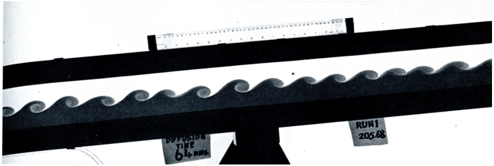
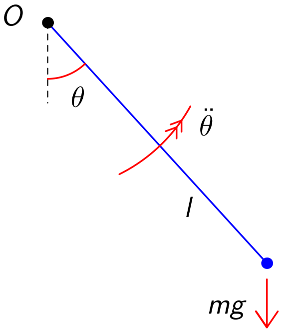
1. Introduction

- The Navier-Stokes equations present a challenging problem, but there are some exact solutions.
- We consider special cases: simple geometry, simple flows.
- Sometimes the simple flow solution is observed in experiments.
- But sometimes other, possibly unsteady/turbulent, flows are observed instead:



- The concept of flow stability was introduced to explain these scenarios.
- Add a small disturbance to a flow, determine whether the disturbance grows ($\Rightarrow$ unstable flow) or does not ($\Rightarrow$ stable flow).

The basic idea of stability theory



The frictionless simple pendulum: a mass m attached to a light rod of length l that pivots freely about O.

Newton's laws tell us

$$\ddot{\theta} + \omega^2 \sin\theta = 0 \qquad (1)$$

where $\omega^2 = g/l$.

We assume there is a steady solution θ_0, and we want to find its stability. Let

$$\theta = \theta_0 + \epsilon\theta_1(t) \qquad (2)$$

where $\epsilon \ll 1$.

Linearization

Substitute (2) into (1):

$$\begin{aligned} 0 &= \epsilon\ddot{\theta}_1 + \omega^2 \sin(\theta_0 + \epsilon\theta_1) \\ &= \epsilon\ddot{\theta}_1 + \omega^2 \sin\theta_0 \cos(\epsilon\theta_1) + \omega^2 \cos\theta_0 \sin(\epsilon\theta_1). \end{aligned} \tag{3}$$

Substitute the Taylor expansions

$$\cos(\epsilon\theta_1) \sim 1 - \frac{(\epsilon\theta_1)^2}{2} + \ldots, \quad \sin(\epsilon\theta_1) \sim \epsilon\theta_1 - \frac{(\epsilon\theta_1)^3}{6} + \ldots$$

into (3) and equate coefficients of $O(1)$ and $O(\epsilon)$:

$$\sin\theta_0 = 0 \tag{4}$$

$$\ddot{\theta}_1 + (\omega^2 \cos\theta_0)\theta_1 = 0 \tag{5}$$

Stability of steady pendulum solutions

- The nonlinear equation for steady states (4) has solutions $\theta_0 = 0$ and $\theta_0 = \pi$.
- The disturbance equation (5) is linear and its coefficients depend on the steady solution θ_0.
- Substitute $\theta_0 = 0$ into (5):

$$\ddot{\theta}_1 + \omega^2\theta_1 = 0 \quad \Rightarrow \quad \theta_1 = A\cos\omega t + B\sin\omega t, \tag{6}$$

  θ_1 remains bounded as $t \to \infty$, therefore $\theta_0 = 0$ is stable.
- Substitute $\theta_0 = \pi$ into (5):

$$\ddot{\theta}_1 - \omega^2\theta_1 = 0 \quad \Rightarrow \quad \theta_1 = A\,\mathrm{e}^{\omega t} + B\,\mathrm{e}^{-\omega t}, \tag{7}$$

  $\theta_1 \to \infty$ as $t \to \infty$, therefore $\theta_0 = \pi$ is unstable.

Hydrodynamic stability theory...

...is the application of this approach to the governing equations for fluid motion.

Applying Newton's laws of motion to fluid elements leads to the Navier–Stokes equations:

$$\begin{aligned} \nabla \cdot u &= 0 & (8) \\ \frac{\partial u}{\partial t} + (u \cdot \nabla) u &= -\nabla p + \frac{1}{Re} \nabla^2 u & (9) \end{aligned}$$

(incompressible, isothermal, no free boundaries).

The Reynolds number, Re, is based on convenient velocity and length scales, U_0, L, and the kinematic viscosity, ν,

$$Re = \frac{U_0 L}{\nu}. \tag{10}$$

Basic flow plus disturbance

Substitute

$$
\begin{aligned}
u &= u_0(x, y, z) + \epsilon u_1(x, y, z, t) && (11)\\
p &= p_0(x, y, z) + \epsilon p_1(x, y, z, t) && (12)
\end{aligned}
$$

into (8) and (9), then equating terms of $O(1)$ gives the nonlinear equations for the steady solution:

$$
\begin{aligned}
\nabla \cdot u_0 &= 0 && (13)\\
(u_0 \cdot \nabla) u_0 &= -\nabla p_0 + \frac{1}{Re} \nabla^2 u_0. && (14)
\end{aligned}
$$

Linearized Navier–Stokes equations

Equating terms of $O(\epsilon)$ gives

$$\begin{aligned} \nabla \cdot u_1 &= 0 && (15) \\ \frac{\partial u_1}{\partial t} + (u_1 \cdot \nabla) u_0 + (u_0 \cdot \nabla) u_1 &= -\nabla p_1 + \frac{1}{Re} \nabla^2 u_1 && (16) \end{aligned}$$

to be solved for u_1 after u_0 has been found from (13), (14).

- Although (15), (16) are linear, their solution, in general, would require a numerical approach.
- Nonetheless, analytic progress can be made in certain special cases.
- We hope to understand more of the general problem by studying special cases in detail.

Summary of basic steps of linear theory

- Find steady solutions.
- Expand solution as a sum of a steady flow and small unsteady disturbance.
- Substitute this sum into the full nonlinear equations and linearize in disturbance quantities.
- Solve the resulting linear disturbance equation.
- If all disturbances decay, or remain bounded, then the steady flow is stable.
- If any disturbance grows then the steady flow is unstable.

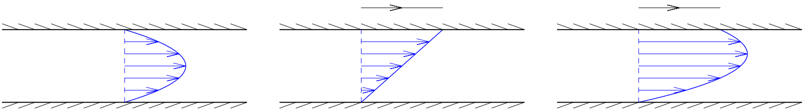
Beyond basic linear theory

- When linear theory predicts stability:
  - Sufficiently small disturbances decay, or remain bounded.
  - But how small do they have to be to decay?
  - Famous examples of stable flows become turbulent, like pipe flow.
  - Disturbances can still grow transiently.
- When linear theory predicts instability:
  - Sufficiently small disturbances grow exponentially.
  - As they grow, they eventually become large enough to invalidate the linear theory. Then what happens?
  - Or the flow could remain laminar in the region of interest if disturbances are swept downstream quickly enough.
- Weakly nonlinear theories predict the next stages of disturbance evolution, if growth rates are weak.
- Absolute/convective instability theory addresses the issue of downstream propagation.

Shear layers

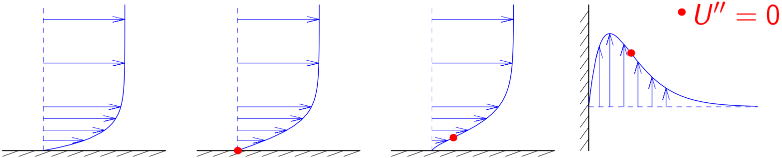
There are many flows that have regions of parallel, or nearly parallel, streamlines where significant shear takes place:

- Channel flows, e.g.



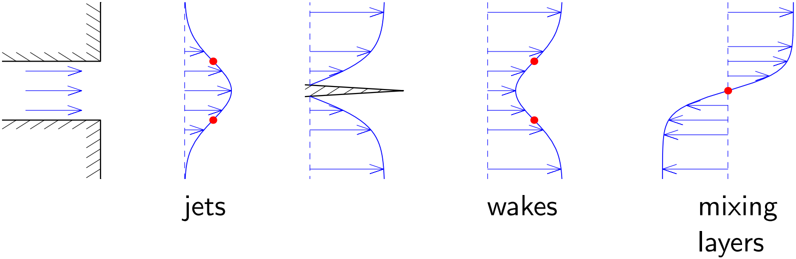
Plane Poiseuille flow | Plane Couette flow | Poiseuille-Couette

- Boundary layers, e.g.



$\partial P/\partial x < 0$ | $\partial P/\partial x = 0$ | $\partial P/\partial x > 0$ | $T_w > T_f$

- Unbounded shear layers, e.g.



- Of these, the channel flows are exactly parallel, the others become ‘more parallel’ as the Reynolds number increases (but needs WKB theory, see later).
- For a parallel flow, or under the parallel flow approximation, the basic flow is described by a single velocity profile:

$$u_0 = U(y)\underline{i}. \tag{17}$$

- There are also axisymmetric versions of some of these flows.

Derivation of disturbance equations

We shall consider two-dimensional disturbances, and work in cartesian coordinates:

$$
\begin{aligned}
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= 0 && (18)\\
\frac{\partial \tilde{u}}{\partial t} + \tilde{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial \tilde{u}}{\partial y} &= -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2}\right) && (19)\\
\frac{\partial \tilde{v}}{\partial t} + \tilde{u}\frac{\partial \tilde{v}}{\partial x} + \tilde{v}\frac{\partial \tilde{v}}{\partial y} &= -\frac{\partial \tilde{p}}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial y^2}\right) && (20)
\end{aligned}
$$

Linearized disturbance equations

The stability of velocity profile $U(y)$ is found by substituting

$$\tilde{u} = U(y) + \epsilon_0 \hat{u}(x, y, t) \tag{21}$$

$$\tilde{v} = \epsilon_0 \hat{v}(x, y, t) \tag{22}$$

$$\tilde{p} = P(x) + \epsilon_0 \hat{p}(x, y, t), \tag{23}$$

where $\epsilon_0 \ll 1$, into (18) – (20), then equating coefficients of $O(\epsilon_0)$:

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = 0 \tag{24}$$

$$\frac{\partial \hat{u}}{\partial t} + U\frac{\partial \hat{u}}{\partial x} + U'\hat{v} = -\frac{\partial \hat{p}}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2}\right) \tag{25}$$

$$\frac{\partial \hat{v}}{\partial t} + U\frac{\partial \hat{v}}{\partial x} = -\frac{\partial \hat{p}}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 \hat{v}}{\partial x^2} + \frac{\partial^2 \hat{v}}{\partial y^2}\right) \tag{26}$$

Normal modes

- Our solution of (24) – (26) relies crucially on the streamlines being parallel, i.e. U is independent of x.
- This makes the solution separable: $\hat{u} = X(x)u(y)T(t)$ etc.
- Attempting a solution of this type, one deduces that X and T are exponential functions of x and t respectively.
- We therefore adopt the normal mode form

$$(\hat{u}, \hat{v}, \hat{p}) = (u(y), v(y), p(y))\mathrm{e}^{\mathrm{i}(\alpha x - \omega t)} \tag{27}$$

  i.e. travelling waves with wavenumber α, frequency ω.
- The linearity of (24) – (26) allows a general solution to be constructed from a superposition of (27) with different α and ω (e.g. by taking Fourier transforms).

Orr-Sommerfeld equation

- Substituting (27) into (24) – (26) gives the ODEs

$$\begin{aligned} \mathrm{i}\alpha u + v' &= 0 && (28) \\ -\mathrm{i}\omega u + \mathrm{i}\alpha U u + U' v &= -\mathrm{i}\alpha p + \frac{1}{Re}\left(u'' - \alpha^2 u\right) && (29) \\ -\mathrm{i}\omega v + \mathrm{i}\alpha U v &= -p' + \frac{1}{Re}\left(v'' - \alpha^2 v\right), && (30) \end{aligned}$$

where $' = \mathrm{d}/\mathrm{d}y$.

- Eliminating u and p gives the Orr-Sommerfeld equation:

$$(U-c)(v''-\alpha^2 v) - U'' v = \frac{1}{\mathrm{i}\alpha Re}\left(v'''' - 2\alpha^2 v'' + \alpha^4 v\right), \quad (31)$$

where $c = \omega/\alpha$.

- This reduction to an ODE is only possible for a parallel flow, otherwise PDEs of the form (24) – (26) must be solved.

Boundary conditions

- No flow through a solid boundary: $v = 0$.
- No slip at a solid boundary: $u = 0 \Rightarrow v' = 0$ from (28).
- If the flow is unbounded as $y \to \infty$ then $v \to 0$ and $v' \to 0$ as $y \to \infty$.
- The four boundary conditions may be summarised as

$$v(y_1) = 0, \quad v'(y_1) = 0, \quad v(y_2) = 0, \quad v'(y_2) = 0, \qquad (32)$$

  where y_1 and/or y_2 could be finite or infinite depending on whether we are considering channel flow, a boundary layer or an unbounded shear layer.
- Nontrivial solutions satisfying these homogeneous boundary conditions are only possible for certain α and ω.
- These values satisfy a relation of the form $\Delta(\alpha, \omega) = 0$ called the dispersion relation.
- Roots of $\Delta(\alpha, \omega) = 0$ are called eigenvalues.

Instability criteria

- Suppose that a Fourier mode with a real α has been chosen giving a complex eigenvalue, $\omega = \omega_r + \mathrm{i}\omega_i$:

$$\mathrm{e}^{\mathrm{i}(\alpha x - \omega t)} = \mathrm{e}^{\mathrm{i}(\alpha x - \omega_r t - \mathrm{i}\omega_i t)} = \mathrm{e}^{\omega_i t}\mathrm{e}^{\mathrm{i}(\alpha x - \omega_r t)}. \tag{33}$$

- Therefore, if for any real α

$$\omega_i > 0 \Rightarrow \text{exponential growth in time} \Rightarrow \text{instability}. \tag{34}$$

- If for all real α

$$\omega_i < 0 \Rightarrow \text{exponential decay in time} \Rightarrow \text{stability}. \tag{35}$$

- Obtaining growth/decay in time is called temporal stability theory.
- Spatial stability theory (real ω, complex α) and spatio-temporal stability theory (complex ω, complex α), i.e. convective/absolute instabilities will be discussed later.

Rayleigh equation

- In the inviscid limit the Orr-Sommerfeld equation (31) reduces to the Rayleigh equation

$$(U - c)(v'' - \alpha^2 v) - U'' v = 0. \tag{36}$$

- The non-slip boundary conditions are dropped for inviscid flow, leaving

$$v(y_1) = 0, \quad v(y_2) = 0 \tag{37}$$

  (the Rayleigh equation is only 2nd order, while the Orr-Sommerfeld equation is 4th order).
- Therefore, when $Re \gg 1$, there will be thin viscous layers near solid boundaries where the non-slip condition is retained.
- Sometimes these viscous layers are important to the stability properties (see later).
- But we start by ignoring them, and just solve the inviscid problem...

2. Inviscid stability theory

- The Rayleigh equation:

$$(U-c)(v''-\alpha^2 v)-U''v=0. \tag{38}$$

- Rayleigh's inflexion point theorem.
- Divide (38) by $(U-c)$, multiply by $\bar{v}$ (the complex conjugate of v) and integrate across the flow domain:

$$\int_{y_1}^{y_2} \bar{v}v''-\left(\frac{U''}{U-c}+\alpha^2\right)|v|^2\,\mathrm{d}y=0. \tag{39}$$

- Integrate the first term by parts:

$$\begin{aligned} \left[\bar{v}v'\right]_{y_1}^{y_2}+\int_{y_1}^{y_2}-\bar{v}'v'-\left(\frac{U''}{U-c}+\alpha^2\right)|v|^2\,\mathrm{d}y &= 0 \quad (40) \\ \Rightarrow\quad \int_{y_1}^{y_2}|v'|^2+\alpha^2|v|^2\,\mathrm{d}y+\int_{y_1}^{y_2}\frac{U''|v|^2}{U-c}\,\mathrm{d}y &= 0. \quad (41) \end{aligned}$$

- Consider the imaginary part of (41), taking α real, with c possibly complex:

$$c_i \int_{y_1}^{y_2} \frac{U''|v|^2}{|U-c|^2}\,\mathrm{d}y = 0. \tag{42}$$

- Therefore, for instability, i.e. $c_i \neq 0$, the integral in (42) must be zero, which can only be true if either $U'' \equiv 0$, or U'' changes sign at least once in the interval $y_1 < y < y_2$.
- This proves Rayleigh's (1880) inflexion point theorem:

A necessary, but not sufficient, condition for instability is that the velocity profile have an inflexion point.

Fjørtoft's theorem

- Let there be an inflexion point at $y = y_I$, and let $U_I = U(y_I)$.
- Note that if $c_i \neq 0$, then (42) implies

$$(c_r - U_I) \int_{y_1}^{y_2} \frac{U''|v|^2}{|U - c|^2} \, \mathrm{d}y = 0. \tag{43}$$

- The real part of (41) is

$$\int_{y_1}^{y_2} \frac{U''(U - c_r)|v|^2}{|U - c|^2} \, \mathrm{d}y = - \int_{y_1}^{y_2} |v'|^2 + \alpha^2 |v|^2 \, \mathrm{d}y. \tag{44}$$

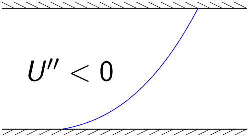
- Adding (43) and (44) leads to

$$\int_{y_1}^{y_2} \frac{U''(U - U_I)|v|^2}{|U - c|^2} \, \mathrm{d}y < 0. \tag{45}$$

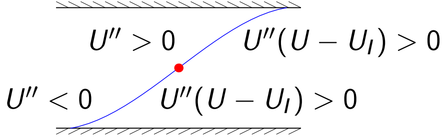
Equation (45) proves Fjørtoft's (1950) theorem:

A necessary, but not sufficient, condition for instability is that $U''(U - U_I) < 0$ somewhere in the flow.

