Basic definitions

- Recall that an *algebra* is a ring which is also a vector space under addition. An algebra is *unital* if it has a multiplicative identity, which we denote by 1 (or 1_A if A is the algebra and we wish to clarify that this is the identity for A). *Unless otherwise stated, all of our algebras are complex!*
- A *Banach algebra* is an algebra which is a Banach space. We assume that the norm is submultiplicative.
- We write $A^\sharp$ for the *algebraic dual* of an algebra A; that is, the collection of all linear functionals; and write A^* for the continuous linear functionals when A is a normed algebra. Note that A^* is a Banach algebra, and we can similarly define A^{**}, etc.

Basic definitions, continued

- A nonzero multiplicative linear functional χ on an algebra A is called a *character*. We write $X(A)$ for the set of characters of A. *This set may be empty!* (consider $M_2(\mathbb{C})$).
- An *(algebra) homomorphism* $\varphi : A \to B$, A, B algebras, is a linear map such that $\varphi(ab) = \varphi(a)\varphi(b)$ (that is, φ preserves algebra structure). If A, B are both unital, then we say that φ is *unital* if $\varphi(1_A) = 1_B$.
- A *representation* of a Banach algebra A is an algebra homomorphism from A into $\mathscr{L}(\mathscr{H})$, the bounded linear operators on a Hilbert space $\mathscr{H}$. The *dimension* of a representation is the dimension of $\mathscr{H}$. An injective representation is said to be *faithful*. Characters are just 1 dimensional representations.

Basic definitions, continued

- A *left ideal* I of an algebra A is a subset of A which is a subgroup under addition, and having the property that $ax \in I$ whenever $a \in A$ and $x \in I$. It is a *right ideal* if instead $xa \in I$, and a *two-sided ideal* (or simply *ideal*) if it is both a left and right ideal. Ideals are assumed to be nontrivial (ie, not $\{0\}$ or A).
- A *maximal ideal* is an ideal which is not properly contained in another ideal. ★ The existence of maximal left or right ideals is a simple application of Zorn's lemma. The set of maximal (two-sided) ideals of an algebra A is denoted by $M(A)$ (this may be empty!).
- The *radical* $\mathrm{rad}(A)$ of an algebra A is the intersection of all maximal left (equivalently, all maximal right) ideals. If the radical is $\{0\}$, the algebra is said to be *simple*.
- A basic result in the theory of commutative unital Banach algebras is that for such an algebra A, the sets $X(A)$ and $M(A)$ are nonempty, and in fact, there is a one to one correspondence between the two (the maximal ideals being kernels of characters).

- If A is not a unital algebra, we can *unitize* it. The unitization $A_1 = A \times \mathbb{C}$ has

$$(a,\alpha)+(b,\beta)=(a+b,\alpha+\beta),$$
$$\beta(a,\alpha)=(\beta a,\beta\alpha),$$
$$(a,\alpha)(b,\beta)=(ab+\beta a+\alpha b,\alpha\beta), \quad \text{and}$$
$$\|(a,\alpha)\|=\|a\|+|\alpha|,$$

  and unit equal to $(0,1)$. It contains A isometrically as an ideal. When A is commutative, so is A_1.

Topologies

- Several different topologies will be of importance to us.
- There is the *norm topology* on a normed algebra. By definition, a Banach algebra is complete in this norm.
- There is the *weak topology*, which is defined as the weakest topology on A such that the linear functionals in A^* are continuous.
- Recall that there is an isometric embedding of a Banach algebra A into A^{**}. The *weak-$*$ topology* is the weakest topology on A^* such that the elements of A, viewed as linear functionals on A^*, are continuous.
- The importance of the weak-$*$ topology lies in the fact that the characters of A form a closed subset of the unit ball of A^*. By Alaoglu's theorem, it follows that $X(A)$ (or $M(A)$ identified with $X(A)$) is relatively weak-$*$ compact.

Topologies continued

- When dealing with subalgebras of $\mathscr{L}(\mathscr{H})$, we define the *strong operator topology* using as a subbase sets $U(T,x,\epsilon) := \{R \in \mathscr{L}(\mathscr{H}) : \|(R-T)x\| < \epsilon\}$, where $T \in \mathscr{L}(\mathscr{H})$, $x \in \mathscr{H}$ and $\epsilon > 0$. We write $A_\alpha \xrightarrow{SOT} A$ if $\lim_\alpha A_\alpha x = Ax$ for all $x \in \mathscr{H}$.
- Similarly, we define the *weak operator topology* as the topology having as a subbase sets of the form $V(T,x,y,\epsilon) := \{R \in \mathscr{L}(\mathscr{H}) : |\langle (R-T)x, y\rangle| < \epsilon\}$, where $T \in \mathscr{L}(\mathscr{H})$, $x, y \in \mathscr{H}$ and $\epsilon > 0$. We write $A_\alpha \xrightarrow{WOT} A$ if $\lim_\alpha \langle A_\alpha x, y\rangle = \langle Ax, y\rangle$ for all $x, y \in \mathscr{H}$.
- The weak operator topology and weak topology are both examples of "weak topologies", though they are in general different.
- In general, weak, strong and norm topologies agree on convex sets but differ otherwise.

Spectrum

Let A be a unital Banach algebra with unit 1. Write $GL(A)$ for the group of invertible elements.

- For $x \in A$, the *spectrum* of x is $\sigma(x) := \{\lambda \in \mathbb{C} : \lambda 1 - x \notin GL(A)\}$.
- The *spectral radius* of $x \in A$ is $r(x) := \sup_{\lambda \in \sigma(x)} |\lambda|$.
- The *resolvent* of x is $\rho(x) = \{\lambda \in \mathbb{C} : \lambda \notin \sigma(x)\} = \{\lambda \in \mathbb{C} : \lambda 1 - x \in GL(A)\}$.
- For $x, y \in A$, $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$.
- The *spectral mapping theorem* states that $\sigma(p(x)) = p(\sigma(x))$ for a nonconstant polynomial p.
- The *Gel'fand-Beurling theorem* states that $\sigma(x)$ is nonempty and compact, $\lambda \mapsto (\lambda - x)^{-1}$ is analytic on $\rho(x)$ and goes to 0 as $\lambda \to \infty$, and $r(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$ (the *spectral radius formula*).
- ★ If A, B are unital and $\varphi : A \to B$ is a unital homomorphism, then $\sigma(\varphi(a)) \subseteq \sigma(a)$.

The Gel'fand transform

- For $a \in A$ define $\widehat{a} \in C(X(A))$, the algebra of continuous functions on $X(A)$ by $\widehat{a}(\chi) = \chi(a)$, $\chi \in X(A)$. The map

$$\Phi : A \to C(X(A)) \quad \text{given by} \quad \Phi(a) = \widehat{a}$$

  is a continuous unital algebra homomorphism called the *Gel'fand transform*.
- If A is not unital, we must assume $X(A) \neq \emptyset$ (ie, A is not a *radical algebra*), and then $X(A)$ will then be locally compact rather than compact. We also replace $C(X(A))$ by $C_0(X(A))$, the continuous functions "vanishing at infinity" (that is, small off of some compact subset).
- The *Gel'fand representation theorem* states that Φ is a contractive (ie, norm decreasing) homomorphism with $\ker \Phi = \operatorname{rad} A$. Furthermore, for $a \in A$, $\sigma(a) = \widehat{a}(X(A))$ or $\widehat{a}(X(A)) \cup \{0\}$, for A unital or nonunital, respectively. In either case, $r(a) = \|\widehat{a}\|_\infty$.

Decompositions of operators

- As mentioned above, we denote by $\mathscr{L}(\mathscr{H},\mathscr{K})$ the bounded operators from a Hilbert space $\mathscr{H}$ to a Hilbert space $\mathscr{K}$. This is a Banach space. When $\mathscr{K}=\mathscr{H}$, we simply write $\mathscr{L}(\mathscr{H})$, and we have a Banach algebra.
- If $T\in\mathscr{L}(\mathscr{H},\mathscr{K})$, the *adjoint* of T, written T^*, is an operator in $\mathscr{L}(\mathscr{K},\mathscr{H})$ satisfying $\langle Tx,y\rangle=\langle x,T^*y\rangle$ for all x,y. It is easy to see that for $\alpha\in\mathbb{C}$, $(\alpha T)^*=\overline{\alpha}T^*$, $(T+R)^*=T^*+R^*$, $(TR)^*=R^*T^*$ and $T^{**}:=(T^*)^*=T$.
- The adjoint map is an example of an *involution*.
- Write $\ker T$ for the kernel of the operator T on a Hilbert space $\mathscr{H}$. This is a closed subspace when T is bounded. One easily checks that $\ker T\perp\operatorname{ran}T^*$ and that $\mathscr{H}=\ker T\oplus\operatorname{ran}T^*$

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- Define the *real* and *imaginary parts* of $T\in\mathscr{L}(\mathscr{H})$ by $\operatorname{Re}T=\frac{1}{2}(T+T^*)$ and $\operatorname{Im}T=\frac{1}{2i}(T-T^*)$. Then $T=\operatorname{Re}T+i\operatorname{Im}T$.

