Category Theory

Johar M. Ashfaque

Introduction

Category theory is the language of much of modern mathematics. It starts from the observation that the collection of all mathematical structures of a certain kind may itself be viewed as a mathematical object a category. This is only an introduction to category theory and therefore the main theme will be universal properties in their various manifestations, one of the most important uses of categories in mathematics.

Contents

1 Categories

When studying groups, we often consider group homomorphisms; when studying sets we often consider functions; when studying topological spaces we often consider continuous maps. The viewpoint of category theory is that we do not just study classes of objects, (groups, sets, topological spaces, etc.) but really we study *categories* of objects together with the relevant ways to compare them and move between them – the homomorphisms or arrows. Taking the common properties of these collections of objects and their arrows gives the definition of a category.

1.1 Definition of a category

A category $\mathcal{C}$ consists of:

- a collection $\mathrm{ob}\,\mathcal{C}$ of objects,
- for each pair A, B of objects, a collection $\mathcal{C}(A, B)$, also written $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and called a Hom-set, of arrows from A to B,
- a composition rule: for each $A, B, C \in \mathrm{ob}\,\mathcal{C}$, for each pair of arrows

  $$A \xrightarrow{f} B \text{ and } B \xrightarrow{g} C$$

  there is a composite arrow $A \xrightarrow{g \circ f} C$. Sometimes we write $g \circ f$ just as gf.
- identity arrows: for each $A \in \mathrm{ob}\,\mathcal{C}$ there is an arrow $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$,

subject to two axioms: composition is associative and the identity arrows act as identities. Specifically, if

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

is a sequence of arrows, then we can compose them as

$$A \xrightarrow{\quad h \circ (g \circ f) \quad} D \text{ or as } A \xrightarrow{\quad (h \circ g) \circ f \quad} D$$

and the two composites are required to be equal.
Given arrows:

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

we must have $1_B \circ f = f = f \circ 1_A$.
We should also require that the Hom-sets are all disjoint and do not intersect the collection of objects.

1.2 Examples of categories: concrete categories

In many categories, the objects are sets with some "structure", and the arrows are the functions which preserve that structure. These are called concrete categories (a precise definition will be given later). A few examples:

- The category **Set**, of sets and functions
- The category **Grp**, of groups and group homomorphisms
- **Ab**, abelian groups and all the group homomorphisms between them
- The category **Top**, of topological spaces and continuous maps
- **Ring**, (commutative, unital) rings and ring homomorphisms
- R-**Mod**, left R-modules for a given ring R
- **Mod**-R, the category of right R-modules
- **Monoid**, the category of monoids
- $\mathbf{Vect}_k$, vector spaces over a given field k, and linear maps
- **CMet**, the category of complete metric spaces and metric-preserving maps

In many cases there is no special name for the arrows, just "homomorphisms".

1.3 More examples of categories

Not all categories look like concrete categories.

(a) There is a category with 2 objects and only one non-identity arrow. It looks like a single arrow $A \xrightarrow{f} B$.

(b) More generally, let P be a category in which every hom-set has at most one arrow. Define a binary relation $\leqslant$ on $\operatorname{ob} P$ by $A \leqslant B$ iff $|\operatorname{Hom}_P(A, B)| = 1$. Then P is just a pre-ordered set, or quasiordered set. If $A \leqslant B$ and $B \leqslant A$ implies $A = B$ then the category P is just a partially ordered set. Conversely, every partially ordered set (poset) or pre-ordered set can be considered as a category.

(c) Posets are degenerate cases of categories. Even more degenerate is a category in which the only arrows are identity arrows, called a *discrete category*. Such a category is essentially the same as its set of objects, so a set is a special kind of category.

(d) A different kind of degeneracy is to consider a category M with only one object. So it has only one Hom-set, which is a set with an associative composition rule and an identity. So M is "the same as" a monoid. (For those who did not know before, this is the definition of a monoid.)

(e) A category with one object in which every arrow is invertible is the same as a group. So each group is a category, and we also consider **Grp**. the category of all groups.

(f) A category in which every arrow is invertible is called a *groupoid*.

(g) There is a category **Toph** of topological spaces and homotopy classes of maps. It looks more like a concrete category, but is not.

1.4 Special kinds of arrows

An arrow $A \xrightarrow{f} B$ is an *isomorphism* iff there is $B \xrightarrow{g} A$ such that $fg = 1_B$ and $gf = 1_A$. This g is necessarily unique and is written f^{-1}. Isomorphisms in concrete categories are what you expect, for example in **Set** they are bijections, in **Top** they are homeomorphisms.

Two useful properties of functions which are weaker than bijectivity are injectivity and surjectivity. They make sense for any concrete category, but not for all categories, so we need a different definition.
An arrow $A \xrightarrow{m} B$ is said to be *monic* or a *monomorphism* iff for all pairs of arrows

$$\begin{tikzcd} D \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, "m"] & B \end{tikzcd}$$

whenever $m \circ f = m \circ g$ then also $f = g$.
Similarly, $A \xrightarrow{e} B$ is said to be *epic* or an *epimorphism* iff for all pairs of arrows

$$\begin{tikzcd} A \arrow[r, "e"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \end{tikzcd}$$

whenever $f \circ e = g \circ e$ then also $f = g$.
It is easy to see that surjective functions are the same as epimorphisms in **Set**, and injective functions are the same as monomorphisms in **Set**. Similarly, in **Grp**, monomorphisms are the same as injective homomorphisms, and epimorphisms are the same as surjective homomorphisms. In other categories, the notions do not agree.
Suppose we have arrows

$$A \xrightarrow{f} B \xrightarrow{g} A$$

such that $g \circ f = 1_A$. So g is a left-inverse for f and f is a right-inverse for g. In geometry (and also elsewhere), a right-inverse for g is often called a section of g, and a left-inverse of f is called a retraction of f. It is easy to see that f is monic and g is epic. In fact we say that f is a *split monic* and g is a *split epic*.

1.5 Special kinds of objects

In **Set**, the empty set $\emptyset$ has the property that there is exactly one function $\emptyset \longrightarrow X$ for any given set X. In **Grp** the one-element group has the same property. We say that $\emptyset$ is an *initial object* for **Set**.
Similarly, an object T in a category $\mathcal{C}$ is called a *terminal object* iff for any $A \in \mathcal{C}$ there is exactly one arrow $A \longrightarrow T$. In **Set**, any 1-point set is a terminal object. In **Grp**, the one-element group is terminal as well as initial. Any object which is both initial and terminal is called a *zero object* or *null object.*
In a poset P considered as a category, an initial object is a least element, and a terminal object is a greatest element.

Exercise Show that, where they exist, initial objects and terminal objects are unique up to (unique) isomorphism.

1.6 Duality

If $\mathcal{C}$ is any category, we form its opposite category, $\mathcal{C}^{\text{op}}$, by reversing the arrows. That is, $\text{ob}\,\mathcal{C}^{\text{op}} = \text{ob}\,\mathcal{C}$ and $\text{Hom}_{\mathcal{C}^{\text{op}}}(A, B) = \text{Hom}_{\mathcal{C}}(B, A)$, with the same composition. For example, if P is a partial order then P^{op} is the same partial order but "upside-down".
By reversing arrows, we can see that certain notions are dual to each other. Monics and epics are dual, as are initial and terminal objects.

1.7 Example: elliptic curves

Often one mathematical structure can be viewed as an object in many different categories. Elliptic curves give a good example.
Consider the complex plane $\mathbb{C}$ as a group under addition. Consider the subgroup of $\mathbb{C}$ generated by 1 and i, which is $\Lambda = \{n + mi \mid n, m \in \mathbb{Z}\}$. Now let E be the quotient group $\mathbb{C}/\Lambda$. Then E is a typical example of a (complex) elliptic curve. We can consider E as a group, so in **Grp**, or in **Ab**. The Euclidean topology on $\mathbb{C}$ gives E the quotient topology, so we can think of it in **Top** or in the subcategory of compact Hausdorff spaces. The group operations are continuous, so in fact E is an object in the category of topological groups (and continuous group homomorphisms). The topology makes E into a differential manifold, and indeed a smooth (infinitely differentiable) manifold. And in fact E is also a complex manifold. Less obviously, E also has the structure of an algebraic variety, so can be viewed in the category of (complex) algebraic varieties and their morphisms. An algebraic geometer might view varieties in a more sophisticated way, so might consider E in the category of schemes. And indeed E is a group variety, and a group scheme as well. That is 12 different categories, and we have not considered number theoretic approaches. In fact for a particular problem, it may be useful to think of an elliptic curve in any one or more of these 12 categories, and maybe others. Category theory makes the process explicit, and gives some tools for translating results from one category to another.
Note: For this category theory course I hope you know what groups and topological spaces are. For your general mathematical background, I hope you have an idea what some of the other categories are. However, knowledge of elliptic curves, varieties, and schemes is not needed for this course.

1.8 Notation and terminology

We write "$A \in \mathcal{C}$" to mean that A is an object of $\mathcal{C}$. Also "f in $\mathcal{C}$" will often mean that f is an arrow of $\mathcal{C}$. Writing "$A \xrightarrow{f} B$ in $\mathcal{C}$" means that A, B are objects of $\mathcal{C}$ and $f \in \text{Hom}_{\mathcal{C}}(A, B)$.
Arrows are also called morphisms. If $A \xrightarrow{f} B$ then A and B are the domain and codomain of f, or sometimes the source and target.

1.9 Remarks on sets and proper classes

A category is *small* if its collection of objects and all its Hom-sets are actually sets, not proper classes. For example, a group is a small category, but the category **Set** is not. This distinction is

important to avoid inconsistencies such as Russell's paradox. In a small category we can form the set of all arrows, but we cannot do this for a large (not small) category.
Although **Set** is not small, it is *locally small*, which means that all its Hom-sets are sets. All the categories listed above are locally small.
We assume the Axiom of Choice throughout the course (and even the stronger axiom of global choice). You may not even notice when we use it. Most books on category theory contain some discussion of foundational issues. They are important in some contexts but rarely important for using category theory in mathematics, and I will not stress them.

1.10 Exercises

Q1.1 Suppose that $A \xrightarrow{f} B$, $B \xrightarrow{g} A$, $B \xrightarrow{h} A$ are arrows in any category, and $fg = fh = 1_B$, $gf = hf = 1_A$. Show that $g = h$, that is, inverses are unique.

Q1.2 Show that an arrow with both a left inverse and a right inverse is an isomorphism, and the two inverses are equal.

Q1.3 Give an example of an arrow which is monic and epic but not an isomorphism.

Q1.4 Find a bijective arrow in **Top** or in another concrete category which is not an isomorphism.

Q1.5 Show that any split epic which is also monic is an isomorphism.

Q1.6 Characterize the epimorphisms and monomorphisms in **Top**, and in **Haus**, the category of Hausdorff topological spaces and continuous maps.

Q1.7 Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$ an object. We can form the *coslice category* $X/\mathcal{C}$, whose objects are arrows $X \xrightarrow{p} A$ in $\mathcal{C}$, and whose arrows are commuting triangles

$$\begin{tikzcd} & A \arrow[d, "f"] \\ X \arrow[ur, "p"] \arrow[r, "p'"'] & A' \end{tikzcd}$$

in $\mathcal{C}$.

(i) Explain how to compose arrows in $X/\mathcal{C}$ (there is only one way which is sensible) and show that $X/\mathcal{C}$ really is a category.

(ii) For any (commutative, unital) ring R, show that $R/$**Ring** is essentially the same as the category R-**Alg** of R-algebras.

(iii) Dually, there is the slice category $\mathcal{C}/X$, whose objects are arrows $A \xrightarrow{p} X$ in $\mathcal{C}$. Following the example of coslice categories, explain what the arrows in $\mathcal{C}/X$ are, and how they compose.

Q1.8 Show that, where they exist, initial objects and terminal objects are unique up to unique isomorphism.

Q1.9 Show that the assertion "Every epimorphism in **Set** is split" is a statement of the Axiom of Choice. Show on the other hand that every monomorphism in **Set** is split (without using Choice), except for the monomorphisms $\emptyset \longrightarrow X$ with $X \neq \emptyset$.

2 Functors

2.1 Definition of functor

The idea behind categories is to study objects by understanding also (or only) the arrows between them. So we need to know what the arrows between categories are.
Given categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* F from $\mathcal{C}$ to $\mathcal{D}$ consists of:

- For each $A \in \mathcal{C}$, an object $FA \in \mathcal{D}$,
- For each $A \xrightarrow{f} B$ in $\mathcal{C}$, an arrow $FA \xrightarrow{Ff} FB$ in $\mathcal{D}$,

such that

- For all $A \in \mathcal{C}$, $F1_A = 1_{FA}$
- Whenever $A \xrightarrow{f} B \xrightarrow{g} C$ is a pair of composable arrows in $\mathcal{C}$, $F(gf) = (Fg)(Ff)$.

Functors can be composed, and there is an identity functor for each category, so we obtain a category **CAT** of all categories and functors. Note that **CAT** is not locally small. For foundational reasons we also consider the category **Cat** of small categories and functors.

2.2 Examples of functors

- Forgetful functors which "forget" part of the structure or a property: $\mathbf{Grp} \xrightarrow{U} \mathbf{Set}$ takes a group to its underlying set, and a homomorphism to its underlying function.
- The inclusion $\mathbf{Ab} \longrightarrow \mathbf{Grp}$ which takes an abelian group and "forgets" that it is abelian.
- Free functors, for example for a field k, $\mathbf{Vect}_k$ is the category of k-vector spaces and linear maps.
$$\mathbf{Set} \xrightarrow{F} \mathbf{Vect}_k$$
takes a set X to the space of formal k-linear combinations of elements of X, a vector space with X as a basis.
- If M, N are monoids, thought of as categories, a functor $M \longrightarrow N$ is just a monoid homomorphism.
- If P, Q are posets, thought of as categories, a functor $P \longrightarrow Q$ is just an order-preserving map.
- The original example: $\mathbf{Top} \xrightarrow{H_n} \mathbf{Ab}$, the n^{th} homology group.
- Let $\mathbf{Top}_*$ be the category of topological spaces with a chosen basepoint, and basepoint-preserving continuous maps. There is a fundamental group functor $\mathbf{Top}_* \xrightarrow{\pi_1} \mathbf{Grp}$.

2.3 Subcategories and full functors

A subcategory $\mathcal{S}$ of $\mathcal{C}$ is a collection of some of the objects and some of the arrows of $\mathcal{C}$, such that $\mathcal{S}$ is itself a category (with the same composition rule from $\mathcal{C}$). If $\mathcal{S}$ is a subcategory of $\mathcal{C}$ then there is an evident inclusion functor $\mathcal{S} \longrightarrow \mathcal{C}$, which takes an object or arrow in $\mathcal{S}$ to itself, considered as an object or arrow of $\mathcal{C}$.
We define a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ to be *full* iff for every $A, B \in \mathcal{C}$, the function of arrows it induces $\mathcal{C}(A,B) \longrightarrow \mathcal{D}(FA, FB)$ is surjective.
In some cases such as $\mathbf{Ab} \longrightarrow \mathbf{Grp}$, the inclusion functor of a subcategory is full. We say **Ab** is a *full subcategory* of **Grp**. To specify a full subcategory it is enough to specify just the objects, and frequently this is done. For example one talks about the subcategory of $\mathbf{Vect}_k$ consisting of all finite-dimensional k-vector spaces, without explicitly mentioning that all linear maps between them are included.
Not all subcategories are full. For example, we can consider the subcategory of **Set** consisting of all sets but just injective functions.

Warning The image of a functor is not necessarily a subcategory of its codomain. For example, let $\mathcal{C}$ have four objects, A, B, C, D, and only two non-identity arrows: $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, and let $\mathcal{D}$ have three objects, X, Y, Z, and non-identity arrows $X \xrightarrow{h} Y \xrightarrow{k} Z$ and the composite kh.
We can define a functor F from $\mathcal{C}$ to $\mathcal{D}$ by $FA = X$, $FB = FC = Y$, and $FD = Z$, and then we must have $Ff = h$ and $Fg = k$. But then $kh = Fg \circ Ff$ is not in the image of F.

2.4 Faithful functors and concrete categories

The inclusion functor of a category is injective on objects and also injective on arrows. In practice a weaker notion is useful:
A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is said to be *faithful* iff for every $A, B \in \mathcal{C}$, the function of arrows it induces $\mathcal{C}(A, B) \longrightarrow \mathcal{D}(FA, FB)$ is injective.
The forgetful functor $\mathbf{Grp} \xrightarrow{U} \mathbf{Set}$ is faithful, because a group homomorphism is determined by its underlying function. Similarly, all the concrete categories we listed earlier come with a forgetful (or underlying) functor to **Set**, which is faithful. We use this idea to make the idea of a concrete category precise.
A *concrete category* consists of a category $\mathcal{C}$ together with a faithful functor (called the underlying functor) $\mathcal{C} \xrightarrow{U} \mathbf{Set}$.
Not all functors are faithful. For example, given a category $\mathcal{C}$ define the shadow of $\mathcal{C}$ to be a new category $S\mathcal{C}$ with the same objects as $\mathcal{C}$, but at most one arrow in each Hom-set. There should be an arrow $A \longrightarrow B$ in $S\mathcal{C}$ precisely when there is at least one such arrow in $\mathcal{C}$. There is a unique functor $\mathcal{C} \xrightarrow{Q} S\mathcal{C}$ which is the identity on objects, which is not faithful unless $\mathcal{C}$ is a pre-ordered set.
Here are two useful pieces of terminology. Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor. Then F *preserves* monics iff whenever $A \xrightarrow{m} B$ is monic in $\mathcal{C}$ then $FA \xrightarrow{Fm} FB$ is monic in $\mathcal{D}$. We say F *reflects* monics iff whenever $A \xrightarrow{m} B$ is monic in $\mathcal{C}$ then $FA \xrightarrow{Fm} FB$ is monic in $\mathcal{D}$.
The words *preserves* and *reflects* are used much more widely. For example, it is clear by analogy what is meant by saying that a functor "preserves isomorphisms", or "reflects initial objects", and so on.

