# What is Geometric Mechanics?

GM is the mechanics on *Lie groups* and the *manifolds* on which they *act*

Three main concepts:

- Lie groups: groups of *transformations*, e.g. rotations or translations
- Manifolds: new configuration spaces generalizing vector spaces
- Group action: a transformation takes a point of a manifold to another one

*This class provides examples of how these concepts are used!*

# 1 Rigid body motion

The main purpose of this section is to write the equations of rigid body motion in geometric form by using symmetry arguments.

## Rotational dynamics

- A **solid rigid body** is a system of at least three non-collinear point masses, constrained so that the distance between any two points remains constant in time.
- If the motion of the body is smooth, then the body can only move by ***combinations of rotations and translations***.
- Moreover, we assume that ***one point of the body remains fixed***, and base our coordinate systems at that point. This means that we are restricting to **consider only rotational motion**.
- A further simplifying assumption is that the ***fixed point is the centre of mass***. This means that **gravity plays no role** (recall that the centre of mass is the centre of gravity).

| |
|---|
| **Inertial (laboratory) frame**: spatial coordinate system with origin at the centre of mass. |

- Position of body particle in the spatial frame at time $t$:

$$\mathbf{x}(t) \in \mathbb{R}^3, \quad \mathbf{x}(0) = \mathbf{x}_0.$$

- The initial condition $\mathbf{x}_0$ for each body particle identifies the (initial) **reference configuration** of the body
- the position of a given particle when the body is in the (initial) reference configuration is sometimes denoted by $\mathbf{X} = \mathbf{x}_0$ and called the **particle label**.
- The configuration of the body particle at time $t$ is given by a rotation matrix that takes the label $\mathbf{x}_0 = \mathbf{X}$ to the current position $\mathbf{x}(t)$:

$$\mathbf{x} = \mathcal{R}(t)\mathbf{x}_0$$

  ($\mathcal{R}$ is a rotation matrix, i.e. $\mathcal{R} \in SO(3)$). The map $\mathbf{x}_0 \mapsto \mathcal{R}\mathbf{x}_0$ is the *body-to-space* map.

## Kinetic energy and Euler-Lagrange equations

Assume the body occupies a region $\mathcal{B} \subset \mathbb{R}^3$ of physical space. Also, denote the (time-independent) **mass distribution** by

$$\rho_0 = \rho_0(\mathbf{x}_0).$$

(the same as in the initial reference configuration).

- The kinetic energy is then given by the sum (integral) of the ***single particle energy*** $|\dot{\mathbf{x}}|^2/2$ over all the particles composing the body with density $\rho$.
- We write:

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho_0 \|\dot{\mathbf{x}}\|^2 \mathrm{d}^3\mathbf{x}_0 = \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{x}_0)\|\dot{\mathcal{R}}(t)\mathbf{x}_0\|^2 \mathrm{d}^3\mathbf{x}_0$$

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- The ***rotation matrix*** $\mathcal{R}$ ***is the Lagrangian coordinate***: this doesn't live in a vector space, not even a surface in $\mathbb{R}^3$! Rotation matrices actually belong to a *differentiable manifold*, which is also a *Lie group*.

- Hamilton's principle $\delta\int_{t_1}^{t_2} K\,\mathrm{d}t = 0$ and $\delta\mathcal{R}(t_1) = \delta\mathcal{R}(t_2) = 0$ yield Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{\mathcal{R}}} - \frac{\partial K}{\partial \mathcal{R}} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{\mathcal{R}}} = 0.$$

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This is a difficult matrix equation $\longrightarrow$ ***let's try a different approach!***

## Introducing symmetry

- Recall that rotation matrices are **orthogonal**:

$$\mathcal{R}^{-1} = \mathcal{R}^T \quad (\text{or} \quad \mathcal{R}\mathcal{R}^T = \mathbf{I}),$$

- so that they ***preserve the dot product***:

$$\mathbf{v} \cdot \mathbf{w} = \mathcal{R}\mathbf{v} \cdot \mathcal{R}\mathbf{w}.$$

- Also, $\mathcal{R}$ orthogonal $\Rightarrow \mathcal{R}^{-1} = \mathcal{R}^T$ is also orthogonal. Therefore

$$\|\dot{\mathcal{R}}\mathbf{x}_0\|^2 = \dot{\mathcal{R}}\mathbf{x}_0 \cdot \dot{\mathcal{R}}\mathbf{x}_0 = \mathcal{R}^{-1}\left(\dot{\mathcal{R}}\mathbf{x}_0\right) \cdot \mathcal{R}^{-1}\left(\dot{\mathcal{R}}\mathbf{x}_0\right) = \|\mathcal{R}^{-1}\dot{\mathcal{R}}\mathbf{x}_0\|^2,$$

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so that the **Lagrangian (kinetic energy) possesses the following symmetry**

$$\boxed{K(\mathcal{R}, \dot{\mathcal{R}}) = K(\mathcal{R}^{-1}\mathcal{R}, \mathcal{R}^{-1}\dot{\mathcal{R}}) =: k(\mathcal{R}^{-1}\dot{\mathcal{R}})}$$

***The matrix $\mathcal{R}^{-1}\dot{\mathcal{R}}$ possesses very important properties!***

## The matrix $\mathcal{R}^{-1}\dot{\mathcal{R}}$ and the hat map

Recall that $\mathcal{R}^{-1}\dot{\mathcal{R}} = \mathcal{R}^T\dot{\mathcal{R}}$.

- If we differentiate the relation $\mathcal{R}\mathcal{R}^T = \mathbf{I}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{R}\mathcal{R}^T\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{I}\right) = 0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{R}^T\mathcal{R}\right).$$

- Since $\mathrm{d}\mathcal{R}^T/\mathrm{d}t = (\dot{\mathcal{R}})^T$ for an arbitrary matrix, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{R}^T\mathcal{R}\right) = \dot{\mathcal{R}}^T\mathcal{R} + \mathcal{R}^T\dot{\mathcal{R}},$$

so that

$$\mathcal{R}^T\dot{\mathcal{R}} = -\left(\mathcal{R}^T\dot{\mathcal{R}}\right)^T,$$

$\Rightarrow \mathcal{R}^{-1}\dot{\mathcal{R}}$ is **skew-symmetric**.

Notice that skew-symmetric matrices do form a ***vector space***: we moved from a (difficult) manifold to a (simple) vector space! Actually, this vector space of skew-symmetric matrices is a ***Lie Algebra***.

Thus, we may introduce

$$\boxed{\mathcal{R}^{-1}\dot{\mathcal{R}} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} =: \widehat{\Omega}}$$

and

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{x}_0)\|\dot{\mathcal{R}}(t)\mathbf{x}_0\|^2 \mathrm{d}^3\mathbf{x}_0 = \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{x}_0)\|\widehat{\Omega}(t)\mathbf{x}_0\|^2 \mathrm{d}^3\mathbf{x}_0$$

Now, $\widehat{\Omega}$ is the only object that depends on time (through $\mathcal{R}$) and thus it is our new generalized coordinate.

An explicit computation yields:

$$\widehat{\Omega}\mathbf{x}_0 = \mathbf{\Omega} \times \mathbf{x}_0$$

where

$$\mathbf{\Omega} := (\Omega_1, \Omega_2, \Omega_3),$$

***Hat Map: to each vector in $\mathbb{R}^3$, we associate a skew-symmetric matrix, and viceversa (isomorphism).***

**Definition** (hat map, *this treatment requires tensors and will be skipped during lectures*)
*Given* $\boldsymbol{\Lambda} \in \mathbb{R}^3$*, the map*

$$\widehat{\Lambda}^i{}_j = -\varepsilon^i{}_{jk}\Lambda^k \quad (\textit{or } \widehat{\Lambda}\boldsymbol{v} = \boldsymbol{\Lambda} \times \boldsymbol{v}, \quad \forall \boldsymbol{v} \in \mathbb{R}^3)$$

*defines an isomorphism between vectors and skew-symmetric matrices.*

The hat map has an inverse. Indeed, one can use the following property of the Levi-Civita symbol

$$\varepsilon^i{}_{jk}\varepsilon^j{}_{is} = -2\delta_{ks}$$

to show that

$$\begin{aligned} \widehat{\Lambda}^i{}_j &= -\varepsilon^i{}_{jk}\Lambda^k \\ \Rightarrow \quad \varepsilon^j{}_{im}\widehat{\Lambda}^i{}_j &= -\varepsilon^j{}_{im}\varepsilon^i{}_{jk}\Lambda^k \\ &= 2\delta_{mk}\Lambda^k = 2\Lambda^\flat{}_m. \end{aligned}$$

Upon applying the inverse metric $\delta^{hm}$ on both sides and by using the properties of the Levi-Civita symbol, we have

$$\delta^{hm}\Lambda^\flat{}_m = \Lambda^h = \frac{1}{2}\delta^{hm}\varepsilon^j{}_{im}\widehat{\Lambda}^i{}_j$$

so that the inverse of the hat map reads

$$\Lambda^h = \frac{1}{2}\delta^{hm}\varepsilon^j{}_{im}\widehat{\Lambda}^i{}_j\,.$$

## The reduced kinetic energy

Substituting into the equation for $K$ we get

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{x}_0)\|\boldsymbol{\Omega} \times \mathbf{x}_0\|^2 \mathrm{d}^3\mathbf{x}_0$$

Now the identity

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$$

gives

$$\begin{aligned} K &= \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{x}_0)\left[\|\boldsymbol{\Omega}\|^2\|\mathbf{x}_0\|^2 - (\boldsymbol{\Omega} \cdot \mathbf{x}_0)^2\right] \mathrm{d}^3\mathbf{x}_0 \\ &= \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{x}_0)\left(\|\mathbf{x}_0\|^2\mathbf{I} - \mathbf{x}_0\mathbf{x}_0^T\right)\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}\, \mathrm{d}^3\mathbf{x}_0 \\ &= \frac{1}{2}\left[\underbrace{\int_{\mathcal{B}} \rho_0(\mathbf{x}_0)\left(\|\mathbf{x}_0\|^2\mathbf{I} - \mathbf{x}_0\mathbf{x}_0^T\right) \mathrm{d}^3\mathbf{x}_0}_{\text{moment of inertia tensor}}\right]\boldsymbol{\Omega} \cdot \boldsymbol{\Omega} \\ &= \frac{1}{2}\mathbb{I}\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}. \end{aligned}$$

where $\mathbf{x}_0\mathbf{x}_0^T$ is a matrix obtained by standard row-by-column product.

## The variational principle

The equations of motion are found by taking variations:

$$\delta \int_{t_1}^{t_2} \left( \frac{1}{2}\, \mathbb{I}\boldsymbol{\Omega} \cdot \boldsymbol{\Omega} \right) \mathrm{d}t = \int_{t_1}^{t_2} \frac{1}{2} \Big( \mathbb{I}\delta\boldsymbol{\Omega} \cdot \boldsymbol{\Omega} + \mathbb{I}\boldsymbol{\Omega} \cdot \delta\boldsymbol{\Omega} \Big) \mathrm{d}t = 0$$

Since $\mathbb{I}$ is symmetric we have

$$\int_{t_1}^{t_2} \mathbb{I}\boldsymbol{\Omega} \cdot \delta\boldsymbol{\Omega} \, \mathrm{d}t = 0.$$

The problem is now to ***take the variation*** $\delta\boldsymbol{\Omega}$.

- Let us vary the definition of the hat map as follows:

$$\boldsymbol{\Omega} \times \mathbf{x}_0 = \widehat{\Omega}\mathbf{x}_0 \quad \Longrightarrow \quad (\delta\boldsymbol{\Omega}) \times \mathbf{x}_0 = (\delta\widehat{\Omega})\mathbf{x}_0 \,.$$

- In turn, we also have

$$\delta\widehat{\Omega} = \delta\big(\mathcal{R}^{-1}\dot{\mathcal{R}}\big) = \delta\big(\mathcal{R}^{-1}\big)\,\dot{\mathcal{R}} + \mathcal{R}^{-1}\delta\big(\dot{\mathcal{R}}\big)\,.$$

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$$\delta\widehat{\Omega} = \delta\big(\mathcal{R}^{-1}\dot{\mathcal{R}}\big) = \delta\big(\mathcal{R}^{-1}\big)\, \dot{\mathcal{R}} + \mathcal{R}^{-1}\delta\big(\dot{\mathcal{R}}\big)\,. \tag{1}$$

- To proceed further, we shall need the following Lemma (prove it by differentiating $\mathcal{R}^{-1}\mathcal{R}$)

  **Lemma 1** *Let $\mathcal{R}$ be an invertible square matrix, then*

$$\delta(\mathcal{R}^{-1}) = -\mathcal{R}^{-1}\,(\delta\mathcal{R})\,\mathcal{R}^{-1}, \qquad \textit{and} \qquad \frac{d}{dt}\big(\mathcal{R}^{-1}\big) = -\mathcal{R}^{-1}\dot{\mathcal{R}}\mathcal{R}^{-1}.$$

- At this point, the variations can be expressed in the matrix form as follows

**Proposition 1 (Expression of the variations)** *Let* $\widehat{\Omega} = \mathcal{R}^{-1}\dot{\mathcal{R}}$*. Then*

$$\delta\widehat{\Omega} = \dot{\widehat{\Lambda}} + \left[\widehat{\Omega}, \widehat{\Lambda}\right]$$

*where* $\widehat{\Lambda} := \mathcal{R}^{-1}\delta\mathcal{R}$ *and* $\left[\widehat{\Omega}, \widehat{\Lambda}\right]$ *is the Lie commutator*

$$\left[\widehat{\Omega}, \widehat{\Lambda}\right] := \widehat{\Omega}\widehat{\Lambda} - \widehat{\Lambda}\widehat{\Omega}.$$

*Moreover, in vector notation, this corresponds to*

$$\delta\mathbf{\Omega} = \dot{\mathbf{\Lambda}} + (\mathbf{\Omega} \times \mathbf{\Lambda}).$$

**Proof.** Substituting the formula for $\delta(\mathcal{R}^{-1})$ into the equation (1) gives

$$\delta\widehat{\Omega} = -\mathcal{R}^{-1}\,(\delta\mathcal{R})\,\mathcal{R}^{-1}\dot{\mathcal{R}} + \mathcal{R}^{-1}\delta\dot{\mathcal{R}}.$$

However, we can write

$$\delta\dot{\mathcal{R}} = \frac{\mathrm{d}}{\mathrm{d}t}(\delta\mathcal{R}).$$

so that

$$\begin{aligned}
\delta\widehat{\Omega} &= -\mathcal{R}^{-1}\,(\delta\mathcal{R})\,\widehat{\Omega} + \mathcal{R}^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\mathcal{R}) \\
&= -\mathcal{R}^{-1}\,(\delta\mathcal{R})\,\widehat{\Omega} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{R}^{-1}\delta\mathcal{R}\right) - \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}^{-1}\right)\delta\mathcal{R}. \\
&= -\mathcal{R}^{-1}\,(\delta\mathcal{R})\,\widehat{\Omega} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{R}^{-1}\delta\mathcal{R}\right) + \mathcal{R}^{-1}\dot{\mathcal{R}}\mathcal{R}^{-1}\delta\mathcal{R} \\
&= -\mathcal{R}^{-1}\,(\delta\mathcal{R})\,\widehat{\Omega} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{R}^{-1}\delta\mathcal{R}\right) + \widehat{\Omega}\mathcal{R}^{-1}\delta\mathcal{R} \\
&= -\widehat{\Lambda}\widehat{\Omega} + \dot{\widehat{\Lambda}} + \widehat{\Omega}\widehat{\Lambda} \\
&= \dot{\widehat{\Lambda}} + \left[\widehat{\Omega}, \widehat{\Lambda}\right]
\end{aligned}$$

and the first statement is proven.

In order to prove the second statement, we use the definition of the hat map, so that, for an arbitrary vector $\mathbf{v}$, we have

$$
\begin{aligned}
\delta\boldsymbol{\Omega} \times \mathbf{v} &= \delta\widehat{\Omega}\mathbf{v} \\
&= \dot{\widehat{\Lambda}}\mathbf{v} + \widehat{\Omega}\widehat{\Lambda}\mathbf{v} - \widehat{\Lambda}\widehat{\Omega}\mathbf{v} \\
&= \dot{\boldsymbol{\Lambda}} \times \mathbf{v} + \widehat{\Omega}\,(\boldsymbol{\Lambda} \times \mathbf{v}) - \widehat{\Lambda}\,(\boldsymbol{\Omega} \times \mathbf{v}) \\
&= \dot{\boldsymbol{\Lambda}} \times \mathbf{v} + \boldsymbol{\Omega} \times (\boldsymbol{\Lambda} \times \mathbf{v}) - \boldsymbol{\Lambda} \times (\boldsymbol{\Omega} \times \mathbf{v}) \\
&= \dot{\boldsymbol{\Lambda}} \times \mathbf{v} + \boldsymbol{\Omega} \times (\boldsymbol{\Lambda} \times \mathbf{v}) + \boldsymbol{\Lambda} \times (\mathbf{v} \times \boldsymbol{\Omega}) \\
&= \dot{\boldsymbol{\Lambda}} \times \mathbf{v} + \mathbf{v} \times (\boldsymbol{\Lambda} \times \boldsymbol{\Omega})
\end{aligned}
$$

(by the Jacobi identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) = 0$)

$$
= \left[\dot{\boldsymbol{\Lambda}} + (\boldsymbol{\Omega} \times \boldsymbol{\Lambda})\right] \times \mathbf{v}.
$$

Thus, since $\mathbf{v}$ is arbitrary, the second statement follows. ■

## Equations of motion

- At this point, we can express the variational principle as

$$\delta \int_{t_1}^{t_2} K(\mathbf{\Omega})\mathrm{d}t = \int_{t_1}^{t_2} \mathbb{I}\mathbf{\Omega} \cdot \delta\mathbf{\Omega}\,\mathrm{d}t$$
$$= \int_{t_1}^{t_2} \mathbb{I}\mathbf{\Omega} \cdot \left[\dot{\mathbf{\Lambda}} + (\mathbf{\Omega} \times \mathbf{\Lambda})\right]\,\mathrm{d}t$$
$$= \mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Lambda}\big|_{t_2} - \mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Lambda}\big|_{t_1} - \int_{t_1}^{t_2} \left(\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{I}\mathbf{\Omega}) \cdot \mathbf{\Lambda}\right)\,\mathrm{d}t + \int_{t_1}^{t_2} \mathbb{I}\mathbf{\Omega} \cdot (\mathbf{\Omega} \times \mathbf{\Lambda})\,\mathrm{d}t$$

(by integrating the first term by parts)

$$= \mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Lambda}\big|_{t_2} - \mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Lambda}\big|_{t_1} - \int_{t_1}^{t_2} \left(\mathbb{I}\dot{\mathbf{\Omega}} - \mathbb{I}\mathbf{\Omega} \times \mathbf{\Omega}\right) \cdot \mathbf{\Lambda}\,\mathrm{d}t$$

(by using $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$).

- The problem is now to deal with the quantities $\mathbb{I}\mathbf{\Omega} \cdot \mathbf{\Lambda}\big|_{t_i}$. However, if we recall the definition $\widehat{\Lambda} := \mathcal{R}^{-1}\delta\mathcal{R}$ and apply the inverse of the hat map, we find that $\mathbf{\Lambda}$ ***actually vanishes at the endpoints*** $t_1$ and $t_2$ because $\delta\mathcal{R}$ does.

