi(\eta) = \eta X^+(\eta) + \int_{X^+(\eta)}^{\infty} \left(\eta - \sqrt{\eta^2 - u_0(X)}\right) dX \,. \tag{55}$$

Semi-classical asymptotics in the IST method

Now we interpret the semi-classical scattering data (51) – (55) in terms of the solution $u(X, T)$ of the small-dispersion KdV equation (47). First of all, the relation (51) implies that the potential $-u_0(X)$ is *asymptotically reflectionless* and, hence, the initial data $u_0(X)$ can be approximated by the N-soliton solution (24), (25),

$$u_0(X) \approx u_N(X/\epsilon) \qquad \text{for} \quad \epsilon \ll 1\,, \tag{56}$$

where $N[u_0] \sim \epsilon^{-1}$ is given by (53) and the discrete spectrum is defined by (52), (54), (55). Now one can use the known N-soliton dynamics for the description of the evolution of an arbitrary initial potential satisfying the condition (46). This observation served as a starting point in the series of papers by Lax, Levermore and Venakides (see their review (1994) and references therein), where the singular zero-dispersion limit of the KdV equation has been introduced and thoroughly studied.

Semi-classical asymptotics in the IST method

While the description of multisolitons at finite T turns out to be quite complicated in the zero-dispersion limit, the asymptotic behaviour as $T \to \infty$ can be easily predicted using formula (26) which implies that the asymptotic as $T \to \infty$ outcome of the evolution will be a 'soliton train' consisting of N free solitons ordered by their amplitudes $a_n = 2\eta_n^2\,,\ n = 1, \dots, N$ and propagating on a zero background. The number of solitons in the train having the amplitude within the interval $(a, a + da)$ is $f(a)da$ where the soliton amplitude distribution function $f(a)$ follows from Weyl's law (52):

$$f(a) = \frac{1}{8\pi\epsilon} \oint \frac{dX}{\sqrt{u_0(X) - a/2}}\,. \tag{57}$$

The formula (57) was obtained for the first time by Karpman (1967).

Semi-classical asymptotics in the IST method

It follows from the Karpman formula that the range of soliton amplitudes in the train is

$$0 < a < 2A, \tag{58}$$

which means that the tallest soliton has the amplitude twice as big as the amplitude of the initial perturbation, $a_{\max} = 2A$. The Karpman formula also yields the spatial distribution of solitons in the soliton train resulting from the initial perturbation $u_0(X)$. Indeed, as the speed of the soliton with the amplitude a moving on a zero background is $c_s = 2a$, its asymptotic position for $T \gg 1$ is $X \cong 2aT$, therefore, the solitons in the soliton train are spatially distributed according to a 'triangle' law

$$a \cong X/2T, \qquad 0 < X/2T < 2A, \quad X, T \gg 1. \tag{59}$$

The number of waves in the interval $(X, X + dX)$ is determined from the balance relationship

$$kdX = f(a)da, \tag{60}$$

where $k(X, T)$ is the spatial density of solitons (the soliton train wavenumber). Then, using (59) we obtain

$$k(X, T) \cong \frac{1}{2T} f\left(\frac{X}{2T}\right). \tag{61}$$

Semi-classical asymptotics in the IST method: concluding remarks

Whereas the long-time multisoliton dynamics in the semi-classical limit is simple enough, the corresponding behaviour at finite times T is quite nontrivial and reveals some remarkable features. As the studies of Lax, Levermore and Venakides showed, there is certain critical time $T_b[u_0(X)]$, after which the multisoliton solution of the small-dispersion KdV equation (47), resulting from the bump-like initial data, asymptotically to the first order in ϵ manifests itself as a *cnoidal wave* with the period and the wavelength scaled as ϵ and with the parameters (mean, amplitude, wavenumber) depending on the slow variables X, T. Moreover, Lax and Levermore obtained the evolution equations for the moments $\overline{u}(X,T)$, $\overline{u^2}(X,T)$, $\overline{u^3}(X,T)$ which turned out to coincide with the modulation equations derived much earlier by Whitham (1965).

Appendix 3: Hamiltonian structures

The connection of the Hamiltonian form (37), (40) with the canonical form of Hamilton equations for finite-dimensional dynamical systems is shown by considering the Fourier series for periodic $u(x,t)$, $u(x,t) = \sum_{-\infty}^{\infty} u_k e^{ikx}$. Then for

$$q_k = \frac{u_k}{k}\,, \quad p_k = u_{-k}\,, \quad H = \frac{i}{2\pi}F \tag{62}$$

we recover familiar Hamilton's equations

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}\,, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k} \tag{63}$$

with the Poisson bracket defined by

$$\{F,G\} = \frac{i}{2\pi}\sum_{k=-\infty}^{\infty} k\frac{\partial F}{\partial u_k}\frac{\partial G}{\partial u_{-k}} \tag{64}$$

10.1 Nonlinear Schrödinger Equation: derivation

Let us consider some dispersive nonlinear wave equation, e.g. Sine-Gordon equation:

$$u_{tt} - u_{xx} = -\sin u \quad (= -u + \frac{1}{6}u^3 - ...) \tag{2}$$

To derive the NLS equation, we look for a solution of (2) in the form of an asymptotic multiple-scales expansion

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + ..., \tag{3}$$

where $0 < \varepsilon \ll 1$ is a small amplitude parameter, and

$$u_1 = A(\tau, \chi, T, X)e^{i(kx - \omega t)} + c.c. \tag{4}$$

Here, $\tau = \varepsilon t, \chi = \varepsilon x, T = \varepsilon^2 t, X = \varepsilon^2 x$ are slow variables. Note, that this solution allows the envelope to have modulations on the time- and space scale of order ε^{-2} as well as ε^{-1} (compare with Lecture 5).

10.1 Nonlinear Schrödinger Equation: derivation

Substituting (10) into (2), and equating equal powers of ε, we obtain

$$O(\varepsilon): \quad u_{1tt} - u_{1xx} + u_1 = 0, \tag{5}$$

$$O(\varepsilon^2): \quad u_{2tt} - u_{2xx} + u_2 = -2(u_{1t\tau} - u_{1x\chi}), \tag{6}$$

$$O(\varepsilon^3): \quad u_{3tt} - u_{3xx} + u_3 = -2(u_{2t\tau} - u_{2x\chi})$$
$$-(u_{1\tau\tau} + 2u_{1tT} - u_{1\chi\chi} - 2u_{1xX}) + \frac{1}{6}u_1^3. \tag{7}$$

Then (11) is a solution of (12) provided $\omega^2 = 1 + k^2$ (dispersion relation), implying $v_g = \frac{d\omega}{dk} = \frac{k}{\omega}$ and $\frac{d^2\omega}{dk^2} = \frac{1-v_g^2}{\omega}$. Then, to avoid secular growth in (6), we require $A_\tau + v_g A_\chi = 0$, and can take $u_2 = 0$. To avoid secular growth in (7) we require $i(A_T + v_g A_X) + \frac{1-v_g^2}{2\omega}A_{\chi\chi} + \frac{1}{4\omega}A^2A^* = 0$. Finally, changing variables, $X \to X - v_g T$, $T \to T$, we obtain the Nonlinear Schrödinger (NLS) equation

$$iA_T + \beta A_{\chi\chi} + \gamma |A|^2 A = 0, \tag{8}$$

where $\beta = \frac{1}{2}\frac{d^2\omega}{dk^2}, \gamma = \frac{1}{4\omega}$.

10.2 Nonlinear Schrödinger Equation: modulational instability

The NLS equation (8) has a simple spatially uniform solution $A_0(T) = a_0 e^{i\gamma|a_0|^2 T}$ (in nonlinear optics it is often called a continuous-wave (CW) solution). Let us consider its perturbation (Stuart and DiPrima 1978):

$$A(\chi, T) = A_0(T)[1 + B(\chi, T)].$$

Linearized equation for B is $iB_T + \beta B_{\chi\chi} + \gamma|a_0|^2(B + B^*) = 0$. Looking for solutions of this equation of the form $B = B_1 e^{i(l\chi + \Omega T)} + B_2 e^{-i(l\chi + \Omega^* T)}$, we obtain a dispersion relation

$$\Omega^2 = \beta l^2(\beta l^2 - 2\gamma|a_0|^2), \tag{9}$$

where $\Omega^2 < 0$ means instability of $A_0(T)$. The simple analysis of (9) shows that if $\beta\gamma > 0$, then the wavenumbers $l < 2\sqrt{|\frac{\gamma}{\beta}|}a_0$ will be unstable. The growth rate (also, modulational-instability gain) is given by

$$g(l) = \mathrm{Im}\,[\Omega(l)] = \sqrt{\beta l^2(2\gamma|a_0|^2 - \beta l^2)}$$

The maximum growth rate is defined as $\max g(l)$.

10.2 Nonlinear Schrödinger Equation: modulational instability

This instability describes the initial stage of formation of modulated waves and is called modulational instability (also, the Benjamin-Feir instability or sideband instability).

From the viewpoint of the IST: the spectral problem for the focusing NLS allows complex eigenvalues, formation of the bright solitons is possible. Eigenvalues in the case of the defocusing NLS are real, and there are no bright solitons.

Remarks.

Stokes waves are the periodic travelling waves of finite amplitude on deep water. In 1966-1967 Benjamin showed that deep-water small (finite) amplitude waves are unstable. Feir performed experiments confirming these results. The NLS equation (8) has been derived by Zakharov (1978).

10.3 Lie symmetries of the NLS equation

In the following we use the following one-parameter groups of symmetries, admitted by the NLS equation (2):

$$t \to t + t_0, \quad x \to x, \quad q \to q \quad \text{(shift in } t\text{)},$$
$$t \to t, \quad x \to x + x_0, \quad q \to q \quad \text{(shift in } x\text{)},$$
$$t \to t, \quad x \to x - ct, \quad q \to q \ \exp[i\frac{c}{2}(x - \frac{c}{2}t)]$$
$$\text{(Galilean transformation)},$$
$$t \to a^2 t, \quad x \to ax, \quad q \to \frac{q}{a} \quad \text{(scaling)}.$$

For example, if $q(x, t)$ is a solution of (2), then due to the Galilean invariance so are

$$q(x - ct, t)\ e^{i\frac{c}{2}(x - \frac{c}{2}t)},$$

and so on.

10.4 Solitary waves of the NLS equation

We look for a solution of the NLS equation (2) of the form

$$q(x,t) = a(x)e^{i\phi(t)}. \tag{10}$$

Substituting (10) into (2), we derive

$$-a\phi_t + a_{xx} + 2\sigma a^3 = 0. \tag{11}$$

Separating variables in (11) gives $\phi_t = \frac{a_{xx}}{a} + 2\sigma a^2 = const$. Then, integrating, we obtain (up to the scaling and the shift in t):

$$\phi = st$$

(we can assume that $s = \pm 1$), and

$$a_{xx} = -2\sigma a^3 + sa. \tag{12}$$

Multiplying (12) by a_x and integrating, we arrive at

$$(a_x)^2 = -\sigma a^4 + sa^2 + C.$$

It turns out that the form of solitary waves depends on the sign of σ (the sign of the nonlinear term in the NLS equation). (As before, more general cnoidal waves can be obtained in terms of elliptic functions.)

10.5 Focusing NLS: bright solitons

Case I ($\sigma = 1$, focusing NLS, in the context of optics - "anomalous dispersion")

$$iq_t + q_{xx} + 2|q|^2 q = 0.$$

In this case, $(a_x)^2 = -a^4 + sa^2 + C$. If $a, a_x \to 0$ as $x \to \pm\infty$, then $C = 0$, and

$$\int \frac{da}{a\sqrt{s - a^2}} = \int dx.$$

For $s = 1$ we obtain the simplest form of the so-called bright soliton $a = \mathrm{sech}\ x, \ \phi = t,$ yielding $q = \mathrm{sech}\ x\ e^{it}$. (Consider the second case, $s = -1$.) Using the scaling and Galilean symmetries, we immediately obtain the two-parameter family of bright solitons:

$$q = A\ \mathrm{sech}\ A(x - ct) \exp\left[i\left(\frac{c}{2}x + \left(A^2 - \frac{c^2}{4}\right)t\right]\right.$$

Note that A and c are independent parameters. (Two more parameters can be added using shifts in x and t, but these parameters are insignificant.)

10.5 Focusing NLS: bright solitons

The larger c, the more rapid the spatial variation in x (see Fig. 1) For sufficiently large values of c the envelope will start to have spatial oscillations of the same period as the carrier wave, and the approximations that led to the NLS equation are no longer valid.

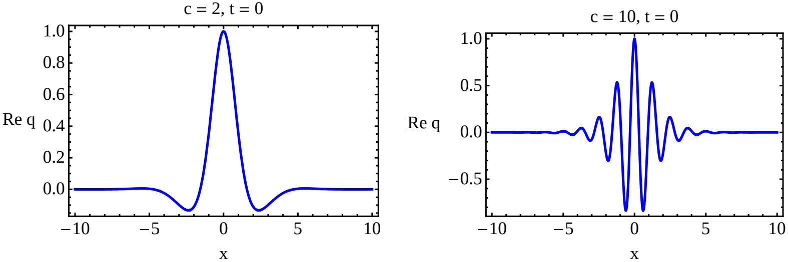


Figure: Bright solitons for $c = 2$ and $c = 10$ at $t = 0$.

The asymptotics of the focusing NLS equation with the rapidly decaying initial data consists of a series of bright solitons and the dispersive radiation (e.g., Novikov et al. 1984).

10.6 Defocusing NLS: dark solitons

Case II ($\sigma = -1$, defocusing NLS, in the context of optics - “normal dispersion”)

$$iq_t + q_{xx} - 2|q|^2 q = 0.$$

In this case, $(a_x)^2 = a^4 + sa^2 + C$, and solitary waves are rather different from those in Case I. For $s = -1$, choosing $C = \frac{1}{4}$ (when the polynomial has repeated roots) we obtain the simplest form of the so-called dark soliton $a = \frac{1}{\sqrt{2}} \tanh \frac{x}{\sqrt{2}}$, $\phi = -t$, yielding

$$q = \frac{1}{\sqrt{2}} \tanh \frac{x}{\sqrt{2}} e^{-it}.$$

(Consider the second case, $s = 1$.) Again, using symmetries, we obtain the two-parameter family of dark solitons:

$$q = \frac{A}{\sqrt{2}} \tanh \frac{A(x - ct)}{\sqrt{2}} \exp[i(\frac{c}{2}x - (A^2 + \frac{c^2}{4})t)].$$

10.6 Defocusing NLS: dark solitons

A dark soliton asymptotes to a periodic wave as $x \to \pm\infty$, and propagates as a reduction in the amplitude (Fig. 2).

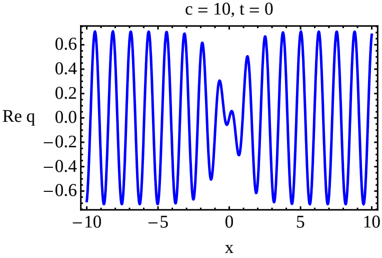


Figure: Dark soliton for $c = 10$ at $t = 0$.

The asymptotics of the initial value problem for the defocusing NLS equation, where the initial condition is a perturbed plane wave with boundary conditions $|q(x,t)| \to |q_0|$ as $x \to \pm\infty$ consists of a series of dark solitons and the dispersive radiation (Zakharov and Shabat (1973)).

10.7 Focusing NLS: breathers

The focusing NLS equation models the evolution of one-dimensional packets of surface gravity waves on sufficiently deep water (Zakharov 1968). Recently, there has been renewed interest in the so-called "breather" solutions of this equation, which have been suggested as models for so-called "freak" waves (also, "rogue" waves). Loosely speaking, a "freak" wave is a single wave or a very short- and short-lived group with a significantly larger steepness than the surrounding waves. One of the breather type solutions for the focusing NLS equation is the so-called Ma-breather (1979). Tthe initial value problem for this equation was considered for the case when the initial condition was a perturbed plane wave with the boundary conditions $|q(x,t)| \to |q_0|$ as $x \to \pm\infty$. It was found that the asymptotic state for his problem consisted of a series of breathers (often called Ma-breathers), given below, and small dispersive radiation:

$$q_M = \frac{\cos(\Omega t - 2ik) - \cosh(k)\cosh(px)}{\cos(\Omega t) - \cosh(\phi)\cosh(px)} e^{2it}.$$

Here, k is the real valued parameter, $\Omega = 2\sinh(2k)$ and $p = 2\sinh(k)$.

10.7 Focusing NLS: breathers

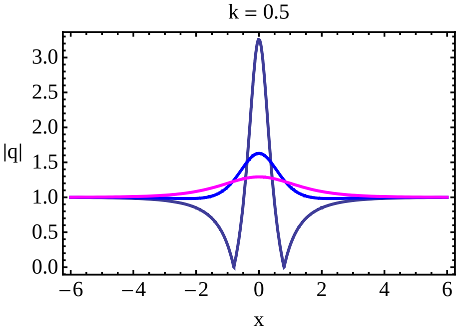


Figure: The envelope of the Ma-breather for $k = 0.5$ at $t = 0; 0.5; 1$.

While for a bright soliton there is always a reference frame where the envelope $|q|$ is stationary, this is not so for breathers ("dynamical solitons").

10.7 Focusing NLS: breathers

Taking the limit $k \to 0$ (i.e. when the breathing period tends to zero), Peregrine (1983) has obtained

$$q_P = \lim_{k \to 0} q_M = [1 - \frac{4(1+4it)}{1+4x^2+16t^2}]\, e^{2it}.$$

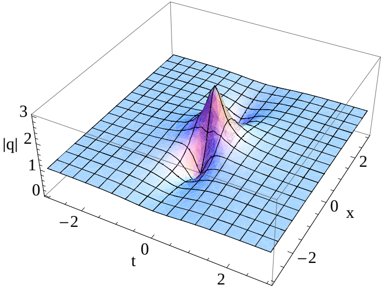


Figure: The envelope of the Peregrin-breather.

Other breather-type solutions have been found by Akhmediev et al. (1987) and Ablowitz and Herbst (1990).

11.1 AKNS scheme

Various solutions of (15) provide compatible linear problems (14) and (15) for different (very special, so-called integrable) PDEs. For example, looking for A, B, C in the form of trancated power series in ζ:

$$A = A_0 + A_1\zeta + A_2\zeta^2,$$
$$B = B_0 + B_1\zeta + B_2\zeta^2,$$
$$C = C_0 + C_1\zeta + C_2\zeta^2,$$

and choosing $r = -\sigma q^*$, where $\sigma = \pm 1$, one can obtain

$$A = -2i\zeta^2 + i\sigma q q^*,$$
$$B = 2q\zeta + iq_x,$$
$$C = -2\sigma q^*\zeta + i\sigma q_x^*,$$

provided $iq_t + q_{xx} + 2\sigma|q|^2 q = 0$. Thus, the linear problem (13) where $r = -\sigma q^*$ is compatible with the equations

$$\psi_{1t} = (-2i\zeta^2 + i\sigma q q^*)\psi_1 + (2q\zeta + iq_x)\psi_2,$$
$$\psi_{2t} = (-2\sigma q^*\zeta + i\sigma q_x^*)\psi_1 + (2i\zeta^2 - i\sigma q q^*)\psi_2, \qquad (16)$$

provided that $q(x, t)$ satisfies the NLS equation (2).

11.2 Remark: AKNS scheme and Bäcklund transformations

Derivation of BTs from AKNS problem?

(Following H-H Chen, Phys. Rev. Lett. 33 (1974) 925-928)

- Having AKNS problem for ψ_1 and ψ_2, introduce $G = \frac{\psi_1}{\psi_2}$ and derive equations for G_x and G_t (turn out to be Riccati equations).
- Eliminating q (solution of the nonlinear PDE) from these equations, one can obtain an equation for G only.
- This equation turns out to be invariant under the transformation $(G, \zeta) \to (-G, -\zeta)$, indicating that there is another solution $\tilde{q}$ of the nonlinear PDE, related to $(-G, -\zeta)$. This gives a second pair of equations for G_x and G_t, containing $\tilde{q}$.
- BTs (relating q and $\tilde{q}$) can be derived from the two pairs of equations for G_x and G_t.

11.3 Direct Scattering Problem for the focusing NLS

Consider the 2x2 spectral problem:

$$\begin{aligned} \psi_{1x} &= -i\zeta\psi_1 + q\psi_2, \\ \psi_{2x} &= i\zeta\psi_2 - q^*\psi_1, \end{aligned} \tag{17}$$

where $q \to 0$ as $x \to \pm\infty$ (sufficiently fast). Here, ψ_1 and ψ_2 are the eigenfunctions, ζ is the eigenvalue.

By taking the complex conjugate of (17) we find that if (ψ_1, ψ_2) is an eigenvector, so is $(\psi_2^*, -\psi_1^*)$. Since $q \to 0$ as $x \to \pm\infty$,

$$(\psi_1, \psi_2) \sim (k_1 e^{-i\zeta x}, k_2 e^{i\zeta x}) \quad \text{as} \quad x \to \pm\infty. \tag{18}$$

Define (ψ_{1+}, ψ_{2+}) to be the solution that satisfies

$$(\psi_{1+}, \psi_{2+}) \sim (e^{-i\zeta x}, 0) \quad \text{as} \quad x \to +\infty. \tag{19}$$

Then, another solution, $(\psi_{2+}^*, -\psi_{1+}^*)$, satisfies

$$(\psi_{2+}^*, -\psi_{1+}^*) \sim (0, -e^{i\zeta x}) \quad \text{as} \quad x \to +\infty. \tag{20}$$

11.3 Direct Scattering Problem for the focusing NLS

Consider the set $\{(\psi_{1+}, \psi_{2+}), (\psi_{2+}^*, -\psi_{1+}^*)\}$ as a basis for solutions of (17). Define (ψ_{1-}, ψ_{2-}) to be the solution that satisfies

$$(\psi_{1-}, \psi_{2-}) \sim (e^{-i\zeta x}, 0) \quad \text{as} \quad x \to -\infty.$$

This solution can be expressed as a linear combination of the basis:

$$\begin{aligned} \psi_{1-} &= a(\zeta)\psi_{1+} + b(\zeta)\psi_{2+}^*, \\ \psi_{2-} &= a(\zeta)\psi_{2+} - b(\zeta)\psi_{1+}^*. \end{aligned} \tag{21}$$

Here, $a(\zeta)$ and $b(\zeta)$ are called the scattering data.
Since $(\psi_1, \psi_2) \sim (k_1 e^{-i\zeta x}, k_2 e^{i\zeta x}) \quad$ as $\quad |\zeta| \to \infty$, we must have

$$\begin{aligned} &(\psi_{1-}, \psi_{2-}) \sim (e^{-i\zeta x}, 0), \\ &(\psi_{1+}, \psi_{2+}) \sim (e^{-i\zeta x}, 0) \quad \text{as} \quad |\zeta| \to \infty. \end{aligned}$$

and $\quad a(\zeta) \to 1, \quad b(\zeta) \to 0 \quad$ as $\quad |\zeta| \to \infty$. Discrete spectrum is introduced as those values of ζ, for which eigenvectors decay both for $x \to +\infty$ and for $x \to -\infty$ (called "bound states"). The discrete spectrum coincides with the zeros of the function $a(\zeta)$ in the upper half-plane.

11.4 Inverse Scattering Problem for the focusing NLS

Given the scattering data, $a(\zeta)$ and $b(\zeta)$, can we reconstruct the function $q(x)$? One begins by writing

$$\psi_{1+} = e^{-i\zeta x} + \int_x^\infty K_1(x,y)e^{-i\zeta y}dy,$$
$$\psi_{2+} = \int_x^\infty K_2(x,y)e^{-i\zeta y}dy \tag{22}$$

with $K_1(x,y) = K_2(x,y) = 0$ for $y < x$. Substituting (22) into (17), one arrives at the boundary value problem

$$\frac{\partial K_1}{\partial x} + \frac{\partial K_1}{\partial y} = qK_2,$$
$$\frac{\partial K_2}{\partial x} - \frac{\partial K_2}{\partial y} = -q^* K_1 \tag{23}$$

subject to $K_1(x,y) \to 0, K_2(x,y) \to 0$ as $y \to \infty, \quad K_2(x,x) = \frac{1}{2}q^*$. It can be shown that the solution of this problem exists and is unique.

11.4 Inverse Scattering Problem for the focusing NLS

Therefore, if we can find $K_2(x,x)$ from our knowledge of the scattering data, we can reconstruct $q(x)$ from

$$q(x) = 2K_2^*(x,x).$$

It can be shown, using the representation (22) (see ZS, AKNS), that $K_2^*(x,y) = K(x,y)$ satisfies the integral equation

$$K(x,z) - f^*(x+z) + \int_x^\infty G(x,y,z)K(x,y)dy = 0, \qquad (24)$$

where $G(x,y,z) = \int_x^\infty f(y+Y)f^*(Y+z)dY$, and

$$f(X) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b(\zeta)}{a(\zeta)}e^{i\zeta X}d\zeta - i\sum_{n=1}^{N} c_n e^{i\zeta_n X}.$$

Here, ζ_n are the N zeros of $a(\zeta)$ in the upper half plane (discrete spectrum), and

$$c_n = \frac{b(\zeta_n)}{a'(\zeta_n)}.$$

By analogy with the IST for the KdV equation, the function $\frac{b(\zeta)}{a(\zeta)}$ is called the "reflection coefficient".