2.5 Hom-functors

Let $\mathcal{C}$ be a locally small category and fix $A \in \mathcal{C}$. For each $B \in \mathcal{C}$ we have a set $\mathcal{C}(A, B)$ and for each $B \xrightarrow{g} B'$ there is an induced map $\mathcal{C}(A, B) \xrightarrow{g_*} \mathcal{C}(A, B')$ given by $f \mapsto g \circ f$. This defines a functor $H^A = \mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$.
Similarly we can look at $H_B = \mathcal{C}(-, B) : \mathcal{C} \to \mathbf{Set}$, where an arrow $A \xrightarrow{f} A'$ in $\mathcal{C}$ induces the map $\mathcal{C}(A', B) \xrightarrow{f_*} \mathcal{C}(A, B)$ given by $g \mapsto g \circ f$. Note that the function of arrows goes the other way. So H_B is not a functor from $\mathcal{C}$ to **Set** the way we have defined it. However, it is a functor $\mathcal{C}^{\mathrm{op}} \xrightarrow{H_B} \mathbf{Set}$.
The functors H^A are called *(covariant) Hom-functors*, and the H_B are called *(contravariant) Hom-functors.*

2.6 Contravariant functors

A functor $\mathcal{C} \longrightarrow \mathcal{D}^{\mathrm{op}}$ is really the same thing as a functor $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}$. Both are known as *contravariant functors* from $\mathcal{C}$ to $\mathcal{D}$. Normal functors are also called *covariant functors.*

- The contravariant Hom-functors $H_B = \mathcal{C}(-, B)$.
- Cohomology functors: $\mathbf{Top}^{\mathrm{op}} \xrightarrow{H^n} \mathbf{Ab}$
- Dual vector space functor $\mathbf{Vect}_k^{\mathrm{op}} \longrightarrow \mathbf{Vect}_k$

For the general theory, it is useful to write a contravariant functor $\mathcal{C} \xrightarrow{F'} \mathcal{D}$ instead as a covariant functor $\mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{D}$. In specific examples such as cohomology and dual spaces, one usually considers contravariant functors directly.

2.7 Example: power set functors

There are two different power set functors $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{Set}$ taking a set X to its power set $\mathcal{P}(X)$:

- A covariant one, taking a function to its *direct image* map;
- A contravariant one, taking a function to its *inverse image* map.

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{P}(X) \arrow[d, "f^{\rightarrow}"] & A \arrow[d, mapsto] \\ Y \arrow[r, mapsto] & \mathcal{P}(Y) & \{f(x) \mid x \in A\} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{P}(X) & \{x \mid f(x) \in B\} \\ Y \arrow[r, mapsto] & \mathcal{P}(Y) \arrow[u, "f^{\leftarrow}"'] & B \arrow[u, mapsto] \end{tikzcd}$$

We could consider a power set not just as a set, but as a partially ordered set, or a boolean algebra etc., which would give rise to more ways of considering the power set operation as a functor.

2.8 Exercises

Q2.1 Show that:

(i) every functor preserves isomorphisms.

(ii) a faithful functor reflects monics and epics.

(iii) a full and faithful functor reflects isomorphisms.

Q2.2 Give an example of a functor which does not reflect all isomorphisms, and an example of a functor which does not reflect all monics and epics.

Q2.3 What are the subcategories of a poset, and of a group?

Q2.4 Is there a functor $\mathbf{Grp} \xrightarrow{Z} \mathbf{Grp}$ taking a group to its centre? [Hint: what would its arrow map be? Consider the groups S_2 and S_3.]

Q2.5 Show that the ring of continuous real-valued functions on a topological space is the object function of a contravariant functor from **Top** to **Ring**.

Q2.6 Let G be a group, and k a field. Show that a functor $G \xrightarrow{\sigma} \mathbf{Set}$ is a left-action of G on a set, and a functor $G \xrightarrow{\rho} \mathbf{Vect}_k$ is a (left, linear)-representation of G. What are contravariant functors from G to **Set**?

Q2.7 If G is a group, what is G^{op}? Show $G \cong G^{\mathrm{op}}$. Show that a monoid may not be isomorphic to its opposite.

Q2.8 Show how the construction of the field of fractions of an integral domain can be regarded as a functor.

Q2.9 Let $\mathbf{1}, \mathbf{2}$, and $\mathbf{3}$ be the totally ordered sets with one, two, and three elements respectively, considered as categories. Show that functors $\mathbf{1} \longrightarrow \mathcal{C}$, $\mathbf{2} \longrightarrow \mathcal{C}$, and $\mathbf{3} \longrightarrow \mathcal{C}$ correspond to objects of $\mathcal{C}$, arrows of $\mathcal{C}$, and composable pairs of arrows of $\mathcal{C}$ respectively.

3 Universal properties

One of the most important ideas in category theory is that of *universal properties*: existence and uniqueness properties which capture the essence of a construction and can be used to define it. We give several examples now. Much of the rest of the course is devoted to formalising the concept of a universal property in different ways.

3.1 Vector space bases

Let W, V be k-vector spaces (k some field), let B be a basis of W and $B \xrightarrow{b} W$ in **Set** be the inclusion map. For each function $B \xrightarrow{f} V$, there is a unique k-linear map $W \xrightarrow{\hat{f}} V$ extending f, that is, such that $\hat{f} \circ b = f$. We can illustrate this via a diagram:

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[dr, "f"'] & W \arrow[d, dotted, "\exists! \hat{f}"] \\ & V \end{tikzcd}$$

You should read this as: given B, W, V, b, and f as shown, there is a unique $\hat{f}$ making the diagram commute.

This diagram does not contain all the information in the words above. In particular, B is a set, and W and V are vector spaces, and b and f are functions, but $\hat{f}$ is a linear map. So really the diagram lives in two different categories. We can give a more complicated diagram reflecting the two different categories by also considering the forgetful functor $\mathbf{Vect}_k \xrightarrow{U} \mathbf{Set}$ which takes a vector space to its underlying set, and a linear map to its underlying function.

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[dr, "f"'] & UW \arrow[d, dotted, "U\hat{f}"] & W \arrow[d, dotted, "\exists! \hat{f}"] \\ & UV & V \end{tikzcd}$$

The triangle on the left is a diagram in the category **Set**, and the arrow on the right is in $\mathbf{Vect}_k$. Even this diagram does not contain all the information in the words. (Most of the quantifiers are still missing.) However, it is a useful visual aid.

3.2 Rational numbers as a field of fractions

The field of rational numbers, $\mathbb{Q}$, is the smallest field in which the ring of integers, $\mathbb{Z}$, embeds. It has a universal property: if $\mathbb{Z} \xrightarrow{f} k$ is a ring embedding of $\mathbb{Z}$ into any field k (embedding, so k has characteristic 0), then there is a unique embedding $\mathbb{Q} \xrightarrow{\hat{f}} k$ extending f.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{Q} \arrow[d, dotted, "\exists! \hat{f}"] \\ & k \end{tikzcd}$$

In the diagram, ι is the inclusion map of $\mathbb{Z}$ into $\mathbb{Q}$.

This universal property can be generalised to the field of fractions of any integral domain, R. The field of fractions FR of R is the smallest field in which R embeds:

$$\begin{tikzcd} R \arrow[r, "\iota"] \arrow[dr, "f"'] & FR \arrow[d, dotted, "\exists! \hat{f}"] \\ & k \end{tikzcd}$$

From the universal property alone, we can prove that FR is unique up to isomorphism. Suppose that $R \xrightarrow{a} A$ and $R \xrightarrow{b} B$ were two inclusions of R into fields, both of which had the universal property of the field of fractions of R. Using the universal property of A, we get:

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[dr, "b"'] & A \arrow[d, dotted, "\exists! \hat{b}"] \\ & B \end{tikzcd}$$

and similarly we get $B \xrightarrow{\hat{a}} A$. Now we have two maps from A to A making the diagram commute:

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[dr, "a"'] & A \arrow[d, shift right, "1_A"'] \arrow[d, shift left, "\hat{a} \circ \hat{b}"] \\ & A \end{tikzcd}$$

The universal property says there is only one such arrow, so $\hat{a} \circ \hat{b} = 1_A$. Similarly, $\hat{b} \circ \hat{a} = 1_B$. So A and B are isomorphic. Even more, the isomorphism is uniquely determined. We could use this universal property as the *definition* of the field of fractions of R, but note that we have not actually proved that the field of fractions exists.

3.3 Tensor Products

Let M and N be R-modules (for example, vector spaces). There is a "universal bilinear map" from the product $M \times N$. That is, an R-module T and a bilinear map $M \times N \xrightarrow{h} T$ such that for any bilinear map $M \times N \xrightarrow{f} W$, there is a unique linear map $T \xrightarrow{\hat{f}} W$ such that $\hat{f} \circ h = f$.

$$\begin{tikzcd} M \times N \arrow[r, "h", "\text{bilinear}"'] \arrow[dr, "f", "\text{bilinear}"'] & T \arrow[d, dotted, "{\exists! \hat{f}, \text{linear}}"] \\ & W \end{tikzcd}$$

The universal property defines T up to isomorphism as an R-module. It is written $M \otimes_R N$, and called the *tensor product* of M and N. This universal property is often used as the definition of the tensor product, but again we would have to prove separately that the tensor product actually exists.

3.4 Products of sets

We take a closer look at the universal property of the Cartesian product of two sets. Given sets X and Y, we can form the product

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

So a point in $X \times Y$ is determined by its X-coordinate and its Y-coordinate, and these are given by the projection functions:

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

given by $p_1(x, y) = x$ and $p_2(x, y) = y$. We can define these projection functions and the product $X \times Y$ just by universal properties of functions, without mentioning the elements at all. The diagram giving the universal property is:

$$\begin{tikzcd} & W \arrow[d, dotted, "\exists! f"] \arrow[ddl, "f_1"'] \arrow[ddr, "f_2"] & \\ & X \times Y \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ X & & Y \end{tikzcd}$$

What it says is the following: suppose W is a set and we have two functions, $W \xrightarrow{f_1} X$ and $W \xrightarrow{f_2} Y$. Then there is a unique function $W \xrightarrow{f} X \times Y$ such that $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$. Of course f is given in terms of elements by $f(w) = (f_1(w), f_2(w))$. However, we can use the universal property as a definition of $X \times Y$, because it defines that set, and the projection maps, uniquely up to unique isomorphism.

3.5 Products in other categories

It is not immediately clear why we might want to think of products of sets via universal properties rather than ordered pairs of elements, because it is not a simplification. The advantage comes with considering products in other categories. The same universal property, with the same diagram, works to define the product $G \times H$ of groups G and H. Now the arrows are group homomorphisms, not any functions. The universal property does not just determine $G \times H$ as a set, but determines it up to isomorphism in the category of groups. So it also gives the group structure. Similarly, in the category of topological spaces and continuous maps, the universal property of a product gives the product topology on the product $X \times Y$ of two topological spaces.
Given two categories $\mathcal{C}$ and $\mathcal{D}$, there is a product category $\mathcal{C} \times \mathcal{D}$ given by:
$\mathrm{ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{ob}\,\mathcal{C} \times \mathrm{ob}\,\mathcal{D}$
$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A,B),(C,D)) = \mathrm{Hom}_{\mathcal{C}}(A,C) \times \mathrm{Hom}_{\mathcal{D}}(B,D)$ with the obvious componentwise definition of composition of arrows.
It is a good exercise to check that this definition of the product of two categories does satisfy the universal property of a product from the diagram above.
In fact the universal property works to define products in any category. We will come back to the definition again.

3.6 Quotients

Let X be a set, and consider $E \subset X \times X$, an equivalence relation. We have two projections maps from E to X, induced from the product. There is a quotient set X/E, of equivalence classes, and a quotient map $X \xrightarrow{q} X/E$, and they have the following universal property:

$$\begin{tikzcd} E \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "q"] \arrow[dr, "f"'] & X/E \arrow[d, dotted, "\exists!\hat{f}"] \\ & & Y \end{tikzcd}$$

If $X \xrightarrow{f} Y$ is any function such that $f \circ p_1 = f \circ p_2$ (that is, such that whenever x, y are points from X which are in the same equivalence class then $f(x) = f(y)$), then there is $X/E \xrightarrow{\hat{f}} Y$ such that $f = \hat{f} \circ q$. We say that f factors uniquely through X/E (or, more precisely, through q).
The universal property determines the set X/E up to unique bijection. In the category of topological spaces and continuous maps, the universal property also gives the quotient topology on X/E.

3.7 Coproducts

The *disjoint union* of two sets X and Y is a set like the union $X \cup Y$ except that the elements are labelled so we know which of X or Y they come from. It is written in different ways: $X \dot{\cup} Y = X \sqcup Y = X \amalg Y$, and one way to define it is $X \amalg Y = (X \times \{1\}) \cup (Y \times \{2\})$. For example, if $X = \{a, b\}$ and $Y = \{a, c\}$ then $X \amalg Y = \{(a,1), (b,1), (a,2), (c,2)\}$. There are obvious functions $X \xrightarrow{p_1} X \amalg Y$ and $Y \xrightarrow{p_2} X \amalg Y$ given by $p_1(x) = (x, 1)$ and $p_2(x) = (x, 2)$. It does not really matter how we do the labelling (and there are other ways in the literature); what matters is the following universal property:
If W is a set and we have two functions $X \xrightarrow{f_1} W$ and $Y \xrightarrow{f_2} W$, then there is a unique function $X \amalg Y \xrightarrow{f} W$ such that $f_1 = f \circ p_1$ and $f_2 = f \circ p_2$. The function f can be given a definition by

cases:

$$f((x, A)) = \begin{cases} f_1(x) & \text{if } A = 1 \\ f_2(x) & \text{if } A = 2 \end{cases}$$

In this case the diagram definitely makes the universal property easier to see:

$$\begin{tikzcd}
X \arrow[dr, "p_1"] \arrow[ddr, "f_1"'] & & Y \arrow[dl, "p_2"'] \arrow[ddl, "f_2"] \\
& X \amalg Y \arrow[d, dotted, "\exists!" ', "f"] & \\
& W &
\end{tikzcd}$$

As in all these examples, the universal property can be used as a definition of a disjoint union. The universal property is dual to that of a product (all the arrows are reversed) so $X \amalg Y$ is called the *coproduct* of X and Y. The same universal property (that is, the universal property with the same diagram) can be used to define coproducts in any category.

3.8 Intersections

Suppose G is a group, and H, J are subgroups, with inclusion maps $H \xrightarrow{h} G$ and $J \xrightarrow{j} G$. The intersection $H \cap J$ has inclusion maps into H and into J, and there is the universal property given by the following diagram:

$$\begin{tikzcd}
K \arrow[dr, dotted, "f"] \arrow[drr, "f_1"] \arrow[ddr, "f_2"'] & & \\
& H \cap J \arrow[r] \arrow[d] & H \arrow[d, "h"] \\
& J \arrow[r, "j"'] & G
\end{tikzcd}$$

Given a group K and homomorphisms $K \xrightarrow{f_1} H$ and $K \xrightarrow{f_2} J$ such that $h \circ f_1 = j \circ f_2$, there is a unique $K \xrightarrow{f} H \cap J$ such that the whole diagram commutes.

3.9 Universal properties as definitions

We have seen several examples of familiar constructions which can be given in terms of universal properties.

Advantages

- The same definitions work in many different categories.
- You can often see exactly what the relevant properties of the construction are, not those which are an accident of your choice of construction.
- Sometimes there is no easy explicit construction, but the universal property can still be easy to work with. For example: Hausdorffification of a topological space, tensor products.

Disadvantages

- Sometimes the abstract nature of universal properties obscures the simplicity of a construction. (In particular, we do not usually teach undergraduates this way.)
- Often a separate existence proof is needed.

3.10 Exercises

Q3.1 Let G be a group and $N \lhd G$ a normal subgroup. Show that the quotient map $G \xrightarrow{q} G/N$ satisfies a universal property, which defines the quotient group and the map q up to unique isomorphism.

Q3.2 Show that completing a metric space can be considered as a universal construction, that is, that the inclusion $X \hookrightarrow \bar{X}$ of a metric space into its completion satisfies a universal property.

Q3.3 For a set X, describe the free monoid with generators X and explain what its universal property is.

Q3.4 Let R be a (commutative, unital) ring, and P a prime ideal in R. The localization of R at P, written R_P, can be described by a universal property: it is the smallest ring extension of R in which every element of $R \setminus P$ is invertible. Make this precise.

Q3.5 Let **Field** be the category of fields and field homomorphisms (embeddings), and let **ACF** be the full subcategory of algebraically closed fields. There is an obvious embedding $k \to k^{\mathrm{alg}}$ of a field into its algebraic closure. Show nonetheless that this embedding does not have a universal property. Show also that there is no natural way (not involving arbitrary choices) to define a functor $\mathbf{Field} \longrightarrow \mathbf{ACF}$ of which the assignment $k \mapsto k^{\mathrm{alg}}$ is the object map. Finally, show that there is no such functor (making arbitrary choices or not) whose restriction to **ACF** is the identity functor. [Hint: consider the two square roots of 2 and the inclusion $\mathbb{Q} \hookrightarrow \mathbb{Q}^{\mathrm{alg}}$.]

Q3.6 In any category with binary products, show directly from the universal property that

$$(A \times B) \times C \cong A \times (B \times C)$$

4 Natural transformations

4.1 Naturality

We start with an example. Let V be a finite dimensional vector space, and V^* its dual space. Then $\dim V = \dim V^*$, so V and V^* are isomorphic. The same argument shows that $V \cong V^{**}$. If we choose a basis for V then there is a dual basis for V^*, which gives rise to a choice of isomorphism. However, if we choose a different basis then the isomorphism changes. However, the isomorphism $V \cong V^{**}$ given by associating a basis vector to its double dual basis vector does not depend on the choice of basis. So it seems that there is no natural or canonical isomorphism $V \cong V^*$, but there is a natural isomorphism $V \cong V^{**}$.