- Therefore, since $\delta\mathcal{R}$ is arbitrary and so is $\mathbf{\Lambda}$, the variational principle

$$\delta\int_{t_1}^{t_2} K\,\mathrm{d}t \;=\int_{t_1}^{t_2}\left(-\mathbb{I}\dot{\mathbf{\Omega}}+\mathbb{I}\mathbf{\Omega}\times\mathbf{\Omega}\right)\cdot\mathbf{\Lambda}\,\mathrm{d}t=0$$

  is satisfied if and only if

$$\boxed{\mathbb{I}\dot{\mathbf{\Omega}}=\mathbb{I}\mathbf{\Omega}\times\mathbf{\Omega},}$$

  which is the (Euler-Poincaré) equation of rigid body dynamics.

## Physical meaning of $\Omega$ and its conjugate momentum

We saw that the rotation matrix $\mathcal{R}$ *maps points in the reference initial frame to points in the spatial lab frame*:

$$\mathbf{x} = \mathcal{R}\mathbf{x}_0, \quad \mathbf{x}_0 = \mathcal{R}^{-1}\mathbf{x}.$$

However, there is also a *non-inertial* frame, which is called the **body frame**. This is defined to *move with the body*, but agrees with the spatial frame when the body is in the reference configuration.

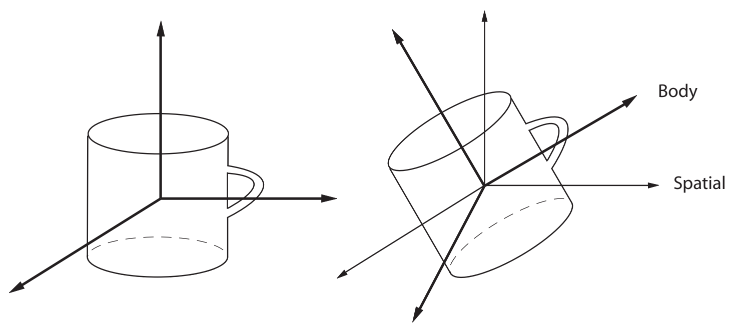


Figure 1: The spatial frame (on the left) and the rotating body frame (on the right). (Courtesy of D.D. Holm, T. Schmah, C. Stoica. From their book *Geometric mechanics and symmetry, Oxford University Press, 2009*)

*In general, left-multiplying by $\mathcal{R}^{-1}$ changes a vector's coordinates from spatial to body. Vice versa, left-multiplying by $\mathcal{R}$ changes from body to spatial.*

- To see this, let's compute

$$\boldsymbol{\Omega} \times \mathbf{x}_0 = \hat{\Omega}\mathbf{x}_0 = \mathcal{R}^{-1}\dot{\mathcal{R}}\mathbf{x}_0 = \mathcal{R}^{-1}\dot{\mathbf{x}}.$$

- However, we know from physics that there is an ***angular velocity*** $\boldsymbol{\omega}$ such that

$$\dot{\mathbf{x}} = \boldsymbol{\omega} \times \mathbf{x}.$$

Indeed,

$$\dot{\mathbf{x}} = \dot{\mathcal{R}}\mathbf{x}_0 = \dot{\mathcal{R}}\mathcal{R}^{-1}\mathcal{R}\mathbf{x}_0 = \dot{\mathcal{R}}\mathcal{R}^{-1}\mathbf{x} =: \boldsymbol{\omega} \times \mathbf{x}$$

since $\dot{\mathcal{R}}\mathcal{R}^{-1}$ is skew symmetric.

- On the other hand, by using standard properties of the cross product, we have

$$\begin{aligned} \boldsymbol{\Omega} \times \mathbf{x}_0 &= \mathcal{R}^{-1}\dot{\mathbf{x}} \\ &= \mathcal{R}^{-1}(\boldsymbol{\omega} \times \mathbf{x}) \\ &= \left(\mathcal{R}^{-1}\boldsymbol{\omega}\right) \times \left(\mathcal{R}^{-1}\mathbf{x}\right). \end{aligned}$$

so that $\mathcal{R}^{-1}$ takes all vectors from spatial to body.

We conclude that

$\boxed{\boldsymbol{\Omega} \textit{ is the angular velocity in the body frame.}}$

We conclude that

> $\boldsymbol{\Omega}$ *is the angular velocity in the body frame.*

- The conjugate variable to $\boldsymbol{\Omega}$ is given as

$$\boldsymbol{m} = \frac{\partial K}{\partial \boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega}$$

and is the ***angular momentum in the body frame***, whose equation is

$$\dot{\boldsymbol{m}} = \boldsymbol{m} \times \mathbb{I}^{-1}\boldsymbol{m}$$

which is the *Euler's equation of a rigid body.*

- Notice that the *body angular momentum is not conserved* because the body frame is a non-inertial frame. However, applying $\mathcal{R}$ takes body quantities into the spatial frame.

- Thus, the **spatial angular momentum** is

$$\mathcal{R}\mathbf{m} = \mathcal{R}\mathbb{I}\boldsymbol{\Omega},$$

and is ***conserved***:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{R}\mathbb{I}\boldsymbol{\Omega}) &= \dot{\mathcal{R}}\mathbb{I}\boldsymbol{\Omega} + \mathcal{R}\mathbb{I}\dot{\boldsymbol{\Omega}} \\
&= \dot{\mathcal{R}}\mathbb{I}\boldsymbol{\Omega} + \mathcal{R}(\mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}) \\
&= \mathcal{R}\mathcal{R}^{-1}\dot{\mathcal{R}}\mathbb{I}\boldsymbol{\Omega} + \mathcal{R}(\mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}) \\
&= \mathcal{R}(\widehat{\boldsymbol{\Omega}}\mathbb{I}\boldsymbol{\Omega}) + \mathcal{R}(\mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}) \\
&= \mathcal{R}(\boldsymbol{\Omega} \times \mathbb{I}\boldsymbol{\Omega}) + \mathcal{R}(\mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}) \\
&= 0.
\end{aligned}
$$

and is ***conserved***:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{R}\mathbb{I}\boldsymbol{\Omega}) &= \dot{\mathcal{R}}\mathbb{I}\boldsymbol{\Omega} + \mathcal{R}\mathbb{I}\dot{\boldsymbol{\Omega}} \\
&= \dot{\mathcal{R}}\mathbb{I}\boldsymbol{\Omega} + \mathcal{R}(\mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}) \\
&= \mathcal{R}\mathcal{R}^{-1}\dot{\mathcal{R}}\mathbb{I}\boldsymbol{\Omega} + \mathcal{R}(\mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}) \\
&= \mathcal{R}(\widehat{\boldsymbol{\Omega}}\mathbb{I}\boldsymbol{\Omega}) + \mathcal{R}(\mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}) \\
&= \mathcal{R}(\boldsymbol{\Omega} \times \mathbb{I}\boldsymbol{\Omega}) + \mathcal{R}(\mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}) \\
&= 0.
\end{aligned}
$$

**What about right symmetry?**

- In all our discussion, we used the left symmetry $K(\mathcal{R}, \dot{\mathcal{R}}) = K(\mathcal{R}^{-1}\mathcal{R}, \mathcal{R}^{-1}\dot{\mathcal{R}}) = K(\boldsymbol{\Omega})$.
- What can we say about the right symmetry? Can we say that $K$ is right-invariant? That is, can we say that
$$K(\mathcal{R}, \dot{\mathcal{R}}) = K(\mathcal{R}\mathcal{R}^{-1}, \dot{\mathcal{R}}\mathcal{R}^{-1}) = K(\boldsymbol{\omega}) \ ?$$
- NO! The presence of $\rho_0(\mathbf{x}_0)$ prevents $K$ from being right-invariant. (exercise!)

## 2 Inviscid fluid flows

- Rigid body dynamics is entirely governed by the rotational symmetry. Here we shall see how fluid dynamics is governed by the ***relabeling symmetry***

- Given the reference configuration $\mathbf{x}_0$ (*label*) for a fluid parcel, a ***smooth invertible map*** $\boldsymbol{\eta}_t$ takes $\mathbf{x}_0$ to the current position $\mathbf{x}(t)$:

$$\mathbf{x} = \boldsymbol{\eta}_t(\mathbf{x}_0) \qquad\qquad (\textit{back-to-labels}\text{ map})$$

- One says $\boldsymbol{\eta}_t$ is a time-dependent ***diffeomorphism***. The space of diffeomorphisms on physical space is denoted by

$$\mathrm{Diff}(\mathbb{R}^3)$$

  This is another example of ***transformation group***, which extends the group $SO(3)$ of rotation matrices.

- In order to emphasize the time dependence, we can write the *back-to-labels* map as

$$\mathbf{x} = \boldsymbol{\eta}(\mathbf{x}_0, t)$$

## Kinetic energy of a fluid

Assume the fluid occupies, in the reference frame, a region $\mathcal{B} \subset \mathbb{R}^3$ of physical space. Also, denote the reference **mass distribution** by

$$\rho_0 = \rho_0(\mathbf{x}_0).$$

- The kinetic energy is then given by the sum (integral) of the ***single particle energy*** $m|\dot{\mathbf{x}}|^2/2$ over all the particles composing the body with density $\rho$.
- We write:

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho_0 \|\dot{\mathbf{x}}\|^2 \mathrm{d}^3\mathbf{x}_0 = \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{x}_0) \|\dot{\boldsymbol{\eta}}(\mathbf{x}_0, t)\|^2 \mathrm{d}^3\mathbf{x}_0$$

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- The ***diffeomorphism*** ***$\eta$ is the Lagrangian coordinate***: this doesn't live in a vector space, not even a surface in $\mathbb{R}^3$! Diffeomorphisms actually belong to a *differentiable manifold*, which is also a *transformation group*. (Notice that this is now *infinite-dimensional*!)
- Hamilton's principle $\delta \int_{t_1}^{t_2} K\,\mathrm{d}t = 0$ and $\delta\boldsymbol{\eta}_{t_1} = \delta\boldsymbol{\eta}_{t_2} = 0$ yield Euler-Lagrange equations

  These are difficult equations on an infinite-dimensional space! $\longrightarrow$ ***let's try symmetry. . .***

## Symmetry and the reduced kinetic energy

- In fluid dynamics, it is customary to express the total energy in terms of the *Eulerian velocity* $\boldsymbol{u}(\mathbf{x},t)$ and mass density $\rho(\mathbf{x},t)$. Here we shall see how this happens.
- The Lagrangian (kinetic energy) ***does not possess the right symmetry***

$$\boxed{K(\boldsymbol{\eta},\dot{\boldsymbol{\eta}}) \neq K(\boldsymbol{\eta}\circ\boldsymbol{\eta}^{-1},\dot{\boldsymbol{\eta}}\circ\boldsymbol{\eta}^{-1}) =: k(\dot{\boldsymbol{\eta}}\circ\boldsymbol{\eta}^{-1}),}$$

where $\circ$ denotes composition of vector functions (this symbol will actually be omitted).

- However, we can still introduce the quantity $\dot{\boldsymbol{\eta}}\boldsymbol{\eta}^{-1}(\mathbf{x},t)$ as follows. (Recall that $\boldsymbol{\eta}\boldsymbol{\eta}^{-1} = \mathrm{Id}$).

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- However, we can still introduce the quantity $\dot{\boldsymbol{\eta}}\boldsymbol{\eta}^{-1}(\mathbf{x},t)$ as follows. (Recall that $\boldsymbol{\eta}\boldsymbol{\eta}^{-1} = \mathrm{Id}$).
- Introduce the ***Lagrange-to-Euler (LE) map*** $\rho = \boldsymbol{\eta}_*\rho_0$, defined as

$$\rho(\mathbf{x},t) := \int_{\mathcal{B}} \rho_0(\mathbf{x}_0)\,\delta(\mathbf{x}-\boldsymbol{\eta}(\mathbf{x}_0,t))\,\mathrm{d}^3\mathbf{x}_0 =: \boldsymbol{\eta}_*\rho_0$$

  This map if of vital importance in geometric fluid dynamics!
- The mass density appears in the kinetic energy as a parameter that ***breaks the right symmetry***, which would hold if $\boldsymbol{\eta}_*\rho_0 = \rho_0 = \text{const}$.

- Then use the properties of the delta function and define $\boldsymbol{u} := \dot{\boldsymbol{\eta}}\boldsymbol{\eta}^{-1}$ to find the fluid kinetic energy in Eulerian variables. Define $K_{\rho_0}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) := K(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}})$ and compute

$$
\begin{aligned}
K_{\rho_0}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) &= \frac{1}{2}\int_{\mathcal{B}} \rho_0(\mathbf{x}_0)\|\dot{\boldsymbol{\eta}}(\mathbf{x}_0, t)\|^2 \mathrm{d}^3\mathbf{x}_0 \\
&= \frac{1}{2}\int_{\mathcal{B}} \mathrm{d}^3\mathbf{x} \left[\int_{\mathcal{B}} \mathrm{d}^3\mathbf{x}_0\, \rho_0(\mathbf{x}_0)\, \delta(\mathbf{x} - \boldsymbol{\eta}(\mathbf{x}_0, t))\right] \left\|\dot{\boldsymbol{\eta}}\boldsymbol{\eta}^{-1}(\mathbf{x}, t)\right\|^2 \\
&= \frac{1}{2}\int_{\mathcal{B}} \rho(\mathbf{x}, t)\, \|\boldsymbol{u}(\mathbf{x}, t)\|^2\, \mathrm{d}^3\mathbf{x} = K_{\boldsymbol{\eta}_*\rho_0}(\boldsymbol{\eta}\boldsymbol{\eta}^{-1}, \dot{\boldsymbol{\eta}}\boldsymbol{\eta}^{-1})
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\end{aligned}
$$

## Mass transport equation

- Taking the derivative of the LE map and pairing with a test function yields mass conservation:

$$
\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0
$$

**Proof.** Omit time dependence for simplicity:

$$
\begin{aligned}
\int_{\mathcal{D}} \mathrm{d}^3\mathbf{x}\, \partial_t \rho(\mathbf{x})\, \varphi(\mathbf{x}) &= -\int_{\mathcal{D}} \mathrm{d}^3\mathbf{x}\, \varphi(\mathbf{x}) \int_{\mathcal{D}} \mathrm{d}^3\mathbf{x}_0\, \rho_0(\mathbf{x}_0) \nabla\delta(\mathbf{x} - \boldsymbol{\eta}(\mathbf{x}_0)) \cdot \dot{\boldsymbol{\eta}}(\mathbf{x}_0) \\
&= \int_{\mathcal{D}} \mathrm{d}^3\mathbf{x}\, \nabla\varphi(\mathbf{x}) \int_{\mathcal{D}} \mathrm{d}^3\mathbf{x}_0\, \rho_0(\mathbf{x}_0) \delta(\mathbf{x} - \boldsymbol{\eta}(\mathbf{x}_0)) \cdot (\dot{\boldsymbol{\eta}}\boldsymbol{\eta}^{-1})(\mathbf{x}) \\
&= -\int_{\mathcal{D}} \mathrm{d}^3\mathbf{x}\, \varphi(\mathbf{x})\, \mathrm{div}\big(\rho(\mathbf{x})\boldsymbol{u}(\mathbf{x})\big)\,,
\end{aligned}
$$

## Compressible fluids: the reduced variational principle

- So far, only the fluid kinetic energy was considered. Let us now denote the ***fluid internal energy*** by $\mathcal{U}$ and assume the fuid is *barotropic*, i.e. $\mathcal{U} = \mathcal{U}(\rho)$.

- Then, the symmetry-reduced Lagrangian can be expressed as

$$\ell(\boldsymbol{u}, \rho) = \frac{1}{2}\int_{\mathcal{B}} \rho\,\|\boldsymbol{u}\|^2\,\mathrm{d}^3\mathbf{x} - \int_{\mathcal{B}} \rho\,\mathcal{U}(\rho)\,\mathrm{d}^3\mathbf{x}$$

- Hamilton's principle leads to computing the variations $\delta\rho$ and $\delta\boldsymbol{u}$:

$$\delta\int_{t_1}^{t_2} \ell(\boldsymbol{u}, \rho)\,\mathrm{d}t = \int_{t_1}^{t_2}\int_{\mathcal{B}} \left[\rho\boldsymbol{u}\cdot\delta\boldsymbol{u} + \left(\frac{1}{2}\|\boldsymbol{u}\|^2 - \mathcal{U}(\rho) - \rho\,\mathcal{U}'(\rho)\right)\delta\rho\right]\mathrm{d}^3\mathbf{x}\,\mathrm{d}t = 0$$

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- The variation $\delta\rho$ is computed by pairing with a test function as follows:

$$\begin{aligned}\int_{\mathcal{D}} \mathrm{d}^3\mathbf{x}\,\delta\rho(\mathbf{x})\,\varphi(\mathbf{x}) &= -\int_{\mathcal{D}} \mathrm{d}^3\mathbf{x}\,\varphi(\mathbf{x})\int_{\mathcal{D}} \mathrm{d}^3\mathbf{x}_0\,\rho_0(\mathbf{x}_0)\nabla\delta(\mathbf{x} - \boldsymbol{\eta}(\mathbf{x}_0))\cdot\delta\boldsymbol{\eta}(\mathbf{x}_0)\\ &= -\int_{\mathcal{D}} \mathrm{d}^3\mathbf{x}\,\varphi(\mathbf{x})\,\mathrm{div}\big(\rho(\mathbf{x})\boldsymbol{w}(\mathbf{x})\big)\,,\end{aligned}$$

  where we defined $\boldsymbol{w} := (\delta\boldsymbol{\eta})\circ\boldsymbol{\eta}^{-1}$. Since $\varphi(\mathbf{x})$ is arbitrary, then $\delta\rho = -\,\mathrm{div}(\rho\boldsymbol{w})$.

- Variation $\delta\boldsymbol{u}$ (no proof):

$$\delta\boldsymbol{u} = \delta(\dot{\boldsymbol{\eta}}\boldsymbol{\eta}^{-1}) = \partial_t\boldsymbol{w} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - (\boldsymbol{w}\cdot\nabla)\boldsymbol{u}$$

## Compressible fluids: equations of motion

- Hamilton's principle becomes

$$\int_{t_1}^{t_2}\int_{\mathcal{B}} \rho\boldsymbol{u}\cdot\Big(\partial_t\boldsymbol{w}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}-(\boldsymbol{w}\cdot\nabla)\boldsymbol{u}\Big)\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t$$
$$-\int_{t_1}^{t_2}\int_{\mathcal{B}}\left(\frac{1}{2}\|\boldsymbol{u}\|^2-\mathcal{U}(\rho)-\rho\,\mathcal{U}'(\rho)\right)\mathrm{div}\big(\rho\boldsymbol{w}\big)\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t=0$$

- Then, upon integrating by parts, we have

$$\int_{t_1}^{t_2}\int_{\mathcal{B}}\Big(\partial_t(\rho\boldsymbol{u})+(\boldsymbol{u}\cdot\nabla)(\rho\boldsymbol{u})+\rho(\mathrm{div}\,\boldsymbol{u})\boldsymbol{u}+\rho\nabla\big(\mathcal{U}+\rho\,\mathcal{U}'\big)\Big)\cdot\boldsymbol{w}\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t$$

- By recalling $\partial_t\rho+\mathrm{div}(\rho\boldsymbol{u})=0$ and upon defining ***pressure*** as $\mathsf{p}:=\rho^2\,\mathcal{U}'(\rho)$, we have

$$\int_{t_1}^{t_2}\int_{\mathcal{B}}\Big(\rho\partial_t\boldsymbol{u}+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla\mathsf{p}\Big)\cdot\boldsymbol{w}\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}t$$

- At this point, notice that $\boldsymbol{w}:=(\delta\boldsymbol{\eta})\circ\boldsymbol{\eta}^{-1}$ vanishes at the endpoints, since $\delta\boldsymbol{\eta}(t_1)=\delta\boldsymbol{\eta}(t_2)=0$.