11.5 "Reflectionless" potential with a single bound state

Consider the case when $\frac{b(\zeta)}{a(\zeta)} = 0$ for ζ real and $a(\zeta)$ has a single zero in the upper half plane at $\zeta = \zeta_1$ with $c_1 = \frac{b(\zeta_1)}{a'(\zeta_1)}$. Then, $f(X) = -ic_1 e^{i\zeta_1 X}$, and

$$G(x,y,z) = -\frac{|c_1|^2}{i(\zeta_1 - \zeta_1^*)} e^{i\zeta_1(y+x)} e^{-i\zeta_1^*(x+z)},$$

$$K(x,z) - ic_1^* e^{-i\zeta_1^*(x+z)} - \frac{|c_1|^2}{i(\zeta_1 - \zeta_1^*)} \int_x^\infty e^{i\zeta_1(y+x)} e^{-i\zeta_1^*(x+z)} K(x,y)dy = 0.$$

Looking for a solution of this integral equation in the form $K(x,z) = L(x)e^{-i\zeta_1^* z}$, one obtains

$$L(x) = \frac{ic_1^*(\zeta_1 - \zeta_1^*)^2 e^{-i\zeta_1^* x}}{(\zeta_1 - \zeta_1^*)^2 - |c_1|^2 e^{2i(\zeta_1 - \zeta_1^*)x}}.$$

Writing $\zeta_1 = \alpha + i\beta$, one finds

$$\begin{aligned} q(x) &= 2L(x)e^{-i\zeta_1^* x} = \frac{4i\beta^2 e^{-2(\beta + i\alpha)x}}{4\beta^2 + |c_1|^2 e^{-4\beta x}} \\ &= ic_1^* e^{-2i\alpha x} \frac{2\beta}{|c_1|} \operatorname{sech}\,(2\beta x - \ln \frac{|c_1|}{2\beta}). \end{aligned} \tag{25}$$

11.6 “Reflectionless” potential with N bound states

Consider the case when $\frac{b(\zeta)}{a(\zeta)} = 0$ for ζ real and $a(\zeta)$ has N zeros in the upper half plane at $\zeta = \zeta_n$ with $c_n = \frac{b(\zeta_n)}{a'(\zeta_n)}$ for $n = 1, ..., N$. Then,

$$f(X) = -i\sum_{n=1}^{N} c_n e^{i\zeta_n X}, \quad G(x,y,z) = i\sum_{n=1}^{N}\sum_{m=1}^{N} \frac{c_n c_m^*}{\zeta_n - \zeta_m^*} e^{i(\zeta_n y - \zeta_m^* z + (\zeta_n - \zeta_m^*)x)},$$

and
$$K(x,z) - i\sum_{n=1}^{N} c_n^* e^{-i\zeta_n^*(x+z)}$$

$$+i\sum_{n=1}^{N}\sum_{m=1}^{N} \frac{c_n c_m^*}{\zeta_n - \zeta_m^*} e^{i(-\zeta_m^* z + (\zeta_n - \zeta_m^*)x)} \int_x^\infty K(x,y) e^{i\zeta_n y}\, dy = 0.$$

Looking for a solution in the form $K(x,z) = \sum_{k=1}^{N} L_k(x) e^{-i\zeta_k^*(x-z)}$, one obtains (equating coefficients of $e^{-i\zeta_k^* z}$)

$$L_k - \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{c_n c_k^*}{(\zeta_n - \zeta_k^*)(\zeta_n - \zeta_m^*)} L_m e^{2i(\zeta_n - \zeta_K^*)x} = i c_k^* e^{-2i\zeta_k^* x}, \quad \text{and}$$

$$q(x) = 2\sum_{k=1}^{N} L_k(x). \tag{26}$$

11.7 Time evolution of the scattering data

So far we have not used the second linear problem (16) (with $\sigma = 1$). Let us now consider these evolution equations:

$$\begin{aligned}\psi_{1t} &= (-2i\zeta^2 + iqq^*)\psi_1 + (2q\zeta + iq_x)\psi_2,\\ \psi_{2t} &= (-2q^*\zeta + iq_x^*)\psi_1 + (2i\zeta^2 - iqq^*)\psi_2.\end{aligned} \tag{27}$$

We had

$$(\psi_{1-}, \psi_{2-}) \sim (a(\zeta, t)e^{-i\zeta x}, -b(\zeta, t)e^{i\zeta x}) \quad \text{as} \quad x \to +\infty. \tag{28}$$

Substituting (28) into (27), and using that $q \to 0$ as $x \to +\infty$, one finds

$$a_t = -2i\zeta^2 a, \quad b_t = 2i\zeta^2 b,$$

which implies

$$a(\zeta, t) = e^{-2i\zeta^2 t}a(\zeta, 0), \quad b(\zeta, t) = e^{2i\zeta^2 t}b(\zeta, 0).$$

Thus, the zeros of $a(\zeta, t)$ (the disctrete spectrum) are independent of t, and

$$c_n(t) = e^{4i\zeta_n^2 t}c_n(0).$$

11.8 N-soliton solutions for the focusing NLS

If we substitute $c_1 = c_1(0)e^{4i\zeta_1^2 t}$ into (25), we obtain the single bright soliton (show that the solution coincides with the one obtained in section 9.3, up to reparametrisation and shifts in t and x),

$$\begin{aligned} q(x,t) &= ic_1^*(0)\exp[i(-2\alpha x + 4(\beta^2-\alpha^2)t)]\frac{2\beta}{|c_1(0)|} \\ &\times \operatorname{sech}\left[2\beta(x+4\alpha t) - \ln\frac{|c_1(0)|}{2\beta}\right]. \end{aligned}$$

Similarly, if we substitute $c_n = c_n(0)e^{4i\zeta_n^2 t}$ into (26), then we obtain solution describing the interaction of N bright solitons:

$$q(x) = 2\sum_{k=1}^{N} L_k(x,t), \quad \text{where}$$

$$L_k(x,t) - \sum_{n=1}^{N}\sum_{m=1}^{N}\frac{c_n(0)c_k^*(0)}{(\zeta_n-\zeta_k^*)(\zeta_n-\zeta_m^*)}L_m(x,t)e^{2i(\zeta_n-\zeta_k^*)(x+2(\zeta_n+\zeta_k^*)t)}$$
$$= ic_k^*(0)e^{-2i\zeta_k^*(x+2\zeta_k^* t)}.$$

(The solution of this linear system can be written in a compact form, as for the KdV equation (e.g., Scott 1999).)

12.1 Perturbed KdV equation

Given the variable-coefficient KdV equation

$$u_t + \alpha(t)uu_x + \beta(t)u_{xxx} = 0. \tag{1}$$

and assuming $\alpha \neq 0$ we introduce the new variables

$$t' = \frac{1}{6}\int_0^t \alpha(\hat{t})d\hat{t}\,, \qquad \lambda(t') = \frac{6\beta}{\alpha}\,, \tag{2}$$

so that, on omitting the superscript for t, equation (1) becomes

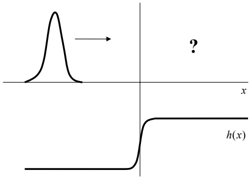
$$u_t + 6uu_x + \lambda(t)u_{xxx} = 0\,, \tag{3}$$

a perturbed KdV equation.

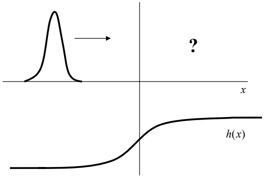
12.1 Perturbed KdV equation

Analytical progress is generally possible in two contrasting cases.

Case 1: Fast depth variation



Case 2: Slow depth variation



+ two similar cases for increasing depth.

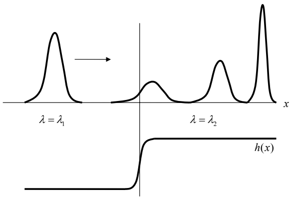
12.2 Case 1. Fast depth change: soliton fission

Propagation of a shallow-water solitary wave over a sloping bottom is modelled by the variable-coefficient KdV equation

$$u_t + 6uu_x + \lambda(t)u_{xxx} = 0\,, \tag{4}$$

Note: Independent variables x and t in (4) are generally **not** physical space and time.

Case 1: Solitary wave advancing into a rapidly decreasing depth region. This is modelled by the following behaviour of $\lambda(t)$ in (4):
$\lambda = \lambda_1$ for $t < 0$; $\quad \lambda = \lambda_2$ for $t > 0$.
Result: soliton fission (Tappert and Zabusky 1971)



12.2 Case 1. Fast depth change: soliton fission

For $t < 0$:

$$u(x,t) = a\,\mathrm{sech}^2[\sqrt{a/(2\lambda_1)}(x - c_s t)]\,, \quad c_s = 2a \tag{5}$$

which is the solitary wave solution of the KdV equation

$$u_t + 6uu_x + \lambda u_{xxx} = 0 \tag{6}$$

with $\lambda = \lambda_1 > 0$.
At $t = 0$ we have $u(x,0) = a\,\mathrm{sech}^2[\sqrt{a/(2\lambda_1)}x]\,,$ which is then considered as the **initial condition** for the KdV equation (6) with $\lambda = \lambda_2$:

Note: For $t > 0$, function (5) is **no longer** a solution of the governing KdV equation (6) so one uses the IST to find the solution. The outcome depends on the sign of λ_2

- $\lambda_2 > 0$: At $t \gg 1$ one has N solitons (discrete spectrum) $u_n(x,0) = a_n\,\mathrm{sech}^2[\sqrt{a_n/(2\lambda_2)}(x - a_n t)]\,,\ n = 1, 2, \ldots, N$ and some linear radiation (continuous spectrum). The amplitudes a_n and the number N of the solitons, as well as the parameters of the radiation, are found from the IST.
- $\lambda_2 < 0$: The initial soliton (5) completely transforms into a linear radiation as $t \to \infty$.

12.3 Case 2. Slow depth change: adiabatic variation of the solitary wave

If the depth h changes slowly with physical coordinate x, one has for the coefficient λ in the corresponding variable coefficient KdV equation (4):

$$\lambda = \lambda(T), \quad T = \epsilon t\,, \quad \epsilon \ll 1. \tag{7}$$

Here ϵ measures the slope. Then, the solution of the KdV equation

$$u_t + 6uu_x + \lambda(T)u_{xxx} = 0 \tag{8}$$

can be sought in the form of the asymptotic expansion

$$u = u_0 + \epsilon u_1 + \dots\,, \tag{9}$$

where the leading term is given by the slowly varying solitary wave solution

$$u_0 = a\ \mathrm{sech}^2\{\gamma(x - \frac{\Phi(T)}{\epsilon})\}\,, \tag{10}$$

so that

$$d\Phi/dT = c_s = 2a = 4\lambda\gamma^2\,. \tag{11}$$

Note that for $\lambda = \text{constant}$ formulae (10), (11) reduce to the standard KdV soliton expression $u(x,t) = a\ \mathrm{sech}^2[\sqrt{a/(2\lambda)}(x - c_s t)]$, $c_s = 2a$.

12.3 Case 2. Slow depth change: adiabatic variation of the solitary wave

The variations of the amplitude a and the inverse half-width parameter γ with the slow time variable T are determined by noticing that the variable-coefficient KdV equation (3) possesses the momentum conservation law

$$\int_{-\infty}^{\infty} u^2 dx = \text{constant}\,. \tag{12}$$

Substitution of (10) into (12) readily shows that

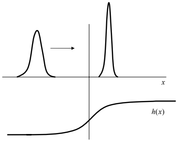
$$\frac{\gamma}{\gamma_0} = \left(\frac{\lambda_0}{\lambda}\right)^{2/3}, \tag{13}$$

where the subscript '0' indicates quantities evaluated at some fixed moment $T = T_0$, say for $\lambda = \lambda_0$.
It follows from (10), (11) and (13) that the slowly-varying solitary wave, u_0 is now completely determined.

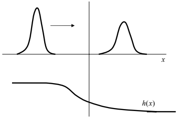
12.3 Case 2. Slow depth change: adiabatic variation of the solitary wave

The outcome: adiabatically varying solitary wave. Generally:

- For a wave advancing into decreasing depth, there is a tendency to increase the amplitude



- For a wave advancing into increasing depth, there is a tendency to decrease the amplitude



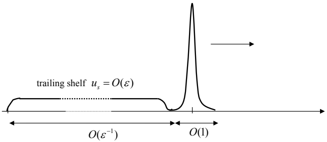
The detailed amplitude variations are determined by the slowly changing bottom profile $h(x)$ which is translated into the function $\lambda(T)$.

12.3 Case 2. Formation of a trailing shelf

So, the slowly-varying solitary wave is completely determined by the leading term of the asymptotic expansion (9) $u = u_0 + \epsilon u_1 + \ldots,$.
However, the variable-coefficient KdV equation (3) also has a conservation law for the 'mass'

$$\int_{-\infty}^{\infty} u dx = \text{constant}, \tag{14}$$

which is not satisfied by $u_0(x,t)$ (**check!!!**). The situation can be remedied by taking into account the next term in the asymptotic expansion (9) and allowing $\int_{-\infty}^{\infty} u_1(x)dx$ be $\mathcal{O}(\epsilon^{-1})$.
More precisely, conservation of mass is assured by the generation of a trailing shelf u_s, such that $u = u_0 + u_s$ where u_s typically has an amplitude $\mathcal{O}(\epsilon)$ and is supported on the interval $0 < x < \Phi(T)/\epsilon$ (Newell). Thus, the shelf stretches over a zone of $\mathcal{O}(\epsilon^{-1})$, and hence carries $\mathcal{O}(1)$ mass.



12.3 Case 2. Determination and evolution of the trailing shelf.

To find u_s, we substitute $u = u_0 + u_s$ into the conservation of mass equation (14) to obtain

$$\int_0^{\Phi(T)/\epsilon} u_s dx + \int_{-\infty}^{\infty} u_0 dx = \text{constant}\,, \tag{15}$$

which, together with the KdV equation $u_t + 6uu_x + \lambda(T)u_{xxx} = 0$ completely determines the dynamics of the shelf $u_s(x,t)$. Indeed,

$$\int_{-\infty}^{\infty} u_0 dx = 2\frac{a}{\gamma} = 4\gamma_0\lambda_0^{\frac{2}{3}}\lambda^{\frac{1}{3}}.$$

Then, differentiating (15) with respect to T we obtain $u_s\dfrac{\Phi_T}{\epsilon} + (4\gamma_0\lambda_0^{\frac{2}{3}}\lambda^{\frac{1}{3}})_T = 0$. This implies, using (11),

$$u_s(\frac{\Phi(T)}{\epsilon}, t) = -\frac{1}{3}\epsilon\gamma_0^{-1}\lambda_0^{-\frac{2}{3}}\lambda^{-\frac{1}{3}}\lambda_T,$$

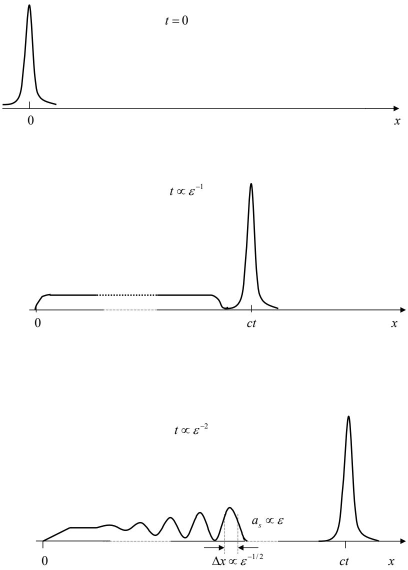
which constitutes a boundary condition for the KdV equation above.

Remarks

- The trailing shelf can have positive or negative polarity depending on the behaviour of the function $\lambda(T)$. Namely, the polarity of the shelf $\sigma = -\mathrm{sgn}(\lambda_T)$ (assuming that $\lambda(T) > 0$).

- If $\sigma > 0$ (elevation) then, according to the IST, the shelf will decompose as $t \to \infty$ into a large number of small-amplitude solitons, which is consistent with the fission scenario for rapidly varying environment. If $\sigma < 0$ (depression) then the shelf asymptotically transforms into a linear dispersing wave packet.

12.3 Case 2. Evolution of the trailing shelf of elevation.



12.4. Higher-order KdV equation: a near-identity transformation

We now consider the higher-order KdV equation:

$$A_t+\alpha AA_x+\beta A_{xxx}+\varepsilon(\alpha_1 A^2A_x+\beta_1 A_{xxxxx}+\gamma_1 AA_{xxx}+\gamma_2 A_xA_{xx})=0. \quad (16)$$

First, notice that an asymptotic near-identity transformation

$$\tilde{A}=A+\varepsilon(aA^2+bA_{xx}) \quad (17)$$

maps equation (16) into the same equation but with the coefficients $\alpha_1,\beta_1,\gamma_1,\gamma_2$ replaced by

$$\tilde{\alpha}_1=\alpha_1-a\alpha,\ \tilde{\beta}_1=\beta_1,\ \tilde{\gamma}_1=\gamma_1,\ \tilde{\gamma}_2=\gamma_2-6a\beta+2b\alpha.$$

This allows one to choose the coefficients $\tilde{\alpha}_1$ and $\tilde{\gamma}_2$ in some convenient way. In particular, the coefficients can be chosen so that the higher-order KdV equation (16) for $\tilde{A}$ is Hamiltonian.

12.4. Higher-order KdV equation: a near-identity transformation

Remark

The KdV equation can be obtained from the variational principle $\delta \int \mathcal{L}\, dxdt = 0$, where the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\phi_t\phi_x + \frac{\alpha}{6}\phi_x^3 - \frac{\beta}{2}\phi_{xx}^2, \quad A = \phi_x$$

(check!). Introducing the momentum density by

$$p = \frac{\partial \mathcal{L}}{\partial \phi_t} = \frac{\phi_x}{2} = \frac{A}{2},$$

one obtains the Hamiltonian density

$$\mathcal{H} = p\phi_t - \mathcal{L} = -\frac{\alpha}{6}\phi_x^3 + \frac{\beta}{2}\phi_{xx}^2 = -\frac{\alpha}{6}A^3 + \frac{\beta}{2}A_x^2.$$

The KdV equation can be written in the canonical Hamiltonian form $A_t = \frac{\partial}{\partial x}\left(\frac{\delta H}{\delta A}\right)$, where the Hamiltonian $H = \int_{-\infty}^{\infty} \mathcal{H} dx$ (total energy of the system).

12.4 A near-identity transformation

For the higher-order KdV equation (16) to be Hamiltonian, one can use the near-identity transformation (17) with

$$a = \frac{\gamma_2 - 2\gamma_1}{6\beta}, \ \ b = 0,$$

which yields an equation

$$A_t = \frac{\partial}{\partial x}\left(\frac{\delta H}{\delta A}\right),$$

where the Hamiltonian $H = \int_{-\infty}^{\infty} \mathcal{H} dx$ has the density

$$\mathcal{H} = -\frac{1}{6}\alpha A^3 + \frac{1}{2}\beta A_x^2 - \varepsilon(\frac{1}{12}\tilde{\alpha}_1 A^4 + \frac{1}{2}\tilde{\beta}_1 A_{xx}^2 - \frac{1}{2}\tilde{\gamma}_1 A A_x^2).$$

When (16) is Hamiltonian then it conserves the mass ($\int_{-\infty}^{\infty} A dx$), the momentum ($\int_{-\infty}^{\infty} A^2 dx$), and the Hamiltonian ($\int_{-\infty}^{\infty} \mathcal{H} dx$), which is desirable both for numerical purposes, and for use of perturbation techniques.

12.5 Asymptotic integrability of the generic higher-order KdV equation

If the leading-order evolution equation is integrable, the underlying physical system is said to be asymptotically integrable to $O(\varepsilon)$. It turns out that for the generic higher-order KdV equation $(\alpha \neq 0, \beta \neq 0)$ it is possible to formally extend the asymptotic integrability to $O(\varepsilon^2)$. Indeed, the generic KdV equation (16) was shown to be asymptotically reducible to an integrable equation (Kodama 1985, Fokas 1995, Fokas and Liu 1996). The following asymptotic near-identity transformation

$$B = A + \varepsilon(aA^2 + bA_{xx} + cA_x \int_{x_0}^{x} A d\tilde{x} + d\ xA_t) \tag{18}$$

where a, b, c, d are constants, can reduce (16) either to the next member of the KdV hierarchy, or even to the KdV equation itself, with accuracy up to $O(\varepsilon^2)$.

12.5 Asymptotic integrability of the generic higher-order KdV equation

Various near-identity transformations reducing the higher-order KdV equation to the KdV equation have been used to obtain particular solutions for the higher-order KdV equation from the known solutions of the KdV equation (e.g., two-soliton solution and an undular bore solution, see Marchant and Smyth 2006).

Remark. An undular bore solution describes the evolution of an initial step. For the KdV equation, it has been obtained by Gurevich and Pitaevsky in 1974. The solution can be interpreted as a slowly varying cnoidal wave. It will be discussed later in our module.

12.6 Degenerate case (small α): eKdV (or Gardner) equation

In some applications the coefficient α can be small or even zero. For example, this can happen in a two-layer fluid, where

$$\alpha = \frac{3c(h_1 - h_2)}{2h_1 h_2},$$

where h_1 and h_2 are the thickness of the upper and lower layers (Kakutani and Yamasaki 1978). If α approaches zero, then it is necessary to take into account the higher-order nonlinear term (the term with the coefficient α_1 in (16)). In this degenerate case, the other higher-order terms in (16) appear in the next order of ε. Thus, the result is the so-called extended KdV (or Gardner) equation, which we write as

$$A_t + \alpha A A_x + \beta A^2 A_x + \lambda A_{xxx} = 0. \tag{19}$$

The coefficient β in (19) can be both positive and negative (in the context of internal waves, this depends on stratification, see Grimshaw 2002). In the case of two-layer fluid it is negative (Kakutani and Yamasaki 1978). It can be positive already in the case of a three-layer fluid (Talipova et al. 1999).

12.6 Degenerate case (small α): eKdV (or Gardner) equation

When $\alpha = 0$, this equation coincides with the modified KdV (mKdV) equation, with the canonical form

$$A_t + 6A^2 A_x + A_{xxx} = 0.$$

For $\alpha \neq 0$ (and $\beta \neq 0$), the Gardner equation (19) can be reduced to mKdV by a simple transformation,

$$B = A + \frac{\alpha}{2\beta}, \quad \tilde{x} = x + \frac{\alpha^2}{4\beta} t, \quad \tilde{t} = t$$

and is integrable (e.g., Ablowitz and Segur 1981).

12.6 Properties of solitons of the Gardner equation

The Gardner equation (19) is widely used for the modeling of internal solitons in the ocean (e.g., Holloway et al. 2002). A train of solitons constitute main part of the asymptotics of any localized initial condition. When $\beta = 0$, the KdV solitons are given by

$$A = a \operatorname{sech}^2 \Lambda(x - vt), \quad v = \frac{1}{3}\alpha a = 4\lambda\Lambda^2. \tag{20}$$

The dispersion coefficient λ is always positive for right-going waves. Therefore, for these solitary waves $v > 0$. They are waves of elevation or depression depending on $\operatorname{sign}(\alpha)$. Note that $\Lambda \sim \sqrt{|a|}$, and larger waves are not only faster, but also narrower.

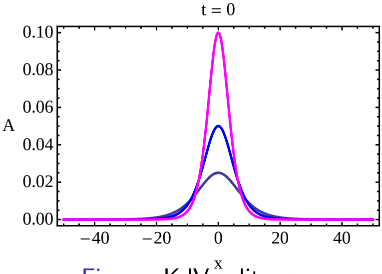


Figure: KdV solitons.

12.6 Properties of solitons of the Gardner equation

For the extended KdV equation (19) solitary waves are given by

$$A = \frac{D}{1 + B\cosh K(x - vt)}, \quad \text{where} \tag{21}$$

$$v = \frac{\alpha D}{6} = \lambda K^2, \;\; B^2 = 1 + \frac{6\lambda\beta K^2}{\alpha^2}. \tag{22}$$

When $\beta \to 0$, the Gardner soliton (21) approaches the KdV soliton (20). Assuming that $\alpha, \beta \neq 0$, the change of variables

$$A \to \frac{|\beta|}{\alpha} A, \quad x \to \sqrt{\frac{\alpha^2}{6\lambda|\beta|}}\, x, \quad t \to \lambda \left(\frac{\alpha^2}{6\lambda|\beta|}\right)^{3/2} t$$

transforms the Gardner equation (19) into the canonical form

$$A_t + 6A(1 + \sigma A)A_x + A_{xxx} = 0,$$

where $\sigma = \mathrm{sign}\;(\beta) = \pm 1$. Then (22) assume the form

$$v = D = K^2, \;\; B^2 = 1 + \sigma K^2.$$

It turns out that solitons of Gardner equation strongly depend on σ.

12.6 Case I (GE-): $\sigma = -1$

The GE- has a single family of solitons such that $0 < B < 1$. When $B \to 1$, the small-amplitude GE- solitons approach the KdV-type solitons. In the limiting case when $B \to 0$ the solution (21) describes the so-called "thick" solitary wave (also, "table-top" soliton), which has a flat crest of amplitude 1. Its slopes are often called "kinks" or "bore-like" solutions.

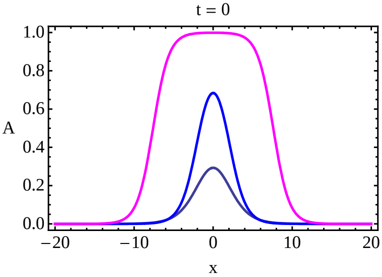


Figure: Solitons of GE-.

12.6 Case II (GE+): $\sigma = 1$

The GE+ has two families of solitons with $B^2 > 1$. One has $1 < B < \infty$ and ranges from small KdV-type solitons when $B \to 1$, to arbitrarily large wave with a "sech"-profile as $B \to \infty$.

