CHAPTER 1

Basic homological algebra

In this chapter we will define cohomology via cochain complexes. We will restrict to considering modules over a ring and to giving a very constructive definition. At the end of the chapter we will look at a more axiomatic apporach. These notes do not contain detailed proofs, although we shall go through them during the lecture. Most of the results in this chapter are fairly standard and can be found in most textbooks on homological algebra. These are, for example, the Springer Graduate Text by Hilton and Stammbach [**9**], the old classic by Rotman [**25**] or the newer and slightly more modern book by Weibel [**29**]. There are two very good places with online lecture notes: Peter Kropholler's notes on cohomology [**10**] and Daniel Murfet's collection of lecture notes [**23**]. Some of the proofs in this chapter follow those of Peter Kropholler and I recommend these as reference.
(I suspect that there are still a number of typos and other inaccuracies in these notes. So, if you find any, please let me know.)

1. Modules

In this section we will quickly review the basic definitions of modules over a ring. In general we denote a ring by R and assume that R has a unit.

Definition 1.1. Let R be a ring. A left R-module is an abelian group $(M,+)$ together with a multiplication

$$\begin{array}{ccc} R\times M & \to & M \\ (r,m) & \mapsto & rm \end{array}$$

satisfying the following axioms:

(M1) $r(m+n) = rm+rn$ for all $r\in R$ and $m,n\in M$
(M2) $(r+s)m = rm+sm$ for all $r,s\in R$ and $m\in M$
(M3) $(rs)m = r(sm)$ for all $r,s\in R$ and $m\in M$
(M4) $1_R m = m$ for all $m\in M$.

We usually write M_R - or M if it is clear which ring is meant. Right R-modules are defined analogously. If R is commutative a left R-module can be made into a right R-module by defining the multiplication by $(m,r)\mapsto rm$.

Example 1.2.

(1) Let k be a field. Then any k-vector space is a k-module.
(2) Any additive abelian group A can be viewed as a $\mathbb{Z}$-module.
(3) *The regular module:* Left multiplication makes any ring R into an R-module by $(r,s)\mapsto rs$. We call R the left regular module.

Definition 1.3. Let M be an R-module. An R-submodule is an abelian subgroup N such that for all $r \in R, n \in N: \ rn \in N$.

Exercise 1.4. Let V be a finite dimensional k-vector space and denote by $End_k(V)$ the ring of endomorphisms. Prove that

(1) V is a left $End_k(V)$-module via

$$\begin{array}{ccc} End_k(V) \times V & \to & V \\ (\phi, v) & \mapsto & \phi(v). \end{array}$$

(2) V has no $End_k(V)$-submodules except 0 and V. Such a module is called *simple*.

Definition 1.5. Let M and N be R-modules. A map $\alpha : M \to N$ is called R-linear or an R-module homomorphism if

- $\alpha(m + m') = \alpha(m) + \alpha(m')$ for all $m, m' \in M$
- $\alpha(rm) = r\alpha(m)$ for all $m \in M, r \in R$.

Let M and N be R-modules. We denote by $\mathrm{Hom}_R(M, N)$ the set of all R-linear maps $\alpha : M \to N$.

Remark 1.6. $\mathrm{Hom}_R(M, N)$ is an abelian group with addition defined pointwise. Furthermore $End_R(M) = \mathrm{Hom}_R(M, M)$ is a ring where multiplication is defined by composition of maps.

Lemma 1.7. *For every R-module M there is a natural isomorphism:*

$$\phi : \qquad \mathrm{Hom}_R(R, M) \longrightarrow M$$

defined by $f \mapsto f(1)$.

Proof: Exercise. □

Naturality means that for every R-module homomorphism $\alpha : M \to N$ the following diagram commutes,

$$\begin{tikzcd} \mathrm{Hom}_R(R, M) \arrow[r, "\phi_M"] \arrow[d, "\alpha_*"'] & M \arrow[d, "\alpha"] \\ \mathrm{Hom}_R(R, N) \arrow[r, "\phi_N"] & N \end{tikzcd}$$

where $\alpha_*(f) = \alpha \circ f$ and $\alpha \circ \phi_M = \phi_N \circ \alpha_*$.

Definition 1.8. **Direct product and direct sum of modules:** Let I be an index set and for each $i \in I$ let M_i be an R-module. Define a new R-module, the direct product of the M_i, by

$$M = \prod_{i \in I} M_i.$$

The elemenets $m \in M$ are families $(m_i)_{i \in I}$, where addition is defined component wise. The R-module structure is given by $r(m_i)_{i \in I} = (rm_i)_{i \in I}$.
Denote by M_0 the submodule of M consisting of those families $(m_i)_{i \in I}$, for which $m_i = 0$ for all but finitely many $i \in I$. We call M_0 the direct sum of the M_i, denoted by

$$M_0 = \bigoplus_{i \in I} M_i.$$

REMARK 1.9. For every $i \in I$ there are natural projections:

$$\begin{array}{rccc} \pi_i : & M & \twoheadrightarrow & M_i \\ & (m_j)_{j\in I} & \mapsto & m_i \end{array}$$

and natural injections

$$\begin{array}{rccc} \iota_i : & M_i & \hookrightarrow & M_0 \\ & m_i & \mapsto & (m_j)_{j\in I}, \end{array}$$

where

$$m_j = \begin{cases} m_i & \text{if} \quad i = j \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 1.10. *Let X be an R-module and let $\phi_i : X \to M_i$ be an R-module homomorphism for every $i \in I$. Then there exists a unique R-module homomorphism $\phi : X \to \prod_{i\in I} M_i$, such that for all $i \in I$ $\pi_i \circ \phi = \phi_i$. In particular,*

$$Hom_R(X, \prod_{i\in I} M_i) \cong \prod_{i\in I} Hom_R(X, M_i).$$

PROPOSITION 1.11. *Let Y be an R-module and let $\psi_i : M_i \to Y$ be an R-module homomorphism for every $i \in I$. Then there is a unique R-module homomorphism $\psi : \bigoplus_{i\in I} M_i \to Y$ such that for every $i \in I$, $\psi \circ \iota_i = \psi_i$. In particular,*

$$Hom_R(\bigoplus_{i\in I}, Y) \cong \prod_{i\in I} Hom_R(M_i, Y).$$

Proof: We leave this an an exercise. Define $\psi((M_i)_{i\in I}) = \sum_{i\in I} \psi_i(m_i)$. □

REMARK 1.12. If I is a finite set, then $\prod_{i\in I} M_i \cong \bigoplus_{i\in I} M_i$.

EXERCISE 1.13. Let M be an R-module and I be a set. Suppose that for each $i \in I$, M_i is a submodule of M. Further assume:

(1) M is generated by the M_i. (i.e. Each $m \in M$ can be expressed as $m = \sum_{i\in I} m_i$, where all but a finite number of the m_i are zero.)
(2) For all $j \in I$ let N_j be the submodule generated by all M_i with $i \neq j$. Then $N_j \cap M_j = \{0\}$. for all $j \in I$.

Show that

$$M \cong \bigoplus_{i\in I} M_i.$$

2. Exact sequences and diagram chasing

Let us begin with some notation and basic facts. Let $\alpha : M \to N$ be an R-module homomorphism. The kernel of α is defined to be the following subset of M: $ker(\alpha)$=$\{m \in M \,|\, \alpha(m) = 0\}$, and the image of α is defined to be the follwing subset of N: $im(\alpha)$=$\{\alpha(m) \,|\, m \in M\}$. Recall, that $ker(\alpha) = \{0\} \iff \alpha$ is a monomorphism, i.e. an injective homomorphism. It is an epimorphism, i.e. a surjective homomorphism if $im(\alpha) = N$. The cokernel of α is defined to be

$$coker(\alpha) = N/im(\alpha).$$

Definition 2.1. A sequence

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\alpha_{i+1}} M_i \xrightarrow{\alpha_i} M_{i-1} \xrightarrow{\alpha_{i-1}} \cdots$$

$(i \in \mathbb{Z})$ of linear maps is called exact at M_i if $im(\alpha_{i+1}) = ker\alpha_i$.
The sequence is called exact if it is exact at every $M_i (i \in \mathbb{Z})$.

Exercise 2.2. Show that:

(1) $0 \longrightarrow L \xrightarrow{\alpha} M$ is exact if and only if α is a monomorphism.
(2) $M \xrightarrow{\beta} N \longrightarrow 0$ is exact if and only if β is an epimorphism.
(3) $0 \longrightarrow L \xrightarrow{\alpha} M \longrightarrow 0$ is exact iff α is an isomomorphism.

Remark 2.3. A short exact sequence is an exact sequence of the form

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0.$$

In particular, α is a monomorphism, β is an epimorphism and $im(\alpha) = ker(\beta)$. Hence $N \cong M/\alpha(L)$. Conversely, if $N \cong M/L$, then there is a short exact sequence

$$L \hookrightarrow M \twoheadrightarrow N.$$

Definition 2.4. A short exact sequence

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

(1) splits at N if there exists an R-module homomorphism $\tau : N \to M$ such that $\beta \circ \tau = id_N$.
(2) splits at L if there exists an R-module homomorphism $\sigma : M \to L$ such that $\sigma \circ \alpha = id_L$.

Theorem 2.5. *Let*

$$\mathrm{E}: \quad 0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

be a short exact sequence. Then the following are equivalent:

(1) *E splits at L;*
(2) *E splits at N;*
(3) *There exist R-module homomorphisms $\sigma : M \to L$ and $\tau : N \to M$ such that $\sigma \circ \alpha = id_L, \beta \circ \tau = id_N$ and $\alpha \circ \sigma + \tau \circ \beta = id_M$.*

Furthermore, any of the above conditions implies

$$M \cong L \oplus N$$

and we say the short exact sequence E splits.

Proof: (3) $\Rightarrow$ (1), (2) is trivial, (1) $\Rightarrow$ (3) we'll do in lecture and (2) $\Rightarrow$ (3) is an exercise. Now assume (3) and define

$$\begin{array}{rccc} \Theta : & M & \to & L \oplus N \\ & m & \mapsto & (\sigma(m), \beta(m)) \end{array}$$

and show this is an isomorphism. $\square$

Let us get back to the groups $\mathrm{Hom}_R(M,N)$: Let $\alpha \in \mathrm{Hom}_R(M,N)$ and let $\xi : N \to X$ be an R-module homomorphism. We then define

$$\xi_* : \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,X)$$

by $\xi_*(\alpha) = \xi \circ \alpha$. In other words, $\mathrm{Hom}_R(M,-)$ is a covariant functor. Now let $\psi : Y \to M$ be an R-module homomorphism. We define

$$\psi^* : \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(Y,N)$$

by $\psi^*(\alpha) = \alpha \circ \psi$. We say $\mathrm{Hom}_R(-,N)$ is a contravariant functor.

Theorem 2.6. *Let X and Y be R-modules and let*

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

be a short exact sequence. Then the following sequences are exact:

(1) $0 \longrightarrow \mathrm{Hom}_R(Y,L) \xrightarrow{\alpha_*} \mathrm{Hom}_R(Y,M) \xrightarrow{\beta_*} \mathrm{Hom}_R(Y,N)$

(2) $0 \longrightarrow \mathrm{Hom}_R(N,X) \xrightarrow{\beta^*} \mathrm{Hom}_R(M,X) \xrightarrow{\alpha^*} \mathrm{Hom}_R(L,X)$.

Proof: We leave (1) as exercise and do (2) in class. □

We say $\mathrm{Hom}_R(-,X)$ and $\mathrm{Hom}_R(Y,-)$ are left exact functors. Neither β_* nor α^* have to be surjective. We'll come back to conditions on X and Y for Hom to be an exact functor.

Example 2.7. Consider the following short exact sequence of abelian groups:

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

and let $X = \mathbb{Z}$. Then $\mathrm{Hom}(\mathbb{Q},\mathbb{Z}) = 0$ but $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \neq 0$ and the map $\mathrm{Hom}(\mathbb{Q},\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z})$ is not surjective.

Let us finish this section with two important results, which we'll come back to and apply a little later. We shall nevertheless prove them now in great detail as the methods used are essential to homological algebra, namely diagram chasing. But first let us say what we mean with a commutative diagram. Consider the following diagram of R-modules and R-module homomorphisms:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

We say this diagram commutes if $\beta \circ \alpha = \delta \circ \gamma$.

Proposition 2.8. **[The 5-Lemma]**
Let

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

be a commutative diagram with exact rows. Then

1. *If β, δ are monomorphisms, α is an epimorphism, then γ is a monomorphism.*
2. *If β, δ are epimorphisms and ϵ is a monomorphism, then γ is an epimorphism.*
3. *If β, δ are isomorphisms, α is an epimorphism and ε is a monomorphism, then γ is an isomorphism.*

Proof: (3) obviously follows from (1) and (2). For (1), see Peter Kropholler's notes [**10**] and part (2) is an exercise, and a very good and useful one. □

Proposition 2.9. **[The Snake-Lemma]** *Let*

$$\begin{tikzcd} & A \arrow[r, "\theta"] \arrow[d, "\alpha"'] & B \arrow[r, "\phi"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\theta'"] & B' \arrow[r, "\phi'"] & C' & \end{tikzcd}$$

be a commutative diagram with exact rows. Then there is a natural exact sequence

$$ker(\alpha) \xrightarrow{\theta_*} ker(\beta) \xrightarrow{\phi_*} \ker(\gamma) \xrightarrow{\delta} coker(\alpha) \xrightarrow{\theta'_*} coker(\beta) \xrightarrow{\phi'_*} coker(\gamma).$$

Moreover, if θ is a monomorphism and ϕ' is an epimorphism, θ_* is a monomorphism and ϕ'_* is an epimorphism.

Before we embark of the proof of this Lemma, let us first note that both kernel and cokernel are functorial, i.e. whenever there's a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

we can insert kernels and cokernels into this diagram and have induced maps γ_* and β_* making the new, bigger diagram commute:

$$\begin{tikzcd} ker(\alpha) \arrow[r, tail] \arrow[d, "\gamma_*"'] & A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[r, two heads] \arrow[d, "\beta"] & coker(\alpha) \arrow[d, "\beta_*"'] \\ ker(\delta) \arrow[r, tail] & C \arrow[r, "\delta"] & D \arrow[r, two heads] & coker(\delta) \end{tikzcd}$$

Proof: The main part is the construction of the connecting map δ, which we will do in detail. The we need to prove exactness at each of 4 places. We also have to check naturality of δ. For a very detailed account see [**10**]. □

3. Projective modules

Projective modules are basically the bread and butter of homological algebra, so let's define them. But first, let's do free modules:

Definition 3.1. Let F be an R-module and X be a subset of F. We say F is free on X if for every R-module A and every map $\xi : X \to A$ there exists a unique R-module homomorphism $\phi : F \to A$ such that $\phi(x) = \xi(x)$ for all $x \in X$.

In other words F is free if there's a unique R-module homorphism ϕ making the following diagram commute:

$$\begin{tikzcd} & F \arrow[dd, dotted, "\phi!"] \\ X \arrow[ur, "i"] \arrow[dr, "\xi"'] & \\ & A \end{tikzcd}$$

A very hard look at this diagram now gives us the following lemma.

Lemma 3.2. *Let F and F' be two modules free, on X. Then $F \cong F'$.*

This gives us uniqueness, i.e. we can talk of *the* free module on X. The following gives us existence and a little bit more.

Exercise 3.3. Let X be a set and consider the R-module

$$E = \bigoplus_{x \in X} R.$$

For each $x \in X$ consider the following map

$$\begin{array}{rcl} s_x : \quad X & \to & R \\ s_x(y) & = & \begin{cases} 1 & \text{x=y} \\ 0 & otherwise \end{cases} \end{array}$$

Let $S = \{s_x \mid x \in X\}$. Show

(1) E is free on S.
(2) For every free module F on X there is an isomorphism $\Theta : E \to F$, such that $\Theta(s_x) = x$ for all $x \in X$.
(3) F is free on X if and only if every element $f \in F$ can be written uniquely as $f = \sum_{x \in X} a_x x$ where $a_x \in R$ and all but a finite number of $a_x = 0$.

Example 3.4.