Some classes of operators

- An operator $T \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ is
  - a *contraction* if for all $x \in \mathscr{H}$, $\langle Tx, Tx\rangle \leq \langle x, x\rangle$ (equivalently, $1 - T^*T \geq 0$);
  - an *isometry* if $\langle Tx, Tx\rangle = \langle x, x\rangle$ (that is, $T^*T = 1$);
  - a coisometry if T^* is an isometry (so $TT^* = 1$);
  - *unitary* if T is both isometric and coisometric;
  - a *partial isometry* if it is isometric on the orthogonal complement of its kernel.
- An operator $T \in \mathscr{L}(\mathscr{H})$ is
  - *selfadjoint* if $T^* = T$;
  - *positive* (written $T \geq 0$) if it is selfadjoint and $\langle Tx, x\rangle \geq 0$ for all $x \in \mathscr{H}$;
  - *normal* if $T^*T = TT^*$ (for example, unitary operators are normal).

The spectral theorem and Douglas' lemma

- If T is selfadjoint, then $\sigma(T) \subset \mathbb{R}$. A selfadjoint T is positive iff $\sigma(T) \subset \mathbb{R}^+$.
- The *spectral theorem* allows us to express a positive / selfadjoint / normal operator as the integral over a positive / real / complex *spectral measure* supported on the spectrum of the operator.
- T is a positive iff $T = R^*R$ for some operator R. In particular, there is an positive operator R such that $T = R^2$, usually written as $T^{1/2}$. If T is not necessarily positive, we write $|T|$ for $(T^*T)^{1/2}$.

★

- *Douglas' lemma* states that $R^*R \leq \alpha^2 T^*T$ for some $\alpha \geq 0$ iff there exists an operator S such that $\|S\| \leq \alpha$ and $R = ST$.
- From Douglas' lemma, it follows that if $R^*R = T^*T$, there is a partial isometry mapping U from $\overline{\operatorname{ran}}\, T$ to $\overline{\operatorname{ran}}\, R$ such that $R = UT$. In particular, we have the *polar decomposition* of an operator, $T = U|T|$, where U is a partial isometry from $\overline{\operatorname{ran}}\, T^*$ to $\overline{\operatorname{ran}}\, T$.

Compact operators

- An operator $T \in \mathscr{L}(\mathscr{H})$ has *finite rank* if $\operatorname{ran} T := T\mathscr{H}$ is finite dimensional (and so closed). Since $T^*\mathscr{H} = T^*T\mathscr{H}$, we see that T^* also has finite rank. Thus if $\mathscr{H}$ is infinite dimensional, $\mathscr{L}_f(\mathscr{H})$, the collection of finite rank operators, is a (two-sided) ideal.
- The norm closure of $\mathscr{L}_f(\mathscr{H})$, $\mathscr{L}_c(\mathscr{H})$, is a norm closed ideal, called the *compact operators*.
- Let $B_{\mathscr{H}}$ be the closed unit ball in $\mathscr{H}$. When $\mathscr{H}$ is infinite dimensional, this will not be compact. However, $T(B_{\mathscr{H}})$ is compact precisely when T is a compact operator; equivalently, the image of any net in $\mathscr{H}$ under T has a convergent subnet.

Compact operators continued

- The quotient $\mathscr{C}(\mathscr{H}) = \mathscr{L}(\mathscr{H})/\mathscr{L}_c(\mathscr{H})$ is called the *Calkin algebra*.
- The part of the spectrum of an operator T which is preserved when mapped to the Calkin algebra is called the *essential spectrum* (written $\sigma_e(T)$). It is the part which is preserved under compact perturbations of the operator.
- An operator is *essentially* selfadjoint, normal, etc, if its image in the Calkin algebra is selfadjoint, normal, etc.
- The *spectral theorem for compact operators* implies the existence of invariant subspaces for compact operators.
- *Lomonosov's theorem* states that a compact operator on an infinite dimensional Hilbert space has a non-trivial *hyper-invariant subspace*; that is, a subspace which is invariant for all operators commuting with the given operator.

C^*-algebras

- Recall that in a Banach algebra A, the norm is submultiplicative; that is, $\|ab\| \le \|a\|\|b\|$ for all $a, b \in A$.
- An *involution* $*$ on A satisfies $(a+b)^* = a^* + b^*$, $(ab)^* = b^* a^*$ and $a^{**} = a$ for all $a, b \in A$. It is conjugate linear if additionally for $\alpha \in \mathbb{C}$, $(\alpha a)^* = \overline{\alpha} a^*$.
- A *Banach-$*$ algebra* is a Banach algebra with a conjugate linear involution. A *C^*-algebra* is a Banach-$*$ algebra in which

$$\|a^* a\| = \|a\|^2 \qquad \text{for all } a \in A.$$

  (We call this the *C^*-property*.)

★
- We may replace the C^*-property by the seemingly weaker condition that

$$\|a^* a\| \ge \|a\|^2 \qquad \text{for all } a \in A.$$

- This rather innocuous looking C^*-property is in fact very powerful.
- By analogy with commutative C^*-algebras, the study of general C^*-algebras can be thought of as the study of noncommutative real analysis, and the linear functionals on C^*-algebras form the basis for noncommutative measure theory.

★
- $\mathscr{L}(\mathscr{H})$, $\mathscr{H}$ a Hilbert space, is a special example of a C^*-algebra.
- Using the norm and involution on a C^*-algebra, we can define various classes of elements of a (unital) C^*-algebra as we did in $\mathscr{L}(\mathscr{H})$, including isometric, coisometric, unitary, selfadjoint and normal. Contractive means norm less than or equal to 1. With a bit more work, we can also define positive elements.

Abstract vs concrete algebras

- The *Gel'fand representation theorem* for commutative Banach algebras allows us to represent abstract commutative Banach algebras concretely as continuous functions on a locally compact Hausdorff space with pointwise operations and the supremum norm.
- Often properties of commutative Banach algebras are much more easily obtained with such a representation (eg, try proving that the spectral radius is submultiplicative in such algebras with and without the Gel'fand representation theorem).
- We will show that commutative C^*-algebras are, perhaps not unsurprisingly, isometrically isomorphic to the continuous functions on a locally compact Hausdorff space. Nevertheless, this has some far reaching implications.

Abstract vs concrete algebras

- What can we say in the noncommutative case? For Banach algebras, not much.
- As we noted, for a Hilbert space $\mathscr{H}$, $\mathscr{L}(\mathscr{H})$ is a C^*-algebra. In fact any norm closed subalgebra which is also closed under taking adjoints is a C^*-algebra. Such algebras are called *concrete C^*-algebras*.
- Remarkably, we shall show that any C^*-algebra is isometrically $*$-isomorphic to such a concrete algebra (Theorem 1.18 – the Gel'fand-Naimark theorem).
- In recent years, there has been great interest in not necessarily selfadjoint subalgebras of $\mathscr{L}(\mathscr{H})$, or simply subspaces of $\mathscr{L}(\mathscr{H})$, or even subspaces of $\mathscr{L}(\mathscr{H})$ which are closed under taking adjoints.
- What about the abstract objects which in concrete form correspond to these? What are the analogues of the Gel'fand-Naimark theorem for these? We explore this in the coming lectures.
- The interplay between the concrete and abstract is the main theme of the course.

Spectral radius of normal elements

For normal (and so in particular, for selfadjoint) elements of a C^*-algebra, the norm and spectral radius are equal.

Lemma 1.1. *Let a be a normal element of a C^*-algebra A. Then $r(a) = \|a\|$.*

Proof.

First observe that

$$\|a^2\|^2 = \|a^{*2}a^2\| = \|(a^*a)^2\| = \|a^*a\|^2 = \|a\|^4.$$

Hence by induction, $\|a^{2^n}\| = \|a\|^{2^n}$. Applying the spectral radius formula, we obtain $r(a) = \|a\|$. □

Corollary 1.2. *There is at most one norm on a Banach $*$-algebra making it into a C^*-algebra.*

Proof.

Let $a \in A$. The result follows since for any C^*-norm, $\|a\|^2 = \|a^*a\| = r(a^*a)$. □

Spectrum of selfadjoint elements

We begin with two observations:

★ ▸ In a C^*-algebra, $\lambda \in \sigma(a)$ iff $\overline{\lambda} \in \sigma(a^*)$. Also if a is invertible, $\lambda \in \sigma(a)$ iff $\lambda^{-1} \in \sigma(a^{-1})$.