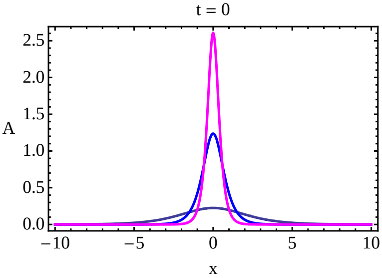


Figure: Solitons of GE+ (first family).

12.6 Case II (GE+): $\sigma = 1$

The other branch has the opposite polarity, exists for $-\infty < B < -1$, and ranges from arbitrarily large waves with a "sech"-profile to a so-called algebraic soliton

$$A = -\frac{2}{1 + x^2}.$$

of amplitude -2. The algrebraic soliton was shown to be structurally unstable (Pelinovsky and Grimshaw 1997). Other solitons of the family are stable.

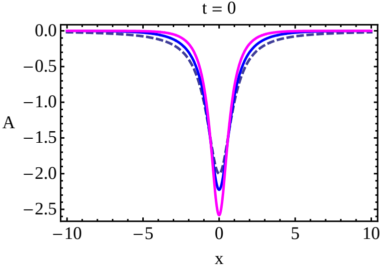


Figure: Solitons of GE+ (second family). Dashed line: an algebraic soliton.

12.7 Breathers of GE+

The GE+ equation admits breathers (Pelinovsky and Grimshaw 1997)

$$A = 2\frac{\partial}{\partial x}\ \tan^{-1}\ \left[\frac{l\cosh(\Psi)\cos(\theta) - k\cos(\Phi)\sinh(\phi)}{l\sinh(\Psi)\sin(\theta) + k\sin(\Phi)\cosh(\phi)}\right],$$

where
$\theta = k(x - wt)$, $\phi = l(x - vt)$, $w = -k^2 + 3l^2$, $v = -3k^2 + l^2$, and $\Phi + i\Psi = \tan^{-1}\ [l + ik]$. Two typical breathers are shown below.

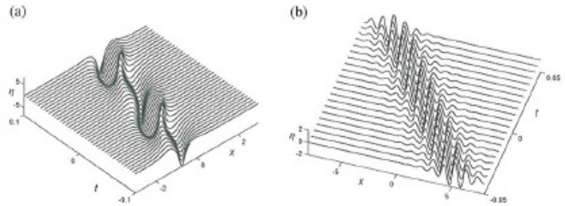


Figure: from Grimshaw (ed.) 2007, p. 100: Two typical breathers of GE+.

Appendix. Gardner equation: solitons in a variable environment

In situations when there is a variable background environment (e.g., a variable topography in the context of surface and internal waves), the constant coefficient Gardner equation (19) is replaced with the variable coefficient extended KdV equation

$$A_t + \alpha A A_x + \beta A^2 A_x + \lambda A_{xxx} + \nu A = 0, \tag{23}$$

where α, β, λ and ν are functions of t. The last term represents nonconservative effects arising from dissipative or forcing terms in the underlying basic state. Extra terms can be added to represent dissipative effects on the wave itself, and Coriolis effects. As usual, asymptotic analysis can be used to study two contrasting situations. In one case the background state changes rapidly from one constant state to another constant state, over a distance much shorter than a typical wavelength. Typically, this change leads to a disintegration of the solitary wave into several solitary waves, in a so-called fissioning process. In the other case, the background state varies slowly relative to a typical wavelength. In this case the dominant effect is a slow adiabatic deformation of the wave, which can be described as a slowly-varying solitary wave.

Fission

We suppose that the background state changes rapidly from one constant state to another, over a distance much shorter than a typical wavelength. We consider the conservative case $\nu = 0$, and suppose that the coefficients in (23) make a transition from the values $\alpha_-, \beta_-, \lambda_-$ in $t < 0$ to the values $\alpha_+, \beta_+, \lambda_+$ in $t > 0$. Then, a solitary wave of (23) with the coefficients $\alpha_-, \beta_-, \lambda_-$ is given by

$$A_0 = \frac{D}{1 + B\cosh K(x - vt)}, \quad \text{where}$$

$$v = \frac{\alpha_- D}{6} = \lambda_- K^2, \; B^2 = 1 + \frac{6\lambda_-\beta_- K^2}{\alpha_-^2}. \tag{24}$$

To leading order, a solitary wave (24) will pass through the transition zone $t = 0$ without change. However, on arrival into the region $t > 0$ it is no longer a permissible solution of (23) with the coefficients $\alpha_+, \beta_+, \lambda_+$. Instead, it now forms an effective initial condition for the new constant-coefficient eKdV equation. Typically, a single incident solitary wave will generate a number of secondary solitons (and some dispersive radiation), and this process is called fission. The main difference with the similar KdV problem is in the number of transmitted solitons (generally, more for a soliton of the same amplitude, Grimshaw et. al 2008).

Slowly-varying solitons

When the background state varies slowly relative to a typical wavelength, the dominant effect is a slow adiabatic deformation of the wave, described as a slowly-varying solitary wave. We suppose that the coefficients are slowly varying, and consider the conservative case, then $\alpha = \alpha(\sigma),\ \beta = \beta(\sigma),\ \lambda = \lambda(\sigma) > 0,\ \nu = 0$, where $\sigma = \kappa t,\ \kappa \ll 1.$
Using an asymptotic multiple-scales expansion

$$A = A_0(\theta, \sigma) + \kappa A_1(\theta, \sigma) + ..., \quad \text{where} \quad \theta = x - \frac{1}{\kappa}\int_0^{\sigma} v(\sigma) d\sigma \quad (25)$$

(assuming that soliton is at $x = 0$ when $t = 0$), one obtains

$$-vA_{0\theta} + \alpha A_0 A_{0\theta} + \beta A_0^2 A_{0\theta} + \lambda A_{0\theta\theta\theta} = 0, \quad (26)$$

$$-vA_{1\theta} + \alpha (A_0 A_1)_\theta + \beta (A_0^2 A_1)_\theta + \lambda A_{1\theta\theta\theta} = -A_{0\sigma}. \quad (27)$$

Again, equation (26) has the solitary wave solution

$$A_0 = \frac{D}{1 + B\cosh K\theta}, \text{ where } v = \frac{\alpha D}{6} = \lambda K^2,\ B^2 = 1 + \frac{6\lambda\beta K^2}{\alpha^2}. \quad (28)$$

When the coefficients are constant, this is just one of the eKdV solitons, described in section 12.6. Here it is a slowly-varying solitary wave, which is now parametrized by a single parameter $B = B(\sigma)$.

Slowly-varying solitons

To find how B varies with σ one either needs to consider equation (27), or equivalently (and more conveniently) to use the conservation law for the momentum

$$\frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} A^2 dx\right) = 0, \tag{29}$$

which holds for the variable-coefficient extended KdV equation. Substitution of (28) into (29) gives

$$\frac{D^2}{K}\int_{-\infty}^{\infty} \frac{du}{(1 + B\cosh u)^2} = const,$$

which can be explicitly integrated, giving different expressions for the cases $B > 1$, $B < -1$ and $0 < B < 1$, discussed earlier.

Slowly-varying solitons

As in the case of the variable coefficient KdV equation, the slowly-varying solitary wave is accompanied by a trailing shelf, in order to conserve total mass:

$$\int_{-\infty}^{\infty} A dx.$$

A trailing shelf κA_s is of small amplitude $O(\kappa)$ but long length-scale $O(\frac{1}{\kappa})$, having $O(1)$ mass, but $O(\kappa)$ momentum. Then,

$$A = A_0 + \kappa A_s, \quad \text{where}$$

$$A_s = A_s(\chi), \quad 0 < \chi = \kappa x < \Phi(\sigma) = \int_0^{\sigma} v d\sigma,$$

and

$$v \; A_s[\Phi(\sigma)] = -M_{0\sigma} \quad (M_0 = \int_{-\infty}^{\infty} A_0 dx).$$

Note, that again, A_s depends on whether $B > 1$, $B < -1$, or $0 < B < 1$.

The change of soliton polarity

The change of soliton polarity is the effect very often observed in the context of applications of the eKdV to the description of internal waves in the coastal zone. The process can be easily explained in terms of the variable coefficient eKdV. Indeed, the polarity of the KdV soliton depends on the sign of the quadratic nonlinearity (in oceanographic applications it depends on stratification), and may change its sign. The degeneration of the quadratic nonlinear term requires the consideration of a higher-order nonlinearity, leading to an extended KdV considered in the previous sections. The analysis of the adiabatic transformations of a solitary wave indicates that one may expect a dramatic change in the wave structure at the critical points where $\alpha = 0$.

When the soliton polarity is defined by the sign of the quadratic nonlinearity (the GE- case or the low-amplitude limit of GE+), a soliton cannot pass the critical point, adiabatically changing its parameters. Typically, a small-amplitude soliton transforms into a dispersive wave packet in the vicinity of the critical point, which later generates a small soliton of the opposite polarity. If a "table-top" soliton passes the critical point, the generated soliton of opposite polarity is again wide, but has a smaller mass due to the radiation.

The change of soliton polarity

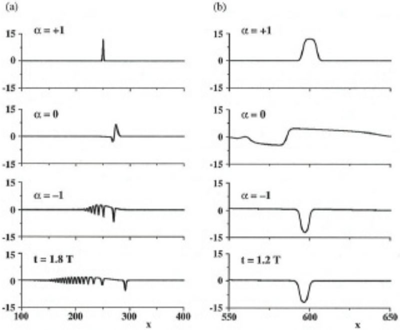


Figure: from Grimshaw (ed.) 2007, p. 105: The change of polarity.

The change of soliton polarity

The GE+ supports intensive solitons of both polarities (see section 12.6). This case is close to the case of mKdV (with $\alpha = 0$), resulting in a relatively small change of soliton parameters. (The small-amplitude case has been considered on previous slides.)

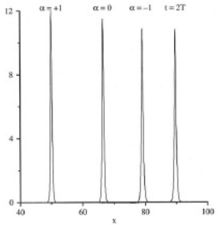


Figure: from Grimshaw (ed.) 2007, p. 106: transformation of an intensive soliton of GE+.

More complicated transformation processes (e.g., generating breathers) due to the change of the nonlinear coefficients of the eKdV have also been simulated (Grimshaw et al. 1999, Nakoulima et al. 2004).

Introduction

Examples of nonlinear hyperbolic systems:

- Ideal shallow water equations:

$$h_t + (hu)_x = 0\,; \qquad u_t + uu_x + h_x = 0\,;$$

- Dispersionless (classical) limit of the KdV equation: ($u = u(x,t)$ – smooth solutions)

$$\lim_{\epsilon \to 0} \{u_t + uu_x + \epsilon^2 u_{xxx} = 0\} = \{u_t + uu_x = 0\}$$

– the **Hopf** (or Riemann-Hopf) equation.

- Semi-classical limit of the KdV equation (weak limit):

$$\lim_{\epsilon \to 0} \{\frac{\partial u_\epsilon}{\partial t} + u_\epsilon \frac{\partial u_\epsilon}{\partial x} + \epsilon^2 \frac{\partial^3 u_\epsilon}{\partial x^3} = 0\} = \{\frac{\partial r_i}{\partial t} + V_i(\mathbf{r}) \frac{\partial r_i}{\partial x} = 0\,, \; i = 1, 2, 3\}$$

– the **Whitham modulation equations.**
Here $u_\epsilon = u(x/\epsilon, t/\epsilon, x, t)$ is a rapidly oscillating KDV solution)

Kinematic waves. Conservation laws

The concept of kinematic waves was introduced by Lighthill and Whitham in 1955.
Consider a continuously distributed quantity characterised by a density $\rho(x,t)$ and a flux (flow per unit time) $q(x,t) = \rho v$, where $v(x,t)$ is the flow velocity. Assume the "mass" is conserved. Then for a section $x_2 \leq x \leq x_1$ we have the conservation equation (integral form)

$$\frac{d}{dt}\int_{x_2}^{x_1} \rho dx + q(x_1,t) - q(x_2,t) = 0 \tag{2}$$

Taking d/dt under the integral sign and passing to the limit as $x_1 \to x_2$ (we need to divide (2) by $(x_1 - x_2)$ before applying the limit) we obtain

$$\rho_t + q_x = 0 \tag{3}$$

provided ρ and q are continuously differentiable (C^1) functions .
Eq. (3) represents the conservation law in the differential form.

Kinematic waves

Consider the differential conservation law

$$\rho_t + q_x = 0$$

Now, if on some theoretical or empirical grounds we assume $q = Q(\rho)$ (this is so to a first approximation in many cases) then (3) becomes

$$\rho_t + c(\rho)\rho_x = 0\,, \tag{4}$$

where $c(\rho) = Q'(\rho)$.
Equation (4) is called the kinematic wave (or simple wave) equation.

- If $c'(\rho) \neq 0$ for all ρ involved, Eq. (4) is genuinely nonlinear.
- If $c(\rho)$ is a differentiable function, equation (4) reduces to the Hopf equation $u_t + uu_x = 0$ for $u = c(\rho)$.

Kinematic waves

Example: Kinematics of linear dispersive waves

The wave conservation law:

$$k_t + \omega_x = 0$$

describes the wavelenth/frequency modulations in a wave packet. Here k is the wavenumber and ω the frequency. For linear harmonic waves one can obtain a dispersion relation in the form

$$\omega = \omega(k)\,.$$

Then the wave conservation law yields the kinematic equation

$$k_t + c(k)k_x = 0\,,$$

where $c(k) = \omega'(k)$ is the linear wave group velocity.

Note: for dispersive waves $c'(k) = \omega''(k) \neq 0$ so kinematics of <u>linear</u> dispersive waves is governed by a nonlinear equation!

Other examples of kinematic waves include traffic flows, flood waves in rivers, chemical exchange processes, waves on glaciers, erosion in mountains, ...(see e.g. Whitham's book "Linear and Nonlinear Waves").

Hopf equation: solution via characteristics

We are going to solve an initial value problem (IVP):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0\,, \quad u(x,0) = u_0(x)\,.$$

The key: note that a linear combination

$$a(u,x,t)\frac{\partial u}{\partial t} + b(u,x,t)\frac{\partial u}{\partial x}\,,$$

can be interpreted as the directional derivative of $u(x,t)$ in the direction (a,b) in the x,t plane.

A characteristic curve or characteristic $\mathcal{C}$ of the equation $au_t + bu_x = f$ is introduced such that the tangent vector at each point of the curve has direction (a,b). For the Hopf equation $a(u,x,t) = 1$, $b(u,x,t) = u$ so that at each point of $\mathcal{C}$ we have

$$\frac{dx}{dt} = \frac{b}{a} = u$$

Now, along $\mathcal{C}$: $u = u(x(t),t)$ and

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = \frac{du}{dt}$$

Hopf equation: solution via characteristics

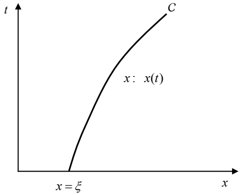
Now, the initial value problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0\,,$$
$$t = 0: \qquad u = u_0(x)\,, \quad -\infty < x < \infty$$

can be re-written in the characteristic form:

$$\text{on } \mathcal{C}: \begin{cases} \dfrac{dx}{dt} = u\,, \quad x(0) = \xi, & (a) \\[2ex] \dfrac{du}{dt} = 0\,, \quad u(0) = u_0(\xi)\,, & (b) \end{cases} \tag{5}$$

where ξ is a parameter (the initial point on the characteristic curve).



Hopf equation: solution via characteristics

Integrating (5a,b) we obtain the characteristic solution on $\mathcal{C}$

$$\begin{aligned} u &= u_0(\xi) && (a) \\ x &= \xi + tu_0(\xi) && (b) \end{aligned} \tag{6}$$

By varying ξ we get the solution in the whole (x, t) region provided $1 + tu_0'(\xi) \neq 0$ (solvability of (6b) for $\xi(x)$ at a given t). It is not difficult to verify by direct calculation that solution $u(x, t)$ specified by (6a), (6b) does solve the IVP – left as an exercise.
Note that characteristics $\mathcal{C}$ given by (6b) are straight lines.

Combining (6a) and (6b)) we obtain the solution $u(x, t)$ of the IVP in an implicit form

$$x = ut + x_0(u) \tag{7}$$

where $x_0(u) = u_0^{-1}(u)$ is the function inverse to $u_0(x)$ (note that by assuming $x_0(u)$ to be an arbitrary function in (7) we get a general solution to the Hopf equation).

An alternative form of the solution (7):

$$u = u_0(x - ut)\,. \tag{8}$$

Characteristic solution for a general simple wave equation

The results obtained for the Hopf equation can be readily generalised to the general simple wave equation

$$\begin{aligned} \rho_t + c(\rho)\rho_x = 0\,, \qquad t > 0\,, \qquad -\infty < x < \infty, \\ t = 0: \quad \rho = \rho_0(x), \quad -\infty < x < \infty, \end{aligned} \tag{9}$$

where $\rho_0(x) \in C^1(\mathbb{R})$, $c(\rho) \in C^1(\mathbb{R})$.
The solution $\rho(x,t)$ is given in the parametric characteristic form:

$$\rho(x,t) = \rho_0(\xi)\,, \qquad x = \xi + tF(\xi)\,, \tag{10}$$

where

$$F(\xi) = c(\rho_0(\xi)).$$

An equivalent implicit algebraic form of the solution (9) is

$$x - c(\rho)t = \rho_0^{-1}(\rho)\,.$$

<u>Remark</u>. When $c(\rho) = c_0 =$ constant, the simple wave equation becomes the linear translation equation $\rho_t + c_0\rho_x = 0$. The characteristic curves are $x = c_0 t + \xi$ and ρ is given by $\rho(x,t) = \rho_0(\xi) = \rho_0(x - c_0 t)$.

Expansion (rarefaction) wave

Consider the IVP

$$u_t + uu_x = 0\,, \quad x \in \mathbb{R},, \quad t > 0, \quad u(x,0) = u_0(x)\,,$$

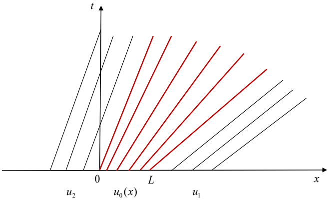
where

$$u_0 = \begin{cases} u_2, & \text{if} \quad x \leq 0 \\ \text{monotonically increasing, if} & 0 \leq x \leq L \\ u_1, & \text{if} \quad x \geq L\,. \end{cases}$$

where $u_1 > u_2$ (i.e. $u_0'(x) > 0$ for $0 \leq x \leq L$). The characteristic solution

$$u = u_0(\xi)\,, \qquad x = \xi + tu_0(\xi)$$

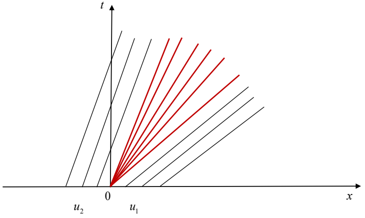
describes then an expansion fan (rarefaction wave).



Since the characteristics do not intersect at $t > 0$, we obtain the solution $u(x,t)$ as a single-valued function for all $t > 0$.

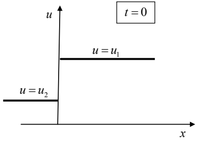
Centred expansion fan

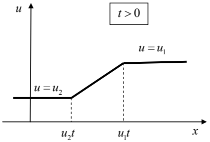
In the limiting case, when $L \to 0$, we obtain a centred fan (all characteristics between u_2 and u_1 pass through the origin).



The centred fan solution $u(x, t)$, $t > 0$ is described explicitly as

$$u = \begin{cases} u_2, & \text{if} \quad x \leq u_2 t \\ x/t & \text{if} \quad u_2 t \leq x \leq u_1 t \\ u_1, & \text{if} \quad x \geq u_1 t\,. \end{cases}$$





Wave breaking: characteristics

We look again at the characteristic solution

$$u = u_0(\xi)\,, \qquad x = \xi + t u_0(\xi)\,,$$

but now assume that the initial profile $u_0(\xi)$ has a section where $u_0'(\xi) < 0$. It implies that the condition $1 + u_0'(\xi)t \neq 0$ of solvability of the characteristic equation $x = \xi + tu_0(\xi)$ for $\xi(x,t)$ will fail for some

$$t = -\frac{1}{u_0'(\xi)} = \frac{1}{|u_0'(\xi)|} > 0.$$

It happens for the first time on the characteristic $\xi = \xi_b$, where $|u_0'(\xi)|$ assumes its maximum value (if $u_0(\xi)$ is smooth, this implies $u_0''(\xi_b) = 0$).

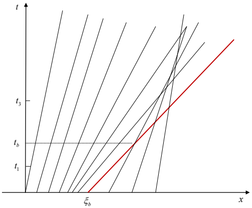
$$t_b = \frac{1}{|u_0'(\xi_b)|}$$

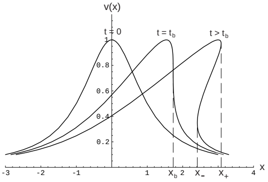
One can see that the derivatives

$$u_t = -\frac{u_0(\xi)u_0'(\xi)}{1 + u_0'(\xi)t}\,, \qquad u_x = \frac{u_0'(\xi)}{1 + u_0'(\xi)t}$$

become infinite at $t = t_b$: a gradient catastrophe, the wave breaking.

Wave breaking: nonlinear deformation of the profile





The described characteristic construction is illustrated in the left Figure.

How is this construction manifested in the solution $u(x,t)$ itself?
From the characteristic form of the Hopf equaion: $u_t + uu_x = 0$:

$$\frac{du}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = u$$

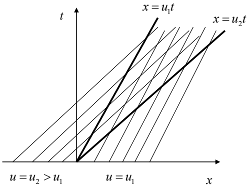
one can conclude that "each value of u propagates with its own speed u", hence the (nonlinear) distortion of the profile $u(x)$. At the wave breaking moment $t = t_b$ one has $\frac{\partial u}{\partial x} = \infty$ – the gradient catastrophe.
For $t > t_b$ – multivalued (three-valued) solution. Generally: non-physical.

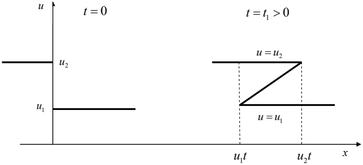
Wave breaking: centred compression wave

Consider the Hopf equation $u_t + uu_x = 0$ with step initial conditions:

$$u_0(x) = \begin{cases} u_2, & \text{if} \quad x < 0 \\ u_1, & \text{if} \quad x > 0\,. \end{cases}$$

where $u_2 > u_1$. Now the wave breaking occurs at $t = 0$. Formal multivalued solution for $t > 0$: centred compression wave with "overhanging":





Wave breaking: analytic construction

Consider an implicit solution of the Hopf equation $u_t + uu_x = 0$ in the form

$$x - ut = f(u)\,, \qquad (11)$$

where $f(u)$ is the inverse of the initial condition $u_0(x)$, all functions are supposed to be smooth. Then, at the breaking point one must have

$$t = t_b : \qquad \frac{\partial x}{\partial u} = 0, \qquad \frac{\partial^2 x}{\partial u^2} = 0 \qquad (12)$$

From (11), (12) we obtain for the breaking time:

$$t_b = -f'(u_b)\,, \qquad f''(u_b) = 0\,, \qquad (13)$$

where u_b is the value of u at the breaking point ($=$ the value at the inflection point of $u_0(x)$) Now, passing to the moving reference frame $\tilde{x} = x - u_b t$, introducing $\tilde{u} = u - u_b$ and shifting the time $\tilde{t} = t - t_b$ we arrive at the same Hopf equation $\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} = 0$ but now the wave breaking occurs at $\tilde{x}_b = \tilde{t}_b = \tilde{u}_b = 0$.

Wave breaking: analytic construction

In the new variables, the solution retains the form (we omit tildes for convenience) $x - ut = f(u)$ but now we have $f(0) = f'(0) = f''(0) = 0$. Expanding $f(u)$ in powers of u and assuming $f'''(0) \neq 0$ we get $f(u) \approx \mu u^3$, where $\mu < 0$ (so that the breaking develops for $t > 0$). The invariance of the Hopf equation with respect to the scaling

$$u \to Cu, \quad x \to xC^{-1/2}, \quad t \to tC^{-3/2} \quad \text{where} \quad C = const\,, \tag{14}$$

enables one w.l.o.g. to set $\mu = -1$ so the 'universal' equation describing the wave breaking reads

$$x - ut = -u^3\,. \tag{15}$$

One can readily show that the curve $u(x,t)$ is single-valued if $u_x(0,t) < 0$, which happens for $t < 0$. In contrast, $u(x,t)$ is triple-valued for $t > 0$.
Solution (15) is a 'generalised similarity solution'. Indeed, it is not difficult to see that (15) implies

$$u = t^{1/2}U(\zeta), \quad \text{where} \quad \zeta = xt^{-3/2}\,, \tag{16}$$

so that for $U(\zeta)$ we obtain an algebraic equation

$$U^3 - U + \zeta = 0 \tag{17}$$

Shock waves: a bit of history

- **1808: S.D Poisson:** *Mémoir sur la théorie du son, J. l'École Polytechnique,* **7,** *319 (1808).*
  Exact solution of the gas dynamics equations.

- **1848: G.G. Stokes:** *On a difficulty in the theory of sound, Phil. Mag.* **33,** *349 (1848)*

  A comment on Poisson's (1808) solution of the gas dynamics equations with regard to its "breaking" after finite time.

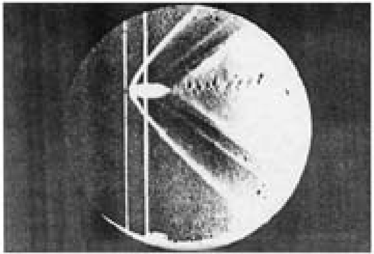
  "... perhaps the most natural supposition to make for trial is that a surface of discontinuity is formed, in passing across through which there is an abrupt change of density and velocity".