One of the original purposes of category theory was to explain what *natural* means in this context, by defining *natural transformations*.

4.2 Definition of natural transformation

The slogan for category theory is *look at the arrows*. If categories are the objects, then the arrows are functors. But we can also take functors as objects and look at arrows between them. These are called *natural transformations*.

Given a parallel pair of functors: $\mathcal{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{D}$, a *natural transformation* α from F to G, consists of an arrow $FC \xrightarrow{\alpha_C} GC$ in $\mathcal{D}$ for each object $C \in \mathcal{C}$, satisfying:

for each $C \xrightarrow{f} C'$ in $\mathcal{C}$, the *naturality square*:

$$\begin{tikzcd} FC \arrow[r, "Ff"] \arrow[d, "\alpha_C"'] & FC' \arrow[d, "\alpha_{C'}"] \\ GC \arrow[r, "Gf"'] & GC' \end{tikzcd}$$

commutes.

We often write a natural transformation with a double arrow (not to be confused with implication): $F \overset{\alpha}{\Rightarrow} G$ or $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$.

4.3 Example: double dual vector space

Let **Vect** be the category of vector spaces and linear maps over some given field k. Given $V \in$ **Vect**, write V^* for the dual space (the space of k-linear maps from V to k). For $V \xrightarrow{f} W$, and $\theta \in W^*$, then $\theta \circ f \in V^*$. Defining $f^*(\theta) = \theta \circ f$ makes $(-)^*$ into a contravariant functor from **Vect** to **Vect**. Applying $(-)^*$ twice, we get the double-dual functor $(-)^{**}$, which is covariant.

Consider the map $V \xrightarrow{\alpha_V} V^{**}$ where for $x \in V$ $\alpha_V(x)$ is "evaluate at x". This defines a natural transformation $\begin{tikzcd} \mathbf{Vect} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "(-)^{**}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{Vect} \end{tikzcd}$ where Id is the identity functor on **Vect**. We must check the naturality condition, that the square

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\alpha_V"'] & W \arrow[d, "\alpha_W"] \\ V^{**} \arrow[r, "f^{**}"'] & W^{**} \end{tikzcd}$$

commutes. This is a simple exercise.

4.4 Determinant of a matrix

We show one way to consider the determinant of a matrix as a natural transformation. Let **CRing** be the category of commutative rings with unit (and ring homomorphisms) and **Grp** the category of groups (and group homomorphisms). For each $R \in \mathbf{CRing}$ and positive integer n, let $\mathrm{GL}_n(R)$ be the group of invertible $n \times n$ matrices with coefficients from R. For a ring homomorphism $R \xrightarrow{f} S$, we can define a map $\mathrm{GL}_n(R) \xrightarrow{\mathrm{GL}_n(f)} \mathrm{GL}_n(S)$ by letting f act separately on each coefficient of the matrix. This makes GL_n into a functor from **CRing** to **Grp**. (You should check that $\mathrm{GL}(f)$ really is a group homomorphism.)

For each $R \in \mathbf{CRing}$, write $\mathbb{G}_\mathrm{m}(R)$ for the multiplicative group of invertible elements of R. The restriction of a ring homomorphism $R \xrightarrow{f} S$ to $\mathbb{G}_\mathrm{m}(R)$ is a group homomorphism $\mathbb{G}_\mathrm{m}(R) \xrightarrow{f} \mathbb{G}_\mathrm{m}(S)$ (we also call it f). This makes $\mathbb{G}_\mathrm{m}$ into a functor from **CRing** to **Grp**.

For any matrix M with coefficients from R, its determinant $\det M$ is an element of R, and furthermore it is invertible precisely when the matrix M is invertible. It is also multiplicative, that is, for each $R \in \mathbf{CRing}$, the determinant defines a homomorphism $\mathrm{GL}_n(R) \xrightarrow{\det_R} \mathbb{G}_\mathrm{m}(R)$. So we can think of the determinant as a transformation from the group $\mathrm{GL}_n(R)$ to $\mathbb{G}_\mathrm{m}(R)$. To show that it is a natural transformation, we must show that

$$\begin{tikzcd}
\mathrm{GL}_n(R) \arrow[r, "\mathrm{GL}_n(f)"] \arrow[d, "\det_R"'] & \mathrm{GL}_n(S) \arrow[d, "\det_S"] \\
\mathbb{G}_\mathrm{m}(R) \arrow[r, "f"'] & \mathbb{G}_\mathrm{m}(S)
\end{tikzcd}$$

commutes. Now f is a ring homomorphism, so it commutes with addition and multiplication, and hence it commutes with all polynomials. In particular, since det is given by a polynomial, it commutes with det. This is enough to show that the naturality square commutes, and so det is a natural transformation.

4.5 An algebraic topology example

Algebraic topology was historically where category theory started, so this example is important from a historical point of view, but too complicated to give full details. Let **Top** be the category of topological spaces and continuous maps, and $\mathbf{Top}_*$ the category of topological spaces with a chosen basepoint, and basepoint-preserving continuous maps. We also have **Grp** and **Ab**, the categories of groups and abelian groups respectively. There are functors between them as follows:

$$\begin{tikzcd}
& \mathbf{Grp} \arrow[dr, "(-)_{ab}"] & \\
\mathbf{Top}_* \arrow[ur, "\pi_1"] \arrow[dr, "U"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \mathbf{Ab} \\
& \mathbf{Top} \arrow[ur, "H_1"'] &
\end{tikzcd}$$

There is an obvious "forgetful" functor $\mathbf{Top}_* \xrightarrow{U} \mathbf{Top}$ which just forgets the basepoint. Any group has an *abelianization* G^{ab}, and this assignment defines a functor from **Grp** to **Ab**. A topological space X with a given basepoint has a *fundamental group* $\pi_1(X)$, which is the group of loops based at that point, up to homotopy equivalence, with composition. We also have the first singular homology group $H_1(X)$ which is an abelian group constructed from maps of the circle into X.

A loop also defines a map from the circle into X, and one can use this idea to define a homomorphism from $\pi_1(X)^{\mathrm{ab}} \xrightarrow{\alpha_X} H_1(X)$. Then one can show that the homomorphisms α_X together form a natural transformation $(-)^{\mathrm{ab}} \circ \pi_1 \overset{\alpha}{\Rightarrow} H_1 \circ U$.

4.6 Functor Categories

Given categories $\mathcal{C}$ and $\mathcal{D}$, functors $F, G, H : \mathcal{C} \to \mathcal{D}$, and natural transformations $F \overset{\alpha}{\Rightarrow} G$ and $G \overset{\beta}{\Rightarrow} H$, there is a composite natural transformation $F \overset{\beta\circ\alpha}{\Rightarrow} H$ given by $(\beta \circ \alpha)_C = \beta_C \circ \alpha_C$ for each $C \in \mathcal{C}$. There is also an identity natural transformation on each functor. So for each pair of categories $\mathcal{C}$ and $\mathcal{D}$, there is a *functor category* $[\mathcal{C}, \mathcal{D}]$, whose objects are functors and whose arrows are natural transformations. Many categories can naturally be seen as functor categories.

4.6.1 Group representations

Recall that a group G can be viewed as a category with one object, in which every arrow is invertible. The category $[G, \mathbf{Vect}_{\mathbb{C}}]$ is the category of (complex) representations of G: a functor from G to $\mathbf{Vect}_{\mathbb{C}}$ is a complex vector space V together with an action of G on V via linear maps.

4.6.2 Order-preserving maps

Let X, Y be partially ordered sets (posets). Recall they can be considered as categories. A functor from X to Y as categories is the same as an order-preserving map. Suppose $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ are order-preserving maps. Then there is a natural transformation $f \overset{\alpha}{\Rightarrow} g$ if and only if for each $x \in X$, $f(x) \leqslant g(x)$, and then α is uniquely determined. So the functor category $[X, Y]$ is the poset of order-preserving maps from X to Y, with the pointwise ordering.

4.7 Natural isomorphisms

A *natural isomorphism* is an isomorphism in a functor category. In fact (see the exercises) a natural transformation $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$ is a natural isomorphism if and only if for each $C \in \mathcal{C}$, α_C is an isomorphism in $\mathcal{D}$.

4.7.1

Consider the forgetful functor $\mathbf{Grp} \xrightarrow{U} \mathbf{Set}$ which sends a group to its underlying set. Also consider the Hom-functor $H^{\mathbb{Z}}$, where $\mathbb{Z}$ is the infinite cyclic group, which sends a group G to $\mathbf{Grp}(\mathbb{Z}, G)$, the set of group homomorphisms from $\mathbb{Z}$ to G.

Define a function $\mathbf{Grp}(\mathbb{Z}, G) \xrightarrow{\alpha_G} UG$ by $\alpha_G(\theta) = \theta(1)$. Then α is a natural transformation $H^{\mathbb{Z}} \overset{\alpha}{\Rightarrow} U$ and in fact a natural isomorphism. We say $H^{\mathbb{Z}} \cong U$ or that $H^{\mathbb{Z}}(G)$ is isomorphic to UG, naturally in $G \in \mathbf{Grp}$.

4.7.2 Dual and double dual vector spaces

Let $\mathbf{FDVect}_k$ be the category of finite dimensional vector spaces over a field k. The dual space V^* of a finite dimensional vector space V has the same dimension as V, so for any fixed vector space V there is an isomorphism $V \xrightarrow{\theta_V} V^*$. Indeed, a basis of V has a dual basis of V^*, and identifying the two gives an isomorphism. However, the isomorphism is not a natural isomorphism. There is no way to choose an isomorphism θ_V for each V such that, if we change V to W by applying a linear map, we always get θ_W. Indeed, since the dual space functor is contravariant, we cannot even make sense of a natural isomorphism from Id to $(-)^*$. So the isomorphism depends on choices we make. However, the natural transformation α from a vector space to its double dual, given earlier, is in fact a natural isomorphism, provided that we restrict to finite dimensional vector spaces. So we say that a finite dimensional vector space is naturally isomorphic to its double dual, or that $V \cong V^{**}$, naturally in $V \in \mathbf{FDVect}_k$.

4.8 Equivalence of categories

Two categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic (in $\mathbf{Cat}$) iff there are functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ such that $FG = 1_{\mathcal{D}}$ and $GF = 1_{\mathcal{C}}$. It turns out that isomorphism of categories is a strong notion which is not very useful. A weaker notion called equivalence is more useful, where we do not require the composites FG and GF to be the identity functors, but only naturally isomorphic to the identity functors.

Precisely: an equivalence between $\mathcal{C}$ and $\mathcal{D}$ consists of functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ and natural isomorphisms $1_{\mathcal{C}} \overset{\eta}{\Rightarrow} GF$ and $FG \overset{\epsilon}{\Rightarrow} 1_{\mathcal{C}}$. We also say that F is an equivalence if some such G, η, and ϵ exist.

4.8.1 Matrices

Let $\mathbf{Mat}_k$ be the category with one object n for each natural number n, such that the arrows from n to m are $n \times m$ matrices with coefficients from the field k. Composition of arrows is given by matrix multiplication. There is a functor $\mathbf{Mat}_k \xrightarrow{F} \mathbf{FDVect}_k$ which takes the object n to k^n and a matrix M to the linear map represented by M with respect to the standard bases.
We can define a functor $\mathbf{FDVect}_k \xrightarrow{G} \mathbf{Mat}_k$ by sending V to $\dim V$, choosing a basis for each V and sending a linear map to the matrix which represents it with respect to the chosen bases.
Then it is easy to check that the composites FG and GF are naturally isomorphic to the identity functors, so the categories $\mathbf{Mat}_k$ and $\mathbf{FDVect}_k$ are equivalent. However, they are not isomorphic categories. For example, $\mathbf{Mat}_k$ is small (it has only a set of objects and arrows), and $\mathbf{FDVect}_k$ is not small.

4.8.2 Sets and cardinals

Let **Card** be the full subcategory of **Set** consisting of the cardinals. The inclusion functor $\mathbf{Card} \hookrightarrow \mathbf{Set}$ is an equivalence. (Both this example and the previous one actually require some set-theoretic assumptions, but we will not concern ourselves with them.)

4.9 Horizontal composition

We have seen one way to compose natural transformations:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[d, equal] \\
\mathcal{C} \arrow[r, "G"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & \mathcal{D} \arrow[d, equal] \\
\mathcal{C} \arrow[r, "H"'] & \mathcal{D}
\end{tikzcd}
\quad \text{gives rise to a \textit{vertical composite}} \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta\circ\alpha"] & \mathcal{D} \arrow[d, equal] \\
\mathcal{C} \arrow[r, "H"'] & \mathcal{D}
\end{tikzcd}$$

The vertical double lines are equalities. My LaTeX diagram package doesn't allow me to draw curved arrows!
We can also define a horizontal composition.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, "F'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha'"] & \mathcal{E} \arrow[d, equal] \\
\mathcal{C} \arrow[r, "G"'] & \mathcal{D} \arrow[r, "G'"'] & \mathcal{E}
\end{tikzcd}
\quad \text{gives rise to} \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, "F'F"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha'*\alpha"] & \mathcal{E} \arrow[d, equal] \\
\mathcal{C} \arrow[r, "G'G"'] & \mathcal{E}
\end{tikzcd}$$

defined by: $(\alpha' * \alpha)_C = \alpha'_{GC} \circ F'\alpha_C$.

4.10 Exercises

Q4.1 Show that a natural transformation $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$ is a natural isomorphism iff for each $C \in \mathcal{C}$, α_C is an isomorphism in $\mathcal{D}$.

Q4.2 For each group H, show that the map $G \mapsto H \times G$ defines a functor $\mathbf{Grp} \xrightarrow{H\times -} \mathbf{Grp}$, and for each group homomorphism $H \xrightarrow{f} K$, there is a natural transformation $H \times - \Rightarrow K \times -$.

Q4.3 A pointed set is a set with a chosen element (basepoint). Let $\mathbf{Set}_*$ be the category of pointed sets and basepoint-preserving functions. Show that $\mathbf{Set}_* \cong 1/\mathbf{Set}$. Show that $\mathbf{Set}_*$ is equivalent to the category of sets and partial functions.

Q4.4 Fix a field k. Let $\mathbf{Mat}_k$ be the category whose objects are natural numbers, with an arrow $n \longrightarrow m$ being an $n \times m$ matrix over k, and composition of arrows being matrix multiplication. Prove that $\mathbf{Mat}_k$ is equivalent to $\mathbf{FDVect}_k$. Is there a canonical choice of an equivalence?

Q4.5 A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is said to be *essentially surjective on objects* iff for every $B \in \mathcal{D}$, there is $A \in \mathcal{C}$ such that $FA \cong B$. Prove that a functor is an equivalence iff it is full, faithful, and essentially surjective on objects.

Q4.6 A *skeleton* of a category $\mathcal{C}$ is a full subcategory with exactly one object from each isomorphism class of $\mathrm{ob}\,\mathcal{C}$. Show that two categories are equivalent iff they have isomorphic skeletons.

Q4.7 Let G, H be groups, and $G \underset{g}{\overset{f}{\rightrightarrows}} H$ group homomorphisms, considered as categories and functors. What is a natural transformation $f \overset{\alpha}{\Rightarrow} g$? What if G, H are just monoids?

Q4.8 Let $\mathcal{C}$ be the category of finite sets and bijections. For $X \in \mathcal{C}$, let $\mathbf{Sym}(X)$ be the set of permutations of X and let $\mathbf{Ord}(X)$ be the set of total orderings of X.

(a) Give definitions of $\mathbf{Sym}$ and $\mathbf{Ord}$ on arrows of $\mathcal{C}$, to make them into functors $\mathcal{C} \longrightarrow \mathbf{Set}$. In each case there is only one sensible way to do it.

(b) Show that for each X, $\mathbf{Sym}(X) \cong \mathbf{Ord}(X)$.

(c) By considering the identity permutations, show there is no natural transformation $\mathbf{Sym} \Rightarrow \mathbf{Ord}$. Hence $\mathbf{Sym}$ and $\mathbf{Ord}$ are not naturally isomorphic, although they are pointwise isomorphic.

Q4.9 (a) For a given set S, define a constant functor $\mathbf{Set} \xrightarrow{S} \mathbf{Set}$ which takes any set to S and any function to 1_S. Check this really is a functor.

(b) Now let $\mathbf{Set} \xrightarrow{F} \mathbf{Set}$ be given by $F(X) = H^S(X) \times S$. Check this is the product functor $F = H^S \times S$.

(c) Let $H^S(X) \times S \xrightarrow{\mathrm{ev}_X} X$ be the evaluation map given by $\mathrm{ev}_X(p, s) = p(s)$, the value of the function p at the point s. Show that ev is a natural transformation $F \Rightarrow 1_{\mathbf{Set}}$.

Q4.10 Recall the definition of the horizontal composite $\alpha' * \alpha$ of two natural transformations: $(\alpha' * \alpha)_C = \alpha'_{GC} \circ F'\alpha_C$.

(a) Show that $\alpha' * \alpha$ is a natural transformation.

(b) There is an alternative way to define $\alpha' * \alpha$, by

$$(\alpha' * \alpha)_C = G'\alpha_C \circ \alpha'_{FC}$$

Show that the two definitions coincide.