(1) Let k be a field. Then every k-module is free. This is nothing other than saying that every k-vector space has a basis. (For the infinite dimensional case we need Zorn's Lemma).
(2) A free $\mathbb{Z}$-module is the same as a free abelian group.
(3) $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

Lemma 3.5. *Every R-module M is the homomorphic image of a free R-module.*

PROPOSITION 3.6. *Let P be an R-module. Then the following statements are equivalent:*

(1) $\mathrm{Hom}_R(P, -)$ *is an exact functor*
(2) P *is a direct summand of a free module.*
(3) *Every epimorphism $M \twoheadrightarrow P$ splits.*
(4) *For every epimorphism $\pi : A \twoheadrightarrow B$ of R-modules and every R-module map $\alpha; P \to B$ there is an R-module homomorphism $\phi : P \to A$ such that $\pi \circ \phi = \alpha$.*

DEFINITION 3.7. Every R-module satisfying the conditions of Proposition 3.6 is called a projective R-module.

REMARK 3.8. Every free R-module is projective, but not every projective is free. Take, for example $R = \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and $P = \mathbb{Z}/2\mathbb{Z}$ but $(0, 1)P = 0$, which contradicts uniqueness of expression of elements in a free module.

LEMMA 3.9. *A direct sum $P = \bigoplus_{i\in I} P_i$ is projective iff each P_i, $i \in I$ is projective.*

REMARK 3.10. Let k be a field and V a vector space of countable dimension. Then there is a k-vector space isomorphism $V \cong V \oplus V$. Hence, for the ring $R = End_k(V)$ we have the following chain of isomorphisms:

$$R = End_k(V) = \mathrm{Hom}_k(V, V) \cong \mathrm{Hom}_k(V \oplus V, V) \cong \mathrm{Hom}_k(V, V) \oplus \mathrm{Hom}_k(V, V) = R \oplus R$$

and there are free modules on a set of n elements which are isomorphic to free modules on a set with m elements where $n \neq m$.

LEMMA 3.11. [**Eilenberg-Swindle**] *Let R be a ring and P be a projective module. Then there exists a free module such that*

$$P \oplus F \cong F.$$

Let us prove one more important result before we return to define cohomology.

DEFINITION 3.12. Let M be an R-module. A projective resolution of M is an exact sequence

$$\cdots \longrightarrow P_{i+1} \xrightarrow{d_i} P_i \xrightarrow{d_{i+1}} \cdots \xrightarrow[1]{d} P_1 \xrightarrow[0]{d} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

where every P_i, $i \geq 0, i \in \mathbb{Z}$, is a projective module.

We also use the short notation

$$\mathbf{P}_* \twoheadrightarrow M.$$

LEMMA 3.13. [**Schanuel's Lemma**] *Let*

$$K \hookrightarrow P \twoheadrightarrow M$$

and

$$K' \hookrightarrow P' \twoheadrightarrow M$$

be two short exact sequences such that P and P' are projective. Then

$$K \oplus P' \cong K' \oplus P.$$

In particular, K is projective if and only if K' is projective.

As a direct consequence of the proof we can now prove inductively the following result about projective resolutions (exercise):

Proposition 3.14. *Let*

$$\mathbf{P}_* \twoheadrightarrow M \quad \text{and} \quad \mathbf{P}'_* \twoheadrightarrow M$$

be two projective resolutions of M *and denote by* $K_n = ker(P_n \to P_{n-1})$ *and* $K' = ker(P'_n \to P'_{n-1})$ *the* n*-th kernels respectively. Then, for all* $n \geq 0$

$$K_n \oplus P'_n \oplus P_{n-1} \oplus ... \cong K'_n \oplus P_n \oplus P'_{n-1} \oplus ....$$

In particular K_n *is projective iff* K'_n *is projective.*

Definition 3.15. Let M be an R-module. We say M has finite projective dimension over R, $\mathrm{pd}_R M < \infty$, if M admits a projective resolution $\mathbf{P}_* \twoheadrightarrow M$ of finite length. In particular, there exists an $n \geq 0$ such that

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

is a projective resulution of n. The smallest such n is called the projective dimension of M.

Remark 3.16. The long Schanuel's Lemma 3.14 implies that if $\mathrm{pd}_R M \leq n$ for some n, then in every projective resolution $\mathbf{Q}_* \twoheadrightarrow M$, the kernel
$K_{n-1} = ker(Q_{n-1} \to Q_{n-2})$ is projective.
Projective modules have projective dimension equal to 0.

4. Cochain complexes

In this section we will give a first definition of the Ext-groups. We will later see a more axiomatic approach. The approach in this section will be very hands-on.
Let $\mathbf{d}_* : \mathbf{P}_* \twoheadrightarrow M$ be a projective resolution of the R-module M and let N be an arbitrary R-module. Apply the functor $\mathrm{Hom}_R(-M)$ to this projective resolution and we obtain a sequence of abelian groups (careful, it's not necessarily exact: see Lemma 2.6):

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(P_0, N) \to \mathrm{Hom}_R(P_1, N) \to \cdots$$
$$\cdots \to \mathrm{Hom}_R(P_i, N) \to \mathrm{Hom}_R(P_{i+1}, N) \to \cdots$$

Exercise 4.1. Let us denote by $\delta_i = d_i^* : \mathrm{Hom}_R(P_i, N) \to \mathrm{Hom}_R(P_{i+1}, N)$ for all $i \geq 0$ and by $\delta_{-1} : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(P_0, N)$. Show that for all $i \geq -1$, $\delta_i \delta_{i-1} = 0$, i.e. composition of two consecutive maps in the above sequence is zero.

Definition 4.2. A cochain complex is a family $\mathbf{C} = (C^q, \delta^q)$ of abelian groups C^q together with homomorphisms $\delta^q : C^q \to C^{q+1}$ such that for all $q \in \mathbb{Z}$,

$$\delta^{q+1}\delta^q = 0.$$

The kernel $Z^q(C) = ker(\delta^q) \subseteq C^q$ is called the group of $q-Cocycles$.
The image $B^q(C) = im(\delta_{q-1}) \subseteq C^q$ is called the group of $q-Coboundaries$

Since, by definition, $\delta^{q+1}\delta^q = 0$, it follows that $B^q \subseteq Z^q \subseteq C^q$ and we can make the following definition:

Definition 4.3. The quotient group

$$\mathrm{H}^q(C) = Z^q(C)/B^q(C)$$

is called the q-th cohomology group of $\mathbf{C}$.

REMARK 4.4. Let $\mathbf{C}$ be exact at C^q. Then $B^q = Z^q$ implying $\mathrm{H}^q(C) = 0$.

We can dow define cohomology of the cochain complex $\mathrm{Hom}_R(\mathbf{P}_*, N)$ above, but it still remains to show that it won't change when we choose a different projective resolution of M.

DEFINITION 4.5. Let $\mathbf{C} = (C^q, \delta_q)$ and $\mathbf{C}' = (C'^q, \delta'_q)$ be two cochain complexes. A cochain map $\mathbf{f} : \mathbf{C} \to \mathbf{D}$ is a system $\mathbf{f} = (f_q)_{q\in\mathbb{Z}}$ of homomorphisms $f_q : C^q \to C'^q$ such that $\delta'_{q-1} f_q = f_{q-1}\delta_q$ for all $q \in \mathbb{Z}$.

That is, we have a ladder of commutative squares:

$$\begin{tikzcd}
\cdots \arrow[r] & C^{q-1} \arrow[r, "\delta_{q-1}"] \arrow[d, "f_{q-1}"] & C^q \arrow[r, "\delta_q"] \arrow[d, "f_q"] & C^{q+1} \arrow[r, "\delta_{q+1}"] \arrow[d, "f_{q+1}"] & \cdots \\
\cdots \arrow[r] & C'^{q-1} \arrow[r, "\delta'_{q-1}"] & C'^q \arrow[r, "\delta'_{q-1}"] & C'^{q+1} \arrow[r, "\delta'_{q+1}"] & \cdots
\end{tikzcd}$$

and we often forget the subscripts to our maps, i.e. we just write

$$f\delta = \delta f.$$

DEFINITION 4.6. Let $\mathbf{f}$ and $\mathbf{g} : (C^q, \delta)_{q\in\mathbb{Z}} \to (C'^q, \delta)'_{q\in\mathbb{Z}}$ be two cochain maps. We say $\mathbf{f}$ and $\mathbf{g}$ are homotopic if there is a system $\mathbf{\Psi} = (\Psi_q)_{q\in\mathbb{Z}}$ of homomorphisms $\Psi_q : C^q \to C'^{q-1}$ such that for all $q \in \mathbb{Z}$:

$$\delta'_{q-1} \circ \Psi_q + \Psi_{q+1} \circ \delta_q = f_q - g_q.$$

In particular, if we have two commutative ladders as above, we can fill in Ψ as follows,

$$\begin{tikzcd}
\cdots \arrow[r] & C^{q-1} \arrow[r, "\delta"] \arrow[dl, dotted, "\Psi"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & C^q \arrow[r, "\delta"] \arrow[dl, dotted, "\Psi"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & C^{q+1} \arrow[r, "\delta"] \arrow[dl, dotted, "\Psi"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & \cdots \arrow[dl, dotted, "\Psi"'] \\
\cdots \arrow[r] & C'^{q-1} \arrow[r, "\delta'"] & C'^q \arrow[r, "\delta'"] & C'^{q+1} \arrow[r, "\delta'"] & \cdots
\end{tikzcd}$$

such that, in short,

$$\delta'\Psi + \Psi\delta = f - g.$$

LEMMA 4.7.

(1) *Every cochain map* $\mathbf{f} : \mathbf{C} \to \mathbf{C}'$ *induces a homomorphism of abelian groups*

$$f_*^q = \mathrm{H}^q(f) : \mathrm{H}^q(C) \to \mathrm{H}^q(C').$$

(2) *Let* $\mathbf{f}$ *and* $\mathbf{g} : \mathbf{C} \to \mathbf{C}'$ *be homotopic cohcain-maps. Then* $f_* = g_*$.

To apply this to projective resolutions, let $\mathbf{P}_* \twoheadrightarrow M$ be a projective resolution and consider the deleted projective resolution

$$\cdots \to P_i \to P_{i-1} \to P_1 \to P_0 \to 0.$$

Note that the sequence is still exact everywhere except at P_0. But note also that $M = coker(P_1 \to P_0)$. We denote by $\mathrm{Hom}_R(P_*, N)$ the cochain complex resulting from applying $\mathrm{Hom}_R(-, N)$ to the deleted resolution.

THEOREM 4.8. *Let* $\mathbf{P}_* \twoheadrightarrow M$ *and* $\mathbf{Q}_* \twoheadrightarrow M$ *be projective resolutions of the* R*-module* M*. Then for all* $n \in \mathbb{Z}$ *and all* R*-modules* N*,*

$$\mathrm{H}^n(\mathrm{Hom}_R(\mathbf{P}_*, N)) \cong \mathrm{H}^n(\mathrm{Hom}_R\, \mathbf{Q}_*, N)).$$

DEFINITION 4.9. Let M and N be R-modules and $\mathbf{P}_* \twoheadrightarrow M$ be a projective resolution of M. We define

$$\mathrm{Ext}^n_R(M,N) \cong \mathrm{H}^n(\mathrm{Hom}_R(\mathbf{P}_*, N)).$$

By the above theorem 4.8, this definition is independent of the choice of projective resolution of M. Please note, that for all $n < 0$, the n-th Ext-group vanishes.

EXERCISE 4.10. Let M and N be R-modules. Prove

(1) $\mathrm{Ext}^0(M,N) \cong \mathrm{Hom}_R(M,N)$.
(2) For every projective R-module P and all $n \geq 1$, $\mathrm{Ext}^n(P,N) = 0$.

5. Long exact sequences in cohomology

Let $\mathbf{C} = (C,d)_{n\in\mathbb{Z}}$ and $\mathbf{C}' = (C',d')_{n\in\mathbb{Z}}$ be two cochain complexes and let $\mathbf{f} : \mathbf{C} \to \mathbf{C}'$ be a cochain map. We say $\mathbf{f}$ is a monomorphism(epimorphism/isomorphism) if for each $n \in \mathbb{Z}$ the maps $f_n : C^n \to C'^n$ are monomorphisms (epimorphisms/isomorphisms). Therefore it makes perfect sense to talk about short exact sequences of cochain complexes. In particular,

$$\mathbf{C}'' \hookrightarrow \mathbf{C} \twoheadrightarrow \mathbf{C}'$$

is a short exaxt sequence of cochain complexes if, for all $n \in \mathbb{Z}$,

$$C''^n \hookrightarrow C^n \twoheadrightarrow C'^n$$

is a short exact sequence of abelian groups.
We could have defined these terms in a more sophisticated manner, noting that the category of cochain complexes is an abelian category and so notions of monomorphisms, epimorphisms and isomorphisms have a category theoretical definition. We would also have to prove that our naive definition agrees with this definition.

THEOREM 5.1. *For every short exact sequence*

$$0 \longrightarrow \mathbf{C}'' \xrightarrow{\alpha} \mathbf{C} \xrightarrow{\beta} \mathbf{C}' \longrightarrow 0$$

of cochain complexes there are natural connecting maps δ such that there is a long exact sequence in cohomology:

$$\cdots \xrightarrow{\delta} \mathrm{H}^n(C'') \xrightarrow{\alpha_*} \mathrm{H}^n(C) \xrightarrow{\beta_*} \mathrm{H}^n(C') \xrightarrow{\delta} \mathrm{H}^{n+1}(C'') \xrightarrow{\alpha_*} \mathrm{H}^{n+1}(C) \longrightarrow \cdots$$

The connecting map $\delta : \mathrm{H}^n(C') \to \mathrm{H}^{n+1}(C'')$ natural means that for every commutative diagram of cochain complexes with exact rows

$$\begin{tikzcd}
\mathbf{C}'' \arrow[r, tail] \arrow[d, "\phi''"] & \mathbf{C} \arrow[r, two heads] \arrow[d, "\phi"] & \mathbf{C}' \arrow[d, "\phi'"] \\
\mathbf{D}'' \arrow[r, tail] & \mathbf{D} \arrow[r, two heads] & \mathbf{C}'
\end{tikzcd}$$

the following diagram commutes for every $n \in \mathbb{Z}$:

$$\begin{tikzcd}
\mathrm{H}^n(C') \arrow[r, "\delta"] \arrow[d, "\phi'_*"'] & \mathrm{H}^{n+1}(C'') \arrow[d, "\delta''_*"] \\
\mathrm{H}^n(D') \arrow[r, "\delta"] & \mathrm{H}^{n+1}(D'').
\end{tikzcd}$$

Exercise 5.2. Let

$$\begin{tikzcd} \mathbf{C}'' \arrow[r, tail] \arrow[d, "\phi''"] & \mathbf{C} \arrow[r, two heads] \arrow[d, "\phi"] & \mathbf{C}' \arrow[d, "\phi'"] \\ \mathbf{D}'' \arrow[r, tail] & \mathbf{D} \arrow[r, two heads] & \mathbf{C}' \end{tikzcd}$$

be a commutative diagram of cochain complexes. Prove that whenever any two of the cochain maps ϕ'', ϕ, ϕ' induce an isomorphism in cohomology, then so does the third.

Next we would like to derive long exact sequences for $\mathrm{Ext}_R(M, N)$. To do this we need to build short exact sequences of projective resolutions, for which the following lemma is an essential step.

Lemma 5.3. **[Horseshoe-Lemma]** *Let $M'' \hookrightarrow M \twoheadrightarrow M'$ be a short exact sequence of R-modules and let $K'' \hookrightarrow P'' \twoheadrightarrow M''$ and $K' \hookrightarrow P' \twoheadrightarrow M'$ be short exact sequences with P'' and P' projective. Then there is a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & K'' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P'' \arrow[r, "\eta"] \arrow[d, two heads, "\pi''"'] & P'' \oplus P \arrow[r, "\mu"] \arrow[d, two heads] & P' \arrow[r] \arrow[d, two heads, "\pi'"] & 0 \\ 0 \arrow[r] & M'' \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & M' \arrow[r] & 0 \end{tikzcd}$$

where $\eta(\pi'') = (p'', 0)$ and $\mu(p'', p') = p'$ are the natural inclusion and projection repectively.