  "The full discussion of the motion which would take place after the change above alluded to, if possible at all, would probably require more pains than the result would be worth".

Shock waves: a bit of history

- Rankine (1870) and Hugoniot (1887) related the formation of a discontinuity with a change of the thermodynamic state of the medium due to which the matching conditions for the flows at two sides of the discontinuity became compatible with conservation of matter, momentum and energy.
- 1st half of 20th century: R. Courant, K. Friedrichs, L. von Mises, T. von Karman — to name a few: different aspects of the shock wave theory.
- P. Lax (1950s): modern mathematical theory of shock waves based on the generalised solutions of hydrodynamic conservation laws.

Shock waves: some classical pictures

Left: Ernst Mach experiment (1881): Generation of oblique shock waves due to supersonic motion of a projectile.

Right: Shock wave around a supersonic jet.

Discontinuous shocks

We shall first try to keep $q = Q(\rho)$ but extend the admissible family of solutions to include discontinuous functions.
We start with the conservation law in the integral form

$$\frac{d}{dt}\int_{x_2}^{x_1} \rho dx + q(x_1, t) - q(x_2, t) = 0\,, \tag{1}$$

which holds even if ρ and q are discontinuous.
Assume that the function $\rho(x, t)$ has a jump discontinuity at $x = s(t)$ where s is a continuously differentiable function of t.
At time t, let $x_1 > s(t) > x_2$ and $\dot{s}(t) \equiv \frac{ds}{dt}$.
The conservation law (1) can now be written as

$$q(x_2, t) - q(x_1, t) = \frac{d}{dt}\left\{\int_{x_2}^{s(t)} \rho dx + \int_{s(t)}^{x_1} \rho dx\right\} =$$

$$= \rho(s^-, t)\dot{s} - \rho(s^+, t)\dot{s} + \int_{x_2}^{s(t)} \rho_t dx + \int_{s(t)}^{x_1} \rho_t dx\,,$$

where $\rho(s^-, t) = \lim_{x\uparrow s(t)} \rho(x, t)$ and $\rho(s^+, t) = \lim_{x\downarrow s(t)} \rho(x, t)$

Discontinuous shocks

Now, noting that ρ_t is bounded on the intervals of the integration and taking the limits $x_1 \to s^+$ and $x_2 \to s^-$ we obtain

$$q(s^-, t) - q(s^+, t) = \{\rho(s^-, t) - \rho(s^+, t)\}\dot{s}.$$

We symbolically write this as

$$-U[\rho] + [q] = 0\,,$$

where $[\cdot]$ denotes the jump across the discontinuity and $U(t) = \dot{s}(t)$. The basic problem can now be written as

$$\rho_t + q_x = 0, \quad \text{at points of continuity}$$
$$-U[\rho] + [q] = 0, \quad \text{at discontinuity points} \qquad (*)$$

A formal correspondence between the differential equation and **the shock (jump) condition** (*):

$$\frac{\partial}{\partial t} \leftrightarrow -U[\cdot]\,, \quad \frac{\partial}{\partial x} \leftrightarrow [\cdot]$$

Note: the shock condition can also be written as

$$U = \frac{q_2 - q_1}{\rho_2 - \rho_1} = \frac{Q(\rho_2) - Q(\rho_1)}{\rho_2 - \rho_1}\,, \quad \text{where} \;\; \rho_{1,2} \equiv \rho(s^\pm, t) \;\; \text{etc.}$$

Discontinuous shocks

Important: the direct association of a jump condition with a differential conservation law is not unique.

Example. Consider

$$\rho_t + \rho\rho_x = 0 \tag{2}$$

This can be written as $\rho_t + (\frac{1}{2}\rho^2)_x = 0$, and the corresponding jump condition is

$$-U[\rho] + [\frac{1}{2}\rho^2] = 0\,. \tag{3}$$

However, equation (2) can also be written as

$$(\frac{1}{2}\rho^2)_t + (\frac{1}{3}\rho^3)_x = 0;$$

the associated shock condition would be

$$-U[\frac{1}{2}\rho^2] + [\frac{1}{3}\rho^3] = 0 \tag{4}$$

Obviously, (3) and (4) are different. Which one is correct?

We have to choose the appropriate jump condition only from the physical considerations of the problem and the original integral form of the conservation law

Discontinuous shocks: Example

The simplest case when breaking occurs:

$$\rho_t + c(\rho)\rho_x = 0\,, \qquad t > 0\,, \qquad -\infty < x < \infty,$$

$$t = 0: \quad \rho = \begin{cases} \rho_2, & \text{if} \quad x < 0 \\ \rho_1, & \text{if} \quad x > 0\,, \quad (\rho_2 > \rho_1) \end{cases}$$

with $c'(\rho) > 0$.
In this case breaking will occur immediately. The resulting discontinuous solution is

$$\rho(x,t) = \begin{cases} \rho_2, & \text{if} \quad x < Ut \\ \rho_1, & \text{if} \quad x > Ut\,, \end{cases}$$

where the shock velocity is

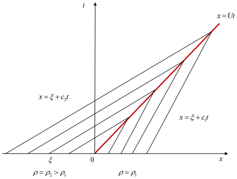
$$U = \frac{Q(\rho_2) - Q(\rho_1)}{\rho_2 - \rho_1}\,, \quad Q(\rho) = \int c(\rho)d\rho$$

For the particular case of the Hopf equation with $c(\rho) = \rho$, $Q(\rho) = \frac{1}{2}\rho^2$ we have

$$U = \frac{\rho_1 + \rho_2}{2}$$

Discontinuous shocks. Entropy condition.

The characteristics behaviour in the obtained discontinuous solution:



Here $c_1 = c(\rho_1)$, $c_2 = c(\rho_2)$. Since $\rho_2 > \rho_1$ and $c'(\rho) > 0$ we have $c_2 > c_1$. Thus we arrive at the inequality

$$c_2 > U > c_1 \tag{5}$$

Now, suppose that $c_2 < c_1$. Formally, the same discontinuous solution still applies. On the other hand, if $c_2 < c_1$ there is a continuous solution in the form of a rarefaction wave. **Non-uniqueness?**

Actually, the discontinuous solution with $\rho_2 > \rho_1$ and $c_2 < c_1$ is unstable. Condition (5) guarantees stability of a shock wave. In gas dynamics it is associated with the requirement that entropy of a gas must increase across the shock. Conditions of the type (5) are often called Lax's entropy conditions (Lax 1957).

Shock conditions for hyperbolic systems

Consider a system of hyperbolic conservation laws

$$\frac{\partial}{\partial t} f_i(x,t,\mathbf{u}) + \frac{\partial}{\partial x} g_i(x,t,\mathbf{u}) = 0\,, \quad i = 1, \ldots, n. \qquad (*)$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ are dependent variables, f_i are the conserved densities and g_i the corresponding fluxes. Assuming that system (*) is derived from integral equations for physical conserved quantities (mass, momentum, energy, etc.) one can introduce discontinuous solutions with the jump conditions across the shock

$$-U[f_i] + [g_i] = 0\,, \quad i = 1, 2, \ldots, n\,, \qquad (6)$$

where U is the velocity of the shock. Often (especially in gas dynamics) conditions (6) are called the Rankine-Hugoniot shock conditions.

Example: Shallow water shock wave (a bore)

The hyperbolic shallow water equations in the dimensionless form are

$$h_t + (hu)_x = 0\,,$$
$$u_t + uu_x + h_x = 0$$

where h is total depth and u is the depth-averaged horisontal velocity. Using "physical" conservation laws for "mass" $\int hdx$ and "momentum" $\int (hu)dx$ we obtain the shock (bore) conditions in the form

$$-U[h] + [uh] = 0$$
$$-U[uh] + [hu^2 + \frac{1}{2}gh^2] = 0$$

Eliminating U from the bore jump conditions one can find a restriction on admissible values of h and u at both sides of the bore

$$u_2 - u_1 = (h_2 - h_1)\sqrt{\frac{h_1 + h_2}{2h_1 h_2}}.$$

(Note that this restriction does not arise for a shock wave in a single hydrodynamic conservation law.) For the bore speed U we obtain

$$U = u_1 + h_2\sqrt{\frac{h_1 + h_2}{2h_1 h_2}},$$

Shallow water tidal bore on Severn river

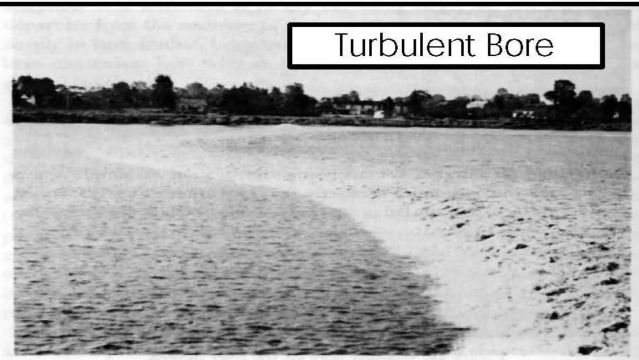


Figure 49. Turbulent bore on another part of the river Severn. The bores in figures 48 and 49 were both photographed by Dr D. H. Peregrine and are reproduced here by his kind permission.

Reproduced from "Waves in Fluids", James Lighthill, CUP (1978)

Fitting a shock in the evolving smooth profile

We now look at the solution $\rho(x, t)$ of an IVP for a single conservation law:

$$\rho_t + q_x = 0, \quad q = Q(\rho), \quad \rho(x, 0) = \rho_0(x)\,,$$

which can be represented implicitly by

$$x = c(\rho)t + P(\rho)\,, \quad \text{where} \quad c(\rho) = Q'(\rho), \quad P(\rho) \equiv \rho_0^{-1}(\rho) \tag{7}$$

— in the regions of continuity and satisfies the shock condition

$$-U[\rho] + [Q(\rho)] = 0 \tag{8}$$

at discontinuity points. We assume $c'(\rho) > 0$.

We now consider $\rho_0(x)$ in the form of an arbitrary smooth function (a smooth hump to be definite). Since at the front part of the hump we have $\rho_0'(x) < 0$, the wave breaking is expected at some $t = t_b > 0$. For $t > t_b$ a multivalued region forms which needs to be replaced by an appropriate shock discontinuity.

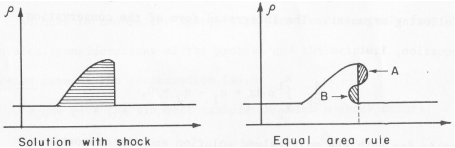
The question: how to fit the shock (8) in the evolving profile (7)?

Fitting in a shock: equal area rule

The key: the integral form of the conservation equation $\frac{d}{dt}\int_{x_2}^{x_1} \rho dx + q(x_1, t) - q(x_2, t) = 0$, holds for both the multivalued solution and the discontinuous solution. If we take a single hump disturbance with $\rho = \rho_0$ on both sides and if we take x_1, x_2 away from the disturbance with $q_1 = q_2 = Q(\rho_0)$, then the area

$$\int_{x_2}^{x_1} \rho dx = \text{constant in time}$$

for both the multivalued and discontinuous solution. Hence the position of the shock must be chosen to give equal areas $A = B$ as shown below.



(see Whitham's book "Linear and Nonlinear Waves" for the analytic implementation of this simple geometric construction).

Asymptotic evolution of a localised disturbance

For simplicity we assume $Q(\rho) = \frac{1}{2}\rho^2$ and consider the initial value problem

$$\rho_t + \rho\rho_x = 0, \qquad \rho(x,0) = \rho_0(x)\,, \quad t > 0, \quad -\infty < x < \infty$$

We now consider the asymptotic behaviour, $t \gg 1$, of a single hump given by

$$\rho_0 = \begin{cases} c_0 & \text{in} \quad x \leq a \\ g(x) & \text{in} \quad a \leq x \leq L \\ c_0 & \text{in} \quad x \geq L\,. \end{cases}$$

where g is continuous in $[a, L]$ with $g(a) = g(L) = c_0$.

The characteristic solution is

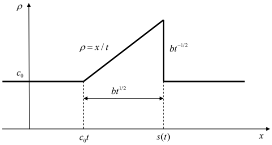
$$\rho = \rho_0(\xi), \qquad\qquad x = \xi + \rho_0(\xi)t\,.$$

Triangular wave

For large $t \gg 1$ to leading order we have the triangular wave

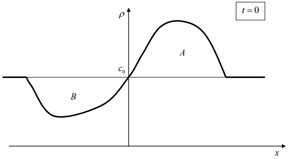
$$\rho = \begin{cases} c_0 & x \le c_0 t \\ x/t & c_0 t \le x \le s(t) \\ c_0 & x \ge s(t)\,, \end{cases}$$

where $x = s(t)$ is the position of the shock still to be determined.
The shock condition gives $U = \frac{\rho_1 + \rho_2}{2}$; therefore, since $\rho_1 = c_0$, $\rho_2 = s(t)/t$ (see the Fig. below) and $U = \dot{s}$, we have $\dot{s} = \frac{1}{2}\{c_0 + s/t\}$.
The solution is easily found to be $s = c_0 t + bt^{1/2}$, where b is a constant.
Thus the shock strength $\rho_2 - \rho_1 = bt^{-1/2}$.

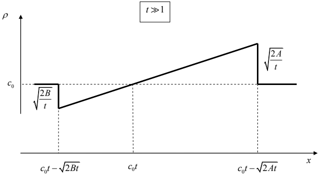


Since the area $A = \int (g(x) - c_0)dx =$ constant in time, we have for the asymptotic triangular wave $A = b^2/2$ and so $b = (2A)^{1/2}$.
Thus only the area of the initial wave appears in the asymptotic solution; all other details are lost. Sharp contrast with linear wave theory.

Asymptotic evolution of a "bipolar" profile is described by the so-called *N − wave* solution. The analysis is analogous to that in the single hump case.

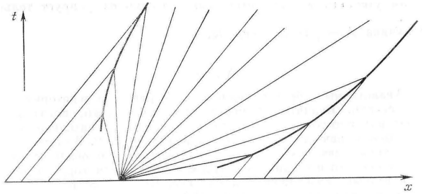


$\longrightarrow$



At $t \gg 1$ the solution between two discontinuities is asymptotically given by

$$\rho \sim \frac{x}{t} \qquad c_0 t - \sqrt{2Bt} < x < c_0 t + \sqrt{2At}$$



Shock waves and weak solutions

The "combined" solution consisting of the continuously differentiable part satisfying equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} = 0 \tag{9}$$

and the discontinuous part satisfying shock condition

$$-U[\rho] + [Q(\rho)] = 0 \tag{10}$$

could be considered as weak solutions of equation (9).
Consider equation

$$-\iint\limits_R \{\rho\phi_t + Q(\rho)\phi_x\} dxdt = 0 \tag{11}$$

where R is an arbitrary rectangle in (x, t) plane and ϕ is an arbitrary "test" function continuously differentiable in R and vanishing at the boundary of R. Using integration by parts one can verify that (11) and (9) are equivalent for $\rho, Q(\rho) \in C^1$ (to prove (9) $\Rightarrow$ (11) part of the equivalence one needs to multiply (9) by ϕ and the integrate over R).

Shock waves and weak solutions

However, equation (11) admits a broader class of solutions than (9) since admissible function $\rho(x,t)$ now do not have to be differentiable. The functions $\rho(x,t)$ satisfying (11) for all test functions ϕ are called weak solutions of (9).

Now let us construct the weak solution $\rho(x,t)$ satisfying (11) such that $\rho(x,t)$ is continuously differentiable in two parts R_1 and R_2 of the rectangle R and having a discontinuity at the boundary S separating R_1 and R_2.

Integrating by parts in each of the regions R_1 and R_2 we obtain from (11):

$$\iint\limits_{R_1} \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} \right\} \phi dxdt + \iint\limits_{R_2} \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} \right\} \phi dxdt + \int\limits_S \{[\rho] l + [Q(\rho)] m\} \phi ds = 0 \,, \tag{12}$$

where (l,m) is a normal to S and $[\rho]$ and $[Q(\rho)]$ are the jumps on S.

Shock waves and weak solutions

The last line integral along S in (12) results from the integration by parts and using the divergence theorem. Since equation (12) must hold for all test function ϕ we conclude that (9) is valid within each region R_1 and R_2 and also

$$[\rho]l + [Q(\rho)]m = 0 \qquad \text{on} \quad S \tag{13}$$

Since the boundary speed $U = -l/m$ (note that $U = dx/dt$ is the gradient of the tangent line to S in (x, t)-plane), equation (13) coincides with the shock condition.

Thus, the weak solution $\rho(x, t)$ constructed above, satisfies (9) at the continuity points and has a discontinuity satisfying (10) as required.

Note: as earlier, uniqueness is ensured by the requirement that (9) must be a physical conservation law following from the original integral formulation.

Dissipative vs dispersive resolution

Linearising the KdV-Burgers equation against constant background $\rho = c_0$ and looking for the solution in the form $\rho - c_0 \sim \exp(i(kx - \omega t))$ we obtain the dispersion relation

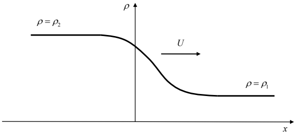
$$\omega = kc_0 - i\nu k^2 - \epsilon^2 k^3 \tag{3}$$

One can see that the term $\nu\rho_{xx}$ (usually arising due to the volume viscosity of the medium) describes the wave dissipation: $\rho - c_0 \sim e^{-\nu k^2 t} \to 0$ as $t \to \infty$ for all x, while $\epsilon^2 \rho_{xxx}$ is the familiar KdV term describing wave dispersion (i.e. dependence of the phase velocity on the wavenumber). The effects of dissipation and dispersion on the wave breaking are drastically different. It is therefore, instructive first to consider them separately. So we shall consider the wave breaking in

- the Burgers equation $\rho_t + \rho\rho_x = \nu\rho_{xx}$,
- the KdV equation $\rho_t + \rho\rho_x + \epsilon^2 \rho_{xxx} = 0$.

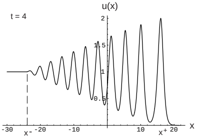
Viscous vs dispersive shock waves

- **The structure of viscous shock wave**



Smooth steady transition propagating with the shock speed $U = \frac{\rho_1 + \rho_2}{2}$. The width of the viscous shock is proportional to ν so we obtain the discontinuous shock profile in the limit as $\nu \to 0$.

- **The structure of dispersive shock wave**



Unsteady nonlinear wavetrain confined to an expanding region $-\Delta \cdot t < x < \frac{2}{3}\Delta \cdot t$, where $\Delta = \rho_2 - \rho_1$. Weak convergence to $\overline{\rho}(x,t)$ as $\epsilon \to 0$, where $\overline{\rho}(x,t)$ is a solution of the **Whitham modulation equations**. No connection with the viscous shock profile/speed !

Viscous shock: Taylor's profile

We consider the Burgers equation

$$\rho_t + \rho\rho_x = \nu\rho_{xx} \tag{4}$$

with constant boundary conditions at infinity:

$$\rho \to \begin{cases} \rho_2, & x \to -\infty, \\ \rho_1, & x \to +\infty\,. \end{cases} \tag{5}$$

Let $\rho_2 > \rho_1$. We shall look for a steady profile moving with constant velocity U, i.e. a travelling wave, $\rho = \rho(x - Ut)$. Then Burgers' equation becomes an ODE:

$$-U\rho_\xi + \rho\rho_\xi = \nu\rho_{\xi\xi}\,, \quad \text{where} \quad \xi = x - Ut\,.$$

Integrating once we obtain

$$\nu\rho_\xi = \tfrac{1}{2}\rho^2 - U\rho + C\,, \tag{6}$$

where C is an arbitrary constant. Using boundary conditions (5) and assuming that $\rho_\xi \to 0$ as $|\xi| \to \infty$ we factorize the polynomial in (6):

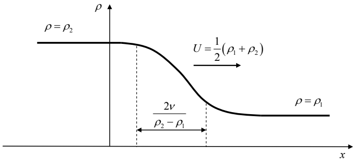
$$\nu\rho_\xi = -\frac{1}{2}(\rho - \rho_1)(\rho_2 - \rho)\,, \tag{7}$$

so that $U = \frac{1}{2}(\rho_1 + \rho_2)$, $C = \frac{1}{2}\rho_1\rho_2$.

Viscous shock: Taylor's profile

Integrating (7) we obtain the so-called Taylor's shock profile

$$\rho(x,t) = \rho_1 + (\rho_2 - \rho_1)\frac{\exp\{-(\rho_2-\rho_1)\xi/(2\nu)\}}{1+\exp\{-(\rho_2-\rho_1)\xi/(2\nu)\}}\,. \qquad (8)$$



One can see that the transition profile (8):

- Propagates with the shock velocity $U = \frac{1}{2}(\rho_1 + \rho_2)$;
- Has the characteristic width $\Delta = \frac{2\nu}{\rho_2 - \rho_1}$, so $\Delta \to 0$ as $\nu \to 0$ for fixed $\rho_{1,2},\ \rho_1 \neq \rho_2$.

Hence the constructed exact solution of the Burgers equation confirms our earlier shock wave theory and the viscous transition layer (8) replaces the discontinuous shock.

Note: for fixed ν, the thickness $\Delta \to \infty$ if one has $\rho_2 - \rho_1 \to 0$.

General solution of the Burgers equation

The Taylor profile represents an important particular solution of the Burgers equation. Now we describe the method to obtain a general solution. This method was developed in 1950-51 independently by Cole and Hopf.

Key idea: By introducing a nonlinear substitution

$$\rho = -2\nu \frac{\phi_x}{\phi}, \tag{9}$$

the Burgers equation $\rho_t + \rho\rho_x = \nu\rho_{xx}$ is reduced to the linear heat equation

$$\phi_t = \nu\phi_{xx} \tag{10}$$

The calculation is done in two simple steps:

1) Introducing $\rho = \psi_x$ in the Burgers equation and integrating it once we obtain

$$\psi_t + \tfrac{1}{2}\psi_x^2 = \nu\psi_{xx}\,. \tag{11}$$

2) Then the transformation

$$\psi = -2\nu \log\phi$$

reduces (11) to the heat equation (10).

General solution of the Burgers equation

The initial condition

$$t = 0 : \quad \rho = \rho_0(x)$$

for the Burgers equation is transformed under the Cole-Hopf substitution (9) into the initial condition for the heat equation (10):

$$t = 0 : \phi = \Phi(x) = \exp\left\{-\frac{1}{2\nu}\int_0^x \rho_0(x')dx'\right\} \tag{12}$$

Then the solution of the heat equation (10) with the initial condition (12) is

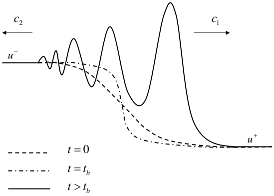
$$\phi = \frac{1}{\sqrt{4\pi\nu t}}\int\limits_{-\infty}^{\infty} \Phi(\eta)\exp\left\{-\frac{(x-\eta)^2}{4\nu t}\right\}d\eta \tag{13}$$

and therefore, by (9)

$$\rho(x,t) = \frac{\int_{-\infty}^{\infty}\frac{x-\eta}{t}\Phi(\eta)\exp\left\{-\frac{(x-\eta)^2}{4\nu t}\right\}d\nu}{\int_{-\infty}^{\infty}\Phi(\eta)\exp\left\{-\frac{(x-\eta)^2}{4\nu t}\right\}d\nu} \tag{14}$$

Purely dispersive resolution: KdV equation

If one takes into account **only** the dispersive correction in the closure condition, $q = Q(\rho) + \epsilon^2 \rho_{xx}$, $\epsilon \ll 1$ the resolution of the breaking singularity occurs through the generation of a nonlinear wavetrain: a dispersive shock wave (DSW) (a.k.a. conservative undular bore).

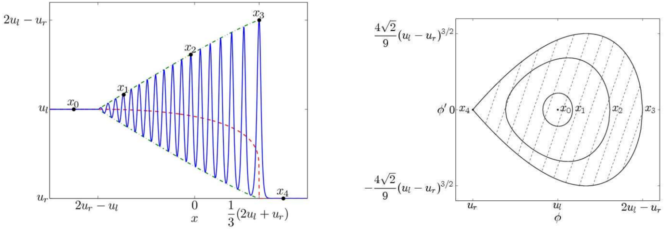


The model equation is the KdV equation $u_t + uu_x + \epsilon^2 u_{xxx} = 0$ and the solution for $t > t_b$ has the form of a modulated **locally periodic** wave. A DSW solution to the KdV equation has the distinctive spatial structure: at the leading edge it exhibits a soliton while in the vicinity of the trailing edge it transforms into a vanishing amplitude linear wave packet.

Important: the DSW expands in time, i.e. no steady transition is formed.

Dispersive shock wave: phase portrait

From: *M. Hoefer and M. Ablowitz (2009) Dispersive shock waves. Scholarpedia, 4(11):5562*



Left: KdV DSW (solid), its envelope (dash-dotted), and its average (dashed). The points marked correspond to phase portraits in the right Figure. **Right**: Example phase portraits of the KdV DSW. Each phase portrait labeled x_j corresponds to a certain point in the DSW solution in the left Figure. As the DSW is traversed from trailing to leading edge, the hashed region is filled by all the phase portraits.

Important: no single trajectory in the phase plane as the exact DSW description for the KdV equation cannot be reduced to an ODE.

Dispersive-dissipative resolution: KdV-Burgers equation

We now return to the resolution of breaking singularity in the full KdV-Burgers (KdV-B) equation $u_t + uu_x + u_{xxx} = \nu u_{xx}$.