(c) Prove the interchange law for natural transformations. That is, given

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, "F'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha'"] & \mathcal{E} \arrow[d, equal] \\
\mathcal{C} \arrow[r, "G"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & \mathcal{D} \arrow[r, "G'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta'"] & \mathcal{E} \arrow[d, equal] \\
\mathcal{C} \arrow[r, "H"'] & \mathcal{D} \arrow[r, "H'"'] & \mathcal{E}
\end{tikzcd}$$

show that $(\beta' * \beta) \circ (\alpha' * \alpha) = (\beta' \circ \alpha') * (\beta \circ \alpha)$.

5 Limits

Limits (and colimits) are the most common examples of universal properties in mathematics. We give some examples before formulating the general definition.

5.1 Products

We have seen that in many categories such as **Set**, **Grp**, **Ring**, **Top**, **Cat** etc., the product $X \times Y$ of objects X and Y comes with projection arrows

$$\begin{tikzcd} & X \\ X \times Y \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & Y \end{tikzcd}$$

and has the following universal property: If

$$\begin{tikzcd} & X \\ A \arrow[ur, "a_1"] \arrow[dr, "a_2"'] & \\ & Y \end{tikzcd}$$

is any object and pair of arrows to X and Y, then there is a unique arrow $A \xrightarrow{\bar{a}} X \times Y$ such that $a_1 = p_1\bar{a}$ and $a_2 = p_2\bar{a}$.

$$\begin{tikzcd} & & X \\ A \arrow[urr, "a_1"] \arrow[drr, "a_2"'] \arrow[r, dotted, "\exists!\bar{a}"] & X \times Y \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & & Y \end{tikzcd}$$

This is the general definition of a binary product in any category.
Similarly, we can define products of any family of objects $(X_j)_{j \in J}$ in a category. The product is written $\prod_{j \in J} X_j$, it has projection arrows $\prod_{j \in J} X_j \xrightarrow{p_i} X_i$ for each $i \in J$ such that if A is any object of the category and $(A \xrightarrow{a_j} X_j)_{j \in J}$ is a family of arrows, then there is an arrow $A \xrightarrow{\bar{a}} \prod_{j \in J} X_j$ such that for all $j \in J$, $a_j = p_j\bar{a}$
Example: in a poset, $\prod_{j \in J} x_j$ is written $\bigwedge_{j \in J} x_j$ and is the infimum of $\{x_j \mid j \in J\}$.

5.2 p-adic integers

We give a construction of the p-adic integers as a ring. Fix a prime p. There is a sequence of rings and ring homomorphisms:

$$\cdots \xrightarrow[h_n]{} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow[h_{n-1}]{} \mathbb{Z}/p^{n-1}\mathbb{Z} \longrightarrow \cdots \xrightarrow[h_1]{} \mathbb{Z}/p\mathbb{Z} \xrightarrow[h_0]{} 1$$

The homomorphisms are the obvious quotients. A p-adic integer is a sequence

$$\ldots, x_n, x_{n-1}, \ldots, x_1, x_0$$

with $x_n \in \mathbb{Z}/p^n\mathbb{Z}$, such that $h_n(x_n) = x_{n-1}$. The set of all p-adic integers forms a ring under pointwise addition and multiplication. It is written $\mathbb{Z}_p$. There are the obvious projection maps

$\mathbb{Z}_p \xrightarrow{q_n} \mathbb{Z}/p^n\mathbb{Z}$, which are ring homomorphisms, and for each $n \in \mathbb{N}^+$, $h_{n-1}q_n = q_{n-1}$. So all triangles in the following diagram commute. $\mathbb{Z}_p$ is called the *inverse limit* or *projective limit* of the sequence.

$$\begin{tikzcd}
\mathbb{Z}_p \arrow[dr, "q_n"'] \arrow[drr, "q_{n-1}"] \arrow[drrrr, "q_1"] \arrow[drrrrr, "q_0"] & & & & & \\
\cdots \arrow[r, "h_n"'] & \mathbb{Z}/p^n\mathbb{Z} \arrow[r, "h_{n-1}"'] & \mathbb{Z}/p^{n-1}\mathbb{Z} \arrow[r] & \cdots \arrow[r, "h_1"'] & \mathbb{Z}/p\mathbb{Z} \arrow[r, "h_0"'] & 1
\end{tikzcd}$$

There is a universal property: suppose that R is any commutative ring and there is a family of homomorphisms $R \xrightarrow{r_n} \mathbb{Z}/p^n\mathbb{Z}$ such that for each $n \in \mathbb{N}^+$, $h_{n-1}r_n = r_{n-1}$. Then there is a unique ring homomorphism $R \xrightarrow{\bar{r}} \mathbb{Z}_p$ such that for every $n \in \mathbb{N}$, $r_n = p_n\bar{r}$. Simplifying the diagram slightly:

$$\begin{tikzcd}
 & & R \arrow[dl, "r_n"'] \arrow[d, dotted, "\bar{r}"] \arrow[dr, "r_{n-1}"] & & \\
 & & \mathbb{Z}_p \arrow[dl, "q_n"] \arrow[dr, "q_{n-1}"'] & & \\
\cdots \arrow[r, "h_n"'] & \mathbb{Z}/p^n\mathbb{Z} \arrow[rr, "h_{n-1}"'] & & \mathbb{Z}/p^{n-1}\mathbb{Z} \arrow[r] & \cdots
\end{tikzcd}$$

5.3 General limits

The notion of a limit in category theory generalises these examples. Let $\mathcal{C}$ be a category, and $\mathbb{J}$ be a small category (the index category). Then a *diagram of shape* $\mathbb{J}$ *in* $\mathcal{C}$ is a functor $\mathbb{J} \xrightarrow{D} \mathcal{C}$.
A *cone* on D consists of

- an object A of $\mathcal{C}$,
- for each $J \in \mathbb{J}$, an arrow $A \xrightarrow{q_J} DJ$ in $\mathcal{C}$,

such that for each arrow $I \xrightarrow{k} J$ in $\mathbb{J}$, the "cone triangle" $\begin{tikzcd}[row sep=small] & DI \arrow[d, "Dk"] \\ A \arrow[ur, "q_I"] \arrow[dr, "q_J"'] & \\ & DJ \end{tikzcd}$ commutes. A cone on D is written as $(A \xrightarrow{q_J} DJ)_{J \in \mathbb{J}}$.
A *limit cone* $(L \xrightarrow{p_J} DJ)_{J \in \mathbb{J}}$ on D is a cone such that for any cone $(A \xrightarrow{q_J} DJ)_{J \in \mathbb{J}}$, there is a unique map $A \xrightarrow{\bar{q}} L$ such that for each $J \in \mathbb{J}$ we have $q_J = p_J \circ \bar{q}$. A diagram illustrating this with one arrow Dk is below, but of course the cone conditions require all such triangles to commute.

$$\begin{tikzcd}
 & & DI \arrow[dd, "Dk"] \\
A \arrow[urr, "q_I"] \arrow[drr, "q_J"'] \arrow[r, dotted, "\exists!\bar{q}"] & L \arrow[ur, "p_I"] \arrow[dr, "p_J"'] & \\
 & & DJ
\end{tikzcd}$$

Sometimes we say that L is the limit of D, and write $L = \varprojlim_{\mathbb{J}} D$. However, note that the projection maps p_J are essential, and the limit is really the whole cone, not just its vertex.
From the universal property of a limit, it follows that limits are unique up to unique isomorphism (if they exist).

5.4 Examples of shapes of limits

5.4.1 Products again

When $\mathbb{J}$ is a discrete category (a category with only identity arrows, essentially just a set of objects), the limit is just a product.

5.4.2 Terminal object

Suppose $\mathbb{J}$ is the empty category, with no objects and no arrows, and D is the unique functor from $\mathbb{J}$ to $\mathcal{C}$. Then a cone on D is just an object of $\mathcal{C}$, and a limit cone is an object L such that for any object $A \in \mathcal{C}$ there is exactly one arrow $A \longrightarrow L$. So L is a terminal object of $\mathcal{C}$.

5.4.3 Inverse limits

The construction of the p-adic integers is a limit where the shape $\mathbb{J}$ is the natural numbers considered as a poset, with the reverse of the usual ordering. Limits where $\mathbb{J}$ is a (downwards) directed poset are often known as inverse limits.

5.4.4 Pullbacks

Let $\mathbb{J}$ be the category with 3 objects and 2 non-identity arrows in the configuration:
$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$
. A diagram of shape $\mathbb{J}$ is then a pair of arrows:
$$\begin{tikzcd} & D_2 \arrow[d, "f_2"] \\ D_1 \arrow[r, "f_1"] & D_3 \end{tikzcd}$$
in $\mathcal{C}$, and a limit cone for D consists of
$$\begin{tikzcd} L \arrow[r, "p_2"] \arrow[d, "p_1"'] & D_2 \\ D_1 & \end{tikzcd}$$
such that $f_1 \circ p_1 = f_2 \circ p_2$ with the universal property. Note that the arrow $L \xrightarrow{p_3} D_3$ from the general definition is not explicitly needed, because it is equal to the composite $f_1 \circ p_1$.

The square
$$\begin{tikzcd} L \arrow[r, "p_2"] \arrow[d, "p_1"'] & D_2 \arrow[d, "f_2"] \\ D_1 \arrow[r, "f_1"] & D_3 \end{tikzcd}$$
is called a *pullback square*, and L is the *pullback* of D. In many contexts it is also known as the *fibre product* and written as $D_1 \times_{D_3} D_2$, but this notation does not include f_1 and f_2 which are essential, so it could be misleading unless the maps f_1 and f_2 are obvious.

5.4.5 Equalizers

Take $\mathbb{J}$ to be $\bullet \rightrightarrows \bullet$, so a diagram is a pair of parallel arrows $D_1 \overset{f_1}{\underset{f_2}{\rightrightarrows}} D_2$. A limit of D is an arrow $L \xrightarrow{p} D_1$ such that $f_1 p = f_2 p$ and universal such. The limit is called the *equalizer* of f_1 and f_2, for reasons we will see in section 5.6.

5.5 Pullbacks in geometry and category theory

Warning In geometry, the word *pullback* is used very widely and not always with the same meaning as given above, but usually there is some universal property at work.

In geometry or topology, a bundle (or fibre bundle) is a surjective continuous map $E \xrightarrow{\pi} B$ (usually with the extra topological property of being locally trivial). B is the base of the bundle, and the subspaces $\pi^{-1}(b)$ of E for $b \in B$ are the fibres.

If $B' \xrightarrow{f} B$ is any map, then we have a diagram

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

and we can form the pullback square in the category theoretic sense:

$$\begin{tikzcd} E' \arrow[r, "e"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

Then $E' \xrightarrow{\pi'} B'$ is a fibre bundle, called the *pullback of* π *along* f. Here the category-theoretic and geometric notions of pullback are the same. The bundle π' may be written as $f^*(\pi)$ in geometry. The expression *pullback along* f is also used in geometry just to mean precomposing with a map f, so if we have maps

$$\begin{tikzcd} C' \arrow[r, "f"] & C \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

then $f^*(\pi) = \pi \circ f$, is called a pullback of π along f, but it is not a pullback in the sense of category theory.
The two different settings are both ways in which some arrow π can be changed to another arrow by somehow combining it with a second arrow, f. These two methods of pullback can be combined with functors in quite complicated ways to make sense of pulling back something other than a continuous map along a map f. For example, if $M \xrightarrow{f} N$ is a smooth map of smooth manifolds, and ω is a differential 1-form on N, then there is a differential 1-form $f^*(\omega)$ on M which is called the pullback of ω along f.
There are (as of October 2010) some reasonable wikipedia articles about pullbacks in category theory and in differential geometry, where you can read more.

5.6 Limits in Set

The notion of a limit makes sense in any category, although of course limits do not necessarily exist in all categories. When the category is **Set**, or certain other concrete categories, then the limits do always exist and there is an easy expression for them. For example, if $D_1 \overset{f_1}{\underset{f_2}{\rightrightarrows}} D_2$ is an equalizer diagram in **Set**, then the equalizer is a subset of D_1 given by $L = \{x \in D_1 \mid f_1(x) = f_2(x)\}$, and the arrow $L \xrightarrow{p} D_1$ is just the inclusion of L into D_1. Hence the name equalizer: it picks out the subset where a certain equality holds.

If $\begin{tikzcd} & D_2 \arrow[d, "f_2"] \\ D_1 \arrow[r, "f_1"] & D_3 \end{tikzcd}$ is a pullback diagram in **Set**, then the pullback is

$$L = \{(d_1, d_2) \in D_1 \times D_2 \mid f_1(d_1) = f_2(d_2)\}$$

with the obvious projection maps.
Both these cases are easy to verify. They generalise to give an expression for the limit of any (small) diagram in **Set**:

$$\lim_{\leftarrow \mathbb{J}} D = \left\{ (x_J)_{J \in \mathbb{J}} \in \prod_{J \in \mathbb{J}} DJ \;\middle|\; \text{for each } I \xrightarrow{k} J \text{ in } \mathbb{J}, (Dk)(x_I) = x_J \right\}$$

again with the obvious projection maps: $L \xrightarrow{p_J} DJ$ is given by $p_J((x_I)_{I \in \mathbb{J}}) = x_J$.
This expression actually also works in other categories like **Grp**, **Ab**, and $\mathbf{Vect}_k$, as can be easily verified.

5.7 Complete categories

A category $\mathcal{C}$ is said to be *complete* if it has all limits; that is, for every small category $\mathbb{J}$ and every diagram $\mathbb{J} \xrightarrow{D} \mathcal{C}$ there is a limit cone for D in $\mathcal{C}$.
We have shown that **Set** is complete by explicitly constructing the limit of a diagram as a subset of a product, the subset being defined by certain equations. Using this idea we can give a criterion for any category to be complete.

Theorem. *A category $\mathcal{C}$ is complete if and only if it has all products and equalizers.*

Proof. The left-to-right direction is clear. Now suppose $\mathcal{C}$ has products and equalizers and D is a diagram in $\mathcal{C}$ of shape $\mathbb{J}$. Consider two products $\prod_{J\in\mathbb{J}} DJ$ and $\prod_{J\xrightarrow{k}K \text{ in } \mathbb{J}} DK$, one indexed by the objects of $\mathbb{J}$ and the other by the arrows. We want to define a parallel pair of arrows

$$\prod_{J\in\mathbb{J}} DJ \underset{v}{\overset{u}{\rightrightarrows}} \prod_{J\xrightarrow{k}K \text{ in } \mathbb{J}} DK$$

which we do using the universal property of the second product; recall that an arrow into a product is determined uniquely by an arrow into each of the factors. So we can define u by $p_k \circ u = p_K$ and v by $p_k \circ v = Dk \circ p_J$, for each $J \xrightarrow{k} K$ in $\mathbb{J}$. Now we have an equalizer diagram, and a cone on the diagram consists of an arrow $A \xrightarrow{a} \prod_{J\in\mathbb{J}} DJ$ such that $u \circ a = v \circ a$. Now, using the universal property of this product, such a cone corresponds to a family of arrows $(A \xrightarrow{a_J} DJ)_{J\in\mathbb{J}}$, and the equalizer property says that these arrows satisfy the equations $a_J \circ Dk = a_K$ for each $J \xrightarrow{k} K$ in $\mathbb{J}$. In other words, a cone on the equalizer diagram corresponds to a cone on D. Since $\mathcal{C}$ has products and equalizers, there is a limit cone for the equalizer diagram, and the universal property transfers to the corresponding cone on D. So D has a limit, and $\mathcal{C}$ is complete. □

5.8 Exercises

Q5.1 (i) Show that a terminal object 1 is a limit of the (unique) diagram of shape $\emptyset$ (the empty category).

(ii) Show that a category has all finite products iff it has a terminal object and binary products.

(iii) Show that a category has all finite limits iff it has all finite products and finite equalizers.

(iv) By considering pullback squares of the form

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[r] & 1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & B \arrow[d] \\ A \arrow[r] & B \times B \end{tikzcd}$$

show that a category has all finite limits iff it has a terminal object and all pullbacks.

Q5.2 Verify that for any diagram $\mathbb{J} \xrightarrow{D} \mathbf{Set}$, the limit is indeed given by

$$\varprojlim_{\mathbb{J}} D = \left\{ (x_J)_{J\in\mathbb{J}} \in \prod_{J\in\mathbb{J}} DJ \;\middle|\; \text{for each } I \xrightarrow{k} J \text{ in } \mathbb{J}, (Dk)(x_I) = x_J \right\}$$

with the obvious projection maps: $L \xrightarrow{p_J} DJ$ is given by $p_J((x_I)_{I\in\mathbb{J}}) = x_J$.

Q5.3 (i) Show that in any category, m is monic iff the left square below is a pullback square.

(ii) Suppose that the right square below is a pullback square and m is monic. Show that g is monic. ["Monics are stable under pullback."]

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"] & A \arrow[d, "m"] \\ A \arrow[r, "m"'] & B \end{tikzcd} \qquad \begin{tikzcd} B \times_A C \arrow[r, "f"] \arrow[d, "g"] & C \arrow[d, "m"] \\ B \arrow[r, "h"'] & A \end{tikzcd}$$

Q5.4 Let $\mathcal{C}$ be a category, $\mathbb{J}$ be a small category, and $A \in \mathcal{C}$. Let $\mathbb{J} \xrightarrow{\Delta A} \mathcal{C}$ be the constant functor taking every object of $\mathbb{J}$ to A and every arrow to 1_A. Show that a cone on a diagram D of shape $\mathbb{J}$ is the same thing as a natural transformation $\Delta A \Rightarrow D$.

Q5.5 Consider the commutative diagram below.

$$\begin{tikzcd} F \arrow[r, "f'"] \arrow[d, "h''"] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \end{tikzcd}$$

(i) Show that if the two squares are pullback squares, then so is the outer rectangle. Deduce that $A \times_C D \cong A \times_B (B \times_C D)$ whenever this makes sense.

(ii) Show that if the outer rectangle and the right square are pullbacks, then so is the left square.