- Equations of motion: $\rho\,(\partial_t\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u})=-\nabla\mathsf{p}$ and $\partial_t\rho+\mathrm{div}(\rho\boldsymbol{u})=0$.

## Equations of motion for arbitrary compressible fluids

- In the most general case, one is given a Lagrangian $\ell(\boldsymbol{u}, \rho)$ that is not exactly of the same form as in the previous case. For example, for electrostatic charged fluids nonlocal terms appear in the potential energy so that the latter is of the type

$$U(\rho) = \int_{\mathcal{B}} \rho\, \mathcal{U}(\rho)\, \mathrm{d}^3\mathbf{x} + \frac{1}{2}\int_{\mathcal{B}} \rho\, G * \rho\, \mathrm{d}^3\mathbf{x}$$

where $G * \rho := \int_{\mathcal{B}} G(\mathbf{x} - \mathbf{x}')\rho(\mathbf{x}')\, \mathrm{d}^3\mathbf{x}$ is a convolution integral. (Tipically, $G = \Delta^{-1}$).

- In the general case, we have to compute variations as

$$0 = \delta \int_{t_1}^{t_2} \ell(\boldsymbol{u}, \rho)\, \mathrm{d}t = \int_{t_1}^{t_2} \left( \frac{\delta \ell}{\delta \boldsymbol{u}} \cdot \delta \boldsymbol{u} + \frac{\delta \ell}{\delta \rho} \delta \rho \right) \mathrm{d}t$$

where we have used the following definition of ***functional derivative***: *given a function(al) $F : V \to \mathbb{R}$ on a space of functions $V$ with inner product, the functional derivative is defined so that*

$$\delta F =: \left\langle \frac{\delta F}{\delta \xi}, \delta \xi \right\rangle \qquad \forall \xi \in V$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $V$ (tipically $L^2$). ***Notice:*** *this is what we already used!*

- At this point, Hamilton's principle becomes

$$\int_{t_1}^{t_2}\int_{\mathcal{B}} \frac{\delta \ell}{\delta \boldsymbol{u}} \cdot \Big(\partial_t \boldsymbol{w} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - (\boldsymbol{w}\cdot\nabla)\boldsymbol{u}\Big)\, \mathrm{d}^3\mathbf{x}\,\mathrm{d}t - \int_{t_1}^{t_2}\int_{\mathcal{B}} \frac{\delta \ell}{\delta \rho}\, \mathrm{div}(\rho \boldsymbol{w})\mathrm{d}^3\mathbf{x}\,\mathrm{d}t = 0$$

- Then, upon integrating by parts, we have

$$\int_{t_1}^{t_2}\int_{\mathcal{B}} \left(\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \boldsymbol{u}} + \mathrm{div}\left(\boldsymbol{u}\,\frac{\delta \ell}{\delta \boldsymbol{u}}\right) + \nabla \boldsymbol{u} \cdot \frac{\delta \ell}{\delta \boldsymbol{u}} - \rho \nabla \frac{\delta \ell}{\delta \rho}\right) \cdot \boldsymbol{w}\, \mathrm{d}^3\mathbf{x}\,\mathrm{d}t$$

- Therefore, the equations of motion are

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \boldsymbol{u}} + (\mathrm{div}\,\boldsymbol{u})\,\frac{\delta \ell}{\delta \boldsymbol{u}} + (\boldsymbol{u}\cdot\nabla)\frac{\delta \ell}{\delta \boldsymbol{u}} + \nabla \boldsymbol{u}\cdot \frac{\delta \ell}{\delta \boldsymbol{u}} = \rho \nabla \frac{\delta \ell}{\delta \rho}\,, \qquad \partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0$$

- At this point, all one needs to do is to take functional derivatives of the Lagrangian. For example, consider

$$\ell(\boldsymbol{u}, \rho) = \frac{1}{2}\int_{\mathcal{B}} \rho\, \|\boldsymbol{u}\|^2\, \mathrm{d}^3\mathbf{x} - \frac{1}{2}\int_{\mathcal{B}} \rho\, G * \rho\, \mathrm{d}^3\mathbf{x}$$

- Set $G = \Delta^{-1} = \|\mathbf{x} - \mathbf{x}'\|^{-1}$. It is easy to compute

$$\frac{\delta \ell}{\delta \boldsymbol{u}} = \rho \boldsymbol{u}\,, \quad \frac{\delta \ell}{\delta \rho} = \rho \|\boldsymbol{u}\|^2 - G*\rho\,, \qquad \Longrightarrow \quad \partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla \boldsymbol{u} = -\nabla(\Delta^{-1}\rho)\,, \quad \partial_t\rho + \mathrm{div}(\rho\boldsymbol{u}) = 0$$

which are the ***Euler-Poisson equations*** for an electrostatic charged fluid.

## Kelvin-Noether theorem

- **Theorem** (no proof). The reduced Lagrangian $\ell(\boldsymbol{u},\rho)$ of an inviscid barotropic fluid moving in a conservative force field yields the following conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\boldsymbol{\eta}(\gamma_0)}\frac{1}{\rho}\frac{\delta\ell}{\delta\boldsymbol{u}}\cdot\mathrm{d}\mathbf{x}=0\,,$$

where $\boldsymbol{\gamma}_0:[0,1]\to\mathcal{B}$ is a fixed loop in $\mathcal{B}$ and the functional derivatives $\delta\ell/\delta\boldsymbol{u}$ and $\delta\ell/\delta\rho$ are defined as

$$\delta\ell:=\int_{\mathcal{B}}\left(\frac{\delta\ell}{\delta\boldsymbol{u}}\cdot\delta\boldsymbol{u}+\frac{\delta\ell}{\delta\rho}\cdot\delta\rho\right)\mathrm{d}^3\mathbf{x}$$

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$$\delta\ell := \int_{\mathcal{B}} \left( \frac{\delta \ell}{\delta \boldsymbol{u}} \cdot \delta\boldsymbol{u} + \frac{\delta \ell}{\delta \rho} \cdot \delta\rho \right) \mathrm{d}^3\mathbf{x}$$

- Therefore, since $\delta\ell/\delta\boldsymbol{u} = \rho\boldsymbol{u}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\boldsymbol{\gamma}} \boldsymbol{u} \cdot \mathrm{d}\mathbf{x} = 0\,,$$

where $\boldsymbol{\gamma} := \boldsymbol{\eta}(\boldsymbol{\gamma}_0)$. This is the ***circulation theorem*** of barotropic fluid flows.

- Notice that the primitive Lagrangian $L(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}})$ is *NOT* symmetric under $\mathrm{Diff}(\mathcal{B})$. However, as we saw, it is symmetric if we restrict to consider the coordinate transformations

$$\mathrm{Diff}_{\rho_0}(\mathcal{B}) := \{ \bar{\boldsymbol{\eta}} \in \mathrm{Diff}(\mathcal{B}) \mid \bar{\boldsymbol{\eta}}_* \rho_0 = \rho_0 \}$$

*The circulation theorem gives the conserved quantity arising from this residual symmetry!*

# 3 Collisionless kinetic theory

Three major approaches to **collective dynamics of many-particle systems**:

- **Particle paths** on phase space (Liouville): traces points $(\mathbf{q}(t), \mathbf{p}(t)) \rightarrow$ ***solves all details***
- **Kinetic theory** (Boltzmann): probability density $f(\mathbf{q}, \mathbf{p}, t) \rightarrow$ ***retains most details***
- **Fluid approach**: local averages (momentum $\mathbf{m}(\mathbf{q}, t)$, density $\rho(\mathbf{q}, t)$) $\rightarrow$ ***forget details***

## From particle motion to kinetic theory

- Particle simulations for $(\mathbf{q}_s, \mathbf{p}_s)$ solve *all* details, but at huge costs. (Particles are identical!)

- **Kinetic theory**: spread particles across phase-space → *probability – statistical approach*

- Averaging processes (BBGKY) lead to the *particle distribution* $f(\mathbf{q}, \mathbf{p})$.

- A ***kinetic equation*** is an evolution equation for $f(\mathbf{q}, \mathbf{p})$.

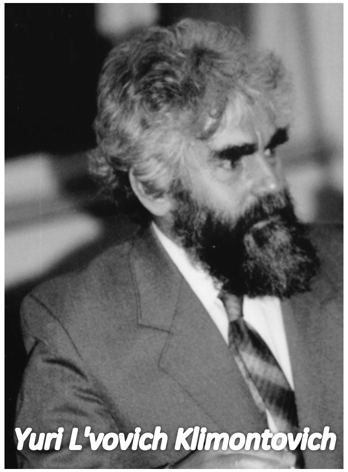
- *Collisional*: no energy conservation → *Boltzmann* ($H$-theorem)

- ***Collisionless***: energy is conserved → **Vlasov** (characteristic equation)

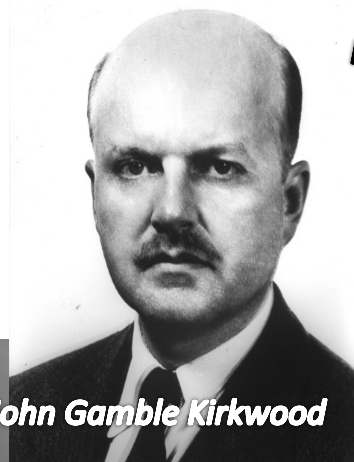
$$\frac{\mathrm{d}}{\mathrm{d}t}\big(f_t\,\mathrm{d}^3\mathbf{x}_t\,\mathrm{d}^3\mathbf{p}_t\big) = 0 \qquad \text{with} \qquad (\dot{\mathbf{q}}_t, \dot{\mathbf{p}}_t) = \left(\frac{\partial H}{\partial \mathbf{p}}, -\frac{\partial H}{\partial \mathbf{q}}\right)$$

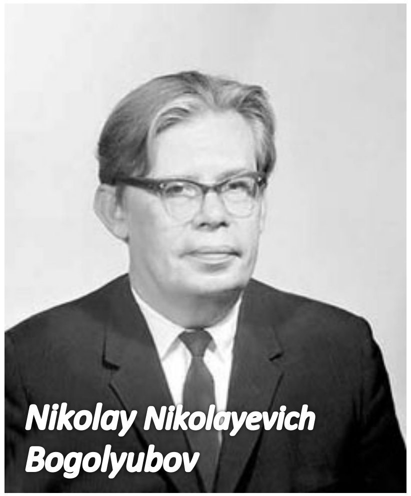
# For more info, look at these guys' work…

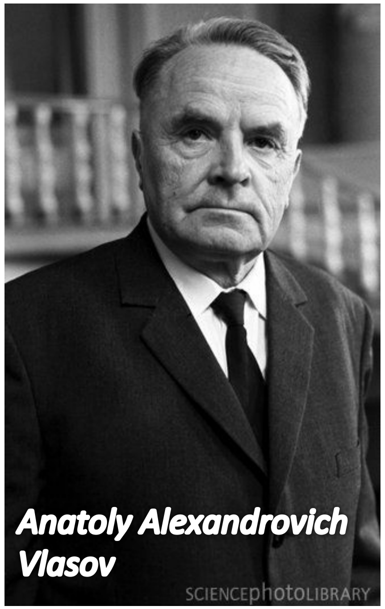










## Euler-Poincaré formulation of Vlasov-type equations

- Consider a Hamiltonian system $(\mathbf{q}(t), \mathbf{p}(t))$ and denote its total energy by $H(\mathbf{q}, \mathbf{p})$. Then, consider its ***Lagrangian on phase-space***:

$$L = \mathbf{p}(t) \cdot \dot{\mathbf{q}}(t) - H(\mathbf{q}(t), \mathbf{p}(t))\,,$$

- Let the coordinates $\mathbf{z} := (\mathbf{q}, \mathbf{p})$ *evolve under the action of the diffeomorphism group* $\mathrm{Diff}(\mathbb{R}^{\mathbf{6}})$:

$$\mathbf{z}(t) = \psi_t(\mathbf{z}_0) := (\mathbf{q}_t(\mathbf{z}_0), \mathbf{p}_t(\mathbf{z}_0))\,, \qquad \text{where} \qquad \psi_t \in \mathrm{Diff}(\mathbb{R}^6)$$

  and the subscript $t$ keeps track of the time dependence.

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  and the subscript $t$ keeps track of the time dependence.

- Set the initial condition $\mathbf{z}_0=\mathbf{s}$ and replace the above relation in the Lagrangian to obtain

$$\begin{aligned}L&=\mathbf{p}_t(\mathbf{s})\cdot\dot{\mathbf{q}}_t(\mathbf{s})-H(\mathbf{q}_t(\mathbf{s}),\mathbf{p}_t(\mathbf{s}))\\&=\int\Big(\mathbf{p}_t(\mathbf{z}_0)\cdot\dot{\mathbf{q}}_t(\mathbf{z}_0)-H(\mathbf{q}_t(\mathbf{z}_0),\mathbf{p}_t(\mathbf{z}_0))\Big)\,\delta(\mathbf{z}_0-\mathbf{s})\,\mathrm{d}^6\mathbf{z}_0\\&=:L_{f_0}(\psi,\dot{\psi})\end{aligned}$$

- Here, we have defined

$$f_0(\mathbf{z}_0)=\delta(\mathbf{z}_0-\mathbf{s})$$

  This encodes all the information on the initial conditions.

- At this point, we adopt the kinetic theory viewpoint: we allow for an arbitrary $f_0$ and we consider it to be a ***phase-space density function*** for a many-particle system, so that

$$f_0(\mathbf{z}_0)\,\mathrm{d}^6\mathbf{z}_0$$

gives the *number of particles in the phase space volume element* $\mathrm{d}^6\mathbf{z}_0$.

- Then, upon defining

$$\mathcal{L}(\psi,\dot{\psi}) := \mathbf{p}_t(\mathbf{z}_0)\cdot\dot{\mathbf{q}}_t(\mathbf{z}_0) - H(\mathbf{q}_t(\mathbf{z}_0),\mathbf{p}_t(\mathbf{z}_0))\,,$$

the Lagrangian

$$L_{f_0}(\psi,\dot{\psi}) := \int f_0(\mathbf{z}_0)\,\mathcal{L}(\psi,\dot{\psi})\,\mathrm{d}^6\mathbf{z}_0$$

has the ***same structure as the fluid kinetic energy***

$$K_{\rho_0}(\boldsymbol{\eta},\dot{\boldsymbol{\eta}}) = \frac{1}{2}\int_{\mathcal{B}}\rho_0(\mathbf{x}_0)\|\dot{\boldsymbol{\eta}}(\mathbf{x}_0,t)\|^2\mathrm{d}^3\mathbf{x}_0$$

with the only difference that $\mathbf{x}_0\in\mathbb{R}^3$ has to be replaced by $\mathbf{z}_0\in\mathbb{R}^6$ and the diffeomorphism $\boldsymbol{\eta}\in\mathrm{Diff}(\mathbb{R}^3)$ has to be replaced by the diffeomorphism $\psi\in\mathrm{Diff}(\mathbb{R}^6)$.

- Then, we notice there is no symmetry $L_{f_0}(\psi,\dot{\psi}) = L_{f_0}(\psi\circ\psi^{-1},\dot{\psi}\circ\psi^{-1})$ (false!) under $\mathrm{Diff}(\mathbb{R}^6)$, although we observe that inserting the Lagrange-to-Euler map does give us

$$L_{f_0}(\psi,\dot{\psi}) = L_{\boldsymbol{\psi_* f_0}}(\psi\circ\psi^{-1},\dot{\psi}\circ\psi^{-1})$$

- Therefore, we compose with $\psi^{-1}$ to obtain

$$\begin{aligned}
L_{\psi_* f_0}(\psi \circ \psi^{-1}, \dot{\psi} \circ \psi^{-1}) &= \iint \Big( (\mathbf{p} \circ \psi^{-1})(\boldsymbol{\zeta}) \cdot (\dot{\mathbf{q}} \circ \psi^{-1})(\boldsymbol{\zeta}) - H(\boldsymbol{\zeta}) \Big) f_0(\mathbf{z}_0)\, \delta(\boldsymbol{\zeta} - \psi(\mathbf{z}_0))\, \mathrm{d}^6\boldsymbol{\zeta}\, \mathrm{d}^6\mathbf{z}_0 \\
&= \int \Big( \mathbf{p} \cdot \mathbf{u}(\mathbf{q}, \mathbf{p}, t) - H(\mathbf{q}, \mathbf{p}) \Big) f(\mathbf{q}, \mathbf{p}, t)\, \mathrm{d}^6\boldsymbol{\zeta} \\
&=: \ell(\mathbf{X}, f)
\end{aligned}$$

  where $\boldsymbol{\zeta} := (\mathbf{q}, \mathbf{p})$ and we dropped the subscript $t$, since these are Eulerian variables (cf. fluids)

- Here, we defined the vector field $\mathbf{X}$ as

$$\mathbf{X}(\mathbf{q}, \mathbf{p}, t) := (\dot{\psi} \circ \psi^{-1})(\mathbf{q}, \mathbf{p}, t) = \big( (\dot{\mathbf{q}}_t, \dot{\mathbf{p}}_t) \circ \psi^{-1} \big)(\mathbf{q}, \mathbf{p}) := \Big( \overbrace{\mathbf{u}(\mathbf{q}, \mathbf{p}, t)}^{\text{Eulerian velocity}}, \underbrace{\mathbf{f}(\mathbf{q}, \mathbf{p}, t)}_{\text{Eulerian force}} \Big)$$

  while the *Lagrange-to-Euler map reads*

$$f = \int f_0(\mathbf{z}_0)\, \delta(\boldsymbol{\zeta} - \psi(\mathbf{z}_0, t))\, \mathrm{d}^6\mathbf{z}_0$$

- Therefore, we compose with $\psi^{-1}$ to obtain

$$
\begin{aligned}
L_{\psi_* f_0}(\psi\circ\psi^{-1},\dot\psi\circ\psi^{-1}) &= \iint\Big((\mathbf{p}\circ\psi^{-1})(\boldsymbol{\zeta})\cdot(\dot{\mathbf{q}}\circ\psi^{-1})(\boldsymbol{\zeta}) - H(\boldsymbol{\zeta})\Big) f_0(\mathbf{z}_0)\,\delta(\boldsymbol{\zeta}-\psi(\mathbf{z}_0))\,\mathrm{d}^6\boldsymbol{\zeta}\,\mathrm{d}^6\mathbf{z}_0 \\
&= \int\Big(\mathbf{p}\cdot\mathbf{u}(\mathbf{q},\mathbf{p},t) - H(\mathbf{q},\mathbf{p})\Big) f(\mathbf{q},\mathbf{p},t)\,\mathrm{d}^6\boldsymbol{\zeta} \\
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$$

while the Lagrange-to-Euler map reads

$$
f = \int f_0(\mathbf{z}_0)\,\delta(\boldsymbol{\zeta}-\psi(\mathbf{z}_0,t))\,\mathrm{d}^6\mathbf{z}_0 =: \psi_* f_0
$$

- Then, we have the ***Euler-Poincaré equations*** (see fluid example)

$$
\frac{\partial}{\partial t}\frac{\delta\ell}{\delta\mathbf{X}} + (\operatorname{div}\mathbf{X})\frac{\delta\ell}{\delta\mathbf{X}} + (\mathbf{X}\cdot\nabla)\frac{\delta\ell}{\delta\mathbf{X}} + \nabla\mathbf{X}\cdot\frac{\delta\ell}{\delta\mathbf{X}} = f\nabla\frac{\delta\ell}{\delta f}\,, \qquad\qquad \partial_t f + \operatorname{div}(f\mathbf{X}) = 0
$$

where all differential operators act on the phase space $\mathbb{R}^6$, so that $\nabla := \partial/\partial\boldsymbol{\zeta}$

## Euler-Poincaré equations of motion

- In order to find the equations of motion, we compute the functional derivatives

$$\frac{\delta l}{\delta \mathbf{X}} = \big(f\mathbf{p}, 0\big) \qquad \text{and} \qquad \frac{\delta l}{\delta f} = \mathbf{p}\cdot\mathbf{u}(\mathbf{q},\mathbf{p},t) - H(\mathbf{q},\mathbf{p})$$

- Then, projecting the first equation on the last three (Eulerian force) components, we obtain

$$\nabla_{\mathbf{p}}\boldsymbol{u}\cdot\mathbf{p} = \nabla_{\mathbf{p}}\big(\mathbf{p}\cdot\boldsymbol{u} - H(\mathbf{q},\mathbf{p})\big) \quad\Longrightarrow\quad \boldsymbol{u}(\mathbf{q},\mathbf{p},t) = \nabla_{\mathbf{p}}H(\mathbf{q},\mathbf{p})$$

  where $\nabla_{\mathbf{p}} := \partial/\partial\mathbf{p}$ and we made use of the continuity equation to eliminate $f$.