★ ▸ If u is a unitary element of a unital C^* algebra, $\|u\| = 1$, and $\sigma(u) \subseteq \mathbb{T}$.

Lemma 1.3. *Let a be a selfadjoint element of a C^*-algebra. Then $\sigma(a) \subset \mathbb{R}$.*

Proof.

We may take A to be unital. For a selfadjoint, define

$$u_t := e^{ita} = \sum_0^\infty \frac{(ita)^n}{n!}, \qquad t \in \mathbb{R}.$$

Then

$$u_t^* = \sum_0^\infty \frac{(\overline{it}a^*)^n}{n!} = \sum_0^\infty \frac{(-ita)^n}{n!} = e^{-ita} = u_t^{-1},$$

so u_t is unitary, and $\|u_t\|^2 = \|u_t^* u_t\| = \|1\| = 1$. Suppose $\lambda \in \sigma(a)$ and set $b = \sum_{n=1}^\infty i^n (a-\lambda)^{n-1}/n!$. Then $b(a-\lambda) = (a-\lambda)b = e^{i(a-\lambda)} - 1$ is not invertible, and so $u - e^{i\lambda} = (e^{i(a-\lambda)} - 1)e^{i\lambda}$ is not invertible either. Hence $e^{i\lambda} \in \sigma(u) \subseteq \mathbb{T}$. Consequently $\lambda \in \mathbb{R}$. □

An algebra homomorphism φ between Banach $*$-algebras is called a *$*$-homomorphism* if it preserves adjoints; that is, $\varphi(a^*) = \varphi(a)^*$.

Theorem 1.4. *Let $\varphi : A \to B$ be a $*$-homomorphism of C^*-algebras. Then φ is contractive.*

Proof.

Suppose φ is a $*$-homomorphism. By passing to the unitizations if necessary, we may assume A, B and φ are unital. Let $a \in A$. Then $\sigma(\varphi(a)) \subseteq \sigma(a)$, and so by Lemma 1.1 and the C^* norm equality,

$$\begin{aligned} \|\varphi(a)\|^2 &= \|\varphi(a)^* \varphi(a)\| = \|\varphi(a^* a)\| = r(\varphi(a^* a)) \\ &\leq r(a^* a) = \|a^* a\| = \|a\|^2, \end{aligned}$$

□

In fact, it can be shown that contractive homomorphisms of C^*-algebras are automatically $*$-homomorphisms. We prove later (Theorem 1.13) that a $*$-homomorphism is isometric onto its range, and so the range of a $*$-homomorphism is closed.

Lemma 1.5. *Let χ be a character for a C^*-algebra A. Then χ is a $*$-homomorphism from A to $\mathbb{C}$.*

Proof.

We may take A to be unital. Recall that if a is selfadjoint, then

$$u_t := e^{ita} = \sum_0^\infty \frac{(ita)^n}{n!}, \qquad t \in \mathbb{R},$$

is unitary and $\|u_t\|^2 = \|u_t^* u_t\| = \|1\| = 1$. Consequently,

$$1 \geq |\chi(u_t)| = \left| \sum_0^\infty \frac{(it\chi(a))^n}{n!} \right| = \left| e^{it\chi(a)} \right| = e^{-t\Im\chi(a)}.$$

Since this holds for all $t \in \mathbb{R}$, we conclude that $\Im\chi(a) = 0$ when a is selfadjoint. For general a, write $a = \Re a + i\Im a$. The claim then follows by linearity of χ. $\square$

- We are now in a position to define what we mean by a positive element of C^*-algebra: We say that a is *positive* (write $a \geq 0$) if it is selfadjoint and $\sigma(a) \subset \mathbb{R}^+ = [0,\infty)$.
- By the spectral mapping theorem, if a is selfadjoint, then $a^2 \geq 0$. Also, if both a and $-a \geq 0$, then $\sigma(a) = \{0\}$, and so $\|a\| = r(a) = 0$; that is, $a = 0$.
- If $a \geq 0$ and $t \geq \|a\|$, then $\sigma(a) \subseteq [0,t]$, and so by the spectral mapping theorem, $\sigma(a - t1) \subseteq [-t, 0]$. Hence $\|a - t1\| \leq t$.
- If a is selfadjoint and $\|a - t1\| \leq t$ for some $t \in \mathbb{R}^+$, then $\sigma(a - t1) = \{x - t : x \in \sigma(a)\} \subseteq [-t, t]$, and so $\sigma(a) \subseteq [0, 2t]$; that is, $a \geq 0$.

Lemma 1.6. *The sum of two positive elements of a C^*-algebra is positive.*

Proof.
Let $a, b \geq 0$ in A, and suppose A is unital. Then $\| a - \|a\|1 \| \leq \|a\|$ and $\| b - \|b\|1 \| \leq \|b\|$. By the triangle inequality, $\| (a+b) - (\|a\| + \|b\|)1 \| \leq \| a - \|a\|1 \| + \| b - \|b\|1 \| \leq \|a\| + \|b\|$, and so $a + b \geq 0$. □

- This shows that the set of positive elements of a C^*-algebra form a *wedge* (in fact a cone since $a \geq 0$ and $-a \geq 0$ implies $a = 0$). We will determine other properties of this cone using the Gel'fand representation theorem.

Theorem 1.7. *Let A be a nonzero commutative C^*-algebra. The Gel'fand transform is an isometric $*$-isomorphism of A onto $C_0(X(A))$.*

Proof.

Begin by assuming A is unital. By Lemma 1.2, all characters of A are $*$-homomorphisms. Hence $\widehat{a^*}(\chi) = \chi(a^*) = \chi(a)^* = \widehat{a}^*(\chi)$, and so $\Phi(a^*) = \Phi(a)^*$. By the Gel'fand representation theorem for commutative Banach algebras, and Lemma 1.1,

$$||\Phi(a)||^2 = ||\Phi(a)^*\Phi(a)|| = ||\Phi(a^*a)|| = r(a^*a) = ||a^*a|| = ||a||^2,$$

so Φ is isometric.

★ We conclude that $\Phi(A)$ is a unital norm closed selfadjoint subalgebra of $C(X(A))$ which separates points. It then follows by the Stone-Weierstrass theorem that Φ is surjective, and so a $*$-isomorphism.

For the nonunital case, embed A in the unitization A_1 as an ideal of codimension 1. Let χ_∞ be the character on A_1 with kernel equal to A. Then $X(A)$ is homeomorphic to $X(A_1) - \{\chi_\infty\}$. The Gel'fand transform on A_1 is then an isometric $*$-isomorphism which takes A to the functions vanishing at χ_∞. □

Continuous functional calculus & spectral mapping for normal elements

Theorem 1.8. *Let a be a normal element of a unital C^*-algebra A. Then $C^*(a)$ is isometrically $*$-isomorphic to $C(\sigma(a))$, the continuous functions on $\sigma(a)$ via the map $a \mapsto z$ where z is the identity function. Furthermore, if $f \in C(\sigma(a))$, then $\sigma(f(a)) = f(\sigma(a))$, and for $g \in C(\sigma(f(a)))$, $(g \circ f)(a) = g(f(a))$.*

Proof.

Let B be the C^*-algebra generated by 1 and a. By the Gel'fand representation theorem (v.o.), $\widehat{a} : X(A) \to \sigma(a)$ is a continuous bijection of compact Hausdorff spaces, and so a homeomorphism. Combining this with the C^* version (theorem 1.7), we can identify the map $\widehat{a}$ with the identity function on $\sigma(a)$ and the algebra $C^*(a)$ is isometrically $*$-isomorphic to $C(\sigma(a))$.

Also by the Gel'fand representation theorem,

$$\sigma(f(a)) = \{\widehat{f(a)}(\chi) : \chi \in X(B)\} = \{f(\chi(a)) : \chi \in X(B)\} = f(\sigma(a)).$$

If C is the C^*-algebra generated by 1 and $f(a)$, then $C \subseteq B$, and therefore, the restriction $\chi \in X(B)$ to C, χ_C, is a character for C. Thus

$\chi((g \circ f)(a)) = g(f(\chi(a))) = g(\chi_C(f(a))) = \chi_C(g(f(a))) = \chi(g(f(a)))$, and so $(g \circ f)(a) = g(f(a))$.

□

The last theorem justifies the terminology of calling the collection of maximal ideals (equivalently, characters) the *spectrum* of a commutative C^*-algebra.

Square roots of positive elements

Theorem 1.9. *Let A be a C^*-algebra, $a \in A$, $a \geq 0$. Then there exists a unique $b \geq 0$ such that $b^2 = a$.*

Proof.