Q5.6 An arrow $A \xrightarrow{m} B$ is called a *regular monic* iff there is an object C and arrows $B \underset{g}{\overset{f}{\rightrightarrows}} C$, such that $A \xrightarrow{m} B \underset{g}{\overset{f}{\rightrightarrows}} C$ is an equalizer diagram. m is a *split monic* iff there is a map $B \xrightarrow{s} A$ such that $sm = 1_A$. We define regular epics and split epics dually.

(i) Show that split monic $\implies$ regular monic $\implies$ monic.

(ii) Show that, in any category, an arrow is an isomorphism iff it is both monic and regular epic.

(iii) In **Ab**, show that all monics are regular but not all monics are split.

(iv) In **Top**, show that not all monics are regular and identify the regular monics.

(v) In the category of graphs, show that regular monics correspond to induced subgraphs.

Q5.7 Suppose that $\mathcal{C}$ is a complete category (that is, it has all limits whose shape is a small category), and suppose also that $\mathcal{C}$ is itself small. Show that $\mathcal{C}$ must be a preorder. [This question assumes some knowledge of set theory.]

6 Colimits

A *colimit* in $\mathcal{C}$ is a limit in $\mathcal{C}^{\mathrm{op}}$. In principle that is sufficient definition, but it is worth giving the details.

6.1 Definition of colimit

So, a *cocone* for a diagram $\mathbb{J} \xrightarrow{D} \mathcal{C}$ consists of

- an object A of $\mathcal{C}$,
- for each $J \in \mathbb{J}$, an arrow $DJ \xrightarrow{q_J} A$ in $\mathcal{C}$,

such that for each arrow $I \xrightarrow{k} J$ in $\mathbb{J}$, the triangle
$$\begin{tikzcd} DI \arrow[d, "Dk"'] \arrow[dr, "q_I"] & \\ DJ \arrow[r, "q_J"'] & A \end{tikzcd}$$
commutes. A cocone on D is written as $(DJ \xrightarrow{q_J} A)_{J \in \mathbb{J}}$.

A *colimit cocone* $(DJ \xrightarrow{p_J} L)_{J \in \mathbb{J}}$ on D is a cocone such that for any cocone $(DJ \xrightarrow{q_J} A)_{J \in \mathbb{J}}$, there is a unique map $L \xrightarrow{\bar{q}} A$ such that for each $J \in \mathbb{J}$ we have $q_J = \bar{q} \circ p_J$. The diagram illustrating this corresponding to the limit diagram earlier looks like this.

$$\begin{tikzcd} DI \arrow[dd, "Dk"'] \arrow[dr, "p_I"] \arrow[drr, "q_I"] & & \\ & L \arrow[r, dotted, "\exists! \bar{q}"] & A \\ DJ \arrow[ur, "p_J"] \arrow[urr, "q_J"'] & & \end{tikzcd}$$

We say that L is the colimit of D, and write $L = \lim\limits_{\to \mathbb{J}} D$. (Note that the arrow goes to the right.)
Again, note that the projection maps p_J are essential, and the colimit is really the whole cocone, not just its vertex.

6.2 Examples

6.2.1 Coproducts

If $\mathbb{J}$ is a discrete category, a diagram $\mathbb{J} \xrightarrow{D} \mathcal{C}$ is just a collection of objects of $\mathcal{C}$ and the colimit is called a *coproduct*, written $\coprod_{J \in \mathbb{J}} DJ$.
In **Set**, coproducts are disjoint unions.
In **Ab** and **Vect**, coproducts are direct sums.
In **Grp**, coproducts are free products.
In a poset, coproducts are suprema.

6.2.2 Direct limits

If $\mathbb{J}$ is the ordered set ω of natural numbers with the usual order, then a diagram of shape ω is a chain

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

The colimit is then known as a *direct limit*.

6.2.3 Pushouts

The dual of a pullback is a *pushout*. Thus a pushout diagram is a pair of arrows:
$$\begin{tikzcd} D_1 \arrow[r, "f_2"] \arrow[d, "f_3"] & D_2 \\ D_3 & \end{tikzcd}$$
and a colimit cocone for D consists of
$$\begin{tikzcd} & D_2 \arrow[d, "p_2"'] \\ D_3 \arrow[r, "p_3"'] & L \end{tikzcd}$$
such that $p_3 \circ f_3 = p_2 \circ f_2$ with the universal property.

6.2.4 Coequalizers

When $\mathbb{J}$ is a parallel pair of arrows $\bullet \rightrightarrows \bullet$, a colimit is called a *coequalizer*.

In **Set**, the coequalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is the set of equivalence classes of elements of B under the equivalence relation generated by $f(x) \sim g(x)$ for each $x \in A$. So whereas the equalizer picks out the largest subset of A where the equation holds, the coequalizer is the largest quotient of B which forces the equation to hold.

In **Grp**, let $N \lhd G$, a normal subgroup, and consider the pair of homomorphisms

$$N \overset{\iota}{\underset{e}{\rightrightarrows}} G$$

where ι is the inclusion of N into G and e is the trivial map. The coequalizer of the diagram is

$$N \overset{\iota}{\underset{e}{\rightrightarrows}} G \xrightarrow{q} G/N$$

where q is the quotient map.

6.2.5 Manifolds

Let $M \in \mathbf{Top}$ be a manifold, considered as a topological space, and let $(U_i)_{i \in I}$ be its atlas of charts. Then we can recognise M as a coequalizer of coproducts:

$$\coprod_{(i,j)\in I^2} (U_i \cap U_j) \rightrightarrows \coprod_{i \in I} U_i \longrightarrow X$$

where the pair of arrows are the inclusions of each $(U_i \cap U_j)$ into U_i and U_j.

6.3 Cocomplete categories

We have seen that **Set** has coproducts (which are disjoint unions) and coequalizers (which are quotients). Thus $\mathbf{Set}^{\mathrm{op}}$ has products and equalizers by duality. So by Theorem 5.7, $\mathbf{Set}^{\mathrm{op}}$ has all limits, and hence by duality, **Set** has all colimits. We say it is *cocomplete*. More generally we have:

Theorem. *If $\mathcal{C}$ has all coproducts and coequalizers then it has all colimits.*

6.4 Colimits in Set

The same idea tells us how to compute colimits in **Set**. For any diagram $\mathbb{J} \xrightarrow{D} \mathbf{Set}$, the colimit is given as $\coprod_{J \in \mathbb{J}} DJ \big/ \sim$ where $\sim$ is the equivalence relation generated by $x \sim (Dk)(x)$ for each arrow $I \xrightarrow{k} J$ of $\mathbb{J}$ and each point $x \in DI$.

Note that although the expression for computing limits in **Set** also works in algebraic categories like **Grp**, but this expression for computing colimits does not work in **Grp**. This is because coproducts in **Grp** (and in other algebraic categories) do not look like disjoint unions.

6.5 Exercises

Q6.1 Prove directly (without using duality) that a category has all colimits iff it has all coproducts and coequalizers.

Q6.2 Give dual versions of the statements in question 5.1.

Q6.3 Let R be a commutative ring (with unit). Show that the coproduct of M and N in the category R-**Mod** of R-modules is the tensor product $M \otimes_R N$. Do coproducts exist in other categories you use in your work? If so, what are they?

Q6.4 Let G be a group and let J_G be the partially ordered set of all finitely generated subgroups of G, ordered by inclusion. Show that G is the colimit of the obvious functor $J_G \longrightarrow \mathbf{Grp}$.

7 Adjunctions

7.1 Vector spaces bases

Recall the universal property of the basis of a vector space. Let $W, V \in \mathbf{Vect}$, the category of k-vector spaces (k some field), let B be a basis of W, $\mathbf{Vect} \xrightarrow{U} \mathbf{Set}$ the forgetful functor, and let $B \xrightarrow{b} UW$ in $\mathbf{Set}$ be the inclusion map.
For each function $B \xrightarrow{f} UV$, there is a unique k-linear map $W \xrightarrow{\hat{f}} V$ extending f, that is, such that $U\hat{f} \circ b = f$.

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[dr, "f"'] & UW \arrow[d, dotted, "U\hat{f}"] & W \arrow[d, dotted, "\hat{f}"] \\ & UV & V \end{tikzcd}$$

So for any $V \in \mathbf{Vect}$, we have a bijection between Hom-sets:

$$\begin{aligned} \mathbf{Vect}(W, V) &\xrightarrow{\beta_V} \mathbf{Set}(B, UV) \\ \varphi &\longmapsto U\varphi \circ b \end{aligned}$$

Now both Hom-sets can be viewed as functors $\mathbf{Vect} \longrightarrow \mathbf{Set}$ in the variable V: $\mathbf{Vect}(W, -)$ is a Hom-functor, and $\mathbf{Set}(B, U-)$ is the composite of a Hom-functor with U. So we can ask if the family $(\beta_V)_{V \in \mathbf{Vect}}$ is a natural isomorphism. We have to check the naturality square: if $V_1 \xrightarrow{\theta} V_2$ is a linear map, then

$$\begin{tikzcd} \mathbf{Vect}(W, V_1) \arrow[r, "\theta^*"] \arrow[d, "\beta_{V_1}"'] & \mathbf{Vect}(W, V_2) \arrow[d, "\beta_{V_2}"] \\ \mathbf{Set}(B, UV_1) \arrow[r, "(U\theta)^*"] & \mathbf{Set}(B, UV_2) \end{tikzcd}$$

should commute, which it does:

$$\begin{tikzcd} \varphi \arrow[r, mapsto] \arrow[dd, mapsto] & \theta \circ \varphi \arrow[d, mapsto] \\ & U(\theta \circ \varphi) \circ b \\ U\varphi \circ b \arrow[r, mapsto] & U\theta \circ (U\varphi \circ b) \end{tikzcd}$$

by associativity and functoriality of U.

7.2 Universal arrows

We generalise this example to give the notion of a universal arrow.
Let $\mathcal{D} \xrightarrow{G} \mathcal{C}$ be a functor, and $C \in \mathcal{C}$. A *universal arrow from C to G* is a pair $(\Delta, C \xrightarrow{u} G\Delta)$, where $\Delta \in \mathcal{D}$ and c in $\mathcal{C}$, such that for every pair $(D, C \xrightarrow{f} GD)$ there is a unique arrow $\Delta \xrightarrow{\hat{f}} D$ in $\mathcal{D}$ such that $G\hat{f} \circ u = f$.

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[dr, "f"'] & G\Delta \arrow[d, dotted, "G\hat{f}"] & \Delta \arrow[d, dotted, "\hat{f}"] \\ & GD & D \end{tikzcd}$$

The proof we did in the special case of vector space bases shows that a universal arrow gives rise to a natural isomorphism $\mathcal{D}(\Delta, -) \overset{\beta}{\Rightarrow} \mathcal{C}(C, G-)$ with β_D taking s to $Gs \circ u$.
In fact, all the universal properties we have considered can be expressed in terms of universal arrows (in some cases, the dual notion).

7.3 Vector spaces bases 2

So far we have considered a fixed vector space W, and basis B of W. In fact, for any set X, we can form a vector space FX such that X is a basis of FX (take formal linear combinations of elements of X). Furthermore F is a functor $\mathbf{Set} \xrightarrow{F} \mathbf{Vect}$, because if $X \xrightarrow{f} Y$ is any function, then it extends uniquely to a linear map $FX \xrightarrow{Ff} FY$, by the universal property of a basis. Let b_X be the inclusion of X into FX. Now our bijection from the universal arrow looks like

$$\begin{aligned} \mathbf{Vect}(FX, V) &\xrightarrow{\beta_{X,V}} \mathbf{Set}(X, UV) \\ \varphi &\longmapsto U\varphi \circ b_X \end{aligned}$$

and we know that $\beta_{X,V}$ is natural in $V \in \mathbf{Vect}$. What about naturality in $X \in \mathbf{Set}$?
We can view $\mathbf{Vect}(F-, V)$ as a contravariant functor: the composite of F with a contravariant Hom-functor. Similarly, $\mathbf{Set}(-, UV)$ is a contravariant Hom-functor. So for a function $X \xrightarrow{f} Y$, we must show that the naturality square

$$\begin{tikzcd} \mathbf{Vect}(FY, V) \arrow[r, "Ff_*"] \arrow[d, "\beta_{Y,V}"'] & \mathbf{Vect}(FX, V) \arrow[d, "\beta_{X,V}"] \\ \mathbf{Set}(Y, UV) \arrow[r, "f_*"] & \mathbf{Set}(X, UV) \end{tikzcd}$$

commutes. Taking $\varphi \in \mathbf{Vect}_{FY,V}$ and chasing it round the top and then the bottom, we must show that $U\varphi \circ UFf \circ b_X = U\varphi \circ b_Y \circ f$ as functions $X \longrightarrow UV$. But it is easy to see that applying the two expressions to any element $x \in X$ gives the same result, so the naturality square does commute. So in fact the bijection $\mathbf{Vect}(FX, V) \overset{\beta_{X,V}}{\Longrightarrow} \mathbf{Set}(X, UV)$ is natural both in $V \in \mathbf{Vect}$ and in $X \in \mathbf{Set}$. This is an example of the general pattern known as an adjunction.

7.4 Definition of adjunction

Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be a pair of categories and functors. We say that F is left adjoint to G, and G is right adjoint to F, and write $F \dashv G$, iff

$$\mathcal{C}(A, GB) \cong \mathcal{D}(FA, B)$$

naturally in $A \in \mathcal{C}$ and in $B \in \mathcal{D}$. An *adjunction* is a choice of natural isomorphism. If we want to name the adjunction we might write $F \overset{\beta}{\dashv} G$, and we also write $\mathcal{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}$.

The naturality conditions are: for any $A_1 \xrightarrow{f} A_2$ in $\mathcal{C}$ and any $B_1 \xrightarrow{g} B_2$ in $\mathcal{D}$,

$$\begin{tikzcd} \mathcal{C}(A_2, GB) \arrow[r, "f^*"] \arrow[d, "\beta_{A_2,B}"'] & \mathcal{C}(A_1, GB) \arrow[d, "\beta_{A_1,B}"] \\ \mathcal{D}(FA_2, B) \arrow[r, "(Ff)^*"'] & \mathcal{D}(FA_1, B) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{C}(A, GB_1) \arrow[r, "(Gg)_*"] \arrow[d, "\beta_{A,B_1}"'] & \mathcal{C}(A, GB_2) \arrow[d, "\beta_{A,B_2}"] \\ \mathcal{D}(FA, B_1) \arrow[r, "g_*"'] & \mathcal{D}(FA, B_2) \end{tikzcd}$$

commute. Note that we can view both $\mathcal{C}(A, GB)$ and $\mathcal{D}(FA, B)$ as functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathbf{Set}$. It can be shown (using the universal property of the product of categories) that naturality in $A \in \mathcal{C}$ and in $B \in \mathcal{D}$ separately is the same as naturality in $(A, B) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$.

7.5 Free and forgetful functors

Many mathematical operations can be seen as adjoint functors. And often, if a functor has an adjoint, then that adjoint will turn out to be a useful operation.

- Our prototypical example: $\mathbf{Vect}_k \underset{F}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{Set}$, where U takes a vector space to its underlying set ("forgetting" the vector space structure), and F takes a set X to a vector space having X as a basis – the free vector space on generators X. The free functor is left adjoint to the forgetful functor.

  There are many similar examples where forgetting some algebraic structure has a left adjoint, which is the operation "forming the free object".

- $\mathbf{Grp} \underset{F}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{Set}$, where FX is the free group on generators X.

- The forgetful functor $\mathbf{Ring} \xrightarrow{U} \mathbf{Monoid}$, where $U(R, +, \cdot) = (R, \cdot)$, forgetting the addition but not the multiplication, has a left adjoint $\mathbf{Monoid} \xrightarrow{\mathbb{Z}\cdot -} \mathbf{Ring}$ where $\mathbb{Z} \cdot M$ is the ring whose elements are formal finite sums of elements of M: $\sum \lambda_i m_i$ ($\lambda_i \in \mathbb{Z}, m_i \in M$). Addition is formal sum, and multiplication is obvious. If M is a group, this is called the *group ring* construction.

- Similarly, if G is an associative algebra (a k-vector space with an associative multiplication) we can form its Lie algebra LG (the same k-vector space with a Lie bracket given by $[x, y] = x \cdot y - y \cdot x$). This functor L has a left adjoint $\mathcal{U}$ which takes a Lie algebra to its *universal enveloping algebra*.

- The forgetful functor $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$ has a left adjoint, which gives a set X the discrete topology. By analogy we might think of this as the free topology on a set. U also has a right adjoint, which gives X the indiscrete topology.

7.6 Reflective and coreflective subcategories

Let $\mathbf{Ab} \xrightarrow{U} \mathbf{Grp}$ be the inclusion functor of the subcategory of abelian groups in the category of groups. Then U has a left adjoint, $(-)^{\mathrm{ab}}$, which takes a group G to its abelianization G^{ab}, which is the largest abelian quotient of G.
A subcategory whose inclusion functor has a left adjoint F is called a *reflective subcategory*. F is called the *reflector*.

- The inclusion $\mathbf{Grp} \xrightarrow{U} \mathbf{Monoid}$ has a left adjoint ("Groupification"), and also a right adjoint (submonoid of invertibles). So it is both a reflective and coreflective subcategory.

- Let $\mathbf{Met}$ be the category of metric spaces and metric-preserving functions, and $\mathbf{CMet}$ the subcategory of complete metric spaces. The inclusion U has a left adjoint which takes a metric space to its completion.

- The inclusion $\mathbf{CHaus} \xrightarrow{U} \mathbf{Top}$ of the subcategory of compact Hausdorff topological spaces into the category of topological spaces has a left adjoint called Stone-Čech compactification.