Proof: Since P' is projective, there exists a $\lambda : P' \to M$ such that $\beta\lambda = \pi'$. Define

$$\begin{array}{rccl} \pi : & P'' \oplus P & \to & M \\ & (p'', p') & \mapsto & \alpha\pi''(p'') + \lambda(p') \end{array}$$

□

Corollary 5.4. *Let $M'' \hookrightarrow M \twoheadrightarrow M'$ be a short exact sequence of R-modules. Then there is a short exact sequence of projective resolutions*

$$\mathbf{P}''_* \hookrightarrow \mathbf{P}_* \twoheadrightarrow \mathbf{P}'_*.$$

And now we can apply Theorem 5.1 to Ext:

Theorem 5.5. *Let $M'' \hookrightarrow M \twoheadrightarrow M'$ be a short exact sequence of R-modules. And let N be an arbitrary R-module. Then there are long exact sequences in cohomology*

(1)

$$\cdots \to \mathrm{Ext}^n(N, M'') \to \mathrm{Ext}^n(N, M) \to \mathrm{Ext}^n(N, M') \to \mathrm{Ext}^{n+1}(N, M'') \to \cdots$$

(2)

$$\cdots \to \mathrm{Ext}^n(M', N) \to \mathrm{Ext}^n(M, N) \to \mathrm{Ext}^n(M'', N) \to \mathrm{Ext}^{n+1}(M', N) \to \cdots$$

LEMMA 5.6. **[Dimension shifting]** *Let $K \hookrightarrow P \twoheadrightarrow M$ be the beginning of a projective resolution of M and let N be an R-module. Then for all $n \geq 1$,*

$$\mathrm{Ext}^n(K,N) \cong \mathrm{Ext}^{n+1}(M,N).$$

Proof: Apply Theorem 5.5 and Exercise 4.10. □

Now let us have a more detailed look at projective dimension. Recall, Definition 3.15 that a module M is said to have $\mathrm{pd}_R M \leq n$ if there is a projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

of length n. We say $\mathrm{pd}_R M = n$ if there is no projective resolution of shorter length. Let's summarise all we know so far:

PROPOSITION 5.7. *Let M be an R-module. Then the following statments are equivalent:*

(1) $\mathrm{pd}_R M \leq n$.
(2) $\mathrm{Ext}^i_R(M,-) = 0$ *for all* $i > n$
(3) $\mathrm{Ext}^{n+1}_R(M,-) = 0$
(4) *Let* $0 \to K_{n-1} \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *be an exact sequence with* P_i *projective for all* $0 \leq i \leq n-1$. *Then* K_{n-1} *is projective.*

EXERCISE 5.8. Let $M'' \hookrightarrow M \twoheadrightarrow M'$ be a short exact sequence of R-modules. Prove the following:

(1) $\mathrm{pd}M' \leq sup\{\mathrm{pd}M, \mathrm{pd}M'' + 1\}$.
(2) $\mathrm{pd}M \leq sup\{\mathrm{pd}M'', \mathrm{pd}M'\}$.
(3) $\mathrm{pd}M'' \leq sup\{\mathrm{pd}M, \mathrm{pd}M' - 1\}$.

(This is an exercise in applying Theorem 5.5)

Let us finish this section with a very useful observation:

PROPOSITION 5.9. *Let M be an R-module such that $\mathrm{pd}M = n$. Then there exists a free R-module F such that*

$$\mathrm{Ext}^n(M,F) \neq 0.$$

6. Injective modules

DEFINITION 6.1. Let I be an R-module. We say I is injective if for every injective R-module homomorphism $\iota : A \hookrightarrow B$ and every R-module homomorphism $\alpha : A \to I$ there exists an R-module homomorphism $\beta : B \to I$ such that $\beta\iota = \alpha$.

PROPOSITION 6.2. *Let I be an R-module. Then the following are equivalent:*

(1) *I is injective.*
(2) $\mathrm{Hom}_R(-,I)$ *is exact.*
(3) *Every injective R-module homomorphism $I \hookrightarrow B$ splits, where B an arbitrary R-module.*

Proof: This is an exercise. □

Proposition 6.3. *Let I be an injective module. Then for every R-module M and all $n \geq 1$,*

$$\mathrm{Ext}^n(M, I) = 0.$$

Proof: Let $\mathbf{P}_* \twoheadrightarrow M$ be a projective resolution of M. Then $\mathrm{Hom}_R(\mathbf{P}_*, I)$ is exact in degree $n > 0$. □

The reader will have noticed that injective modules have dual properties to those of projective modules. We have seen how to construct projective modules (3.3) and that every module has a projective mapping onto it (3.5). Analogous results hold but are slightly more complicated. One can show how to build injectives and that every module maps into an injective. A thorough account of these facts can be found in Rotman's book [**25**, pages 65–71].

Exercise 6.4. Prove the following:

(1) Let $\{E_j \mid j \in J\}$ be a family of injective modules, then $\prod_{j\in J} E_j$ is injective.
(2) Every summand of an injective module is injective.

7. An axiomatic approach to cohomology

We have seen before that we can set up cohomology using category theoretical language, such that Ext is just an example. Let us begin by recalling a few of the definitions in category theory. A detailed account of all the main results we might be needing later can be found in Hilton-Stammbach [**9**, Chapter II, Sections 1–6.].
A category $\mathfrak{C}$ consists of three sets of data:

- A class of objects $A, B, C, \ldots$
- To each pair of objects A, B of $\mathfrak{C}$ a set of morphisms $\mathfrak{C}(A, B)$ $(f : A \to B)$ from A to B
- to each triple of objects A, B, C of $\mathfrak{C}$ a law of composition $\mathfrak{C}(A, B) \times \mathfrak{C}(B, C) \to \mathfrak{C}(A, C)$ $((f, g) \mapsto g \circ f)$.

satisfying the following three axioms:

(1) The sets $\mathfrak{C}(A, B)$ and $\mathfrak{C}(A', B')$ are disjoint unless $A = A'$ and $B = B'$.
(2) given $f : A \to B$, $g : B \to C$ and $h : C \to D$ then $h(gf) = (hg)f$.
(3) To each object A there is a morphism $1_A : A \to A$ such that for any $f : A \to B$ and $g : C \to A$, $f1_A = f$ and $1_A g = g$.

Here is a small, by no means exhaustive list of categories:

- The category $\mathfrak{S}$ of sets and functions
- The category $\mathfrak{G}$ of groups and group homomorphisms
- The category $\mathfrak{Ab}$ of abelian groups and homomorphisms
- The category $\mathfrak{Top}$ of topological spaces and continuous functions
- The category $\mathfrak{V}_k$ of vector spaces over a field k and linear transformations
- The category $\mathfrak{R}$ of rings and ring homomorphisms
- The category $\mathfrak{Mod}_R$ of (left) R-modules and linear maps.

We now need a transformation from one category to another. This is called a *functor*. A *functor* $F : \mathfrak{C} \to \mathfrak{D}$ is a rule associating to each object $A \in \mathfrak{C}$ an object $FA \in \mathfrak{D}$ and to every morphism $f \in \mathfrak{C}(A, B)$ a morphism $Ff \in \mathfrak{D}(FA, FB)$ such that

$$F(fg) = (Ff)(Fg), \qquad F(1_A) = 1_{FA}.$$

Let us look at a few examples:

- The embedding of a subcategory $\mathfrak{C}_0$ into $\mathfrak{C}$ is a functor.
- Underlying every R-module M there is a set. Hence we get the forgetful functor $\mathfrak{Mod}_R \to \mathfrak{S}$.
- $\mathfrak{Mod}_R(A,B) = \mathrm{Hom}_R(A,B)$ can be given the structure of an abelian group. Fix A, then we obtain a functor $\mathrm{Hom}_R(A,-) : \mathfrak{Mod}_R \to \mathfrak{Ab}$ by $\mathrm{Hom}_R(A,-)(B) = \mathrm{Hom}_R(A,B)$.
- Similarly, we have functors $\mathrm{Ext}^n_R(A,-) : \mathfrak{Mod}_R \to \mathfrak{Ab}$.

A quick check shows that $\mathrm{Hom}_R(-,A)$ is not a functor, but we can repair this easily. For every category $\mathfrak{C}$ define the opposite category $\mathfrak{C}^{opp}$, which has the same objects as $\mathfrak{C}$ but the morhisms sets are defined to be $\mathfrak{C}^{opp}(A,B) = \mathfrak{C}(B,A)$. Now we can see, that both $\mathrm{Hom}_R(-,A)$ and $\mathrm{Ext}^n_R(-,A)$ are functors from the opposite category $\mathfrak{Mod}_R^{opp}$ to $\mathfrak{Ab}$. We also say these are contravariant functors from $\mathfrak{Mod}_R$ to $\mathfrak{Ab}$.
Let us finally come to the notion of a natural transformation. Naturality is an important concept we already have spent some time explaining before:
Let F and G be two functors $\mathfrak{C} \to \mathfrak{D}$. Then a *natural transformation* $t : F \to G$ is a rule assigning to each object $A \in \mathfrak{C}$ a morphism $t_A : FA \to GA$ such that for any morphism $f : A \to B$ in $\mathfrak{C}$ the following diagram commutes:

$$\begin{tikzcd} FA \arrow[r, "t_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "t_B"] & GB. \end{tikzcd}$$

We have seen a natural transformation in Lemma 1.7: For every R-module M there is a natural isomorphism: $\phi : \mathrm{Hom}_R(R,M) \to M$ defined by $f \mapsto f(1)$. The map ϕ is a natural transformation from $\mathrm{Hom}_R(R,-)$ to the identity functor.

From now on let R and S denote two rings and we consider the two categories $\mathfrak{Mod}_R$ and $\mathfrak{Mod}_S$. In most of our applications $S = \mathbb{Z}$ and $\mathfrak{Mod}_S = \mathfrak{Ab}$.

Definition 7.1. A cohomological functor from $\mathfrak{Mod}_R$ to $\mathfrak{Mod}_S$ is a family $(U^n)_{n\in\mathbb{Z}}$ of functors $U^n : \mathfrak{Mod}_R \to \mathfrak{Mod}_S$ together with natural connecting maps $\delta : U^n(M') \to U^{n+1}(M'')$ for each short exact sequence $M" \hookrightarrow M \twoheadrightarrow M'$ and each $n \in \mathbb{Z}$ such that the following axiom holds:

AXIOM (Long exact sequence)
For each short exact sequence $0 \longrightarrow M'' \xrightarrow{\iota} M \xrightarrow{\pi} M' \longrightarrow 0$ there is a long exact sequence

$$\cdots \xrightarrow{\delta} U^n(M'') \xrightarrow{\iota_*} U^n(M) \xrightarrow{\pi_*} U^n(M') \xrightarrow{\delta} U^{n+1}(M'') \xrightarrow{\iota_*} \cdots$$

We also require the following optional axiom

AXIOM (Coeffaceability)
U^n is zero for all $n < 0$ and $U^n(I) = 0$ for all injective R-modules and all $n \geq 1$.

It is extremely important to understand the significance of the maps δ to be natural. Let

$$\begin{tikzcd} M'' \arrow[r, tail, "\iota"] \arrow[d, "\phi''"'] & M \arrow[r, two heads, "\pi"] \arrow[d, "\phi"'] & M' \arrow[d, "\phi'"] \\ N'' \arrow[r, tail, "\eta"] & N \arrow[r, two heads, "\rho"] & N' \end{tikzcd}$$

be a commutative diagram of R-modules with exact rows. Then the naturality of δ ensures we get a commutative ladder

$$\begin{tikzcd} \cdots \arrow[r, "\delta"] & U^n(M'') \arrow[r, "\iota_*"] \arrow[d, "\phi''_*"'] & U^n(M) \arrow[r, "\pi_*"] \arrow[d, "\phi_*"'] & U^n(M') \arrow[r, "\delta"] \arrow[d, "\phi'_*"'] & U^{n+1}(M'') \arrow[r, "\iota_*"] \arrow[d, "\phi''_*"'] & \cdots \\ \cdots \arrow[r, "\delta"] & U^n(N'') \arrow[r, "\eta_*"] & U^n(M) \arrow[r, "\rho_*"] & U^n(M') \arrow[r, "\delta"] & U^{n+1}(M'') \arrow[r, "\eta_*"] & \cdots \end{tikzcd}$$

Commutativity of the squares not involving δ follows directly from functoriality of each U^n.

Example 7.2. $\mathrm{Ext}^*(M,-)$ is a coeffaceable cohomological functor from $\mathfrak{Mod}_R$ to $\mathfrak{Ab}$. We have shown the long exact sequence axiom in Theorem 5.5 and coeffaceability follows from Proposition 6.3.

Theorem 7.3. *Let U^* and V^* be two cohomological functors from $\mathfrak{Mod}_R$ to $\mathfrak{Mod}_S$ such that U^* is coeffaceable. Then any natural map $\nu^0 : U^0 \to V^0$ extends uniquely to a natural map $\nu^* : U^* \to V^*$*

Exercise 7.4. Use Theorem 7.3 to show that the definition of $\mathrm{Ext}^*(M,-)$ is independent of the choice of projective resolution of M. Hint: Use the fact that $\mathrm{H}^0(\mathrm{Hom}(\mathbf{P}_*, N)) \cong \mathrm{Hom}_R(M, N)$ for any projective resolution $\mathbf{P}_* \twoheadrightarrow \mathbf{M}$.

Exercise 7.5. Use Theorem 7.3 to show that composition of maps

$$\mathrm{Hom}_R(B,C) \times \mathrm{Hom}_R(A,B) \to \mathrm{Hom}_R(A,C)$$

extends to a biadditive product, the *Yoneda product*

$$\mathrm{Ext}^n_R(B,C) \times \mathrm{Ext}^m(A,B) \to \mathrm{Ext}^{m+n}_R(A,C).$$

8. Chain complexes

A chain complex (C_*, d) of R-modules is a family $(C_n)_{n\in\mathbb{Z}}$ of R-modules together with maps $d : C_n \to C_{n-1}$ such that composition of two consecutive maps is zero, i.e. $dd = 0$. We write $Z_n = ker(C_n \to C_{n-1})$ for the n-cycles and $B_n = im(C_{n+1} \to C_n)$ for the n-cycles and the n-th homology of $\mathbf{C}_*$ is defined to be

$$H_n(\mathbf{C}_*) = Z_n/B_n.$$

Alternatively, we may say that a chain complex (C_*, d) of R-modules is a family $(C_n)_{n\in\mathbb{Z}}$ of R-modules together with maps $d : C_n \to C_{n-1}$ such that $(\hat{C}^n)_{n\in\mathbb{Z}} = (C_{-n})_{n\in\mathbb{Z}}$ is a cochain complex and $H_n(\mathbf{C}_*) = H^{-n}(\hat{\mathbf{C}}_*)$. All theorems we have established for cohomology work for homology without requiring seperate proof. In particular, every short exact sequence of chain complexes

$$\mathbf{C}''_* \hookrightarrow \mathbf{C}_* \twoheadrightarrow \mathbf{C}'_*$$

gives rise to a long exact sequence in homology with natural connecting maps:

$$\cdots \to H_n(C'') \to H_n(C) \to H_n(C') \to H_{n-1}(C'') \to H_{n-1}(C) \to \cdots .$$

CHAPTER 2

Group cohomology

1. The Group Ring

Throughout we denote a group by G. Let $\mathbb{Z}G$ denote the free $\mathbb{Z}$-module with basis the elements of G. In particular, every $x \in \mathbb{Z}G$ can be written in a unique way as

$$x = \sum_{g \in G} n_g g$$

where $n_g \in \mathbb{Z}$and almost all $n_g = 0$. Define a multiplication on $\mathbb{Z}G$ as follows:

$$xy = (\sum_{g\in G} n_g g)(\sum_{h \in G} n_h h) = \sum_{g,h\in G} n_g n_h (gh).$$

this makes $\mathbb{Z}G$ into a ring, the **integral group ring.**

Example 1.1. (1) Let $G = \langle x \rangle$ be infinite cyclic. Then $\mathbb{Z}G$ has $\mathbb{Z}$-basis $\{x^i \mid i \in \mathbb{Z}\}$ and can be identified with the ring $\mathbb{Z}[x, x^{-1}]$ of Laurent polynomials $\sum_{i\in\mathbb{Z}} a_i x^i$, where almost all $a_i = 0$.