Let $C^*(a)$ be the C^*-subalgebra of A generated by a. This is a commutative ★ C^*-algebra, so the Gel'fand representation theorem (Theorem 1.7) applies. The function $\Phi(a)$ is a positive valued function on $X(C^*(a))$, the character space of $C^*(a)$, and so there is a unique continuous positive function which is its square root. Let ★ $b = \Phi^{-1}(\Phi(a)^{1/2})$. Then $b \geq 0$, and since Φ is an isometric $*$-isomorphism, b is the unique positive element such that $b^2 = a$. □

- We denote the unique positive square root of $a \geq 0$ by $a^{1/2}$.
- ★ For selfadjoint a, define $|a| := (a^2)^{1/2}$, and set $a_\pm = \frac{1}{2}(|a| \pm a)$. Then $a_\pm \geq 0$, $a = a_+ - a_-$ and $a_+ a_- = a_- a_+ = 0$.

Squares as positive elements

Theorem 1.10. *Let a be an element of a C^*-algebra. Then $a^*a \geq 0$.*

Proof.

Suppose $-a^*a \geq 0$. Since $\sigma(-aa^*) \cup \{0\} = \sigma(-a^*a) \cup \{0\}$, we have $-aa^* \geq 0$. Write $a = b + ic$, b, c selfadjoint. Then $a^*a + aa^* = 2b^2 + 2c^2$, and thus $a^*a = 2b^2 + 2c^2 + (-aa^*) \geq 0$. Consequently $\sigma(a^*a) = \{0\}$, and so $\|a\|^2 = \|a^*a\| = r(a^*a) = 0$. Hence $a = 0$.

In general $b = a^*a$ is selfadjoint, and so we decompose it as on the previous slide as $b = b_+ - b_-$ where $b_\pm \geq 0$ and $b_+b_- = b_-b_+ = 0$. Set $c = ab_-$. Then

★
$$-c^*c = -b_-a^*ab_- = -b_-(b_+ - b_-)b_- = b_-^3 \geq 0.$$

Hence by the previous paragraph, $b_- = 0$, and so $a^*a = b_+ \geq 0$. □

The following are mostly applications of the Gel'fand representation theorem.

★
- The set $\{a^*a : a \in A\}$ equals A^+, the positive elements of A;
- If a, b are selfadjoint and $a \leq b$ (ie, $b - a \geq 0$), then for all $c \in A$, $c^*ac \leq c^*bc$;
- If $0 \leq a \leq b$, then $\|a\| \leq \|b\|$;
- If A is unital and $0 < a \leq b$, then $0 < b^{-1} \leq a^{-1}$.
- If A is unital and $0 < a \leq b$, then $a(1+a)^{-1} \leq b(1+b)^{-1}$.

Wedges and cones

- A set P in a vector space X which is closed under vector addition and positive scalar multiplication is called a *wedge*. Wedges are convex and order X by $x \leq y$ iff $y - x \in P$. If $P \cap -P = \emptyset$, P is called a *cone*.

★ - We have $X = P - P$ iff for all $x, y \in X$, there exists $z \in X$ such that $x \leq z$ and $y \leq z$.

- In this context, a linear functional τ is *positive* iff $\tau(x) \geq 0$ for all $x \in P$. Write P^* for the collection of positive linear functionals on X.
- We work exclusively on topological vector spaces, though some of the theorems on the next slide may be stated more generally.

A digression: the Hahn-Banach theorem and its allies

- *Hahn-Banach theorem:* Let M be a linear subspace of X, and let ρ be a seminorm on X. If $\varphi \in M^*$ with $|\varphi(\cdot)| \leq \rho|M(\cdot)$. Then there exists an extension $\tilde{\varphi} \in X^*$ of φ such that $|\tilde{\varphi}(\cdot)| \leq \rho(\cdot)$.

- We state three more results which follow from the Hahn-Banach theorem.

★

- *Krein-Rutman theorem:* Let M be a linear subspace of X ordered by a wedge P with $\mathrm{int}(P) \neq \emptyset$. If $P_M := \mathrm{int}(P) \cap M \neq \emptyset$, then any element of P_M^* has an extension to P^*.
- *Separation theorem:* Let A and B be disjoint nonempty convex subsets of X with A closed and B compact. Then A and B can be strictly separated by a closed hyperplane (that is, there exists $\tau \in X^*$ and $\alpha \in \mathbb{R}$ such that $\tau(A) > \alpha$ and $\tau(B) < \alpha$).
- *Support theorem:* A closed convex subset C with nonempty interior in a real TVS is supported at every boundary point by a closed hyperplane (that is, for any $x \in \partial C$, there exists $0 \neq \tau \in X^*$ and $\alpha \in \mathbb{R}$ such that $\tau(C) \geq \alpha$ and $\tau(x) = \alpha$).

The cone of positive elements in a C^*-algebra

★
- Given a positive element a of a unital C^*-algebra A, we have $\|a\| = \inf\{\alpha \geq 0 : \alpha 1 - a \geq 0\}$.
- For general $a \in A$, it follows by Theorem 1.10 that

$$\|a\| = \inf\{\alpha \geq 0 : \alpha^2 1 - a^* a \geq 0\}.$$

- In particular, a is a contraction iff $1 - a^* a \geq 0$.

★
- The cone A^+ of positive elements of A is norm closed.
- The set A_{sa} of selfadjoint elements of the C^*-algebra A form a *real* vector space. The cone A^+ spans A_{sa} and defines a partial order on A_{sa}.

The archimedean property for the cone of positive elements

- A positive cone on an ordered vector space is said to be *archimedean* if it has an *order unit*; that is, an element e in the cone such that for any a in the space, there is a constant $\beta \geq 0$ such that $\beta e \pm a$ is in the cone.
- In the ordered vector space A_{sa} in a unital C^*-algebra A, the positive cone A^+ is norm closed and archimedean with order unit 1. It has nonempty interior (for example, it contains the open unit ball about 1 in A_{sa}).
- ★ The element $\|a\|^2 1 - a^* a \geq 0$ is on ∂A^+. Hence by the support theorem, there is a nonzero positive linear functional τ such that $\|a\|^2 \tau(1) = \tau(a^* a)$.

Approximate units

- An *approximate unit* for a C^*-algebra A is an increasing net $(u_\lambda)_{\lambda \in \Lambda}$ of positive elements bounded in norm by 1 such that for all $a \in A$,

$$\lim_\lambda u_\lambda a = \lim_\lambda a u_\lambda = a.$$

★ - In fact, the first equality is automatic.

- **Example:** Let $\mathscr{H}$ be an separable infinite dimensional Hilbert space with ★ orthonormal basis $(e_n)_{n=1}^\infty$. The C^*-algebra of compact operators $\mathscr{C}(\mathscr{H})$ is nonunital. Let p_n be the orthonormal projection onto the space spanned by the first n basis elements. Claim that $(p_n)_n$ is an approximate unit for $\mathscr{C}(\mathscr{H})$. We only need to show that $\lim_n p_n a$ for any $a \in \mathscr{C}(\mathscr{H})$ where a has finite rank, since such elements are dense in $\mathscr{C}(\mathscr{H})$. We can write

$$a = \sum_{k=1}^m x_k \otimes y_k := \sum_{k=1}^m \langle \cdot, y_k \rangle x_k.$$

Since for any $x \in \mathscr{H}$, $\lim_n p_n x = x$, it then follows that $\lim_n p_n a = a$.

Theorem 1.11. *Every C^*-algebra admits an approximate unit.*

Indeed, if we take Λ to be the set of all positive elements of A of norm less than one with the usual order, then Λ will be a directed set, and with $u_\lambda = \lambda$, $(u_\lambda)_\lambda$ will be the *canonical approximate unit*.

Proof of Theorem 1.11.

We show that for any $a \in A$, $\lim_\lambda u_\lambda a = a$ where $(u_\lambda)_\lambda$ is the canonical approximate unit. Since the elements of $(u_\lambda)_\lambda$ span A, it suffices to consider $a \in (u_\lambda)_\lambda$.

Let $1 > \epsilon > 0$ and $\Phi : C^*(a) \to C_0(\sigma(a) \backslash \{0\})$ be the Gel'fand representation (nonunital case!). Since $\sigma(a)$ is compact and Hausdorff, it is T_4. Also $z = \Phi(a)$ is continuous so $K = \{t \in \mathbb{R} : z(t) \geq \epsilon\}$ is compact and thus closed. Hence we may apply Urysohn's lemma with K and $\{0\}$ to find continuous $g : \sigma(a) \to [0,1]$ with $g(K) = 1$ and $g(\{0\}) = 0$. Restricting to $\sigma(a) \backslash \{0\}$, this means that $g \in C_0(\sigma(a) \backslash \{0\})$.