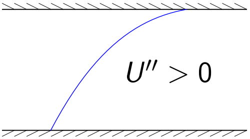
Examples of Rayleigh's and Fjørtoft's results:



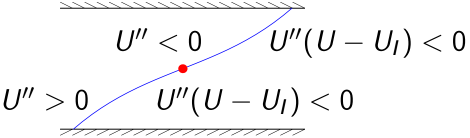
Stable by Rayleigh



Stable by Fjørtoft



Stable by Rayleigh



Could be unstable

Howard's semi-circle theorem

- This is a result placing bounds on the phase velocity and the maximum growth rate of any unstable wave.
- Howard's semi-circle theorem is derived in a similar manner to Rayleigh's and Fjørtoft's, but from the adjoint of the Rayleigh equation.
- The adjoint equation has the same eigenvalues as the Rayleigh equation.
- First derive the adjoint of the Rayleigh equation:
- Multiply (38) by a function w satisfying the same homogeneous boundary conditions as v and integrate:

$$\int_{y_1}^{y_2} \left[(U-c)(v''-\alpha^2 v) - U''v\right] w \, \mathrm{d}y = 0. \qquad (46)$$

- Integrating the v'' term in (46) by parts twice and applying homogeneous boundary conditions gives

$$\int_{y_1}^{y_2} \left[(U-c)(w''-\alpha^2 w)+2U'w'\right] v \,\mathrm{d}y = 0. \tag{47}$$

- Let

$$(U-c)(w''-\alpha^2 w)+2U'w'=0. \tag{48}$$

- (48) is the adjoint of the Rayleigh equation (38).
- It may be verified that if v is a solution of (38), then $w=v/(U-c)$ is a solution of (48).
- The eigenvalues of (48) are the same as the eigenvalues of (38) because w satisfies the same boundary conditions as v.
- (A bi-orthogonality condition exists between the eigenfunctions of an equation and its adjoint, which allows coefficients to be calculated in eigenfunction expansions, which is useful in solving initial value problems).

- Multiply the adjoint Rayleigh equation (48) by $(U - c)$ and write the result as

$$\left[(U-c)^2 w'\right]' - \alpha^2 (U-c)^2 w = 0. \tag{49}$$

- Multiply (49) by $\bar{w}$ and integrate:

$$\int_{y_1}^{y_2} \bar{w} \left[(U-c)^2 w'\right]' - \alpha^2 (U-c)^2 |w|^2 \, \mathrm{d}y = 0. \tag{50}$$

- Integrating the first term once by parts gives

$$\int_{y_1}^{y_2} (U-c)^2 Q \, \mathrm{d}y = 0 \tag{51}$$

where $Q = |w'|^2 + \alpha^2 |w|^2$ is positive definite.

- Equating the imaginary parts of (51) gives

$$\int_{y_1}^{y_2} UQ \, \mathrm{d}y = c_r \int_{y_1}^{y_2} Q \, \mathrm{d}y. \tag{52}$$

- Equating the real parts of (51), and making use of (52), gives

$$\int_{y_1}^{y_2} U^2 Q \, \mathrm{d}y = \left(c_r^2 + c_i^2\right) \int_{y_1}^{y_2} Q \, \mathrm{d}y. \tag{53}$$

- Let U_{min} and U_{max} be the minimum and maximum velocities respectively of the basic flow, then clearly

$$\int_{y_1}^{y_2} (U - U_{min})(U - U_{max}) Q \, \mathrm{d}y \leq 0. \tag{54}$$

- Multiplying out the brackets in (54) and substituting in (52) and (53) gives

$$\left[c_r^2 + c_i^2 - (U_{min} + U_{max})c_r + U_{min}U_{max}\right] \int_{y_1}^{y_2} Q \, \mathrm{d}y \leq 0. \tag{55}$$

- Therefore

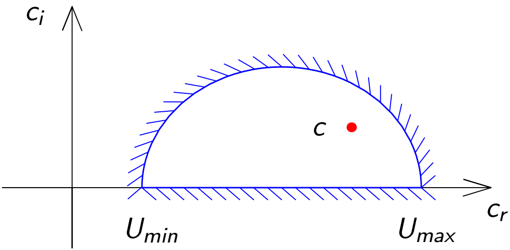
$$\left[c_r^2 + c_i^2 - (U_{min} + U_{max})c_r + U_{min}U_{max}\right] \leq 0, \tag{56}$$

which can be arranged to give

$$\left[c_r - \frac{1}{2}(U_{min} + U_{max})\right]^2 + c_i^2 \leq \left[\frac{1}{2}(U_{max} - U_{min})\right]^2. \tag{57}$$

Equation (57) proves Howard's (1961) semi-circle theorem for unstable waves:

The complex phase velocity c lies inside, or on, the semi-circle centred on $(U_{max} + U_{min})/2$ with radius $(U_{max} - U_{min})/2$:

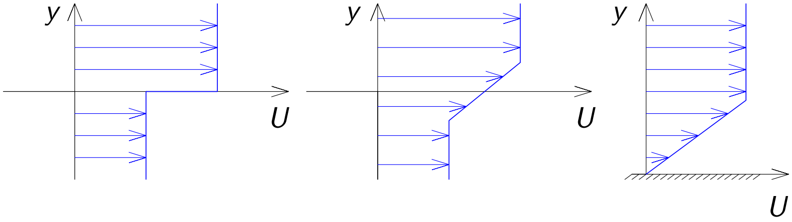


Piecewise-linear profiles

- In general, the difficulty in solving the Rayleigh equation (38) lies in the y-dependence of the coefficients.
- However, for profiles made up of linear segments, $U'' = 0$, and then (38) reduces to

$$v'' - \alpha^2 v = 0 \tag{58}$$

- This suggests a method for estimating the stability of smooth profiles by approximating them by piecewise-linear profiles, e.g.



Jump conditions

- Solutions to (58) in neighbouring segments of the basic profile must be related to one another.
- Let either U or U' be discontinuous at $y = y_0$.
- Let

$$\Delta f = \lim_{\epsilon \to 0} \{ f(y_0 + \epsilon) - f(y_0 - \epsilon) \} \tag{59}$$

  be the jump in a quantity f across y_0.
- Note that the Rayleigh equation (38) can be written

$$\left[(U - c)v' - U'v \right]' - \alpha^2 (U - c)v = 0. \tag{60}$$

- Integrate (60) across the discontinuity from $y_0 - \epsilon$ to $y_0 + \epsilon$:

$$\left[(U-c)v' - U'v\right]_{y_0-\epsilon}^{y_0+\epsilon} - \alpha^2 \int_{y_0-\epsilon}^{y_0+\epsilon} (U-c)v \, \mathrm{d}y = 0. \quad (61)$$

- In the limit $\epsilon \to 0$ the integral term in (61) vanishes, giving the first jump condition:

$$\Delta \left[(U-c)v' - U'v\right] = 0. \quad (62)$$

- Note that eliminating u between (28) and (29) (with $1/Re \to 0$) gives

$$\mathrm{i}\alpha p = (U-c)v' - U'v. \quad (63)$$

- Therefore (62) corresponds to continuous pressure across the discontinuity (dynamic boundary condition).

- Divide (63) by $(U-c)^2$:

$$\frac{\mathrm{i}\alpha p}{(U-c)^2} = \frac{v'}{U-c} - \frac{U'v}{(U-c)^2} = \left[\frac{v}{U-c}\right]'. \tag{64}$$

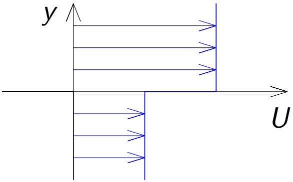
- Integrating (64) across the discontinuity from $y_0 - \epsilon$ to $y_0 + \epsilon$, taking the limit $\epsilon \to 0$ and noting that the integral on the LHS vanishes, gives the second jump condition

$$\Delta\left[\frac{v}{U-c}\right] = 0. \tag{65}$$

- When c is real, $v/(U-c)$ is the ratio of vertical to horizontal velocity in a frame of reference moving at c, and so gives the slope of the streamlines in this frame.
- Therefore, (61) implies that the streamlines on one side of the discontinuity are parallel to those on the other (corresponds to a kinematic boundary condition).

Kelvin-Helmholtz instability

- Our first example is the simplest model of a mixing layer:



where $$U = \begin{cases} U_1 & \text{if } y > 0 \\ U_2 & \text{if } y < 0 \end{cases} .$$

- In each layer the Rayleigh eqn (38) reduces to $v'' - \alpha^2 v = 0$.
- Let

$$v = \begin{cases} v_1 & \text{if } y > 0 \\ v_2 & \text{if } y < 0 \end{cases} . \tag{66}$$

- The solutions satisfying homogeneous boundary conditions $v_1 \to 0$ as $y \to \infty$, and $v_2 \to 0$ as $y \to -\infty$, are

$$v_1 = A\mathrm{e}^{-\alpha y}, \quad v_2 = B\mathrm{e}^{\alpha y} \tag{67}$$

where $\alpha > 0$.

- Applying the first jump condition (62) to the solutions (67) gives

$$\begin{aligned} (U_1 - c)v_1'(0) &= (U_2 - c)v_2'(0) \qquad (68) \\ \Rightarrow \quad (U_1 - c)(-\alpha)A &= (U_2 - c)\alpha B \qquad (69) \end{aligned}$$

- Applying the second jump condition (65) to (67) gives

$$\begin{aligned} \frac{v_1(0)}{(U_1 - c)} &= \frac{v_2(0)}{(U_2 - c)} \qquad (70) \\ \Rightarrow \quad \frac{A}{(U_1 - c)} &= \frac{B}{(U_2 - c)} \qquad (71) \end{aligned}$$

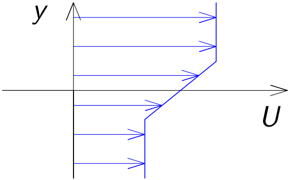
- Eliminating A/B between (69) and (71) leads to the dispersion relation for Kelvin-Helmholtz instability on a vortex sheet:

$$c = \frac{1}{2}(U_1 + U_2) \pm \frac{\mathrm{i}}{2}(U_1 - U_2). \tag{72}$$

- There is instability for all $U_1 \neq U_2$.
- The phase velocity is the mean of U_1 and U_2.
- In fact, c lies at the top of Howard's semi-circle.
- The flow is nondispersive: c is independent of α.
- The growth rate $\omega_i = \alpha c_i = \alpha |U_1 - U_2|$, i.e. the shorter the wave the greater the growth rate.
- This nonphysical growth arises because there is no natural length scale in the basic flow.

Mixing layer with finite thickness

- Consider a mixing layer of thickness h:



where

$$U = \begin{cases} U_1 & \text{if } y > h/2 \\ (U_1 + U_2)/2 + (U_1 - U_2)y/h & \text{if } -h/2 < y < h/2 \\ U_2 & \text{if } y < -h/2 \end{cases} . \tag{73}$$

- Let

$$v = \begin{cases} v_1 & \text{if } y > h/2 \\ v_2 & \text{if } -h/2 < y < h/2 \\ v_3 & \text{if } y < -h/2 \end{cases} . \tag{74}$$

- In each layer the Rayleigh eqn (38) reduces to $v'' - \alpha^2 v = 0$.
- The solutions satisfying homogeneous boundary conditions are

$$v_1 = A\mathrm{e}^{-\alpha y} \quad (75)$$

$$v_2 = B\mathrm{e}^{-\alpha y} + C\mathrm{e}^{\alpha y} \quad (76)$$

$$v_3 = D\mathrm{e}^{\alpha y}. \quad (77)$$

- Applying the first jump condition (62) to solutions (75) and (76) at $y = h/2$ gives

$$-(U_1 - c)\alpha A\mathrm{e}^{-\alpha h/2} = (U_1 - c)\left(-\alpha B\mathrm{e}^{-\alpha h/2} + \alpha C\mathrm{e}^{\alpha h/2}\right) - \frac{U_1 - U_2}{h}\left(B\mathrm{e}^{-\alpha h/2} + C\mathrm{e}^{\alpha h/2}\right). \quad (78)$$

- Applying the second jump condition (65) to solutions (75) and (76) at $y = h/2$ gives

$$\frac{A\mathrm{e}^{-\alpha h/2}}{U_1 - c} = \frac{B\mathrm{e}^{-\alpha h/2} + C\mathrm{e}^{\alpha h/2}}{U_1 - c}. \tag{79}$$

- Applying jump conditions (62) and (65) to solutions (76) and (77) at $y = -h/2$ gives

$$\begin{aligned}(U_2 - c)\alpha D\mathrm{e}^{-\alpha h/2} &= (U_2 - c)\left(-\alpha B\mathrm{e}^{\alpha h/2} + \alpha C\mathrm{e}^{-\alpha h/2}\right) - \\ &\quad \frac{U_1 - U_2}{h}\left(B\mathrm{e}^{\alpha h/2} + C\mathrm{e}^{-\alpha h/2}\right) && (80)\\ \frac{D\mathrm{e}^{-\alpha h/2}}{U_2 - c} &= \frac{B\mathrm{e}^{\alpha h/2} + C\mathrm{e}^{-\alpha h/2}}{U_2 - c}. && (81)\end{aligned}$$

- Eliminating A, B, C and D from (78), (79), (80) and (81) gives the dispersion relation

$$c = \frac{1}{2}(U_1 + U_2) \pm \frac{U_1 - U_2}{2\alpha h}\left[(1-\alpha h)^2 - \mathrm{e}^{-2\alpha h}\right]^{1/2}. \qquad (82)$$

- In the long-wave limit, i.e. $\alpha h \ll 1$, (82) reduces to

$$c \sim \frac{1}{2}(U_1 + U_2) \pm \frac{\mathrm{i}}{2}(U_1 - U_2). \qquad (83)$$

- Therefore, for waves that are long compared with the shear layer thickness there is instability, and the Kelvin-Helmholtz vortex sheet is a good model.

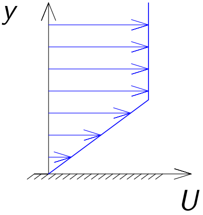
- In the short-wave limit, i.e. $\alpha h \gg 1$, (82) reduces to

$$c \sim U_1 - \frac{U_1 - U_2}{2\alpha h}, \quad U_2 + \frac{U_1 - U_2}{2\alpha h}. \tag{84}$$

- Therefore waves that are short compared with the shear layer thickness are stable and have phase velocities close to the free stream velocities (the joins in the profile act as waveguides).
- The change from stable to unstable behaviour takes place for wavelengths that are comparable to the shear layer thickness, i.e. $\alpha h = O(1)$.
- The neutral wavenumber is $\alpha h = 1.278$.
- The most unstable wave is on the same scaling: its wavenumber is $\alpha h = 0.7968$ and the strongest growth rate is $0.2012(U_1 - U_2)/h$.

Model boundary layer

- Consider a model boundary layer of thickness δ:



where
$$U = \begin{cases} U_1 & \text{if } y > \delta \\ U_1 y/\delta & \text{if } 0 < y < \delta \end{cases} . \tag{85}$$

- Let
$$v = \begin{cases} v_1 & \text{if } y > \delta \\ v_2 & \text{if } 0 < y < \delta \end{cases} . \tag{86}$$

- Solutions to the reduced Rayleigh equation (38), $v'' - \alpha^2 v = 0$, satisfying homogeneous boundary conditions:

$$v_1 = A\mathrm{e}^{-\alpha y} \tag{87}$$
$$v_2 = B\left(\mathrm{e}^{-\alpha y} - \mathrm{e}^{\alpha y}\right) . \tag{88}$$

- Applying the jump conditions (62) and (65) to the solutions (87) and (88) at $y = \delta$ gives

$$\begin{aligned} -(U_1 - c)\alpha A \mathrm{e}^{-\alpha\delta} &= -(U_1 - c)\alpha B \left(\mathrm{e}^{-\alpha\delta} + \mathrm{e}^{\alpha\delta}\right) \\ &\quad -\frac{U_1}{\delta} B \left(\mathrm{e}^{-\alpha\delta} - \mathrm{e}^{\alpha\delta}\right) \qquad (89) \\ \frac{A\mathrm{e}^{-\alpha\delta}}{U_1 - c} &= \frac{B(\mathrm{e}^{-\alpha\delta} - \mathrm{e}^{\alpha\delta})}{U_1 - c} \qquad (90) \end{aligned}$$

- Eliminating A and B from (89) and (90) gives the dispersion relation

$$c = \frac{U_1}{2\alpha\delta} \left(\mathrm{e}^{-2\alpha\delta} - 1 + 2\alpha\delta\right), \qquad (91)$$

showing this boundary layer is stable for all α.

- In the long-wave limit, i.e. $\alpha\delta \ll 1$, we have

$$c \sim \alpha\delta U_1. \qquad (92)$$

Tollmien's critical point solutions

- In fact, basic flow curvature, U'', can make an important contribution to the (in)stability of smooth velocity profiles.
- Tollmien (1929) showed that this contribution comes from critical points y_c defined by $U(y_c) = c$.
- Critical points are regular singular points of the Rayleigh equation (38).
- Seek series solutions about y_c: assume

$$v = \sum_{n=0} a_n (y - y_c)^n. \tag{93}$$

- Expand the basic flow about y_c:

$$U = c + (y - y_c)U'_c + \frac{1}{2}(y - y_c)^2 U''_c + \dots \tag{94}$$

where $U'_c = U'(y_c) \neq 0$, $U''_c = U''(y_c)$ etc.

- Substituting expansions (93) and (94) into the Rayleigh equation (38), and equating coefficients of powers of $(y - y_c)$ gives

$$a_0 = 0, \quad a_2 = \frac{U_c''}{2U_c'} a_1, \quad a_3 = \frac{1}{6}\left(\frac{U_c'''}{U_c'} + \alpha^2\right) a_1, \ldots \tag{95}$$

- The constant a_1 is arbitrary since (38) is a linear homogeneous equation.
- We can choose to normalize this solution such that $a_1 = 1$, and denote it by

$$v_1 = (y - y_c) + \frac{U_c''}{2U_c'}(y - y_c)^2 + \frac{1}{6}\left(\frac{U_c'''}{U_c'} + \alpha^2\right)(y - y_c)^3 + \ldots \tag{96}$$

- To find a second independent solution to the Rayleigh equation (38), consider $a_0 \neq 0$.
- The leading terms in (38) near $y = y_c$ are now

$$U_c'(y - y_c)\left(v'' - \alpha^2 a_0\right) - U_c'' a_0 = 0. \tag{97}$$

- Balance at leading order now requires

$$v'' \sim (y - y_c)^{-1} \quad \Rightarrow \quad v \sim (y - y_c)\ln(y - y_c). \tag{98}$$

- Therefore, let

$$v = 1 + b_1 v_1 \ln(y - y_c) + \sum_{n=2} b_n (y - y_c)^n \tag{99}$$

where we have chosen the normalization $a_0 = 1$, there is no coefficient of $O(y - y_c)$ since this is contained in the v_1 solution, and the use of v_1 in the coefficient of $\ln(y - y_c)$ is acceptable since $v_1 \sim (y - y_c)$.

- Substitute (99) into (38): the coefficient of $\ln(y - y_c)$ vanishes since v_1 satisfies (38).
- Equating coefficients of powers of $(y - y_c)$ gives the b_n. Calling this solution v_2, we find

$$
\begin{aligned}
v_2 &= 1 + \frac{U_c''}{U_c'} v_1 \ln(y - y_c) \\
&\quad + \frac{1}{2}\left[\frac{U_c'''}{U_c'} - \left(\frac{2U_c''}{U_c'}\right)^2 + \alpha^2\right](y - y_c)^2 + \ldots
\end{aligned}
\tag{100}
$$

- v_1, (96), is the regular solution and v_2, (100), is the singular solution.
- The general solution to the Rayleigh equation (38) near a critical point y_c is

$$
v = Av_1 + Bv_2 \tag{101}
$$

where A and B are arbitrary constants.