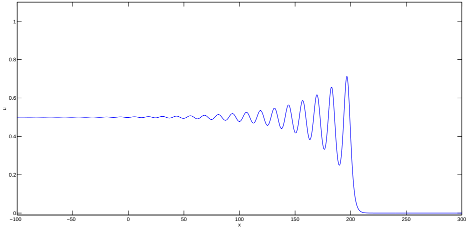


Figure: Numerical solution of the KdV-B equation with $\nu = 0.05$ and initial condition $u(x,0) = (1 - \tanh(x))/4$ describing weakly dissipative DSW (viscous DSW – vDSW).

Nontrivial!: Despite apparent qualitative similarity of the numerical plots, the analytical nature of the DSW solutions in 'pure' KdV and KdV-B equations is completely different . Moreover, the long-time asymptotic solution for the step resolution in the KdV-B equation is actually simpler than the corresponding solution of the pure KdV equation!

We consider the KdVB equation

$$u_t + uu_x + u_{xxx} = \nu u_{xx} \tag{15}$$

with constant boundary conditions at infinity:

$$u \to \begin{cases} u_2 & x \to -\infty, \\ u_1 & x \to +\infty, \end{cases} \tag{16}$$

where $u_2 > u_1$ We shall look for a steady profile moving with constant velocity U, i.e. a travelling wave $u = u(\xi)$, where $\xi = x - Ut$. Then the KdVB equation becomes an ODE:

$$-Uu_\xi + uu_\xi + u_{\xi\xi\xi} = \nu u_{\xi\xi},$$

Integrating once we obtain an ODE for a nonlinear oscillator with damping

$$u_{\xi\xi} + u^2/2 - Uu + C = \nu u_\xi, \tag{17}$$

where U, C are found from boundary conditions (16) (assuming $u_\xi, u_{\xi\xi} \to 0$ as $|\xi| \to \infty$):

$$U = \frac{1}{2}(u_1 + u_2), \quad C = \frac{1}{2}u_1 u_2, \tag{18}$$

i.e. the oscillatory solution moves as a whole with the classical shock speed!

Phase plane analysis

We assume for simplicity that $u_2 = 1$, $u_1 = 0$. Then introducing $w = u_\xi$ we have in the phase plane (u, w)

$$w_\xi = \nu w - \tfrac{1}{2}u(u-1)\,, \qquad u_\xi = w\,. \tag{19}$$

The integral curve

$$\frac{dw}{du} = \frac{\nu w - \frac{1}{2}u(u-1)}{w} \tag{20}$$

connects two singular points in the (u, w) plane: $(0, 0)$ and $(1, 0)$.

- $(0, 0)$. Linearising about $(u = 0, w = 0)$ we obtain from (20):

$$w \sim \lambda_1 u\,, \quad u \sim e^{\lambda_1 \xi}\,, \quad \lambda_1 = \tfrac{1}{2}(\nu - \sqrt{\nu^2 + 2}) < 0$$

  — a saddle point (note that $u \to 0$ when $\xi \to \infty$)
- $(1, 0)$. Linearising about $(u = 1, w = 0)$ we obtain from (20):

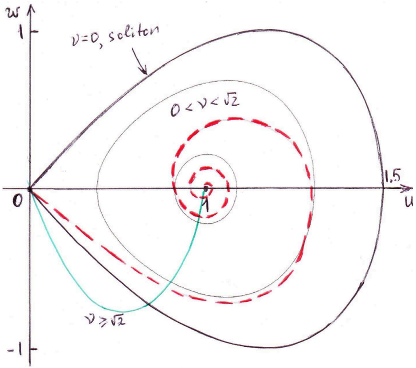
$$w \sim \lambda_2(u-1)\,, \quad (u-1) \sim e^{\lambda_2 \xi}\,, \quad \lambda_2 = \tfrac{1}{2}(\nu \pm \sqrt{\nu^2 - 2})\,:$$

$$\begin{cases} \text{a stable node if} & \nu \geq \sqrt{2} \quad \text{— strong damping, monotonic decay} \\ \text{a stable focus if} & \nu < \sqrt{2} \quad \text{— weak damping, decay with oscillations} \end{cases}$$

  (note that in both cases $u \to 1$ when $\xi \to -\infty$)

Phase trajectory (integral curve $w(u)$ starting from $(0,0)$):

- $\nu = 0$: a homoclinic orbit (a soliton in (ξ, u)-plane);
- $0 < \nu < \sqrt{2}$: starts at $(0,0)$ and spirals to $(1,0)$ (a weakly dissipative DSW/undular bore in (ξ, u)-plane);
- $\nu \geq \sqrt{2}$: monotonic curve connecting $(0,0)$ and $(1,0)$ (a monotonic transition in (ξ, u)-plane);
- $\nu \to \infty$: phase trajectory corresponding to the Taylor shock profile.



Weakly dissipative DSW: speeds and amplitudes

The detailed analysis in Johnson (1970) shows that for sufficiently small damping the leading wave in the vDSW is asymptotically close to the solitary wave solution of the KdV-B equation with $\nu = 0$, i.e. the standard KdV soliton. The corresponding relationship between the KdV soliton speed c_s and its amplitude a_s is $c_s = a/3$. Now, since the speed of the vDSW is $U = (u_2 + u_1)/2 = \Delta/2$ where $\Delta = u_2$ is the jump of u across the DSW (assuming $u_1 = 0$), we find by equating $c_s = U$ that $a_s = 1.5\Delta$ (does not depend on the value of ν – a very general result!)

However, this result cannot be applied to a purely conservative DSW, when $\nu = 0$. Actually, we shall see that for the KdV equation $a_s = 2\Delta$. As was mentioned earlier, the case $\nu = 0$ is intrinsically unsteady so that the steady solution considered above is never realised and a completely different theory based on the analysis of nonlinear PDEs (the so-called Whitham equations) is required.

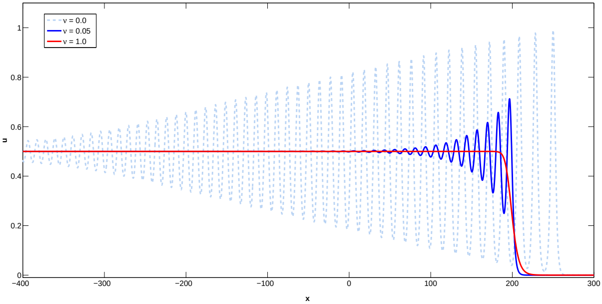
Wave profiles at $\nu = 0$, $0 < \nu < \nu_c$ and $\nu > \nu_c$

On the plot, numerical solutions of the IVP problem

$$u_t + uu_x + u_{xxx} = \nu u_{xx}\,, \qquad u(x,0) = (1 - \tanh(x))/4\,.$$

are presented at $t = 800$ for

- $\nu = 0$: no dissipation, unsteady conservative DSW (dashed line);
- $\nu = 0.05$: weak dissipation, steady vDSW (blue line);
- $\nu = 1.0$: strong dissipation, steady monotonic shock transition (red line).



Shallow water: simple waves

The shallow water system can be represented as a system of conservation equations

$$\partial_t \rho_i + \partial_x q_i = 0 \quad i = 1, 2\,, \tag{3}$$

where

$$\rho_1 = h\,, \quad q_1 = \rho_2 = hu\,, \quad q_2 = hu^2 + \frac{1}{2}h^2. \tag{4}$$

In the theory of kinematic waves we require, on the physical or empirical grounds, that $q = Q(\rho)$, which leads to the simple wave equation. Similar requirement for system (3), (4) would be say $q_1 = uh = Q_1(h)$ where $Q_1(h)$ is now provided not from outside physical observations etc. but from the second shallow-water equation. Or, we could require that $q_2 = hu^2 + \frac{1}{2}h^2 = Q_2(hu)$. The two requirements are equivalent and come down to the question whether there are solutions in which, say

$$h = h(u)\,. \tag{5}$$

Substituting this into the shallow water system we get

$$u_t + uu_x + \frac{h(u)}{h'(u)}u_x = 0\,, \quad u_t + uu_x + h'(u)u_x = 0 \tag{6}$$

Shallow water: simple waves

For consistency one must require

$$h'(u) = \frac{h(u)}{h'(u)}\,, \tag{7}$$

which implies

$$h'(u) = \pm\sqrt{h}\,. \tag{8}$$

Taking the positive sign in (8) and integrating we obtain

$$u - 2\sqrt{h} = -2\sqrt{h_0}\,, \quad \text{where} \quad h_0 = h(0)\,. \tag{9}$$

Then both equations (6) become the same simple wave equation for the right-propagating wave:

$$u_t + c_+(u)u_x = 0\,, \quad c_+(u) = u + \sqrt{h(u)}\,. \tag{10}$$

If we take the negative sign in (8) we obtain the equation for the left-propagating simple wave

$$u_t + c_-(u)u_x = 0\,, \quad c_-(u) = u - \sqrt{h(u)} \tag{11}$$

Note: equation (9) is the functional relation equivalent to $q = Q(\rho)$ in the kinematic waves theory.

Method of characteristics for a system: $N = 2$

As already was mentioned, the simple wave solutions are limited to the waves moving in one direction only. We now consider the waves moving in both directions and interacting with each other.
The shallow-water equations

$$h_t + uh_x + hu_x = 0\,; \qquad u_t + uu_x + h_x = 0\,; \tag{12}$$

contain information about the rates of change of h and u in different directions of the (x, t)-plane. If the directions were the same we could apply the method of characteristics as for a single simple-wave equation.
Consider a linear combination

$$\{u_t + uu_x + h_x\} + m\{h_t + uh_x + hu_x\} = 0\,, \tag{13}$$

where m is determined by the condition that equation (13) assumes the "unidirectional" characteristic form

$$\{u_t + vu_x\} + m\{h_t + vh_x\} = 0\,. \tag{14}$$

Comparing (14) and (13) we see that this happens if

$$u + mh = u + 1/m = v\,, \tag{15}$$

which gives $m = \pm h^{-1/2}$ so that $v = v_\pm = u \pm \sqrt{h}$.

Riemann invariants

Taking $m = h^{-1/2}$ in (14) we obtain

$$(u + 2\sqrt{h})_t + (u + \sqrt{h})(u + 2\sqrt{h})_x = 0 \tag{16}$$

We consider the characteristic $\mathcal{C}_+$ defined by $\frac{dx}{dt} = u + \sqrt{h}$. On $\mathcal{C}_+$ equation (16) becomes $\frac{d}{dt}(u + 2\sqrt{h}) = 0$, which implies

$$u + 2\sqrt{h} = constant \quad \text{on} \quad \mathcal{C}_+ \,. \tag{17}$$

Similarly, taking $m = -h^{-1/2}$ we obtain

$$(u - 2\sqrt{h})_t + (u - \sqrt{h})(u - 2\sqrt{h})_x = 0 \,. \tag{18}$$

The characteristic $\mathcal{C}_-$ is $\frac{dx}{dt} = u - \sqrt{h}$. On $\mathcal{C}_-$ equation (18) becomes $\frac{d}{dt}(u - 2\sqrt{h}) = 0$, which implies

$$u - 2\sqrt{h} = constant \quad \text{on} \quad \mathcal{C}_- \,. \tag{19}$$

The quantities $r_\pm = u \pm 2\sqrt{h}$, which remain constant along the corresponding characteristics are called Riemann invariants.

Riemann invariants and simple waves

Introducing the Riemann invariants $r_\pm = u \pm 2\sqrt{h}$ instead of h and u in the shallow water equations we arrive at the diagonal (Riemann form) system

$$\partial_t r_+ + V_+(r_-, r_+)\partial_x r_+ = 0\,, \qquad \partial_t r_- + V_-(r_-, r_+)\partial_x r_- = 0\,, \tag{20}$$

where the characteristic velocities $V_\pm$ are (these are nothing but $v_\pm$ expressed in terms of the Riemann invariants)

$$V_+ = \tfrac{3}{2} r_+ + \tfrac{1}{2} r_-\,, \qquad V_- = \tfrac{3}{2} r_- + \tfrac{1}{2} r_+\,. \tag{21}$$

System (20) has two families of integrals: $r_- = constant$ and $r_+ = constant$ (note that these are global integrals, i.e. for all x, t).
(i) Let $r_- = r_-^0$ be fixed. Then the second equation (20) is satisfied identically while the first one becomes the simple wave equation for r_+

$$\partial_t r_+ + V_-(r_-^0, r_+)\partial_x r_+ = 0\,. \tag{22}$$

(ii) Setting $r_+ = r_+^0 = const$ we similarly obtain the simple-wave equation for r^-: $\partial_t r_- + V_+(r_-, r_+^0)\partial_t r_- = 0\,.$

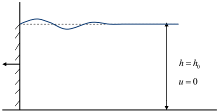
These simple wave equations for r^+ and r^- are equivalent to equations (10) and (11) obtained earlier by the substitution $h = h(u)$.

Simple waves: Example

Piston problem

We consider a piston 'wavemaker' moving parallel to the x-axis in the negative direction with given velocity. Initially, when the piston is at rest, the water is at rest, i.e.

$$t = 0: \qquad u = 0\,, \quad h = h_0\,. \tag{23}$$



The movement of the piston is represented in the (x, t)-plane by the given curve $x = X(t)$ on which we require that

$$x = X(t): \quad u = \dot{X}(t) < 0\,. \tag{24}$$

Far from the piston the water is undisturbed so we have

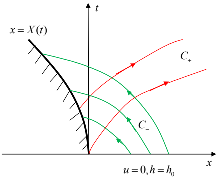
$$x \to \infty: \quad u = 0\,, \quad h = h_0 \;\text{ for }\; x \geq 0\,. \tag{25}$$

We need to find solution of the shallow-water equations with initial and boundary conditions (23), (24) and (25).

We know that

$$\begin{aligned} r_- &= u - 2\sqrt{h} \quad \text{is constant on} \quad \mathcal{C}_- : \ \tfrac{dx}{dt} = u - \sqrt{h} \\ r_+ &= u + 2\sqrt{h} \quad \text{is constant on} \quad \mathcal{C}_+ : \ \tfrac{dx}{dt} = u + \sqrt{h} \end{aligned} \tag{26}$$

Then, since $u - \sqrt{h} \leq u$ one can show that the $\mathcal{C}_-$ characteristics cover the whole region $\{(x,t) : t \geq 0, \ x \geq X(t)\}$



From the initial condition (23) we have that $r_-(x,0) = -2\sqrt{h_0}$ for all $x \geq 0$. Thus each $\mathcal{C}_-$ transfers the same value of r_-. Therefore, $r_- = -2\sqrt{h_0}$ everywhere. At the same time, each $\mathcal{C}_+$ characteristic, starting on the piston, transfers its 'own' value of r_+. Thus the shallow-water system reduces to the simple wave equation for r_+:

$$\partial_t r_+ + (\tfrac{3}{2} r_+ - \sqrt{h_0}) \partial_x r_+ = 0 \,. \tag{27}$$

From $r_- = u - 2\sqrt{h} = -2\sqrt{h_0}$ we find $\sqrt{h} = \frac{u}{2} + \sqrt{h_0}$ and so $r_+ = u + 2\sqrt{h} = 2(u + \sqrt{h_0})$. Substituting this into equation (27) for r_+ we obtain an equivalent simple wave equation for u

$$u_t + (\tfrac{3}{2}u + \sqrt{h_0})u_x = 0 \tag{28}$$

with the piston boundary condition

$$x = X(t): \quad u = \dot{X}(t). \tag{29}$$

Integrating by characteristics we obtain solution in a parametric form

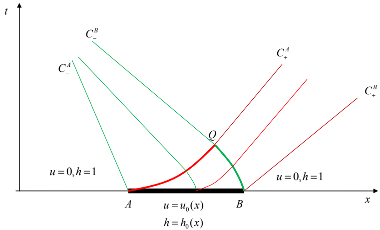
$$u = \dot{X}(\xi), \qquad x = X(\xi) + \{\sqrt{h_0} + \tfrac{3}{2}\dot{X}(\xi)\}(t - \xi). \tag{30}$$

This solution is global (i.e. exists for all t and $x > X(t)$) if $\dot{X} < 0$ and $\ddot{X}(t) \leq 0$.

If the piston moves forward ($\dot{X} > 0$) or if $\ddot{X} > 0$, then the wave breaking will occur at some $t = t_b > 0$ so solution (30) ceases to be valid for $t > t_b$.

General solution: hodograph transformation

A simple wave represents a disturbance propagating along a single characteristic family. For a general Cauchy problem, i.e. if the initial conditions $u(x,0) = u_0(x)$, $h(x,0) = h_0(x)$ do NOT satisfy any of the simple wave relations $u_0(x) - 2\sqrt{h_0(x)} = const$ or $u_0(x) + 2\sqrt{h_0(x)} = const$), both families of characteristics carry nontrivial disturbances so we need more general solutions than those described so far.



Outside ABQ there are two separate simple waves propagating in opposite directions. Within ABQ the simple waves interact and cannot be described individually. Integration along characteristics can be used to obtain solution numerically.

We are going to describe the interacting simple waves analytically. The equations are:

$$h_t + uh_x + hu_x = 0\,, \qquad u_t + uu_x + h_x = 0\,. \tag{31}$$

Note that the coefficients are functions of the dependent variables only. We try to make use of that fact by interchanging the role of dependent and independent variables. Such a transformation is called the hodograph transformation.

We have $u = u(x,t)$, $h = h(x,t)$ and, assuming that the mapping $(x,t) \mapsto (u,h)$ is invertible, consider the inverse functions

$$x = x(u,h)\,, \qquad t = t(u,h)\,. \tag{32}$$

Now, using implicit differentiation or otherwise we obtain the relations between the derivatives

$$h_t = -x_u/J \quad u_t = x_h/J, \quad h_x = t_u/J, \quad u_x = -t_h/J\,, \tag{33}$$

where $J = \frac{\partial(x,t)}{\partial(u,h)} = x_u t_h - x_h t_u$ is the Jacobian of the transformation. It is clear that the hodograph transformation requires that $J \neq 0$, $J^{-1} \neq 0$.

Substituting (33) into (31) we obtain the linear system (the Jacobian cancels thanks to the absence of undifferentiated terms!)

$$x_u - ut_u + ht_h = 0, \quad x_h - ut_h + t_u = 0\,. \tag{34}$$

Cross-differentiation yields a single second-order linear PDE for t:

$$t_{uu} - ht_{hh} = 2t_h \tag{35}$$

Introducing $T(c, v)$ instead of $t(h, u)$ by the relations $t = \frac{1}{c}\frac{\partial T}{\partial c}\,, \quad c = \sqrt{h}\,, \quad v = u/2$, one reduces (35) to the cylindrical wave equation

$$T_{vv} = T_{cc} + \frac{1}{c}T_c\,, \tag{36}$$

whose general solution can be represented as

$$T = \frac{F(v + c) + G(v - c)}{c}\,, \tag{37}$$

where F and G are arbitrary functions. Note that $v \pm c = u/2 \pm \sqrt{h}$ are the Riemann invariants!

Hodograph transform: restrictions of the method

- Consider an IVP for the shallow-water system

$$t = 0 : \qquad h = \mathcal{H}(x) \quad u = \mathcal{U}(x) \tag{38}$$

  These expressions, in principle, parametrically define a curve in the hodograph (u, h)-plane, where one specifies $t = 0$ and x. These are the boundary conditions for the hodograph equations. Unfortunately, in most cases these boundary conditions turn out to be quite awkward so the hodograph solutions are not often used in gas and fluid dynamics. There are, however, some exceptional configurations when hodograph solutions are extremely valuable. A classical example: nonlinear shallow-water waves on a sloping beach (Carrier and Greenspan 1958)

- When wave breaking occurs, the Jacobian of the hodograph transformation $J = 0$, corresponding to multivaluedness, and fitting in shocks may be difficult in (h, u)-plane.

- For simple waves (i.e. when $h = h(u)$), the hodograph transform is degenerate, $J = 0$ (left as an exercise), so the simple wave solutions are not captured by the hodograph method.

Generalisation to $N > 2$: Characteristics

Consider an N-component quasilinear system

$$u^i_t + A^i_j(\mathbf{u}, x, t)u^j_x = 0\,, \quad j = 1, \ldots, N\,, \tag{39}$$

(summation over repeating indices is assumed). Also, for simplicity we assume $A(\mathbf{u}, x, t) = A(\mathbf{u})$. We introduce a linear combination

$$l_i\{u^i_t + A^i_j(\mathbf{u})u^j_x\} = 0\,, \tag{40}$$

where the vector $\mathbf{l} = \mathbf{l}(\mathbf{u})$.
A PDE (40) assumes the characteristic form:

$$l_i \frac{du^i}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = \lambda \tag{41}$$

$$\text{provided} \quad l_i A^i_j = \lambda l_j\,, \tag{42}$$

so that $\mathbf{l}$ is the left eigenvector of the matrix A and the characteristic speeds $\lambda = \lambda^k(\mathbf{u})$, $k = 1, 2, \ldots, N$ are the corresponding eigenvalues satisfying

$$|A^j_i - \lambda \delta^j_i| = 0. \tag{43}$$

System (39) is hyperbolic if the eigenvectors $\mathbf{l}^k$ form a basis.

Generalisation to $N > 2$: Riemann invariants

Each equation in a characteristic form introduces its own linear combination of derivatives $l_i du^i/dt$. In some cases it is possible to find an integrating factor $\mu(\mathbf{u})$ such that

$$\mu l_i du^i = dr \tag{44}$$

where $r = r(\mathbf{u})$. Equation (44) is equivalent to the requirement that

$$\mu l_i = \frac{\partial r}{\partial u^i}\,, \quad i = 1, \ldots, N. \tag{45}$$

If $N = 2$ and one has only two equations (45), so one can always find a single equation for μ. However for $N > 2$ it is generally not possible as the system for μ becomes overdetermined (and, generally, inconsistent!). If it is possible to introduce r^k for each of N characteristic forms, then the system of characteristic equations becomes

$$\frac{dr^k}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = V^k(\mathbf{r}), \tag{46}$$

where $V^k(\mathbf{r}) = \lambda^k(\mathbf{u}(\mathbf{r}))$. Eqs. (46) are equivalent to a diagonal system:

$$r^k_t + V^k(\mathbf{r}) r^k_x = 0, \quad k = 1, \ldots, N\,. \tag{47}$$

(no summation over k!). Thus r^k are the Riemann invariants.

Generalisation to $N > 2$: Hodograph transform

Theorem (Generalised Hodograph Method) (Tsarev 1985):
The general local smooth nonconstant, $r^i_x \neq 0$, solution of the diagonal hyperbolic hydrodynamic type system

$$r^k_t + V^k(\mathbf{r})r^k_x = 0, \quad k = 1, \dots, N \tag{48}$$

has the form

$$x - V^i(\mathbf{r})t = W^i(\mathbf{r})\,, \tag{49}$$

where N functions $W^i(\mathbf{r})$ satisfy **linear** overdetermined system of PDEs

$$\frac{\partial_i W^j}{W^i - W^j} = \frac{\partial_i V^j}{V^i - V^j}\,, \quad i,j = 1, \dots, N, \quad i \neq j, \quad \partial_i \equiv \frac{\partial}{\partial r^i} \tag{50}$$

provided the characteristic velocities $V^i(\mathbf{r})$ satisfy the following set of conditions (the semi-Hamiltonian property)

$$\partial_j \frac{\partial_k V^i}{V^k - V^i} = \partial_k \frac{\partial_j V^i}{V^j - V^i}, \quad i \neq j \neq k\,. \tag{51}$$

semi-Hamiltonian $=$ Integrable

Characteristics: some important properties

Below we present several important consequences of the definitions of characteristics and Riemann invariants. The proofs are not difficult and can be found in any good book on fluid mechanics/nonlinear waves (e.g. Whitham's book "Linear and nonlinear waves" or "Fluid mechanics" by Landau and Lifshitz)

- The discontinuities of derivatives (i.e. weak discontinuities) of dependent variables can occur only on characteristics. One can say that weak discontinuities propagate along characteristics with the corresponding characteristic speed.

- The line separating two analytically different solutions of a hyperbolic system of hydrodynamic type must be a characteristic.

- **The adjacent flow theorem**: the flow adjacent to a constant flow is either a constant flow or a simple wave. Note that this statement is valid only for the flows described by (2×2) hyperbolic systems.

- Care should be taken when specifying boundary conditions on a characteristic.

17.1. The Whitham Method: Intoduction

One of the most important applied aspects of the Whitham theory is analytic description of formation and evolution of dispersive shock waves (DSWs). A DSW represents a nonlinear wave train replacing a traditional viscous shock when the resolution of a breaking singularity is dominated by dispersion rather than dissipation. These DSWs are quite ubiquitous and manifest themselves as undular bores on rivers (like famous Severn bore in England) and in density-stratified waters of coastal ocean (e.g. internal waves caused by Strait of Gibraltar). They occur in the atmosphere as striking wave-forms with associated cloud formation in the atmospheric boundary layer (Australian Morning Glory), as collisionless shocks in rarefied plasmas and as nonlinear diffraction patterns in laser optics. Very recently DSWs have been observed in Bose-Einstein condensates, one of the most intriguing states of matter.

There is a number of important connections between the Whitham theory, the IST and the general theory of integrable hydrodynamic type systems. Some of them will be considered in the remaining part of this course.