- Moreover, by proceeding analogously, inserting the latter in the projection on the first three (Eulerian velocity) components gives

$$\big(\boldsymbol{u}\cdot\nabla_{\mathbf{q}} + \mathbf{f}\cdot\nabla_{\mathbf{p}}\big)\mathbf{p} + \nabla_{\mathbf{q}}\boldsymbol{u}\cdot\mathbf{p} = \nabla_{\mathbf{q}}\big(\mathbf{p}\cdot\boldsymbol{u} - H(\mathbf{q},\mathbf{p})\big) \quad\Longrightarrow\quad \mathbf{f} = -\nabla_{\mathbf{q}}H(\mathbf{q},\mathbf{p})$$

  so that we obtain the ***Hamiltonian vector field***

$$\mathbf{X}(\mathbf{q},\mathbf{p}) = \big(\nabla_{\mathbf{p}}H(\mathbf{q},\mathbf{p}), -\nabla_{\mathbf{q}}H(\mathbf{q},\mathbf{p})\big) \qquad \text{such that} \qquad \operatorname{div}\mathbf{X} = 0$$

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- In conclusion, we obtain the Vlasov equation for an arbitrary Hamiltonian

$$\partial_t f + \nabla_{\mathbf{p}}H(\mathbf{q},\mathbf{p})\cdot\nabla_{\mathbf{q}}f - \nabla_{\mathbf{q}}H(\mathbf{q},\mathbf{p})\cdot\nabla_{\mathbf{p}}f = 0 \qquad \text{or, equivalently} \qquad \boxed{\partial_t f + \{H, f\} = 0}$$

## The Poisson-Vlasov system for electrostatic plasmas (I)

- Notice that so far *we did NOT consider any collective feature, i.e. interactions among particles are invisible* in this treatment. All we started with is single particle dynamics!

- However, let us now consider the following Hamiltonian for *one charge moving in an ensemble of identical charged particles* (assume $m = q = 1$ for simplicity)

$$H = \frac{1}{2}|\mathbf{p}|^2 + \varphi(\mathbf{q})$$

  where $\varphi(\mathbf{q})$ is the electrostatic potential so that $\mathbf{E} = -\nabla\varphi$.

- ***Gauß' law*** for electrostatics gives (upon setting the dielectric constant to 1)

$$\nabla \cdot \mathbf{E} = \rho$$

  where $\rho(\mathbf{q}, t)$ is the ***charge density of the system***

- Then, we need to express the charge density $\rho$ (i.e. the same as the particle density since $q = 1$) on the configuration space in terms of the particle density $f$ on phase space. This is done by ***projecting out (averaging) all the information about the particle momentum***, that is

$$\rho(\mathbf{q}, t) = \int f(\mathbf{q}, \mathbf{p}, t)\, \mathrm{d}^3\mathbf{p}$$

- Then, Vlasov dynamics must be coupled to Gauß' law: the ***Poisson-Vlasov system*** reads

$$\partial_t f + \mathbf{p} \cdot \nabla_{\mathbf{q}} f - \nabla \varphi \cdot \nabla_{\mathbf{p}} f = 0\,, \qquad \Delta \varphi(\mathbf{q}, t) = -\int f(\mathbf{q}, \mathbf{p}, t)\, \mathrm{d}^3\mathbf{p}$$

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- Upon inverting the Laplace operator, the ***Poisson-Vlasov Lagrangian*** can be written as

$$\ell(\mathbf{X},f) = \int\!\Big(\mathbf{p}\cdot\mathbf{u}(\mathbf{q},\mathbf{p},t) - \frac{1}{2}\,|\mathbf{p}|^2 + \frac{1}{2}\,\Delta^{-1}\!\int f(\mathbf{q},\mathbf{p},t)\,\mathrm{d}^3\mathbf{p}\Big) f(\mathbf{q},\mathbf{p},t)\,\mathrm{d}^6\boldsymbol{\zeta}\,,$$

where the last term involves a factor $1/2$ to avoid double counting.

- Then, the Euler-Poincaré equations result in a ***nonlocal kinetic equation***, i.e.

$$\mathbf{X} = \left(\mathbf{p},\nabla\!\left(\Delta^{-1}\!\int f\,\mathrm{d}^3\mathbf{p}\right)\right)\,, \qquad \partial_t f + \mathbf{p}\cdot\nabla_{\mathbf{q}} f + \nabla\!\left(\Delta^{-1}\!\int f\,\mathrm{d}^3\mathbf{p}\right)\cdot\nabla_{\mathbf{p}} f = 0$$

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## Remarks

- Notice that the ***factor 1/2 was inserted*** **ad hoc** in the expression of the Lagrangian.
- Moreover, Hamilton's principle ***does NOT return Gauß' law in a natural fashion***. Rather, Gauß' law has to be known *a priori*.

*Can we solve these problems? Is there a better treatment?*

## The Poisson-Vlasov system for electrostatic plasmas (II)

- Previous problems can be solved by letting $\varphi$ ***be a Lagrangian variable*** on its own right.
- In order to understand how this works, let us write

$$L_{f_0}(\psi,\dot\psi,\varphi,\dot\varphi) := \int f_0(\mathbf{z}_0)\left(\mathbf{p}_t(\mathbf{z}_0)\cdot\dot{\mathbf{q}}_t(\mathbf{z}_0) - \frac{1}{2}\left\|\mathbf{p}_t(\mathbf{z}_0)\right\|^2 - \varphi(\mathbf{q}_t(\mathbf{z}_0))\right) \mathrm{d}^6\mathbf{z}_0 + \frac{1}{2}\int \left\|\nabla\varphi(\mathbf{r},t)\right\|^2 \mathrm{d}^3\mathbf{r}$$

where the last term is the ***electrostatic energy***, i.e. $1/2\int\left\|\mathbf{E}(\mathbf{r},t)\right\|^2\mathrm{d}^3\mathbf{r}$

- At this point, we pass to Eulerian coordinates according to the relation

$$L_{f_0}(\psi,\dot\psi,\varphi,\dot\varphi) = L_{\psi_* f_0}(\psi\circ\psi^{-1},\dot\psi\circ\psi^{-1},\varphi,\dot\varphi) =: \ell(\mathbf{X},f,\varphi,\dot\varphi)$$

where we notice that *the diffeomorphism* $\psi\in\mathrm{Diff}(\mathbb{R}^6)$ *does NOT act on the electrostatic potential* $\varphi\in C^\infty(\mathbb{R}^3)$.

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where we notice that *the diffeomorphism* $\psi \in \mathrm{Diff}(\mathbb{R}^6)$ *does NOT act on the electrostatic potential* $\varphi \in C^\infty(\mathbb{R}^3)$.

- Upon integrating by parts the electrostatic energy term, the reduced Lagrangian is

$$\ell(\mathbf{X}, f, \varphi, \dot\varphi) = \int\left(\mathbf{p}\cdot\mathbf{u}(\mathbf{q},\mathbf{p},t) - \frac{1}{2}|\mathbf{p}|^2 - \varphi(\mathbf{q})\right)f(\mathbf{q},\mathbf{p},t)\,\mathrm{d}^6\boldsymbol{\zeta} - \frac{1}{2}\int\varphi(\mathbf{q},t)\Delta\varphi(\mathbf{q},t)\mathrm{d}^3\mathbf{q}$$

- Then, Hamilton's principle requires the variations

$$\delta\mathbf{X} = \partial_t\mathbf{W} + (\mathbf{X}\cdot\nabla)\mathbf{W} - (\mathbf{W}\cdot\nabla)\mathbf{X}\,, \qquad \delta f = -\,\mathrm{div}(f\mathbf{W})\,, \qquad \delta\varphi \ \textit{arbitrary}$$

where $\mathbf{W} := \delta\psi\circ\psi^{-1}$.

- At this point, it is easy to see that the variational principle

$$\delta \int_{t_1}^{t_2} \ell(\mathbf{X}, f, \varphi, \dot{\varphi}) = 0$$

yields a coupled system of Euler-Poincaré (for $(\mathbf{X}, f)$) and Euler-Lagrange (for $(\varphi, \dot{\varphi})$) equations:

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \mathbf{X}} + (\operatorname{div} \mathbf{X})\,\frac{\delta \ell}{\delta \mathbf{X}} + (\mathbf{X} \cdot \nabla)\frac{\delta \ell}{\delta \mathbf{X}} + \nabla \mathbf{X} \cdot \frac{\delta \ell}{\delta \mathbf{X}} = f \nabla \frac{\delta \ell}{\delta f} \\
&\partial_t f + \operatorname{div}(f\mathbf{X}) = 0 \\
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta \ell}{\delta \dot{\varphi}} - \frac{\delta \ell}{\delta \varphi} = 0
\end{aligned}$$

- At this point, it is easy to see that the variational principle

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$$\partial_t f+\operatorname{div}(f\mathbf{X})=0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\ell}{\delta\dot\varphi}-\frac{\delta\ell}{\delta\varphi}=0$$

- Therefore, while the first two (Euler-Poincaré) equations yield

$$\mathbf{X}=(\mathbf{p},-\nabla_{\mathbf{q}}\varphi)\;,\qquad \partial_t f+\mathbf{p}\cdot\nabla_{\mathbf{q}}f-\nabla\varphi\cdot\nabla_{\mathbf{p}}f=0\,,$$

the third (Euler-Lagrange) equation gives the coupling:

$$0=-\frac{\delta\ell}{\delta\varphi}=\Delta\varphi+\int f\,\mathrm{d}^3\mathbf{p}\qquad\Longrightarrow\qquad -\Delta\varphi=\int f\,\mathrm{d}^3\mathbf{p}$$

- `http://socrates.berkeley.edu/%7Efajans/2Dfluid/2Dfluid.html`
`http://sdpha2.ucsd.edu/index.html`

## Kelvin-Noether theorem

- For kinetic theories, the Kelvin-Noether theorem reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\psi(\gamma_0)}\frac{1}{f}\frac{\delta\ell}{\delta\mathbf{X}}\cdot\mathrm{d}\boldsymbol{\zeta}=0$$

  where $\gamma_0$ is an arbitrary loop in phase space

- In the cases treated above, one has

$$\frac{1}{f}\frac{\delta\ell}{\delta\mathbf{X}}\cdot\mathrm{d}\boldsymbol{\zeta}=(\mathbf{p},0)\cdot(\mathrm{d}\mathbf{q},\mathrm{d}\mathbf{p})=\mathbf{p}\cdot\mathrm{d}\mathbf{q}$$

- In conclusion, one obtains that the following quantity is constant:

$$\oint_{\psi(\gamma_0)}\mathbf{p}\cdot\mathrm{d}\mathbf{q}\,,$$

  which is nothing else than ***Poincaré integral invariant*** of Hamiltonian dynamics!

## *Kinetic approaches are expensive!*


**Better forget details? Fluid approaches are often convenient!**

## 4 Fluid dynamics from kinetic theory – the moment method

- One of the greatest success of kinetic theory was that it could reproduce fluid motion.
- Indeed, fluid motion arises by considering the hierarchy for the quantities

$$A_n(\mathbf{q},t) = \int \mathbf{p}^{\otimes n}\, f(\mathbf{q},\mathbf{p},t)\, \mathrm{d}^3\mathbf{p}$$

These symmetric tensors are called ***kinetic moments*** of the phase-space density $f$.

KINETIC THEORY ON PHASE SPACE

$$\partial_t f + p\cdot\partial_x f - \nabla\Phi\cdot\partial_p f = 0$$

*Micro-scales govern dynamics*

$$\xrightarrow[\int p^n f\, dp]{\textit{Moments}}$$

FLUID MODELS

$$\rho\left(\partial_t u + u\cdot\nabla u\right) = -\nabla p,$$

$$\partial_t\rho + \nabla\cdot\left(\rho u\right) = 0$$

*Neglects small scales*

- The dynamics of each moment involves higher-order moments in an ***endless hierarchy***, which is subject of ongoing research in mathematical analysis.
- Fluid variables arise from the first two moments

$$(\rho,\mathbf{m}) = \left(\int f\, \mathrm{d}^3\mathbf{p},\, \int \mathbf{p} f\, \mathrm{d}^3\mathbf{p}\right)$$

The first is the ***fluid density*** $\rho$, while the second is the ***fluid momentum*** $\mathbf{m} = \rho\boldsymbol{v}$.

*How can we formulate the moment method in geometric mechanics?*

## Euler-Poincaré formulation of moment dynamics

- Consider our Vlasov Lagrangian
$$\ell(\mathbf{X}, f) = \int \Big(\mathbf{p}\cdot\mathbf{u}(\mathbf{q},\mathbf{p},t) - H_f(\mathbf{q},\mathbf{p})\Big) f(\mathbf{q},\mathbf{p},t)\,\mathrm{d}^6\boldsymbol{\zeta}$$
where the subscript in $H_f$ reminds that the Hamiltonian $H$ can depend on the density $f$.
- For the purpose of simplicity, we rectrict to consider a 2D phase space, i.e. $\boldsymbol{\zeta} = (q,p) \in \mathbb{R}^2$:
$$\ell(\mathbf{X}, f) = \int \Big(pu(q,p,t) - H_f(q,p)\Big) f(q,p,t)\,\mathrm{d}^2\boldsymbol{\zeta}$$
- Now we assume that both $\mathbf{X}$ and $H_f$ are analytical in the momentum coordinate:
$$\mathbf{X}(q,p,t) = (u(q,p,t), \mathrm{f}(q,p,t)) = \sum_{n=0}^{\infty} (p^n U_n(q,t), p^n F_n(q,t)) =: \sum_{n=0}^{\infty} p^n \boldsymbol{\Xi}_n(q,t)$$
Analogously, $H_f(q,p) = \sum_{n=0}^{\infty} p^n \beta_n(q)$.
- Replacing in the Lagrangian yields ***the moment Lagrangian***
$$\begin{aligned}\ell(\{\boldsymbol{\Xi}\}, f) =& \sum_{n=0} \int \Big(p^{n+1} U_n(q,t) - p^n \beta_n(q)\Big) f(q,p,t)\,\mathrm{d}^2\boldsymbol{\zeta} \\ =& \sum_{n=0} \int \Big(A_{n+1} U_n(q,t) - A_n \beta_n(q)\Big)\,\mathrm{d}q =: \ell(\{\boldsymbol{\Xi}\}, \{A\})\end{aligned}$$

- At this point, the problem is taking variations

$$\delta \boldsymbol{\Xi} = \ldots\,, \qquad \delta A_n = \ldots$$

- As for the sequence $\{\boldsymbol{\Xi}\}$, we know that

$$\delta \mathbf{X} = \partial_t \mathbf{W} + (\mathbf{X}\cdot\nabla)\mathbf{W} - (\mathbf{W}\cdot\nabla)\mathbf{X}$$

and it is convenient to expand as

$$\mathbf{W}(q,p,t) = \left(\sum_{n=0}^{\infty} p^n V_n(q,t), \sum_{n=0}^{\infty} p^n \Phi_n(q,t)\right) =: \sum_{n=0}^{\infty} p^n \boldsymbol{\Omega}_n(q,t)$$

- Then, upon denoting $\partial/\partial q$ by prime, we have

$$\sum_{n=0}^{\infty} p^n \delta\boldsymbol{\Xi}_n = \sum_{n=0}^{\infty} p^n \dot{\boldsymbol{\Omega}}_n + \sum_{n,m=0}^{\infty} p^{n+m}\left(U_n \boldsymbol{\Omega}_n' - V_m \boldsymbol{\Xi}_n'\right) + \sum_{n,m=0}^{\infty} p^{n+m-1}\left(m F_n \boldsymbol{\Omega}_m - n\Phi_m \boldsymbol{\Xi}_n\right)$$

so that

$$\sum_{n=0}^{\infty} p^n \delta U_n = \sum_{n=0}^{\infty} p^n \dot{V}_n + \sum_{n,m=0}^{\infty} p^{n+m}\left(U_n V_n' - V_m U_n'\right) + \sum_{n,m=0}^{\infty} p^{n+m-1}\left(m F_n V_m - n\Phi_m U_n\right)$$

- Also, upon expanding $\mathbf{W}$, one has

$$\delta f = -\operatorname{div}(f\mathbf{W}) \overset{\int \mathrm{d}p\, p^n \ldots}{\overbrace{\Longrightarrow}} \delta A_n = \sum_{m=0}^{\infty}\Big(n\Phi_m A_{n+m-1} - (V_m A_{n+m})'\Big)$$

- Replacing these relations in the variational principle yields a very complicated dynamical system. The first equation is obtained by setting the variation in $\delta\{\Phi\}$ to zero, i.e.

$$\sum_n \left(A_{n+m}U_n - nA_{n+m-1}\beta_n\right) = 0 \qquad \forall m \in \mathbb{N}$$

which is the $m$th moment of our previous equation $U = \partial H/\partial p$, as seen by expanding the $A$'s:

$$\sum_n \int p^m f\left(u - \frac{\partial H}{\partial p}\right)\mathrm{d}p = 0$$

- Analogously, upon setting the variation in $\delta\{V\}$ to zero, one has $F = -\partial H/\partial q$ in the form

$$\sum_n A_{n+m-1}\left(F_n + \beta_n'\right) = 0 \qquad \forall m \in \mathbb{N}\,,$$

where we have used the $n$th moment of the auxiliary equation as follows

$$\partial_t f = -\operatorname{div}(f\mathbf{X}) \overset{\int \mathrm{d}p\, p^n \ldots}{\overbrace{\Longrightarrow}} \partial_t A_n = \sum_{m=0}^{\infty}\Big(nF_m A_{n+m-1} - (U_m A_{n+m})'\Big)$$

- Eventually, the moment hierarchy becomes

$$\boxed{\partial_t A_n = -\sum_{m=0}^{\infty}\Big(n\beta_m' A_{n+m-1} + m(A_{n+m-1}\beta_m)'\Big)\,, \qquad \text{where} \qquad \beta_n(q) := \frac{1}{n!}\frac{\partial H}{\partial p}\bigg|_{p=0}}$$

## Back to fluid dynamics

*Let us now go back to fluid dynamics...*

- For the particular case of $H = p^2/2 + \varphi(q)$, we have $\beta_n = 0$, except $\beta_2 = 1/2$ and $\beta_0 = \varphi(q)$:

$$\begin{aligned}\partial_t A_n &= -\sum_{m=0}^{\infty}\Big((n+m)\beta'_m A_{n+m-1} + mA'_{n+m-1}\beta_m\Big)\\ &= -\,A'_{n+1} - n\varphi' A_{n-1}\end{aligned}$$

  As we can see the $n$th equation involves the $(n+1)$th moment, in an endless hierarchy.