Path around a critical point

- The presence of the logarithm raises the question of which branch to use when $y < y_c$ when y_c is real.
- I.e. should we take $\ln(y - y_c) = \ln(y_c - y) + \mathrm{i}\pi$ or $\ln(y - y_c) = \ln(y_c - y) - \mathrm{i}\pi$?
- This question was first answered by looking at the viscous version of the problem, i.e. the Orr-Sommerfeld equation.
- Viscosity removes the singularity in the solution. Its effects were originally understood using WKB theory, though matched asymptotic expansions can also be used. The branch is chosen that corresponds to the viscous solution.
- This question can also be answered without appeal to viscosity, but by considering the inviscid initial value problem.

- When $\alpha > 0$, it turns out (Lin's critical point rule) that

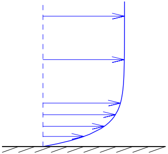
$$\begin{aligned} U_c' > 0 \quad &\Rightarrow \quad \text{solution path passes below critical point} \\ &\Rightarrow \quad \ln(y - y_c) = \ln(y_c - y) - \mathrm{i}\pi \quad \text{for } y < y_c \end{aligned} \tag{102}$$

- and

$$\begin{aligned} U_c' < 0 \quad &\Rightarrow \quad \text{solution path passes above critical point} \\ &\Rightarrow \quad \ln(y - y_c) = \ln(y_c - y) + \mathrm{i}\pi \quad \text{for } y < y_c. \end{aligned} \tag{103}$$

- The imaginary term generated by the logarithm, $\pm \mathrm{i}\pi$, (the 'phase jump' at the critical point) can be stabilizing or destabilizing depending on the sign of U_c'', since this multiplies the logarithm in (100).
- For a neutral wave, with a real dispersion relation, when there is a single critical point, we must have $U_c'' = 0$, i.e. the critical point lies at the inflexion point. The eigenfunction is then regular.
- If two critical points are present, a neutral wave can be constructed by requiring the stabilizing contribution from one critical point to cancel the destabilizing contribution of the other.

Long-wave inviscid boundary layer stability



- Consider a smooth boundary layer velocity profile $U(y)$ that increases monotonically from $U(0) = 0$ (at the wall) to $U = 1$ in the freestream.
- Let the boundary layer thickness (e.g. displacement thickness, momentum thickness, or distance from the wall to where U is, say 95% of its freestream value) be of $O(1)$.
- Consider disturbances whose wavelengths are long compared with the boundary layer thickness, so let $\alpha = \alpha_0 \epsilon$, $\epsilon \ll 1$.
- Far from the boundary layer, $y \gg 1$, we have $U = 1$, and so in this region the solution takes the form

$$v = A\mathrm{e}^{-\alpha y} = A\mathrm{e}^{-\alpha_0 \epsilon y} \qquad (104)$$

as in (87).

- Therefore, the length scale over which the disturbance decays outside the boundary layer is $O(\epsilon^{-1})$, i.e. of order the disturbance wavelength.
- Behaviour of v inside the boundary layer: at leading order as $\epsilon \to 0$, the Rayleigh equation (38) reduces to

$$(U - c)v'' - U''v = 0. \tag{105}$$

- By inspection,

$$v = U - c \tag{106}$$

  is a solution of (105).
- A second solution can be found by substituting $v = (U - c)V$ into (105):

- giving

$$V'' + \frac{2U'}{U-c} V' = 0 \tag{107}$$

- (107) can be solved using an integrating factor for V' leading to a second solution of (105):

$$v = (U-c) \int \frac{\mathrm{d}y}{(U-c)^2}. \tag{108}$$

- But this integral has a non-integrable singularity at $y = y_c$.
- Note that the Laurent series of the integrand about y_c is

$$\frac{1}{(U-c)^2} \sim \frac{1}{(U_c')^2 (y-y_c)^2} - \frac{U_c''}{(U_c')^3 (y-y_c)} + O(1). \tag{109}$$

- We can extract the singular part from the integral:

$$\begin{aligned}
\int \frac{\mathrm{d}y}{(U-c)^2} &= \int \Bigg[\frac{1}{(U-c)^2} - \frac{1}{(U_c')^2(y-y_c)^2} + \frac{U_c''}{(U_c')^3(y-y_c)}\Bigg] \\
&\quad + \frac{1}{(U_c')^2(y-y_c)^2} - \frac{U_c''}{(U_c')^3(y-y_c)}\,\mathrm{d}y \\
&= \int \frac{1}{(U-c)^2} - \frac{1}{(U_c')^2(y-y_c)^2} + \frac{U_c''}{(U_c')^3(y-y_c)}\,\mathrm{d}y \\
&\quad - \frac{1}{(U_c')^2(y-y_c)} - \frac{U_c''}{(U_c')^3}\ln(y-y_c) \qquad (110)
\end{aligned}$$

- where the integrand in (110) is regular.

- We choose the second solution to (105) to be

$$v = (U-c)\left[\int_0^y \frac{1}{(U-c)^2} - \frac{1}{(U_c')^2(t-y_c)^2} + \frac{U_c''}{(U_c')^3(t-y_c)}\,\mathrm{d}t - \frac{1}{(U_c')^2(y-y_c)} - \frac{U_c''}{(U_c')^3}\ln(y-y_c)\right] \tag{111}$$

  where the integrand is regular.
- The solution in the main part of the boundary layer is a linear superposition of (106) and (111).
- (106) is the long-wave version of the regular Tollmien solution (96), and (111) has the logarithmic behaviour of the singular Tollmien solution (100).

- In the long-wave limit the piecewise-linear boundary layer had $c = O(\alpha)$, see (92) (when $\delta U_1 = O(1)$, as here).
- Does c follow the same scaling for a smooth profile?
- Try

$$c = c_0 \epsilon^a \tag{112}$$

where a is some constant to be determined, and $c_0 = O(1)$.
- If $a > 0$, then c is small, and the critical point is close to the wall because there U is also small.
- The Taylor series for U is

$$U = U_0' y + U_0'' \frac{y^2}{2} + \dots \tag{113}$$

where $U_0' = U'(0)$, etc.

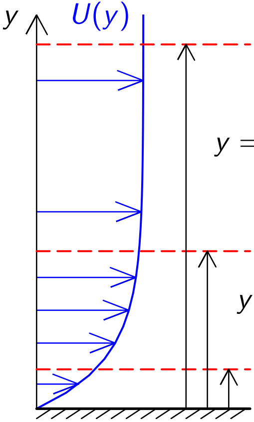
- Setting $c = U(y_c)$, using (112) and the leading term of (113), gives

$$y_c \sim \frac{c_0 \epsilon^a}{U_0'}. \tag{114}$$

- The regular part of the solution, (106), varies over the same scale as U, i.e. on the scale $y = O(1)$.
- The singular part of the solution, (111), varies most sharply when y is comparable to y_c, i.e. on the scale $y = O(\epsilon^a)$.
- And outside the boundary layer, the disturbance decays on the scale $y = O(\epsilon^{-1})$, see (104).

Inviscid triple-decked disturbances

In summary, the disturbance varies on three different length scales in different parts of the flow:



$y = O(\epsilon^{-1})$, upper layer, exponential behaviour

$y = O(1)$, main layer

$y = O(\epsilon^{a})$, critical layer, logarithmic behaviour

Matched asymptotic expansions

- The method of matched asymptotic expansions involves obtaining solutions in each layer, then matching them to produce a solution uniformly valid across all layers (we shall use van Dyke's matching rule).
- We shall work directly with the inviscid version of the linearized continuity and momentum equations (28) – (30), rather than the Rayleigh equation:

$$\begin{aligned} \mathrm{i}\alpha u + v' &= 0 && (115) \\ -\mathrm{i}\omega u + \mathrm{i}\alpha U u + U' v &= -\mathrm{i}\alpha p && (116) \\ -\mathrm{i}\omega v + \mathrm{i}\alpha U v &= -p'. && (117) \end{aligned}$$

- These equations take different, simplified forms in each layer.
- The thickness of the critical layer, $O(\epsilon^a)$, is determined as part of the solution process.

- The idea of dominant balance is central to the method.
- The order of magnitudes (in terms of powers of ϵ) of u, v and p can be different in each layer.
- These various orders of magnitude, and the critical layer thickness, are all determined in the leading order calculation.
- We shall start with the leading order calculation, then add higher order terms later.
- Note that

$$\omega = c\alpha = c_0\alpha_0\epsilon^{a+1} \tag{118}$$

using $\alpha = \alpha_0\epsilon$ and (112).

Upper layer

- Define an upper-layer variable $y = y_u/\epsilon$, where $y_u = O(1)$ in the upper layer.
- In the upper layer $U = 1 \Rightarrow U' = 0$.
- Let

$$u(y) = u_{u0}(y_u), \quad v(y) = v_{u0}(y_u), \quad p(y) = p_{u0}(y_u). \tag{119}$$

  be the upper layer variables to be solved for.
- Note, e.g.,

$$v' = \frac{\mathrm{d}v}{\mathrm{d}y} = \frac{\mathrm{d}v_{u0}}{\mathrm{d}y} = \frac{\mathrm{d}v_{u0}}{\mathrm{d}y_u}\frac{\mathrm{d}y_u}{\mathrm{d}y} = \epsilon v'_{u0}. \tag{120}$$

- Substitute (118), (119), $U = 1$ and $\alpha = \alpha_0\epsilon$ into (115) – (117). At $O(\epsilon)$:

$$\begin{aligned} \mathrm{i}\alpha_0 u_{u0} + v'_{u0} &= 0 && (121)\\ \mathrm{i}\alpha_0 u_{u0} + \mathrm{i}\alpha_0 p_{u0} &= 0 && (122)\\ \mathrm{i}\alpha_0 v_{u0} + p'_{u0} &= 0. && (123) \end{aligned}$$

- Eliminating u_{u0} and v_{u0} from (121) – (123) gives

$$p''_{u0} - \alpha_0^2 p_{u0} = 0. \tag{124}$$

- The solution to (124) that decays far from the wall, and hence v_{u0} from (123), is

$$p_{u0} = P_{u0}\mathrm{e}^{-\alpha_0 y_u} \tag{125}$$
$$v_{u0} = -\mathrm{i}P_{u0}\mathrm{e}^{-\alpha_0 y_u} \tag{126}$$

  where P_{u0} is a constant.

- These upper-layer solutions will be matched to the main-layer solutions.

Main layer

- The main-layer variables are

$$u = \epsilon^{-1} u_{m0}(y), \quad v = v_{m0}(y), \quad p = p_{m0}(y). \qquad (127)$$

- The magnitudes of v and p in the main layer are the same as in the upper layer so they can be matched with the upper layer.
- The principle of dominant balance requires $u = O(\epsilon^{-1})$, so that u appears at leading order in the continuity equation:
- Substituting (118), (127) and $\alpha = \alpha_0 \epsilon$ into (115) – (117) gives at leading order

$$\begin{aligned} \mathrm{i}\alpha_0 u_{m0} + v'_{m0} &= 0 && (128)\\ \mathrm{i}\alpha_0 U u_{m0} + U' v_{m0} &= 0 && (129)\\ p'_{m0} &= 0. && (130) \end{aligned}$$

- Therefore, in the main layer the velocity and pressure fields decouple.
- The solutions of (128) – (130) are

$$p_{m0} = P_{m0} \tag{131}$$
$$v_{m0} = A_{m0}U \tag{132}$$

where A_{m0} and P_{m0} are constants.
- These main-layer solutions are to be matched with both the upper layer and the critical layer.
- Note that $v_{m0} \sim A_{m0}U_0'y$ for small y, and hence $v_{m0} = O(\epsilon^a)$ in the critical layer.

Critical layer

- Define a critical layer variable $y = \epsilon^a Y$, where $Y = O(1)$ in the critical layer.
- In the critical layer, the basic flow takes the form

$$U = \epsilon^a U_0' Y + \epsilon^{2a} U_0'' \frac{Y^2}{2} + \ldots, \quad U' = U_0' + \epsilon^a U_0'' Y + \ldots \quad (133)$$

- Let

$$u = \epsilon^{-1} u_{c0}(Y), \quad v = \epsilon^a v_{c0}(Y), \quad p = p_{c0}(Y). \quad (134)$$

- The magnitudes of v and p are chosen so they can be matched to the main-layer solutions.
- Dominant balance requires $u = O(\epsilon^{-1})$ so that u_{c0} appears at leading order in the continuity equation.

- Substituting (118), (133), (134) and $\alpha = \alpha_0\epsilon$ into (115) – (117), and neglecting the unambiguously smaller terms, gives

$$\begin{aligned} \mathrm{i}\alpha_0 u_{c0} + v'_{c0} &= 0 \quad (135)\\ -\mathrm{i}c_0\alpha_0\epsilon^a u_{c0} + \mathrm{i}\alpha_0\epsilon^a U'_0 Y u_{c0} + \epsilon^a U'_0 v_{c0} + \mathrm{i}\alpha_0\epsilon p_{c0} &= 0 \quad (136)\\ p'_{c0} &= 0. \quad (137) \end{aligned}$$

- The principle of dominant balance gives $a = 1$.
- This choice of a couples the velocity and pressure fields at leading order.
- $a = 1 \Rightarrow c = O(\epsilon) \Rightarrow c = O(\alpha)$, i.e. the same scaling as for the piecewise-linear boundary layer, (92).

- With $a = 1$, (135) – (137) become

$$\begin{aligned} \mathrm{i}\alpha_0 u_{c0} + v'_{c0} &= 0 \qquad (138) \\ -\mathrm{i}c_0\alpha_0 u_{c0} + \mathrm{i}\alpha_0 U'_0 Y u_{c0} + U'_0 v_{c0} + \mathrm{i}\alpha_0 p_{c0} &= 0 \qquad (139) \\ p'_{c0} &= 0. \qquad (140) \end{aligned}$$

- The general solution to (138) – (140) is

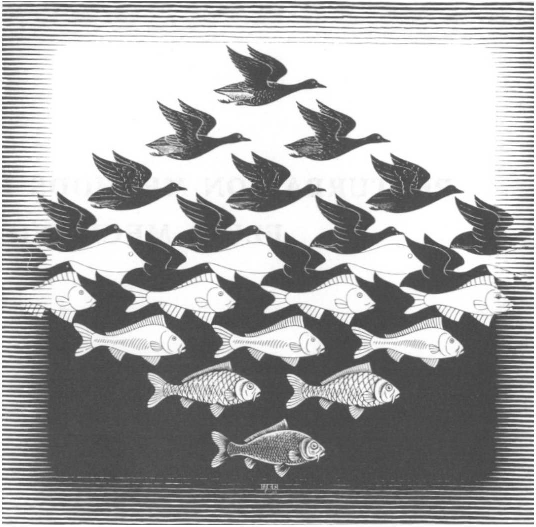
$$\begin{aligned} p_{c0} &= P_{c0} \qquad (141) \\ v_{c0} &= B_{c0}\left(U'_0 Y - c_0\right) - \frac{\mathrm{i}\alpha_0 P_{c0}}{c_0} Y \qquad (142) \end{aligned}$$

where P_{c0} and B_{c0} are a constants.

Matching between layers

- We have found the thickness of the critical layer and the solutions in all three layers.
- Now match between the layers to find relations between the constants P_{u0}, A_{m0}, P_{m0}, B_{c0} and P_{c0}.
- Essentially, this requires the solutions in adjacent layers to merge into one-another in an over-lap region between the two layers.
- We shall use van Dyke's (1964) matching rule.

‘Imperceptible blending’ – M.C. Escher



- In van Dyke's matching rule, the solution in each layer is written in terms of the variable in the next layer, e.g. $v_{u0}(y_u) = v_{u0}(\epsilon y)$, and then expanded as a series as $\epsilon \to 0$.
- We write the matching of v between the upper and main layers as

$$H_0\{v_{u0}(\epsilon y)\} = H_0\{v_{m0}(y_u/\epsilon)\} \tag{143}$$

where $H_n\{\cdot\}$ means expand its argument in powers of ϵ up to and including $O(\epsilon^n)$.

- Substituting (126) and (132) into (143) we have

$$\begin{aligned} H_0\{v_{u0}(\epsilon y)\} &= H_0\{-\mathrm{i}P_{u0}\mathrm{e}^{-\alpha_0\epsilon y}\} \\ &= H_0\{-\mathrm{i}P_{u0}(1-\epsilon\alpha_0 y+\ldots\} = -\mathrm{i}P_{u0} \quad (144) \\ H_0\{v_{m0}(y_u/\epsilon)\} &= H_0\{A_{m0}U(y_u/\epsilon)\} = H_0\{A_{m0}\} = A_{m0} \quad (145) \\ \Rightarrow \quad -\mathrm{i}P_{u0} &= A_{m0}. \quad (146) \end{aligned}$$

- To match pressures between upper and main layers, substitute (125) and (131) into

$$
\begin{aligned}
H_0\{p_{u0}(\epsilon y)\} &= H_0\{p_{m0}(y_u/\epsilon)\} \qquad (147)\\
\Rightarrow \quad P_{u0} &= P_{m0}. \qquad (148)
\end{aligned}
$$

- To match pressures between main and critical layers, substitute (131) and (141) into

$$
\begin{aligned}
H_0\{p_{m0}(\epsilon Y)\} &= H_0\{p_{c0}(y/\epsilon)\} \qquad (149)\\
\Rightarrow \quad P_{m0} &= P_{c0}. \qquad (150)
\end{aligned}
$$

- To match velocities between main and critical layers, substitute (132) and (142) into

$$
\begin{aligned}
H_1\{v_{m0}(\epsilon Y)\} &= H_0\{\epsilon v_{c0}(y/\epsilon)\} && (151)\\
\Rightarrow \quad H_1\{A_{m0}U(\epsilon Y)\} &= H_0\left\{\epsilon\left[B_{c0}\left(U_0'\frac{y}{\epsilon}-c_0\right)-\frac{\mathrm{i}\alpha_0 P_{c0}}{c_0}\frac{y}{\epsilon}\right]\right\}\\
\Rightarrow \quad A_{m0}U_0'\epsilon Y &= B_{c0}U_0'y-\frac{\mathrm{i}\alpha_0 P_{c0}}{c_0}y\\
\Rightarrow \quad A_{m0}U_0' &= B_{c0}U_0'-\frac{\mathrm{i}\alpha_0 P_{c0}}{c_0}. && (152)
\end{aligned}
$$

- Eliminating the constants P_{u0}, A_{m0} and P_{m0} from (146), (148), (150) and (152) gives

$$B_{c0}=\mathrm{i}P_{c0}\left(\frac{\alpha_0}{U_0'c_0}-1\right). \qquad (153)$$

- Substituting (153) into (142) gives

$$v_{c0}=-\mathrm{i}P_{c0}\left(U_0'Y-c_0+\frac{\alpha_0}{U_0'}\right). \qquad (154)$$

- (154) can be interpretted as a linear combination of the first terms from Tollmien's solutions (96) and (100).
- Van Dyke's matching rule gives the particular linear combination that decays exponentially as $y \to \infty$.
- Applying the wall boundary-condition $v_{c0}(0) = 0$ to (154) gives the leading-order inviscid dispersion relation:

$$c_0 = \frac{\alpha_0}{U_0'} \quad \Rightarrow \quad c = \frac{\alpha}{U_0'}. \tag{155}$$

- The piecewise-linear boundary layer (85) has a velocity gradient in the lower segment of $U_1/\delta = U_1' \Rightarrow \delta = U_1/U_1'$, hence the long-wave result (92) can be written $c \sim \alpha U_1^2/U_1'$.
- (155) was derived for a boundary layer with $U = 1$ in the freestream, corresponding to $U_1 = 1$, and the piecewise-linear long-wave result is then $c \sim \alpha/U_1'$.
- Therefore, for long waves, piecewise-linear theory agrees best with a smooth $U(y)$ if velocity gradients at the wall are equal.