17.2 Linear modulated waves

Let us first consider the linearised KdV equation

$$u_t + 6u_0 u_x + u_{xxx} = 0\,, \tag{1}$$

where u_0 is a constant. We know that there is a harmonic solution

$$u = ae^{i(kx-\omega t)} + c.c.\,, \tag{2}$$

where the wave amplitude a and the wavenumber k are constants and the frequency ω is related to the wavenumber by the dispersion relation

$$\omega = 6ku_0 - k^3\,. \tag{3}$$

Now let us look for the **approximate** solution in the form of a slowly modulated linear wave. For that, we consider the same solution (2) but allow slow variations for the parameters a, k, ω:

$$a = a(X,T), \quad k = k(X,T), \quad \omega = \omega(X,T)\,, \tag{4}$$

where $X = \epsilon x$, $T = \epsilon t$ and $\epsilon \ll 1$ is a small parameter.

Now the general approach is to assume $u = u(\theta, X, T)$, where $\theta = \dfrac{S(X,T)}{\epsilon}$ is the "fast" phase variable.
Explicitly, we look for the solution of the linearised KdV equation in the form

$$u = a(X,T)e^{i\frac{S(X,T)}{\epsilon}} + \text{c.c}\,. \tag{5}$$

We define the **local** wavenumber $k(X,T)$ and **local** frequency $\omega(X,T)$ as

$$k = S_X\,, \qquad \omega = -S_T\,. \tag{6}$$

Cross-differentiating we obtain the "wave conservation" law

$$k_T + \omega_X = 0 \tag{7}$$

Then the derivatives are transformed as follows:

$$\begin{gathered}\frac{\partial}{\partial t} = -\omega\frac{\partial}{\partial\theta} + \epsilon\frac{\partial}{\partial T}\,, \qquad \frac{\partial}{\partial x} = k\frac{\partial}{\partial\theta} + \epsilon\frac{\partial}{\partial X}\\ \frac{\partial^3}{\partial x^3} = k^3\frac{\partial^3}{\partial\theta^3} + \epsilon(3k\frac{\partial k}{\partial X}\frac{\partial^2}{\partial\theta^2} + 3k^2\frac{\partial}{\partial X}\frac{\partial^2}{\partial\theta^2}) + O(\epsilon^2)\end{gathered} \tag{8}$$

Now, explicitly, using $u = a(X, T)e^{i\theta}$, we have

$$\frac{\partial u}{\partial t} = [-i\omega a + \epsilon \frac{\partial a}{\partial T}]e^{i\theta}, \qquad \frac{\partial u}{\partial x} = [-i\omega a + \epsilon \frac{\partial a}{\partial X}]e^{i\theta}$$
$$\frac{\partial^3 u}{\partial x^3} \simeq [-ik^3 a - 3\epsilon\{ka\frac{\partial k}{\partial X} + k^2 \frac{\partial a}{\partial X}\}]e^{i\theta} \tag{9}$$

Substituting (9) into the linearised KdV equation (1) we obtain **to first order in** ϵ:

$$[-i\omega + 6iku_0 - ik^3] + \epsilon[\frac{\partial a}{\partial T} + (6u_0 - 3k^2)\frac{\partial a}{\partial X} - 3ka\frac{\partial k}{\partial X}] = 0 \tag{10}$$

To leading order we obtain

$$\omega = 6ku_0 - k^3 \tag{11}$$

which is the linear dispersion relation (3) which now is shown to connect the **local** wavenumber and **local** frequency.
The first order in *epsilon* yields the amplitude equation,

$$\frac{\partial a}{\partial T} + (6u_0 - 3k^2)\frac{\partial a}{\partial X} - 3ka\frac{\partial k}{\partial X} = 0, \tag{12}$$

Noticing that $6u_0 - 3k^2 = \omega'(k)$ (the group velocity) and $-3k = \frac{1}{2}\omega''(k)$ equation (12) can be cast in a conservative (wave energy)form:

$$\frac{\partial}{\partial T} a^2 + \frac{\partial}{\partial X}(a^2\omega'(k)) = 0 \,. \tag{13}$$

Equation (13) together with the "conservation of waves" law

$$k_T + \omega_X = 0 \tag{14}$$

form the **modulation system** describing propagation of a **linear dispersive** wavepacket characterised by the dispersion relation (11).

It turns out that system (13), (14) is a **universal** modulation system for linear dispersive waves (of course, each dispersive equation is characterised by its own relation $\omega = \omega(k)$).

One can also show that

$$a(X,T)e^{i\frac{S(X,T)}{\epsilon}} \to a(X,T)e^{i[k(X,T)x - \omega(X,T)t]} \quad \text{as} \quad \epsilon \to 0$$

(up to an arbitrary phase shift, or "initial phase" which does not contribute to the modulation equations) so one can use the original representation (2) with the fast phase linearly depending on x and t for the derivation of the modulation system (13), (14).

17.3. Slow modulations of the nonlinear periodic wave

Now we try to derive modulation equations for nonlinear periodic waves.

The single-phase travelling wave solution $u = u(kx - \omega t)$ of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{15}$$

satisfies the ODE

$$-\omega u_\theta + 6kuu_\theta + k^3 u_{\theta\theta\theta} = 0\,, \quad \theta = kx - \omega t\,, \tag{16}$$

which can be integrated twice to give

$$\tfrac{1}{2}k^2(u_\theta)^2 = -(u - b_1)(u - b_2)(u - b_3)\,, \quad b_1 \le b_2 \le b_3\,. \tag{17}$$

Here $c = \omega/k = 2(b_1 + b_2 + b_3)$ is the phase velocity.
In the periodic wave $b_2 \le u \le b_3$ and the spatial period (wavelength) $L = 2\pi/k$ is found from the condition of 2π-periodicity of u with respect to the phase θ:

$$2\pi = \int_0^{2\pi} d\theta = 2\int_{b_2}^{b_3} \frac{du}{u_\theta}\,. \tag{18}$$

Using (17) we obtain

$$L(b_1, b_2, b_3) = \sqrt{2}\int_{b_2}^{b_3} \frac{du}{\sqrt{-(u - b_1)(u - b_2)(u - b_3)}} = \frac{2\sqrt{2}K(m)}{(b_3 - b_1)^{1/2}}\,, \tag{19}$$

where $K(m)$ is the complete elliptic integral of the first kind, $m = (b_3 - b_2)/(b_3 - b_1)$ is the modulus, $0 \le m \le 1$.

Modulation equations via multiple-scale expansions

The solution of (17) can be expressed in terms of the Jacobi elliptic $cn(\xi, m)$ as

$$u = b_2 + (b_3 - b_2)\,\mathrm{cn}^2\left(\sqrt{2(b_3 - b_1)}(x - ct - x_0); m\right), \qquad (20)$$

The solution (20) is often referred to as the cnoidal wave solution. We shall use the notation

$$u(x,t) = U_0(\theta; \mathbf{b}), \quad \mathbf{b} = (b_1, b_2, b_3) \qquad (21)$$

for the periodic solution (20).

We now introduce a slowly modulated periodic wave by letting the constant parameters b_j be functions of x and t on a large spatio-temporal scale, i.e. $b_j = b_j(X,T)$, where $X = \epsilon x$, $T = \epsilon t$, and $\epsilon \ll 1$ is a small parameter. Now function $U_0(\theta; \mathbf{b})$ (21) is no longer an exact solution of the KdV equation (15). One can, however, require that $U_0(\theta, \mathbf{b}(X,T))$ satisfies the KdV equation asymptotically, i.e. to first order in ϵ. This requirement leads to a set of restrictions for the slowly varying functions $b_j(X,T)$, which are called modulation equations.

We shall seek for an asymptotic solution of the KdV equation in the form

$$u = u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \dots, \tag{22}$$

where the leading term $u^{(0)}$ has the form (21) of the periodic travelling wave solution but with slowly varying parameters b_1, b_2 and b_3. To this end, we introduce an auxiliary phase function $S(X, T)$ and represent the terms $u^{(n)}$ of the expansion(22) in the form

$$u^{(n)} = U_n(S(X, T)/\epsilon;\ \mathbf{b}(X, T)), \qquad n = 1, 2, \dots \tag{23}$$

where $U_n(S(X, T))$ are 2π-periodic functions, which depend smoothly on X, T.
Then, for the leading term in (22) to have the form of the travelling wave (21), i.e. $u^{(0)} \to U_0(kx - \omega t; \mathbf{b})$ as $\epsilon \to 0$, one must require (**check!!!**)

$$S_X = k(\mathbf{b}(X, T)), \qquad S_T = -\omega(\mathbf{b}(X, T)). \tag{24}$$

The compatibility condition $S_{XT} = S_{TX}$ yields the wave conservation law

$$k_T + \omega_X = 0. \tag{25}$$

This is one of the nonlinear modulation equations.

17.4. Modulation equations via multiple-scale expansions

To obtain the remaining two modulation equations we substitute the asymptotic expansion (22) into the KdV equation (15) with the account of the form of the leading periodic term U_0 (21). As a result, collecting the terms $\mathcal{O}(\epsilon)$, we arrive at the ODE

$$-\omega(U_1)_\theta + 6k(U_0U_1)_\theta + k^3(U_1)_{\theta\theta\theta} = -(U_0)_T - 6U_0(U_0)_X \,. \qquad (26)$$

Since the right-hand side of equation (26) is a 2π-periodic function in θ, an unbounded growth of the solutions is expected due to resonances with the eigenfunctions of the linear operator on the left-hand side. To eliminate this unbounded growth, we impose the orthogonality conditions,

$$\int_0^{2\pi} y_\alpha R d\theta = 0\,, \qquad \alpha = 1, 2\,, \qquad (27)$$

where $R(U_0, \partial_T U_0, \partial_X U_0)$ is the right-hand side of equation (26), and y_α are the 2π-periodic solutions of the adjoint equation (cf. (17))

$$-\omega y_\theta + 6kU_0 y_\theta + k^3 y_{\theta\theta\theta} = 0\,. \qquad (28)$$

One can see that there are indeed just two periodic solutions of the equation (28): $y_1 = 1$ and $y_2 = U_0$. Equations (25) and (27) provide then the full set of the modulation equations for $b_{1,2,3}(X, T)$.

17.5. The Whitham Method 1: Averaging the conservation laws

There is a convenient alternative to the direct multiple-scale perturbation procedure outlined above. This alternative (but, of course, equivalent) formal method was proposed by Whitham in 1965.

The **Whitham method** for obtaining modulation equations prescribes the averaging of a certain number of dispersive conservation laws over the period of the travelling wave solution.

The "certain" number of conservation laws required for the averaging is equal to the number of constants characterising the periodic travelling wave solution (we only ignore the initial phase θ_0, which in majority of problems in not essential). Say, for the KdV equation this number equals three so we need to consider any three independent conservation laws of the KdV equation

$$\partial_t P_j + \partial_x Q_j = 0, \qquad j = 1, 2, 3 . \tag{29}$$

KdV: single-phase averaging

We repeat here the ODE (17) for the single-phase travelling wave solution $u = u(kx - \omega t)$ of the KdV equation:

$$\frac{1}{2}k^2(u_\theta)^2 = -(u - b_1)(u - b_2)(u - b_3)\,, \quad b_1 \leq b_2 \leq b_3, \tag{30}$$

where $\theta = kx - \omega t$, $c = \omega/k = 2(b_1 + b_2 + b_3)$.
The period (wavelength) is given by

$$L(b_1, b_2, b_3) = \sqrt{2}\int_{b_2}^{b_3} \frac{du}{\sqrt{-(u - b_1)(u - b_2)(u - b_3)}}\,. \tag{31}$$

For any function $F = F(u)$ one may introduce the average over the periodic family (30):

$$\overline{F}(b_1, b_2, b_3) = \frac{1}{2\pi}\int_0^{2\pi} F(u(\theta))d\theta = \frac{\sqrt{2}}{L}\int_{b_2}^{b_3} \frac{F(u)}{\sqrt{-(u - b_1)(u - b_2)(u - b_3)}}du$$

KdV: single-phase averaging

Let the parameters b_j of the periodic solution be slowly varying functions of x, t, i.e. $b_j = b_j(X, T)$, where $X = \epsilon x$, $T = \epsilon t$, $\epsilon \ll 1$.

Consider the KdV equation in the conservation form

$$\partial_t[u] + \partial_x[3u^2 + u_{xx}] = 0 \tag{32}$$

and average it, $(1/\Delta)\int_0^\Delta ...dx$, over a large interval $[0, \Delta]$, where $L \ll \Delta \ll L/\epsilon$, **on the periodic family** (30). Then to leading order in ϵ we obtain

$$\partial_T \overline{u} + \partial_X[\overline{3u^2 + u_{xx}}] = 0\,. \tag{33}$$

Explicitly (note: $\overline{u_{xx}} = 0$),

$$\partial_T\left[\frac{1}{L}\oint\frac{u}{\sqrt{-(u-b_1)(u-b_2)(u-b_3)}}du\right]+\partial_X\left[\frac{1}{L}\oint\frac{3u^2}{\sqrt{-(u-b_1)(u-b_2)(u-b_3)}}du\right]=0$$

which is a PDE for $b_1(X, T)$, $b_2(X, T)$, $b_3(X, T)$.

Two more equations are obtained by averaging the next two KdV conservation laws.

Averaged conservation laws: remarks.

- Modulation system does not depend on the choice of local conservation laws used for averaging.
- The averaging method is equivalent to multiple-scale expansions.
- The "wave conservation" equation $k_T + \omega_X = 0$ must be consistent with the modulation system so can be used instead of any of the local averaged conservation laws. Hence only $N - 1$ local conservation laws are necessary for the derivation of the closed modulation system.
- One can choose any 3 independent combinations of the parameters b_1, b_2, b_3 as the modulation variables, e.g. $\overline{u}, k, a$.
- The averaging procedure can be applied to any nonlinear dispersive equation provided the periodic solution and necessary number of local conservation laws are available.

17.6. Some properties of the Whitham equations

The general Whitham system:

$$\frac{\partial}{\partial T}\overline{P}_j(\mathbf{b}) + \frac{\partial}{\partial X}\overline{Q}_j(\mathbf{b}) = 0\,, \quad j = 1, 2, \ldots, N\,, \tag{34}$$

where $\mathbf{b} = (b_1, b_2, \ldots, b_N)^T$ – can be represented in a standard form of a **system of hydrodynamic type**.

$$\mathbf{b}_t + \mathrm{A}(\mathbf{b})\mathbf{b}_x = 0\,, \tag{35}$$

provided $det[\partial\overline{P}_i/\partial b^j] \neq 0$. Of course, one can expect that the matrix $\mathrm{A}(\mathbf{b})$ would possess some special properties, especially if the original equation, like the KdV equation, is an integrable system. On a more immediate level, one can note that the Whitham system (35):

- describes evolution of an 'envelope' of a nonlinear dispersive wave
- is itself a dispersionless hydrodynamic type system (i.e. linear with respect to first derivatives).

17.7. The Whitham Method 2: Variational Approach to Modulation Theory

Another method to derive modulation equations was also proposed by Whitham (1966) and is based on the averaged variational principle.

Let the evolution wave equation can be represented as an Euler-Lagrange equation, that is, as a consequence of the variational principle

$$\delta \int \int \Lambda(u, u_x, u_t) dxdt = 0 \tag{36}$$

with some Lagrangian Λ depending on the field variable u and its spatial and temporal derivatives.
Thus, the corresponding evolution equation is given by $\delta\Lambda/\delta u = 0$, i.e

$$\frac{\partial}{\partial t}\frac{\partial \Lambda}{\partial u_t} + \frac{\partial}{\partial x}\frac{\partial \Lambda}{\partial u_x} - \frac{\partial \Lambda}{\partial u} = 0\,. \tag{37}$$

The Whitham Method 2: Variational Approach to Modulation Theory

Now we consider the family of periodic solutions

$$u(x,t) = U(kx - \omega t, a)\,, \quad U(\theta + 2\pi, a) = U(\theta, a)\,, \tag{38}$$

a being the wave amplitude (for simplicity of presentation we do not introduce other modulation parameters than k, ω and a). Next we introduce the averaged Lagrangian by

$$\mathfrak{L}(\omega, k, a) = \frac{1}{2\pi}\int_0^{2\pi} \Lambda(U, kU, -\omega U, a)d\theta. \tag{39}$$

Now, we assume that the parameters of the periodic solution are slowly varying functions, i.e. we introduce $X = \epsilon x$, $T = \epsilon t$, $\epsilon \ll 1$ and represent the periodic solution (38) in the form

$$U = U(\frac{S(X,T)}{\epsilon}; a, k, \omega),$$

where $\omega = \omega(X,T)$, $k = k(X,T)$, $a = a(X,T)$, and

$$k = \partial S/\partial X\,, \qquad \omega = -\partial S/\partial T\,. \tag{40}$$

The Whitham Method 2: Variational Approach to Modulation Theory

Now, it is postulated that the modulation equations can be obtained from the **averaged variational principle** (Whitham 1966):

$$\delta \int \int \mathfrak{L}(\omega, k, a) dX dT = 0\,, \quad \text{where} \quad k = S_X, \quad \omega = -S_T \tag{41}$$

The averaged Lagrangian $\mathfrak{L}$ depends on two functions $a(X, T)$ and $S(X, T)$ so the variational equations have the form $\delta\mathfrak{L}/\delta a = 0$ and $\delta\mathfrak{L}/\delta S = 0$, i.e.

$$\delta a: \qquad \mathfrak{L}_a = 0\,, \tag{42}$$

$$\delta S: \qquad \frac{\partial}{\partial T}\mathfrak{L}_\omega - \frac{\partial}{\partial X}\mathfrak{L}_k = 0\,. \tag{43}$$

Complementing this with the wave conservation law $S_{XT} = S_{TX}$, i.e.

$$k_T + \omega_X = 0, \tag{44}$$

we obtain a closed modulation system for k and ω, where equation (42) represents a nonlinear dispersion relation $\omega = \omega(k, a)$. When $a \to 0$ we recover the linear dispersion relation $\omega = \omega_0(k)$.

The Whitham system for the KdV equation

It was discovered by Whitham (1965) that, upon introducing symmetric combinations

$$r_1 = \frac{b_1 + b_2}{2}\,,\ r_2 = \frac{b_1 + b_3}{2}\,,\ r_3 = \frac{b_2 + b_3}{2}\,, \tag{4}$$

the KdV-Whitham system assumes the diagonal (Riemann) form

$$\frac{\partial r_j}{\partial T} + V_j(r_1, r_2, r_3)\frac{\partial r_j}{\partial X} = 0\,, \quad j = 1, 2, 3\,, \tag{5}$$

where no summation over the repeated indices is assumed and the characteristic velocities have the form

$$V_1 = 2(r_1 + r_2 + r_3) - 4(r_2 - r_1)\frac{K(m)}{K(m) - E(m)}\,,$$

$$V_2 = 2(r_1 + r_2 + r_3) - 4(r_2 - r_1)\frac{(1 - m)K(m)}{E(m) - (1 - m)K(m)}\,,$$

$$V_3 = 2(r_1 + r_2 + r_3) + 4(r_3 - r_1)\frac{(1 - m)K(m)}{E(m)}.$$

Here $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and the second kind respectively.
The variables r_j are the Riemann invariants.

The structure of the characteristic velocities

Consider the wave conservation law (3)

$$k_T + \omega_X = 0\,, \qquad (*)$$

where now $k = k(\mathbf{r})$, $\omega = \omega(\mathbf{r})$. Introduce the Riemann invariants in (*) explicitly:

$$\sum_{i=1}^{3}\left\{\frac{\partial k}{\partial r_j}\frac{\partial r_j}{\partial T} + \frac{\partial \omega}{\partial r_j}\frac{\partial r_j}{\partial X}\right\} = 0\,, \qquad (**)$$

We know that equation (*), being part of the modulation system, must be consistent with its Riemann form (5)

$$\frac{\partial r_j}{\partial T} + V_j(r_1, r_2, r_3)\frac{\partial r_j}{\partial X} = 0\,, \quad j = 1, 2, 3\,. \qquad (***)$$

Now substituting (***) into (**) yields

$$\sum_{j=1}^{3}\left\{\frac{\partial \omega}{\partial r_j} - V_j\frac{\partial k}{\partial r_j}\right\}\frac{\partial r_j}{\partial X} = 0 \quad \Rightarrow \quad V_i = \frac{\partial\omega/\partial r_j}{\partial k/\partial r_j}\,, \quad i = 1, 2, 3.$$

(the derivatives $\partial r_j/\partial X$ are independent). So we have arrived at a nonlinear generalization of the linear group velocity notion $\partial\omega_0/\partial k$.

The structure of the characteristic velocities

Using $k = 2\pi/L$ and $\omega = kc$ we get

$$V_i = \frac{\partial_i \omega}{\partial_i k} = (1 - \frac{L}{\partial_i L}\partial_i)c\,, \qquad \partial_i \equiv \frac{\partial}{\partial r_i} \tag{6}$$

Now expressing L and c in terms of r_j explicitly we get

$$L = \int_{r_1}^{r_2} \frac{d\lambda}{\sqrt{(\lambda - r_1)(r_2 - \lambda)(r_3 - \lambda)}} \qquad c = 2(r_1 + r_2 + r_3). \tag{7}$$

Calculating L in terms of the complete elliptic integral of the first kind $K(m)$ we get

$$L = \frac{2K(m)}{(r_3 - r_1)^{1/2}}\,, \quad m = \frac{r_2 - r_1}{r_3 - r_1}\,. \tag{8}$$

Substituting (8) into (6) we obtain explicit formulae for the characteristic velocities $V_j(\mathbf{r})$ in terms of the complete elliptic integrals.

We shall call the representation (6) for the characteristic velocities a "potential" representation, the phase velocity c being the potential.
It is instructive to study the behaviour of the Whitham system in two important limiting cases : $m = 0$ (harmonic limit) and $m = 1$ (soliton limit)

The structure of the Whitham equations: two limits

- **Harmonic limit,** $m \to 0$
  Direct calculation shows: $V_3 \to 6r_3$; $V_2 \to V_1 \to (12r_1 - 6r_3)$ so the Whitham system reduces to

$$r_2 = r_1\,, \quad \frac{\partial r_3}{\partial T} + 6r_3\frac{\partial r_3}{\partial X} = 0\,, \quad \frac{\partial r_1}{\partial T} + (12r_1 - 6r_3)\frac{\partial r_1}{\partial X} = 0\,. \quad (9)$$

- **Soliton limit,** $m \to 1$
  Now we have: $V_2 \to V_3 \to (2r_1 + 4r_3)$; $V_1 \to 6r_1$ so that the Whitham system reduces to

$$r_2 = r_3\,, \quad \frac{\partial r_1}{\partial T} + 6r_1\frac{\partial r_1}{\partial X} = 0\,, \quad \frac{\partial r_3}{\partial T} + (2r_1 + 4r_3)\frac{\partial r_3}{\partial X} = 0\,. \quad (10)$$

Thus, in both limits, one of the Whitham equations converts into the Hopf equation $r_T + 6rr_X = 0$ (the dispersionless limit of the KdV equation). Using explicit limiting expressions for $k = 2\pi/L$ and c (7), (8), one can show that the multiple characteristic velocity in (9) $V_2 = V_1$ coincides with the linear group velocity $\partial\omega_0/\partial k$ while the multiple characteristic velocity $V_2 = V_3$ in (10) coincides with the soliton velocity, the soliton amplitude being $2(r_3 - r_1)$. This is left as an exercise.

The structure of the KdV-Whitham equations

Analysis of the characteristic velocities V_j shows that

- the characteristic velocities are real and distinct so the KdV-Whitham system is hyperbolic. Hence:
  1. modulational stability.
  2. classical theory of characteristics for the hydrodynamic type systems is applicable.

- $\frac{\partial V_i}{\partial r_i} > 0$, therefore the KdV-Whitham system is genuinely nonlinear. Thus one can expect the wave breaking effects and complications with the existence of global solution.

Integrability of the KdV-Whitham equations: 1. Simple wave solutions

First, one can observe that the Riemann form $\frac{\partial r_j}{\partial T} + V_j(r_1, r_2, r_3)\frac{\partial r_j}{\partial X} = 0\,, \quad j = 1, 2, 3\,,$ implies that any $r_j = \text{constant}$ is an exact solution of the Whitham equations. This enables one to consider exact reductions of the Whitham system and find some important particular solutions.

Simple wave reduction.

We consider the reduction of the Whitham system when two of the Riemann invariants, say r_2 and r_3, are constant, $r_2 = r_{20}$, $r_3 = r_{30}$. Then the Whitham system reduces to a single "simple-wave" equation

$$\frac{\partial r_1}{\partial T} + V_1(r_1, r_{20}, r_{30})\frac{\partial r_1}{\partial X} = 0$$

which is integrated using characteristics to give an exact solution

$$X - V_1(r_1, r_{20}, r_{30})T = W(r_1) \tag{11}$$

Here $W(r_1)$ is an arbitrary function. In the solution of an initial-value problem $W(r_1)$ has the meaning of the inverse to the initial function $r_1(X, 0)$.

Integrability of the KdV-Whitham equations: 2. Hodograph Solutions

Let $r_3 = r_{30} = \text{constant}$. Then for the remaining two $r_{1,2}(X, T)$, one has a 2×2 system, which can be solved (linearised) using the classical hodograph transformation provided $r_{1X} \neq 0$, $r_{2X} \neq 0$.

This is achieved through the "swap" of the dependent and independent variables $(r_1, r_2) \leftrightarrows (X, T)$. For that, we write the differentials

$$dr_1 = \frac{\partial r_1}{\partial T} dT + \frac{\partial r_1}{\partial X} dX$$

$$dr_2 = \frac{\partial r_2}{\partial T} dT + \frac{\partial r_2}{\partial X} dX$$

and solve this system for dX and dT to obtain the derivatives

$$\frac{\partial X}{\partial r_1} = \frac{1}{J} \frac{\partial r_2}{\partial T}, \quad \frac{\partial X}{\partial r_2} = -\frac{1}{J} \frac{\partial r_1}{\partial T}, \quad \frac{\partial T}{\partial r_1} = -\frac{1}{J} \frac{\partial r_2}{\partial X}, \quad \frac{\partial T}{\partial r_2} = \frac{1}{J} \frac{\partial r_1}{\partial T}$$

where

$$J = \left\{ \frac{\partial r_1}{\partial X} \frac{\partial r_2}{\partial T} - \frac{\partial r_1}{\partial T} \frac{\partial r_2}{\partial X} \right\} \neq 0$$

is the Jacobian of the transformation $(r_1, r_2) \mapsto (X, T)$.