- The fluid equations for $\rho = A_0$ and $m = A_1 = \rho v$ read

$$\boxed{\partial_t m = -\mathbb{P}' - \rho\varphi'\,, \qquad \partial_t \rho = -m'}$$

  where $\mathbb{P} := A_2$ is the ***absolute stress tensor***.

- The momentum equation acquires a more familiar form upon introducing the relative pressure

$$\widetilde{\mathbb{P}}(q,t) = \int\left(p - \frac{A_1}{A_0}\right)^2 f\,\mathrm{d}p = \int (p - v(q,t))^2 f(q,p,t)\mathrm{d}p$$

  which is measured in the reference of the mean flow, with velocity $v = A_1/A_0$

- Upon using the relation
$$\widetilde{\mathbb{P}}' = \mathbb{P}' - A_1 v' - v A_1' \,,$$
the fluid equation for the momentum $m = A_1 = \rho v$ becomes
$$\rho(\partial_t v + v v') = -\widetilde{\mathbb{P}}\,' - \rho\varphi'$$
where $\widetilde{\mathbb{P}}$ possesses our own equation of motion.

- The system is not closed because it involves infinitely many dynamical variables. A ***moment closure*** is an approximation that expresses higher moments in terms of $A_0$ and $A_1$.

- The ***cold-plasma closure*** arises from the relation
$$f(q,p,t) = A_0(q,t)\,\delta\left(p - \frac{A_1(q,t)}{A_0(q,t)}\right) \quad \Longrightarrow \quad A_n(q,t) = A_0(q,t)\left(\frac{A_1(q,t)}{A_0(q,t)}\right)^n$$
which yields *pressureless fluid dynamics*. Replacing $\delta$ by a step gives the ***waterbag closure***.

## Physical arguments

- In physics, closing the system requires an ***equation of state*** for the eigenvalues of $\widetilde{\mathbb{P}}$, which are related to temperature and other thermodynamic quantities.
- In the isotropic case, the eigenvalues coincide with the fluid scalar pressure, which is then expressed in terms of a given *internal energy* $\mathcal{U}(\rho)$.
- The step above (equation of state) is where the ***geometric approach reaches its limits*** to leave more space for physical reasoning.
- Geometrically, the moment $A_1$ ***generates fluid motion***. When higher order moments are included, these take the overall dynamics *beyond ordinary fluid dynamics*.
- All this requires no dissipation. ***Viscosity*** comes about when dealing with collisional kinetic theories (Boltzmann). Collisions determine another limit of the geometric approach.

## More on Vlasov moments: application to particle accelerators

- Particle accelerators accelerate a particle beam, possessing its own density on phase space
- The moment method is used in this context too, although another type of moments is involved
- ***Statistical moments*** are symmetric tensors defined as

$$X_n(t) := \frac{1}{n!}\int \boldsymbol{\zeta}^{\otimes n} f(\boldsymbol{\zeta}, t)\, \mathrm{d}^6\boldsymbol{\zeta}$$

- They obey another endless hierarchy, which is now a hierarchy of ODE's
- The simplest description involves the moment $X_2$. This is simply a symmetric matrix and it is related to the group of symplectic matrices governing ***linear Hamiltonian dynamics***!
- The Euler-Poincaré formulation of statistical moments can be found again by using ***Taylor expansions in $\boldsymbol{\zeta}$***, as opposed to the Taylor expansions in $\mathbf{p}$ generating kinetic moments

# 5 Elements of differential geometry

## Dual spaces and covariant vectors

**Definition:** Let $V$ be a finite dimensional vector space on $\mathbb{R}$. The dual vector space of $V$ is defined as

$$V^* := L(V, \mathbb{R})$$

where $L$ denotes the space of *linear functionals* on $V$ (ie. linear mappings from $V$ to the real line). Elements of $V^*$ are called *covectors* (covariant-vectors).

Hence

$$\boldsymbol{\mu}(\mathbf{v}) \in \mathbb{R} \qquad\qquad \forall \boldsymbol{\mu} \in V^*, \quad \mathbf{v} \in V$$

**Pairing notation:** The operation $\boldsymbol{\mu}(\mathbf{v})$ defines a non-degenerate bilinear operation

$$\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}\,, \qquad (\boldsymbol{\mu}, \mathbf{v}) \mapsto \langle \boldsymbol{\mu}, \mathbf{v} \rangle := \boldsymbol{\mu}(\mathbf{v}) \in \mathbb{R}$$

This operation $\langle \cdot, \cdot \rangle$ between a covector and a vector is called *pairing*. Essentially, $\langle \cdot, \cdot \rangle$ means applying a linear function from $V^*$ to a vector in $V$ to obtain an output in $\mathbb{R}$.

**Proposition:** Let $(\mathbf{e}_1, ..., \mathbf{e}_m)$ be a basis in $V$ (where $dim(V) = m$). Then the relation

$$\langle \mathbf{e}^i, \mathbf{e}_j \rangle = \mathbf{e}^i(\mathbf{e}_j) = \delta^i{}_j$$

defines a basis of $V^*$ called the **dual basis**. Therefore $dim(V^*) = dim(V)$

Observe subscripts are used to index elements of $V$ whilst superscripts are used for those of $V^*$. To visualise the equivalence of dimensions, note that a linear function acting on the column $\mathbf{v}$ must itself be a row vector of the same length.

Thus, for any vector $\boldsymbol{\mu} \in V^*$ we can write

$$\boldsymbol{\mu} = \sum_{i=1}^{m} \mu_i \mathbf{e}^i \quad \Longrightarrow \quad \boldsymbol{\mu}(\mathbf{e}_j) \ = \ \sum_{i=1}^{m} \mu_i \mathbf{e}^i(\mathbf{e}_j) = \sum_{i=1}^{m} \mu_i \delta^i{}_j = \mu_j$$

Similarly $\forall \mathbf{u} \in V$

$$\mathbf{u} = \sum_i u^i \mathbf{e}_i \quad \Longrightarrow \quad \mathbf{e}^j(\mathbf{u}) = \sum_i u^i \mathbf{e}^j(\mathbf{e}_i) = \sum_i u^i \delta^j{}_i = u^j$$

where upper indexes are used to denote components of a vector so that they are distinguished from basis elements (carrying lower indexes).

**Remark 1 (Practical rule for real spaces: dual vectors are transposes)** *As a general rule, given a vector space $V$ of matrices with real coefficients, the dual elements in the corresponding dual space $V^*$ may be constructed by taking the transpose of the matrices in $V$. For example, consider the space of $n \times 1$ matrices with real coefficients (a.k.a. column vectors): the dual space is the space of row vectors. Also, consider* ***the space*** $\mathfrak{so}(n, \mathbb{R})$ ***of real antisymmetric matrices****: taking the transpose of an antisymmetric matrix $A$ yields another antisymmetric matrix $A^T = -A$. Therefore, $\mathfrak{so}(n, \mathbb{R})^* \simeq \mathfrak{so}(n, \mathbb{R})$. A similar argument holds for the space* $\mathrm{Sym}(n, \mathbb{R})$ *of symmetric matrices with real coefficients.*

**Einstein Summation Convention:** The summation symbol $\sum$ will be omitted for repeated indexes. More precisely this can occur if the same index of summation appears twice in any term; once as a subscript and once as a superscript. For example

$$\sum_{i=1}^{m} v^i \mu_i = v^i \mu_i \qquad \sum_i \mu_i \mathbf{e}^i = \mu_i \mathbf{e}^i \qquad \sum_i A^j{}_i v^i = A^j{}_i v^i$$

**Theorem:** (Canonical isomorphism) Let $V$ be a vector space with $dim(V) < \infty$. There exists a canonical (base-independent) isomorphism between $V$ and $(V^*)^* := V^{**}$ given by

$$(.)^D : V \to V^{**}$$

where $(\mathbf{v})^D(\boldsymbol{\mu}) := \boldsymbol{\mu}(\mathbf{v})$, ie. $\langle (\mathbf{v})^D, \boldsymbol{\mu} \rangle := \langle \boldsymbol{\mu}, \mathbf{v} \rangle$, $\forall \boldsymbol{\mu} \in V^*$, $\mathbf{v} \in V$.

## Manifolds

In our work on the rigid body and fluids (or kinetic theory), our Lagrangian coordinates were a time-dependent orthogonal matrix $\mathcal{R}(t)$ and a time-dependent diffeomorphism $\boldsymbol{\eta}_t$, respectively.

Although we could work with orthogonal matrices (or diffeomorphisms) by hand, their geometric properties are completely mysterious. For simple rotations $\mathcal{R}(t)$, three main problems appeared:

1. ***Orthogonal matrices are not a vector space***: the sum of two orthogonal matrices is in general not orthogonal. So what kind of entity is $\mathcal{R}$?

2. ***The time derivative $\dot{\mathcal{R}}$ is not orthogonal*** and the same question arises.

3. ***The matrix $\mathcal{R}^{-1}\dot{\mathcal{R}}$ is skew-symmetric and thus it belongs to a vector space***. However, $\mathcal{R}^{-1}\dot{\mathcal{R}}$ has some nice extra properties which lead to the matrix commutator $[\cdot,\cdot]$. Then, we wonder what interesting geometric properties characterize $\mathcal{R}^{-1}\dot{\mathcal{R}}$?

It is clear that our mathematical knowledge appears somewhat limited and new concepts have to be introduced. The answers to the above points will turn out to be

1. $\mathcal{R}$ belongs to a manifold which is also a Lie group.

2. $\dot{\mathcal{R}}$ is a tangent vector (in a generalized sense) to the same manifold.

3. $\mathcal{R}^{-1}\dot{\mathcal{R}}$ is an element of a Lie algebra which is associated to the Lie group containing $\mathcal{R}$.

**Definition (Smooth manifold):** *A* ***smooth (i.e. differentiable) manifold*** $M$ *is a set of points together with a finite (or countable) set of subsets* $U_\alpha \subset M$ *and one-to-one mappings* $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ *such that*

1. $\bigcup_\alpha U_\alpha = M$.
2. *For every nonempty intersection* $U_\alpha \cap U_\beta$*, the set* $\phi_\alpha(U_\alpha \cap U_\beta)$ *is an open subset of* $\mathbb{R}^n$ *and the one-to-one mapping* $\phi_\beta \circ \phi_\alpha^{-1}$ *is a smooth function on* $\phi_\alpha(U_\alpha \cap U_\beta)$.

## Nomenclature

- The sets $U_\alpha$ are called **charts**.
- The mappings $\phi_\alpha$ are called **local coordinates**.
- The superscript $n$ on $\mathbb{R}^n$ identifies the **dimension** of the manifold.

As a general guideline, we say that *a smooth n-dimensional manifold is a set that is* ***locally isomorphic to*** $\mathbb{R}^n$, so that the general rules of calculus can be applied in the local coordinates.

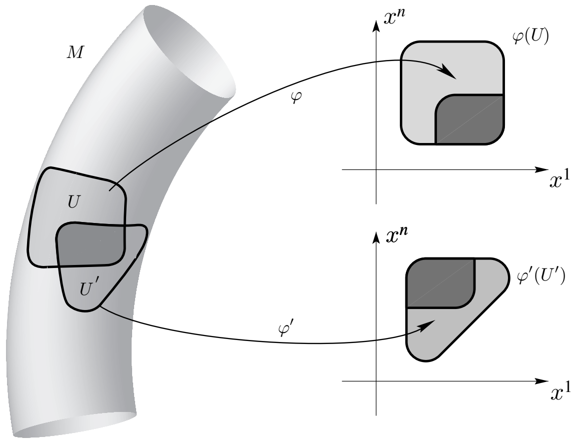


Figure 2: A manifold, two of its charts and their local coordinate systems. (Courtesy of T.S. Ratiu. From *J.E. Marsden, T.S. Ratiu, Introduction to Mechanics and Symmetry, Springer, 2004*).

## Examples

1. $\mathbb{R}^n$ is a manifold consisting of one chart. Any finite-dimensional vector space $V$ is a manifold.

2. Surfaces in $\mathbb{R}^3$ (e.g. $S^2$ sphere) are 2-dimensional manifolds when they appear as level sets:
$$M = \{\mathbf{x} \in \mathbb{R}^3 \mid f(\mathbf{x}) = 0\}.$$

3. The vector space $M_{n\times m}$ of $n \times m$ matrices over $\mathbb{R}$ is a manifold with dimension $nm$.

4. The set $\mathrm{SL}(m)$ of (special) changes of basis is given by square invertible $n \times n$ matrices with unit determinant. This is an $(n^2 - 1)$-dimensional manifold appearing as a level set
$$\{\mathcal{E} \in M_{n\times n} \mid \det \mathcal{E} = 1\}.$$

5. The set $\mathrm{SO}(3)$ of special orthogonal matrices is also a manifold of dimension 3.

## Tangent vectors & tangent spaces

The velocity $\dot{q}(t)$ of some curve $q(t) \in M$ is a **tangent vector** to $M$ based at the point $q(t)$.

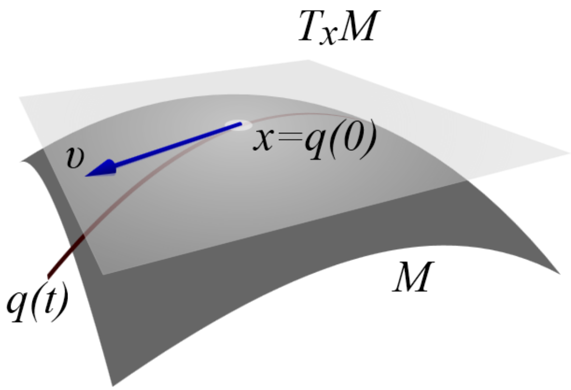


Figure 3: A trajectory $q(t)$ on a manifold $M$ and its tangent vector $\dot{q}(0)$ at the point $x = q(0)$.

The set of all tangent vectors based at a given point $\mathbf{x}$, corresponding to all possible paths in $M$ through $\mathbf{x}$, is the **tangent space** to $M$ at $\mathbf{x}$. [1]

**Definition (tangent vector)**: *Let $M$ be a manifold and $q : \mathbb{R} \to M$ an arbitrary path such that $q(0) = x \in M$. Then $v := \dot{q}(0)$ is a* ***tangent vector*** *to $M$ at the base point $x \in M$.*

[1] Strictly speaking, the following definition applies only to the case when the differentiable manifold $M$ is a sub-manifold of $\mathbb{R}^n$, e.g. it is given as a level set (see example 2 above). The generalization of this definition to arbitrary manifolds requires using local coordinate charts and equivalence classes and it can be found in textbooks.

The set of all tangent vectors at x is called the **tangent space** at $x \in M$, denoted $T_xM$:

**Definition (tangent space):** *The tangent space to $M$ at the point $x \in M$ is defined as*

$$T_xM = \{v \colon v = \dot{q}(0) \textit{ for an arbitrary path } q(t) \in M \textit{ with } q(0) = x\}$$

**Property:** Every tangent space is a vector space of ***dimension*** $\dim M$.

**Remark:** Notice that if $v(t)$ is a curve in a vector space $V$, its derivative $\dot{v}(t)$ still belongs to $V$. It is easy to see this when $V = \mathbb{R}^n$. Therefore, $T_vV \simeq V$.

Upon collecting all the tangent spaces at all base points on $M$, we obtain another manifold:

**Definition (tangent bundle):** *The tangent bundle $TM$ of a smooth manifold $M$ is the smooth manifold defined by the disjoint union of the tangent spaces to $M$ at the points $x \in M$, i.e.*

$$TM = \bigcup_{x \in M} T_xM$$

*Points in the tangent bundle are denoted by $(x, v) \in TM$.*

**Property:** The tangent bundle is a manifold of dimension $2 \dim M$.

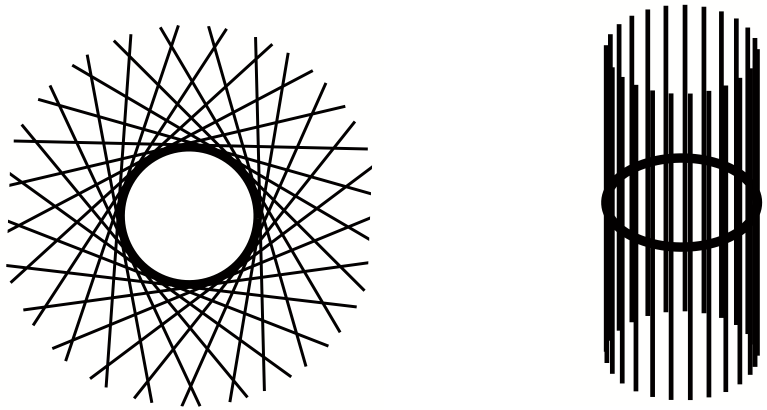

Figure 4: The tangent bundle of the circle $S^1$ is obtained by considering all the tangent spaces (left), and joining them together in a smooth and non-overlapping manner (right).

In order to understand how this concept emerges, consider a curve $q(t) \in M$. The vectors $\dot{q}(t)$ belong to different spaces at different times, i.e.

$$\dot{q}(t_1) \in T_{q(t_1)}M \neq T_{q(t_2)}M \ni \dot{q}(t_2).$$

However, both $(q(t_1), \dot{q}(t_1))$ and $(q(t_2), \dot{q}(t_2))$ *do* belong to the *same* tangent bundle $TM$ since

$$(q(t), \dot{q}(t)) \in TM \qquad \forall t \in \mathbb{R}.$$

**Example** (Lagrangians). **Lagrangians** are defined as functions on the tangent bundle:

$$L = L(q, \dot{q})$$
$$L : TM \to \mathbb{R}$$

## Cotangent bundle

Once the notion of a tangent vector space has been introduced, it is natural to take its dual.

**Definition (cotangent space):** *At each point $x \in M$, the dual space $T_x^* M := (T_x M)^*$ is called the **cotangent space**.*

**Definition (cotangent bundle):** *The disjoint union*

$$T^*M := \bigcup_{x \in M} T_x^* M$$

*is called the **cotangent bundle**.*

**Example** (Hamiltonians). Take a curve $q(t) \in M$, then $\dot{q}(t)$ is tangent at $q(t)$ (i.e. $\dot{q}(t) \in T_{q(t)}M$). A covector in $T^*_{q(t)}M$ is denoted by $p(t)$ and

$$(q(t), p(t)) \in T^*M.$$

A **Hamiltonian** is a function $H : T^*M \to \mathbb{R}$.

## Vector fields & differential one-forms

Consider a fluid flowing in physical space: to each point $\mathbf{x}$, we associate the fluid velocity $\mathbf{u}(\mathbf{x})$. Being a velocity, one expects $\mathbf{u}(\mathbf{x})$ to be a tangent vector. However, the tangent vector $\mathbf{u}(\mathbf{x})$ depends on $\mathbf{x}$. This is an example of a **vector field**[2].

**Definition (vector field):** *A* ***vector field*** *on $M$ is a map*

$$X : M \to TM$$
$$X : x \mapsto X(x) \in T_xM.$$

**Remark**. The space of vector fields on $M$ is a vector space, denoted by $\mathfrak{X}(M)$.

The dual concept of a vector field is the following

**Definition (differential one-form):** *A* ***differential one-form*** *on $M$ is a map*

$$\alpha : M \to T^*M$$
$$\alpha : x \mapsto \alpha(x) \in T_x^*M.$$

[2]Notice that the following is a heuristic definition. A more precise definition requires *sections*...