(2) Let G be cyclic order n and t be a generator for G. $\{1, t, t^2, ..., t^{n-1}\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}G$ and $t^n - 1 = 0$ hence

$$\mathbb{Z}G \cong \mathbb{Z}[T]/T^n - 1.$$

Definition 1.2. Let M be an abelian group and let G act on M

$$\begin{array}{ccc} G \times M & \to & M \\ (g,m) & \mapsto & gm \end{array}$$

such that for all $m, n \in M$ and $g, h \in G$:

- $1_G m = m$
- $(gh)m = g(hm)$
- $g(m + n) = gm + gn$

we say that M is a G-module.

A G-module can be made in a $\mathbb{Z}G$-module by "linearly extending" the action, i.e. $xm = (\sum_{g\in G} n_g g)m = \sum_{g\in G} n_g(gm)$. Furthermore, G is a subgroup of the multiplicative group $\mathbb{Z}G^*$ and hence there's the follwing universal property:
Let R be a ring and $f : G \to R^*$ be a group homomorphism. Then f can be extended uniquely to a ring homomorphism $\mathbb{Z}G \to R$. Hence

$$Hom_{rings}(\mathbb{Z}G, R) \cong Hom_{groups}(G, R^*)$$

and a G-module is nothing but a $\mathbb{Z}G$-module.

Example 1.3. Every abelian group A is a trivial G-module with the action defined by $ag = a$ for all $a \in A, g \in G$. Hence for $x = \sum_{g\in G} n_g g$ it follows that $xa = \sum_{g\in G} n_g a$.

For every group G there is a ring homomorphism

$$\varepsilon : \mathbb{Z}G \to \mathbb{Z}$$

defined by $\varepsilon(g) = 1$. for all $g \in G$. Hence for $x = \sum_{g\in G} n_g g$, $\varepsilon(x) = \sum_{g\in G} n_g$. The kernel of ε is called the **augmentation ideal** and is denoted by $\mathfrak{g}$ or IG.

LEMMA 1.4. *$\mathfrak{g}$ is a free $\mathbb{Z}$-module with basis*

$$X = \{g - 1 \mid 1 \neq g \in G\}.$$

ε is a G-module homomorphism and $\mathfrak{g}$ is a G-module.

LEMMA 1.5.
(1) *Let S be generating set for G. Then $\mathfrak{g}$ is generated as a G-module by*

$$S - 1 = \{s - 1 \mid s \in S\}.$$

(2) *Let S be a set of elements of G such that $S-1$ generates $\mathfrak{g}$ as a G-module. Then S generates the group G.*

Proof: We do (1) in class and leave (2) as an exercise. □

Now let Ω be a G-set and consider the free abelian group $\mathbb{Z}\Omega$ on Ω. The operation of G on Ω can be extended to a $\mathbb{Z}$-linear operation of G on $\mathbb{Z}\Omega$. Hence $\mathbb{Z}\Omega$ is a G-module, the so called **Permutation module**.

EXAMPLE 1.6.
(1) Let $H \leq G$ be a subgroup and let G/H be the set of left cosets. Then $\mathbb{Z}[G/H]$ is a permutation module.
(2) Let $\Omega = \sqcup_{i\in I}\Omega_i$ (disjoint union). Then $\mathbb{Z}\Omega = \oplus_{i\in I}\mathbb{Z}\Omega_i$.

In particular, every permutation module can be expressed as

$$\mathbb{Z}\Omega = \bigoplus_{\omega\in\Omega^0} \mathbb{Z}[G/G_\omega],$$

where Ω^0 is a system of representatives of the orbits of the G-action and $G_\omega = \{g \in G \mid g\omega = \omega\}$ is the stabiliser (or isotropy group) of ω. We say G acts **freely** on Ω if all stabilisers are trivial.

LEMMA 1.7. *Let Ω be a free G-set and let Ω^0 be a system of representatives for the G-orbits. Then $\mathbb{Z}\Omega$ is a free G-module with basis Ω^0.*

LEMMA 1.8. *Let $H \leq G$ be a subgroup of G. Then $\mathbb{Z}G$ is free as a left H-module.*

Now let us define the cohomology groups:

DEFINITION 1.9. Let G be a group. Then the n-th cohomology group of G with coefficients in the G-module M is defined to be

$$\mathrm{H}^n(G, M) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M).$$

In chapter one we have determined the zeroth cohomology group Ext^0. Hence

$$\mathrm{H}^0(G, M) \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \cong M^G,$$

where M^G denote the G-fixed points of M. We have, so far, computed cohomology via projective resolutions and defined the projective resolution of a module M to be the shortest length of a projective resolution of M. One theme of this course will be cohomological finiteness conditions for groups, so let's make first definition.

Definition 1.10. Let G be a group. The cohomological dimension of G, denoted $\mathrm{cd}G$ is defined to be

$$\mathrm{cd}G = \mathrm{pd}_{\mathbb{Z}G}\mathbb{Z}.$$

The above Lemma 1.8 implies directly:

Proposition 1.11. *Let $H \leq G$ be a subgroup of G. Then*

$$\mathrm{cd}H \leq \mathrm{cd}G.$$

Remark 1.12. One can, of course always define the group ring RG for any ring R. $\mathrm{H}^*_R(G, -)$ and $\mathrm{cd}_R G$ are defined analogously. Something more here, adjoint functors?

We shall now spend some time on finding projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$. Let us begin with two easy examples:

Example 1.13. (1) Let $G = \langle x \rangle$ be an infinite cyclic group. Then

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{*(x-1)} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a projective (free) resolution of $\mathbb{Z}$.

(2) Let G be a cyclic group of order n generated by t. Then, as seen before, $\mathbb{Z}G \cong \mathbb{Z}[t]/(T^n - 1)$. Now $T^n - 1 = (T-1)(T^{n-1} + T^{n-2} + ... + T^o)$ and hence for each $x \in \mathbb{Z}G$ it follows that

$$(t-1)x = 0 \iff x = (t^{n-1} + ... + t + t^0)y = Ny \quad \text{some } y \in \mathbb{Z}G.$$

Hence there is a projective (free) resolution of $\mathbb{Z}$ of infinite length:

$$\cdots \xrightarrow{*(t-1)} \mathbb{Z}G \xrightarrow{*N} \mathbb{Z}G \xrightarrow{*(t-1)} \cdots \xrightarrow{*N} \mathbb{Z}G \xrightarrow{*(t-1)} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

We will see later that G has no projective resolution of finite length.

2. Resolutions via Topology

In this section we shall see that we can construct resolutions once we have constructed models for classifying spaces. We shall introduce very quickly the basic topological notions used later. We shall, however introduce classifying spaces in a more general way than initially used. We will see how to construct classifying spaces for proper actions.

2.1. CW-complexes. In this section we only briefly introduce the concept of a CW-complex. The interested reader can find all detail in most Algebraic Topology textbooks, such as for example Hatcher's book [**10**], appendix.

A CW-complex can be thought of as built by the following proceedure:

(1) Start with a discrete set X^0, whose points are regarded as 0-cells. (This is the 0-skeleton).
(2) Inductively, from the $(n-1)$-skeleton X^{n-1} build the n-skeleton X^n by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. (This means that X^n is the quotient space of the disjoint union $X^{n-1} \sqcup_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identification $x \sim \varphi_\alpha x$ for $x \in D^n_\alpha$. Thus, as a set $X^n = X^{n-1} \sqcup_\alpha e^n_\alpha$, where each e^n_α is an open n-disk.)
(3) Put $X = \bigcup_n X^n$ where X is given the weak topology: A set $A \subset X$ is open if and only if $A \cap X^n$ is open for all n.

Example 2.1. A 1-dimensional CW-comples is just a graph with vertices the 0-cells and edges the 1-cells.

Example 2.2. $X = \mathbb{R}^2$ is a 2-dimensional CW-complex with $\mathbb{Z} \times \mathbb{Z}$ as the 0-cells, the open intervalls as the 1-cells and the interior of the unit squares as 2-cells.

Example 2.3. The sphere S^n has the structure of a CW-complex with one 0-cell and one n-cell.

Example 2.4. The real projective plane, $\mathbb{R}P^2$ can be seen as D^2 with antipodal points of $S^1 = \delta D^2$ identified. Hence $\mathbb{R}P^2 = e^0 \cup e^1 \cup e^n$.

Exercise 2.5. How can we see that $\mathbb{R}P^n$ has a CW-structure, $e^0 \cup e^1 \cup ... \cup e^n$?

Exercise 2.6. How can we see that a closed orientable surface M_g of genus g ($M_1 = T$, the torus) has a CW-structure given by: $e^0 \cup e^1_1 \cup e^1_2 \cup ... \cup e^1_{2g} \cup e^2$, i.e. has one 0-cell, $2g$ 1-cells and one 2-cell? (Identify edges on a regular $4g$-gon.)

2.2. *G*-spaces. In this course, all our groups are discrete groups. One can, however, define classifying spaces for proper actions for arbitrary topological groups. For detail see tomDieck's book on transformation groups [**6**].

Definition 2.7. A G-space is a topological space X with a (continuous) left G-action

$$G \times X \to X, \qquad (g, x) \mapsto gx$$

satisfying

(1) $ex = x$ for all $x \in X$ and $e = e_g$ the identity of G.
(2) (gh)x = g(hx) for all $x \in X$ and all $g, h \in G$.

Example 2.8. (a) Let G be the infinite cyclic group with generator g, i.e. $G = \langle g \rangle$ and $X = \mathbb{R}$. X is a G-space with G acting by translation $g^i x = x + i$.
(b) Let $G = \mathbb{Z} \times \mathbb{Z}$. $X = \mathbb{R}^2$ is a G-space with G acting by translation.
(c) Let $\mathbb{H}$ be the upper half plane model of the hyperbolic plane,

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} \,|\, y > o\}.$$

$Sl_2(\mathbb{Z}) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,|\, a, b, c, d \in \mathbb{Z}, det(A) = 1\}$ acts on $\mathbb{H}$ by Möbius-transformations, i.e $Az = \frac{az+b}{cz+d}$.
(Check this really makes $\mathbb{H}$ into a $Sl_2(\mathbb{Z})$-space)
The kernel of this action consists of scalar multiples in $Sl_2(\mathbb{Z})$ of the identity matrix I. Hence $\mathbb{H}$ is a G space for $G = PSl_2(\mathbb{Z}) = Sl_2(\mathbb{Z})/\{\pm I\}$.

Definition 2.9. The stabilizer $G_x \leq G$ of a point $x \in X$ is the subgroup $\{g \in G \,|\, gx = x\}$.

Let us note that the Cartesian product $X \times Y$ of two G-spaces X and Y is again a G-space via the diagonal action $g(x, y) = (gx, gy)$ for all $x \in X, y \in Y$ and $g \in G$.

Definition 2.10. Let $H \subseteq G$ be a subgroup of G. Write X^H for the subspace of H-fixed points

$$X^H = \{x \in X \,|\, hx = x, \forall h \in H\}$$

and X/H for the space of H-orbits,

$$X/H = \{Hx \mid x \in X\}.$$

Let $N_G(H)$ denote the normalizer of H in G:

$$N_G(H) = \{g \in G \mid gH = hg\}.$$

Then the G-action on X restricts to an $N_G(H)$-action on X^H with H acting trivally. Hence X^H is a $N_G(H)/H$-space.

DEFINITION 2.11. Let $H \subseteq G$ be a subgroup. $N_G(H)/H = W(H)$ is called the Weyl-group of H.

EXERCISE 2.12. Let $H \subseteq G$ be a finite subgroup and denote by $C_G(H)$ the centralizer of H in G, i.e

$$C_G(H) = \{g \in G \mid gh = hg \, \forall h \in H\}.$$

Prove that the index $|N_G(H) : C_G(H)|$ is finite.

EXAMPLE 2.13. The space of left cosets G/H is a G-space via $(g, kH) \mapsto gkH$ for all $g, k \in G$.
(Fact: Every discrete G-space is a disjoint union of such G-spaces.)
Let $K \leq G$ a subgroup. Then $(G/H)^K$ consists of all cosets gH such that $KgH = gH \iff g^{-1}Kg \leq H$.

DEFINITION 2.14. A G-CW-complex consists of a G-space X together with a filtration

$$X^0 \subset X^1 \subset X^2 \subset ... \subset X$$

by G-subcomplexes such that

(1) Each X^n is closed in X.
(2) $\bigcup_{n \in \mathbb{N}} X^n = X$.
(3) X^0 is a discrete subspace of X.
(4) For each $n \geq 1$ there exists a discrete G-space Δ_n together with G-maps $F : S^{n-1} \times \Delta_n \to X^{n-1}$ and $\hat{f} : D^n \times \Delta_n \to X^n$ such that the following diagramme is a push-out:

$$\begin{array}{ccc} S^{n-1} \times \Delta_n & \to & X^{n-1} \\ \downarrow & & \downarrow \\ D^n \times \Delta_n & \to & X^n \end{array}$$

(5) A subspace Y of X is open if and only if $Y \cap X^n$ is open for all $n \geq 0$.

A map $f : X \to Y$ of G-CW-complexes is a G-map if $f(gx) = gf(x)$ for all $g \in G$, $x \in X$. If $G = \{e\}$ the trivial group, then a G-CW-complex is just a CW-complex as in Chapter 1. All our examples in 2.8 are G-CW-complexes.

EXAMPLE 2.15. Let $G = C_2$ be the cyclic group of order 2. Then the sphere, S^2 is a G-CW-complex with G acting by the antipodal map.

DEFINITION 2.16. (1) A G-CW-complex is called finite dimensional if $X^n = X$ for some $n \geq 0$. The least such n is called the dimension of X. (In case $dim(X) < \infty$, Axiom 5 above is redundant.)
(2) A G-CW-complex is said to be of finite type, if there are finitely many G-orbits in each dimension. (Equivalently, as X/G is a CW-complex, X/G only has finitely many cells in each dimension.)

(3) A G-CW-complex is called cocompact if X is finite dimensional and of finite type. (Equivalently, X/G is a finite CW-complex.)

All the examples in this chapter are cocompact. Before we can move on to defining classifying spaces, we need to have a quick look at an important construction, the join construction:

DEFINITION 2.17. For all $n \geq 0$ let

$$\sigma^n = \{(t_0, t_1, ..., t_n) \mid \sum t_i = 1, t_i \geq 0\}$$

denote the standard n-simplex in $\mathbb{R}^{n+1}$. Let $X_1, ..., X_n$ be G-CW-complexes. We define the join of $X_1, ..., X_n$ to be:

$$X_1 * X_2 * ... * X_n = (\sigma^n \times X_1 \times X_2 \times ... \times X_n)/\ ,$$

where $(t_0, ..., t_n, x_1, ..., x_n)\ (t'_0, ..., t'_n, x'_1, ..., x'_n) \iff$ for all i either $(t_i, x_i) = (t'_i, x'_i)$ or $t_i = t'_i = 0$.

Hence the dimension of $X_1 * X_2 * ... * X_n$ is equal to $n + \sum dim(X_i)$. Furthermore, the join of two G-spaces is again a G-space with diagonal G-action. One can also show that the join of two G-CW-complexes is again a G-CW-comples.

EXAMPLE 2.18. (1) $X * \{pt\} = CX$ the cone on X.
(2) $X * S^0 = \Sigma X$ the suspension on X.
(3) The n-fold join $\{pt\} * ... * \{pt\}$ is a $n-1$-simplex

LEMMA 2.19. [**23**] *Let X be a non-empty and Y be a n-connected space. Then $X * Y$ is $n+1$-connected. In particular, the infinite join of non-empty G-CW-complexes is contractible.*

A space X is called 0-connected if it is non-empty and path-connected; it is callen n-connected if X is 0-connected and for each $1 \leq i \leq n$, the homotopy group $\pi_i(X)$ is trivial. For detail on connectedness and higher homotopy groups see [**28**, Chapter 11].