Second-order long-wave theory

- Having obtained the leading-order long-wave solution, and established the matched asymptotic framework for the problem, it is straightforward, in principle, to extend the analysis to higher order.
- We shall carry out the second-order calculation to capture the phase jump at the critical layer.
- This phase jump controls the stability/instability — it is a qualitatively new effect, and so is worth the effort...
- Making use of the result $a = 1$, and in anticipation of the logarithmic behaviour of (111), we try the expansion

$$\omega = c_0\alpha_0\epsilon^2 + c_{1L}\alpha_0\epsilon^3 \ln \epsilon + c_1\alpha_0\epsilon^3 + \dots \qquad (156)$$

- We follow the same steps as before, but there are more terms to manipulate.

Upper layer

- The upper-layer variables (119) are now expanded as

$$u = u_{u0} + u_{u1L}\epsilon \ln \epsilon + u_{u1}\epsilon + \dots \qquad (157)$$

$$v = v_{u0} + v_{u1L}\epsilon \ln \epsilon + v_{u1}\epsilon + \dots \qquad (158)$$

$$p = p_{u0} + p_{u1L}\epsilon \ln \epsilon + p_{u1}\epsilon + \dots \qquad (159)$$

  where all the new variables are functions of y_u.
- Sub. $\alpha = \alpha_0\epsilon$, (156) – (159), $U = 1$, into (115) – (117).
- At leading order (121) – (123) are reproduced.
- At next order we find

$$\mathrm{i}\alpha_0 u_{u1L} + v'_{u1L} = 0 \qquad (160)$$

$$\mathrm{i}\alpha_0 u_{u1L} + \mathrm{i}\alpha_0 p_{u1L} = 0 \qquad (161)$$

$$\mathrm{i}\alpha_0 v_{u1L} + p'_{u1L} = 0. \qquad (162)$$

- Equations (160) – (162) are essentially the same as (121) – (123), so their solutions can be written down immediately:

$$p_{u1L} = P_{u1L}\mathrm{e}^{-\alpha_0 y_u} \tag{163}$$

$$v_{u1L} = -\mathrm{i}P_{u1L}\mathrm{e}^{-\alpha_0 y_u} \tag{164}$$

where P_{u1L} is a constant.

- At the next order in the expansion we find forced equations

$$\mathrm{i}\alpha_0 u_{u1} + v'_{u1} = 0 \tag{165}$$

$$\mathrm{i}\alpha_0 u_{u1} + \mathrm{i}\alpha_0 p_{u1} = \mathrm{i}c_0\alpha_0 u_{u0} \tag{166}$$

$$\mathrm{i}\alpha_0 v_{u1} + p'_{u1} = \mathrm{i}c_0\alpha_0 v_{u0}. \tag{167}$$

- Eliminating u_{u1} and v_{u1} from (165) – (167), making use of (121) – (123) as necessary, gives

$$p''_{u1} - \alpha_0^2 p_{u1} = 0. \tag{168}$$

- The solution to (168) that decays far from the wall, and the solution for v_{u1}, from (126) and (167), are

$$p_{u1} = P_{u1}\mathrm{e}^{-\alpha_0 y_u} \tag{169}$$

$$v_{u1} = -\mathrm{i}(P_{u1} + c_0 P_{u0})\mathrm{e}^{-\alpha_0 y_u} \tag{170}$$

where P_{u1} is a constant.

Main layer

- The main-layer variables (127) are now expanded as

$$u = u_{m0}\epsilon^{-1} + u_{m1L}\ln\epsilon + u_{m1} + \dots \quad (171)$$

$$v = v_{m0} + v_{m1L}\epsilon\ln\epsilon + v_{m1}\epsilon + \dots \quad (172)$$

$$p = p_{m0} + p_{m1L}\epsilon\ln\epsilon + p_{m1}\epsilon + \dots \quad (173)$$

  where all the new variables are functions of y.
- Substitute $\alpha = \alpha_0\epsilon$, (156), (171) – (173) into (115) – (117).
- At leading order (128) – (130) are reproduced.
- At the next order, the equations for u_{m1L}, v_{m1L} and p_{m1L} have the same form as (128) – (130).

- Therefore, their solutions have the same form as (131) and (132):

$$
\begin{aligned}
p_{m1L} &= P_{m1L} &\quad (174)\\
v_{m1L} &= A_{m1L} U &\quad (175)
\end{aligned}
$$

  where A_{m1L} and P_{m1L} are constants.

- At the next order, forced equations appear:

$$
\begin{aligned}
\mathrm{i}\alpha_0 u_{m1} + v'_{m1} &= 0 &\quad (176)\\
\mathrm{i}\alpha_0 U u_{m1} + U' v_{m1} &= \mathrm{i}c_0\alpha_0 u_{m0} - \mathrm{i}\alpha_0 p_{m0} &\quad (177)\\
p'_{m1} &= -\mathrm{i}\alpha_0 U v_{m0}. &\quad (178)
\end{aligned}
$$

- Substituting the leading order solutions (131) and (132) into (176) – (178), and eliminating u_{m1} gives

$$\begin{aligned} v'_{m1} - \frac{U'}{U} v_{m1} &= c_0 A_{m0} \frac{U'}{U} + \frac{\mathrm{i}\alpha_0 P_{m0}}{U} && (179) \\ p'_{m1} &= -\mathrm{i}\alpha_0 A_{m0} U^2. && (180) \end{aligned}$$

- A solution to (180) can be written

$$p_{m1} = -\mathrm{i}\alpha_0 A_{m0} \int U^2 \, \mathrm{d}y. \qquad (181)$$

- However, the integral in (181) is divergent as $y \to \infty$, and the behaviour of p_{m1} in this limit will be needed when matching to the upper-layer solution.

- Therefore, we remove the divergent part of the integral, and choose instead

$$p_{m1} = P_{m1} - \mathrm{i}\alpha_0 A_{m0} \left(\int_0^y U^2 - 1 \, \mathrm{d}t + y \right). \qquad (182)$$

- A solution to (179) can be obtained using an integrating factor:

$$v_{m1} = A_{m1} U - c_0 A_{m0} + \mathrm{i}\alpha_0 P_{m0} U \int \frac{\mathrm{d}y}{U^2}. \qquad (183)$$

- However, the integral in (183) is divergent both as $y \to 0$ and $y \to \infty$, and we will need the behaviour of v_{m1} in both these limits when matching with the critical-layer solutions and upper-layer solutions.

- Therefore, we remove the divergent parts of the integral, and choose instead

$$
\begin{aligned}
v_{m1} \quad = \quad & A_{m1} U - c_0 A_{m0} + \mathrm{i}\alpha_0 P_{m0} U \left[\int_0^y \frac{1}{U^2} - \frac{1}{(U_0')^2 t^2} \right. \\
& \left. + \frac{U_0''}{(U_0')^3 t} - 1 \,\mathrm{d}t - \frac{1}{(U_0')^2 y} - \frac{U_0''}{(U_0')^3} \ln(y) + y \right]
\end{aligned} \tag{184}
$$

(similar to the treatment of the integral in (111)).

- This solution captures the logarithmic behaviour of the singular Tollmien solution (100).

Critical layer

- The critical-layer variables (134) are now expanded as

$$
\begin{aligned}
u &= u_{c0}\epsilon^{-1} + u_{c1L}\ln\epsilon + u_{c1} + \ldots && (185)\\
v &= v_{c0}\epsilon + v_{c1L}\epsilon^{2}\ln\epsilon + v_{c1}\epsilon^{2} + \ldots && (186)\\
p &= p_{c0} + p_{c1L}\epsilon\ln\epsilon + p_{c1}\epsilon + \ldots && (187)
\end{aligned}
$$

  where all the new variables are functions of Y, where $y = \epsilon Y$.
- The basic flow in the critical layer is

$$
U = \epsilon U_0' Y + \epsilon^2 U_0'' \frac{Y^2}{2} + \ldots, \quad U' = U_0' + \epsilon U_0'' Y + \ldots \quad (188)
$$

- Substitute $\alpha = \alpha_0\epsilon$, (156) and (185) – (188) into (115) – (117).
- At leading order (138) – (140) are reproduced.

- At the order where $\ln \epsilon$ appears, the equations are

$$
\begin{aligned}
\mathrm{i}\alpha_0 u_{c1L} + v'_{c1L} &= 0 && (189) \\
-\mathrm{i}c_0\alpha_0 u_{c1L} + \mathrm{i}\alpha_0 U'_0 Y u_{c1L} + U'_0 v_{c1L} + \mathrm{i}\alpha_0 p_{c1L} &= \mathrm{i}\alpha_0 c_{1L} u_{c0} && (190) \\
p'_{c1L} &= 0. && (191)
\end{aligned}
$$

- Substituting the leading order solution (154) into (189) – (191) gives solutions

$$
\begin{aligned}
p_{c1L} &= P_{c1L} && (192) \\
v_{c1L} &= B_{c1L}\left(U'_0 Y - c_0\right) - \frac{\mathrm{i}(\alpha_0 P_{c1L} - U'_0 c_{1L} P_{c0})}{c_0} Y. && (193)
\end{aligned}
$$

where P_{c1L} and B_{c1L} are constants.

- At the next order basic flow curvature terms appear:

$$
\begin{aligned}
\mathrm{i}\alpha_0 u_{c1} + v'_{c1} &= 0 && (194)\\
-\mathrm{i}c_0\alpha_0 u_{c1} + \mathrm{i}\alpha_0 U'_0 Y u_{c1} + U'_0 v_{c1} + \mathrm{i}\alpha_0 p_{c1} &= \mathrm{i}c_1\alpha_0 u_{c0} - \\
&\quad \frac{\mathrm{i}\alpha_0 U''_0 Y^2}{2} u_{c0} - U''_0 Y v_{c0} && (195)\\
p'_{c1} &= 0. && (196)
\end{aligned}
$$

- Substituting the leading-order result (154) into (195), we obtain the general solution

$$
\begin{aligned}
p_{c1} &= P_{c1} && (197)\\
v_{c1} &= B_{c1}\left(U_0'Y - c_0\right) - \frac{\mathrm{i}(\alpha_0 P_{c1} - U_0' c_1 P_{c0})}{c_0} Y \\
&\quad - \frac{\mathrm{i} U_0'' P_{c0}}{(U_0')^2}\left[\frac{(U_0'Y)^2}{2} - \alpha_0 Y\right. \\
&\quad \left. + \alpha_0 \left(Y - \frac{c_0}{U_0'}\right) \ln\left(Y - \frac{c_0}{U_0'}\right)\right] && (198)
\end{aligned}
$$

where P_{c1} and B_{c1} are constants.

Matching upper and main layers

- From (159) and (173), van Dyke's matching rule for the pressure for these layers is

$$
\begin{aligned}
H_1 \left\{ p_{u0}(\epsilon y) + p_{u1L}(\epsilon y)\epsilon \ln \epsilon + p_{u1}(\epsilon y)\epsilon \right\} &= \\
H_1 \left\{ p_{m0}(y_u/\epsilon) + p_{m1L}(y_u/\epsilon)\epsilon \ln \epsilon + p_{m1}(y_u/\epsilon)\epsilon \right\}. \quad &(199)
\end{aligned}
$$

- Substitute (125), (163), (169), (131), (174) and (182) into (199) and equate coefficients of powers of ϵ.
- At leading order we reproduce the leading order matching result (148), and at next orders find

$$
\begin{aligned}
P_{u1L} &= P_{m1L} \quad &(200)\\
P_{u1} &= P_{m1} - \alpha_0 I_1 P_{c0} \quad &(201)
\end{aligned}
$$

where $I_1 = \int_0^\infty U^2 - 1 \, \mathrm{d}y$, and leading order constants have been expressed in terms of P_{c0} via the leading order matching results (146), (148), (150) and (152).

- From (158) and (172), van Dyke's matching rule for the velocity for these layers is

$$\begin{aligned} H_1\left\{v_{u0}(\epsilon y)+v_{u1L}(\epsilon y)\epsilon\ln\epsilon+v_{u1}(\epsilon y)\epsilon\right\} &= \\ H_1\left\{v_{m0}(y_u/\epsilon)+v_{m1L}(y_u/\epsilon)\epsilon\ln\epsilon+v_{m1}(y_u/\epsilon)\epsilon\right\}. \end{aligned} \quad (202)$$

- Substitute (126), (164), (170), (132), (175) and (184) into (202) and equate coefficients of powers of ϵ.
- At leading order we reproduce the leading order matching result (146), and at next orders find

$$-\mathrm{i}P_{u1L} = A_{m1L} \quad (203)$$

$$-\mathrm{i}P_{u1} = A_{m1}+\mathrm{i}P_{c0}(2c_0+\alpha_0 I_2) \quad (204)$$

where

$$I_2=\lim_{y\to\infty}\left[\int_0^y \frac{1}{U^2}-\frac{1}{(U_0')^2t^2}+\frac{U_0''}{(U_0')^3t}-1\,\mathrm{d}t-\frac{U_0''}{(U_0')^3}\ln(y)\right]. \quad (205)$$

Matching main and critical layers

- From (173) and (187), van Dyke's matching rule for the pressure for these layers is

$$\begin{aligned} H_1\left\{p_{m0}(\epsilon Y)+p_{m1L}(\epsilon Y)\epsilon\ln\epsilon+p_{m1}(\epsilon Y)\epsilon\right\} &= \\ H_1\left\{p_{c0}(y/\epsilon)+p_{c1L}(y/\epsilon)\epsilon\ln\epsilon+p_{c1}(y/\epsilon)\epsilon\right\}. & \qquad (206)\end{aligned}$$

- Substitute (131), (174), (182), (141), (192) and (197) into (206) and equate coefficients of powers of ϵ.
- At leading order we reproduce the leading order matching result (150), and at next orders find

$$P_{m1L} = P_{c1L} \qquad (207)$$

$$P_{m1} = P_{c1}. \qquad (208)$$

- From (172) and (186), van Dyke's matching rule for the velocity for these layers is

$$\begin{aligned} H_2\left\{v_{m0}(\epsilon Y) + v_{m1L}(\epsilon Y)\epsilon \ln \epsilon + v_{m1}(\epsilon Y)\epsilon\right\} &= \\ H_1\left\{v_{c0}(y/\epsilon)\epsilon + v_{c1L}(y/\epsilon)\epsilon^2 \ln \epsilon + v_{c1}(y/\epsilon)\epsilon^2\right\}. & \quad (209) \end{aligned}$$

- Note from (184) that v_{m1} has a term of order $y \ln y$ for small y, and from (198) that v_{c1} has a term of order $Y \ln Y$ for large Y.
- Converting between y and Y generates a $\ln \epsilon$ term:

$$y \ln y = \epsilon Y \ln(\epsilon Y) = \epsilon Y \ln \epsilon + \epsilon Y \ln Y. \qquad (210)$$

- The existence of this $\ln \epsilon$ term forces the presence of the $\ln \epsilon$ terms in the expansions (156), (157) – (159), (171) – (173) and (185) – (187).

- Substitute (132), (154), (175), (184), (193) and (198) into (209) and equate coefficients of powers of ϵ.
- At leading order we reproduce the leading order matching result (152), and at next orders find

$$A_{m1L} U_0' = B_{c1L} U_0' - \frac{\mathrm{i}(\alpha_0 P_{c1L} - c_{1L} U_0' P_{c0})}{c_0} + \frac{\mathrm{i}\alpha_0 U_0'' P_{c0}}{(U_0')^2} \tag{211}$$

$$A_{m1} U_0' = B_{c1} U_0' - \frac{\mathrm{i}(\alpha_0 P_{c1} - c_1 U_0' P_{c0})}{c_0} + \frac{3\mathrm{i}\alpha_0 U_0'' P_{c0}}{2(U_0')^2}. \tag{212}$$

Dispersion relations

- Eliminating P_{u1L}, P_{m1L}, A_{m1L} and P_{c1L} from (200), (203), (207) and (211), and using (155) for c_0, gives

$$B_{c1L} = -\frac{\mathrm{i}U_0'}{\alpha_0} P_{c0} \left(c_{1L} + \frac{\alpha_0^2 U_0''}{(U_0')^4} \right). \tag{213}$$

- Substituting (213) into (193) gives

$$v_{c1L} = -\mathrm{i}U_0' Y \left(P_{c1L} + \frac{\alpha_0 U_0'' P_{c0}}{(U_0')^3} \right) + \mathrm{i}P_{c0} \left(c_{1L} + \frac{\alpha_0^2 U_0''}{(U_0')^4} \right). \tag{214}$$

- Applying the wall boundary condition $v_{c1L}(0) = 0$ then gives the inviscid dispersion relation

$$c_{1L} = -\frac{\alpha_0^2 U_0''}{(U_0')^4}. \tag{215}$$

- Eliminating P_{u1}, P_{m1}, A_{m1} and P_{c1} from (201), (204), (208) and (212), and using (155) for c_0, gives

$$B_{c1} = -\mathrm{i}\alpha_0 P_{c0}\left(\frac{U_0'}{\alpha_0^2}c_1 - I_1 + I_2 + \frac{2}{U_0'} + \frac{3U_0''}{2(U_0')^3}\right). \quad (216)$$

- Substituting (216) into (198) gives

$$\begin{aligned} v_{c1} &= \mathrm{i}\alpha_0^2 P_{c0}\left(\frac{c_1}{\alpha_0^2} - \frac{I_1 - I_2}{U_0'} + \frac{2}{(U_0')^2} + \frac{3U_0''}{2(U_0')^4}\right) \\ &\quad -\mathrm{i}U_0' Y\left[P_{c1} - \left(I_1 - I_2 - \frac{2}{U_0'} - \frac{3U_0''}{2(U_0')^3}\right)\alpha_0 P_{c0}\right] - \\ &\frac{\mathrm{i}U_0'' P_{c0}}{(U_0')^2}\left[\frac{(U_0' Y)^2}{2} - \alpha_0 Y + \alpha_0\left(Y - \frac{\alpha_0}{(U_0')^2}\right)\ln\left(Y - \frac{\alpha_0}{(U_0')^2}\right)\right]. \end{aligned} \quad (217)$$

- Applying the wall boundary condition $v_{c1}(0) = 0$ to (217), and using (102), then gives the inviscid dispersion relation

$$c_1 = \alpha_0^2 \left\{ \frac{I_1 - I_2}{U_0'} - \frac{2}{(U_0')^2} - \frac{U_0''}{(U_0')^4} \left[\frac{3}{2} + \ln\left(\frac{\alpha_0}{(U_0')^2} \right) - \mathrm{i}\pi \right] \right\}. \tag{218}$$

- Prandtl's boundary layer equations show that $U_0'' = \partial P / \partial x$, where P is the basic flow pressure in the freestream (see section 3).