Choose $1 - \epsilon < \delta < 1$. Then $\|z - \delta g z\| < \epsilon$, so if $\lambda_0 = \Phi^{-1}(\delta g)$, then $\lambda_0 \in \Lambda$ and $\|a - u_{\lambda_0} a\| < \epsilon$. On the other hand, if $\lambda > \lambda_0$, then in A_1, $1 - u_\lambda \leq 1 - u_{\lambda_0}$ and so $a(1 - u_\lambda)a \leq a(1 - u_{\lambda_0})a$. Hence

$$\begin{aligned}\|a - u_\lambda a\|^2 &= \|(1-u_\lambda)^{1/2}(1-u_\lambda)^{1/2} a\|^2 \leq \|(1-u_\lambda)^{1/2} a\|^2 = \|a(1-u_\lambda)a\| \\ &\leq \|a(1-u_{\lambda_0})a\| \leq \|(1-u_{\lambda_0})a\| < \epsilon,\end{aligned}$$

and so $a = \lim_\lambda u_\lambda a$. □

Separable C^*-algebras and sequential approximate units

- A C^*-algebra is *separable* if it is a separable topological space in the norm topology (that is, has a countable dense subset).
- *If a C^*-algebra is separable, it admits a sequential approximate unit:* Let $(F_n)_n$ a sequence of nested finite sets such that $F = \bigcup_n F_n$ is dense in A, and let $(u_\lambda)_\lambda$ be an approximate unit for A. For fixed F_n, there exist λ_n such that $\|a - au_\lambda\| < 1/n$ for all $a \in F_n$ and $\lambda \geq \lambda_n$. Since the F_ns are nested, we may choose $\lambda_n < \lambda_{n+1}$ for all n. Hence $\lim_{n\to\infty} \|a - au_{\lambda_n}\| = 0$ for all $a \in F$, and so since F is dense, for all $a \in A$. Thus $(u_{\lambda_n})_n$ is a sequential approximate unit.

- Unless otherwise noted, ideals in C^*-algebras are assumed to be two-sided and closed.

Lemma 1.12. *Every ideal I of a C^*-algebra A is selfadjoint, and so a C^*-algebra. Furthermore, A/I is a C^*-algebra, and if $(u_\lambda)_\lambda$ is an approximate unit for I, then*

$$\|a+I\| = \lim_\lambda \|a - u_\lambda a\| = \lim_\lambda \|a - a u_\lambda\|.$$

Proof.

The ideal I is a norm closed subalgebra of the C^*-algebra A, and so $B := I \cap I^*$ is a C^*-subalgebra of A. Let $(u_\lambda)_\lambda$ be an approximate unit for B. If $b \in I$, then $bb^* \in B$ and

$$\lim_\lambda \|b^* - b^* u_\lambda\|^2 = \lim_\lambda \|(bb^* - bb^* u_\lambda) - u_\lambda(bb^* - bb^* u_\lambda)\| = 0.$$

Since $u_\lambda \in I$ and I is norm closed, it follows that $b^* \in I$. Hence $I = I^*$.

Proof continued

Let $\epsilon > 0$, $a \in A$. Then there exists $x \in I$ such that $\|a + x\| \leq \|a + I\| + \epsilon/2$, and since $x \in I$, there exists λ_0 such that for all $\lambda \geq \lambda_0$, $\|x - u_\lambda x\| < \epsilon/2$. Thus for such λ,

$$\|a - u_\lambda a\| \leq \|(1 - u_\lambda)(a + x)\| + \|x - u_\lambda x\| \leq \|a + x\| + \|x - u_\lambda x\| < \|a + I\| + \epsilon.$$

Hence $\|a + I\| = \lim_\lambda \|a - u_\lambda a\|$, and since $\|a^* + I\| = \|a + I\|$, $\|a + I\| = \lim_\lambda \|a - a u_\lambda\|$ as well.

Let $a \in A$ and $x \in I$. Then

$$\begin{aligned}
\|a + I\|^2 &= \lim_\lambda \|a - u_\lambda a\|^2 = \lim_\lambda \|(1 - u_\lambda) a^* a (1 - u_\lambda)\| \\
&\leq \sup_\lambda \|(1 - u_\lambda)(a^* a + x)(1 - u_\lambda)\| + \lim_\lambda \|(1 - u_\lambda) x (1 - u_\lambda)\| \\
&\leq \|a^* a + x\| + \lim_\lambda \|x - u_\lambda x\| \\
&= \|a^* a + x\|.
\end{aligned}$$

Since x is arbitrary, we have $\|a + I\|^2 \leq \|a^* a + I\|$, and so A/I is a C^*-algebra. $\square$

Theorem 1.13. *Let $\varphi : A \to B$ be a $*$-homomorphism and $q : A \to A/\ker\varphi$ the quotient map. Then $\varphi(A)$ is a C^*-subalgebra of B and the induced map $\tilde{\varphi} : A/\ker\varphi \to B$ given by $\tilde{\varphi}(q(a)) = \varphi(a)$ is isometric.*

Proof.

By Theorem 1.1, φ is continuous, and so $\ker\varphi$ is closed. It is clearly an ideal, so by Lemma 1.12, $C := A/\ker\varphi$ is a C^*-algebra. Since $\|a\|^2 = \|a^*a\|$ and $\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\|$, it suffices to show that φ is isometric when restricted to $C^*(a)$, a selfadjoint.

The map $\tilde{\varphi}$ given by $\tilde{\varphi}(q(a)) = \varphi(a)$ is well defined and injective. Extending if ★ necessary, we take C, B and $\tilde{\varphi}$ to be unital. If $\chi \in X(B)$, then $\tilde{\chi} := \chi \circ \tilde{\varphi} \in X(C)$, and the map $\varphi' : X(B) \to X(C)$ by $\varphi'(\chi) = \tilde{\chi}$ is continuous. Hence $\varphi'(X(B))$ is a compact and so closed subspace of the T_4 space $X(C)$. If $\varphi'(X(B)) \neq X(C)$, by Urysohn's lemma, there is a nonzero continuous function g on $X(C)$ vanishing on $\varphi'(X(B))$. By the Gel'fand representation theorem, $g = \widehat{c}$ for some $c \in C$. Thus for each $\chi \in X(B)$, $\chi(\tilde{\varphi}(c)) = \widehat{c}(\tilde{\chi}) = 0$, and so $\tilde{\varphi}(c) = 0$. We conclude that $c = 0$. But then $g = 0$, giving a contradiction. Hence $\tilde{\varphi}$ is surjective. Finally, for any $c \in C$,

$$\|c\| = \|\widehat{c}\|_\infty = \sup_{\tilde{\chi} \in X(C)} |\tilde{\chi}(c)| = \sup_{\chi \in X(B)} |\chi(\tilde{\varphi}(c))| = \|\tilde{\varphi}(c)\|,$$

so $\tilde{\varphi}$ is isometric. $\square$

Corollary 1.14. *Let A be a C^*-algebra, I an ideal and B a C^*-subalgebra of A. Then $B+I$ is a C^*-algebra and the map*

$$\varphi : B/(B\cap I) \to (B+I)/I \quad \textit{by} \quad b+(B\cap I) \mapsto b+I$$

is a $*$-isomorphism.

Proof.

Let $q: A \to A/I$ be the quotient map on A, $q' : B \to A/I$ the restriction of this map to B. Since q' is a $*$-homomorphism, by Theorem 1.13 its range is a C^*-algebra and hence is closed. The map q is continuous, hence $q^{-1}(q'(B)) = B+I$ is closed, and so a C^*-subalgebra of A containing the ideal I. Let $\widehat{q}: B+I \to (B+I)/I$ be the quotient map and set q'' to be the restriction of $\widehat{q}$ to B. Since $\ker \widehat{q} = I$ and $\widehat{q}$ is surjective, it follows that q'' is also surjective.

The space $\ker q'' = B\cap I$ is an ideal of B. So if $\tilde{q}: B \to B/(B\cap I)$ is the quotient map, the map $\varphi : B/(B\cap I) \to (B+I)/I$ given by $\varphi \circ \tilde{q} = q''$ is a well defined $*$-homomorphism. If $\varphi(\tilde{q}(b)) = 0$, then $q''(b) = 0$, and so $\tilde{q}(b) = 0$. Thus φ is injective, and since q'' is surjective, φ is as well. Hence by Theorem 1.13, φ is isometric $*$-isomorphism. □

Tensor products of vector spaces

- Let $\mathscr{H}$ and $\mathscr{K}$ be vector spaces, and consider the cartesian product as a vector space $\mathscr{F}$ freely generated by the elements of $\mathscr{H} \times \mathscr{K}$. Let R be the following set of relations on $\mathscr{F}$:

$$
\begin{aligned}
(h_1 + h_2) \times k &\sim h_1 \times k + h_2 \times k \\
h \times (k_1 + k_2) &\sim h \times k_1 + h \times k_2 \\
\alpha(h \times k) &\sim (\alpha h) \times k \sim h \times (\alpha k).
\end{aligned}
$$

- Define the (algebraic) *tensor product* of $\mathscr{H}$ and $\mathscr{K}$ to be $\mathscr{H} \otimes \mathscr{K} = \mathscr{F}/R$.
- If $\mathscr{H}$ and $\mathscr{K}$ are normed vector spaces, there are in general numerous ways in which we can put a norm on $\mathscr{H} \otimes \mathscr{K}$. We generally require though that the norm be a *cross norm*, in the sense that $\|h \otimes k\| = \|h\| \|k\|$. In most cases this only partially specifies the norm!