**Example: the differential**. Take a curve $q(t) \in M$ and a function $f : M \to \mathbb{R}$. For simplicity, take $M = \mathbb{R}^n$. Now compute

$$\frac{\mathrm{d}}{\mathrm{d}t} f(q(t)) = \frac{\partial f}{\partial q^i} \frac{\mathrm{d}q^i}{\mathrm{d}t} = \boldsymbol{\nabla} f \cdot \dot{\mathbf{q}}.$$

However, $\dot{\mathbf{q}}$ is a tangent vector. Thus it can only be paired with a covector to write

$$\frac{\mathrm{d}}{\mathrm{d}t} f(q(t)) = \langle \mathrm{d}f, \dot{q} \rangle,$$

where we have replaced the gradient $\boldsymbol{\nabla} f$ by the differential $\mathrm{d}f = \boldsymbol{\nabla} f \cdot \mathbf{e}^i$ in order to be base-independent. In the more general case of an arbitrary manifold $M \neq \mathbb{R}^n$, the differential reads

$$\mathrm{d}f = \frac{\partial f}{\partial q^i} \mathrm{d}q^i.$$

At this point, we ***interpret*** $\{\boldsymbol{d}q^i\}$ ***as the basis on*** $T_q^* M$***, so that*** $\boldsymbol{d}f$ ***is a differential one-form on*** $M$ ***with components*** $\partial f / \partial q^i$.

| |
|---|
| *The differential is the main example of a differential one-form.* |

**Remark (The basis of tangent and cotangent spaces):** It can be shown that vectors in $T_qM$ are equivalent to derivations, so that

$$\dot{q} = \dot{q}^a \frac{\partial}{\partial q^a}$$

and $\left\{ \frac{\partial}{\partial q^a} \right\}$ is identified with the natural basis on $T_qM$. Then, since we interpreted $\{\mathrm{d}q^i\}$ as the basis on $T_q^*M$, we write

$$\left\langle \mathrm{d}q^i, \frac{\partial}{\partial q^j} \right\rangle = \delta^i{}_j.$$

## Lie groups

**Definition (group)** A group $G$ is a set of elements possessing

- a binary product $G \times G \to G$ such that
  - the product of $g$ and $h$ is written $gh$;
  - the product is associative: $(gh)k = g(hk)$;
- an identity element, denoted by $e$, such that $ge = eg = g \qquad \forall g \in G$;
- an inverse operation $G \to G$ such that $g^{-1}g = gg^{-1} = e$.

**Definition (Lie group)** A **Lie group** is a smooth manifold which is also a group, so that the binary product and inversion are smooth functions.

**Examples**

- The manifold of invertible square $n \times n$ real matrices is a Lie group denoted by $\mathrm{GL}(n, \mathbb{R})$.
- The manifold of invertible square $n \times n$ matrices with unit determinant is a Lie group denoted by $\mathrm{SL}(n, \mathbb{R})$ and called the **special linear group**.
- The manifold of rotation matrices in $n$ dimensions is a Lie group denoted by $\mathrm{SO}(n, \mathbb{R})$ and called the **special orthogonal group**.

## Lie algebras

If $G$ is a Lie group, then $T_eG$ (the tangent space at the identity) is an interesting vector space possessing a remarkable structure called **Lie algebra** structure.

**Definition (Lie algebra)** A **Lie algebra** is a vector space $V$ endowed with a **commutator** (or *Lie bracket*), that is, a bilinear map

$$[\,\cdot\,,\,\cdot\,] \,:\, V \times V \to V$$

such that

1. $[B, A] = -[A, B] \qquad \forall A, B \in V$ (the commutator is skew-symmetric)
2. $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \quad \forall A, B, C \in V.$ (the **Jacobi identity**)

**Theorem** (without proof) *Let $G$ be a matrix Lie group. Then $T_eG$ is a Lie algebra (denoted by $\mathfrak{g}$), with commutator given by the matrix commutator.*

**Examples**

1. Lie algebra $\mathfrak{gl}(n,\mathbb{R}) := T_e\mathrm{GL}(n,\mathbb{R})$ comprises real square $n \times n$ matrices (with commutator).

2. The Lie algebra $\mathfrak{sl}(n,\mathbb{R}) := T_e\mathrm{SL}(n,\mathbb{R})$ is the vector space of real *traceless* square matrices.

   **Proof**: Take $g(t) \in \mathrm{SL}(n,\mathbb{R})$; then $\det g(t) = 1$. Now take $g(t)$ such that $g(0) = e$ and $\dot{g}(0) = \xi \in \mathfrak{sl}(n,\mathbb{R})$. Then, by using the formula for the derivative of the determinant,

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\det g(t)\Big)\Big|_{t=0} \\
&= \det g(0)\ \mathrm{Tr}\,\big(g(0)^{-1}\dot{g}(0)\big) \\
&= \mathrm{Tr}\ \xi.
\end{aligned}
$$

3. The Lie algebra $\mathfrak{so}(3) = T_e\mathrm{SO}(3)$ is the vector space of skew-symmetric matrices.

**Important properties**: remember the body angular velocity $\hat{\Omega} := \mathcal{R}^{-1}\dot{\mathcal{R}}$ and the fluid velocity $\boldsymbol{u} = \boldsymbol{\eta}_t^{-1}\dot{\boldsymbol{\eta}}$? Turns out they are Lie algebra elements! Indeed, take a path $g(t) \in G$, then

$$
\dot{g} \in T_gG \begin{cases} \Rightarrow & (1)\quad g^{-1}\dot{g} \in T_eG \\ \Rightarrow & (2)\quad \dot{g}g^{-1} \in T_eG \end{cases}
$$

# 6 Lie group actions and Lie algebra actions

## Actions of a Lie group on itself and its Lie algebra

**Definition (conjugation action)** Let $g \in G$. Then the operation

$$\begin{aligned} I_g : G &\to G \\ h &\mapsto ghg^{-1} \qquad \forall h \in G \end{aligned}$$

is called the **conjugation action** of $G$ on itself.

Take an arbitrary curve $h(t) \in G$ such that $h(0) = e$. Then, upon denoting

$$\xi = \dot{h}(0) \in T_eG$$

we define

$$\mathrm{Ad}_g\, \xi := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 I_g h(t) = g\xi g^{-1} \in T_eG.$$

**Definition (adjoint and coadjoint actions** of $G$ on $\mathfrak{g}$ and $\mathfrak{g}^*$) *The* ***adjoint action*** *of the matrix group $G$ on $\mathfrak{g}$ is a map*

$$\begin{aligned} Ad:\ G \times \mathfrak{g} &\to \mathfrak{g} \\ Ad_g\, \xi &= g\xi g^{-1}. \end{aligned}$$

*The dual map*

$$\langle Ad_g^*\, \mu,\ \xi \rangle = \langle \mu,\ Ad_g\, \xi \rangle$$

*is called the* ***coadjoint action*** *of $G$ on $\mathfrak{g}^*$.*

Take $g(t) \in G$ such that $g(0) = e$ and denote

$$\eta = \dot{g}(0) \in T_e G.$$

Then, define

$$\mathrm{ad}_\eta\, \xi := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{g(t)}\, \xi \quad \forall \xi \in \mathfrak{g}.$$

**Definition (adjoint and coadjoint action** of $\mathfrak{g}$ on $\mathfrak{g}$ and $\mathfrak{g}^*$) *The* ***adjoint action*** *of the matrix Lie algebra on itself is given as a map*

$$ad : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\,, \qquad\qquad ad_\eta\, \xi = [\eta, \xi].$$

*The dual map*

$$\langle ad^*_\eta\, \mu,\ \xi \rangle = \langle \mu,\ ad_\eta\, \xi \rangle$$

*is the* ***coadjoint action*** *of* $\mathfrak{g}$ *on* $\mathfrak{g}^*$.

## Actions of a Lie group on a manifold

**Definition (Left Lie group action)**
Let $M$ be a manifold and let $G$ be a Lie group. A **(left) action** of the Lie group $G$ on $M$ is a smooth mapping $\phi : G \times M \to M$ such that

1. $\phi(e, x) = x \quad \forall x \in M$.
2. $\phi(g, \phi(h, x)) = \phi(gh, x) \quad \forall g, h \in G, \forall x \in M$.
3. For every $g \in G$ the map

$$\begin{aligned}\phi_g :& M \to M \\ \phi_g(x) =& \phi(g, x)\end{aligned}$$

   is a diffeomorphism (i.e. smooth and invertible).

*Concatenation notation*: We write $gx$ for $\phi(g, x)$. Then, (2) becomes $g(hx) = (gh)x$.

**Example (Matrices acting on physical space)** Let $G \subseteq GL(n, \mathbb{R})$ and $M = \mathbb{R}^n$. Then, the standard action

$$\phi(A, \mathbf{v}) = A\mathbf{v}$$

is a left action. Notice that (2) is satisfied, since

$$\phi(A, \phi(B, \mathbf{v})) = A(B\mathbf{v}) = (AB)\mathbf{v} = \phi(AB, \mathbf{v}).$$

**Definition (Right Lie group action)** A **right Lie group action** of $G$ on $M$ satisfies (1) and (3), while (2) is replaced by

$$\phi(g, \phi(h, x)) = \phi(hg, x) \quad \forall g, h \in G, \forall x \in M.$$

*Concatenation notation*: $\phi(g, x)$ is denoted by $xg$, and (2) becomes $(xh)g = x(hg)$.

**Example**: Lagrange-to-Euler map $\Phi(\boldsymbol{\eta}, \rho) = \boldsymbol{\eta}_* \rho$ is a right action of the diffeomorphism group

**Example (Left vs right)** Any left action $(g, x) \mapsto gx$ produces a right action by $(g, x) \mapsto g^{-1}x$.

**Actions of a Lie group on itself:**

- *Left multiplication*: $L_g : G \to G : h \mapsto gh = L_g(h)$
- *Right multiplication*: $R_g : G \to G : h \mapsto hg = R_g(h)$
- *Conjugation action*: $I_g : G \to G : h \mapsto ghg^{-1} = R_{g^{-1}} \circ L_g = I_g(h)$.

**Definition (symmetry)** A function $F$ is **invariant** under the $G$-action $\phi$ if

$$F \circ \phi_g = F.$$

Then $G$ is called a **symmetry group** of $F$.

**Example (Invariant Lagrangians)**
Let $G$ act on $M = TQ$, for some manifold $Q$. Then, consider a Lagrangian on $M$:

$$L = L(q, \dot{q}).$$

Then, this is said to be invariant iff

$$L(\phi_g(q, \dot{q})) = L(q, \dot{q})$$

For example, if $Q = \mathbb{R}^n$ and $G = GL(n, \mathbb{R})$, then $TQ \simeq \mathbb{R}^n \times \mathbb{R}^n$ and the Lagrangian $L$ is invariant if

$$L(A\mathbf{q}, A\dot{\mathbf{q}}) = L(\mathbf{q}, \dot{\mathbf{q}})$$

## Actions of a Lie algebra on a manifold

So far, we showed how groups act on manifolds (and thus also on themselves). Also, a group acts on its Lie algebra by the adjoint action $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ and on its dual by the coadjoint action $\mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$. In this section, we show how Lie algebras act on manifolds to produce tangent vectors, or, more precisely, vector fields.

**Definition (Infinitesimal generator, a.k.a. Lie algebra action)** Let $g(t) \in G$ be such that $g(0) = e$ and $\dot{g}(0) = \xi \in \mathfrak{g}$. Let $\phi : G \times M \to M$ be a $G$-action. Then the **infinitesimal generator** of the $G$-action corresponding to $\xi$ at the point $x \in M$ is

$$
\begin{aligned}
\xi_M(x) &:= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_{g(t)}(x) \\
\left(\textit{by the concatenation notation}\right) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(g(t)x\big) \in T_xM.
\end{aligned}
$$

(This is also known as the **Lie algebra action** of $\xi$ on $M$).

Notice that the map $\xi_M : M \to TM$, for all $x \in M$ together, is a vector field.

**Example (Generator of rotations)**
Consider $\chi(t) \in SO(3)$ and $\mathbf{n}(t) = \chi(t)\mathbf{n}_0$ and denote $\hat{\omega} = \dot{\chi}(0)$. Then

$$\begin{aligned}\omega_{\mathbb{R}^3}(\mathbf{n}_0) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Big(\chi(t)\mathbf{n}_0\Big) \\ &= \dot{\chi}(0)\mathbf{n}_0 \\ &= \hat{\omega}\mathbf{n}_0 \\ &= \boldsymbol{\omega} \times \mathbf{n}_0.\end{aligned}$$

Therefore the infinitesimal generator of the $SO(3)$-action on $\mathbb{R}^3$ is the cross product.

**Example (Infinitesimal left matrix multiplication)** Take the left multiplication on $GL(n, \mathbb{R})$. Take $g(t) \in GL(n, \mathbb{R})$ with $g(0) = e$ and $\dot{g}(0) = \xi \in \mathfrak{gl}(n, \mathbb{R})$.

$$\xi_{TGL(n,\mathbb{R})}(h, h') = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Big(g(t)h, g(t)h'\Big) = (\xi h, \xi h').$$

**Example (Infinitesimal generator of the conjugation action)** Consider the conjugation action on a Lie group $G \subseteq GL(n,\mathbb{R})$:

$$I_g : G \to G\,, \qquad I_g(h) = ghg^{-1}$$

Then, upon taking a curve $g(t) \in G$ such that $g(0) = e$ and $\dot{g}(0) = \xi \in \mathfrak{g}$, its infinitesimal generator is computed as follows:

$$\begin{aligned}
\xi_G(h) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} I_{g(t)}h \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g(t)hg^{-1}(t) \\
&= \dot{g}(t)hg^{-1}(t)\big|_{t=0} + \Big(g(t)h\frac{d}{dt}\big(g^{-1}(t)\big)\Big)\Big|_{t=0} \\
&= \dot{g}(t)hg^{-1}(t)\big|_{t=0} - g(t)hg^{-1}(t)\dot{g}(t)g^{-1}(t)\big|_{t=0} \\
&= \dot{g}(0)h - h\dot{g}(0) \\
&= [\dot{g}(0), h] \\
&= [\xi, h].
\end{aligned}$$

Therefore, the infinitesimal generator of the conjugation action on $GL(n,\mathbb{R})$ is $\xi_G(h) = [\xi, h]$ for any $\xi \in \mathfrak{g}$ and any $h \in G$.

# 7 Euler-Poincaré theory with symmetry breaking

In previous examples (compressible fluids, kinetic theories), we saw that there always was a physical object which was responsible for symmetry breaking.

| Physical example | Symmetry breaking parameter | |
|---|---|---|
| Rigid body in the body frame | NONE | |
| Compressible fluids | $\rho = \boldsymbol{\eta}_* \rho_0$ | (mass density) |
| Kinetic theory | $f = \boldsymbol{\psi}_* f_0$ | (particle density on phase-space) |

Despite the symmetry breaking phenomena, this class of problems can still be approached by symmetry techniques to describe the dynamics on the Lie algebra rather than staying on the tangent bundle. The remainder of this section formulates the Euler-Poincaré theory for systems with symmetry breaking.

**Theorem (Euler-Poincaré reduction with symmetry-breaking parameter)**

- *Consider a Lie group $G$ and a left action on a manifold $M$. For a given $a_0 \in M$, let*

$$L_{a_0} : TG \to \mathbb{R}$$

*be a Lagrangian with symmetry-breaking parameter $a_0$.*

- *Then, define the unique function (if it exists)*

$$\begin{aligned} L : TG \times M &\to \mathbb{R} \\ L(g, \dot{g}, a_0) : &= L_{a_0}(g, \dot{g}) \quad \forall (g, \dot{g}) \in TG. \end{aligned}$$

*such that $L$ is invariant under the left action*

$$\begin{aligned} G \times (TG \times M) &\to TG \times M \\ \big(h, (g, \dot{g}, a_0)\big) &\mapsto (hg, h\dot{g}, ha_0). \end{aligned}$$

- *Also, define*

$$L\left(g^{-1}g,\ g^{-1}\dot{g},\ g^{-1}a_0\right) =: \ell(\xi, a),$$

*where*

$$a := g^{-1}a_0, \qquad \xi := g^{-1}\dot{g}.$$

*Then, the following are equivalent:*

1. *Hamilton's principle holds for* $\delta g(t_1) = \delta g(t_2) = 0$*:*

$$\delta \int_{t_1}^{t_2} L_{a_0}(g, \dot g)\, dt = 0$$

2. $g(t)$ *satisfies the Euler-Lagrange equations.*

3. *The reduced Hamilton's principle holds on* $\mathfrak{g} \times M$*:*

$$\delta \int_{t_1}^{t_2} \ell(\xi, a)\, dt = 0\,, \qquad \text{with} \qquad \delta\xi = \dot\eta + ad_\xi \eta, \qquad \delta a = -\eta_M(a),$$

*and* $\eta(t_1) = \eta(t_2) = 0$.

4. *The Euler-Poincaré equations*

$$\boxed{\frac{d}{dt}\frac{\partial \ell}{\partial \xi} = \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi} - a \diamond \frac{\partial \ell}{\partial a} \qquad \dot a = -\xi_M a}$$

*hold on* $\mathfrak{g} \times V$*, where*

$$\boxed{\left\langle \frac{\partial \ell}{\partial a}\,,\ \zeta_M a \right\rangle =: \left\langle a \ \diamond \ \frac{\partial \ell}{\partial a}\,, \zeta \right\rangle \qquad \forall \zeta \in \mathfrak{g}.}$$

**Proof.** We shall prove only that point 3 is equivalent to point 4. We already computed

$$\delta\xi = \dot{\eta} + [\xi, \eta] \quad \text{with } \eta = g^{-1}\delta g.$$

Also, we have

$$\delta a = \delta(g^{-1}a_0) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}(g^{-1}a_0) = -\eta_M(a).$$

Then

$$\begin{aligned}
0 &= \delta\int_{t_1}^{t_2}\ell(\xi\,,\,a)\mathrm{d}t\\
&= \int_{t_1}^{t_2}\left(\left\langle\frac{\partial\ell}{\partial\xi}\,,\,\delta\xi\right\rangle + \left\langle\frac{\partial\ell}{\partial a}\,,\,\delta a\right\rangle\right)\mathrm{d}t\\
&= \int_{t_1}^{t_2}\left(\left\langle\frac{\partial\ell}{\partial\xi}\,,\,\dot{\eta} + \mathrm{ad}_\xi\eta\right\rangle - \left\langle\frac{\partial\ell}{\partial a}\,,\,\eta_M a\right\rangle\right)\mathrm{d}t\\
&= \int_{t_1}^{t_2}\left(\left\langle -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\ell}{\partial\xi} + \mathrm{ad}^*_\xi\frac{\partial\ell}{\partial\xi}\,,\,\eta\right\rangle - \left\langle a\diamond\frac{\partial\ell}{\partial a}\,,\,\eta\right\rangle\right)\mathrm{d}t,
\end{aligned}$$

which proves the first equation.

The equation of motion for the parameter $a$ follows from

$$\dot{a} = \frac{\mathrm{d}}{\mathrm{d}t}(g^{-1}a_0) = -\xi_M a\,. \qquad \blacksquare$$

**Remark (The diamond momentum map).** The diamond operator " $\diamond$ " defines the **momentum map**

$$J\colon T^*M \to \mathfrak{g}^* \qquad J\left(a\,,\,\frac{\partial \ell}{\partial a}\right) := a \diamond \frac{\partial \ell}{\partial a}.$$

associated with (cotangent) lifts of Lie algebra actions. It is a key object in geometric mechanics.