- Therefore,

$$\begin{aligned}
\text{Accelerating freestream} \quad &\Rightarrow \quad \frac{\partial P}{\partial x} < 0 \ \Rightarrow \ U_0'' < 0 \\
&\Rightarrow \quad \text{stable}. \\
\text{Decelerating freestream} \quad &\Rightarrow \quad \frac{\partial P}{\partial x} > 0 \ \Rightarrow \ U_0'' > 0 \\
&\Rightarrow \quad \text{unstable}.
\end{aligned}$$

- Note $U'' < 0$ for large y.
- Accelerating boundary layers have no inflexion points and are stable in agreement with Rayleigh's inflexion point theorem.
- Decelerating boundary layers have an inflexion point and are unstable.
- Over a typical wing, the flow accelerates near the leading edge, and decelerates further downstream.
- Laminar flow can exist near the leading edge, but transition to turbulence occurs further downstream.

The Blasius boundary layer

- The case of constant freestream velocity, corresponding to flow over a flat plate aligned with the freestream (Blasius flow), has $U_0'' = 0$, and so is neutral at this order in the calculation.
- It can be shown that near the wall the y-dependence of Blasius flow takes the form

$$U = U_0' y - \frac{(U_0')^2}{48} y^4 + \dots \tag{219}$$

  (using suitable non-dimensional variables, see section 3).
- The singular Tollmien solution (100) shows that the coefficient of the phase jump $\mathrm{i}\pi$ produced by the logarithm is always proportional to U_c''.
- (219) shows that $U'' = O(y^2) = O(\epsilon^2)$ in the critical layer for Blasius flow.

- Therefore, (156) is replaced by

$$\omega = c_0\alpha_0\epsilon^2 + c_1\alpha_0\epsilon^3 + c_2\alpha_0\epsilon^4 + c_{3L}\alpha_0\epsilon^5 \ln \epsilon + c_3\alpha_0\epsilon^5 + \dots \tag{220}$$

- Our interest lies in the stability characteristics, which are due to the critical-layer phase-jump $\mathrm{i}\pi$ produced by the logarithm, which will appear in c_3.
- After much algebra, it can be shown that

$$c_3 = \dots - \frac{\mathrm{i}\pi\alpha_0^4}{4(U_0')^6}. \tag{221}$$

- Therefore, the Blasius boundary layer is stable.
- In agreement with Rayleigh's theorem, since the inflexion point is at the wall, $U_0'' = 0$.
- But the decay rate is very small for long waves: $\mathrm{Im}(\omega) = O(\epsilon^5)$.

3. Viscous stability theory

- Viscous disturbances are governed by the Orr-Sommerfeld equation:

$$(U - c)(v'' - \alpha^2 v) - U'' v = \frac{1}{\mathrm{i}\alpha Re}\left(v'''' - 2\alpha^2 v'' + \alpha^4 v\right), \tag{222}$$

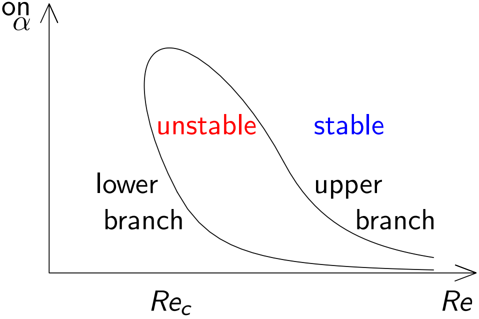
  see (31), where $U = U(y)$ and $' = \mathrm{d}/\mathrm{d}y$.
- Viscosity regularizes the inviscid solution at $U = c$.
- Viscosity dissipates kinetic energy, so it is stabilizing?
- Bizarrely, some stable inviscid flows are destabilized by viscosity!
- The inviscid solution has a slip velocity at the wall.
- Small viscosity is a singular perturbation to the inviscid problem (it raises the order of the differential equations).
- The viscous problem satisfies nonslip boundary conditions.

Viscous wall layer

- Introducing viscosity produces a viscous layer near the wall that brings this inviscid slip velocity down to zero at the wall.
- This is called boundary-layer behaviour.
- Viscosity can be included in the framework of matched asymptotic expansions that we have established for the inviscid problem, by including a viscous layer next to the wall.
- Taylor (1915) and Prandtl (1921) both suspected that the inclusion of a viscous wall layer could be destabilizing.
- But it would be another 40 years until systematic methods of matched asymptotic expansions were developed...

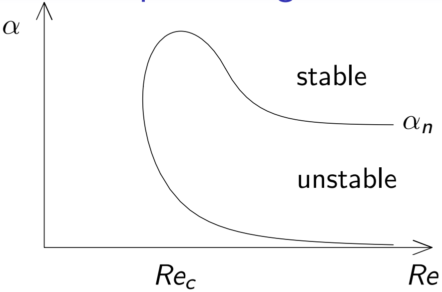
Viscous instability

- Tollmien (1929) found asymptotic solutions to the Orr-Sommerfeld equation (222) for profiles with no inflection point predicting instability:



- Flow is stable for $Re < Re_c$.
- The neutral curve has upper and lower branches.
- Questionable assumptions made, results not widely accepted...
- ...until Schubauer & Skramstad (1947) verified this behaviour in wind tunnel experiments on boundary layers in the US.

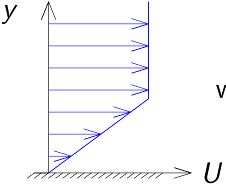
Adverse pressure gradients



- When there is an inflexion point, $\alpha \to \alpha_n$ along the upper branch as $Re \to \infty$.
- α_n can be found from the Rayleigh equation.
- α_n is such that the critical point lies at the inflexion point.
- Small viscosity stabilizes very long waves.
- Small viscosity destabilizes waves near α_n.
- Re_c(adverse pres. grad.) $<$ Re_c(favourable pres. grad.)

A simple example illustrating viscous instability

- Consider a piecewise-linear model boundary-layer profile:



$$\text{where} \quad U = \begin{cases} 1 & \text{if } y > 1 \\ y & \text{if } 0 < y < 1 \end{cases} . \qquad (223)$$

- Assume that viscosity only acts in a thin layer next to the wall, where $y \ll 1$.
- The solution for $y > 1$ is $v_1 = \mathrm{e}^{-\alpha y}$, as before, see (86), where we have chosen the normalization $A = 1$.
- However, instead of applying the inviscid wall boundary condition, $v_2(0) = 0$, as in (87), we choose the general inviscid solution $v_2 = B\mathrm{e}^{-\alpha y} + C\mathrm{e}^{\alpha y}$ when $0 < y < 1$.
- This is because now the wall boundary conditions will be applied in the viscous wall layer, and this viscous solution will be matched to the inviscid solution v_2.

- The constants B and C appearing in v_2 are obtained by applying the jump conditions (62) and (65) at $y = 1$, giving

$$v_2 = \frac{1}{2\alpha(1-c)} \left\{ [2\alpha(1-c) - 1]\mathrm{e}^{-\alpha y} + \mathrm{e}^{\alpha y - 2\alpha} \right\}. \qquad (224)$$

- In preparation for matching to the viscous wall layer solution, note that for small y

$$v_2 = v_2(0) + v_2'(0)y + O(y^2) \qquad (225)$$

where

$$\begin{aligned} v_2(0) &= \frac{1}{2\alpha(1-c)} \left[\mathrm{e}^{-2\alpha} - 1 + 2\alpha(1-c) \right] & (226) \\ v_2'(0) &= \frac{\alpha}{2\alpha(1-c)} \left[\mathrm{e}^{-2\alpha} + 1 - 2\alpha(1-c) \right]. & (227) \end{aligned}$$

- Applying the inviscid boundary condition, $v_2(0) = 0$, to (226) reproduces the inviscid dispersion relation (91):

$$c = \frac{1}{2\alpha} \left(\mathrm{e}^{-2\alpha} - 1 + 2\alpha \right) \quad \Rightarrow \quad c \sim \alpha + O(\alpha^2). \qquad (228)$$

The viscous wall layer

- Introduce a thin viscous wall layer whose thickness scales as some power of Re, where $Re \gg 1$, and assume that it produces some small correction to the inviscid eigenvalue: let

$$y = Re^{-a} y_v, \quad c = c_0 + Re^{-b} c_1 \tag{229}$$

where $a > 0$, $b > 0$, and c_0 is an inviscid eigenvalue.

- Write the viscous solution

$$v(y) = v_v(y_v). \tag{230}$$

- Substitute (229), (230) and $U = y$ into the OS equation (222) to give:

$$\begin{aligned}\left(Re^{-a} y_v - c_0 - Re^{-b} c_1\right)\left(Re^{2a} v_v'' - \alpha^2 v_v\right) &= \\ \frac{1}{\mathrm{i}\alpha Re}\left(Re^{4a} v_v'''' - 2\alpha^2 Re^{2a} v_v'' + \alpha^4 v_v\right). \end{aligned} \tag{231}$$

- The leading order terms on the LHS and RHS of (231) are

$$-c_0 Re^{2a} v_v'' = \frac{1}{\mathrm{i}\alpha} Re^{4a-1} v_v''''. \tag{232}$$

- The principle of dominant balance is invoked, which requires

$$2a = 4a - 1 \quad \Rightarrow \quad a = \frac{1}{2}. \tag{233}$$

- The viscous wall layer is thus characterized by a balance between unsteady and viscous terms.
- Substituting (233) into (232) gives

$$v_v'''' + \mathrm{i}\alpha c_0 v_v'' = 0 \tag{234}$$

with general solution

$$v_v = k_1 + k_2 y_v + k_3 \exp\left(-\sqrt{-\mathrm{i}\alpha c_0} y_v\right) + k_4 \exp\left(\sqrt{-\mathrm{i}\alpha c_0} y_v\right). \tag{235}$$

- Viscous effects do not grow exponentially outside the wall layer, so $k_4 = 0$ ($\sqrt{\cdot}$ denotes the root with positive real part).
- The viscous boundary conditions are $v_v(0) = v_v'(0) = 0$ giving

$$v_v = k_3 \left[\exp\left(-\sqrt{-\mathrm{i}\alpha c_0} y_v \right) - 1 + \sqrt{-\mathrm{i}\alpha c_0} y_v \right]. \qquad (236)$$

- Following van Dyke's matching rule, write the viscous solution, (236), in terms of the inviscid variable, y, and expand:

$$v_v \sim k_3 \left[-1 + \sqrt{-\mathrm{i}\alpha c_0} Re^{1/2} y \right]. \qquad (237)$$

- And write the inviscid solution, (225), in terms of the viscous variable, y_v, and expand:

$$v_2 \sim v_2(0) + v_2'(0) Re^{-1/2} y_v. \qquad (238)$$

- Having truncated this series for v_2 at two terms, re-write in terms of y:

$$v_2 \sim v_2(0) + v_2'(0) y \qquad (239)$$

- Comparing the linear terms in y in (237) and (239) shows that $k_3 = O(Re^{-1/2})$ when $v_2'(0) = O(1)$.
- Matching the constants in (237) and (239) then gives $v_2(0) = O(Re^{-1/2})$.
- Note that $v_2(0) = 0$ when $c = c_0$, since c_0 is an eigenvalue of the inviscid problem.
- We can therefore obtain $v_2(0) = O(Re^{-1/2})$ by choosing

$$b = \frac{1}{2} \tag{240}$$

  in (229).
- Instead of using (226) and (227) to give $v_2(0)$ and $v_2'(0)$, we simplify the algebra by using long-wave approximations, $\alpha \ll 1$, and substitute c from (229) and (240), with $c_0 = \alpha$ from (228), into (226) and (227) to give

$$\begin{aligned} v_2(0) &\sim -c_1 Re^{-1/2} + \ldots \qquad (241) \\ v_2'(0) &\sim 1 + \ldots . \qquad (242) \end{aligned}$$

- Substituting $c_0 = \alpha$ into (237) and (241) and (242) into (239) gives

$$v_v \sim k_3 \left[-1 + \sqrt{-\mathrm{i}} \alpha Re^{1/2} y \right] \tag{243}$$

$$v_2 \sim -c_1 Re^{-1/2} + y \tag{244}$$

respectively.

- Matching the viscous to the inviscid solution is then achieved by setting the RHS of (243) equal to the RHS of (244).
- Equating the constant terms and the coefficients of y, and eliminating k_3, gives the viscous correction

$$c_1 = \frac{1}{\sqrt{-\mathrm{i}}\alpha} = \frac{1+\mathrm{i}}{\sqrt{2}\alpha} \tag{245}$$

which is destabilizing because $\mathrm{Im}(c_1) > 0$.

A distinguished limit

- Substituting the leading order long-wave inviscid result, $c_0 = \alpha$, from (228), the scaling result (240), and (245) into (229) gives
$$c \sim \alpha + \frac{1+\mathrm{i}}{\sqrt{2}\alpha} Re^{-1/2} + \dots. \tag{246}$$
- Does viscosity also destabilize smooth profiles, or is the piecewise-linear flow atypical?
- We have been a bit vague about the relative sizes of α and $Re^{-1/2}$.
- In fact, the viscous correction becomes comparable to the leading inviscid term in (246) when
$$\alpha \sim \frac{Re^{-1/2}}{\alpha} \quad \Rightarrow \quad \alpha \sim Re^{-1/4}. \tag{247}$$
- This balance gives a distinguished limit where viscosity enters the dispersion relation at leading order.

Triple-deck scalings

- The scaling $\alpha \sim Re^{-1/4}$ suggests that $c \sim Re^{-1/4}$ by (246).
- These scalings alter the thickness of the viscous wall layer: Substitute

$$\alpha = \alpha_0 Re^{-1/4}, \quad c = c_0 Re^{-1/4}, \quad y = Re^{-a} y_v, \tag{248}$$

$$v(y) = v_v(y_v) \tag{249}$$

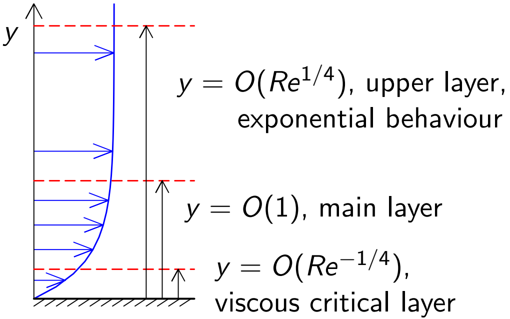
and $U = y$ into the OS equation (222) to give:

$$\left(Re^{-a} y_v - c_0 Re^{-1/4}\right)\left(Re^{2a} v_v'' - \alpha_0^2 Re^{-1/2} v_v\right) = \frac{1}{\mathrm{i}\alpha_0 Re^{3/4}}\left(Re^{4a} v_v'''' - 2\alpha_0^2 Re^{2a-1/2} v_v'' + \alpha_0^4 Re^{-1} v_v\right). \tag{250}$$

- The leading order unsteady term on the LHS of (250) balances the leading order viscous term on the RHS of (250) when

$$2a - \frac{1}{4} = 4a - \frac{3}{4} \quad \Rightarrow \quad a = \frac{1}{4}. \tag{251}$$

- Now the basic flow term (the y_v term) enters at the same order as the unsteady term (the c_0 term), so this wall layer is a viscous critical layer.
- We now show that these scalings also apply to smooth profiles.
- The disturbance structure is



Viscous disturbance equations

- Let $\epsilon = Re^{-1/4}$.
- Recall that the disturbance equations (28) - (30) are

$$
\begin{aligned}
\mathrm{i}\alpha u + v' &= 0 && (252)\\
-\mathrm{i}\omega u + \mathrm{i}\alpha U u + U' v &= -\mathrm{i}\alpha p + \frac{1}{Re}\left(u'' - \alpha^2 u\right) && (253)\\
-\mathrm{i}\omega v + \mathrm{i}\alpha U v &= -p' + \frac{1}{Re}\left(v'' - \alpha^2 v\right). && (254)
\end{aligned}
$$

- Eliminate u, substitute $\omega = \alpha c$, and $c = \epsilon c_0$ and $\alpha = \epsilon\alpha_0$ from (248), and also $1/Re = \epsilon^4$, to give

$$
\begin{aligned}
-(U - \epsilon c_0)v' + U' v &= -\mathrm{i}\epsilon\alpha_0 p - \frac{\epsilon^3}{\mathrm{i}\alpha_0}\left(v''' - \epsilon^2\alpha_0^2 v'\right) && (255)\\
\mathrm{i}\epsilon\alpha_0(U - \epsilon c_0)v &= -p' + \epsilon^4\left(v'' - \epsilon^2\alpha_0^2 v\right). && (256)
\end{aligned}
$$

- Solve (255), (256) in each layer and match.

Upper layer

- The upper layer variables are

$$v(y) = v_u(y_u), \quad p(y) = p_u(y_u), \quad y = y_u/\epsilon. \tag{257}$$

- Substituting $U = 1$ and (257) into (255), (256) gives at $O(\epsilon)$:

$$-v_u' = -\mathrm{i}\alpha_0 p_u \tag{258}$$

$$\mathrm{i}\alpha_0 v_u = -p_u', \tag{259}$$

with solutions

$$v_u = -\mathrm{i}P_u \exp(-\alpha_0 y_u), \quad p_u = P_u \exp(-\alpha_0 y_u) \tag{260}$$

satisfying homogeneous boundary conditions, where P_u is a constant.