Integrability of the KdV-Whitham equations: 2. Hodograph solutions

Now, substituting the expressions for $\partial_T r_{1,2}$ and $\partial_X r_{1,2}$ into the Whitham system we obtain a system of two *linear* equations for $X(r_1, r_2)$, $T(r_1, r_2)$:

$$\partial_1 X - V_2(r_1, r_2, r_{30})\partial_1 T = 0\,, \qquad \partial_2 X - V_1(r_1, r_2, r_{30})\partial_2 T = 0\,, \qquad (*)$$

where $\partial_j \equiv \partial/\partial r_j$. Next, we introduce in (*) substitutions

$$W_1(r_1, r_2) = X - V_1 T\,, \qquad W_2(r_1, r_2) = X - V_2 T\,, \qquad (**)$$

to cast it in the form of a symmetric system for $W_{1,2}$:

$$\frac{\partial_1 W_2}{W_1 - W_2} = \frac{\partial_1 V_2}{V_1 - V_2}\,; \qquad \frac{\partial_2 W_1}{W_2 - W_1} = \frac{\partial_2 V_1}{V_2 - V_1}. \qquad (***)$$

Now, any solution of the linear system (***) will generate, via (**), a *local smooth* solution $\{r_1(X, T), r_2(X, T), r_{30}\}$ of the Whitham system. One can see that analogous systems can be obtained for any two pairs of the Riemann invariants, provided the third invariant is constant. Note:

- the requirement $J \neq 0$ is essential
- the hodograph solution (**) has the form similar to the simple wave solution (11). The crucial difference is that the functions $W_{1,2}$ in (**) are not arbitrary.

Integrability of the KdV-Whitham equations: 3. Generalised Hodograph Transform

In 1985 Tsarev showed that even if *all three* Riemann invariants vary, any smooth non-constant solution of the Whitham system (5) can be obtained from the algebraic system

$$X - V_j(r_1, r_2, r_3)T = W_j(r_1, r_2, r_3)\,, \quad i = 1, 2, 3, \tag{12}$$

where the functions W_j are found from the overdetermined system of linear partial differential equations,

$$\frac{\partial_i W_j}{W_i - W_j} = \frac{\partial_i V_j}{V_i - V_j}\,, \quad i, j = 1, 2, 3\,, \quad i \neq j\,. \tag{13}$$

Now, the condition of integrability of the nonlinear diagonal system (5) is reduced to the condition of consistency for the overdetermined linear system (13), which has the form (see Lecture 16)

$$\partial_i\left(\frac{\partial_j V_k}{V_j - V_k}\right) = \partial_j\left(\frac{\partial_i V_k}{V_i - V_k}\right), \quad i \neq j,\ i \neq k,\ j \neq k\,. \tag{14}$$

It is not difficult to show that the characteristic velocities (6) satisfy (14) so the KdV-Whitham system (5) is integrable via the **generalised hodograph transform** (15), (13).

Reduction to the Euler-Poisson-Darboux equation

Consider the linear Tsarev system

$$\frac{\partial_i W_j}{W_i - W_j} = \frac{\partial_i V_j}{V_i - V_j}\,, \quad i,j = 1,\ldots,3, \quad i \neq j, \tag{15}$$

where for V_i we have the potential representation

$$V_i = (1 - \frac{L}{\partial_i L}\partial_i)c\,, \quad i = 1,2,3 \tag{16}$$

It can be readily seen that unknown functions $W_i(\mathbf{r})$ admit analogous representation

$$W_i = (1 - \frac{L}{\partial_i L}\partial_i)g\,, \quad i = 1,2,3. \tag{17}$$

where $g(r_1, r_2, r_3)$ is a new unknown (scalar) function. Substituting (16), (17) into (15) we obtain an overdetermined system of
Euler-Poisson-Darboux (EPD) equations

$$2(r_i - r_j)\frac{\partial^2 g}{\partial r_i \partial r_j} = \frac{\partial g}{\partial r_i} - \frac{\partial g}{\partial r_j}\,, \quad i,j = 1,\ldots,3, \quad i \neq j\,. \tag{18}$$

It is not difficult to show that system (18) is consistent.

General solution

Combining the above results, the general local solution of the KdV-Whitham system is given by the implicit formula

$$X - V_i(\mathbf{r})T = g - \frac{L}{\partial_i L}\partial_i g \ , \tag{19}$$

where $g(r_1, r_2, r_3)$ satisfies the EPD system.
One can see by direct verification that the function

$$G(\lambda, r_1, r_2, r_3) = \frac{\phi(\lambda)}{\sqrt{(\lambda - r_1)(\lambda - r_2)(\lambda - r_3)}}, \tag{20}$$

where $\phi(\lambda)$ is an arbitrary function, satisfies the EPD system (18) identically (i.e. for any λ). Thus $g(\lambda, r_1, r_2, r_3)$ is the generating function for the solutions to the EPD system and, therefore, via (19), to the (local) solutions of the KdV-Whitham system.
General solution to the EPD system can be represented in the form (Eisenhart 1918)

$$g(r_1, r_2, r_3) = \sum_{i=1}^{3} \int_0^{r_i} \frac{\phi_i(\lambda) d\lambda}{\sqrt{(\lambda - r_3)(\lambda - r_2)(\lambda - r_1)}} \, , \tag{21}$$

where $\phi_i(\lambda)$, $i = 1, 2, 3$ are arbitrary functions.

Similarity solutions

Choosing $\phi(\lambda) = 4\lambda^{3/2}$ and expanding the generating function $G(\lambda, r_1, r_2, r_3)$ for $\lambda \gg 1$ we obtain

$$G = 1 + \frac{g_1}{\lambda} + \frac{g_2}{\lambda^2} + \dots \tag{22}$$

where $g_1 = 2s_1\,, \quad g_2 = -2s_2 + \frac{3}{2}s_1^2, \quad g_3 = \dots, \; s_1 = r_1 + r_2 + r_3,$ $s_2 = r_1r_2 + r_1r_3 + r_2r_3,\; s_3 = \dots$ are the symmetric polynomials. Each $g_\alpha(\mathbf{r})$, $\alpha = 1, 2, 3\dots$ is a symmetric homogeneous function of r_1, r_2, r_3 with the homogeneity index α, i.e.

$$g_\alpha(Cr_1, Cr_2, Cr_3) = C^\alpha g_j(r_1, r_2, r_3)$$

NB: $W_j^\alpha = g_\alpha - \frac{L}{\partial_j L}\partial_j g_\alpha$ are the characteristic speeds of the averaged KdV hierarchy.

Theorem: *Each homogeneous solution*

$$g = g_\alpha(r_1, r_2, r_3)\,, \quad \alpha = 3, 5, 7, \dots$$

of the Euler-Poisson-Darboux system

$$2(r_i - r_j)\partial^2_{ij} g = \partial_i g - \partial_j g\,, \qquad i, j = 1, 2, 3\,, \quad i \neq j\,.$$

gives rise, via

$$X - V_i(\mathbf{r})T = W_j^\alpha(\mathbf{r})\,,$$

to the similarity solution

$$r_j = \frac{1}{T^\gamma} R_j(\frac{X}{T^{\gamma+1}})\,, \quad \gamma = \frac{1}{\alpha - 1}\,, \quad j = 1, 2, 3.$$

of the KdV-Whitham system.

Integrability of the KdV-Whitham equations: 4. Connection with the spectral problem

There is a deep connection between the Whitham equations (5) and the spectral problem associated with the original KdV equation. This connection was discovered and thoroughly studied in the paper by Flaschka, Forest and McLaughlin (FFM)(1979).
Let us consider the cnoidal wave solution

$$u_{cn}(x,t) = b_2 + (b_3 - b_2)\,\mathrm{cn}^2\left(\sqrt{2(b_3 - b_1)}(x - ct - x_0); m\right) \tag{23}$$

taken with negative sign, as a potential in the linear Schrödinger equation in the associated spectral problem,

$$\left(-\partial_{xx}^2 - u_{cn}\right)\phi = \lambda\phi$$

It is well known that the spectrum of the periodic Schrödinger operator generally consists of an *infinite number* of disjoint intervals called bands. Correspondingly, the ‘forbidden’ zones between the bands are called gaps.

Integrability of the KdV-Whitham equations: 4. Connection with the spectral problem

The unique property of the cnoidal wave solution (23) is that its spectrum contains only one finite band. To be exact, the spectral set for the potential $-u_{cn}(x)$ is $S = \{\lambda : \lambda \in [\lambda_1, \lambda_2] \bigcup [\lambda_3, \infty)\}$.

This fact had been known long before the creation of the soliton theory in connection with the so-called *Lamè* potentials. The soliton studies showed that the cnoidal wave solutions of the KdV equation represent the simplest case of potentials belonging to a general class of the so-called finite-gap potentials discovered by Novikov (1974) and Lax (1975). These finite-gap potentials can be expressed in terms of Riemann theta-functions and give rise to *multiphase* almost periodic solutions of the KdV equation.

Integrability of the KdV-Whitham equations: 4. Connection with the spectral problem.

It is clear that the cnoidal wave solution can be parametrised by three spectral parameters $\lambda_1, \lambda_2, \lambda_3$ instead of the roots of the polynomial b_1, b_2 and b_3 (see equation (4)). The remarkable general fact established by FFM is that the Riemann invariants of the Whitham system (5) coincide with the endpoints of the spectral bands of finite-gap potential. In particular, for the single-gap solution (the cnoidal wave), $r_1 = \lambda_1$, $r_2 = \lambda_2$, $r_3 = \lambda_3$. Thus, the spectral problem provides one with the most convenient set of modulation parameters (the Riemann invariants) and, therefore, the Whitham equations (5) describe slow evolution of the *spectrum* of periodic (generally – quasiperiodic) KdV solutions.

The general theory of finite-gap integration and the spectral theory of the Whitham equations are quite technical. However, in the case of the single-phase waves, which is the most important from the viewpoint of the fluid dynamics applications, a simple universal method has been developed by Kamchatnov (2000), enabling one to construct periodic solutions and the Whitham equations directly in Riemann invariants for a broad class of integrable nonlinear dispersive wave equations.

Integrability of the KdV-Whitham equations: 5. Commuting hydrodynamic flows (symmetries)

We have shown that integrability of the KdV equation is inherited by the averaged (Whitham) system. Integrability of the KdV equation is based on the Lax pair $\mathbf{L}, \mathbf{A}$ and implies existence of an infinite number of commuting flows –"higher" KdV equations generated by the operators commuting with $\mathbf{L}$. One can expect that the KdV-Whitham system will also have an infinite number of commuting flows (symmetries).
Let us consider a hydrodynamic type system

$$\frac{\partial r_j}{\partial \tau} + W_j(r_1, r_2, r_3)\frac{\partial r_j}{\partial X} = 0\,, \quad j = 1, 2, 3\,, \tag{24}$$

where τ is a new "time" variable and $W_j(\mathbf{r})$ are some characteristic velocities. Now we require that system (24) commutes with the KdV-Whitham system (5), i.e. we impose the condition

$$\partial_\tau \partial_T r_j = \partial_T \partial_\tau r_j\,. \tag{25}$$

Then (25) leads to a number of constraints on functions $W_i(\mathbf{r})$, which turn out to coincide with the generalised hodograph equations (13). So the generalised hodograph equations (13) are also equations for the hydrodynamic symmetries of the KdV-Whitham system.

Appendix. Asymptotic expansions of the complete elliptic integrals

Asymptotic expansions of the complete elliptic integrals (see, for instance, Abramowitz & Stegun 1965):

$$m \ll 1 :$$

$$K(m) = \frac{\pi}{2}\left(1 + \frac{m}{4} + \frac{9}{64}m^2 + \ldots\right), \qquad E(m) = \frac{\pi}{2}\left(1 - \frac{m}{4} - \frac{3}{64}m^2 + \ldots\right);$$

$$(1-m) \ll 1 :$$

$$K(m) \approx \frac{1}{2}\ln\frac{16}{1-m}, \qquad E(m) \approx 1 + \frac{1}{4}(1-m)\left(\ln\frac{16}{1-m} - 1\right).$$

Dispersive shock waves. Introduction

Although often the mathematical modelling of dispersive shock waves (DSWs) in nature requires taking into account weak dissipation (Benjamin & Lighthill 1954; Sagdeev 1956), which stabilises the expansion of the oscillatory zone, it is now customary to use the terms 'undular bore' and 'dispersive shock wave' for any wave-like transition between two different smooth flows in solutions of nonlinear dispersive systems. The significance of the study of purely conservative, unsteady DSWs is twofold:

- Conservative DSWs represent an ubiquitous wave phenomenon realised in its "pure" form in a number of fluid dynamic applications, nonlinear optics and Bose-Einstein condensates.
- Generation of a DSW is a universal mechanism of soliton generation out of non-oscillatory initial or boundary conditions in conservative dispersive systems.

In fact, the studies of purely conservative dispersive shock waves have stimulated a number of important discoveries in modern nonlinear wave theory and in mathematical physics in general.

DSWs/undular bores in Nature and laboratory experiments

- **Ocean and atmosphere.** There is the ever increasing number of observations revealing the ubiquity of internal undular bores in the atmosphere, where they may influence local weather events (e.g. thunderstorm initiation) as well as in the coastal oceans where they have a role in mixing processes and in the energy budgets of tidal flows. Also, there are observations of tsunamis propagating as undular bores.
- **Bose-Einstein condensates.** Dispersive shock waves have been observed in recent groundbreaking experiments on Bose-Einstein condensates (BECs): a unique supercold state of matter demonstrating quantum properties on a macroscopic scale. The generation of DSWs in supersonic BEC flow past obstacles is one of the fundamental mechanisms of the brakdown of superfluidity.
- **Laser optics** The propagation of powerful laser beam through an optical crystal/fiber is often accompanied by the wave breaking phenomenon and formation of an optical shock, which represents a dispersive shock wave. There is an important (though harmful) role of optical shocks in the propagation of light pulses through digital communication fiber optic systems.

Undular bores in the atmosphere and on shallow water

- Left: Atmospheric undular bore: Morning Glory in the Gulf of Carpentaria (Australia); Right: undular bore on river Severn (England)

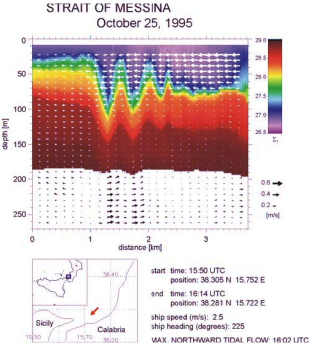
,

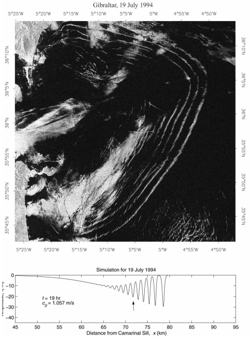

- Undular bore of the 2004 Indian Ocean tsunami reaching the island of Koh Jum, Thailand

Internal udular bores in the ocean

Oceanic undular bores are frequently observed in the coastal oceans and seas

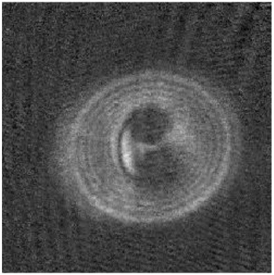
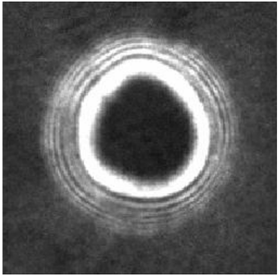
Left: Right: internal undular bore in Straight of Messina; Right; a satellite image of the internal undular bore in the Straight of Gibraltar.





Dispersive shocks in Bose - Einstein condensates

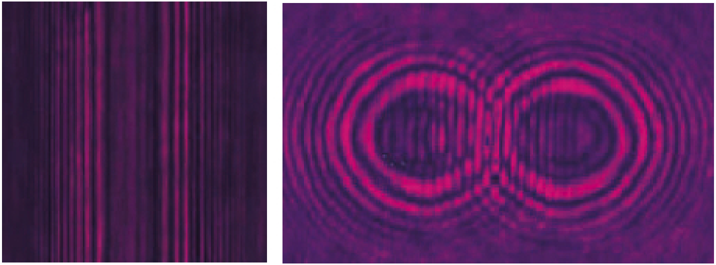
Cornell's JILA group experiments
http://jilawww.colorado.edu/bec/papers.html

Blast waves in a BEC: radially expanding DSWs

M.A. Hoefer, M.J. Ablowitz, I. Coddington, E.A. Cornell, P. Engels, and V. Schweikhard (2006)

Optical dispersive shocks



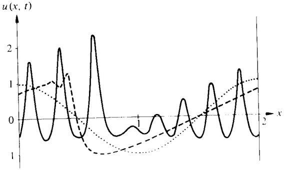
Wan, Jia, and Fleischer (2007)

Zabusky & Kruskal (1965) numerical experiment revisited

The Korteweg - de Vries (KdV) equation

$$u_t + uu_x + \epsilon^2 u_{xxx} = 0\,, \qquad \epsilon = 0.022 \tag{1}$$

with initial conditions $u(x,0) = \cos \pi x$, $0 \le x \le 2$ and u, u_x, u_{xx} periodic on $[0,2]$ for all t.



- generation of solitons occurs via the wave breaking and formation of the dispersive shock wave!

This shows the fundamental role of dispersive shock waves in the soliton theory and in the theory of nonlinear waves in general.

19.2. Formation of a dispersive shock wave

We consider the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \,. \tag{2}$$

with large-scale initial conditions

$$u(x,0) = u_0(X), \quad X = \epsilon x, \quad \epsilon \ll 1 \,. \tag{3}$$

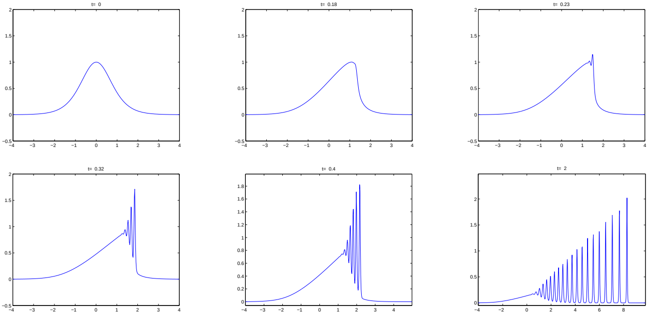
To be definite, we consider $u_0(X)$ in the form of a smooth step with single inflection point, $u_0(-\infty) = u^- \,, \quad u_0(+\infty) = u^+ \,, \quad u^- > u^+$
During the initial stage of the evolution, $|u_x| \sim \epsilon$, $|u_{xxx}| \sim \epsilon^3$, hence $|u_{xxx}| \ll |uu_x|$ and one can neglect the dispersive term u_{xxx} in (2). The evolution at this stage is approximately described by the dispersionless limit of the KdV equation (the Hopf equation):

$$u_t + 6uu_x = 0 \,, \tag{4}$$

The evolution (4), (3) leads to a gradient catastrophe, which occurs at the inflection point at a certain wave breaking time:
$t \to t_b, x \to x_b : \quad u_x \to -\infty, \;\; u_{xx} \to 0.$

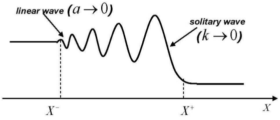
19.2 Formation of a dispersive shock wave

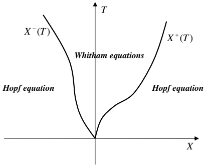
As was described in the Introduction, the dispersive resolution of the wave breaking singularity occurs through the generation of nonlinear wavetrains – dispersive shock waves (DSWs). Indeed, for $t > t_b$, one can no longer neglect the KdV dispersive term u_{xxx} in the vicinity of the wave breaking point.

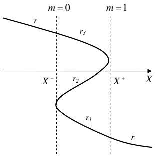


19.3 Gurevich-Pitaevskii (1973) problem

Key assumption: the DSW can be asymptotically described by a slowly modulated single-phase travelling wave solution. The modulation provides a gradual change of the waveform from the linear wave ($m = 0$) at the trailing edge $X = X^-(T)$ to the solitary wave ($m = 1$) at the leading edge $X = X^+(T)$. At $X^\pm$ the DSW solution must match with the solution of the Hopf equation $r(X, T)$, which is valid outside the DSW region.







Thus, the problem is reduced to the integration of the Whitham equations for $r_1 \le r_2 \le r_3$:

$$\partial_T r_j + V_j(\mathbf{r})\partial_X r_j = 0, \qquad j = 1, 2, 3,$$

with the free-boundary matching conditions (recall that $m = (r_2 - r_1)/(r_3 - r_1)$):

$$\begin{aligned} &\text{at} \quad X = X^-(T): \qquad r_2 = r_1\,, \quad r_3 = r\,, \\ &\text{at} \quad X = X^+(T): \qquad r_2 = r_3\,, \quad r_1 = r\,, \end{aligned} \tag{5}$$

where $r(X, T)$ is the solution of the Hopf equation: $r = u_0(\xi)$, $X = \xi + u_0(\xi)T$.

19.4 Boundaries of the DSW

It follows from conditions (5) and the limiting properties of the Whitham velocities described in Lecture 18, that the boundaries $X^{\pm}(T)$ are the multiple characteristics of the Whitham system for $m = 0$ ($X = X^{-}(T)$) and $m = 1$ ($X = X^{+}(T)$). Indeed

- when $m = 0$: $V_2 = V_1 = 12r_1 - 6r_3$
- when $m = 1$: $V_2 = V_3 = 2r_1 + 4r_3$

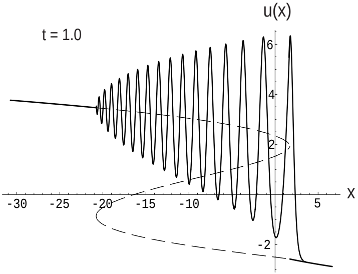
Thus the boundaries of the dispersive shock wave are are found by integrating the ordinary differential equations

$$dX^{-}/dT = (12r_1 - 6r_3)|_{X=X^{-}}\,, \qquad dX^{+}/dT = (2r_1 + 4r_3)|_{X=X^{+}} \quad (6)$$

defined *on the solution* $\{r_j(X,T)\}$ of the Gurevich-Pitaevskii problem.

19.5. Gurevich-Pitaevskii problem: concluding remarks

The output of the Gurevich-Pitaevskii problem is the modulation solution $r_1(X,T),\ r_2(X,T),\ r_3(X,T)$ depending on the initial function $u_0(X)$ for the KdV equation. One substitutes this modulation solution into the cnoidal wave solution
$u_{cn}(x,t) = r_1 + r_2 - r_3 + 2(r_2 - r_1)\,\mathrm{cn}^2[2(r_3 - r_1)^{1/2}(x - ct)|m]$ to obtain the oscillating dispersive shock wave profile:



Dashed line: the modulation solution $r_{1,2,3}$ which does not coincide with the formal three-valued solution of the Hopf equation after the wave breaking time.

19.6 Decay of an initial discontinuity

As an important example, where a simple analytical representation of a dispersive shock wave can be obtained, we consider the decay of an initial discontinuity problem. We take the initial data in the form of a sharp step,

$$t = 0: \qquad u = A > 0 \quad \text{if} \quad x < 0, \qquad u = 0 \quad \text{for} \quad x > 0, \tag{7}$$

which implies immediate breaking and formation of a dispersive shock wave. We now make use of the Gurevich-Pitaevskii problem formulation. First we observe that, both initial data (7) and the modulation equations

$$\frac{\partial r_j}{\partial T} + V_j(\mathbf{r})\frac{\partial r_j}{\partial X} = 0 \tag{8}$$

– are invariant with respect to the scaling $x \to Cx$, $t \to Ct$, where $C = \text{constant}$. Thus the modulation variables must be functions of a self-similar variable $s = X/T$ alone, i.e. $r_j = r_j(s)$. As a result, the Whitham system (8) reduces to the system of ODEs:

$$\frac{dr_j}{ds}(V_j - s) = 0, \qquad j = 1, 2, 3. \tag{9}$$

19.6 Decay of an initial discontinuity

The Gurevich-Pitaevskii matching conditions (5) then assume the form

$$\begin{array}{ll} s = s^- : & r_2 = r_1\,, \quad r_3 = A\,, \\ s = s^+ : & r_2 = r_3\,, \quad r_1 = 0\,, \end{array} \tag{10}$$

where $s^\pm$ are the (unknown) speeds of the undular bore edges, $X^\pm = s^\pm T$.

It can be readily seen that the boundary-value problem (9), (10) has the solution in the form of a centred simple wave in which all but one Riemann invariants are constant:

$$r_1 = 0\,, \quad r_3 = A\,, \quad V_2(0, r_2, A) = s\,, \tag{11}$$

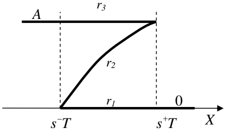
Using an explicit expression for $V_2(r_1, r_2, r_3)$ (see Lecture 17) one represents the last equation in (11) in the form

$$2A\left\{1 + m - \frac{2(1-m)m}{E(m)/K(m) - (1-m)}\right\} = \frac{X}{T}\,, \tag{12}$$

where $m = (r_2 - r_1)/(r_3 - r_1) = r_2/A$.