**Remark (Euler-Poincaré equations for right actions)** It is easy to prove that, if the unreduced Lagrangian $L : TG \times M \to \mathbb{R}$ is invariant under a right action

$$\begin{aligned} G \times (TG \times M) &\to TG \times M \\ \big(h, (g, \dot{g}, a_0)\big) &\mapsto (gh, \dot{g}h, a_0 h), \end{aligned}$$

then, the corresponding Euler-Poincaré equations read

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \ell}{\partial \xi} = -\,\mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi} + a \diamond \frac{\partial \ell}{\partial a}\,, \qquad \dot{a} = \xi_M a\,.$$

where all signes are appear changed in the right hand sides.

## Noether's theorem (with broken symmetry)

*Let $\xi = g^{-1}\dot{g}$ be a solution of the Euler-Poincaré equations with parameter $a = g^{-1}a_0$. Then, upon denoting*

$$\mu(t) = \left(\frac{\partial \ell}{\partial \xi}\right)(t),$$

*we have*

$$\boxed{\frac{d}{dt}\operatorname{Ad}^*_{g(t)^{-1}}\mu(t) = -\operatorname{Ad}^*_{g(t)^{-1}}\left(a \diamond \frac{\partial \ell}{\partial a}\right).}$$

**Proof**

For any $\zeta \in \mathfrak{g}$, we have

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \mathrm{Ad}_{g^{-1}(t)}\zeta &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \mathrm{Ad}_{g^{-1}(t)} \circ \mathrm{Ad}_{g(t_0)}\Big(\mathrm{Ad}_{g^{-1}(t_0)}\zeta\Big) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \mathrm{Ad}_{g^{-1}(t)g(t_0)}\Big(\mathrm{Ad}_{g^{-1}(t_0)}\zeta\Big) \\
&= -\mathrm{ad}_{\xi(t_0)}\Big(\mathrm{Ad}_{g^{-1}(t_0)}\zeta\Big),
\end{aligned}$$

where the first two steps use the fact that $\mathrm{Ad}_g$ is an action, while the last step follows from

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} g(t)^{-1}g(t_0) = \Big(-g(t_0)^{-1}\dot{g}(t_0)g(t_0)^{-1}\Big)g(t_0) = -\xi(t_0).$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{g(t)^{-1}}\zeta = -\mathrm{ad}_{\xi}\Big(\mathrm{Ad}_{g(t)^{-1}}\zeta\Big)$$

So we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \mathrm{Ad}^*_{g(t)^{-1}}\mu(t)\,,\,\zeta\right\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \mu(t)\,,\,\mathrm{Ad}_{g(t)^{-1}}\zeta\right\rangle \\
\textit{(by using the previous result)} &= \left\langle \dot{\mu}\,,\,\mathrm{Ad}_{g(t)^{-1}}\zeta\right\rangle - \left\langle \mu(t)\,,\,\mathrm{ad}_\xi\left(\mathrm{Ad}_{g(t)^{-1}}\zeta\right)\right\rangle \\
\textit{(by using the definition of } \mathrm{ad}^*\textit{)} &= \left\langle \dot{\mu}\,,\,\mathrm{Ad}_{g(t)^{-1}}\zeta\right\rangle - \left\langle \mathrm{ad}^*_\xi\mu\,,\,\mathrm{Ad}_{g(t)^{-1}}\zeta\right\rangle \\
\textit{(by using the definition of } \mathrm{Ad}^*\textit{)} &= \left\langle \mathrm{Ad}^*_{g(t)^{-1}}\dot{\mu}\,,\,\zeta\right\rangle - \left\langle \mathrm{Ad}^*_{g(t)^{-1}}\left(\mathrm{ad}^*_\xi\mu\right)\,,\,\zeta\right\rangle \\
&= \left\langle \mathrm{Ad}^*_{g(t)^{-1}}\left(\dot{\mu} - \mathrm{ad}^*_\xi\mu\right)\,,\,\zeta\right\rangle \\
\textit{(by using the equations of motion)} &= \left\langle -\mathrm{Ad}^*_{g(t)^{-1}}\left(a \diamond \frac{\partial \ell}{\partial a}\right)\,,\,\zeta\right\rangle. \qquad \blacksquare
\end{aligned}$$

**Corollary (Noether's theorem)**

If $\partial\ell/\partial a = 0$, then the Lagrangian

$$L_{a_0} = L_{a_0}(g, \dot{g}) \qquad L_{a_0} : TG \to \mathbb{R}$$

is invariant (i.e. $a_0$ is absent) and the following quantity is conserved:

$$\mathrm{Ad}^*_{g(t)^{-1}} \frac{\partial \ell}{\partial \xi}(t) = \mathrm{const}$$

thereby recovering the Noether's theorem arising from the case of unbroken symmetry.

**Remark (Coadjoint motion).** Notice that, since $\mathrm{Ad}^*$ is an action, we can apply $\mathrm{Ad}^*_{g(t)}$ on both sides, so that

$$\mathrm{Ad}^*_{g^{-1}(t)} \frac{\partial \ell}{\partial \xi}(t) = \mathrm{const} \quad \Longrightarrow \quad \frac{\partial \ell}{\partial \xi}(t) = \mathrm{Ad}^*_{g(t)}\ (\mathrm{const}) = \mathrm{Ad}^*_{g(t)} \frac{\partial \ell}{\partial \xi}(0).$$

where we have defined $\partial\ell/\partial\xi(0) := \mathrm{const}$. A motion of the type

$$\mu(t) = \mathrm{Ad}^*_{g(t)} \mu(0)$$

is called a **coadjoint motion**.

# 8 One more example: charged rigid body in a magnetic field

This section shows how the Euler-Poincaré equations of motion of a mechanical system with (broken) symmetry can be written down by simply using the corresponding Euler-Poincaré theorem. More particularly, we shall consider the example of a charged rigid body in a constant magnetic field.

- Upon writing the usual relation $\mathbf{x}(\mathbf{x}_0) = \mathcal{R}(t)\mathbf{x}_0$ for the dynamics of a single particle in the body, the Lagrangian reads

$$L = \int_{\mathcal{B}} \rho(\mathbf{x}_0) \left( \frac{1}{2} |\mathbf{x}(\mathbf{x}_0)|^2 + \alpha \dot{\mathbf{x}}(\mathbf{x}_0) \cdot \mathbf{A}(\mathbf{x}(\mathbf{x}_0)) \right) \mathrm{d}^3\mathbf{x}_0,$$

  where $\alpha = e/m$ is the charge-to-mass ratio of an arbitrary particle in the body and $\mathbf{A}(\boldsymbol{x})$ is the magnetic potential associated to the magnetic field $\mathbf{B}(\boldsymbol{x}) := \nabla \times \mathbf{A}(\boldsymbol{x})$. If $\mathbf{B}$ is constant in space, we can write

$$\mathbf{A}(\boldsymbol{x}) = \mathbf{B} \times \boldsymbol{x}.$$

- Then, upon assuming $\alpha = 1$ for simplicity, the previous Lagrangian is written on the tangent bundle $TSO(3)$ as

$$L = \int_{\mathcal{B}} \rho(\mathbf{x}_0) \left( \frac{1}{2} |\mathcal{R}\mathbf{x}_0|^2 + \mathbf{B} \cdot (\mathcal{R}\mathbf{x}_0) \times (\dot{\mathcal{R}}\mathbf{x}_0) \right) \mathrm{d}^3\mathbf{x}_0 =: L_{\mathbf{B}}(\mathcal{R}, \dot{\mathcal{R}})$$

- We observe that the above Lagrangian is *not* left invariant, i.e.

$$L_{\mathbf{B}}(\mathcal{R}, \dot{\mathcal{R}}) \neq L_{\mathbf{B}}(\chi\mathcal{R}, \chi\dot{\mathcal{R}}), \qquad \chi \in SO(3)$$

(actually, the above Lagrangian is not right-invariant either). However, we easily check that the left-invariance is recovered if we also act on the magnetic field $\mathbf{B}$, i.e.

$$L_{\mathbf{B}}(\mathcal{R}, \dot{\mathcal{R}}) = L_{\chi\mathbf{B}}(\chi\mathcal{R}, \chi\dot{\mathcal{R}})\,, \qquad \forall\chi \in SO(3).$$

This means that the Euler-Poincaré theorem can be applied upon setting $g = \mathcal{R} \in SO(3)$ and $a_0 = \mathbf{B} \in \mathbb{R}^3$. In particular, one may apply the Euler-Poincaré reduction theorem for left actions.

## The hat map as a Lie Algebra isomorphism $(\mathfrak{so}(3),[\cdot,\cdot]) \cong (\mathbb{R}^3, \cdot\times\cdot)$

The Lie Algebra of $SO(3)$ is the space of skew-symmetric matrices $\mathfrak{so}(3)$. Therefore, upon denoting $\widehat{\Omega} = \mathcal{R}^{-1}\dot{\mathcal{R}}$ and $\mathbf{b} = \mathcal{R}^{-1}\mathbf{B}$, the Euler-Poincaré equations are naturally written as[3]

$$\frac{d}{dt}\frac{\partial \ell}{\partial \widehat{\Omega}} - \mathrm{ad}^*_{\widehat{\Omega}}\frac{\partial \ell}{\partial \widehat{\Omega}} = -\mathbf{b} \diamond \frac{\partial \ell}{\partial \mathbf{b}}$$

where we define $\langle A, B\rangle = \mathrm{Tr}(A^T B)$ (this is a general definition for matrix spaces) so that

$$\left\langle \mathrm{ad}^*_{\widehat{\Omega}}\Pi, \widehat{\omega}\right\rangle = \left\langle \Pi, [\widehat{\Omega}, \widehat{\omega}]\right\rangle = \mathrm{Tr}(\Pi^T[\widehat{\Omega}, \widehat{\omega}]) = \mathrm{Tr}([\Pi^T, \widehat{\Omega}]\,\widehat{\omega}) = -\mathrm{Tr}([\widehat{\Omega}, \Pi]^T\widehat{\omega}) = \left\langle [\Pi, \widehat{\Omega}], \widehat{\omega}\right\rangle$$

and thus

$$\mathrm{ad}^*_{\widehat{\Omega}}\Pi = [\Pi, \widehat{\Omega}]\,,$$

since $\widehat{\omega}$ is arbitrary.

However, the *hat map* is a Lie Algebra isomorphism, i.e.

$$[\widehat{\Omega}, \widehat{\omega}] = \widehat{\boldsymbol{\Omega}\times\boldsymbol{\omega}}$$

$$\widehat{\ } : (\mathbb{R}^3, \cdot\times\cdot) \to (\mathfrak{so}(3), [\cdot,\cdot])$$

and the Euler-Poincaré equations may easily be written on the dual of the Lie Algebra $(\mathbb{R}^3, \cdot\times\cdot)$ (like we did for the rigid body).

[3] Recall that $\mathfrak{so}^*(3) \cong \mathfrak{so}(3)$ so that $\partial\ell/\partial\widehat{\Omega} \in \mathfrak{so}^*(3)$ has to be antisymmetric.

## Euler-Poincaré reduction and equations of motion

The equations of motion can be derived by simply following the prescriptions of the Euler-Poincaré theorem, under the identification $(\mathbb{R}^3, \cdot\times\cdot) \simeq (\mathfrak{so}(3), [\cdot,\cdot])$.

- By introducing the body-frame quantities

$$\widehat{\Omega} = \mathcal{R}^{-1}\dot{\mathcal{R}}, \quad \mathbf{b} = \mathcal{R}^{-1}\mathbf{B},$$

  we define

$$\ell(\mathbf{\Omega}, \mathbf{b}) := L_{\mathcal{R}^{-1}\mathbf{B}}(\mathcal{R}^{-1}\mathcal{R}, \mathcal{R}^{-1}\dot{\mathcal{R}}),$$

  so that $\mathbf{b}$ is the magnetic field as seen in the body frame. An easy computation shows that

$$\ell(\mathbf{\Omega}, \mathbf{b}) = \frac{1}{2}\mathbf{\Omega}\cdot\mathbb{I}\mathbf{\Omega} + \mathbf{b}\cdot\mathbb{I}\mathbf{\Omega}$$

  with the usual definition of the moment of inertia tensor

$$\mathbb{I} := \int_{\mathcal{B}} \rho(\mathbf{x}_0)\left(|\mathbf{x}_0|^2\mathbf{I} - \mathbf{x}_0\mathbf{x}_0^T\right)\mathrm{d}^3\mathbf{x}_0,$$

- Again, we recognize that $\mathbf{b}$ is to be considered as a parameter that breaks the symmetry of $L_{\mathbf{B}}(\mathcal{R}, \dot{\mathcal{R}})$ under left multiplication, so that the Euler-Poincaré reduction theorem can be applied directly. For example, the relationship

$$\dot{a} = -\xi_M a$$

becomes

$$\dot{\mathbf{b}} = -\widehat{\Omega}_{\mathbb{R}^3}\mathbf{b} = -\boldsymbol{\Omega} \times \mathbf{b}.$$

(recall the definition of a Lie algebra action). Also, we know from the Euler-Poincaré reduction theorem that the equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \ell}{\partial \boldsymbol{\Omega}} - \mathrm{ad}^*_{\boldsymbol{\Omega}}\frac{\partial \ell}{\partial \boldsymbol{\Omega}} = -\mathbf{b} \diamond \frac{\mathrm{d}\ell}{\mathrm{d}\mathbf{b}}.$$

so that the ***expressions of*** $\boldsymbol{ad^*_{\Omega}}$ ***and*** $\diamond$ ***become necessary*** to write the above equation in its explicit form.

- In order to compute the infinitesimal coadjoint action, let $\boldsymbol{\nu} \in \mathfrak{so}(3) \simeq \mathbb{R}^3$ be arbitrary and compute

$$\left\langle \mathrm{ad}^*_{\boldsymbol{\Omega}}\frac{\partial \ell}{\partial \boldsymbol{\Omega}}, \boldsymbol{\nu} \right\rangle = \left\langle \frac{\partial \ell}{\partial \boldsymbol{\Omega}}, \mathrm{ad}_{\boldsymbol{\Omega}}\boldsymbol{\nu} \right\rangle = \frac{\partial \ell}{\partial \boldsymbol{\Omega}} \cdot \boldsymbol{\Omega} \times \boldsymbol{\nu} = -\boldsymbol{\Omega} \times \frac{\partial \ell}{\partial \boldsymbol{\Omega}} \cdot \boldsymbol{\nu} = \left\langle -\boldsymbol{\Omega} \times \frac{\partial \ell}{\partial \boldsymbol{\Omega}}, \boldsymbol{\nu} \right\rangle$$

$$\therefore \qquad \mathrm{ad}^*_{\boldsymbol{\Omega}}\frac{\partial \ell}{\partial \boldsymbol{\Omega}} = -\boldsymbol{\Omega} \times \frac{\partial \ell}{\partial \boldsymbol{\Omega}}.$$

Also, the computation of the diamond momentum map proceeds as follows:

$$\left\langle \mathbf{b} \diamond \frac{\partial \ell}{\partial \mathbf{b}}, \boldsymbol{\nu} \right\rangle := \left\langle \frac{\partial \ell}{\partial \mathbf{b}}, \nu_{\mathbb{R}^3}\mathbf{b} \right\rangle = \frac{\partial \ell}{\partial \mathbf{b}} \cdot \boldsymbol{\nu} \times \mathbf{b} = \left\langle \mathbf{b} \times \frac{\partial \ell}{\partial \mathbf{b}}, \boldsymbol{\nu} \right\rangle,$$

$$\therefore \quad \mathbf{b} \diamond \frac{\partial \ell}{\partial \mathbf{b}} = \mathbf{b} \times \frac{\partial \ell}{\partial \mathbf{b}}.$$

and finally, by applying the usual relation

$$\delta \ell = \left\langle \frac{\partial \ell}{\partial \boldsymbol{\Omega}}, \delta \boldsymbol{\Omega} \right\rangle + \left\langle \frac{\partial \ell}{\partial \mathbf{b}}, \delta \mathbf{b} \right\rangle,$$

we find

$$\frac{\partial \ell}{\partial \boldsymbol{\Omega}} = \mathbb{I}(\boldsymbol{\Omega} + \mathbf{b})\,, \qquad \frac{\partial \ell}{\partial \mathbf{b}} = \mathbb{I}\boldsymbol{\Omega}$$

Consequently, the Euler-Poincaré equations read

$$\boxed{\mathbb{I}(\dot{\boldsymbol{\Omega}} + \dot{\mathbf{b}}) + \boldsymbol{\Omega} \times \mathbb{I}(\boldsymbol{\Omega} + \mathbf{b}) = -\mathbf{b} \times (\mathbb{I}\boldsymbol{\Omega})\,, \qquad \dot{\mathbf{b}} = \mathbf{b} \times \boldsymbol{\Omega}.}$$

which provide the body description of a charged rigid body in a magnetic field.

## Noether's conserved quantity and angular momentum

The main advantage of using Euler-Poincaré reduction is that this procedure makes conservation laws as transparent as possible.

- Indeed, conserved quantities can be found by direct application of Noether's theorem, which is written in the form
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Ad}^*_{\mathcal{R}^{-1}}\frac{\partial \ell}{\partial \boldsymbol{\Omega}}\right) = -\mathrm{Ad}^*_{\mathcal{R}^{-1}}\left(\mathbf{b}\diamond\frac{\partial \ell}{\partial \mathbf{b}}\right)$$
where we recall that, in the case under consideration, diamond coincides with the cross product.

- Now, one finds that $\mathrm{Ad}^*_{\mathcal{R}^{-1}}\,\boldsymbol{\Pi} = \mathcal{R}\boldsymbol{\Pi}$, so we can write the above general form of Noether's theorem in the more explicit form:
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{R}\frac{\partial \ell}{\partial \boldsymbol{\Omega}}\right) = -\mathcal{R}\left(\mathbf{b}\times\frac{\partial \ell}{\partial \mathbf{b}}\right)$$

- Interestingly enough, taking the dot product of the above relation with the spatial magnetic field $\mathbf{B}$ yields
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{B}\cdot\mathcal{R}\frac{\partial \ell}{\partial \boldsymbol{\Omega}}\right) = -\mathbf{B}\cdot\mathcal{R}\left(\mathbf{b}\times\frac{\partial \ell}{\partial \mathbf{b}}\right)$$
so that the left-hand side identifies the time rate of the spatial angular momentum $\mathcal{R}(\partial\ell/\partial\boldsymbol{\Omega})$ in the direction of the magnetic field $\mathbf{B}$. Upon recalling that the dot product is left-invariant

under $SO(3)$, we have

$$\mathbf{B}\cdot\mathcal{R}\left(\mathbf{b}\times\frac{\partial\ell}{\partial\mathbf{b}}\right)=\mathbf{b}\cdot\mathbf{b}\times\frac{\partial\ell}{\partial\mathbf{b}}=0.$$

- Therefore, we obtain the following conservation law:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{b}\cdot\frac{\partial\ell}{\partial\boldsymbol{\Omega}}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\Big((\boldsymbol{\Omega}+\mathbf{b})\cdot\mathbb{I}\mathbf{b}\Big)=0\,,}$$

  so that the spatial angular momentum in the direction of the magnetic field is conserved by the dynamics, regardless of the explicit expression of the Lagrangian.