Main layer

- The main layer variables are

$$v(y) = v_m(y), \quad p(y) = p_m(y). \tag{261}$$

- Substituting (261) into (255), (256) gives at $O(1)$:

$$-Uv_m' + U'v_m = 0 \tag{262}$$
$$0 = -p_m' \tag{263}$$

with solutions

$$v_m = A_m U, \quad p_m = P_m, \tag{264}$$

where A_m and P_m are constants.

Lower layer

- The lower layer variables are

$$v(y) = \epsilon v_v(y_v), \quad p(y) = p_v(y_v), \quad y = \epsilon y_v, \tag{265}$$

where $v(y) = O(\epsilon)$ to match with v_m.

- Substituting $U = U_0' y$, where $U_0' = U'(0)$, and (265) into (255), (256) gives at leading order:

$$\begin{aligned} -(U_0' y_v - c_0)v_v' + U_0' v_v &= -\mathrm{i}\alpha_0 p_v - \frac{1}{\mathrm{i}\alpha_0} v_v''' \quad (266) \\ 0 &= -p_v'. \quad (267) \end{aligned}$$

- If we differentiate (266) w.r.t. y_v to eliminate p_v, we obtain an Airy type equation for v_v'':

$$-(U_0' y_v - c_0)v_v'' = -\frac{1}{\mathrm{i}\alpha_0} v_v'''' \tag{268}$$

which can be recognized as the dominant terms in the OS equation found in (250) when $a = 1/4$.

- Clearly,

$$p_v = P_v, \tag{269}$$

where P_v is a constant, satisfies (267).

- If we substitute

$$v_v(y_v) = V(\xi), \quad \xi = (\mathrm{i}\alpha_0 U_0')^{1/3} \left(y_v - \frac{c_0}{U_0'} \right) \tag{270}$$

into (268) we obtain

$$V'''' - \xi V'' = 0. \tag{271}$$

- The solutions to (271) that do not grow exponentially with distance from the wall, and that satisfy $v_v(0) = v_v'(0)$, are

$$\begin{aligned} V(\xi) &= A_v \int_{\xi_0}^{\xi} \int_{\xi_0}^{\theta_1} \mathrm{Ai}(\theta_2)\, \mathrm{d}\theta_2\, \mathrm{d}\theta_1 \\ &= A_v \xi \int_{\xi_0}^{\xi} \mathrm{Ai}(\theta)\, \mathrm{d}\theta - A_v \mathrm{Ai}'(\xi) + A_v \mathrm{Ai}'(\xi_0) \end{aligned} \tag{272}$$

where A_v is a constant, and $\xi_0 = \xi(y_v = 0)$.

Matching

- Match velocities and pressures between upper and main layers, and between main and lower layers:

$$
\begin{aligned}
H_0\{v_u(\epsilon y)\} &= H_0\{v_m(y_u/\epsilon)\} \quad \Rightarrow \quad -\mathrm{i}P_u = A_m && (273)\\
H_0\{p_u(\epsilon y)\} &= H_0\{p_m(y_u/\epsilon)\} \quad \Rightarrow \quad P_u = P_m && (274)\\
H_0\{p_m(\epsilon y_v)\} &= H_0\{p_v(y/\epsilon)\} \quad \Rightarrow \quad P_m = P_v && (275)\\
H_1\{v_m(\epsilon y_v)\} &= H_0\{\epsilon v_v(y/\epsilon)\} \\
\Rightarrow A_m U_0' &= A_v(\mathrm{i}\alpha_0 U_0')^{1/3} \int_{\xi_0}^{\infty} \mathrm{Ai}(\theta)\,\mathrm{d}\theta && (276)
\end{aligned}
$$

  using (260), (264), (269) and (272).
- The relationship between A_v and P_v is obtained by substituting (269), (270) and (272) into (266):

$$U_0' \mathrm{Ai}'(\xi_0) A_v = -\mathrm{i}\alpha_0 P_v. \qquad (277)$$

Triple-deck dispersion relation

- Eliminating P_u, P_m, P_v, A_m and A_v from (273) – (277) gives the dispersion relation:

$$\frac{U_0' c_0}{\alpha_0} = -\frac{\xi_0 \int_{\xi_0}^{\infty} \mathrm{Ai}(\theta)\,\mathrm{d}\theta}{\mathrm{Ai}'(\xi_0)}, \quad \xi_0 = -(\mathrm{i}\alpha_0 U_0')^{1/3}\frac{c_0}{U_0'}. \tag{278}$$

- The dependence on U_0' can be scaled out using

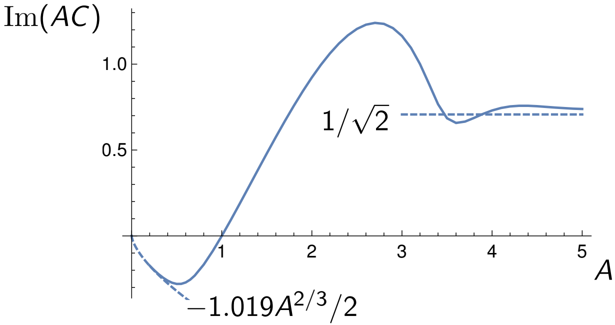
$$\alpha_0 = A(U_0')^{5/4}, \quad c_0 = C(U_0')^{1/4}, \tag{279}$$

and substituting (279) into (278) gives

$$\frac{C}{A} = -\frac{\xi_0 \int_{\xi_0}^{\infty} \mathrm{Ai}(\theta)\,\mathrm{d}\theta}{\mathrm{Ai}'(\xi_0)}, \quad \xi_0 = -(\mathrm{i}A)^{1/3} C. \tag{280}$$

- In general, this complex dispersion relation requires numerical treatment.

Triple-deck growth rates



- There is one real value of A that makes C real:

$$A = 1.0005, \quad C = 2.2968. \tag{281}$$

- The maximum growth rate occurs at

$$A = 2.7159, \quad C = 3.0321 + 0.4568\mathrm{i}. \tag{282}$$

Long and short wave limits

- In the long-wave limit, $A \ll 1$, we find that $C = O(A^{-1/3})$ and $\xi_0 = O(1)$.
- Substitute these scalings into (280), then at leading order the dispersion relation reduces to

$$\mathrm{Ai}'(\xi_0) = 0 \quad \Rightarrow \quad C = -\zeta_n \mathrm{e}^{-\mathrm{i}\pi/6} A^{-1/3} \tag{283}$$

  where $\zeta_n = -1.019, -3.248, -4.820, \ldots$ are the zeros of $\mathrm{Ai}'(\zeta_n) = 0$.
- In the short-wave limit, $A \gg 1$, we find that $C = O(A)$ and $\xi_0 = O(A^{4/3})$.
- This limit requires knowledge of the large-argument asymptotics of the derivative, and integral, of the Airy function, Ai.

- The required behaviour can be found by considering the ODE

$$y''' - xy' = 0 \tag{284}$$

with a solution that decays as $x \to \infty$ given by

$$y = \int_{\infty}^{x} \mathrm{Ai}(\theta)\, \mathrm{d}\theta. \tag{285}$$

- We are interested in $x \gg 1$, so let $x = X/\epsilon$ where $\epsilon \ll 1$, and try a WKB solution:

$$y = \mathrm{e}^{-f(X)/\epsilon^n}. \tag{286}$$

- Substituting (286) into (284) gives

$$\epsilon^{3-n} f''' + 3\epsilon^{3-2n} f' f'' + \epsilon^{3-3n} (f')^3 - \epsilon^{-n} X f' = 0. \tag{287}$$

- Apply the principle of dominant balance: the last two terms in (287) balance, and are the largest terms, when

$$3 - 3n = -n \quad \Rightarrow \quad n = 3/2 \tag{288}$$

and then keeping only the leading terms in (287) gives

$$(f')^3 - Xf' = 0 \quad \Rightarrow \quad f = \frac{2}{3}X^{3/2}. \tag{289}$$

- We can find further corrections by substituting

$$y = \left[g_0(X) + \epsilon^{3/2}g_1(X) + \dots\right]\mathrm{e}^{-2x^{3/2}/3} \tag{290}$$

into (284), and solving for g_0, g_1, etc, to obtain

$$y = kx^{-3/4}\left(1 - \frac{41}{48x^{3/2}} + \dots\right)\mathrm{e}^{-2x^{3/2}/3} \tag{291}$$

where k is an arbitrary constant (provided $-\pi < \arg x < \pi$).

- Therefore, comparing (285) and (291):

$$\int_{\infty}^{x} \mathrm{Ai}(\theta)\,\mathrm{d}\theta \sim kx^{-3/4}\left(1 - \frac{41}{48x^{3/2}} + \dots\right)\mathrm{e}^{-2x^{3/2}/3}. \quad (292)$$

- Differentiating (292) twice w.r.t. x gives

$$\mathrm{Ai}'(x) \sim kx^{1/4}\left(1 + \frac{7}{48x^{3/2}} + \dots\right)\mathrm{e}^{-2x^{3/2}/3}. \quad (293)$$

- Substituting (292) and (293) into (280) gives

$$\frac{C}{A} \sim \frac{1 - 41/(48\xi_0^{3/2})}{1 + 7/(48\xi_0^{3/2})} \sim 1 - \frac{1}{\xi_0^{3/2}}. \quad (294)$$

- Care must be taken over the root taken of $\xi_0^{3/2}$, which must be the same as one taken in $x^{3/2}$ in the exponents of (292) and (293).
- This root decays for large $|x|$ when $|\arg x| < \pi/3$, and grows when $\pi/3 < |\arg x| < \pi$.

- $\xi_0 = -(\mathrm{i}A)^{1/3}C = \mathrm{e}^{-5\mathrm{i}\pi/6}A^{1/3}C$, so the root with negative real part is chosen:

$$\xi_0^{3/2} = \mathrm{e}^{3\mathrm{i}\pi/4}A^{1/2}C^{3/2}. \tag{295}$$

- Substituting (295) into (294) gives

$$\frac{C}{A} \sim 1 - \mathrm{e}^{-3\mathrm{i}\pi/4}A^{-1/2}C^{-3/2}. \tag{296}$$

- Solving (296) asymptotically for C for $A \gg 1$ gives

$$C \sim A + \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{A}, \tag{297}$$

  and thus the asymptote $\mathrm{Im}(AC) = 1/\sqrt{2}$ in the last figure.
- Note also that (297) is the same as (246) when we write $\alpha = \alpha_0\epsilon = ARe^{-1/4}$ and $c = c_0\epsilon = CRe^{-1/4}$ when $U_0' = 1$.

Another distinguished limit

- As A increases, eventually this destabilizing viscous correction becomes of the same order as the stabilizing inviscid phase jump produced by the logarithm at $O(\alpha^2)$ in the inviscid theory.
- Recalling (246):

$$c \sim \alpha + \frac{1+\mathrm{i}}{\sqrt{2}\alpha} Re^{-1/2} + \dots, \tag{298}$$

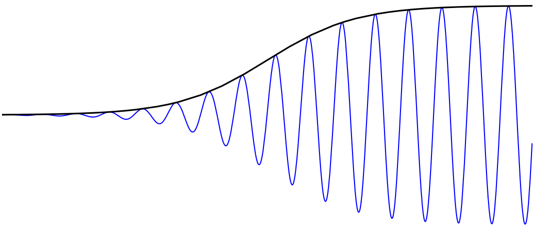
the viscous term is $O(\alpha^2)$ when

$$\frac{Re^{-1/2}}{\alpha} \sim \alpha^2 \quad \Rightarrow \quad \alpha = O(Re^{-1/6}), \quad c = O(Re^{-1/6}). \tag{299}$$

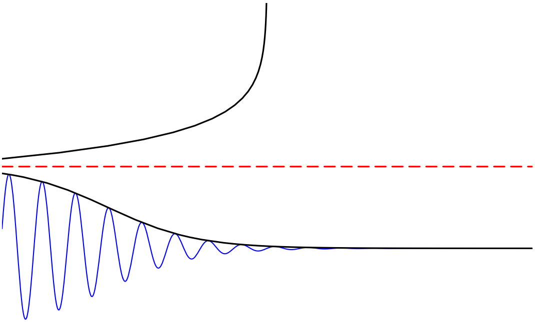
- When the imaginary parts of these terms are equal, we reach the upper branch of the neutral curve, and these scalings provide the starting point for such a calculation.

4. Weakly nonlinear theory

- This describes the next stages of disturbance evolution when amplitudes start to become too large for linear theory to apply.
- Nonlinearity might be stabilizing, and could lead to a saturated nonlinear equilibrium state of finite amplitude:



- Or nonlinearity might be destabilizing, and could lead to threshold behaviour:



- These roles of nonlinearity, and others, can be determined by using perturbation theory, i.e. weakly nonlinear theory.
- This requires both small amplitudes and weak growth rates.

Solvability conditions

- When we include nonlinear effects we will encounter disturbance equations with right hand sides, e.g. forced Orr-Sommerfeld equations.
- Forced equations do not necessarily have solutions satisfying homogeneous boundary conditions.
- The solvability of such equations is crucial to the development of weakly nonlinear theory.
- The essence of the matter can be illustrated using this simple ODE:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \omega^2 y = \sin \Omega x, \quad y(0) = 0, \quad y(L) = 0 \tag{300}$$

where $\omega \neq \pm\Omega$.

- The general solution of (300) is

$$y = A\cos\omega x + B\sin\omega x + \frac{1}{\omega^2 - \Omega^2}\sin\Omega x. \tag{301}$$

- Applying the boundary conditions to (301) gives

$$\begin{aligned} 0 &= A && (302)\\ 0 &= B\sin\omega L + \frac{1}{\omega^2 - \Omega^2}\sin\Omega L. && (303) \end{aligned}$$

- If $\sin\omega L \neq 0$, then (303) has a solution:

$$B = -\frac{\sin\Omega L}{(\omega^2 - \Omega^2)\sin\omega L}. \tag{304}$$

- However, if $\sin \omega L = 0$, then ω is an eigenvalue of the homogeneous (unforced) problem, and (303) may have no solution.
- Therefore, a solution of the forced problem exists if ω is not an eigenvalue.
- If ω is an eigenvalue, then a solution to (303) only exists if

$$\sin \Omega L = 0. \tag{305}$$

- (305) is the solvability condition required for (300) when ω is an eigenvalue.

Adjoint equations

- The solvability condition can be obtained even when we do not have the general solution.
- Consider

$$y'' + \omega^2 y = f(x), \quad y(0) = 0, \quad y(L) = 0 \tag{306}$$

where ω is an eigenvalue of the homogeneous problem.

- What is the solvability condition on f?
- Multiply (306) by a function $g(x)$, which satisfies the same homogeneous boundary conditions as y, i.e. $g(0) = 0$ and $g(L) = 0$, and integrate from $x = 0$ to $x = L$:

$$\int_0^L g\left(y'' + \omega^2 y\right) \mathrm{d}x = \int_0^L gf \,\mathrm{d}x. \tag{307}$$

- Integrate by parts using the boundary conditions on y and g:

$$\begin{aligned}\int_0^L g y'' \mathrm{d}x &= \left[g y'\right]_0^L - \int_0^L g' y' \mathrm{d}x \\ &= -\left[g' y\right]_0^L + \int_0^L g'' y \, \mathrm{d}x \\ &= \int_0^L g'' y \, \mathrm{d}x. \qquad (308)\end{aligned}$$

- Substitute (308) into (307):

$$\int_0^L y \left(g'' + \omega^2 g\right) \mathrm{d}x = \int_0^L g f \, \mathrm{d}x. \qquad (309)$$

- Choose g to satisfy

$$g'' + \omega^2 g = 0. \tag{310}$$

- The equation for g obtained in this way, (310), is called the adjoint equation.
- This example happens to be self-adjoint (the adjoint equation here is the same as the original equation).
- Substituting (310) into (309) gives the solvability condition for (306):

$$\int_0^L gf \,\mathrm{d}x = 0. \tag{311}$$

- In the previous example, $f(x) = \sin \Omega x$, and $g = \sin \omega x$ satisfies the adjoint equation (310) and $g(0) = g(L) = 0$ since $\omega = n\pi/L$ is an eigenvalue.
- Substituting these f and g into the solvability condition (311) gives

$$
\begin{aligned}
0 &= \int_0^L \sin \omega x \, \sin \Omega x \, \mathrm{d}x \\
&= \int_0^L \frac{1}{2}[\cos(\omega - \Omega)x - \cos(\omega + \Omega)x] \, \mathrm{d}x \\
&= \frac{\omega}{\Omega^2 - \omega^2} \cos \omega L \, \sin \Omega L \\
\Rightarrow \quad \sin \Omega L &= 0
\end{aligned}
$$

which reproduces (305).

A hydrodynamic stability example

- We shall consider the weakly nonlinear theory for plane Poiseuille flow, $U(y) = 1 - y^2$, between plates at $y = \pm 1$.
- We shall consider two-dimensional disturbances, and hence use a streamfunction, ψ, where

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{312}$$

- The Navier-Stokes equations are now

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi = \frac{1}{Re}\nabla^4\psi. \tag{313}$$

Finding the form of the expansion

- We shall consider waves that grow or decay slowly compared with their period of oscillation.
- To a first approximation the frequency is then real.
- This occurs near a neutral curve.
- We can determine the form of the perturbation expansion by first adding a small neutral wave to the basic flow:

$$\psi = \int U \,\mathrm{d}y + \epsilon\psi_{11}(y)E + \epsilon\bar{\psi}_{11}(y)E^{-1} \tag{314}$$

  where $\bar{\psi}_{11}$ is the complex conjugate of ψ, $E = \exp \mathrm{i}(\alpha x - \omega t)$, α and ω are real, and ϵ characterises the small amplitude of the disturbance.
- Adding the complex conjugate of the wavy term ensures that ψ is real.

Substitute (314) into (313) and equate coefficients of powers of ϵ and E:

At ϵE the Orr-Sommerfeld equation is obtained:

$$(U-c)(\psi_{11}''-\alpha^2\psi_{11})-U''\psi_{11}-\frac{1}{\mathrm{i}\alpha Re}(\psi_{11}''''-2\alpha^2\psi_{11}''+\alpha^4\psi_{11})=0 \tag{315}$$

where $c=\omega/\alpha$, (at this order we reproduce linear theory).

At $\epsilon^2 E^0$:

$$(\bar{\psi}_{11}\psi_{11}'-\psi_{11}\bar{\psi}_{11}')''=0. \tag{316}$$

At $\epsilon^2 E^2$:

$$\psi_{11}'\psi_{11}''-\psi_{11}\psi_{11}'''=0. \tag{317}$$

ψ_{11} is determined by solving (315), leaving (316) and (317) unsatisfied.