Tensor products of Hilbert spaces

- If $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces, we can make $\mathscr{H} \otimes \mathscr{K}$ into a pre-Hilbert space by defining the inner product as

$$\langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle = \langle h_1, h_2 \rangle \langle k_1, k_2 \rangle,$$

  and extending by linearity. The completion of $\mathscr{H} \otimes \mathscr{K}$ in the associated cross norm is called the *Hilbert space tensor product* and is denoted by $\mathscr{H} \hat{\otimes} \mathscr{K}$.

★

- If E_1 and E_2 are bases for $\mathscr{H}$ and $\mathscr{K}$, respectively, then $E_1 \otimes E_2 = \{e_1 \otimes e_2 : e_1 \in E_1,\ e_2 \in E_2\}$ is a basis for $\mathscr{H} \otimes \mathscr{K}$, so $\dim(\mathscr{H} \otimes \mathscr{K}) = \dim(\mathscr{H}) \cdot \dim(\mathscr{K})$. If $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces and E_1 and E_2 are orthonormal bases, then $E_1 \otimes E_2$ is an orthonormal basis for $\mathscr{H} \otimes \mathscr{K}$.

Tensor products of algebras

- If A and B are algebras, we define the tensor product as for vector spaces. To make $A \otimes B$ into an algebra, we set $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ and extend by linearity. This multiplication is uniquely defined, and $A \otimes B$ is called the *algebra tensor product* of A and B.
- If A and B are unital algebras, the following relation is often useful:

$$(a \otimes 1)(1 \otimes b) = (1 \otimes b)(a \otimes 1) = (a \otimes b).$$

- If A and B are $*$-algebras, we can define an involution on $A \otimes B$ by $(a \otimes b)^* = a^* \otimes b^*$ (*note the order!*). One can show that this is well-defined and unique.

Tensor products of C^*-algebras

- If A and B are C^*-algebras, it is natural to want the tensor product to be a C^*-algebra. There are several problems we face.
  - The first is to define a norm on the algebraic tensor product which is a cross norm.
  - Secondly, the resulting norm should satisfy the C^*-property.
  - Finally, $A \otimes B$ will probably not be complete in the resulting norm, so we will need to complete.
- As it happens, there are a number of ways we can norm $A \otimes B$ when A and B are C^*-algebras.
- We construct one, called the spatial tensor product, by first constructing such a norm on the tensor product of concrete C^*-algebras, and then pulling the norm back to the abstract setting via the Gel'fand-Naĭmark theorem.

$\mathcal{B}(\mathcal{H})\hat{\otimes}\mathcal{B}(\mathcal{K})$

- We identify $\mathcal{B}(\mathcal{H})\hat{\otimes}\mathcal{B}(\mathcal{K})$ with $\mathcal{B}(\mathcal{H}\hat{\otimes}\mathcal{K})$ as follows. For $a \in \mathcal{B}(\mathcal{H})$, $b \in \mathcal{B}(\mathcal{K})$, set

$$(a\otimes b)(h\otimes k) = a(h)\otimes b(k) \qquad \text{for all } h\in\mathcal{H},\ k\in\mathcal{K},$$

  and extend linearly.

★ - If $z \in \mathcal{H}\otimes\mathcal{K}$, we may write $z = \sum x_j \otimes y_j$, where $\{y_1,\dots,y_n\}$ is an orthonormal set. Then $x_j\otimes y_j$ and $x_k\otimes y_k$ are orthogonal for $k\neq j$, and so $\|z\|^2 = \sum \|x_j\|^2$. Hence

$$\begin{aligned}\|(a\otimes 1)(z)\|^2 &= \|\sum a(x_j)\otimes y_j\|^2 = \sum \|a(x_j)\otimes y_j\|^2 = \sum \|a(x_j)\|^2\|y_j\|^2\\ &= \sum \|a(x_j)\|^2 \le \|a\|^2 \sum \|x_j\|^2 = \|a\|^2\|z\|^2.\end{aligned}$$

  Similarly, $\|(1\otimes b)(z)\| \le \|b\|\|z\|$, and so $\|(a\otimes b)(z)\| \le \|a\|\|b\|\|z\|$. Thus $\|a\otimes b\| \le \|a\|\|b\|$.

- Let $\epsilon > 0$, $a, b \neq 0$. Choose unit vectors x, y such that $0 < \|a\| - \epsilon < \|a(x)\|$, $0 < \|b\| - \epsilon < \|b(y)\|$. Then

$$\|a\otimes b\| \ge \|(a\otimes b)(x\otimes y)\| = \|a(x)\otimes b(y)\| = \|a(x)\|\|b(y)\| > (\|a\|-\epsilon)(\|b\|-\epsilon).$$

  Since ϵ is arbitrary, we find $\|a\otimes b\| \ge \|a\|\|b\|$, and so equality holds.

- Now extend $a\otimes b$ to $\mathcal{H}\hat{\otimes}\mathcal{K}$ and call the resulting operator $a\hat{\otimes}b$.

★ - The maps $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}\hat{\otimes}\mathcal{K})$ given by $a \mapsto a\hat{\otimes}1$ and $\mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H}\hat{\otimes}\mathcal{K})$ given by $b \mapsto 1\hat{\otimes}b$ are injective $*$-homomorphisms, so by Theorem 1.13 they are isometric.

Theorem 1.19. *Let A, B be C^*-algebras with representations $(\varphi, \mathscr{H})$ and $(\psi, \mathscr{K})$, respectively. Then there is a $*$-homomorphism $\varphi\hat{\otimes}\psi : A \otimes B \to \mathscr{B}(\mathscr{H}\hat{\otimes}\mathscr{K})$ such that for all $a \in A$, $b \in B$, $(\varphi\hat{\otimes}\psi)(a \otimes b) = \varphi(a)\hat{\otimes}\psi(b)$. This map is injective if φ and ψ are faithful.*

Proof.

The maps $\varphi' : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H}\hat{\otimes}\mathscr{K})$ given by $\varphi'(a) = \varphi(a)\hat{\otimes}1$ and $\psi' : \mathscr{B}(\mathscr{K}) \to \mathscr{B}(\mathscr{H}\hat{\otimes}\mathscr{K})$ given by $\psi'(b) = 1\hat{\otimes}\psi(b)$ are commuting $*$-homomorphisms ★ which are injective if φ and ψ are injective. Hence there is a unique $*$-homomorphism on $A \otimes B$ defined on elementary tensors by

$$(\varphi\hat{\otimes}\psi)(a \otimes b) := \varphi'(a)\psi'(b) = \varphi(a)\hat{\otimes}\psi(b), \qquad a \in A, \;\; b \in B.$$

If the representations φ and ψ are faithful, they are isometric (Theorem 1.13). Hence if $c = \sum a_j \otimes b_j \in \ker \varphi\hat{\otimes}\psi$ where $\{b_1, \dots, b_n\}$ are chosen to be linearly independent, then $\{\psi(b_1), \dots, \psi(b_n)\}$ are linearly independent since ψ is injective. Thus the elements of $\{\varphi(a_j) \otimes \psi(b_j)\}$, where j ranges over the set where $\varphi(a_j) \neq 0$, are linearly independent. Since $0 = \varphi\hat{\otimes}\psi(c) = \sum \varphi(a_j) \otimes \psi(b_j)$ it follows that for all j, $\varphi(a_j) = 0$, and so by faithfulness that $a_j = 0$. We conclude that $c = 0$; that is, $\varphi\hat{\otimes}\psi$ is injective. □

- Let A and B be C^*-algebras with universal representations $(\varphi, \mathscr{H})$ and $(\psi, \mathscr{K})$, respectively. By Theorem 1.19, $\varphi \hat{\otimes} \psi$ is an injective $*$-homomorphism of $A \otimes B$ into $\mathscr{B}(\mathscr{H} \hat{\otimes} \mathscr{K})$.
- We can therefore norm $A \otimes B$ by setting $\|z\| = \|(\varphi \hat{\otimes} \psi)(z)\|$. This is a cross norm, since

$$\|a \otimes b\| = \|(\varphi \hat{\otimes} \psi)(a \otimes b)\| = \|\varphi(a) \otimes \psi(b)\| = \|\varphi(a)\| \|\psi(b)\| = \|a\| \|b\|.$$

- The completion of $A \otimes B$ in this norm is denoted by $A \hat{\otimes} B$, and is called the ★ *spatial tensor product* of A and B. This norm is clearly a C^*-norm.