2.3. Classifying spaces for proper actions. Let $\mathfrak{F}$ denote a family of subgroups of a group G. This is a collection of subgroups closed under conjugation and finite intersection. The following are examples of such families:

- $\mathfrak{F} = \mathfrak{All}$, the family of all subgroups of G
- $\mathfrak{F} = \mathfrak{Fin}$, the family of all finite subgroups of G
- $\mathfrak{F} = \mathfrak{VC}$, the family of all virtually cyclic subgroups of G. (A group is virtually cyclic if it has a cyclic subgroup of finite index)
- $\mathfrak{F} = \{e\}$, the family consisting only of the trivial subgroup.

For this course we will only be concerned with $\mathfrak{F} = \mathfrak{F}in$, the class of all finite subgroups, but most constructions work equally well for any family of subgroups.

DEFINITION 2.20. A G-CW-complex X is called a classifying space for proper actions, or a model for $\underline{\mathrm{EG}}$, if for each subgroup $H \leq G$, the following holds:

$$X^H \simeq \begin{cases} * & \text{if} \quad \text{H} \quad \text{finite} \\ \emptyset & \text{otherwise} \end{cases}$$

THEOREM 2.21. *For each group G there exists a model for $\underline{\mathrm{EG}}$.*

Proof To prove existence one could follows either Milnor's [**22**] or Segal's [**29**] construction of EG, the classifying space for free actions. We shall follow Milnor's model here: Let

$$\Delta = \bigsqcup_{H \in \mathfrak{F}} G/H$$

be the discrete G-CW-complex as in example 2.13. Now form the n-fold join

$$\Delta_n = \underbrace{\Delta * ... * \Delta}_{n}$$

and put

$$X = \bigcup_{n \in \mathbb{N}} \Delta_n.$$

Example 2.13 now implies that $\Delta^H = \emptyset \iff H \notin \mathfrak{F}$. Furthermore, since

$$\Delta^H * ... * \Delta^H = (\Delta * ... * \Delta)^H,$$

Lemma 2.19 implies that $X^H \simeq *$ for $H \in \mathfrak{F}$ and $X^H = \emptyset$ otherwise and X is therefore a model for $\underline{\text{EG}}$. $\square$

This construction, however gives us an infinite dimensional model, which is not of finite type. In this course we will try to find "nice" models.

Remark 2.22. Let G be torsion-free and X be a model for $\underline{\text{EG}}$. Then X is contractible and G acts freely ($X^{\{e\}} \simeq *$ and $X^H = \emptyset$ for all $\{e\} \neq H \leq G$). Hence X is a model for EG, the classifying space for free actions, or equivalently the universal cover of a $K(G,1)$, an Eilenberg-Mac Lane space.

Example 2.23. (Examples for torsion-free groups)

(a) $G = \mathbb{Z}$. Then $\mathbb{R}$ is a model for EG by Example 2.8 (a)
(b) $G = \mathbb{Z} \times \mathbb{Z}$ and $\mathbb{R}^2$ is a model for EG by Example 2.8 (b).
(c) Let G be the free group on 2 generators, $G = \langle x, y \rangle$. Then the Cayley-graph is a tree, which is a model for EG .

Example 2.24. (Examples for groups with torsion)

(a) If G is a finite group, then $\{*\}$ is a model for $\underline{\text{EG}}$.
(b) Let $G = D_\infty$ be the infinite dihedral group. Then $\mathbb{R}$ is a model for $\underline{\text{EG}}$, where the generator for the infinite cyclic group acts by translation and the generator of order two acts by reflection.
(c) Let G be a wallpaper group, i.e. an extension of $\mathbb{Z} \times \mathbb{Z}$ with a finite subgroup of O_2, the group of 2×2 orthogonal matrices. Then $\mathbb{R}^2$ is a model for $\underline{\text{EG}}$.
(d) Let $G = PSL_2(\mathbb{Z})$. We've seen in example 2.8 (c) that G acts by Möbius transformations on $\mathbb{H}$ the upper half plane. This is a 2-dimensional model for $\underline{\text{EG}}$.

    Considering the two generators, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ we can see that $G \cong C_2 * C_3$ the free product of a cyclic group of order 2 and a cyclic group of order 3. Hence, the dual tree T is a 1-dimensional model for $\underline{\text{EG}}$.

REMARK 2.25. One can apply the same construction to define classifying spaces for arbitrary families $\mathfrak{F}$ of subgroups: A G-CW complex X is called a classifying space for the family $\mathfrak{F}$ of subgroups if

$$X^H \simeq \begin{cases} * & \text{if} \quad H \in \mathfrak{F} \\ \emptyset & \text{otherwise} \end{cases} .$$

For the interested reader I will include a very brief overview of some of the homotopy theory behind the above construction:

DEFINITION 2.26. A G-space X is called proper of for each pair of points $x, y \in X$ there are open neighbourhoods V_x of x and V_y of y such that the closure of $\{g \in G \mid gV_x \cap V_y \neq \emptyset\}$ is a compact subset of G.

If G is discrete this means that the above set is finite. Hence a G-CW complex X is proper if and only if all stabilizers are finite.

THEOREM 2.27. *(**J.H.C. Whitehead**, see* [**20**], *Chapter I)*
A G-map $f : X \to Y$ between two G-CW-complexes is a G-homotopy equivalence if for all $H < G$ and all $x_0 \in X^H$ the induced map

$$\pi_*(X^H, x_0) \to \pi_*(Y^H, f(x_0))$$

is bijective.

Now, the following theorem explains why we call $\underline{\mathrm{EG}}$ the classifying space for proper actions:

THEOREM 2.28. *(See* [**24**, Theorem 2.4]*) Let X be a proper G-CW-complex. Then, up to G-homotopy, there is a unique G-map $X \to \underline{\mathrm{EG}}$.*

EXERCISE 2.29. Show that any two models for $\underline{\mathrm{EG}}$ are G-homotopy equivalent.

2.4. Projective resolutions. In this section we will construct projective resolutions by considering classifying spaces. We will look at EG, the classifying space for free actions. But let us begin with the augemented cellular chain complex.

Let X be a G-CW-complex. It's augmentes cellular chain complex is a chain complex of G-modules

$$\cdots \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_o(X) \twoheadrightarrow \mathbb{Z},$$

in which each $C_i(X)$ is the free abelian group on the orbits of i-cells. Hence

$$C_i(X) \cong \bigoplus_{orbit-reps\,\sigma} \mathbb{Z}[G/G_\sigma].$$

Recall that a G-CW-complex X is a model for EG if X is contractible and G acts freely on X.

PROPOSITION 2.30. *Let X be a model for* EG*. Then the augmented cellular chain complex is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

REMARK 2.31. Let $\mathbf{C}$ be a chain complex. We say $\mathbf{C}$ is acyclic of and only if $H_*(\mathbf{C}) = \mathbf{0}$. A G-CW-complex X is called acyclic if it has the homology of a point. Hence an acyclic G-CW-complex with a free G-action would also give us a free resolution of $\mathbb{Z}$.

DEFINITION 2.32. We say a group G has finite geometric dimension (gd$G < \infty$) if it admits a finite dimensional model for EG. The smallest such dimension is called the geomtric dimension of G.

We can now state the following corollary to Proposition 2.30:

COROLLARY 2.33. *For each group G:*

$$\mathrm{cd}G \leq gdG.$$

The converse is almost true and involves rather more than we can cover here. It comes in two parts, which were proved using very different methods.

THEOREM 2.34. [**30, 31**][**Stallings-Swan**] *Let G be a group. Then*

$$\mathrm{cd}G = 1 \iff \mathrm{gd}G = 1 \iff G \textit{is free.}$$

THEOREM 2.35. [**8**][**Eilenberg-Ganea**] *Let G be a group such that* cd$G \geq 3$. *Then*

$$\mathrm{cd}G = \mathrm{gd}3.$$

This leaves the case when G is a group with cd$G = 2$. It is still unknown whether there is a group G, which does not admit a 2-dimensional model for EG, although there are some candidates for examples [**1**] (Bestvina).

EXAMPLE 2.36. Let G be a free group on S. We now construct a model X for EG . We take a fixed vertex x_0 and we then have a unique orbit of vertices . Hence

$$C_0(X) = \mathbb{Z}V = \mathbb{Z}G.$$

As basis for $C_1(X) = \mathbb{Z}E$ we take for each $s \in S$ an oriented 1-cell e_s. We assume the initial vertex is x_0. Otherwise we translate by a suitable g. Hence the initial vertex of e_s is x_0 and the terminal vertex is sx_0. and we get a map

$$\begin{array}{rccl} \delta : & \mathbb{Z}E & \to & \mathbb{Z}V \\ & e_s & \mapsto & sx_0 - x_0 = (s-1)x_0 \end{array}$$

$\mathbb{Z}E = C_i(X) = \mathbb{Z}G^{(S)}$ and we have a free resolution of length 1:

$$0 \longrightarrow \mathbb{Z}G^{(S)} \xrightarrow{\delta} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

(a) Let $G = \langle t \rangle$ be infinite cyclic. Then $X = \mathbb{R}$ and we recover the resolution in section 1:

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

(b) We can also view X as a G-CW-comples with one orbit of 0-cells and s orbits of one cells $\{g, gs\}$ with the action induced by left translation of G on itself. E.g for $G = \langle s, t \rangle$ we get the following tree:

EXAMPLE 2.37. Let G be a finite cyclic group, i.e. $G = \langle t \,|\, t^n - 1 \rangle$. The circle S^1 is a G-CW-complex with n vertices and n 1-cells. There is one orbit of vertices $\{v, tv, ..., t^{n-1}v\}$ and one orbit of 1-cells $\{e, te, ..., t^{n-1}e\}$ and $(t-1)(e + te + ... + t^{n-1}e) = 0$. Consider

$$\mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0\ .$$

Hence $H_1(S^1)$ is generated by 1 element $e + te + .. + t^{n-1}e = Ne$ and we get an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\eta} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $\eta(1) = N$. We now splice these sequences together to obtain a free resolution

$$\cdots \longrightarrow \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

of infinite length.

PROPOSITION 2.38. *Let G be a finite cyclic group. Then*

$$\mathrm{H}^{2k}(G, \mathbb{Z}) \neq 0)$$

for all $k > 0$. In particular

$$\mathrm{cd}G = \infty.$$

COROLLARY 2.39. *Let G be a group of finite cohomological dimension. Then G is torsion-free.*

CHAPTER 3

Cohomological dimension

1. Induction and Coinduction

In this section we are goint to definie the tensor product of R-modules and then introduce Induction and Coinduction functors and shall return to group cohomology towards the end of this section.
Let A be a right R-module, B a left R-module and let T be a $\mathbb{Z}$-module. A *R-pairing* from A and B to $\mathbb{Z}$ is a map

$$f : A \times B \to T$$

satisfying the following:

$$\begin{array}{lcl} f(a_1 + a_2, b) & = & f(a_1, b) + f(a_2, b) \\ f(a, b_1 + b_2) & = & f(a, b_1) + f(a, b_2) \\ f(ar, b) & = & f(a, rb) \end{array}$$

for all $a, a_1, a_2 \in A, b, b_1, b_2 \in B, r \in R$.

Definition 1.1. A pair (T, f) consisting of a $\mathbb{Z}$-module T and an R-pairing f from A and B to T is called **Tensor product** from A and B to T if for every R-pairing $A \times B \xrightarrow{g} S$ there us a unique $\mathbb{Z}$-module homomorphism $\varphi : T \to S$ such that $\varphi \circ f = g$.

Remark 1.2. Tensor products are unique up to isomorphism.

Proposition 1.3. **Existence of Tensor product** *Let $\bar{A}$ be a set bijective to A and $\bar{B}$ a set bijective to B. Let E denote the free $\mathbb{Z}$-module over $\bar{A} \times \bar{B}$ and by N the $\mathbb{Z}$-submodule generated by the elements*

- $(\overline{a_1 + a_2}, \bar{b}) - (\overline{a_1}, \bar{b}) - (\overline{a_2}, \bar{b})$,
- $(\overline{a}, \overline{b_1 + b_2}) - (\overline{a}, \overline{b_1}) - (\overline{a}, \overline{b_2})$,
- $(\overline{ar}, \bar{b}) - (\overline{a}, \overline{rb})$.

Then

$$\begin{array}{rccl} f : & A \times B & \to & E/N \\ & (a, b) & \mapsto & (\bar{a}, \bar{b}) + N \end{array}$$

is an R-pairing from A and B to E/N and $(E/N, f)$ is a tensor product.

We write $(a, b) \mapsto a \otimes_R b$ and $T = A \otimes_R B$.
Now let $f : A \to A'$ and $g : B \to B'$ be R-module homomorphisms. Then

$$\begin{array}{ccc} A \times B & \to & A' \otimes_R B' \\ (a, b) & \mapsto & f(a) \otimes g(b) \end{array}$$

is an R-pairing and hence by 1.1 there is a unique $\mathbb{Z}$-module homomorphism

$$\begin{array}{rccc} \varphi : & A \otimes_R B & \to & A' \otimes_R B' \\ & a \times b & \mapsto & f(a) \otimes f(b). \end{array}$$

We write
$$\varphi = f \otimes g.$$

LEMMA 1.4. (1) $id_a \otimes id_B = id_{A\otimes B}$

(2) *Let* $A' \xrightarrow{f'} A \xrightarrow{f''} A''$ *and* $B \xrightarrow{g'} B \xrightarrow{g''} B''$ *be* R*-module homomorphisms. Then*
$$(f'' \otimes g'')(f' \otimes g') = (f''f' \otimes g''g')$$

(3) *Let* $f, f' \in \mathrm{Hom}_R(A, A')$ *and* $g, g' \in \mathrm{Hom}_R(B, B')$*. Then*
$$\begin{aligned}(f + f') \otimes g &= f \otimes g + f' \otimes g\\ f \otimes (g + g') &= f \otimes g + f \otimes g'\end{aligned}$$

This can be proved using the universal property 1.1.

LEMMA 1.5. *Let* R *and* S *be rings,* B *a left* R*-module and* A *a right* R*-module and a left* S*-module. Then* $A \otimes_R B$ *is a left* S*-module.*

Analogously, if A is a right R-module and B is a left R-module and a right S-module, it follows that $A \otimes_R B$ is a right S-module. If R is commutative, then always $A \otimes_R B$ is a left and right R-module.

PROPOSITION 1.6. *Let* R *and* S *be rings,* A *a right* S*-module,* B *a left* R*-module and a right* S*-module and* C *a right* R*-module. Then there is a natural isomorphism*
$$\mathrm{Hom}_R(A \otimes_S B, C) \cong \mathrm{Hom}_S(A, \mathrm{Hom}_R(B, C)),$$
the so called adjoint isomorphism.

$\mathrm{Hom}_R(B, C)$ is a right S-module via $(\varphi s)(b) = \varphi(sb)$. Contravariance of $\mathrm{Hom}_R(-, C)$ leads to this 'switch from right to left'.

LEMMA 1.7. *There are natural isomorphisms*
$$A \otimes_R R \cong A \quad \textit{and} \quad R \otimes_R B \cong B.$$

LEMMA 1.8. *Let* $(A_i)_{i\in I}$ *and* $(B_j)_{j\in J}$ *be families of right and left* R*-modules respectively. Then*
$$(\bigoplus_{i\in I} A_i) \otimes_R (\bigoplus_{j\in J} B_j) \cong \bigoplus_{i\in I, j\in J} (A_i \otimes_R B_j).$$

But note, that generally
$$(\prod_{i\in I} A_i) \otimes (\prod_{j\in J} B_j) \to \prod_{i\in I, j\in J} (A_i \otimes B_j)$$
is neither injective nor projective.

PROPOSITION 1.9. *Let* $f : A \to A'$ *and* $g : B \to B'$ *be surjective homomorphisms of right and left* R*-modules respectively. Then* $f \otimes g : A \otimes_R B \to A' \otimes B'$ *is a surjective* $\mathbb{Z}$*-module homomorphism whose kernel is generated as a* $\mathbb{Z}$*-module by all elements* $k \otimes b$ *and* $a \otimes l$ *with* $a \in A, b \in B, k \in ker(f)$ *and* $l \in ker(g)$*. In particular there's an exact sequence*
$$(ker(f) \otimes B) \oplus (A \otimes ker(g)) \to A \otimes B \twoheadrightarrow A' \otimes B'.$$

COROLLARY 1.10. *Let* $A' \hookrightarrow A \twoheadrightarrow A''$ *and* $B' \hookrightarrow B \twoheadrightarrow B''$ *short exact sequences of right and left* R*-modules respectively. Then the following sequences are exact:*

(1) $A' \otimes B \to A \otimes B \twoheadrightarrow A'' \otimes B$;
(2) $A \otimes B' \to A \otimes B \twoheadrightarrow A \otimes B''$.