7.7 Composition of adjunctions

There are diagrams of forgetful functors and their left adjoints:

$$
\begin{tikzcd}
\mathbf{Ab} \arrow[r] \arrow[dr] & \mathbf{Grp} \arrow[d] \\
& \mathbf{Set}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Ab} & \mathbf{Grp} \arrow[l, "(-)^{\mathrm{ab}}"'] \\
& \mathbf{Set} \arrow[u, "F"'] \arrow[ul, "F'"]
\end{tikzcd}
$$

where F takes a set X to the free group on generators X and F' takes X to the free abelian group on those generators. In fact the diagrams commute, so $F' = (-)^{\mathrm{ab}} \circ F$. This is a general pattern.

Lemma. *If* $\mathcal{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}$ *and* $\mathcal{D} \underset{G'}{\overset{F'}{\underset{\perp}{\rightleftarrows}}} \mathcal{E}$ *are pairs of adjoint functors, then the composite functors are also adjoint:* $F'F \dashv GG'$*.*

Proof. Compose the natural isomorphisms:

$$\mathcal{E}(F'FA, B) \cong \mathcal{D}(FA, G'B) \cong \mathcal{C}(A, GG'B)$$

and recall that the composite of natural isomorphisms is a natural isomorphism. □

For any category $\mathcal{C}$ there is an identity functor $1_{\mathcal{C}}$, and we have $\mathcal{C} \underset{1_C}{\overset{1_{\mathcal{C}}}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}$. So in fact there is a category **Adj** whose objects are small categories and whose arrows are adjunctions. By convention, the adjunction is an arrow in the direction of the left adjoint functor.

7.8 Unit and Counit of an adjunction

Fix an adjunction $\mathcal{C}(A, GB) \xrightarrow[\cong]{\beta_{A,B}} \mathcal{D}(FA, B)$.
Putting $B = FA$, we get an arrow $A \xrightarrow{\eta_A} GFA$ given by $\eta_A = \beta^{-1}_{A,FA}(1_{FA})$. The naturality condition of the adjunction implies that $1_{\mathcal{C}} \overset{\eta}{\Rightarrow} GF$ is a natural transformation, called the *unit* of the adjunction.
Similarly, putting $A = GB$, we have $FGB \xrightarrow{\epsilon_B} B$ with $\epsilon_B = \beta_{GB,B}(1_{GB})$ giving a natural transformation $FG \overset{\epsilon}{\Rightarrow} 1_{\mathcal{D}}$, called the *counit* of the adjunction.

Lemma (Triangle Identities)**.** *The following two triangles involving the unit and counit of an adjunction commute:*

$$\begin{tikzcd} FA \arrow[r, "F\eta_A"] \arrow[dr, "1_{FA}"'] & FGFA \arrow[d, "\epsilon_{FA}"] \\ & FA \end{tikzcd} \qquad \begin{tikzcd} GB \arrow[r, "\eta_{GB}"] \arrow[dr, "1_{GB}"'] & GFGB \arrow[d, "G\epsilon_B"] \\ & GB \end{tikzcd}$$

Proof. Exercise □

In our example $\mathbf{Vect}_k \underset{F}{\overset{U}{\underset{\top}{\rightleftarrows}}} \mathbf{Set}$, the unit is a function $X \xrightarrow{\eta_X} UFX$. Recall that FX is the vector space of formal linear combinations of elements of X, and UFX is the same thing considered just as a set. In fact η_X is just the inclusion of X into UFX, which is the universal arrow we earlier called b_X.
The counit is a linear map $FUV \xrightarrow{\epsilon_V} V$ which takes a formal finite linear combination $\sum \lambda_i v_i$ of elements of V and evaluates it as an element of V.
From the point of view of mathematical logic, in this and similar algebraic examples we can see the unit and counit as important operations relating the syntax of the algebra with the semantics.

7.9 Equivalence of categories

Suppose that $\mathcal{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}$ is an adjunction in which both the unit and counit are invertible, so they are natural isomorphisms $1_{\mathcal{C}} \overset{\eta}{\Rightarrow} GF$ and $FG \overset{\epsilon}{\Rightarrow} 1_{\mathcal{C}}$. This is precisely the definition of an equivalence of categories.

7.10 Summary of adjunctions

Adjunctions crop up in many different guises, and it is useful to be able to recognise them in any of their guises. The following theorem gives six equivalent formulations.

Theorem. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An adjunction* $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\beta\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$ *is determined by any one of the following six things.*

(a) The functors F and G, and the isomorphism $\mathcal{C}(A, GB) \xrightarrow[\cong]{\beta_{A,B}} \mathcal{D}(FA, B)$, natural in $A \in \mathcal{C}$ and $B \in \mathcal{D}$.

(b) The functors F and G and natural transformations $1_{\mathcal{C}} \xrightarrow{\eta} GF$ and $FG \xrightarrow{\epsilon} 1_{\mathcal{D}}$ satisfying the triangle identities.

(c) The functors F and G and a natural transformation $1_{\mathcal{C}} \xrightarrow{\eta} GF$ such that, for each $A \in \mathcal{C}$, $A \xrightarrow{\eta_A} GFA$ is a universal arrow from A to G.

(d) The functors F and G and a natural transformation $FG \xrightarrow{\epsilon} 1_{\mathcal{D}}$ such that, for each $B \in \mathcal{D}$, $FGB \xrightarrow{\epsilon_B} B$ is a universal arrow from F to B.

(e) The functor G and for each object $A \in \mathcal{C}$, an object $F_0A \in \mathcal{D}$ and an arrow $A \xrightarrow{\eta_A} GF_0A$ which is a universal arrow from A to G.

(f) The functor F and for each object $B \in \mathcal{D}$, an object $G_0B \in \mathcal{C}$ and an arrow $FG_0B \xrightarrow{\epsilon_B} B$ which is a universal arrow from F to B.

Note that in (e) we do not need to know in advance that F is a functor, we only need its object function. Similarly in (f) we only need the object function of G.

For the proof, see p83 of Mac Lane's Categories for the Working Mathematician. We have seen how to prove most of it already.

7.11 Exercises

Q7.1 Show that if F and F' are both left adjoints to G then there is a natural isomorphism $F \cong F'$. So adjoints are unique up to isomorphism.

Q7.2 Show that if $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is an equivalence of categories, then there is a functor $\mathcal{D} \xrightarrow{G} \mathcal{C}$ such that F is a left adjoint of G, and F is also a right adjoint of G.

Hint: use the characterization of an equivalence by F being full, faithful, and essentially surjective on objects.

Q7.3 **Universal arrows as initial objects** Consider categories and functors $\mathcal{A} \xrightarrow{T} \mathcal{C}$ and $\mathcal{B} \xrightarrow{S} \mathcal{C}$. The *comma category* written $(T \Rightarrow S)$ or $(T \downarrow S)$ [originally written (T, S), hence the name] has

- objects: triples (A, h, B) with $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $TA \xrightarrow{h} SB$ in $\mathcal{C}$
- arrows $(A, h, B) \longrightarrow (A', h', B')$ are pairs $(f, g) \in \mathcal{A}(A, A') \times \mathcal{B}(B, B')$ such that

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "h"'] & TA' \arrow[d, "h'"] \\ SA \arrow[r, "Sg"'] & SA' \end{tikzcd}$$ commutes.

(i) Show that the slice and coslice constructions (see Question 1.7(iii)) are special cases of comma categories.

(ii) Show that $(\Delta, C \xrightarrow{u} G\Delta)$ is a universal arrow from C to G iff (C, u, Δ) is an initial object in the comma category $(C \Rightarrow G)$. [Here C is the functor $\mathbf{1} \longrightarrow \mathcal{C}$ with value C.]

Q7.4 Show that the forgetful functor R-**Mod** $\xrightarrow{U}$ **Ab** has a left adjoint, $A \mapsto R \otimes A$ and also a right adjoint, $A \mapsto \mathbf{Ab}(R, A)$.

Q7.5 Show that the following full subcategories are reflective:

(i) the subcategory of **Ab** given by the torsion-free abelian groups,

(ii) the subcategory of partially ordered sets in the category **Quoset** of quasi-ordered sets (pre-ordered sets) and order-preserving functions,

(iii) the subcategory of T_0-spaces in **Top**. [A T_0-space is one which satisfies the T_0-separation property: if x, y are points contained in exactly the same open sets, then $x = y$.]

Q7.6 Show that the forgetful functor **Ring** $\xrightarrow{U}$ **Set** has a left adjoint F, and describe it. Show that the adjunction $F \dashv U$ can be written as a composite adjunction in two ways: when U factors through **Ab** and when U factors through **Monoid**.

Q7.7 (i) Let $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ be a pair of order-preserving maps between posets. Show that there is at most one adjunction $f \dashv g$ and describe explicitly when there is one. [Hint: use the triangle identities.]

(ii) Let $A \xrightarrow{p} B$ be a function in **Set**, and let $\mathcal{P}B \xrightarrow{p^*} \mathcal{P}A$ be the induced map of powersets, considered as posets. Give left and right adjoints of p^*.

Q7.8 Let **Cat** $\xrightarrow{O}$ **Set** be the functor taking a small category to its set of objects (and a functor to its object map). Exhibit a chain of adjoints

$$C \dashv D \dashv O \dashv I$$

and prove that C has no left adjoint and I has no right adjoint.

Q7.9 **Galois connections.** We consider posets X, Y and order-reversing maps giving an adjunction

$$X \underset{g}{\overset{f}{\underset{\perp}{\rightleftarrows}}} Y^{\mathrm{op}}.$$

(i) Let $K \subseteq L$ be a field extension, and consider the poset of intermediate fields and the poset of subgroups of $\mathrm{Gal}(L/K)$. Show the maps "invariant subgroup" and "fixed field" are adjoint. (This is the fundamental theorem of Galois theory.)

(ii) Let K be an algebraically closed field, and $n \in \mathbb{N}$. Consider the poset of subsets of K^n and the poset of subsets of the ring $K[X_1, \ldots, X_n]$. Describe obvious order-reversing maps in each direction, and thus show that the Hilbert Nullstellensatz is an example of a Galois connection.

8 Limit Preservation

8.1 Examples and definition

If $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are the cyclic groups with two and three elements, then the product $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ in the category **Grp** (or in **Ab**) is a group with 6 elements, and its underlying set is the product of the two underlying sets. Similarly for any product of groups. So we say that the forgetful functor $\mathbf{Grp} \xrightarrow{U} \mathbf{Set}$ *preserves* products. (In fact, we will see that it preserves all limits.)
By contrast, the coproduct $\mathbb{Z}/2\mathbb{Z} \amalg \mathbb{Z}/3\mathbb{Z}$ in the category **Grp** is an infinite group, but the coproduct in **Set** of the underlying sets is just the disjoint union, which has 5 elements. The coproduct in the category **Ab** is the direct sum $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ (isomorphic to the product $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$) whose underlying set has 6 elements. So we see that the forgetful functors $\mathbf{Grp} \xrightarrow{U} \mathbf{Set}$, $\mathbf{Grp} \xrightarrow{U} \mathbf{Ab}$ and $\mathbf{Ab} \xrightarrow{U} \mathbf{Set}$ do not preserve coproducts.

Now the general definition: a functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is said to *preserve limits* of shape $\mathbb{J}$ iff whenever $\mathbb{J} \xrightarrow{D} \mathcal{A}$ is a diagram in $\mathcal{A}$ and $(L \xrightarrow{p_J} DJ)_{J\in\mathbb{J}}$ is a limit cone of D, then $(FL \xrightarrow{Fp_J} FDJ)_{J\in\mathbb{J}}$ is a limit cone of the diagram $\mathbb{J} \xrightarrow{FD} \mathcal{B}$. A functor which preserves all (small) limits is called *continuous.*
Dually one obtains the definition of F preserving colimits.
Informally, we write $F(\lim\limits_{\leftarrow\mathbb{J}} D) = \lim\limits_{\leftarrow\mathbb{J}} FD$, but F must preserve the whole limit cone, not just its vertex.

8.2 Right adjoints preserve limits

Note that the forgetful functors above which preserve products have left adjoints. This is a general pattern.

Theorem. *Let* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *be an adjunction* ($F \dashv G$)*. Then F preserves colimits and G preserves limits.*

Proof. It is enough to show that G preserves limits. Then, by duality, F preserves colimits.
Let $\mathbb{J} \xrightarrow{D} \mathcal{D}$ be a diagram, and suppose $(L \xrightarrow{p_J} DJ)_{J\in\mathbb{J}}$ is a limit cone. We must show that $(GL \xrightarrow{Gp_J} GDJ)_{J\in\mathbb{J}}$ is a limit cone for $\mathbb{J} \xrightarrow{GD} \mathcal{C}$. It is a cone, since the functor G preserves the commutativity of the necessary cone triangles: $Gp_J = GDk \circ Gp_I$ for any $I \xrightarrow{k} J$ in $\mathbb{J}$.
So let $(A \xrightarrow{a_J} GDJ)_{J\in\mathbb{J}}$ be any cone on GD. Apply the adjunction isomorphism

$$\mathcal{C}(A, GDJ) \xrightarrow{\simeq} \mathcal{D}(FA, DJ)$$

to get a family of arrows $(FA \xrightarrow{\bar{a}_J} DJ)_{J\in\mathbb{J}}$. (We write a bar to denote the image under the adjunction.) Consider the following diagram, where the left double lines are equalities, inserted to spread the diagram out and make it easier to read.

$$\begin{tikzcd}
FA \arrow[rr, "\bar{a}_I"] \arrow[dd, equal] & & DI \arrow[dddd, "Dk"] \\
& FGDI \arrow[ur, "\epsilon_{DI}"'] \arrow[dd, "FGDk"] & \\
FA \arrow[ur, "Fa_I"] \arrow[dr, "Fa_J"'] \arrow[dd, equal] & & \\
& FGDJ \arrow[dr, "\epsilon_{DJ}"] & \\
FA \arrow[rr, "\bar{a}_J"'] & & DJ
\end{tikzcd}$$

The triangle in the left middle of the diagram commutes because it is the image under F of a cone triangle of the cone $(A \xrightarrow{a_J} GDJ)_{J\in\mathbb{J}}$. The trapesium in the right middle is the naturality square of

the counit ϵ. By definition, $\bar{a}_I$ is the image of a_I under the adjunction isomorphism, which is equal to the composite $\epsilon_{DI} \circ Fa_I$, using the counit as a universal arrow. So the top triangle commutes, and similarly the bottom one. So the outside "triangle" commutes, which shows that $(FA \xrightarrow{\bar{a}_J} DJ)_{J\in\mathbb{J}}$ is a cone on D.
Since $(L \xrightarrow{p_J} DJ)_{J\in\mathbb{J}}$ is a limit cone for D, there is a unique arrow $FA \xrightarrow{\sigma} L$ such that $\bar{a}_J = \sigma \circ p_J$, for each $J \in \mathbb{J}$. Thus $a_J = \overline{\sigma \circ p_J}$. By naturality of the adjunction in the second variable, $a_J = Gp_J \circ \bar{\sigma}$ as is shown by the following diagram.

$$\begin{tikzcd}
\sigma \arrow[rrr, mapsto] \arrow[ddd] & & & \bar{\sigma} \arrow[dd] \\
& \mathcal{D}(FA, L) \arrow[r] \arrow[d, "(p_J)_*"] & \mathcal{C}(A, GL) \arrow[d, "(Gp_J)_*"] & \\
& \mathcal{D}(FA, DJ) \arrow[r] & \mathcal{C}(A, GDJ) & Gp_J \circ \bar{\sigma} \\
p_J \circ \sigma \arrow[rrr, mapsto] & & & \overline{p_J \circ \sigma}
\end{tikzcd}$$

Thus we have the arrow $A \xrightarrow{\bar{\sigma}} GL$ we need to show that GL is the limit of GD. We should still show that it is unique. So suppose $A \xrightarrow{\theta} GL$ is another arrow such that $a_J = Gp_J \circ \theta$, for all $J \in \mathbb{J}$. Then, applying the adjunction again, $\bar{a}_J$ must factor as

$$FA \xrightarrow{\bar{\theta}} L \xrightarrow{p_J} DJ$$

for all $J \in \mathbb{J}$, so $\bar{\theta} = \sigma$ by the universal property of the limit L, and thus $\theta = \bar{\sigma}$. So $\bar{\sigma}$ is unique.
We have shown that $(GL \xrightarrow{Gp_J} GDJ)_{J\in\mathbb{J}}$ is a limit cone for $\mathbb{J} \xrightarrow{GD} \mathcal{C}$, as required. □

8.3 Applications

(a) The forgetful functor **Grp** $\xrightarrow{U}$ **Set** has a left adjoint (the free group functor), so U preserves all limits: products, equalizers, kernels, inverse limits, etc.

Similarly, replacing **Grp** by another category of algebraic structures which has free objects, such as **Ab**, **Ring**, R-**Mod**, etc.

We have see that U does not preserve colimits (for example the initial object: $U1 = \{1\} \neq \emptyset$), so we can deduce that U does not have a right adjoint.

(b) In the category **Ab** of abelian groups, each Hom-set $\mathbf{Ab}(A, B)$ naturally carries the structure of an abelian group (with pointwise addition). So the covariant Hom-functor $\mathbf{Ab}(A, -)$ can be considered as a functor $\mathbf{Ab} \longrightarrow \mathbf{Ab}$. It has a left adjoint, tensoring with A. So we have an adjunction:

$$\begin{tikzcd}
\mathbf{Ab} \arrow[r, shift left, "A \otimes -"] \arrow[r, phantom, "\perp"] & \mathbf{Ab} \arrow[l, shift left, "\mathbf{Ab}(A, -)"]
\end{tikzcd}$$

which says that group homomorphisms $C \longrightarrow \mathbf{Ab}(A, B)$ correspond to homomorphisms $A \otimes C \longrightarrow B$. So $A \otimes -$ preserves colimits. In particular, $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$.