Tensor products of C^*-algebras, continued

- As an example, consider $A = M_n(\mathbb{C})$ and B any C^*-algebra, then $A \hat{\otimes} B$ is $M_n(B)$, which we saw earlier.
- In general there are many C^*-norms on $A \otimes B$. It can be shown that the spatial norm is the minimal C^*-norm, and there is also a maximal C^*-norm.
- A C^*-algebra A with the property that for any other C^*-algebra B, there is only one C^*-norm on $A \otimes B$ is said to be *nuclear*.
- ★ Finite dimensional C^*-algebras like $M_n(\mathbb{C})$ are nuclear.
- A more difficult statement to prove is that abelian C^*-algebras are nuclear.

Double Centralizers

- The most interesting case in what follows is when A is a nonunital C^*-algebra.
- A pair of maps (L,R), $L,R: A \to A$, is called a *double centralizer* if $R(a)b = aL(b)$ for all $a,b \in A$. We write $D(A)$ to denote the set of all double centralizers of A.
- If $c \in A$ and $L_c(a) = ca$, $R_c(a) = ac$, then (L_c, R_c) is a double centralizer.
- However, *a priori* we do not assume that L or R are bounded, or even linear.

Theorem 1.20. *Let $(L,R)\in D(A)$. Then the following hold:*

- $L(ab)=L(a)b$ *and* $R(ab)=aR(b)$*;*
- L *and* R *are linear;*
- L *and* R *are bounded with* $\|L\|=\|R\|$*.*

Set $\alpha(L_1,R_1)+\beta(L_2,R_2)=(\alpha L_1+\beta L_2,\alpha R_1+\beta R_2)$*,* $(L_1,R_1)(L_2,R_2)=(L_1L_2,R_2R_1)$*,* $(L,R)^*:=(R^*,L^*)$ *where* $L^*(a):=(L(a^*))^*$ *and* $R^*(a):=(R(a^*))^*$*,* $\|(L,R)\|:=\|L\|=\|R\|$*, and* $1:=(L_1,R_1)$ *where* $L_1(a)=R_1(a)=a$*. Then* $D(A)$ *is a unital* C^**-algebra containing* A *isometrically isomorphically as an ideal via* $c\mapsto(L_c,R_c)$*, and* $A\cong D(A)$ *if* A *is unital.*

We cheated earlier in claiming a C^*-algebra A has a unitization A_1, since the norm we chose for A_1 might not be a C^*-norm. We can correct this by taking A_1 to be the unital subalgebra generated by A and the identity in $D(A)$.

In general $D(A)$ will be much bigger than A_1. For example, if Ω is a locally compact Hausdorff space, then the unitization of $C_0(\Omega)$ corresponds to $C(\Omega_1)$, Ω_1 the one point compactification of Ω, while $D(A)$ corresponds to $C(\beta\Omega)$, $\beta\Omega$ the Stone-Čech compactification of Ω.

Proof of Theorem 1.20

Proof.

- Throughout, we let $(u_\lambda)_\lambda$ be an approximate unit for the C^*-algebra A. If A is unital, we take $u_\lambda = 1$ for all λ.
- We show that $L(ab) = L(a)b$ and $R(ab) = aR(b)$. Let $a, b \in A$. Then

$$u_\lambda L(ab) = R(u_\lambda)ab = u_\lambda L(a)b,$$

  and so in the limit we get $L(ab) = L(a)b$. Similarly, $R(ab) = aR(b)$.
- Now take $\alpha, \beta \in \mathbb{C}$. Then

$$u_\lambda L(\alpha a + \beta b) = R(u_\lambda)[\alpha a + \beta b] = \alpha R(u_\lambda)a + \beta R(u_\lambda)b = u_\lambda[\alpha L(a) + \beta L(b)],$$

  which in the limit gives linearity of L. Linearity of R is proved likewise.
- We now show L is bounded using the closed graph theorem. Suppose $a_n \to a$, $L(a_n) \to c$. Then for $b \in A$,

$$\begin{aligned} \|b[L(a) - c]\| &\leq \|b[L(a) - L(a_n)]\| + \|b[L(a_n) - c]\| \leq \|R(b)[a - a_n]\| + \|b\|\|L(a_n) - c\| \\ &\leq \|R(b)\|\|a - a_n\| + \|b\|\|L(a_n) - c\| \to 0. \end{aligned}$$

  In particular then, $u_\lambda L(a) = u_\lambda c$, and so in the limit, $L(a) = c$. Likewise for R.

Proof of Theorem 1.20, continued

- Compare $\|L\|$ and $\|R\|$.

$$\|L\|^2 = \sup_{a,\|a\|=1} \|L(a)\|^2 = \sup_{a,\|a\|=1} \|L(a)^* L(a)\| = \sup_{a,\|a\|=1} \|R(L(a)^*)a\|$$
$$\leq \|R\| \sup_{a,\|a\|=1} \|L(a)^*\| = \|R\|\|L\|,$$

  so $\|L\| \leq \|R\|$. Likewise $\|R\| \leq \|L\|$, and so we have equality. The calculation also shows that the norm is a C^*-norm.

★ - The remaining statements are left as an exercise. □

Multipliers

- An ideal I of a C^*-algebra A is termed *essential* iff $I \cap I' \neq 0$ for any other ideal $I' \subset A$.

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- Equivalently, I is essential iff $I^{\perp} := \{a \in A : aI = 0\} = 0$.
- A representation $\pi : A \to \mathscr{B}(\mathscr{H})$ of a C^*-algebra A is said to be *nondegenerate* iff for $x \in \mathscr{H}$, $\pi(A)x = 0$ implies $x = 0$.
- Equivalently, $\pi(A)\mathscr{H}$ is dense in $\mathscr{H}$. For suppose $x \perp \pi(A)\mathscr{H}$. Then for any $y \in \mathscr{H}$, $a \in A$, $0 = \langle \pi(a)y, x \rangle = \langle y, \pi(a^*)x \rangle$, and so $x = 0$. On the other hand, suppose $\overline{\pi(A)\mathscr{H}} = \mathscr{H}$, $x \in \mathscr{H}$, $\|x\| = 1$. Then there is a $y \in \mathscr{H}$, $a \in A$ such that $\|x - \pi(a)y\| < 1/2$, and so $\|\pi(a)y\| > 1/2$. Hence

$$1/4 > \langle x - \pi(a)y, x - \pi(a)y \rangle > 1 - 2\Re\langle x, \pi(a)y \rangle + 1/4,$$

  and so $\langle x, \pi(a)y \rangle \neq 0$. Thus $\pi(a^*)x \neq 0$.
- If π is not nondegenerate, then $\mathscr{K} := \overline{\pi(A)\mathscr{H}}$ reduces $\pi(A)$, and so we can view π as a nondegenerate representation into $\mathscr{B}(\mathscr{K})$.
- Let $\pi : A \to \mathscr{B}(\mathscr{H})$ be a faithful nondegenerate representation of a C^*-algebra A. An operator $m \in \mathscr{B}(\mathscr{H})$ is called a *(two-sided) multiplier* of A iff for all $a \in A$, $m\pi(a) \in \pi(A)$ and $\pi(a)m \in \pi(A)$. We write $M(A)$ for the set of multipliers of A.

★
- It is easy to see that $M(A)$ is an selfadjoint subalgebra of $\mathscr{B}(\mathscr{H})$.

- At first sight, it appears that the definition of $M(A)$ depends on the representation chosen. In fact, this is not the case.
- For the time being, we view A as a concrete C^*-algebra acting nondegenerately on a Hilbert space $\mathscr{H}$. Otherwise, we replace A by a nondegenerate faithful representation of itself. We then write ma for $m\pi(a)$, etc.
- We state the following without proof.

Proposition 1.21. *The algebra $M(A)$ is a unital C^*-algebra containing A as an essential ideal. Furthermore, $M(A) \subseteq A''$, $A_1 \subseteq M(A)$ if A is nonunital, and $M(A) = A$ if A is unital.*

Theorem 1.22. *Let A be a C^*-algebra, $\pi : A \to \mathscr{B}(\mathscr{H})$ a nondegenerate faithful $*$-representation. The map $\mu : M(A) \to D(A)$ given by $\mu(x) = (L_x, R_x)$ where $L_x(a) = \pi^{-1}(x\pi(a))$ and $R_x(a) = \pi^{-1}(\pi(a)x)$, is an isometric $*$-isomorphism. In particular, the definition of $M(A)$ is independent (up to isometric $*$-isomorphism) on the choice of representation.*

Operator algebras, systems and spaces

- A *(concrete) operator algebra* is a norm closed subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.
- A *(concrete) operator space* is a closed subspace of $\mathscr{B}(\mathscr{H})$.
- A *(concrete) operator system* is an operator space which is selfadjoint and contains the identity.
- Since by assumption an operator system S contains the identity, $S^+ = S \cap \mathscr{B}(\mathscr{H})^+ \neq \emptyset$, where $\mathscr{B}(\mathscr{H})^+$ is the positive cone of $\mathscr{B}(\mathscr{H})$. Consequently, for $f \in S$,

$$f = \tfrac{1}{2}(\|f\| \cdot 1 + f) - \tfrac{1}{2}(\|f\| \cdot 1 - f)$$

  gives a decomposition of f as the difference of two elements of S^+; that is, $S = S^+ - S^+$.