- (316) and (317) can be solved by introducing new terms proportional to $\epsilon^2 E^0$ and $\epsilon^2 E^2$ to the expansion (314).
- These new terms produce terms of $O(\epsilon^3)$, so terms proportional to $\epsilon^3 E$ and $\epsilon^3 E^3$ must also be included:

$$\begin{aligned} \psi \;=\; & \int U \,\mathrm{d}y + \epsilon\psi_{11}(y)E + \epsilon^2\psi_{02} + \epsilon^2\psi_{22}E^2 + \epsilon^3\psi_{13}E \\ & +\epsilon^3\psi_{33}E^3 + \epsilon\bar{\psi}_{11}(y)E^{-1} + \epsilon^2\bar{\psi}_{22}E^{-2} + \epsilon^3\bar{\psi}_{13}E^{-1} \\ & + \epsilon^3\bar{\psi}_{33}E^{-3} \end{aligned} \tag{318}$$

  where complex conjugates have been added to make ψ real.
- Substitute (318) into (313) and equate powers of ϵ and E:

- At ϵE:

$$L_1(\psi_{11}) = 0 \tag{319}$$

where we introduce L_n to denote the Orr-Sommerfeld operator for wavenumber and frequency $(n\alpha, n\omega)$, i.e. (319) is the same as (315).

- At $\epsilon^2 E^0$:

$$\frac{\psi_{02}''''}{\mathrm{i}\alpha Re} = (\bar{\psi}_{11}\psi_{11}' - \psi_{11}\bar{\psi}_{11}')''. \tag{320}$$

- At $\epsilon^2 E^2$:

$$L_2(\psi_{22}) = \psi_{11}'\psi_{11}'' - \psi_{11}\psi_{11}''' \tag{321}$$

- At $\epsilon^3 E$:

$$
\begin{aligned}
L_1(\psi_{13}) \quad = \quad & \psi_{11}''\psi_{02}' - \psi_{11}(\alpha^2\psi_{02}' + \psi_{02}''') + \bar{\psi}_{11}\psi_{22}''' \\
& -\bar{\psi}_{11}''\psi_{22}' + 2\bar{\psi}_{11}'\psi_{22}'' - 2(3\alpha^2\bar{\psi}_{11}' + \bar{\psi}_{11}''')\psi_{22} \\
& -3\alpha^2\bar{\psi}_{11}\psi_{22}' \qquad (322)
\end{aligned}
$$

- At $\epsilon^3 E^3$:

$$
L_3(\psi_{33}) = \dots \qquad (323)
$$

- As before, (319) is solved for ψ_{11}.
- (320) is then solved for the mean flow correction term ψ_{02}.

- (321) is solved for ψ_{22}, because, in general, if (α, ω) are eigenvalues of the OS equation, then $(2\alpha, 2\omega)$ will not be eigenvalues.
- Similarly, (323) can be solved for ψ_{33}, because, in general, if (α, ω) are eigenvalues of the OS equation, then $(3\alpha, 3\omega)$ will not be eigenvalues.
- However, an important difficulty arises when we consider (322).
- (α, ω) are eigenvalues of the left hand side of (322), so there is no solution, unless the right hand side satisfies a solvability condition similar to (311).
- In general, the solvability condition for (322) will not be satisfied.
- Therefore, our perturbation expansion needs to be modified...

Multiple scales theory

- It was shown by Stuart (1960, JFM) that this difficulty can be resolved if one allows the amplitude of the fundamental wave to vary on an appropriate slow time scale.
- In fact, this assumption is implicit in our earlier figures showing nonlinear saturation and threshold behaviour: the envelope of the oscillation varies slowly compared with the time scale of the oscillation.
- The fundamental wave is written

$$\epsilon A(T)\psi_{11}(y)E \tag{324}$$

where the slow time scale for the amplitude evolution is

$$T = \epsilon^2 t. \tag{325}$$

- Furthermore, we need no longer consider exactly neutral waves: we can let the eigenvalue for a given real α be

$$\omega = \omega_r + \mathrm{i}\epsilon^2 \omega_i \tag{326}$$

(e.g. by being appropriately close to the neutral curve).

- The term that depends on the fast time scale is now

$$E = \exp \mathrm{i}(\alpha x - \omega_r t). \tag{327}$$

- Differentiating (324) with respect to t gives

$$\begin{aligned} \frac{\partial}{\partial t}[\epsilon A(T)\psi_{11}(y)E] &= \epsilon \frac{\mathrm{d}A}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}t}\psi_{11}E - \mathrm{i}\omega_r \epsilon A \psi_{11} E \\ &= -\mathrm{i}\omega\epsilon A\psi_{11}E + \epsilon^3\left(\frac{\mathrm{d}A}{\mathrm{d}T} - \omega_i A\right)\psi_{11}E. \end{aligned} \tag{328}$$

- The slow scale, (325), and the smallness of the imaginary part of frequency, (326), were chosen to produce terms proportional to $\epsilon^3 E$ in (328).
- These terms would therefore appear in (322), where a solvability condition is needed.
- We choose $A(T)$ to vary so as to satisfy the solvability condition.
- Therefore, we replace (318) by

$$\begin{aligned}\psi \quad = \quad & \int U \,\mathrm{d}y + \epsilon A(T)\psi_{11}(y)E + \epsilon^2 |A|^2 \psi_{02} + \epsilon^2 A^2 \psi_{22} E^2 \\ & + \epsilon^3 \psi_{13} E + \epsilon^3 \psi_{33} E^3 + \epsilon \bar{A} \bar{\psi}_{11} E^{-1} + \epsilon^2 \bar{A}^2 \bar{\psi}_{22} E^{-2} \\ & + \epsilon^3 \bar{\psi}_{13} E^{-1} + \epsilon^3 \bar{\psi}_{33} E^{-3}. \end{aligned} \tag{329}$$

- Substitute (329) into the Navier-Stokes equations (313) and at $\epsilon^3 E$ we get:

$$
\begin{aligned}
L_1(\psi_{13}) \quad = \quad & \left[\left(3\mathrm{i}\alpha U + \mathrm{i}\frac{U''}{\alpha} - 2\mathrm{i}\omega + \frac{4\alpha^2}{Re}\right)\psi_{11}\right. \\
& \left. - \left(\mathrm{i}\frac{U}{\alpha} + \frac{4}{Re}\right)\psi_{11}''\right]\left(\frac{\mathrm{d}A}{\mathrm{d}T} - \omega_i A\right) \\
& + \left[\psi_{11}''\psi_{02}' - \psi_{11}(\alpha^2\psi_{02}' + \psi_{02}''') + \bar{\psi}_{11}\psi_{22}'''\right. \\
& -\bar{\psi}_{11}''\psi_{22}' + 2\bar{\psi}_{11}'\psi_{22}'' - 2(3\alpha^2\bar{\psi}_{11}' + \bar{\psi}_{11}''')\psi_{22} \\
& \left. -3\alpha^2\bar{\psi}_{11}\psi_{22}'\right] A|A|^2 \\
= \quad & f(y)\left(\frac{\mathrm{d}A}{\mathrm{d}T} - \omega_i A\right) + h(y)A|A|^2. \qquad (330)
\end{aligned}
$$

An amplitude equation

- We apply the solvability condition (311), i.e. multiply (330) by the adjoint of the Orr-Sommerfeld equation, $g(y)$, and integrate across the flow:

$$0 = \int_{-1}^{1} g(y) f(y) \mathrm{d}y \left(\frac{\mathrm{d}A}{\mathrm{d}T} - \omega_i A \right) + \int_{-1}^{1} g(y) h(y) \mathrm{d}y A|A|^2 \tag{331}$$

which can be rearranged to give

$$\frac{\mathrm{d}A}{\mathrm{d}T} = \omega_i A + \lambda A|A|^2 \tag{332}$$

where

$$\lambda = -\frac{\int_{-1}^{1} g(y) h(y) \mathrm{d}y}{\int_{-1}^{1} g(y) f(y) \mathrm{d}y}. \tag{333}$$

- The form of (332) was conjectured by Landau in 1944, but it was not derived until Stuart in 1960.
- (332) is called the Landau equation, or Stuart-Landau equation.
- It is an example of an amplitude equation.
- λ is called the Landau coefficient.
- The sign of $\mathrm{Re}(\lambda)$ determines whether nonlinearity is stabilizing or destabilizing.
- The adjoint eigenfunction, $g(y)$, is found by a similar procedure as that leading to (310): multiply (315) by g, integrate across the flow and use integration by parts to transfer derivatives from ψ_{11} to g.

- The adjoint of the Orr-Sommerfeld equation is found to be

$$(U-c)(g''-\alpha^2 g)+2U'g'-\frac{1}{\mathrm{i}\alpha Re}(g''''-2\alpha^2 g''+\alpha^4 g)=0, \tag{334}$$

  cf. the adjoint of the Rayleigh equation (48).
- We can analyse the Landau equation (332) by considering the magnitude and phase of A.
- Substitute $A = r\exp\mathrm{i}\theta$ into (332):

$$\frac{\mathrm{d}A}{\mathrm{d}T}=\frac{\mathrm{d}r}{\mathrm{d}T}\exp\mathrm{i}\theta+\mathrm{i}\frac{\mathrm{d}\theta}{\mathrm{d}T}r\exp\mathrm{i}\theta=\omega_i r\exp\mathrm{i}\theta+(\lambda_r+\mathrm{i}\lambda_i)r^3\exp\mathrm{i}\theta \tag{335}$$

  where λ_r and λ_i are the real and imaginary parts of λ.

- Dividing (335) by $\exp \mathrm{i}\theta$ then equating real and imaginary parts gives

$$\frac{\mathrm{d}r}{\mathrm{d}T} = \omega_i r + \lambda_r r^3 \tag{336}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}T} = \lambda_i r^2. \tag{337}$$

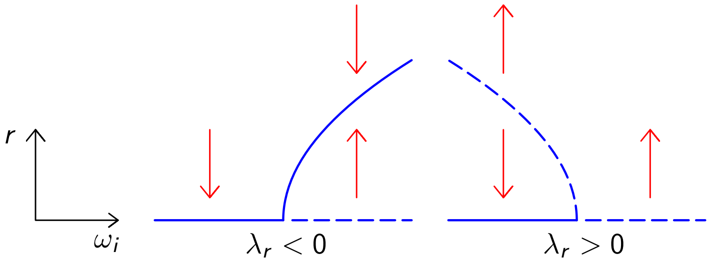
- The condition for an equilibrium amplitude is

$$\frac{\mathrm{d}r}{\mathrm{d}T} = 0 \quad \Rightarrow \quad r = 0, \quad r = \left(-\frac{\omega_i}{\lambda_r}\right)^{1/2} \tag{338}$$

where the nontrivial solution exists when $\omega_i/\lambda_r < 0$.

- The stability of these equilibria can be found by considering the sign of $\mathrm{d}r/\mathrm{d}T$ for various r.

Bifurcation diagrams



- Solid lines are stable; dashed lines are unstable.
- For small r, $\mathrm{d}r/\mathrm{d}T \sim \omega_i r$, so the $r = 0$ solution follows linear theory, i.e. unstable if $\omega_i > 0$ and stable if $\omega_i < 0$.
- At large r, $\mathrm{d}r/\mathrm{d}T \sim \lambda_r r^3$, so the nonlinear solution is unstable if $\lambda_r > 0$ (threshold behaviour) and stable if $\lambda_r < 0$ (nonlinear saturation).

Wave interaction

- Nonlinearity does not only modify the growth rate of a single wave, e.g. to create nonlinear equilibrium solutions.
- If more than one wave is present, then nonlinearity causes them to interact with each other (the superposition principle does not apply to nonlinear equations).
- Consider a pair of waves with wavy parts

$$E_1 = \exp \mathrm{i}(\alpha_1 x - \omega_1 t), \quad E_2 = \exp \mathrm{i}(\alpha_2 x - \omega_2 t) \tag{339}$$

  where (α_1, ω_1) are eigenvalues, and so are (α_2, ω_2), that do not satisfy resonance conditions, e.g. $\omega_2 \neq n\omega_1$ when $\alpha_2 = n\alpha_1$ for some integer n.
- By following a similar procedure to that for a single wave, we find that an expansion of the form

$$\psi = \int U \,\mathrm{d}y + \epsilon A_1(T)\psi_{111}(y)E_1 + \epsilon A_2(T)\psi_{211}(y)E_2 + \dots \tag{340}$$

- ...leads to two solvability conditions, giving a pair of coupled amplitude equations:

$$\frac{\mathrm{d}A_1}{\mathrm{d}T} = \omega_{1i}A_1 + \lambda_1|A_1|^2A_1 + a_1|A_2|^2A_1 \qquad (341)$$

$$\frac{\mathrm{d}A_2}{\mathrm{d}T} = \omega_{2i}A_2 + \lambda_2|A_2|^2A_2 + a_2|A_1|^2A_2 \qquad (342)$$

- The Landau coefficients λ_1 and λ_2 govern the evolution of A_1 and A_2 if each is present by itself, i.e. if $A_1 = 0$ or $A_2 = 0$.
- By letting $A_1 = r_1 \exp \mathrm{i}\theta_1$ and $A_2 = r_2 \exp \mathrm{i}\theta_2$ we find that the real parts of the interaction coefficients a_1 and a_2 determine how energy is transfered from one wave to the other.

Resonant wave interaction

- If the dispersion relation is such that it admits roots satisfying

$$\alpha_2 = 2\alpha_1 \quad \text{and} \quad \omega_2 = 2\omega_1 \tag{343}$$

  or

$$\alpha_2 = 3\alpha_1 \quad \text{and} \quad \omega_2 = 3\omega_1 \tag{344}$$

  then the form of the amplitude equations changes qualitatively.
- We shall consider the case (343).
- It turns out that solvability conditions now first occur at $O(\epsilon^2)$, instead of the $O(\epsilon^3)$ that we encountered before.

- This means that the slow time scale is now $T = \epsilon t$, and the imaginary parts of ω_1 and ω_2 are $O(\epsilon)$.
- The solvability conditions give amplitude equations

$$\frac{\mathrm{d}A_1}{\mathrm{d}T} = \omega_{1i}A_1 + b_1 A_2 \bar{A}_1 \tag{345}$$
$$\frac{\mathrm{d}A_2}{\mathrm{d}T} = \omega_{2i}A_2 + b_2 A_1^2. \tag{346}$$

- The direction of energy transfer between the waves now depends on the relative phase between the waves.
- A detuning parameter can be introduced to acount for the case when the real parts of the eigenvalues don't satisfy (343) exactly, but differ by an amount $O(\epsilon)$.
- This adds an imaginary coefficient to either A_1 or A_2.

Craik resonant triads

- Craik (1971, JFM) showed that a $2:1$ resonance like (343) can be satisfied in typical boundary layer dispersion relations, for any given plane wave and a particular pair of oblique waves with suitably chosen spanwise wavenumber:

$$A_1 \exp \mathrm{i}(\alpha x - \omega t), \quad A_2 \exp \mathrm{i}\left(\frac{\alpha}{2}x + \beta z - \frac{\omega}{2}t\right),$$

$$A_3 \exp \mathrm{i}\left(\frac{\alpha}{2}x - \beta z - \frac{\omega}{2}t\right). \tag{347}$$

- This is often called a subharmonic resonance.
- The resonant amplitude equations are in the form

$$\dot{A}_1 = a_1 A_1 + b_1 A_2 A_3 \tag{348}$$

$$\dot{A}_2 = a_2 A_2 + b_2 A_1 \bar{A}_3 \tag{349}$$

$$\dot{A}_3 = a_3 A_3 + b_3 A_1 \bar{A}_2. \tag{350}$$

- For example, long waves near the upper branch of the neutral curve have leading order inviscid dispersion relation given by $c = \alpha/U_0'$, see (155), and the generalization to 3d waves proportional to $\exp \mathrm{i}(\alpha x + \beta z - \omega t)$ is

$$c = \frac{\sqrt{\alpha^2 + \beta^2}}{U_0'} \tag{351}$$

where $c = \omega/\alpha$.

- The phase velocities of the waves in (347) are all equal.
- Therefore, for a 2d wave with wavenumber α_0, the 3d waves satisfying (347) and (351) are given by

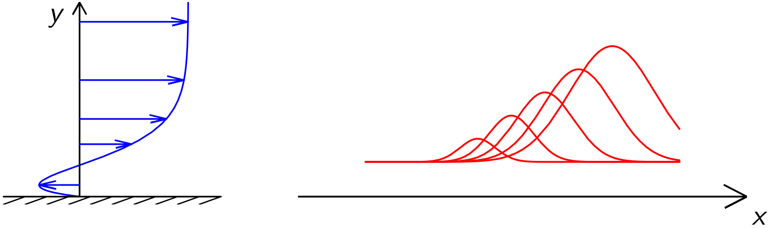
$$c = \frac{\alpha_0}{U_0'} = \frac{\sqrt{(\alpha_0/2)^2 + \beta_0^2}}{U_0'} \quad \Rightarrow \quad \beta_0 = \pm\frac{\sqrt{3}}{2}\alpha_0. \tag{352}$$

- The wave-angle of these resonant oblique waves is $\theta = \pi/3$, where $\tan\theta = \beta/\alpha = (\sqrt{3}\alpha_0/2)/(\alpha_0/2) = \sqrt{3}$.
- This wave-angle prediction is independent of α_0 and U_0', and can be tested in experiments.

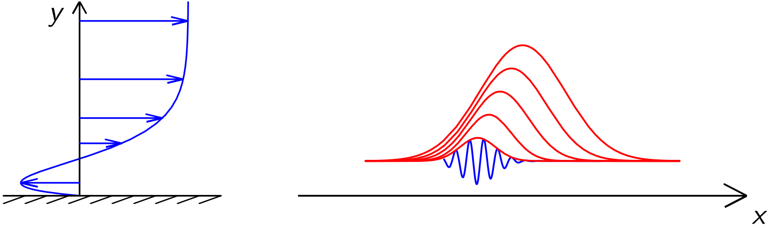
- Resonant waves develop nonlinear behaviour more quickly than non-resonant waves because they evolve on the faster (less slow) time scale $T = \epsilon t$, compared to the non-resonant time scale $T = \epsilon^2 t$.
- Therefore, from broadband initial background disturbances, those waves that experience resonant nonlinear growth may quickly dominate the dynamics.
- Resonant interaction can be detected in experiments by checking for phase dependence, e.g. by changing the sign of the input disturbance: (341) and (342) are invariant under $A_1 \to -A_1$ and $A_2 \to -A_2$, but (345) and (346), and (348) – (350), behave differently under a change of sign.
- Experiments have confirmed that Craik resonant triads are important in generating three-dimensionality in the early stages of the breakdown to turbulence.
- But predictions for the wave-angle are not very accurate.