(c) In **Set**, we have an adjunction $A \times - \dashv \mathbf{Set}(A, -)$. So the operation "taking the product with A" preserves colimits. In **Ab**, things are different. The terminal object is the 0 group (with one element) and $A \times 0 = A \oplus 0 \cong A \not\cong 0$ in general. So $A \times -$ does not preserve all colimits, and hence it does not have a right adjoint.

8.4 Exercises

Q8.1 Let $\mathcal{C}$ be a category with finite products and suppose we have made a choice of product $A \times B$ for each pair of objects in $\mathcal{C}$. Show directly that

$$\mathcal{C}(A, B \times C) \cong \mathcal{C}(A, B) \times \mathcal{C}(A, C)$$

naturally in $A, B, C \in \mathcal{C}$. Show how this is an example of a functor preserving limits.

Q8.2 Suppose $\mathcal{C}$ is a complete category and $\mathcal{C} \xrightarrow{F} \mathcal{D}$ preserves products and equalisers. Show that F is continuous.

Q8.3 Consider the functor $\mathbf{Set} \xrightarrow{F} \mathbf{Ab}$ which sends a set X to the free abelian group generated by X. Show that F does not preserve all limits.

Q8.4 Prove directly that left adjoint functors preserve initial objects, and right adjoint functors preserve terminal objects.

Q8.5 This is filling in details from the lectures.

1. Give the definition of the tensor product of abelian groups $A \otimes B$.
2. Show that $A \otimes -$ is left-adjoint to $\mathbf{Ab}(A, -)$ when both are considered as functors $\mathbf{Ab} \to \mathbf{Ab}$.

Q8.6 Show that for any category $\mathcal{C}$ and any $A \in \mathcal{C}$, the Hom-functor H^A preserves limits.

9 More on Limits and Adjunctions

9.1 Limit creation

Consider again the forgetful functor $\mathbf{Grp} \xrightarrow{U} \mathbf{Set}$, and suppose $\mathbb{J} \xrightarrow{D} \mathbf{Grp}$ is a diagram. Then the composite UD has a limit given by

$$L = \left\{ (x_J)_{J\in\mathbb{J}} \in \prod_{J\in\mathbb{J}} UDJ \;\middle|\; \text{for each } I \xrightarrow{k} J \text{ in } \mathbb{J}, (UDk)(x_I) = x_J \right\}$$

and the projection arrows of the limit cone are given by $p_I((x_J)_{J\in\mathbb{J}}) = x_I$. We would like to see this limit cone in **Set** as the image under U of a limit cone for D in **Grp**. For that, each p_J would need to be a group homomorphism $L \xrightarrow{p_J} DJ$. There is a unique way to put a group structure on L such that each p_J is a homomorphism, namely pointwise multiplication: $(x_J)_{J\in\mathbb{J}} \cdot (y_J)_{J\in\mathbb{J}} = (x_J \cdot y_J)_{J\in\mathbb{J}}$. In the case of products and equalizers it is easy to check that this does give us a limit cone on D, so in fact it does in general. What we have done here is computed the limit of D in **Grp** by first applying the functor U, then computing the limit of UD, then pulling that limit cone back to **Grp**. We say that U *creates limits.*
The formal definition:
A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is said to *reflect limits* (of shape $\mathbb{J}$) iff for all diagrams $\mathbb{J} \xrightarrow{D} \mathcal{C}$ and all cones $(A \xrightarrow{q_J} DJ)_{J\in\mathbb{J}}$ on D, if $(FA \xrightarrow{Fq_J} FDJ)_{J\in\mathbb{J}}$ is a limit cone on FD then $(A \xrightarrow{q_J} DJ)_{J\in\mathbb{J}}$ is a limit cone on D.
F is said to *create limits* (of shape $\mathbb{J}$) iff F reflects limits (of shape $\mathbb{J}$) and also whenever FD has a limit in $\mathcal{D}$ then then there is a cone $(A \xrightarrow{p_J} DJ)_{J\in\mathbb{J}}$ in $\mathcal{C}$ such that $(FA \xrightarrow{Fp_J} FDJ)_{J\in\mathbb{J}}$ is a limit cone in $\mathcal{D}$.

Example The construction of the p-adic integers given earlier is exactly of this form, and works because the forgetful functor $\mathbf{Ring} \xrightarrow{U} \mathbf{Set}$ creates limits.

Lemma. *If $\mathcal{C} \xrightarrow{F} \mathcal{D}$ creates limits and $\mathcal{D}$ has limits, then $\mathcal{C}$ also has limits (and F preserves them).*

Proof. Immediate from the definitions. □

The point of limit creation is that it tells you how to compute limits in one category ($\mathcal{C}$ above) if you already know how to compute them in another category $\mathcal{D}$. We have seen explicitly how to compute limits in **Set**, and most often the idea is used when F is a forgetful functor from some category to **Set**.

9.2 General adjoint functor theorem

We have shown that if $\mathcal{D} \xrightarrow{G} \mathcal{C}$ has a left adjoint then G preserves limits. Now we want to prove the converse, that if G preserves limits then it has a left adjoint. This need not be true. For example, if $\mathcal{D}$ does not have (many) limits, then the statement that G preserves them is not very strong. So we should also assume that $\mathcal{D}$ is complete. There is one other problem which is set-theoretic: we need to distinguish between sets and proper classes, and ensure that certain things really are sets.

Theorem (General Adjoint Functor Theorem)**.** *Let $\mathcal{D} \xrightarrow{G} \mathcal{C}$ be a functor such that*

- *$\mathcal{D}$ is complete and locally small (all Hom-sets are sets, not proper classes);*
- *G preserves limits; and*
- *for all $A \in \mathcal{C}$, there is a set of arrows $(A \xrightarrow{w_\lambda} G(W_\lambda))_{\lambda\in\Lambda}$ such that for any arrow $A \xrightarrow{f} GB$ there exists $\lambda \in \Lambda$ and an arrow $W_\lambda \xrightarrow{g} B$ such that*
$$\begin{tikzcd} A \arrow[r, "w_\lambda"] \arrow[dr, "f"'] & GW_\lambda \arrow[d, "Gg"] \\ & GB \end{tikzcd}$$
commutes. (This is called the solution set condition.)

Then G has a left adjoint.

The proof can be found in Mac Lane, page 121.

9.3 Examples of GAFT

9.3.1 Free groups

The functor $\mathbf{Grp} \xrightarrow{U} \mathbf{Set}$ creates limits, and **Set** is complete, so **Grp** is complete and U preserves limits. Also, **Grp** is locally small. For any set A and group B, a function $A \xrightarrow{f} UB$ factors as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & UB \\ & \langle f(A) \rangle \arrow[ur] & \end{tikzcd}$$

where $\langle f(A) \rangle$ is the subgroup of B generated by the image of f. The cardinality of $\langle f(A) \rangle$ is bounded by $|A| + \aleph_0$. Let $\mathbb{W}$ be a set containing one representative of each isomorphism class of groups of cardinality up to $|A| + \aleph_0$, and let $(A \xrightarrow{w_\lambda} U(W_\lambda))_{\lambda \in \Lambda}$ be the set of all functions from A to some $W \in \mathbb{W}$. This really is a set, not a proper class, because it has cardinality at most $2^{|A|+\aleph_0}$. So the solution set condition holds, and U has a left adjoint, the "free group" functor.
Of course we already knew this, because we knew how to construct a free group. However, we have now proved abstractly that free groups exist, without actually constructing them via words. While this may not be a more economical or enlightening approach for groups, it may be easier for constructing free objects in more complicated categories.

9.3.2 Hausdorff spaces

Consider the inclusion $\mathbf{Haus} \xrightarrow{U} \mathbf{Top}$ of the full subcategory of Hausdorff topological spaces into **Top**. The category **Top** is complete, and products and equalizers of Hausdorff spaces are easily seen to be Hausdorff, so **Haus** is complete, and U preserves limits. Also **Haus** is locally small. Any continuous map $A \xrightarrow{f} B$ for Hausdorff B factorises via its image as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & f(A) \arrow[ur] & \end{tikzcd}$$

so we can take $\mathbb{W}$ to be a set with one representative of each isomorphism class of Hausdorff spaces of cardinality at most $|A|$ to prove the solution set condition. Thus U has a left adjoint which takes a topological space X to its largest Hausdorff quotient, called its Hausdorffification. It is more difficult to describe this space explicitly.

9.3.3 Compact Hausdorff spaces

We can also consider the inclusion $\mathbf{CHaus} \xrightarrow{U} \mathbf{Top}$ of compact Hausdorff spaces. The above argument does not work. To prove the solution set condition we used the fact that all subspaces of Hausdorff spaces are Hausdorff, so $f(A)$ is Hausdorff. This fails for compact spaces. Indeed, one cannot use the General adjoint functor theorem to show that U has a left adjoint. Nonetheless, it does, and one can prove it using the Special adjoint functor theorem (see Mac Lane, p128). The left adjoint takes a space X to its *Stone-Čech compactification*, βX, which can be described explicitly using ultrafilters.

9.4 Equational classes of algebras

The example of free groups can be generalized to any *equational class* of algebras.

For example, a *monoid* is a set M with functions $M \times M \xrightarrow{\mu} M$ and $1 \xrightarrow{e} M$, where 1 is a singleton set, such that

$$\begin{tikzcd}
M \times (M \times M) \arrow[rr, "\cong"] \arrow[d, "1_M \times \mu"'] & & (M \times M) \times M \arrow[d, "\mu \times 1_M"] \\
M \times M \arrow[r, "\mu"] & M & M \times M \arrow[l, "\mu"']
\end{tikzcd}$$

commutes (the associativity law) and both squares of

$$\begin{tikzcd}
M \times 1 \arrow[d, "\langle 1_M, e\rangle"'] & M \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "1_M"] & 1 \times M \arrow[d, "\langle e, 1_M\rangle"] \\
M \times M \arrow[r, "\mu"'] & M & M \times M \arrow[l, "\mu"]
\end{tikzcd}$$

commute (the identity laws). No elements are used in the definition!

Any class of structures such as groups, abelian groups, rings, R-modules, R-algebras, etc. which can be defined in terms of having certain functions (addition, multiplication, scalar multiplication, etc.) and certain equations between terms in those functions which hold for all elements of the algebra, can be defined by commutative diagrams in a similar way. We call such classes of algebraic structures *equational classes.*

Note that the class of fields is exceptional: fields have a multiplicative inverse for all elements except 0, and they cannot be described in any other way to make them an equational class.

Any equational class forms a category: as arrows, take "homomorphisms": just those functions which preserve all the specified functions. There is a forgetful functor to **Set** which creates limits. One can formalize this to prove:

Theorem. *Any equational class has free objects.*

This is a very useful device in some algebraic contexts. For example, the free R-algebra on a set X is the ring of polynomials $R[X]$, with indeterminates from the set X. It is surprisingly difficult to give a precise definition of polynomials, but this is one way to do it.

For a more complicated example: an *exponential ring* is a ring R equipped with a homomorphism from its additive group to its multiplicative group (of invertible elements). It is easy to describe exponential rings as an equational class. The theorem tells us that there are free objects (called exponential polynomial rings). Describing them explicitly takes several pages.

9.5 Monoid and Group objects in categories

Note that the definition of a monoid via commutative diagrams as above makes sense in any category $\mathcal{C}$ with a terminal object T and chosen binary products.

Thus we define a *monoid in* $\mathcal{C}$ to be an object $C \in \mathcal{C}$ with arrows $C \times C \xrightarrow{\mu} C$ and $T \xrightarrow{e} C$, such that the diagrams above commute. Similarly one can define groups, rings, etc, in a category with chosen finite products. For example: a *topological group* is, by definition, a group object in **Top**. Similarly we have notions of group manifold, group variety, group scheme, and so forth.

9.6 Cones as natural transformations

Since a diagram of shape $\mathbb{J}$ in $\mathcal{C}$ is the same thing as a functor $\mathbb{J} \longrightarrow \mathcal{C}$, all the diagrams of shape $\mathbb{J}$ in $\mathcal{C}$ form a category, the functor category $[\mathbb{J}, \mathcal{C}]$.

For $A \in \mathcal{C}$, let $\mathbb{J} \xrightarrow{\Delta A} \mathcal{C}$ be the constant functor which sends every object to A and every arrow to 1_A. Now observe that a cone on D in $\mathcal{C}$ with vertex A is the same thing as a natural transformation $\Delta A \Rightarrow D$.

9.7 Limits and colimits as adjunctions

For any category $\mathcal{C}$ there is a unique functor $\mathcal{C} \xrightarrow{!} 1$, where 1 is the category with one object and only the identity arrow. A left adjoint $I \dashv\, !$ is an initial object of $\mathcal{C}$. A right adjoint $! \dashv T$ is a terminal object of $\mathcal{C}$.

Now let $\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \times C$ be the diagonal functor which takes an object A to (A, A) and an arrow $A \xrightarrow{f} B$ to $(A, A) \xrightarrow{(f,f)} (B, B)$. A left adjoint to Δ is a functor $\coprod$ which takes (A, B) to the coproduct $A \coprod B$. A right adjoint to Δ takes (A, B) to the product $A \times B$.

More generally, let $\mathbb{J}$ be any small category and let $\Delta_{\mathbb{J}}$ be the functor

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Delta_{\mathbb{J}}} & [\mathbb{J}, \mathcal{C}] \\ A & \longmapsto & \Delta_{\mathbb{J}} A \\ f\Big\downarrow & \longmapsto & \Big\downarrow \Delta_{\mathbb{J}} f \\ B & \longmapsto & \Delta_{\mathbb{J}} B \end{array}$$

where $\Delta_{\mathbb{J}} f$ is the natural transformation such that $(\Delta_{\mathbb{J}} f)_J = f$ for every J in $\mathbb{J}$. The functor $\Delta_{\mathbb{J}}$ is called the *diagonal embedding* of $\mathcal{C}$ in $[\mathbb{J}, \mathcal{C}]$.

Suppose that $\mathcal{C}$ has all limits of shape $\mathbb{J}$. Then the diagonal functor $\Delta_{\mathbb{J}}$ has a right adjoint $[\mathbb{J}, \mathcal{C}] \xrightarrow{\lim_{\leftarrow \mathbb{J}}} \mathcal{C}$ which takes a diagram D to a chosen limit of D.
The counit of the adjunction is a natural transformation $\Delta_{\mathbb{J}} \lim\limits_{\leftarrow \mathbb{J}} \stackrel{\epsilon}{\Rightarrow} 1_{[\mathbb{J}, \mathcal{C}]}$. That is, for each $D \in [\mathbb{J}, \mathcal{C}]$, it consists of an arrow $\Delta_{\mathbb{J}} \lim\limits_{\leftarrow \mathbb{J}} D \xrightarrow{\epsilon_D} D$. However, $[\mathbb{J}, \mathcal{C}]$ is a functor category, so ϵ_D is itself a natural transformation from the constant functor $\Delta_{\mathbb{J}} \lim\limits_{\leftarrow \mathbb{J}} D$ to the functor D. By the remarks above, this natural transformation is actually a cone on D with vertex $\lim\limits_{\leftarrow \mathbb{J}} D$, and in fact it is a choice of limit cone for D.
So limits can be seen as a special case of adjunctions, at least when all limits of a certain shape exist in a category.

9.8 Exercises

Q9.1 Suppose $\mathcal{D} \xrightarrow{U} \mathcal{C}$ creates limits and $\mathcal{C}$ is complete. Show that $\mathcal{D}$ is complete and U preserves limits.

Q9.2 Work through a proof of the General Adjoint Functor Theorem from a book.

Q9.3 Prove that the subcategory of spaces with the T_1-separation property is reflective in **Top**, both using GAFT and directly.

Q9.4 Prove that there is a free ring on any set of generators without constructing it.

Q9.5 Fill in the details in the definition of equational classes, and show using the GAFT that they always have free objects.

Q9.6 Fill in the details to prove that if $\mathcal{C}$ is a category with all limits of shape $\mathbb{J}$, then the limits can be given as a right adjoint of the diagonal embedding $\mathcal{C} \xrightarrow[\mathbb{J}]{\Delta} [\mathbb{J}, \mathcal{C}]$.

Q9.7 A *differential ring* is a commutative ring R equipped with a *derivation*, which is a function $R \xrightarrow{\partial} R$ such that for all $x, y \in R$, $\partial(x + y) = \partial x + \partial y$ and $\partial(xy) = x\partial y + y\partial x$ (the Leibniz rule). For example, take R to be the ring of all holomorphic functions on $\mathbb{C}$, with the obvious derivation. There is an obvious category of differential rings and their homomorphisms. Let **D$\mathbb{Q}$-Alg** be the full subcategory of differential rings which are also $\mathbb{Q}$-algebras. There is an obvious forgetful functor **D$\mathbb{Q}$-Alg** $\xrightarrow{U}$ **$\mathbb{Q}$-Alg**.

(i) Show that U has a left adjoint (differential ring of polynomials) and also a right adjoint (differential ring of power series).

(ii) What are the units and counits of these adjunctions?

(iii) Can you say anything similar about several commuting derivations?

(iv) Why do we have to restrict to $\mathbb{Q}$-algebras? In the case of a general ring, or in particular a field of positive characteristic, find a generalization of the notion of derivation which gets around the problem.

Q9.8 Show how colimits can be seen as left adjoints of the diagonal functor $\Delta_{\mathbb{J}}$.

Q9.9 Show that a monoid in **Monoid** is a commutative monoid. [This is called the Eckmann-Hilton argument.]