19.6 Decay of an initial discontinuity

The obtained solution for the Riemann invariants is schematically shown below



The solution (12) represents a characteristic fan in the X, T plane. So it never breaks up for $T > 0$ and, therefore, is global. The speeds s^- and s^+ of the trailing and leading edges of the dispersive shock wave are found by assuming $m = 0$ and $m = 1$ respectively in the solution $s(m)$ (12):

$$s^- = s(0) = -6A\,, \quad s^+ = s(1) = 4A\,. \tag{13}$$

Thus, the dispersive shock is confined to an expanding zone $-6AT \leq X \leq 4AT$.

19.7. Behaviour of physical parameters in DSW

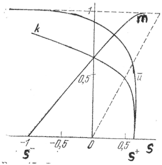
Let us extract some important physical parameters from the obtained modulation solution for the decay of an initial discontinuity of the magnitude A. Apart from the already determined locations of the dispersive shock boundaries $X^+ = 4AT$ and $X^- = -6AT$ we are interested in the value of the amplitude a^+ of the leading soliton as well as in the behaviour of the mean $\overline{u}$ and the wavelength L (or wavenumber k) within the oscillation zone.

- The amplitude of the cnoidal wave is $a = 2(r_2 - r_1)$ (see Lecture 17). In the soliton limit, $r_2 = r_3$, so the soliton amplitude $a_s = 2(r_3 - r_1)$. Now, for the obtained modulation solution, $r_1 = 0$, $r_3 = A$ so the amplitude of the leading soliton in the dispersive shock wave is $a^+ = 2A$, i.e. is twice the value of the initial step
- The expressions for the mean $\overline{u}$ and the wavenumber k in terms of the Riemann invariants r_j (see Lecture 17):

$$\overline{u} = r_2 + r_1 - r_3 + 2(r_3 - r_1)\frac{E(m)}{K(m)}, \quad k = \frac{2K(m)}{(r_3 - r_1)^{1/2}}$$

which, together with the modulation solution (11) gives dependencies $\overline{u}(s)$, $k(s)$ within the dispersive shock wave.

19.7. Behaviour of physical parameters in DSW



Using asymptotic expansions of the complete elliptic integrals $K(m)$ and $E(m)$ for $m \ll 1$ and $1 - m \ll 1$ (see Appendix to Lecture 18) we obtain the asymptotic behaviour of the modulus m near the boundaries of the dispersive shock. Near the trailing edge $s = s^-$ we have

$$m \simeq (s - s^-)/9 \ll 1\,,$$

which also describes the amplitude variations since $m = a/A$ for the solution under study. Near the leading edge $s = s^+$, we get:

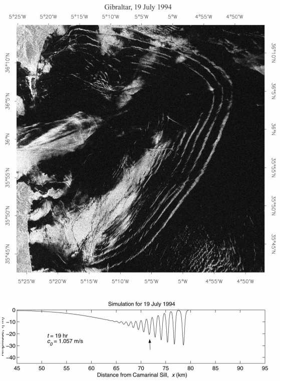
$$1 - m \simeq (s^+ - s)/2\ln(1/(s^+ - s)) \ll 1\,,$$

which, in particular, yields the asymptotic behaviour

$$\bar{u} \simeq 12k\,, \qquad k \simeq 2\pi/\ln(1/(s^+ - s))$$

for the mean value and the wavenumber respectively.

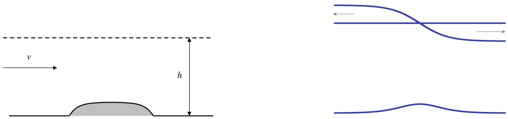
19.8 Applications 1. Internal undular bores in the ocean



Modeled by the KdV equation (J.P. Apel, A New Analytical Model for Internal Solitons in the Ocean, Journ Phys Ocean. **33** (2003) 2247 - 2269)

Application 2:
Resonant shallow-water flow past topography

Shallow water flow over a ridge:



Modeled by the **forced KdV equation**

$$-A_t - \Delta A_x + 6AA_x + A_{xxx} + G_x(x) = 0\,.$$

Here A is the wave elevation, $\Delta = v/\sqrt{gh} - 1$ measures the degree of criticality, and the function $G(x)$ describes the topography variations.

The key is the existence, in the transcritical regime, $\Delta_- < \Delta < \Delta_+$, of a localized steady "hydraulic" transition in the forcing region. This is characterized by an upstream constant state $A_-(>0)$ and a downstream constant state $A_+(<0)$. These are resolved back to the equilibrium state $A = 0$ by undular bores, propagating upstream and downstream respectively (*Grimshaw & Smyth 1986*).

Resonant shallow-water flow past topography

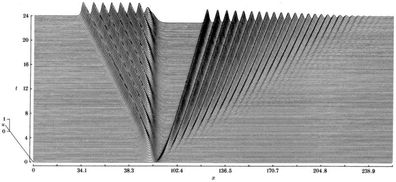
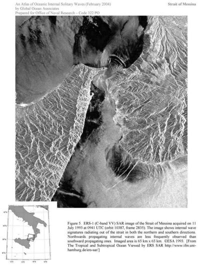


FIGURE 1. Numerical solution of (1.1) for $\Delta = 0$ and a broad obstacle (centred at $x = 85$).


An Atlas of Oceanic Internal Solitary Waves (February 2004)
by Global Ocean Associates
Prepared for Office of Naval Research – Code 322 PO

Strait of Messina

Figure 5. ERS-1 (C-band VV) SAR image of the Strait of Messina acquired on 11 July 1993 at 0941 UTC (orbit 10387, frame 2835). The image shows internal wave signatures radiating out of the strait in both the northern and southern directions. Northwards propagating internal waves are less frequently observed than southward propagating ones. Imaged area is 65 km x 65 km. ©ESA 1993. [From The Tropical and Subtropical Ocean Viewed by ERS SAR http://www.ifm.uni-hamburg.de/ers-sar/]

Left: Numerical solution of the resonant forced KdV (Grimshaw and Smyth (1986);
Right: Resonant generation of internal undular bores in the Straight of Messina.

Appendix: Connection of Gurevich-Pitaevskii problem with the Inverse Scattering Transform

To take advantage of the inverse scattering transform formalism for the KdV equation one needs to consider the initial conditions $u(x,0) = u_0(x)$ in the form of a large-scale *decaying* function: $u_0(x) \to 0$ as $|x| \to \infty$. Lax and Levermore (1983) considered the semi-classical zero-dispersion limit of the KdV equation with positive decaying initial conditions

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0\,, \qquad u(x,0) = u_0(x) > 0\,, \qquad \epsilon \to 0\,.$$

and proved that, after the breaking time $t = t_b$, the weak limits as $\epsilon \to 0$ of the solution $u(x,t,\epsilon)$, i.e. the moments $\overline{u}(x,t), \overline{u^2}(x,t), \overline{u^3}(x,t)$ – in a certain x,t- region – are governed by the Whitham equations. The Gurevich-Pitaevskii matching conditions for the Riemann invariants arise as a natural by-product of the Lax-Levermore construction. Similar result for negative, solitonless, initial conditions was obtained by Venakides (1985).

Thus, the Lax-Levermore-Venakides construction can be considered as a formal justification of the validity of the direct Gurevich-Pitaevskii formulation of the problem in terms of the Whitham equations.

20.1 Hydrodynamic form of the defocusing NLS equation

We introduce in (1) the change of variables (the so-called Madelung transformation) $\psi \mapsto \{\rho, u\}$:

$$\psi = \sqrt{\rho}\exp(i\theta), \qquad u = \theta_x$$

Separating real and imaginary parts (it is technically convenient to first introduce the "polar" substitution $\psi = Ae^{i\theta}$ and only then to pass to the hydrodynamic valuables $\rho = A^2$ and $u = \theta_x$), we arrive at the dispersive-hydrodynamic form of the defocusing NLS equation

$$\begin{gathered} \rho_t + (\rho u)_x = 0\,, \\ u_t + uu_x + \rho_x + \left(\frac{(\rho_x)^2}{8\rho^2} - \frac{\rho_{xx}}{4\rho}\right)_x = 0 \end{gathered} \tag{2}$$

Here $\rho > 0$ can be interpreted as the "fluid" density and u as velocity. The "blue" terms describe the dispersionless limit of the NLS equation (the classical shallow-water system) and the dispersive terms are shown in red.

Remark: In Bose-Einsten condensate, ρ and u are actual condensate density and velocity.

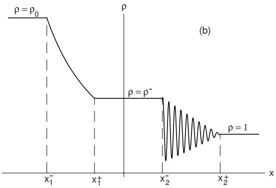
Some major differences between the KdV and NLS decay of a step (Riemann) problems

- The NLS equation is a two-wave equation while the KdV equation describes uni-directional wave propagation
- The NLS equation describes nonlinear waves with positive dispersion, i.e. short waves propagate faster than long waves and the solitons have negative polarity.

We consider initial conditions for the NLS equation (2) in the form

$$\rho = \begin{cases} \rho_0 & \text{for} \quad x < 0, \\ 1 & \text{for} \quad x > 0; \end{cases} \qquad u = \begin{cases} u_0 & \text{for} \quad x < 0, \\ 0 & \text{for} \quad x > 0, \end{cases} \tag{3}$$

As a result, the general wave pattern of the decay of a step involves two waves (either of which could be a dispersive shock wave or a rarefaction wave depending on the parameters of the initial step). A typical configuration looks like that:



20.2 Defocusing NLS equation: periodic solutions

The periodic travelling wave solution of the NLS equation is sought in the form $\rho = \rho(\theta)$, $u = u(\theta)$, where $\theta = x - ct$, the phase velocity c being a constant. After some standard algebra (left as an exercise) one arrives at the ODE for ρ:

$$\frac{1}{4}(\rho')^2 = (\rho - \rho_1)(\rho - \rho_2)(\rho - \rho_3), \qquad \rho_3 \geq \rho_2 \geq \rho_1 \geq 0\,, \tag{4}$$

which is integrated in terms of Jacobi elliptic functions cn to give

$$\rho = \rho_2 - (\rho_3 - \rho_2)\mathrm{cn}^2(\sqrt{\rho_3 - \rho_1}(x - ct - x_0)|m)\,, \tag{5}$$

where the modulus $m = (\rho_2 - \rho_1)/(\rho_3 - \rho_1)$.
The wavelength is defined by

$$L = \int_{\rho_1}^{\rho_2} \frac{d\rho}{\sqrt{(\rho - \rho_1)(\rho_2 - \rho)(\rho_3 - \rho)}} = \frac{2K(m)}{\sqrt{\rho_3 - \rho_1}}\,, \tag{6}$$

$K(m)$ being he complete elliptic integral of the first kind.
The "velocity" u in the periodic wave is connected with the "density" ρ by the relation

$$u = c + \frac{A}{\rho}\,, \qquad A = \rho_1\rho_2\rho_3\,. \tag{7}$$

20.3 Periodic solutions: NLS vs KdV

One can observe a number of similarities and differences between the periodic travelling wave solutions for the KdV and the NLS equations.

- Both periodic solutions are characterised by cubic "potential curve" and are described by very similar expressions in terms of the roots of the potential curve
- The periodic solutions of the KdV equations are characterised by three constants while the NLS periodic solution is parametrised by four constants: ρ_1, ρ_2, ρ_3 and c.
- Unlike in the KdV periodic wave solution, the phase velocity c of the NLS travelling wave is an independent parameter and is not related to the roots of the "potential curve" ρ_1, ρ_2, ρ_3
- There is an important restriction on the possible values of the NLS parameters ρ_1, ρ_2, ρ_3: the density ρ should be positive. This restriction does not appear in the KdV theory.

Two limits: linear waves of modulations and dark solitons

Similar to the KdV theory, there are two special limits of the periodic travelling wave solutions of the defocusing NLS equation

- $m \to 0$: **linear waves of modulations** For the waves of infinitesimally small amplitude $a = \rho_2 - \rho_1 \ll 1$ we obtain

$$\rho = \rho_0 + a^2 \sin^2(kx - \omega_0(k)t)\,, \tag{8}$$

  where $\rho_0 = \rho_1$ is the background density, and the linear dispersion relation is given by

$$\omega_0(k) = ku_0 \pm k\sqrt{\rho_0 + k^2/4} \tag{9}$$

- $m \to 1$: **dark solitons**

$$\rho = \rho_0 - a_s \mathrm{sech}^2(\sqrt{a}(x - c_s t))\,, \tag{10}$$

  where $\rho_0 = \rho_3$ is the background density, $a_s = \rho_0 - \rho_1$ is the soliton amplitude and $c_s = u_0 \pm \sqrt{\rho_1}$ is the dark soliton velocity.

Note: important restriction, which does not appear in the KdV theory: $\rho_1 > 0$.

20.4 Modulation system for the defocusing NLS equation

Straightforward route: (i) introduce slow variables $X = \varepsilon x$ and $T = \varepsilon t$, $\varepsilon \ll 1$; (ii) average three conservation laws of the NLS equation (there is an infinite number of them as the NLS equation is a completely integrable system) over the periodic solution; and (iii) close the system with the "wave conservation law" $k_T + (kc)_X = 0$.

More advanced (and practical!) method: to use the underlying spectral structure and introduce the endpoints of spectral bands as the modulation variables .

The periodic travelling wave solution (5) spectrum consists of **two bands** $[\lambda_1, \lambda_2]$ and $[\lambda_3, \lambda_4]$ separated by a **gap** (λ_2, λ_3) (see Lecture 9 for the stationary self-adjoint spectral problem associated with the defocusing NLS equation):

Two relevant questions:

- How are these spectral endpoints $\lambda_1 < \lambda_2 < \lambda_3, < \lambda_4$ related to the roots ρ_1, ρ_2, ρ_3 of the polynomial in the ODE (4) for the travelling wave solution?
- What is the special role of λ_j's in the modulation equations?

Modulation system for the defocusing NLS equation

Pavlov (1987) and (Forest and Lee 1987), using finite-gap integration method and generalisation of the Flaschka-Forest-McLaughlin theory, showed that

- $$\begin{gathered}\rho_1 = \frac{1}{4}(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)^2, \quad \rho_2 = \frac{1}{4}(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)^2, \\ \rho_3 = \frac{1}{4}(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)^2, \quad c = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\end{gathered} \tag{11}$$

- λ_j are the Riemann invariants of the NLS-Whitham system:

$$\frac{\partial \lambda_j}{\partial T} + V_j(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\frac{\partial \lambda_j}{\partial X} = 0\,, \quad j = 1, 2, 3, 4. \tag{12}$$

and derived explicit expressions for the characteristic velocities in terms of the complete elliptic integrals. Kamchatnov (1990) developed a more straightforward method of deriving (11), (12). In particular, he showed that relationships (11) follow from the fact that the "potential curve" $R(\rho) = (\rho - \rho_1)(\rho - \rho_2)(\rho - \rho_3)$ of the travelling wave solution is a cubic resolvent of the spectral curve $P(\lambda) = \prod_{j=1}^{4}(\lambda - \lambda_j)$

Modulation system for the defocusing NLS equation

Now, that we know the relations between $\rho_1, \rho_2, \rho_3, c$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ we easily find the representation of the travelling wave solution in terms of the Riemann invariants λ_j.
In particular (and most importantly) we have for the wavelength

$$L = \frac{1}{2} \oint \frac{d\lambda}{\sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda_4 - \lambda)}} \tag{13}$$

Now, considering the wave conservation law $k_T + (kc)_X = 0$, where $k = 2\pi/L$, $c = \sum_{i=1}^{4} \lambda_i$ as a consequence of the Whitham system in the Riemann form $\partial_T \lambda_j + V_j \partial_X \lambda_j = 0$ (see analogous calculations for the KdV case in Lecture 18) we obtain the"potential" representation for the NLS-Whitham characteristic velocities (cf. () in Lecture)

$$V_i = \frac{\partial_i (kc)}{\partial_i k} = c - \frac{L}{\partial_i L} \partial_i c \, , \quad i = 1, 2, 3, 4 \, , \tag{14}$$

where $\partial_j \equiv \partial / \partial \lambda_j$.

Modulation system for the defocusing NLS equation

Analysis of the characteristic velocities of the defocusing NLS-Whitham system shows that

- the characteristic velocities are real, i.e. the defocusing NLS-Whitham system is hyperbolic. Hence:
  1. modulational stability.
  2. classical theory of characteristics for the hydrodynamic type systems is applicable;

- $\frac{\partial V_i}{\partial \lambda_i} > 0$, therefore the defocusing NLS-Whitham system is genuinely nonlinear. Thus one can expect the wave breaking effects and complications with the existence of global solution.

The properties above are similar to the properties of the characteristic velocities of the KdV-Whitham system.

20.5. NLS: Dispersive shock waves.

One can now extend the Gurevich-Pitaevskii problem to the case of the defocusing NLS equation (Gurevich and Krylov 1987, El et al 1995). The ingredients:

- travelling wave solution $\rho = \rho(x - ct;\ \lambda_1, \lambda_2, \lambda_3, \lambda_4)$
- the NLS-Whitham equations $\partial_T \lambda_j + V_j(\lambda)\partial_X \lambda_j = 0, \quad j = 1, \dots, 4$
- the matching conditions for the Riemann invariants λ_j at the (unknown) boundaries of the dispersive shock $X^{\pm}(T)$

Remark: The “external” equations in the matching problem are the shallow-water equations in the Riemann invariant form (see Lecture 16)

$$\frac{\partial \lambda_+}{\partial T} + (\frac{3}{2}\lambda_+ + \frac{1}{2}\lambda_-)\frac{\partial \lambda_+}{\partial X} = 0\,, \qquad \frac{\partial \lambda_-}{\partial T} + (\frac{3}{2}\lambda_- + \frac{1}{2}\lambda_+)\frac{\partial \lambda_-}{\partial X} = 0$$

where

$$\lambda_{\pm} = \frac{1}{2}u \pm \sqrt{\rho}\,,$$

NLS: Dispersive shock waves

For the decay of a step problem, we introduce the similarity variable $s = X/T$ (see the analogous consideration for the KdV case) and transform the NLS-Whitham system into the system of ODEs:

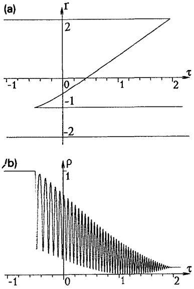
$$(V_j - s)\frac{d\lambda_j}{ds} = 0\,, \qquad j = 1, 2, 3, 4.$$

which has two physically meaningful solutions:

- $V_3 = s, \qquad \lambda_{1,2,4} = \text{constant}_{1,2,4}$ for the right-propagating dispersive shock wave
- $V_2 = s, \qquad \lambda_{1,3,4} = \text{constant}_{1,3,4}$ for the left-propagating dispersive shock wave

The constants are determined by the initial step (3) via the matching conditions for Riemann invariants — see details in El et. al., Physica D (1995) - can be downloaded from:
http://www-staff.lboro.ac.uk/ mage2/publications.htm

NLS: Dispersive shock waves



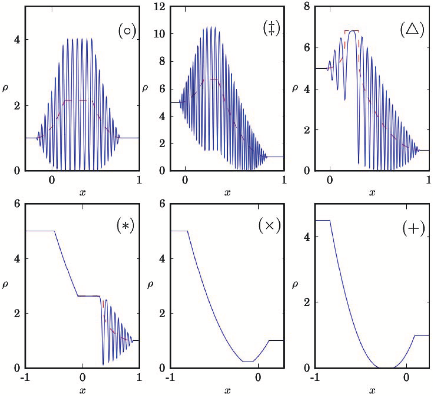
The behaviour of the Riemann invariants, $\lambda_4 > \lambda_3 > \lambda_2 > \lambda_1$ (top) and the corresponding oscillatory profile $\rho(x)$ of the right-propagating dispersive shock wave at some $t > 0$ (bottom).

Riemann problem for the NLS equation: full classification

Complete classification of the Riemann problem solutions for the defocusing NLS equation:

- G.A. El, V.V. Geogjaev, A.V. Gurevich and A.L. Krylov (1995), Physica D **87**, 186-192..

See also: M. Hoefer and M. Ablowitz (2009) Dispersive shock waves. Scholarpedia, 4(11):5562



Lecture 20 References

Taken from the main list of references

[1] Kamchatnov, A.M. 2000 *Nonlinear Periodic Waves and Their Modulations* Singapore, World Scientific.

Additional references

[2] A.V. Gurevich & A.L. Krylov, Dispersionless shock wave in medium with positive dispersion, *Sov. Phys. JETP* **65**, 944 (1987).

[3] El, G.A., Geogjaev, V.V., Gurevich, A.V. & Krylov, A.L. 1995 Decay of an initial discontinuity in the defocusing NLS hydrodynamics. *Physica D* **87**, 186-192.

[4] M. Hoefer and M. Ablowitz (2009) Dispersive shock waves. Scholarpedia, 4(11):5562

Appendix: Bose-Einstein condensates and Matter waves

- All matter behaves like waves. De Broglie wavelength:

$$\lambda_{dB} = \frac{2\pi\hbar}{mv} \sim T^{-1/2}$$

(extremely small for objects with a reasonable mass at ordinary temperatures).

- The Schrödinger equation for the wavefunction $\Psi(\mathbf{r}, t)$ of a quantum particle in an external field $V(\mathbf{r})$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V(\mathbf{r})\Psi\,.$$

$|\Psi|^2$ is the probability density, $\int |\Psi|^2 d^3r = 1$

Linear theory.

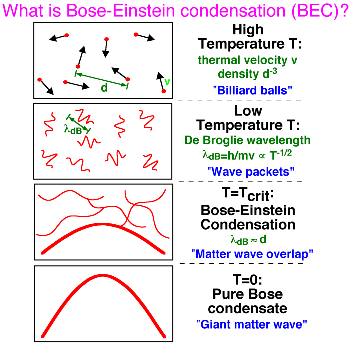
Fermions and Bosons

Many-particle systems: quantum statistics.

Fermions	half-integer spin	only one per state	Examples: electrons, protons, neutrons, quarks, neutrinos
Bosons	integer spin	many can occupy the same state	Examples: photons, ^{4}He atoms, gluons

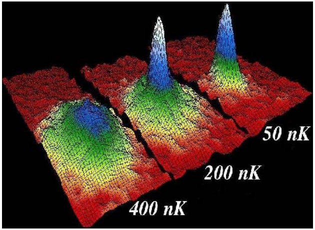
Bose-Einstein condensation: formation of a giant matter wave

A **Bose-Einstein condensate** (BEC) is a state of matter of a dilute gas of weakly interacting bosons confined in an external potential and cooled to temperatures very near to absolute zero (0 K or $-273.15\ ^{o}$C). Under such conditions, a large fraction of the bosons occupy the lowest quantum state of the external potential, at which point quantum effects become apparent on a macroscopic scale.



Bose-Einstein condensate: history

- First predicted by Satyendra Nath Bose and Albert Einstein in 1924-25.
- The first gaseous condensate was produced by Eric Cornell and Carl Wieman in June 1995 at the University of Colorado at Boulder NIST-JILA lab, using a gas of rubidium atoms cooled to 170 nK (1.7×10^{-7} K) and by Wolfgang Ketterle in Sept 1995 at MIT using a gas of sodium atoms.
- Cornell, Wieman, and Ketterle received the 2001 Nobel Prize in Physics for the creation of a BEC.



Bose-Einstein condensate: properties

- 5-th state of matter (gas + liquid + solid + plasma + BEC);
- Coherent monochromatic matter wave (BEC is to matter what laser is to light);
- A superfluid, i.e. no friction of any kind for sufficiently slow motions of objects in it;
- Supports a plethora of linear and nonlinear wave patterns: dispersive Cherenkov radiation, solitons, quantized vortices, dispersive shock waves ...;
- Can be manipulated with unprecedented precision;
- There is a mathematical model describing dynamics of a BEC with high accuracy: the Gross-Pitaevskii equation;

BEC: a unique testing ground to study nonlinear wave effects.

Foreseeable applications: atom lasers, atom chips, high precision measurements ...

Mathematical model: GP equation

At $T = 0$ the dynamics of a one-component BEC in the mean-field approximation is described by the Gross-Pitaevskii (GP) equation for the **macroscopic** single-particle wavefunction Ψ:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V(\mathbf{r})\Psi + g|\Psi|^2\Psi,$$

Here: $V(\mathbf{r})$ is the potential of external forces (magnetic trap or an "obstacle"); g is the coupling constant: $g > 0$ for attractive inter-atomic interactions in a BEC, $g < 0$ for repulsive interactions.

- nonlinear wave equation — **no superposition principle**;
- no "small-amplitude" approximation in the derivation!
- the wave function normalisation: $\int |\Psi|^2 dV = mN$, where N is the number of bosons in the BEC. $\Rightarrow$ $n = |\Psi|^2$ is the BEC density;
- characteristic ("healing") length: $\xi = \hbar/\sqrt{2mn_0|g|}$.

Mathematical model: GP equation

The GP equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V(\mathbf{r})\Psi + g|\Psi|^2\Psi,$$

can be viewed as

- Linear non-stationary Schrödinger equation with an added nonlinear term $g|\Psi|^2\Psi$ due to inter-atomic interactions

or

- Nonlinear Schrödinger (NLS) equation with an added term $V(\mathbf{r})\Psi$ due to BEC interaction with an external potential

The NLS equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + g|\Psi|^2\Psi,$$

- Integrable in 1D (exact periodic and quasiperiodic solutions, solitons, IST);
- Non-integrable in 2D and 3D. Integrable 2D asymptotic reductions : (KP-I ($g < 0$) and KP-II ($g > 0$)).