Hence $- \otimes_R B$ *and* $A \otimes_R -$ *are right exact functors from* $\mathfrak{Mod}_R \to \mathcal{A}b$.

DEFINITION 1.11. A R-module B is called **flat** if for every short exact sequence $A' \hookrightarrow A \twoheadrightarrow A''$ of right R-modules there is an exact sequence

$$A' \otimes B \hookrightarrow A \otimes B \twoheadrightarrow A'' \otimes B$$

of $\mathbb{Z}$-modules.

PROPOSITION 1.12. *Projective modules are flat.*

REMARK 1.13. Tensor products are associative and if R is commutative then $A \otimes_R B \cong B \otimes_R A$.

REMARK 1.14. Let $\alpha : S \to R$ be a ring homomorphism. Then every R-module M can be viewed as an S-module via $sm = \alpha(s)m$ for all $s \in S, m \in M$. This is called **Restriction of scalars**.

REMARK 1.15. **Extension of Scalars** Let $\alpha : S \to R$ be a ring homomorphism. As above, R can be viewed as a left S-module vial $sr = \alpha(s)r$ for all $s \in S, r \in R$. Now let M be a right R-module and form a $\mathbb{Z}$-module

$$M \otimes_S R.$$

The right action of R on itself commuted with the left action of S. Hence $M \otimes_S R$ can be viewed as a right R-module via

$$(m \otimes r)r' = m \otimes rr'.$$

We now apply the adjoint isomorphism 1.6 to obtain a natural isomorphism

$$\mathrm{Hom}_R(M \otimes_S R, N) \cong \mathrm{Hom}_S(M, N).$$

We say extension of scalars is left adjoint to restriction of scalars.

REMARK 1.16. **Coextension of scalars** This construction is dual to that in 1.15. Let M be a right S-module. Then

$$\mathrm{Hom}_S(R, M)$$

is a right R-module via $f^{r'}(r) = f(r'r)$. Now it follows from 1.6 that for all R-modules N and S-modules M there is a natural isomorphism

$$\mathrm{Hom}_R(N, \mathrm{Hom}_S(R, M)) \cong \mathrm{Hom}_S(M, N).$$

We say Coextension of scalars is right adjoint to restriction of scalars.

EXAMPLE 1.17. Let $S = \mathbb{Z}G$ for a group G and $R = \mathbb{Z}$. Consider the augmentation map $\epsilon \mathbb{Z}G \twoheadrightarrow \mathbb{Z}$, which is a ring homomorphism. extension of scalars sends a G-module M to

$$M \otimes_{\mathbb{Z}G} \mathbb{Z} \cong M_G,$$

where $M_G = M/L$, where L is the submodule generated by all $mg - m$. Also not, that

$$M_G \cong M/\mathfrak{g}.$$

On the other hand, extension of scalars gives $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) = M^G = \mathrm{H}^o(G, M)$.

We will be insterested under which circumstances these constructions preserve exactness, send projectives to projectives or injectives to injectives. Note, that so far, it is only clear that restriction preserves exactness.

Lemma 1.18. (1) *Extension of scalars sends projective S-modules to projective R-modules.*
(2) *Coextension of scalars sends injective S-modules to injective R-modules.*
(3) *Let R be flat as an S-module. Then under restriction, injective R-modules become injective S-modules.*
(4) *Let R be projective as an S-module. Then under restriction projective mR-modules become projective S-modules.*

Lemma 1.19. *Let G be a group. Then every right G-module can be viewed as a left G-module and vice versa. The operation is given by $gm = mg^{-1}$ for all $g \in G, m \in M$.*

From now on let's consider group rings again. Let $H \leq G$ be a subgroup. Then the inclusion induces a ring-homomorphism

$$\mathbb{Z}H \hookrightarrow \mathbb{Z}G.$$

Extension of scalars becomes **Induction from H to G.** Let M be an H-module. Then.

$$Ind_H^G M = M \otimes_{\mathbb{Z}H} \mathbb{Z}G = M\uparrow_H^G$$

Coextension of scalars becomes **Coinduction from H to G.** Let M be an H-module. Then:

$$Coind_H^G M = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M).$$

Let N be a G-module. Then restriction of scalars is usually denoted by

$$Res_H^G N = N\downarrow_H^G .$$

Proposition 1.20. *The G-module $M\uparrow_H^G$ contains M as a H-submodule. Furthermore,*

$$M\uparrow_H^G \cong \bigoplus_{g\in E} Mg$$

where E is a system of representatives for the right cosets Hg.

Note that $\mathbb{Z}\uparrow_H^G \cong \mathbb{Z}[H\backslash G]$ is a permutation module.

Proposition 1.21. **Frobenius reciprocity** *Let $H \leq G$ be a subgroup of the group G. Let M be an H-module and N be a G-module. Then there is an isomorphism of G-modules*

$$N \otimes M\uparrow_H^G \cong (N\downarrow_H^G \otimes M)\uparrow_H^G .$$

This implies that for every H-module N

$$N \otimes \mathbb{Z}[H\backslash G] \cong N \otimes_{\mathbb{Z}H} \mathbb{Z}G,$$

where on the left we have a diagonal G-action, wheras on the right hand side the G-action only comes from the action on $\mathbb{Z}G$. In particular, if M is a G-module with underlying abelian group M_0 then

$$M \otimes \mathbb{Z}G \cong M_0 \otimes \mathbb{Z}G.$$

In particular, if M_0 is a free abelian group, $M \otimes \mathbb{Z}G$ is a free G-module.

PROPOSITION 1.22. **Mackey's formula** *Let $H \leq G$ and $K \leq G$ and let E denote a system of representatives for the double cosets KgH. For each K-module M there is a K-module isomorphism:*

$$(M\uparrow_H^G)\downarrow_K^G \cong \bigoplus_{g\in E}(Mg\downarrow_{K\cap H^g}^{H^g})\uparrow_{K\cap H^g}^K .$$

In particular, if N is a normal subgroup of G then

$$(M\uparrow_H^G)\downarrow_H^G \cong \bigoplus_{g\in H\backslash G} Mg.$$

We can identify $Mg\downarrow_{K\cap H^g}^{H^g}$ with $M\downarrow_{K\cap H^g}^{H}$ whereby the second restriction is with respect to the map: $K\cap g^{-1}Hg \to H$ mapping $k \mapsto gkg^{-1}$.

PROPOSITION 1.23. *Let $|G:H| < \infty$. Then*

$$Ind_H^G M \cong Coind_H^G M$$

for every H-module M.

EXERCISE 1.24. (1) Show that induction is invariant under conjugation,i.e. show that for every H-module M and $g \in G$

$$M\uparrow_H^G \cong Mg\uparrow_{H^g}^G .$$

(2) Let $|G:H| = \infty$. Show that for any H-module M:

$$(M\uparrow_H^G)^G = 0.$$

THEOREM 1.25. **Eckmann-Shapiro Lemma** *Let $H \leq G$ and let M be an H-module. Then*

$$\mathrm{H}^*(H,M) \cong \mathrm{H}^*(G, Coind_H^G M).$$

REMARK 1.26. Let $|G:H| < \infty$. Then

(1) $\mathrm{H}^*(H,\mathbb{Z}) \cong \mathrm{H}^*(G,\mathbb{Z}[H\backslash G])$ and
(2) $\mathrm{H}^*(H,\mathbb{Z}H) \cong \mathrm{H}^*(G,\mathbb{Z}G)$.

Finally we will make a remark on the exactness of induction:

PROPOSITION 1.27. *Let $A \hookrightarrow B \twoheadrightarrow C$ be a short exact sequence of $\mathbb{Z}H$-modules. Then*

$$A\uparrow_H^G \hookrightarrow B\uparrow_H^G \twoheadrightarrow C\uparrow_H^G$$

is an exact sequence of $\mathbb{Z}G$-modules.

EXERCISE 1.28. Let k be a field and let G be a finite group. Prove that a kG-module is projective if and only if it is injective. (Hint: every k-module is free).

2. Cohomological dimension

Recall the definition of cohomological dimension from Chapter 2.1:

$$\begin{aligned}\mathrm{cd}G &= \mathrm{pd}_{\mathbb{Z}G}\mathbb{Z}\\ &= inf\{n \,|\, \mathbb{Z} \text{ has a projective resolution of length } n\}\\ &= inf\{n \,|\, \mathrm{H}^i(G,-) = 0 \ \forall I > n\}\\ &= sup\{n \,|\, \exists M \ s.t. \ \mathrm{H}^n(G,M) \neq 0.\}\end{aligned}$$

We have further seen that if $\mathrm{cd}G = n$ there exists a free module F such that $\mathrm{H}^n(G,F) \neq 0$.
Let us recall a few more facts:

(1) Let G be a finite group. Then $\mathrm{cd}G = \infty$. In particular groups of finite cohomological dimension are torsion-free.
(2) Let $H \leq G$. Then $\mathrm{cd}H \leq \mathrm{cd}G$.
(3) $\mathrm{cd}G = 0 \iff G = \{e\}$.
(4) Let G be a free group (in particular if G is infinite cyclic). Then $\mathrm{cd}G = 1$. The converse is also true and due to Stallings and Swan.

THEOREM 2.1. *Let G be a group of finite cohomological dimension and let $H \leq G$ be a subgroup of finite index. Then*

$$\mathrm{cd}G = \mathrm{cd}H.$$

REMARK 2.2. One can make an even stronger statement not having to assume that G has finite cohomological dimension. Let $|G : H| < \infty$ Then $\mathrm{cd}G = \mathrm{cd}H$. This is known as **Serre's Lemma** and the proof relies on the above theorem and on building a model for EG from a product of models for EH.

THEOREM 2.3. *Let $H \hookrightarrow G \twoheadrightarrow Q$ be a short exact sequence of groups. Then*

$$\mathrm{cd}G \leq \mathrm{cd}H + \mathrm{cd}Q.$$

This follows directly from the Hochschild-Serre spectral sequence. We shall, however, give an elementary proof.

EXERCISE 2.4. (1) Let G be a free abelian group of finite rank n. Then $\mathrm{cd}G = n$.
(2) Supoose that G is a group of cohomological dimension n. Show that $\mathbb{Z}$ has a free resolution of length n.

Actually let's prove the result for abelian groups. We need some preparation, though:

LEMMA 2.5. *Let G and G' be two groups. Then there is a $G \times G'$-module isomorphism*

$$\mathbb{Z}G \otimes \mathbb{Z}G' \cong \mathbb{Z}[G \times G'].$$

DEFINITION 2.6. Let (A_*, d) and (B_*, d') be two chain complexes. Then there we define a complex $(A \otimes B, \delta)$ as follows

$$(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j.$$

and the chain map $\delta :$ is defined via:

$$\begin{array}{ccl} A_i \otimes B_j & \to & (A \otimes B)_{n-1} \\ (a_i, b_j) & \mapsto & da_i \otimes b_j + (-1)^i a_i \otimes d' b_j. \end{array}$$

where $da_0 = d'b_0 = 0$.

EXERCISE 2.7. Show that $(A \otimes B)_*, \delta)$ is a chain complex.

LEMMA 2.8. *Let $P_* \twoheadrightarrow \mathbb{Z}$ and $P'_* \twoheadrightarrow \mathbb{Z}$ be two exact sequences of $\mathbb{Z}G$- and $\mathbb{Z}G'$-modules respectively. Then $(P \otimes P')_* \twoheadrightarrow \mathbb{Z}$ is an exact sequence of $\mathbb{Z}[G \times G']$-modules. If these are, in addition, projective resolutions, then $(P \otimes P')_* \twoheadrightarrow \mathbb{Z}$ is a projective resolution.*

THEOREM 2.9. *Let G be a free abelian group of rank n. Then*

$$\mathrm{cd}G = n,$$

for all $i < n$

$$\mathrm{H}^i(G, \mathbb{Z}G) = 0$$

and

$$\mathrm{H}^n(G, \mathbb{Z}G) \neq 0.$$

CHAPTER 4

Cohomological finiteness conditions

We already did see one cohomological finiteness condition, the cohomological dimension of a group. The main purpose of this chapter is a discussion of the notion of groups of type FP_n, which can be viewed as a generalization of finite generation (at least as long as $n \geq 2$).

1. Modules of type FP_n

DEFINITION 1.1. (1) Let M be an R-module. We say M is of type FP_n if there is a projective resolution $P_* \twoheadrightarrow M$ with P_i finitely generated for all $i \leq n$.
(2) M is of type FP_∞ if there is a projective resolution $P_* \twoheadrightarrow M$ with P_i finitely generated for all $n \geq 0$.
(3) M is of type FP if M is of type FP_∞ and $\mathrm{pd}_R M < \infty$.

REMARK 1.2. (1) M is of type FP_0 if and only if M is finitely generated.
(2) M is of type FP_1 if and only if M is finitely presented.
(3) Let M be of type FP_n . Then there is a free resolution $F_* \twoheadrightarrow M$ with each F_i finitely generated for all $i \leq n$.

We say a module is of type FL if there is a finite length free resolution $F_* \twoheadrightarrow M$ where all F_i are finitely generated. It is obvious that modules of type FL are of type FP but the converse is not necessarily true.

To look at a few more properties of modules of type FP_n we will need yet another construction, a direct limit of modules: Let I be a quasiordered set. A direct system in $\mathfrak{Mod}_R$ is a functor

$$F : I \to \mathfrak{Mod}_R.$$

Let's be a bit more detailed. For all $i \in I$ there exist R-modules M_i and for all $i, j \in I$ such that $i \leq j$ there exist morphisms $\varphi^i_j : M_i \to M_j$ such that

(1) $\varphi^i_i = id_{M_i}$ for all $i \in I$.
(2) For all $i \leq j \leq k$: $\varphi^i_k = \varphi^i_k \varphi^i_j$.

DEFINITION 1.3. Let $M = \{M_i \,|\, \varphi^i_j\}$ be a direct system in $\mathfrak{Mod}_R$. The direct limit of this system, denoted $\varinjlim M_i$ is an R-module and a family of R-module homomorphisms

$$\alpha_i : M_i \to \varinjlim M_i$$

where $\alpha_i = \alpha_j \varphi^i_j$ for all $i \leq j \in I$ such that for all $X \in \mathfrak{Mod}_R$ and maps $f_i : M \to X$ such that $f_i = f_j \varphi^i_j$ there exists a unique R-module homomorphism $\beta : \varinjlim M_i \to X$ such that $\beta\alpha_i = f_i$ for all $i \in I$.

REMARK 1.4. $\varinjlim M_i$ is unique up to isomorphism.

Proposition 1.5. *The direct limit of a direct system of R-modules exists.*

Example 1.6. (1) The direct limit of the constant system M is M.
(2) Let I have the trivial quasi-order. Then $\varinjlim M_i = \oplus M_i$.
(3) Let $I = \{1, 2, 3\}$ with the quasi-order $1 < 2$ and $1 < 3$. Then $\varinjlim M_i$ is the push-out.

Proposition 1.7. *$\varinjlim M_i$ is exact: Let $M_i' \hookrightarrow M_i \twoheadrightarrow M_i''$ be short exact sequences for all $i \in I$. Then*

$$\varinjlim M_i' \hookrightarrow \varinjlim M_i \twoheadrightarrow \varinjlim M_i''$$

is exact.

Exercise 1.8. (1) Assume that $I = \mathbb{Z}_{\geq 0}$ and that there is an ascending chain of R-modules $M_1 \subset M_2 \subset \ldots$ Prove that

$$\varinjlim M_i = \bigcup_{i \in I} M_i.$$

(2) Prove that every module is the direct limit of its finitely generated submodules.