10 The category of Sets

The first example of a category we gave was **Set**, the category of sets and functions. However, one might legitimately ask whether this is well-defined. There is a general problem in the foundations of mathematics of saying what the collection (or class, or category) of Sets looks like. Unlike say the ring of integers $\mathbb{Z}$, we do not have a definition which pins down the category of sets precisely, up to isomorphism. For example: is there a set X which is infinite and uncountable (that is, there is no epimorphism (surjection) from the set of natural numbers to X) but from which there is no epimorphism onto the set of real numbers? This is the continuum hypothesis problem - it turns out that the usual axioms of set theory do not tell you what the answer is. In fact, it follows from Gödel's second incompleteness theorem that whatever axioms for set theory you write down, there will be more questions which are not answered by those axioms. The usual convention in mathematics is to take the ZFC axioms (Zermelo-Fraenkel with the Axiom of Choice). Then by **Set** we mean to choose any model of the ZFC axioms, and take all the sets and functions according to that model. Furthermore we expect only to consider questions which do not depend on the model we chose, so they depend only on the ZFC axioms. These axioms are adequate for most mathematics, but not for example for deciding the continuum problem.

The ZFC axioms are not category-theoretic in nature. In this section we will see an alternative list of axioms for **Set**, of a category-theoretic nature, which are also adequate for much of mathematics but have a very different "flavour".

10.1 Exponentials

Given objects A and B in a category $\mathcal{C}$, we have a *set* of arrows $\mathcal{C}(A, B)$. When $\mathcal{C}$ is the category **Set**, that means the Hom-sets correspond to objects of the category itself. This correspondence can be explained in category-theoretic terms.

Consider **Set** equipped with chosen products. Then for each $B \in \mathbf{Set}$, we have a functor $\mathbf{Set} \xrightarrow{-\times B} \mathbf{Set}$ taking A to $A \times B$. We also have the Hom-functor $\mathbf{Set} \xrightarrow{H^B} \mathbf{Set}$ taking C to $\mathbf{Set}(B, C) = C^B$. There is an obvious isomorphism

$$\mathbf{Set}(A \times B, C) \cong \mathbf{Set}(A, C^B)$$

given by taking a function $(a, b) \mapsto f(a, b)$ of two variables to the function $a \mapsto [b \mapsto f(a, b)]$ of one variable. It is easy to see that this isomorphism is natural in A and C in **Set**, so the exponential H^B is right adjoint to $- \times B$.

A *Cartesian closed category* is a category $\mathcal{S}$ which has a terminal object, binary products, and exponentials. That is, each of the functors $\mathcal{S} \xrightarrow{!} 1$, $\mathcal{S} \xrightarrow{\Delta} \mathcal{S} \times \mathcal{S}$, and $\mathcal{S} \xrightarrow{-\times B} \mathcal{S}$ has a right adjoint.

By part (d) of Theorem 7.10 characterizing adjunctions, an adjunction

$$- \times B \dashv -^B$$

is specified by its counit: $A^B \times B \xrightarrow{\epsilon_A} A$ which is a function taking (f, b) to $f(b)$. This is just the *evaluation map.*

Example **Cat** is Cartesian closed, with functor categories $C^B = [B, C]$ as exponentials.

10.2 Subobjects

In any category $\mathcal{C}$, a *subobject* of $A \in \mathcal{C}$ is (provisionally) a monic arrow $B \overset{m}{\rightarrowtail} A$. Two subobjects $B_1 \overset{m_1}{\rightarrowtail} A$, $B_2 \overset{m_2}{\rightarrowtail} A$ are isomorphic iff there is $B_1 \xrightarrow[\cong]{\theta} B_2$ such that

$$\begin{tikzcd} B_1 \arrow[dr, "m_1"] \arrow[d, "\theta"'] & \\ B_2 \arrow[r, "m_2"'] & A \end{tikzcd}$$

commutes.
We identify isomorphic subobjects, and properly define a subobject to be an equivalence class of monics under the equivalence relation of isomophism. However, in practice, we usually work with some chosen monic from the equivalence class and call that a subobject.

Examples

1. Subobjects in **Set** correspond to subsets, in **Grp** to subgroups, in **Vect** to subspaces, etc.
2. Subobjects of a graph are subgraphs, not necessarily induced subgraphs.
3. Subobjects of a topological space are not necessarily subsets with the subspace topology, rather they are subsets with a topology refining the subspace topology. (Subspaces are *regular subobjects*, see Exercise 6.3.)

10.3 Characteristic functions

Let 1, 2 be sets with 1 and 2 elements, and let $1 \overset{t}{\rightarrowtail} 2$ be a function "true". For any monic in **Set**, $S \overset{m}{\rightarrowtail} A$,

$$\begin{tikzcd} S \arrow[r, "!"] \arrow[d, tail, "m"'] & 1 \arrow[d, tail, "t"] \\ A \arrow[r, "\chi_m"'] & 2 \end{tikzcd}$$

is a pullback square, where χ_m is the *characteristic function* of m and ! is the unique function from S to 1.
A *subobject classifier* in a category $\mathcal{S}$ is an object $\Omega \in \mathcal{S}$ and a monic $1 \overset{t}{\rightarrowtail} \Omega$ from a terminal object such that every monic $S \overset{m}{\rightarrowtail} A$ in $\mathcal{S}$ is a pullback of t in a unique way. That is, there is a unique $A \xrightarrow{\chi_m} \Omega$ such that

$$\begin{tikzcd} S \arrow[r, "!"] \arrow[d, tail, "m"'] & 1 \arrow[d, tail, "t"] \\ A \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

is a pullback square. The arrow χ_m is called the *characteristic arrow* of the subobject, S.

Examples

1. In $\mathcal{S} = \mathbf{Set} \times \mathbf{Set}$, $1 \times 1 \overset{(t,t)}{\rightarrowtail} 2 \times 2$ is a subobject classifier. The characteristic arrow of $(S_1, S_2) \subseteq (A_1, A_2)$ is the pair (χ_{S_1}, χ_{S_2}), where each χ_{S_i} is the characteristic funcion of S_i in A_i.
2. Writing **2** for the category with 2 objects and one non-identity arrow between them, consider $\mathcal{S} = \mathbf{Set}^{\mathbf{2}}$. Then $\mathcal{S}$ is the category whose objects are functions $X \xrightarrow{f} Y$, and whose arrows are commuting squares
$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "a_X"'] & Y_1 \arrow[d, "a_Y"] \\ X_2 \arrow[r, "f_2"'] & Y_2 \end{tikzcd}$$
. A subobject of $X \xrightarrow{f} Y$ is $S \xrightarrow{f\restriction_S} T$ where $S \subseteq X$ and $T \subseteq Y$. There are three sorts of elements of X: $x \in S$, $x \notin S$ with $f(x) \in T$, and x such that $f(x) \notin T$. The subobject classifier is
$$\begin{tikzcd} 1 \arrow[r, "!"] \arrow[d, tail, "0"'] & 1 \arrow[d, tail, "0"] \\ \{0,1,2\} \arrow[r, "j"'] & \{0,2\} \end{tikzcd}$$
where $j(0) = j(1) = 0$ and $j(2) = 2$.

10.4 Toposes

A *topos* (or *elementary topos*) is a category $\mathcal{S}$ such that

1. $\mathcal{S}$ has all finite limits;
2. $\mathcal{S}$ has a subobject classifier; and
3. $\mathcal{S}$ is Cartesian closed.

Examples

1. **Set** is the motivating example, as we have already seen.
2. **FinSet**: the category of finite sets and functions is easily seen to be a subtopos of **Set**.
3. **Set**$\times$**Set** and $\mathbf{Set}^{\mathbf{2}}$ are easily seen to be toposes.
4. For any group G, the category G-**Set** of sets with an action of G.
5. For any small category $\mathbb{C}$, the functor categories $\mathbf{Set}^{\mathbb{C}}$ and $\hat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$ are toposes. $\hat{\mathbb{C}}$ is called the *presheaf category of* $\mathbb{C}$ and all the previous examples of toposes are examples of presheaf categories.
6. The category $Sh(X)$ of sheaves on a fixed topological space X. These were the original geometric examples of toposes. The slightly more general notion of a category of sheaves on a site is called a Grothendieck topos, and sometimes the word topos is used to mean just Grothendieck topos.

Facts about toposes

1. Every topos has finite colimits.
2. Every topos has *power objects* corresponding to power sets in **Set**.
3. There is an *internal logic* whose truth values are given by Ω.
4. One can do reasoning about the topos inside the topos, just as set theory has the power to describe itself.

10.5 Natural Numbers Object

The category **FinSet** of finite sets and functions is a topos, so the axioms of toposes do not include an analogue of the axiom of infinity, which says "there is an infinite set". The category-theoretic version is the existence of a *natural numbers object*.
A natural numbers object (NNO) is an object N and arrows $1 \xrightarrow{0} N \xrightarrow{s} N$ (1 the terminal object) such that for any $1 \xrightarrow{h} X \xrightarrow{r} X$, there is a unique arrow $N \xrightarrow{f} X$ such that

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "h"'] & N \arrow[r, "s"] \arrow[d, "f"] & N \arrow[d, "f"] \\ & X \arrow[r, "r"] & X \end{tikzcd}$$

commutes.
In **Set**, $N = \mathbb{N}$ and s is the successor function $n \mapsto n+1$, so this is an axiomatization of the concept of *definition of a function by recursion* via a universal property.

10.6 Final axioms of Set

Any presheaf category $\hat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$ has a Natural Numbers Object, so we need more axioms to isolate the important properties of **Set**.
Note that for any set A, the elements (or points) $a \in A$ are in correspondence with the arrows $1 \xrightarrow{a} A$. A function $A \xrightarrow{f} B$ is determined by A and B and the values $f(a)$ for all the elements $a \in A$, or equivalently the functions $f \circ a$ when the points are considered as arrows.

A category $\mathcal{S}$ with a terminal object 1 is *well-pointed* iff, whenever $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in $\mathcal{S}$ with $f \neq g$, there is $1 \xrightarrow{p} A$ such that $fp \neq gp$.
As mentioned above, every topos has an internal logic with truth values corresponding to the subobject classifier Ω. In some toposes the logic is *intuitionistic*, not classical. However, in a well-pointed topos it is classical, which means that the truth values lie in a Boolean algebra, and furthermore that Boolean algebra has to be the one with only two elements (truth values).
Recall that an equivalent, category-theoretic statement of the Axiom of Choice is that every epimorphism splits. We have the following definition of **Set** which is adequate for most purposes:
Set is a well-pointed topos with a natural numbers object, such that every epimorphism splits.

10.7 Skeletal categories of Sets

A property of categories is said to be *categorial* or *category-theoretic* iff it is preserved under equivalence of categories. All the properties defining **Set** are categorial. In particular, a skeletal subcategory $\mathcal{S}$ of **Set** satisfies the axioms of **Set**. Note that $\mathcal{S}$ has one set of each cardinal size, so in particular only one set of size 1. This leads to an apparent paradox: the set of size 2 has 2 subsets of size 1, but there is only one set of size 1! How can that be? Surely they should have different elements, so be different sets?
In fact this is not a paradox at all. In the category-theoretic approach, a subset of A is *not a set*, but instead it is a function $S \overset{m}{\rightarrowtail} A$. An element $a \in A$ is an arrow $1 \xrightarrow{a} A$. So the two subsets of 2 of size 1 do have different elements – the element is the same arrow as the subset in this case. This approach to sets is very different from the usual approach. In particular, the axiom of extension, which says that two sets are equal if and only if they have the same elements, is not meaningful for sets in the category-theoretic approach (although it is meaningful for subsets of a given set).

10.8 Exercises

Q10.1 What axioms of a topos does **Top** satisfy?

Q10.2 Suppose $\mathcal{C} \simeq \mathcal{D}$, that is, $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories.

(i) Show that if $\mathcal{C}$ has finite limits and a subobject classifier then so does $\mathcal{D}$. Show also that if $\mathcal{C}$ is a topos then so is $\mathcal{D}$.

(ii) Show that being well-pointed, every epimorphism splits (the axiom of choice) and the existence of a Natural Numbers Object are all preserved under equivalence of categories.

(iii) Deduce that a skeleton $\mathcal{S}$ of the category **Set** also satisfies all these axioms. In $\mathcal{S}$, there is only one set of size 1. Explain how the set of size 2 can still have two distinct subsets of size 1 in $\mathcal{S}$.

11 Miscellaneous exercises

Q11.1 Let $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$ be an adjunction with unit η and counit ϵ. Let $\mathrm{Fix}(GF)$ be the full subcategory of $\mathcal{C}$ whose objects are those $A \in \mathcal{C}$ such that η_A is an isomorphism, and dually $\mathrm{Fix}(FG) \subseteq \mathcal{D}$. Prove that the adjunction $F \dashv G$ restricts to an equivalence between $\mathrm{Fix}(GF)$ and $\mathrm{Fix}(FG)$.

Q11.2 **Idempotent adjunctions.** Let $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$ be an adjunction with unit η and counit ϵ.

(i) Show that the following conditions are equivalent:

a) $F\eta$ is a natural isomorphism
b) ϵF is a natural isomorphism
c) $G\epsilon F$ is a natural isomorphism

d) $GF\eta = \eta GF$

e) $GF\eta G = \eta GFG$

f)-j) the duals of a)-e), e.g. i) says that $FG\epsilon = \epsilon FG$.

[Hint: do the implications in cyclic order i) $\Rightarrow$ ii) $\Rightarrow \cdots$.] An adjunction satisfying these conditions is called an *idempotent adjunction*.

(ii) Let $\mathcal{I}(F)$ be the *essential full image of* F, the full subcategory of $\mathcal{D}$ consisting of those $B \in \mathcal{D}$ such that $B \cong FA$ for some $A \in \mathcal{C}$. Using the previous question show that, in any idempotent adjunction, $\mathcal{I}(F)$ is equivalent to $\mathcal{I}(G)$.

(iii) Show that every adjunction between quasi-ordered sets is idempotent. Can you see the main theorem of Galois theory in these terms?

Q11.3 **Quantifiers as adjoints** [This question assumes the basic definitions of predicate logic.]

Consider a fixed, formal, first-order language, L. Let L_n be the set of formulas of L with free variables from $\{x_1, \dots x_n\}$. Consider L_n as a quasiordered set by the order $\varphi(\bar{x}) \models \theta(\bar{x})$, meaning that for any L-structure A, and any tuple $\bar{a}$ from A, if $A \models \varphi(\bar{a})$ then $A \models \theta(\bar{a})$.

There are order-preserving inclusions $L_n \hookrightarrow L_{n+1}$ for all $n \in \mathbb{N}$. For any n, there are also order-preserving maps

$$L_{n+1} \xrightarrow{\forall x_{n+1}} L_n \qquad \text{and} \qquad L_{n+1} \xrightarrow{\exists x_{n+1}} L_n.$$

Show that $\exists x_{n+1}$ is left adjoint to the inclusion, and $\forall x_{n+1}$ is right adjoint. So quantification in mathematics is just an example of applying adjoint functors!

Q11.4 **The Stone Representation Theorem.** This question gives an example of an idempotent adjunction which is important in logic, particularly in model theory. I give the outline and leave you to fill in the details. Let **Bool** be the category of Boolean algebras.

(i) Show there is a functor $\mathbf{Top} \xrightarrow{L} \mathbf{Bool}^{\mathrm{op}}$ which takes a topological space to its Boolean algebra of clopen subsets, and a continuous map to its inverse image map.

(ii) Given $A \in \mathbf{Bool}$, explain what filters and ultrafilters on A are. Let SA be the set of all ultrafilters on A, and show how to make SA into a topological space.

(iii) Show how a homomorphism $A_1 \xrightarrow{f} A_2$ of Boolean algebras gives rise to a map $SA_2 \longrightarrow SA_1$, show this map is continuous, and deduce that S is a functor $\mathbf{Bool} \xrightarrow{S} \mathbf{Top}^{\mathrm{op}}$.

(iv) Show that for every $A \in \mathbf{Bool}$, SA is compact, Hausdorff, and zero-dimensional (a Stone space).

(v) Show that, for any Boolean algebra A, there is an isomorphism $A \xrightarrow{\eta_A} LSA$ and that the isomorphism is natural in $A \in \mathbf{Bool}$.

(vi) For $X \in \mathbf{Top}$, define $X \xrightarrow{\psi_X} SLX$ by $\psi_X(x) = \{U \in LX \mid x \in U\}$. Show this is a well-defined, continuous map, and that it is a homeomorphism iff X is a Stone space. Show also that ψ is natural in $X \in \mathbf{Top}$.

(vii) Show that L and S are adjoint functors. [Note, these are contravariant functors, so they are both left adjoints, and there are two units, rather than a unit and counit. One can also have a contravariant adjunction with two counits, which is a dual notion.]

(viii) Consider what this means for the Lindenbaum algebras and the Stone spaces of types of a first-order theory.

Q11.5 **The Yoneda Lemma.** [You may have to look up most of this question in a book.]
For any small category $\mathbb{C}$, let $\hat{\mathbb{C}}$ be the functor category $\mathbf{Set}^{\mathbb{C}^{\mathrm{op}}} = [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$.

(i) For any $A \in \mathbb{C}$, we have the contravariant Hom-functor $H_A \in \hat{\mathbb{C}}$. Show that $A \mapsto H_A$ is the object function of a functor $\mathbb{C} \xrightarrow{H_\bullet} \hat{\mathbb{C}}$.

(ii) Prove the Yoneda Lemma: if $\mathcal{C}$ is locally small then the map $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, X) \to X(A)$ given by $\alpha \mapsto \alpha_A(1_A)$, is an isomorphism which is natural in $A \in \mathcal{C}$ and in $X \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

(iii) Deduce that $H_\bullet$ is full and faithful. So we can regard it as embedding $\mathbb{C}$ into $\hat{\mathbb{C}}$.

(iv) Show that $\hat{\mathbb{C}}$ is a complete category.

(v) Functors of the form H_A, or naturally isomorphic to them, are called *representable functors*. Show that every $X \in \hat{\mathbb{C}}$ is a colimit of representable functors. We say that $\mathbb{C}$ is *dense* in $\hat{\mathbb{C}}$, so we may regard $\hat{\mathbb{C}}$ as some sort of category-theoretic completion of $\mathbb{C}$.