Does this conservation arise from some symmetry? Yes - indeed, the Lagrangian

$$L=\int_{\mathcal{B}}\rho(\mathbf{x}_0)\left(\frac{1}{2}|\mathcal{R}\mathbf{x}_0|^2+\mathbf{B}\cdot(\mathcal{R}\mathbf{x}_0)\times(\dot{\mathcal{R}}\mathbf{x}_0)\right)\mathrm{d}^3\mathbf{x}_0=:L_{\mathbf{B}}(\mathcal{R},\dot{\mathcal{R}})$$

is invariant with respect to transformations in the group

$$\mathcal{G}_{\mathbf{B}}:=\{\mathcal{R}\in SO(3)\,\big|\,\mathcal{R}\mathbf{B}=\mathbf{B}\},$$

which comprises rotations that leave the $\mathbf{B}$-axis fixed (say $\mathbf{B}=b\mathbf{e}_3$), that is, $SO(2)$-rotations in the $xy$-plane. Angular momentum conservation along $\mathbf{B}$ arises from the $SO(2)$ symmetry of $L_{\mathbf{B}}(\mathcal{R},\dot{\mathcal{R}})$.

# 9 Schrödinger and Von Neumann equations

In quantum mechanics, the dynamics of a pure quantum state $\psi$ (a vector on a complex Hilbert space $\mathcal{H}$) is given by Schrödinger's equation

$$i\hbar\dot{\psi}(t) = H\psi(t) \tag{2}$$

where $\hbar$ is Planck's constant, $H$ is a self-adjoint matrix on a complex Hilbert space $\mathcal{H}$ and represents the Hamiltonian of the system. Here, we shall set $\hbar = 1$ for cenvenience. Notice that $\psi$ (the state of the system) is defined up to a multiplicative complex number: it is convenient to normalize the quantum state so that that $\|\psi\|^2 = 1$.

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**Remark 2 (Inner product and pairing)** *Recall that the Hermitian* ***inner product*** *on complex square matrices is defined as*

$$(A, B) := \mathrm{Tr}\big(AB^\dagger\big) \in \mathbb{C}$$

*where dagger denotes conjugate transpose. On the other hand, the* ***real-valued pairing*** *reads*

$$\langle A, B\rangle := \Re\,(A, B) = \Re\,\mathrm{Tr}\big(AB^\dagger\big) \in \mathbb{R}$$

*where $\Re$ denotes the real part, so that $\Re(z) = \Re(\bar{z})$ for any complex number $z$ (here, the bar symbol denotes complex conjugate). Upon recalling that $\overline{\mathrm{Tr}\,A} = \mathrm{Tr}\,A^\dagger$, we have the property*

$$\Re\,\mathrm{Tr}\big(AB^\dagger\big) = \Re\,\mathrm{Tr}\big(A^\dagger B\big)\,.$$

## Euler-Poincaré reduction

- An explicit verification shows that Schrödinger's equation (3) can be recovered from the ***Dirac-Frenkel Lagrangian***

$$L(\psi,\dot\psi) = \langle \psi, i\dot\psi - H\psi\rangle\,. \tag{4}$$

This is seen by using the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\psi} - \frac{\partial L}{\partial\psi} = 0 \tag{5}$$

**Exercise.** *Show that (4) yields Schrödinger equation (3)*

- Now, let

$$\psi(t) = U(t)\psi_0 \tag{6}$$

where $U(t)\in\mathcal{U}(n)$ and $\psi_0$ is the initial state. Then (4) can be written as

$$L_{\psi_0} : TU(n)\to\mathbb{R} \tag{7}$$

$$L_{\psi_0}(U,\dot U) = \langle U\psi_0, i\dot U\psi_0 - HU\psi_0\rangle\,, \tag{8}$$

which in turn can be written as

$$L_{\rho_0}(U,\dot{U}) = \langle U\rho_0, i\dot{U} - HU\rangle \tag{9}$$

where $\rho_0 = \psi_0\psi_0^\dagger$.

**Exercise.** *Show that (8) equals (9) and show that the latter is not right-invariant.*

- In order to apply the Euler-Poincaré theorem, we can define

$$L : TU(n) \times M \to \mathbb{R} \tag{10}$$

$$L(U,\dot{U},\rho_0) := L_{\rho_0}(U,\dot{U}), \quad \forall (U,\dot{U}) \in TU(n) \tag{11}$$

where $M$ is the space of Hermitian matrices. Then, we can verify that $L(U,\dot{U},\rho_0)$ is right-invariant if we act on $\rho_0$ by the action

$$\phi(U,\rho_0) = U^\dagger \rho_0 U \tag{12}$$

**Exercise.** *Show that the action (12) is a group action on $M$. Is it left or right?*

Then, the reduced Lagrangian is

$$l(\xi,\rho) = \langle \rho, (i\xi - H)\rangle \tag{13}$$

where $\xi = \dot{U}U^{-1} \in \mathfrak{u}(n)$ and $\rho \in M$ is the symmetry breaking parameter.

**Exercise.** *Show that the Lagrangian (13) arises from reduction of the Lagrangian* $L(U, \dot{U}, \rho_0) = \langle U\rho_0, i\dot{U} - HU \rangle$.

## The Euler-Poincaré equations

Recall the Euler-Poincaré equations for (right) symmetry breaking systems

$$\frac{d}{dt}\frac{\partial l}{\partial \xi} = -\operatorname{ad}^*_\xi \frac{\partial l}{\partial \xi} - \rho \diamond \frac{\partial l}{\partial \rho} \tag{14}$$

$$\dot{\rho} = -\xi_M(\rho) \tag{15}$$

where $\xi_M(\rho)$ is the infinitesimal generator.

- Take variations in the reduced Lagrangian (13)

$$\delta l\,(\xi, \rho) = \langle \delta\rho, (i\xi - H) \rangle + \langle \rho, i\delta\xi \rangle \tag{16}$$

so that

$$\frac{\partial l}{\partial \xi} = -i\rho, \qquad \frac{\partial l}{\partial \rho} = (i\xi - H) \tag{17}$$

- Moreover, from previous section we know that

$$\operatorname{ad}^*_\xi \frac{\partial l}{\partial \xi} = \left[\frac{\partial l}{\partial \xi}, \xi\right] \implies \operatorname{ad}^*_\xi \frac{\partial l}{\partial \xi} = [i\xi, \rho]\ . \tag{18}$$

- Also, the inifitesimal generator is given by its definition:

$$\xi_M\left(\rho_0\right) = \frac{d}{dt}\bigg|_{t=0} \phi(U, \rho_0) = -[\xi, \rho_0]\,. \tag{19}$$

Hence, we have

$$\dot{\rho} = [\xi, \rho]\,. \tag{20}$$

**Exercise.** *Verify the second equality in* (19)*.*

- The diamond operator is found by applying its definition. For an arbitrary $\zeta \in \mathfrak{u}(n)$, we have

$$\left\langle \rho \diamond \frac{\delta l}{\delta \rho}, \zeta \right\rangle = \left\langle \frac{\partial l}{\partial \rho}, \zeta_M(\rho) \right\rangle = \left\langle \left[\rho, \frac{\partial l}{\partial \rho}\right], \zeta \right\rangle \tag{21}$$

and thus

$$\rho \diamond \frac{\partial l}{\partial \rho} = \left[\rho, \frac{\partial l}{\partial \rho}\right] = [\rho, i\xi - H]\ . \tag{22}$$

**Exercise.** *Show that* $\rho \diamond \eta = [\rho, \eta]$*.*

- At this point, we can rewrite (14) in the form

$$-\, i\dot{\rho} = -i[\xi, \rho] + [i\xi - H, \rho]\,. \tag{23}$$

Combining with (20), the above relation gives

$$i[\xi, \rho] = [H, \rho]\,, \tag{24}$$

thereby recovering ***Von Neumann equation***

$$\dot{\rho} = -i[H, \rho]$$

- In conclusion, Euler-Poincaré reduction does NOT reproduce Schrödinger equation. Rather, it gives the Von Neuman equation for the ***density matrix*** $\rho = \psi\psi^\dagger$.

## 10 The symplectic group and linear Hamiltonian systems

- In this section we shall show that symplectic matrices govern the dynamics of Hamiltonian systems, whose Hamilton's equations are linear in the phase-space coordinates.
- Denote $\mathbf{z} := (q, p)$ and let the configuration manifold be $Q = \mathbb{R}^n$, so that $\mathbf{z} \in T^*Q = \mathbb{R}^{2n}$. Then Hamilton's equations are written as

$$\dot{\mathbf{z}} = \mathbb{J}\nabla_{\mathbf{z}} H(\mathbf{z}), \qquad \mathbb{J} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

**Definition.** *A **linear Hamiltonian system** with constant coefficients is a set of* $2n$ *Hamilton's equations associated to a quadratic Hamiltonian of the form*

$$H = \frac{1}{2}\mathbf{z} \cdot S\mathbf{z} = \frac{1}{2}\,\mathrm{Tr}(\mathbf{z}^T S\mathbf{z}),$$

*where* $S \in Sym(2n, \mathbb{R})$*, so that*

$$\dot{\mathbf{z}} = \mathbb{J}S\mathbf{z}.$$

**Example (Harmonic oscillator).** The simplest example of linear motion in classical mechanics is the harmonic oscillator:

$$S = \begin{pmatrix} \alpha \mathbb{1} & 0 \\ 0 & \beta \mathbb{1} \end{pmatrix} \quad \Longrightarrow \quad H(q,p) = \frac{\beta}{2}|p|^2 + \frac{\alpha}{2}|q|^2.$$

**Lie algebraic structure:** from ODE theory, we notice that

$$\dot{\mathbf{z}} = \mathbb{J} S \mathbf{z} \quad \Longrightarrow \quad \mathbf{z}(t) = e^{t\mathbb{J}S}\mathbf{z}(0),$$

*What's the algebraic structure of the matrix $\mathbb{J}S$?*

**Example (Harmonic oscillator).** The simplest example of linear motion in classical mechanics is the harmonic oscillator:

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*What's the algebraic structure of the matrix* $\mathbb{J}S$*?*

We notice that, upon defining the *tilde map* $\widetilde{S} = -\mathbb{J}S$, we have

1. $\widetilde{S}^T\mathbb{J} + \mathbb{J}\widetilde{S} = 0$, so that $\widetilde{S}$ is in the ***Lie algebra*** $\mathfrak{sp}(2n)$ ***of Hamiltonian matrices***
2. Lie bracket relation: $\left[\widetilde{S}, \widetilde{S'}\right] = -\mathbb{J}(S'\mathbb{J}S - S\mathbb{J}S') \quad \forall S, S' \in \mathrm{Sym}(2n, \mathbb{R})$
3. Upon defining $[S, S']_{\mathbb{J}} := S'\mathbb{J}S - S\mathbb{J}S'$, we have $[\widetilde{S}, \widetilde{S'}] = \widetilde{[S, S']_{\mathbb{J}}}$
4. ***Lie algebra isomorphism***: $(\mathfrak{sp}(2n), [\cdot,\cdot]) \simeq (\mathrm{Sym}(2n), [\cdot,\cdot]_{\mathbb{J}})$

**Symplectic group:** the space Hamiltonian matrices is the Lie algebra underlying the symplectic group

$$Sp(2n, \mathbb{J}) = \{g \in GL(2n, \mathbb{R}) \mid g^T \mathbb{J} g = \mathbb{J}\}.$$

***This group governs the evolution of all linear Hamiltonian systems***

**Symplectic group:** the space Hamiltonian matrices is the Lie algebra underlying the symplectic group

$$Sp(2n, \mathbb{J}) = \{g \in GL(2n, \mathbb{R}) \mid g^T \mathbb{J} g = \mathbb{J}\}.$$

***This group governs the evolution of all linear Hamiltonian systems***

**Proposition** *If* $\tilde{S} \in \mathfrak{sp}(2n, \mathbb{J})$*, then*

$$g(t) := e^{t\tilde{S}} \in Sp(2n, \mathbb{J}) \quad \forall t \in \mathbb{R}.$$

**Proof.** We must show that $(e^{t\tilde{S}})^T \mathbb{J} e^{t\tilde{S}} = \mathbb{J}$. However, we have $(e^{t\tilde{S}})^T = e^{t(\tilde{S}^T)}$ and $(e^{t\tilde{S}})^{-1} = e^{-t\tilde{S}}$. Then, we need to prove

$$(e^{t\tilde{S}})^T \mathbb{J} = \mathbb{J} e^{-t\tilde{S}}.$$

Indeed,

$$\begin{aligned} e^{t\tilde{S}^T} \mathbb{J} &= \sum_0^\infty \frac{t^n}{n!} (\tilde{S}^T)^{n-1} \tilde{S}^T \mathbb{J} \\ &= \sum_0^\infty \frac{t^n}{n!} (\tilde{S}^T)^{n-1} (-\mathbb{J}\tilde{S}) \\ &= \sum_0^\infty \frac{t^n}{n!} \mathbb{J} (-1)^n \tilde{S}^n = \mathbb{J} e^{-t\tilde{S}} \qquad \blacksquare \end{aligned}$$

This shows that the evolution relation

$$z(t) = e^{t\mathbb{J}S} z(0)$$

is entirely determined by the symplectic transfer matrix $e^{t\mathbb{J}S}$.

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## *What happens in the Euler-Poincaré framework?*

## Euler-Poincaré formulation of linear Hamiltonian systems

Consider the Lagrangian $L : T\mathbb{R}^{2n} \to \mathbb{R}$ defined by

$$L(\mathbf{z}, \dot{\mathbf{z}}) := \frac{1}{2}\dot{\mathbf{z}} \cdot \mathbb{J}\mathbf{z} - \frac{1}{2}\mathbf{z} \cdot \Sigma\mathbf{z}\,.$$

where $\Sigma$ is a fixed symmetric $2n \times 2n$ matrix

- Verify that the Euler-Lagrange equations imply Hamilton's equations on $\mathbb{R}^{2n}$.

We compute

$$\delta L = \frac{1}{2}\delta\dot{\mathbf{z}} \cdot \mathbb{J}\mathbf{z} + \frac{1}{2}\dot{\mathbf{z}} \cdot \mathbb{J}\delta\mathbf{z} - \Sigma\mathbf{z} \cdot \delta\mathbf{z}$$

Therefore, the Euler-Lagrange equations give

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\mathbf{z}}} - \frac{\partial L}{\partial \mathbf{z}} = \frac{1}{2}\mathbb{J}\dot{\mathbf{z}} - \left(-\frac{1}{2}\mathbb{J}\dot{\mathbf{z}} - \Sigma\mathbf{z}\right) = \mathbb{J}\dot{\mathbf{z}} + \Sigma\mathbf{z}$$

which gives, upon using $\mathbb{J}^{-1} = -\mathbb{J}$,

$$\dot{\mathbf{z}} = \mathbb{J}\Sigma\mathbf{z}\,.$$

• Let

$$\mathbf{z}(t) = g(t)\mathbf{z}_0 \,, \qquad \text{with} \qquad g(t) \in \mathrm{Sp}(2n, \mathbb{J})$$

and verify that the Lagrangian $L(\mathbf{z}, \dot{\mathbf{z}})$ can be written in the Euler-Poincaré form

$$\ell(S, \mathbf{z}) = -\frac{1}{2}\mathbf{z} \cdot (S + \Sigma)\mathbf{z}$$

where $S$ is a symmetric matrix.

We have

$$L(\mathbf{z}, \dot{\mathbf{z}}) := \frac{1}{2}\dot{g}\mathbf{z}_0 \cdot \mathbb{J}\mathbf{z} - \frac{1}{2}\mathbf{z} \cdot \Sigma\mathbf{z} = \frac{1}{2}\widetilde{S}\mathbf{z} \cdot \mathbb{J}\mathbf{z} - \frac{1}{2}\mathbf{z} \cdot \Sigma\mathbf{z} = -\frac{1}{2}\mathbf{z} \cdot \mathbb{J}\widetilde{S}\mathbf{z} - \frac{1}{2}\mathbf{z} \cdot \Sigma\mathbf{z}$$

where $\widetilde{S} = \dot{g}g^{-1}$. However, $\widetilde{S}$ is Hamiltonian and thus $\mathbb{J}\widetilde{S}$ is symmetric because

$$(\mathbb{J}\widetilde{S})^T = -\widetilde{S}^T\mathbb{J} = \mathbb{J}\widetilde{S} \,.$$

Therefore $S = \mathbb{J}\widetilde{S}$. (*Inverse of the tilde map!*)

- Define the Lie bracket (arising from commutator of Hamiltonian matrices)

$$[S, S']_{\mathbb{J}} = S\mathbb{J}S' - S'\mathbb{J}S = \mathrm{ad}_S\, S'$$

as well as the infinitesimal generator (arising from $\widetilde{S}_{\mathbb{R}^{2n}}(\mathbf{z}) = \widetilde{S}\mathbf{z}$)

$$S_{\mathbb{R}^{2n}}(\mathbf{z}) = -\mathbb{J}S\mathbf{z}\,.$$

Then, write the following Euler-Poincaré equations in explicit form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \ell}{\partial S} + \mathrm{ad}^*_S\,\frac{\partial \ell}{\partial S} = -\mathbf{z} \diamond \frac{\partial \ell}{\partial \mathbf{z}}\,, \qquad \dot{\mathbf{z}} = -\mathbb{J}S\mathbf{z}\,,$$

Applying the definition of $\mathrm{ad}^*$, we have

$$\begin{aligned}
\langle \mathrm{ad}^*_S Y, S' \rangle &= \langle Y, [S, S']_{\mathbb{J}} \rangle \\
&= \mathrm{Tr}(YS\mathbb{J}S' - YS'\mathbb{J}S) \\
&= \mathrm{Tr}((YS\mathbb{J} - \mathbb{J}SY)S') \\
&= \langle YS\mathbb{J} - \mathbb{J}SY, S' \rangle
\end{aligned}$$

and analogously, for the diamond,

$$\begin{aligned}\left\langle \mathbf{z} \diamond \boldsymbol{\kappa}, S \right\rangle =& \boldsymbol{\kappa} \cdot (-\mathbb{J}S\mathbf{z}) \\ =& - \operatorname{Tr}(\boldsymbol{\kappa}^T \mathbb{J} S \mathbf{z}) \\ =& - \operatorname{Tr}(\mathbf{z}\boldsymbol{\kappa}^T \mathbb{J} S) \\ =& - \frac{1}{2}\operatorname{Tr}((\mathbf{z}\boldsymbol{\kappa}^T\mathbb{J} - \mathbb{J}\boldsymbol{\kappa}\mathbf{z}^T)S) \\ =& \left\langle \frac{1}{2}(\mathbb{J}\boldsymbol{\kappa}\mathbf{z}^T - \mathbf{z}\boldsymbol{\kappa}^T\mathbb{J}), S \right\rangle\end{aligned}$$

where we have projected against the symmetric part. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \ell}{\partial S} + \frac{\partial \ell}{\partial S} S\mathbb{J} - \mathbb{J}S\frac{\partial \ell}{\partial S} = -\frac{1}{2}\left(\mathbb{J}\frac{\partial \ell}{\partial \mathbf{z}}\mathbf{z}^T - \mathbf{z}\frac{\partial \ell}{\partial \mathbf{z}}^T\mathbb{J}\right)\,, \qquad \dot{\mathbf{z}} = -\mathbb{J}S\mathbf{z}\,,$$

and in conclusion, taking the derivatives of $\ell$ takes the first equation to the form

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{z}\mathbf{z}^T\right) - \frac{1}{2}\mathbf{z}\mathbf{z}^T S\mathbb{J} + \frac{1}{2}\mathbb{J}S\mathbf{z}\mathbf{z}^T = \frac{1}{2}\left(\mathbb{J}(S+\Sigma)\mathbf{z}\mathbf{z}^T - \mathbf{z}\mathbf{z}^T(S+\Sigma)\mathbb{J}\right)\,.$$

Then, upon defining $X = \mathbf{z}\mathbf{z}^T$, we have

$$\dot{X} = \mathbb{J}\Sigma X - X\Sigma\mathbb{J} = -\mathrm{ad}^*_\Sigma X$$