Example 1.9. Let $I = \mathbb{N}$ with the usual ordering and for all $i \in I$ let $M_i = \mathbb{Z}$ and φ_{i+1}^i be multiplication by p. Then

$$\varinjlim M_i = \mathbb{Z}[\frac{1}{p}].$$

From now on let F be a functor usually $\mathfrak{Mod}_R \to \mathcal{A}b$. By applying the universal property for $\varinjlim F(M_i)$ we see that there is a unique map

$$\varinjlim F(M_i) \to F(\varinjlim M_i).$$

Definition 1.10. We say F is continuous (or we say F commutes with $\varinjlim$) if for all direct systems M_* the canonical map

$$\varinjlim F(M_i) \to F(\varinjlim M_i)$$

is an isomorphism.

Proposition 1.11. *Let M be an R-module. Then $\mathrm{Hom}_R(M, -)$ is continuous if and only if M is finitely presented.*

Proposition 1.12. *Let A be an R-module of type FP_n with $0 \leq n \leq \infty$. Then for every direct system M_*, the natural homomorphism*

$$\varinjlim \mathrm{Ext}_R^k(A, M_i) \to \mathrm{Ext}_R^k(A, \varinjlim M_i)$$

is an isomorphism for all $k < n$ and a monomorphism for $k = n$.

Theorem 1.13. *Let A be an R-module. Then the following are equivalent:*
1. *A is of type FP_n;*
2. *The natural homomorphism $\varinjlim \mathrm{Ext}_R^k(A, M_i) \to \mathrm{Ext}_R^k(A, \varinjlim M_i)$ is an isomorphism for all $k < n$ and a monomorphism for $k = n$;*
3. *For all direct systems M_* with $\varinjlim M_i = 0$ one has $\varinjlim \mathrm{Ext}_R^k(A, M_i) = 0$ for all $k \leq n$.*

COROLLARY 1.14. *An R-module M is of type* FP_∞ *if and only if* $\mathrm{Ext}^k_R(M,-)$ *is continuous for all $k \geq 0$.*

PROPOSITION 1.15. *Let $A' \hookrightarrow A \twoheadrightarrow A''$ be a short exact sequence of R-modules. Then*

(1) *If A' is of type* FP_{n-1} *and A is of type* FP_n*, then A'' is of type* FP_n *.*
(2) *If A is of type* FP_{n-1} *and A'' is of type* FP_n*, then A' is of type* FP_{n-1} *.*
(3) *If A' and A'' are of type* FP_n *then so is A.*

Proof. Exercise.

2. Groups of type FP_n .

DEFINITION 2.1. A group G is said to be of type FP_n if $\mathbb{Z}$ is a $\mathbb{Z}G$-module of type FP_n .

REMARK 2.2. Every group is of type FP_0, since the augmentation map $\epsilon : \mathbb{Z}G \twoheadrightarrow \mathbb{Z}$ gives the beginning of a projective resolution and $\mathbb{Z}G$ is a finitely generated $\mathbb{Z}G$-module.

PROPOSITION 2.3. *A group G is of type* FP_1 *if and only if G is finitely generated.*

The description of groups of type FP_2 is already a lot more complicated. A group is called almost finitely presented if there is an exact sequence of groups $K \hookrightarrow F \twoheadrightarrow G$ where F is finitely generated free and $K/[K,K]$ is finitely generated as a G-module. Finitely presented groups are almost finitely presented but the converse is not true in general, see the examples by Bestvina and Brady [**2**]. Bieri [**3**] has shown that the property FP_2 is equivalent to the group being almost finitely presented.

Now let's have a look at finite extensions. We cannot make any more general statements as even finite generation is in general not a subgroup-closed property.

PROPOSITION 2.4. *Let $G' \leq G$ be a subgroup of finite index. Then G is of type* FP_n *if and only if G' is of type* FP_n *.*

Recall:

DEFINITION 2.5. (1) A group G is of type FP iff G is of type FP_∞ and $\mathrm{cd}G < \infty$.
(2) A group is of type FL if G has a finite length finitely generated free resolution.

Obviously does FL imply FP but the converse is not known. Let P be a projective module in the top dimension of a projective resolution of $\mathbb{Z}$. Suppose F is a finitely generated free module such that $P \oplus F$ is free. Then on can construct a finitely generated free resolution

$$F \hookrightarrow P \oplus F \to F_{n-1} \to ... \to F_0 \twoheadrightarrow \mathbb{Z}$$

. We say such a P is **stably free.**

PROPOSITION 2.6. *Let G be a group of type* FP *Suppose that*

$$0 \to P \to F_{n-1} \to ... \twoheadrightarrow \mathbb{Z}$$

is a finitely generated resolution with F_i finitely generated for all $i \leq n-1$. Then G is of type FL *if and only of P is stable free.*

Hence the question whether FL implies FP reduces to the question whether there are projectives that are not stably free. Over general rings the answer can be Yes. There are even examples over group rings $\mathbb{Z}G$ where $G = \mathbb{Z}_{23}$ due to Milnor [**25**, Chapter 3]. These groups, however have infinite cohomological dimension.

Let is conclude this chapter with some topological remarks. We have already defined finite type and finitelt generated G-CW-complexes. A G-CW-complex X is finitely dominated if there exists a finite complex K such that Y is a homotopy retract, i.e. there are maps $i : X \to K$ and $r : K \to X$ such that $ri \simeq id_X$.

PROPOSITION 2.7. (1) *Let G admit a finite type model for* EG*. Then G is of type* FP_∞ .

(2) *Let G admit a cocompact model for* EG*. Then G is of type* FL .
(3) *Let G admit a finitely dominated model for* EG*. Then G is of type* FP .

The converse to the above is also true if we also assume that G is finitely presented. This is due to Eilenberg-Ganea and Wall.

CHAPTER 5

Groups acting on trees

In this chapter we will look at two very important group constructions, free products with amalgamation and HNN-extensions. Both groups act on certain trees and we shall begin by looking at an alternative way of defining group cohomology, namely via the cochain complex consisting of the abelian groups

$$C^n(G,M) = \{f : \underbrace{G \times \cdots \times G}_{n-times} \to M \mid \text{f a function}\}.$$

To do this we have to consider the standard resolution and the Bar resolution.

1. The Bar-resolution

The standard resolution
We define a free resolution

$$F_* \twoheadrightarrow \mathbb{Z}$$

as follows: Let

$$F_i = \mathbb{Z}G^{n+1}$$

with the G-action defined diagonally:

$$(g_0, g_1, ..., g_n)g = (g_0 g, g_1 g, ..., g_n g).$$

The chain maps $\delta : F_n \to F_{n-1}$ are defined as by

$$\begin{array}{rccc} d_i : & F_n & \to & F_{n-1} \\ & (g_0, ..., g_i, ...g_n) & \mapsto & (g_0, ..., \hat{g}_i, ..., g_n) \end{array}$$

($\hat{g}_i$ denotes omitting this term) and then

$$\delta = \sum_{i=0}^{n} (-1)^i d_i.$$

Show that $\delta\delta = 0$. To see that $F_* \twoheadrightarrow \mathbb{Z}$ is exact , we note that there is a contracting homotopy (not a G-map)

$$\begin{array}{rccc} h : & F_n & \to & F_{n-1} \\ & (g_o, ..., g_n) & \mapsto & (1, g_0, ..., g_n). \end{array}$$

This resolution is called the standard resolution.

The Bar resolution Now define free G-modules

$$Q_n = \mathbb{Z}(G^n \times G)$$

where G acts on $G^n \times G$ as follows:

$$(g_0, ..., g_{n-1}, g_n)g = (g_o, ..., g_{n-1}, g_n g).$$

For all $n \geq 0$ there is G-module isomorphism

$$Q_n \cong F_n.$$

LEMMA 1.1. $\mathrm{Hom}_{\mathbb{Z}G}(Q_n, M) \cong C^n(G, M)$, *where* $C^n(G, M)$ *denotes the set of all functions* $\varphi : G^n \to M$.

REMARK 1.2. (1) $C^n(G, M)$ is an abelian group.
(2) As $\mathbb{Z}G$ module Q_n has basis $G^n \times \{1\}$.
(3) Induced by the cochain maps $\mathrm{Hom}_{\mathbb{Z}G}(F_n, M) \to \mathrm{Hom}_{\mathbb{Z}G}(F_{n+1}, M)$ there is a cochain map

$$D : C^n(G, M) \to C^{n+1}(G, M)$$

defined by $(D\varphi)(g_0, ..., g_n) = \varphi(g_1, ..., g_n) - \varphi(g_0g_1, g_2, ..., g_n) + \varphi(g_0, g_1g_2, g_3, ..., g_n) - ... + (-1)^n(\varphi(g_0, ..., g_{n-1}g_n) + (-1)^{n+1}\varphi(g_0, ..., g_{n-1})g_n$.
Show $DD = 0$. As before, suppose $\varphi : G^n \to M$ such that $D\varphi = 0$ we call φ a n-cocycle.

DEFINITION 1.3. A 1-cocyle is called a **derivation**.

For $\varphi : G \to M$ and for all $g, h \in G$, $0 = (D\varphi)(g, h) = \varphi(h) - \varphi(gh) + \varphi(g)h$ implies

$$\varphi(gh) = \varphi(h) + \varphi(g)h.$$

REMARK 1.4. Let $\varphi : G \to M$ be a derivation. Then
(1) $\varphi(1) = 0$;
(2) $\varphi(g^{-1}) = -\varphi(g)g^{-1}$.

LEMMA 1.5. *Let M be a G-module. for every $m \in M$ and $g \in G$ the function $g \mapsto mg - m$ is a derivation. Such derivations are called* **inner derivations**.

A 2 cocyle is sometimes called a **factor set**.

EXERCISE 1.6. Let $\varphi : G \times G \to M$ be a factor set. Show that $\varphi(g, 1) = \varphi(1, 1)$ and $\varphi(1, k) = \varphi(1, 1)k$. for all $g, k \in G$.

EXAMPLE 1.7. Let G be an abelian group and let A be a trivial G-module. Then every multilinear map

$$\varphi : G \times ... \times G \to A$$

is a cocylce.
(1) Let $G = \mathbb{R}^n$ Then $det : \underbrace{\mathbb{R}^n \times ... \times \mathbb{R}^n}_{n} \to \mathbb{R}$ is a n-cocylce.
(2) Let $G = \mathbb{R}^n$ and $\langle v, w \rangle = \sum_{i=0}^{n} v_i w_i$ with $v, w \in \mathbb{R}^n$ is a 2-cocycle.

As shown above, the definition

$$\mathrm{H}^n(G, M) = \mathrm{H}^n(C^*(G, M))$$

is consistent with our previous definition of group cohomology via projective resolutions.

THEOREM 1.8. *Let G be a finite group and let M be a G-module. Then every element of $\mathrm{H}^n(G, M)$ for $n > 0$ has finite order dividing the order of G. In particular, for all $n > 0$.*

$$|G| \, \mathrm{H}^n(G, M) = 0.$$

2. Trees

Definition 2.1. Let X be a set. A free group on X is a group F together with a function $\iota : X \to F$ such that for any group G and any function $\phi : X \to G$ there is a unique homomorphism $\theta : F \to G$ such that $\theta\iota = \phi$.

Remark 2.2. Any two free groups on X are isomorphic and for every set X there exists a free group on X. Let F be the set of all reduced words in $X \cup X^{-1}$.

Definition 2.3. Let G be a group. A G-graph $\Gamma = (\Gamma, V, E, \iota, \tau)$ consists of two G-sets V (vertices) and E (edges) and G-maps $\iota, \tau : E \to V$.

We call ι the initial vertex (function) and τ the terminal vertex (function). In case $\iota(e) = \tau(e)$ for a $e \in E$, then e is a loop.

Definition 2.4. Let G be a group and let X be a subset of G. The Cayley-graph $\Gamma = \Gamma(G, X)$ with respect to X is the G-graph Γ defines as follows:

$$V = G \quad \text{and} \quad E = X \times G.$$

and $\iota(x, g) = g$ and $\tau(x, g) = xg$ for all $x \in X, g \in G$.

Let Γ be a G-graph. We define new initial and terminal vertex functions. Or, in other works we define edges e^1 and e^{-1} with an orientation. Let $e \in E$. now $\iota(e^1) = \iota(e)$ and $\tau(e^1) = \tau(e)$ whereas $\iota(e^{-1}) = \tau(e)$ and $\tau(e^{-1}) = \iota(e)$. A **path** in Γ is a finite sequence

$$v_0 e_1^{\epsilon_1} v_1 e_2^{\epsilon_2} ... v_{n-1} e_n^{\epsilon_n} v_n$$

where $\epsilon_i \in \{1, -1\}, v_i \in V, e_i \in E$ and $\iota(e^{\epsilon_i}) = v_{i-1}$ and $\tau(e^{\epsilon_i}) = v_i$. We shorten this to

$$p = e_1^{\epsilon_1} e_2^{\epsilon_2} ... e_n^{\epsilon_n}$$

with $\iota(p) = v_0$ and $\tau(p) = v_n$. The inverse path is

$$p^{-1} = e_n^{-\epsilon_n} ... e_1^{-\epsilon_1}$$

Let q be a path such that $\iota(q) = \tau(p)$. We form a new path (p, q) by gluing.
A path is called **reduced** if for all $i = 1, ..., n-1$

$$e_i^{\epsilon_i} \neq e_{i+1}^{-\epsilon_{i+1}}.$$

A path is called a **tree** if for all vertices v, w there is a unique reduced path p such that $\iota(p) = v$ and $\tau(p) = w$. Such a path is called a geodesic.
A path p is called closed at the vertex v if $\iota(p) = \tau(p) = v$. p is simply closed if there is no repetition of vertices. A graph Γ is called a **forest** if there are no simply closed paths.
A graph Γ is connected if for all $v, v' \in V$ there is a path connecting v and v'.

Proposition 2.5. *A graph Γ is a tree if and only if Γ is a connected forest.*

The augmented cellular chain complex $C_*(\Gamma) \twoheadrightarrow \mathbb{Z}$ of a G-graph $\Gamma = (\Gamma, V, E, \iota.\tau)$ is the following sequence of G-modules

$$0 \longrightarrow \mathbb{Z}E \xrightarrow{d} \mathbb{Z}V \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

where $d(e) = \tau(e) - \iota(e)$ and $\epsilon(e) = 1$. This is always a complex as $d\epsilon = 0$. Also ϵ is onto if and only if V is non-empty.

Lemma 2.6. (1) *A non-empty graph is connected if and only if*

$$\mathbb{Z}E \to \mathbb{Z}V \twoheadrightarrow \mathbb{Z}$$

is exact.

(2) *A graph is a forest if and only if*

$$0 \to \mathbb{Z}E \to \mathbb{Z}V$$

is exact.

(3) *A non-empty graph Γ is a tree if and only if*

$$0 \to \mathbb{Z}E \to \mathbb{Z}V \to \mathbb{Z} \to 0$$

is exact.

Let us now consider a little aside:

Definition 2.7. Let M be a G-module. The split extension $M \rtimes G$ of M over G is the set $M \times G$ together with multiplication

$$(m, g)(n, h) = (mh + n, gh).$$

Remark 2.8. (1) $M \rtimes G$ is a group with identity $(0, 1)$.

(2) The map $\pi : M \rtimes G \to G$ defined by $\pi(m, g) = g$ is a homomorphism.

Lemma 2.9. *There is a bijection between the set of group homomorphisms $\theta : G \to M \rtimes G$ satisfying $\pi\theta = id_G$ and the set of derivations $\varphi : G \to M$.*

Now back to trees:

Theorem 2.10. *Let F be a free group on X. Then the Cayley graph $\Gamma(F, X)$ is a tree.*

Exercise 2.11. Let G be a group and suppose the the Cayley graph $\Gamma(G, X)$ is connected. Show that G is generated by X.

Definition 2.12. **Free product with amalgamation** Let G_1, G_2 and A be groups and let $\alpha_1 : A \to G_1$ and $\alpha_2 : A \to G_2$ be group homomorphisms. The free amalgamated product of G_1 and G_2 over A is the group G satisfying:

There is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & G_1 \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_1} \\ G_2 & \xrightarrow{\beta_2} & B \end{array}$$

satisfying the following universal property

Let H be a group with homomorphisms $\gamma_i : G_i \to H$ $(i = 1, 2)$ such that $\gamma_1\alpha_1 = \gamma_2\alpha_2$ then there exists a unique homomorphism $\phi : G \to H$ such that $\phi\beta_i = \gamma_i$ $(i = 1, 2)$.

