Linear Algebra
A Lightening Review

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I. VECTOR SPACES AND LINEAR MAPS

A vector space V over a field k is a set with a consistent way to take linear combinations of elements with coefficients in k. We will only be considering $k = \mathbb{R}$ and $k = \mathbb{C}$ so such finite-dimensional V will just be $\mathbb{R}^n$ or $\mathbb{C}^n$. Choosing a basis (a set of linearly independent vectors) $\{e_j\}$, an arbitrary vector $v \in V$ can be written as

$$v = \sum_{j=1}^{n} v_j e_j$$

giving an explicit identification of V with n-tuples v_j of real or complex numbers which we will usually write as column vectors

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{pmatrix}.$$

The choice of basis also allows us to express the action of a linear operator L on V

$$L : v \in V \to Lv \in V$$

as multiplication by an $n \times n$ matrix

$$\begin{pmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{pmatrix} \to \begin{pmatrix} L_{11} & L_{12} & \cdot & \cdot & \cdot & L_{1n} \\ L_{21} & L_{22} & \cdot & \cdot & \cdot & L_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ L_{n1} & L_{n2} & \cdot & \cdot & \cdot & L_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{pmatrix}.$$

Note that when working with vectors as linear combinations of basis vectors we can use matrix notation to write a linear transformation a follows

$$v \to Lv = \begin{pmatrix} e_1 & \cdot & \cdot & \cdot & e_n \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} & \cdot & \cdot & \cdot & L_{1n} \\ L_{21} & L_{22} & \cdot & \cdot & \cdot & L_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ L_{n1} & L_{n2} & \cdot & \cdot & \cdot & L_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{pmatrix}.$$

II. DUAL VECTOR SPACES

To any vector space V, we can associate a new vector space, its dual:

Definition II.1 *For V a vector space over a field k, the dual vector space V^* is the vector space of all linear maps $V \to k$ that is*

$$V^* = \{l : V \to k | l(\alpha v + \beta w) = \alpha l(v) + \beta l(w)\}$$

for α, $\beta \in k$ and v, $w \in V$.

Given a linear transformation L acting on V, we can define

Definition II.2 *The transpose of L is the linear transformation*

$$L^t : V^* \to V^*$$

given by

$$(L^t l)(v) = l(Lv)$$

for $l \in V^*$ and $v \in V$.

For any choice of basis of V, there is a dual basis of V^* that satisfies

$$e_j^*(e_k) = \delta_{jk}.$$

Coordinates on V with respect to a basis are linear functions and thus elements of V^*. The coordinate function v_j can be identified with the dual basis vector e_j^* since

$$e_j^*(v) = e_j^*\left(\sum_{j=1}^{n} v_i e_j\right) = v_j.$$

It can be easily seen that the elements of the matrix for L in the basis e_j are given by

$$L_{jk} = e_j^*(Le_k)$$

and the matrix for the transpose map with respect to the dual basis is the matrix transpose

$$L^T_{jk} = L_{kj}.$$

III. CHANGE OF BASIS

Any invertible transformation A on V can be used to change the basis e_j of V to a new basis e'_j by taking

$$e_j \to e'_j = Ae_j.$$

The matrix for a linear transformation L transforms under this change of basis as follows

$$\begin{aligned} L_{jk} = e_j^*(Le_k) \to (e'_j)^*(Le'_k) &= (Ae_j)^*(LAe_k) \\ &= (A^T)^{-1}(e_j^*)(LAe_k) \\ &= e_j^*(A^{-1}LAe_k) \\ &= (A^{-1}LA)_{jk} \end{aligned}$$

In the second step, we are using the fact that elements of the dual basis transform as the dual representation. This what is needed to ensure the relation

$$(e'_j)^*(e'_k) = \delta_{jk}.$$

The change of basis formula shows that if two matrices L_1 and L_2 are related by conjugation by a third matrix A

$$L_2 = A^{-1}L_1A$$

then they represent the same linear transformation with respect to two different choices of basis.

IV. INNER PRODUCTS

An inner product on a vector space V is an additional structure that provides a notion of length for vectors, of angle between two vectors and identifies $V^* \simeq V$. In the real case:

Definition IV.1 *An inner product space on a real vector space V is a symmetric map*

$$(.,.) : V \times V \to \mathbb{R}$$

that is non-degenerate and linear in both variables.

The real inner products are usually positive definite. In the complex case:

Definition IV.2 *A Hermitian inner product on a complex vector space V is a map*

$$\langle .,. \rangle : V \times V \to \mathbb{C}$$

that is conjugate symmetric

$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$

non-degenerate in both variables, linear in the second variable, anti-linear in the first variable.

An inner product gives a notion of length-squared $||.||^2$ for vectors with

$$||v||^2 = \langle v, v \rangle.$$

Note that whether to specify anti-linearity in the first or second variable is a matter of convention. An inner product also provides an isomorphism $V \simeq V^*$ by the map

$$v \in V \to l_v \in V^*$$

where l_v is defined by

$$l_v(w) = (v, w)$$

in the real case, and

$$l_v(w) = \langle v, w \rangle$$

in the complex case (where this is a complex anti-linear rather than linear isomorphism).

V. ADJOINT OPERATORS

When V is a vector space with inner product, the adjoint of L can be defined as

Definition V.1 *The adjoint of a linear operator $L : V \to V$ is an operator $L^\dagger$ satisfying*

$$\langle Lv, w \rangle = \langle v, L^\dagger w \rangle$$

for all v, w/inV.

Note that mathematicians tend to favour L^* as the notation for the adjoint of L. In terms of explicit matrices since l_{Lv} is the conjugate-transpose of Lv, the matrix for $L^\dagger$ will be given by the conjugate-transpose $\overline{L^T}$ of the matrix for L

$$L^\dagger_{jk} = \overline{L_{kj}}.$$

In the real case, the matrix for the adjoint is just the transpose matrix. We will say that a linear transformation is self-adjoint if $L^\dagger = L$, skew-adjoint if $L^\dagger = -L$.

VI. EIGENVALUES AND EIGENVECTORS

We have seen that the matrix for a linear transformation L of a vector space V changes by conjugation, when we change our choice of basis of V. To get basis-independent information about L one considers the eigenvalues of the matrix. Complex matrices behave in much simpler fashion than their real analogues since in the complex case the eigenvalue equation

$$\det(\lambda \mathbf{1} - L) = 0$$

can always be factored into linear factors. For an arbitrary $n \times n$ complex matrix, there will be n solutions (counting repeated eigenvalues with multiplicity). A basis will exist for which the matrix will be in upper triangular form.

The case for the self-adjoint matrices L is much more constrained since transposition relates matrix elements. This leads us to the spectral theorem for self-adjoint matrices

Theorem VI.1 *Given a self-adjoint complex $n \times n$ matrix L there exists a unitary matrix U such that*

$$ULU^{-1} = D$$

where D is a diagonal matrix with entries $D_{jj} = \lambda_j$, $\lambda_j \in \mathbb{R}$.

Given L, its eigenvalues λ_j are the solutions to the eigenvalue equation

$$\det(\lambda \mathbf{1} - L) = 0$$

and U is determined by the eigenvectors. For distinct eigenvalues, the corresponding eigenvectors are orthogonal. The spectral theorem here is a theorem about finite-dimensional vector spaces and matrices but there are analogous theorems for self-adjoint operators on infinite-dimensional state spaces.