5. Absolute and convective instabilities

Envelope of impulsive disturbance:
in a convectively unstable flow

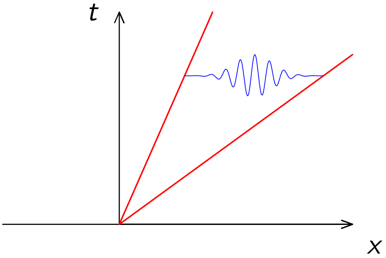


in an absolutely unstable flow



Space-time diagrams for parallel flows

Convective instability

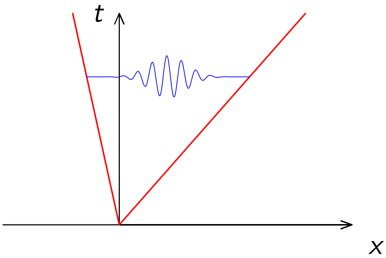


Disturbance grows as it propagates away, eventually leaving flow undisturbed.

Flow acts as a spatial amplifier of transients.

Can use spatial theory.

Absolute instability



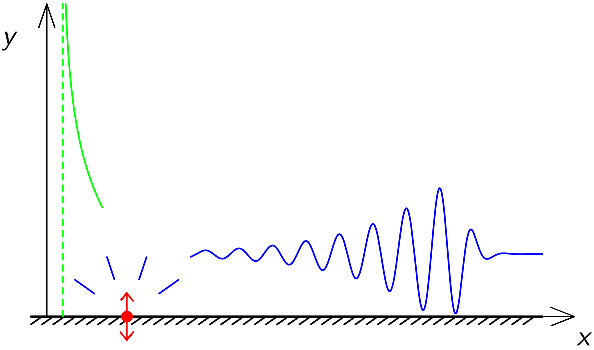
Disturbance grows in time everywhere.

Flow acts as a self-excited oscillator.

What is the temporal growth rate in the rest frame?

Spatial stability theory

The vibrating ribbon experiment:



The disturbance is time-periodic, but grows/decays downstream from the exciter (Schubauer & Skramstad, 1947).

Spatial stability theory

- Temporal theory, with real α and complex ω, was used:

$$\mathrm{e}^{\mathrm{i}(\alpha x-\omega t)}=\mathrm{e}^{\omega_i t}\mathrm{e}^{\mathrm{i}(\alpha x-\omega_r t)}. \tag{353}$$

- Temporal growth rate was converted into a spatial growth rate by choosing a downstream convection velocity.
- Gaster (1962) argued that spatial theory, with real ω and complex α, should be used instead:

$$\mathrm{e}^{\mathrm{i}(\alpha x-\omega t)}=\mathrm{e}^{-\alpha_i x}\mathrm{e}^{\mathrm{i}(\alpha_r x-\omega t)}. \tag{354}$$

- He showed that by assuming ω is an analytic function of α and integrating a Cauchy-Riemann relation:

$$\frac{\partial\omega_i}{\partial\alpha_i}=\frac{\partial\omega_r}{\partial\alpha_r}\quad\Rightarrow\quad\int_{\omega_i}^{0}\mathrm{d}\omega_i=\int_0^{\alpha_i}\frac{\partial\omega_r}{\partial\alpha_r}\mathrm{d}\alpha_i\quad\Rightarrow\quad\omega_i\approx-\frac{\partial\omega_r}{\partial\alpha_r}\alpha_i,$$

  the lower integration limit is temporal theory (real α, complex ω), the upper limit is spatial theory (real ω, complex α).

The signalling problem

- Landau (1946) suggested the idea of ‘spatial instability’.
- Watson (1962) developed a weakly nonlinear spatial theory.
- But α_i alone does not tell us if a wave is stable or unstable:
- Growth/decay factor is $\exp(-\alpha_i x)$, (354), so if $\alpha_i < 0$, then:
- (i) if wave propagates downstream there is growth as $x \to \infty$.
- (ii) if wave propagates upstream there is decay as $x \to -\infty$.
- Finding the direction of propagation is called the signalling problem.
- Also, people found the idea of a mode proportional to $\exp(-\alpha_i x)$ difficult to accept because it has infinite energy over the domain $-\infty < x < \infty$.
- In fact, all these difficulties can be resolved by considering the initial value problem.

A model initial value problem

- The flow is taken to be initially undisturbed, and then an impulsive disturbance is introduced.
- Its progress can be followed upstream/downstream.
- At any subsequent finite time the disturbance only extends over a finite part of the flow, so its energy remains finite.
- The impulse problem is solved using integral transforms, and the principle of causality must be applied.
- The key ideas can be illustrated by giving a pendulum that is hanging at rest, a sharp knock:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \Omega^2 y = v_0 \delta(t) \tag{355}$$

  where $\delta(t)$ is the Dirac delta function, v_0 is a constant, and the constant Ω is the natural frequency.
- The impulse transfers finite momentum to the pendulum, and causes it to start swinging.

Solution by elementary methods

- The impulse response can be obtained by considering

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \Omega^2 y = v_0 f(t), \quad f(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1/\epsilon & \text{for } 0 \leq t \leq \epsilon \\ 0 & \text{for } t \geq \epsilon \end{cases} \tag{356}$$

  as $\epsilon \to 0$.
- For $t \leq 0$, $y = 0$ (pendulum undisturbed before the impulse).
- For $0 \leq t \leq \epsilon$:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \Omega^2 y = \frac{v_0}{\epsilon}, \quad y(0) = y'(0) = 0, \tag{357}$$

  with solution

$$y = \frac{v_0}{\epsilon \Omega^2}(1 - \cos \Omega t). \tag{358}$$

- For $t \geq \epsilon$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \Omega^2 y = 0, \quad y(\epsilon) = \frac{v_0}{\epsilon\Omega^2}(1 - \cos\Omega\epsilon), \quad y'(\epsilon) = \frac{v_0}{\epsilon\Omega}\sin\Omega\epsilon, \tag{359}$$

  where the initial conditions are obtained from (358) evaluated at $t = \epsilon$.

- The solution to (359) is

$$y = -\frac{v_0}{\epsilon\Omega^2}(1 - \cos\Omega\epsilon)\cos\Omega t + \frac{v_0}{\epsilon\Omega^2}\sin\Omega\epsilon\sin\Omega t \tag{360}$$

  and in the limit $\epsilon \to 0$:

$$y \sim \frac{v_0}{\Omega}\sin\Omega t. \tag{361}$$

Solution by transform methods

- Now solve the knocked pendulum problem (355) by taking the Fourier transform of the equation:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\mathrm{e}^{\mathrm{i}\omega t} + \Omega^2 y \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t = \int_{-\infty}^{\infty} v_0 \delta(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t. \tag{362}$$

- Let

$$\tilde{y}(\omega) = \int_{-\infty}^{\infty} y(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t \tag{363}$$

  be the Fourier transform of y.
- We need (363) to converge when $y = 0$ and $y' = 0$ for all $t < 0$.
- If y remains bounded for all $t > 0$, then (363) converges when $\omega_i > 0$.

- Integrate the derivative term twice by parts:

$$\begin{aligned}\int_{-\infty}^{\infty} y'' \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t &= \left[y'\mathrm{e}^{\mathrm{i}\omega t}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \mathrm{i}\omega y' \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t \\ &= -\mathrm{i}\omega\left[y\mathrm{e}^{\mathrm{i}\omega t}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (\mathrm{i}\omega)^2 y \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t \\ &= -\omega^2 \tilde{y} \qquad (364)\end{aligned}$$

  using $y, y' = 0$ for $t < 0$, $\omega_i > 0$ and (363).
- Substituting (363) and (364) into (362), and evaluatng the integral on the RHS gives

$$-\omega^2\tilde{y} + \Omega^2\tilde{y} = v_0 \quad \Rightarrow \quad \tilde{y} = \frac{v_0}{\Omega^2 - \omega^2}. \qquad (365)$$

- The solution for $y(t)$ is obtained from the inverse Fourier transform of $\tilde{y}(\omega)$:

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{y}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}\omega \qquad (366)$$

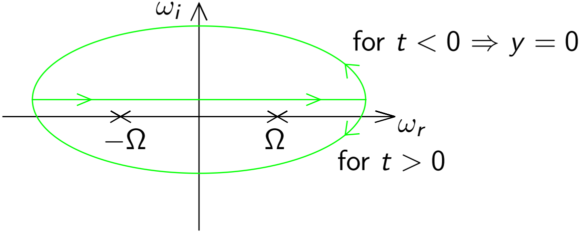
  integrated along a path in the complex ω plane with $\omega_i > 0$.

- Substituting (365) into (366) gives

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{v_0 \mathrm{e}^{-\mathrm{i}\omega t}}{\Omega^2 - \omega^2}\mathrm{d}\omega = \frac{v_0}{4\pi\Omega}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega + \Omega} - \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega - \Omega}\mathrm{d}\omega \tag{367}$$

using partial fractions.

- By closing the path we can use the residue theorem:



- The solution for $t > 0$ is

$$y = \frac{v_0}{4\pi\Omega}\left(-2\pi\mathrm{i}\mathrm{e}^{\mathrm{i}\Omega t} + 2\pi\mathrm{i}\mathrm{e}^{-\mathrm{i}\Omega t}\right) = \frac{v_0}{\Omega}\sin\Omega t. \tag{368}$$

Summary of transform method for IVP

- The impulsive disturbance excites an oscillation at the natural (unforced) frequency.
- There are poles in the complex ω plane at the natural frequency.
- The principle of causality is respected by placing the integration contour above all the poles in the complex ω plane.
- The application of this approach to shear layer instabilities involves Fourier transforms, and inverse Fourier transforms in both space and time.
- In this spatio-temporal theory both wavenumber and frequency may be complex.

The initial value problem in a shear layer

- Flow is initially undisturbed everywhere.
- A localized unsteady forcing is switched on at $t = 0$:

$$\hat{v}(x, 0, t) = \delta(x)\hat{f}(t), \tag{369}$$

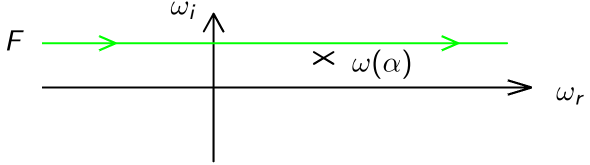
  where $\hat{f} = 0$ for $t < 0$.
- Physical solution is obtained from inverse Fourier transforms:

$$\hat{v}(x, y, t) = \frac{1}{4\pi^2}\int_F\int_A \frac{f(\omega)}{\Delta(\alpha,\omega)} v(y) \exp \mathrm{i}(\alpha x - \omega t)\, \mathrm{d}\alpha\, \mathrm{d}\omega. \tag{370}$$

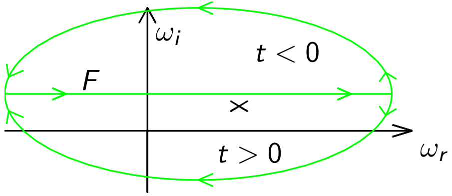
- $\Delta = 0$ is the dispersion relation: Δ appears in denominator and produces poles because roots of $\Delta = 0$ give the natural oscillations of the unforced problem.
- Integration contours F and A run from $-\infty$ to $+\infty$.
- But not necessarily along the real axes...

The Bromwich contour

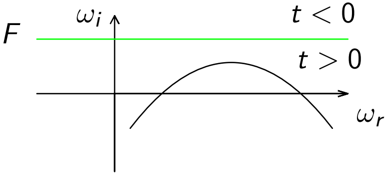
- The poles in the complex ω plane depend on α, and if the flow is unstable, these will be in the upper half-plane for some α.
- Respect causality (ensures convergence of Fourier transforms), by placing integration contour above all poles:

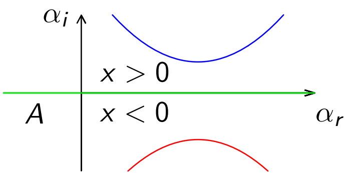


- Close the F-contour in either upper or lower half-plane depending on $t < 0$ or $t > 0$, so that $\exp(\omega_i t)$ is small:



- Similar considerations apply to the A-contour:





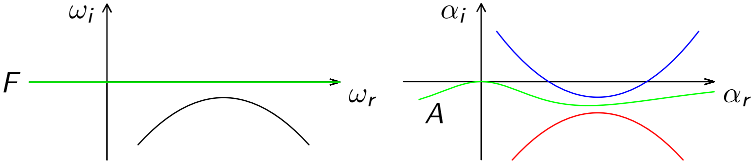
- A can be placed on real axis because disturbance is spatially localised, so transforms are convergent — in effect a temporal stability theory: ω complex and α real.
- The A-contour can be closed in the upper half-plane when $x > 0$, or the lower half-plane when $x < 0$, so that $\exp(-\alpha_i x)$ is small in each case.
- Therefore, any poles in the upper half of the α-plane (blue line) produced by F give downstream propagating waves.
- Any poles in the lower half of the α-plane (red line) produced by F give upstream propagating waves.
- Black line in the ω-plane is a locus of poles produced by A.

Deformation of integration contours

- The integration countours F and A have been placed in their complex planes in accordance with the principle of causality.
- They can then be moved in their complex planes provided they do not cross any singularities.
- Any movement of F causes the spatial branches (blue and red lines) to move.
- Any movement of A causes the temporal branches (black lines) to move.
- The simultaneous deformation of contours maintains causality provided no singularities are crossed.

Convective instability

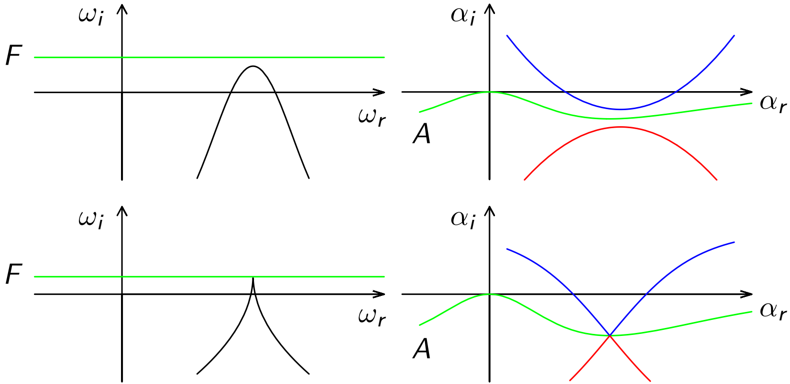
- It may be possible to lower F to the real axis, e.g.



- If this causes a spatial branch of downstream travelling waves to cross the real axis (as shown) then A must be moved below real axis.
- This corresponds to converting a temporal instability description of disturbances into a spatial theory (real ω, complex α).
- This is called convective instability.
- The example shown establishes that there are waves travelling downstream that grow with downstream distance.
- A spatial branch of upstream travelling waves crossing the real axis would imply unstable upstream travelling waves.

Absolute instability

Sometimes F cannot be lowered to the real ω-axis:



- The A-contour is said to be pinched by the spatial branches.
- This is Briggs' (1964) method for finding absolute instability.
- $\mathrm{Im}(\omega)$ at the pinch point gives the growth rate in time of the absolute instability.

Summary of Briggs' method

- Place wavenumber contour on real axis (Fourier transform is convergent because disturbance is localised).
- Place frequency contour above all singularities (ensures causality).
- Any branches of the dispersion relation in upper half of wavenumber plane are downstream propagating waves.
- Any branches of the dispersion relation in lower half of wavenumber plane are upstream propagating waves.
- If frequency contour can be lowered to real axis of frequency plane and a spatial branch crosses real axis of wavenumber plane then flow is convectively unstable.
- If frequency contour can not be lowered to real axis of frequency plane then flow is absolutely unstable.
- $\omega = \omega_0$ at the pinch point is the absolute frequency: $\mathrm{Im}(\omega_0)$ is the growth rate in the rest frame.

Saddle-point method

- An alternative interpretation based on the same calculations and diagrams.
- Consider an impulsive disturbance $f(\omega) = 1$.
- Use residue theorem to evaluate the ω-integral of (370) first.
- This leaves an α-integral of the form

$$\int_A -\frac{2\pi\mathrm{i}}{\Delta_\omega} v(y) \exp \mathrm{i}[\alpha x - \omega(\alpha) t] \, \mathrm{d}\alpha = \int_A -\frac{2\pi\mathrm{i}}{\Delta_\omega} v(y) \exp \phi t \, \mathrm{d}\alpha \tag{371}$$

where

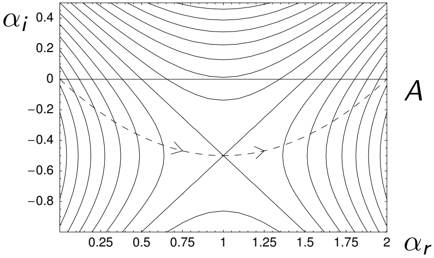
$$\phi(\alpha) = \mathrm{i}\left[\alpha \frac{x}{t} - \omega(\alpha)\right]. \tag{372}$$

- This is a superposition of normal modes satisfying the dispersion relation. The factor Δ_ω represents the receptivity to disturbances of different frequencies.

- In the limit $t \to \infty$ this integral is dominated by the contribution from the saddle-point, at which

$$\frac{\mathrm{d}\phi}{\mathrm{d}\alpha} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\omega}{\mathrm{d}\alpha} = \frac{x}{t}. \tag{373}$$

- This is because away from the saddles the integrand is highly oscillatory, leading to substantial cancelation. At a saddle the phase of ϕ is stationary $\Rightarrow$ non-oscillatory integrand.
- Locate saddle points of ϕ by plotting contours of constant $\mathrm{Re}(\phi)$ in the complex wavenumber plane:



- The saddles that contribute to the solution are those that have valleys along the real wavenumber axis.
- The A integration contour through such a saddle then follows steepest descent paths from the saddle point.
- The large-t behaviour in different frames of reference are found by choosing different values of x/t in (373).
- The large-t behaviour in the rest frame comes from

$$\frac{x}{t} = 0 \quad \Rightarrow \quad \frac{\partial \omega}{\partial \alpha} = 0, \tag{374}$$

  which corresponds loosely to zero group velocity.
- When $x/t = 0$, $\phi = -\mathrm{i}\omega$, so contours of $\mathrm{Re}(\phi)$ = contours of $\mathrm{Im}(\omega)$, which correspond to roots of $\Delta = 0$ for horizontal F-integration contours, i.e. spatial branches in the α-plane.
- Therefore, locating saddles and steepest-descent valleys corresponds to finding the spatial branches that pinch the integration contour in Brigg's method.