We write

$$G = G_1 *_A G_2.$$

Exercise 2.13. Let $G = G_1 *_A G_2$ with $A \to G_1$ and $A \to G_2$ not necessarily monomorphisms. Denote by $\bar{G_1}$ and $\bar{G_2}$ the images of G_1 and G_2 in G. Prove that $G = \bar{G_1} *_A \bar{G_2}$.

We therefore assume from now on that $A \hookrightarrow G_i$ and $G_i \hookrightarrow G$ for $i = 1, 2$.

LEMMA 2.14. *Let $G = K *_H L$. for every G-module M and derivations $\delta' : K \to M$ and $\delta'' : L \to M$ such that $\delta'|_H = \delta''|_H$. Then there is a unique derivation $\delta : G \to M$ such that $\delta|_K = \delta'$ and $\delta|_L = \delta''$.*

REMARK 2.15. Let $G = K *_H L$. Then

(1) $H = K \cap L$;
(2) G is generated by K and L.

THEOREM 2.16. *Let $G = K *_H L$ and define a G-graph $\Gamma = \Gamma(E, V, \iota, \tau)$ as follows:*

$$E = G/H \quad \textit{and} \quad V = G/K \sqcup G/L,$$

with $\iota(gH) = gK$ and $\tau(gH) = gL$. Then Γ is a tree.

DEFINITION 2.17. **HNN-extensions** Let $H \leq K \leq G$ be groups ad let $t \in G$ such that $H^t \subseteq K$. G is an HNN-extension with respect to (K, H, t) if is satisfying the following universal property: Let G_1 be a group, $t_1 \in G$ and $\theta : K \to G$ be a homomorphism such that for all $h \in H$. $\theta(h^t) = \theta(h)^{t_1}$. Then there is a unique homomorphism $\hat{\theta} G \to G_1$ such that $\hat{\theta}|_K = \theta$ and $\hat{\theta}(t) = t_1$. We write

$$G = K *_{H,t} .$$

THEOREM 2.18. *Let $G = K *_{H,t}$ be an HNN-extension. Then the G-graph $\Gamma = \Gamma(E, V, \iota, \tau)$ defined by*

$$V = G/K \quad \textit{and} \quad E = G/H$$

with $\iota(gH) = gK$ and $\tau(gH) = gtK$ is a tree.

Let T be a G-tree with one orbit of edges. Then there are one or two orbits of vertices. Let e be an edge. Then every vertex is in the orbit of $\iota(k)$ or in the orbit of $\tau(k)$.
One can also show that if $H \leq G$ and T a G-tree such that

(1) There is only one orbit of edges (with $H = G_e$ where $e \in E$.
(2) For every vertex v there is a $g \in G$ such that $gv \neq v$.

Then either $G = G_1 *_H G_2$ with $G_1 \neq G_2 \neq G$ or G is an HNN-extension with $G = K *_{H,t}$.

THEOREM 2.19. *Let $H \leq G_1 *_A G_2$ be a finite subgroup. Then H is subconjugated to a subgroup of G_1 or G_2.*

THEOREM 2.20. **Mayer Vietoris sequence:** *Let $G = K *_H L$ be a fee product with amalgamation and let M be a G-module. Then the following sequence in cohomology is exact:*

$$\cdots \to \mathrm{H}^n(G, M) \to \mathrm{H}^n(L, M) \oplus \mathrm{H}^n(K, M) \to \mathrm{H}^n(H, M) \to \mathrm{H}^{n+1}(G, M) \to \cdots .$$

COROLLARY 2.21. *Let $G = K *_H L$ be a fee product with amalgamation. Then*

(1) $\mathrm{cd}G \leq sup\{\mathrm{cd}H, \mathrm{cd}K, \mathrm{cd}L\} + 1$.
(2) *If H, K and L are of type FP_∞ or FP respectively, then so is G.*

EXERCISE 2.22. State and prove an analogous result to 2.21 (2) for H, K, L of types $\mathrm{FP}_l, \mathrm{FP}_n$ and FP_m respectively.

CHAPTER 6

Cohomology of soluble groups

1. Soluble and polycyclic groups

Definition 1.1. A group G is called soluble if there is a chain of normal subgroups
$$\{e\} = G_0 \lhd G_1 \lhd \ldots \lhd G_{n-1} \lhd G_n = G$$
such that G_i/G_{i-1} is abelian for all $i = 1, \ldots, n$.

(1) Abelian groups are soluble.
(2) Finite p-groups are soluble.
(3) A_n for $n \geq 5$ is not soluble as simple.
(4) S_3, A_4 and S_4 are soluble.

Definition 1.2. Let G be a soluble group. The length of the shortest abelian series is called the **derived length** of G.

G has derived length 0 if and only if G is trivial. It has derived length 1 if and only if G is abelian. Groups of derived length 2 are called metabelian.

Proposition 1.3. (1) *Subgroups of soluble groups are soluble.*
(2) *Quotients (homomorphic images) of soluble groups are soluble.*
(3) *Extensions of soluble groups are soluble.*

Let G be a group and let $g, h \in G$. We denote by
$$[g, h] = g^{-1}h^{-1}gh = g^{-1}h^g = (h^{-1})^g h$$
the **commutator** of g and h. Let H, K be subgroups of G. Then
$$[G, H] = \langle [h, k] \mid h \in H, k \in K \rangle$$
is called the commutator subgroup.

Lemma 1.4. *Let H, K be characteristic subgroups of G. Then $[H, K]$ is characteristic in G.*

Definition 1.5. Denote by $G' = [G, G]$, $G^{n+1} = (G^n)'$ and $G^0 = G$. We call G' the **derived subgroup** of G. By 1.4 it follows that $G^{n+1} \trianglelefteq G^n$ and we obtain a normal series, the **derived series** of G :
$$G = G^0 \trianglerighteq G^1 \trianglerighteq G^2 \trianglerighteq \ldots$$

Proposition 1.6. *Let G be a group. Then the following are equivalent:*
(1) *G is soluble of derived length d.*
(2) *There is a normal series*
$$\{1\} = N_0 \lhd N_1 \lhd \ldots \lhd N_d = G$$
with each N_{i+1}/N_i abelian.

(3) $G^d = \{1\}$.

Definition 1.7. A group G is called **nilpotent** if it has a central series, i.e. a normal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq ... \trianglelefteq G_n = G$$

such that $G_{i+1}/G_i \subseteq \xi(G/G_i)$ for all $i = 0, ..., n-1$. The length c of the shortest central series is called the nilpotency class of G.

Nilpotent groups are obviously soluble but, for example, S_3 is soluble but not nilpotent as it has trivial centre. Finite p-groups are nilpotent.

Proposition 1.8.
(1) *Subgroups of nilpotent groups are nilpotent.*
(2) *Quotients of nilpotent groups are nilpotent.*
(3) *(Fitting's Lemma) Let M and N be two nilpotent normal subgroups of a group G. Then MN is nilpotent.*

The property nilpotent is not closed under group extensions. S_3 is an extension of $A_3 \cong C_3$ with cyclic quotient of order 2 but is not nilpotent.

Definition 1.9. A group G is said to satisfy the property **max** if one of the following equivalent conditions holds:

(1) Every family of subgroups has a maximal member.
(2) Every strictly ascending chain $H_0 < H_1 < ...$ of subgroups of G is finite.
(3) Every subgroup of G is finitely generated.

Lemma 1.10. *Suppose $N \triangleleft G$. Then G has max if and only if both N and G/N have max.*

Example 1.11.
(1) All finite groups have max
(2) C_∞ the infinite cyclic group has max.
(3) Until 'recently' only groups built up from these using the above lemma where known to have max. In 1978 Rips and Olshanskii found infinite simple groups satisfying max.

Definition 1.12. A group G is said to be **polycyclic** if there is a chain of subgroups

$$1 = G_0 \triangleleft G_1 \triangleleft ... \triangleleft G_n = G$$

where each factor G_{i+1}/G_i is cyclic.

Lemma 1.13. *An abelian group is polycyclic if and only if it is finitely generated.*

Theorem 1.14. *A soluble group has max if and only if it is polycyclic.*

Not every finitely generated soluble group is polycyclic, take for example the wreath product of two infinite cyclic groups. But here is a list of known results:

(1) (Hall, Baer) Let G be finitely generated and nilpotent. Then G is polycyclic.
(2) (Malcev) Soluble subgroups of $Gl_n(\mathbb{Z})$ are polycyclic.
(3) (Auslander) Polycylic groups are $\mathbb{Z}$-linear.
(4) (Malcev) Let G be soluble such that all abelian subgroups have max. Then G has max.

Definition 1.15. Let G be a soluble group with an abelian series $\{1\} = G_0 \lhd G_1 \lhd ... \lhd G_n = G$. We define the **Hirsch length** $\mathrm{h}(G)$ as follows:

$$\mathrm{h}(G) = \sum_{i=0}^{n-1} dim(G_{i+1}/G_i \otimes \mathbb{Q}).$$

Note that for an abelian group A, the tensor product $A \otimes \mathbb{Q}$ is a $\mathbb{Q}$-vector space. The above definition is independent of the choice of series. By the Schreier refinement theorem any two series have equivalent refinements. Furthermore, this definition agrees with the, probably better known, definition of Hirsch length for polycyclic groups, which is defined to be the number of infinite cyclic factors of the polycyclic group.

Proposition 1.16. *Let G be a soluble group.*

1. *Let H be a subgroup of G. Then $\mathrm{h}(H) \leq \mathrm{h}(G)$.*
2. *Let $N \lhd G$. Suppose $\mathrm{h}(N) < \infty$ and $\mathrm{h}(G/N) < \infty$. Then*
$$\mathrm{h}(G) = \mathrm{h}(N) + \mathrm{h}(G/N).$$
3. *Let $N \lhd G$. Then $\mathrm{h}(G/N) \leq \mathrm{h}(G)$. In particular, $\mathrm{h}(G/N) = \mathrm{h}(G) \iff |G : N| < \infty$.*

Theorem 1.17. *(R. Bieri) Every torsion-free soluble group of finite Hirsch length is countable.*

2. Homological dimension of soluble groups

We shall begin with a few general remarks regarding homological dimension. This dimension can be defined via flat resolutions:

$$.. \to L_n \to L_{n-1} \to ... \to L_0 \to \mathbb{Z} \to 0,$$

were each L_i is a flat $\mathbb{Z}G$-module. Such resolutions always exist as we have shown before that projective modules are flat. The **homological dimension,** $\mathrm{hd}G$ of the group G is now defined to be the smallest such N that there is a flat resolution

$$0 \to L_n \to L_{n-1} \to ... \to L_0 \to \mathbb{Z} \to 0$$

of length n.

Lemma 2.1. *Let G be a group. Then*

$$\mathrm{hd}G \leq \mathrm{cd}G.$$

Proposition 2.2. *Let G be a group.*

1. *Let $H \leq G$ be a subgroup. Then $\mathrm{hd}H \leq \mathrm{hd}G$.*
2. *Let $H \hookrightarrow G \twoheadrightarrow Q$ bea group extension. Then $\mathrm{hd}G \leq \mathrm{hd}H + \mathrm{hd}Q$.*
3. *$\mathrm{hd}G = 0$ if and only if G is the trivial group.*
4. *Groups of finite homological dimension are torsion-free.*
5. *(Serre's Lemma) Let G be torsion-free and $H \leq G$ of finite index. Then $\mathrm{hd}G = \mathrm{hd}H$.*
6. *Let $\{G_i \,|\, i \in I\}$ be a direct system of groups and $G = \varinjlim G_i$ Then*
$$\mathrm{hd}G = sup\{\mathrm{hd}G_i\}.$$

The proof of (6) uses the fact that $\varinjlim$ commutes with direct sums and is exact. Let's take a free resolution $F_* \twoheadrightarrow \mathbb{Z}$. Then this can be viewed as the direct limit of induced frees. It remains to check that the n-th kernel is indeed the direct limit of induced flat modules.

The following result plays an important role in what is to follow:

THEOREM 2.3. *(D. Lazard) Let R be a ring. An R-module is flat if and only if it a direct limit of finitely generated free R-modules.*

COROLLARY 2.4. *Let R be a ring. Then*

(1) *Let L be countable flat. Then* $\mathrm{pd}_R L \leq 1$.
(2) *Every finitely presented flat R-module is projective.*

THEOREM 2.5. *Let G be a countable group. Then*

$$\mathrm{hd}G \leq \mathrm{cd}G \leq \mathrm{hd}G + 1.$$

The proof of the second inequality is an application of the Corollary to Lazard's theorem. Suppose $\mathrm{hd}G = n$. Then in the standard resolution $F_* \twoheadrightarrow \mathbb{Z}$ the n-th kernel is a countable flat G-module and hence has projective dimension less or equal to one.

THEOREM 2.6. *(Stammbach) Let G be a torsion-free soluble group. Then*

$$\mathrm{hd}G = \mathrm{h}G.$$

Stammbach actually proved the following stronger result. Let G be a group with no R-torsion. Then $\mathrm{hd}_R G = \mathrm{h}G$.

3. Soluble groups of type FP_∞.

Im the previous section, there's still one question remaining: How do we characterize groups where $\mathrm{cd}G = \mathrm{hd}G$. It follows from the proof of Theorem 2.5. That groups of type FP_∞ satisfy $\mathrm{cd}G = \mathrm{hd}G$ and the problem for soluble groups has been solved completely. We have already mentioned that polycyclic groups are of type FP_∞. Furthermore

THEOREM 3.1. *(Gruenberg) Let G be a nilpotent group such that* $\mathrm{cd}G = \mathrm{h}G$ *is finite. Then G is finitely generated.*

Let us now consider the class of **constructible groups**. This is the smallest class of groups containing the trivial group, which is closed under finite extensions and under HNN-extensions in which the base group and associated subgroups are constructible. Constructible groups are finitely presented [**2**] (Baumslag-Bieri). And it is not too hard to see that constructible soluble groups are of type FP_∞. Also, the Mayer Vietoris sequence and Serre's Lemma show that torsion-free soluble constructible groups satisfy $\mathrm{cd}G = \mathrm{h}G < \infty$. They admit a finite model for EG, or in other words are of type F. Furthermore one can also show that $\mathrm{cd}G = \mathrm{h}G < \infty$ implies G is torsion-free and $\mathrm{cd}_{\mathbb{Q}}G = \mathrm{h}G < \infty$. Hence Guildenhauys-Strebel [**11**] conjectured and partially answered that a torsion-free soluble group with $\mathrm{cd}_{\mathbb{Q}}G = \mathrm{h}G < \infty$ is constructible. This was then answered bu Kropholler [**14**]. Kropholler [**15**] also proved, using completely different methods that torsion-free soluble groups of type FP_∞ have finite Hirsch length. Let us summarize:

THEOREM 3.2. *Let G be a torsion-free soluble group. Then the following are equivalent:*

(1) *G is constructible*
(2) *G is of type* FP_∞
(3) *G is of type* FP
(4) $\mathrm{cd}G = \mathrm{h}G < \infty$
(5) $\mathrm{cd}_{\mathbb{Q}}G = \mathrm{h}G < \infty$.

Let us finish with a quick remark on soluble groups with torsion. Kropholler's result actually says that soluble groups of type FP_∞ have a finite index torsion-free subgroup satisfying all the above. We have seen a more general construction, the classifying space for proper actions $\underline{\mathrm{E}}\mathrm{G}$, which allows for torsion in a group. There is an algebraic analogue, the so called Bredon cohomology with dimensions and finiteness conditions defined in a similar fashion. Very recently a close analogue to Theorem 3.2 for Bredon-cohomology and $\underline{\mathrm{E}}\mathrm{G}$ has been shown. In particular:

THEOREM 3.3. [**16**]*(Nov.2007) Let G be a soluble group. Then the following are equivalent:*

(1) *G is of type* FP_∞
(2) *G has a cocompact (finite) model for* $\underline{\mathrm{E}}\mathrm{G}$.