Operator algebras, systems and spaces, continued

- Operator algebras, systems and spaces have so far only been defined *concretely*.
- Our goals is to mimic what we have done with C^*-algebras, to defining abstract versions of these which are (completely) isometrically isomorphic to the concrete version.
- Because of the Gel'fand-Naimark theorem (Theorem 1.18), we could equally well have defined these objects as subsets of a C^*-algebra, and we will often view them in this way.
- As usual, we will write $M_n(S)$ for the collection of $n \times n$ matrices with entries in S. For a map φ defined on S, we retain the notation $\varphi^{(n)}$ for the ampliation of φ to $M_n(S)$; that is, the application of φ to every entry of elements of $M_n(S)$.
- Obviously, if S is an operator algebra / space / system in a C^*-algebra A, then $M_n(S)$ is an object of the same type in $M_n(A)$.

- Throughout, A and B will be C^*-algebras. We use $X \subseteq A$ and $Y \subseteq B$ to denote operator spaces, and $S \subseteq A$ and $T \subseteq B$ operator systems. All maps are assumed to be linear.
- A map $\varphi : X \to Y$ is *bounded* if it is bounded as a map between X and Y viewed as Banach subspaces of A and B. Write $\mathscr{B}(X, Y)$ for the space of bounded maps.
- The map $\varphi : X \to Y$ is *completely bounded* (*cb* for short) if the set $\{\|\varphi^{(n)}\|\}_n$ is bounded. In this case, we define the *cb norm* of φ to be

$$\|\varphi\|_{cb} = \sup_n \|\varphi^{(n)}\|.$$

  The space of cb maps is denoted by $\mathscr{C}\mathscr{B}(X, Y)$.

Completely contractive / positive maps; complete (isometric / order) isomorphisms

- A cb map is *completely contractive* if it has cb norm less than or equal to 1.
- A map φ is a *complete isometry* if $\varphi^{(n)}$ is isometric for all n.
- A map $\varphi : S \to T$ is *positive* if $\varphi(S^+) \subset T^+$. It is *completely positive* if $\varphi^{(n)}$ is positive for all n.
- Two operator spaces X and Y are *completely isomorphic* if there is a linear isomorphism $\varphi : X \to Y$ such that φ and φ^{-1} are completely bounded. Call φ a *complete isomorphism*.
- If φ and φ^{-1} are completely isometric, then we say that X and Y are *completely isometrically isomorphic*, or simply *completely isometric*.
- Finally, if φ is a completely positive isomorphism, then it is called a *complete order isomorphism* and the spaces are *completely order isomorphic*

Banach spaces vs operator spaces

- Operator spaces, being closed subspaces of a Banach algebra, are Banach spaces.
- Furthermore, every Banach space X is a closed subspace of some C^*-algebra: set Ω_X be the norm closed unit ball of X^* with the weak-$*$ topology. Then X embeds isometrically in $X^{**} \subset C(\Omega_X) \subset \mathscr{B}(\mathscr{H})$, where $\mathscr{H} = \ell_2(\Omega_X)$.
- As we shall see, what distinguishes the category of Banach spaces from the category of operator spaces is the set of morphisms.
- For the category of Banach spaces, the morphisms are bounded maps, for the category of operator spaces, the morphisms are the completely bounded maps.
- In fact, Banach space theory can be viewed as a part of operator space theory, since if we embed Banach spaces X and Y into commutative C^*-algebras $C(\Omega_X)$ and $C(\Omega_Y)$ as above, then a map $\varphi : X \to Y$ is necessarily bounded iff it is completely bounded (with $\|\varphi\|_{cb} = \|\varphi\|$).
- Indeed, we can say even more whenever the range of a bounded or positive map is contained in a commutative C^*-algebra.

Maps into $C_0(\Omega)$

- Let A be a C^*-algebra. Note that for $a=(a_{ij})\in M_n(A)$,

$$\begin{pmatrix} \overline{\beta_1}1 & \cdots & \overline{\beta_n}1 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} a \begin{pmatrix} \alpha_1 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n 1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} \sum_{ij}\alpha_i a_{ij}\overline{\beta_j} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

- Hence, if $\sum_j |\alpha_j|^2 \le 1$ and $\sum_j |\beta_j|^2 \le 1$, then $\|a\| \ge \left\|\sum_{ij}\alpha_i a_{ij}\overline{\beta_j}\right\|$. Furthermore, if $A=\mathbb{C}$, we have equality for some choice of α_js and β_js.
- Suppose Ω is a locally compact Hausdorff space, $A=C_0(\Omega)$. Then we can identify $M_n(A)$ with $C_0(\Omega, M_n(\mathbb{C}))$, the continuous $n\times n$ complex matrix valued functions on Ω vanishing at infinity. The norm $\|F\|=\sup_{\omega\in\Omega}\|F(\omega)\|$ is a C^*-norm on $C_0(\Omega, M_n(\mathbb{C}))$, and this norm is unique (Corollary 1.4).

Proposition 2.1. *Let X and Y be operator spaces, $Y\subseteq C_0(\Omega)$ for some locally compact Hausdorff space Ω, and $\varphi: X\to Y$ be bounded. Then φ is completely bounded, and $\|\varphi\|_{cb}=\|\varphi\|$.*

Proof.
For each $\omega\in\Omega$, define $\varphi_\omega: X\to\mathbb{C}$ by $\varphi_\omega(x)=\varphi(x)(\omega)$. Then

$$\|\varphi^{(n)}\| = \sup_{x\in X,\|x\|=1}\|\varphi^{(n)}(x)\| = \sup_{\omega\in\Omega}\sup_{x\in X,\|x\|=1}\|\varphi^{(n)}(x)(\omega)\| = \sup_{\omega\in\Omega}\|\varphi_\omega^{(n)}\| = \sup_{\omega\in\Omega}\|\varphi_\omega\| = \|\varphi\|. \qquad \square$$

Some useful positivity results

★ **Lemma 2.2.** *Let A, B be C^*-algebras, $\rho: A \to B$ a positive linear map. Then ρ is bounded and $\rho(b^* a^* a b) \leq \|a^* a\| \rho(b^* b)$.*

★ **Lemma 2.3.** *Let A be a unital C^*-algebra. Then $\begin{pmatrix} 1 & a \\ a^* & b \end{pmatrix} \geq 0$ in $M_2(A)$ iff $b \geq a^* a$.*

Note in particular that by the last lemma, a is contractive iff the matrix is positive with $b = 1$.

- The proof of the first is very much like the proof that a positive linear functional is bounded (See the slide titled *Positive linear functionals, states and pure states*) and the proof of Lemma 1.16.
- For the second, use the factorization

$$\begin{pmatrix} 1 & a \\ a^* & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^* & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b - a^* a \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

and the fact that $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ is invertible, with inverse $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$ (see also the slide titled *The archimedean property for the cone of positive elements*).

Lemma 2.4 (Cauchy-Schwarz inequality for 2-positive maps). *Let A, B be unital C^*-algebras, $\varphi : A \to B$ a linear map which is 2-positive (that is, φ and $\varphi^{(2)}$ are positive). Then for all $a \in A$,*

$$\varphi(a)^* \varphi(a) \leq \|\varphi(1)\| \varphi(a^* a).$$

Proof.

Apply Lemma 2.3 to $\begin{pmatrix} 1 & 0 \\ a^* & 0 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ a^* & a^* a \end{pmatrix}$. Since B is a C^*-algebra and $\varphi(1)$ is positive, $\|\varphi(1)\| 1 \geq \varphi(1) \geq 0$. We use 2-positivity of φ to get $\varphi(a^*) = \varphi(a)^*$. □

Lemma 2.5. *Let A, B be unital C^*-algebras, $S \subseteq A$ an operator system. Suppose that $\varphi : S \to B$ is completely positive. Then φ is completely bounded with $\|\varphi\|_{cb} = \|\varphi\| = \|\varphi(1)\|$.*

Proof.

Clearly, $\|\varphi(1)\| \leq \|\varphi\| \leq \|\varphi\|_{cb}$, so it suffices to prove that $\|\varphi\|_{cb} \leq \|\varphi(1)\|$. Let 1_n be the unit for $M_n(A)$. Then if $a \in M_n(A)$ with $\|a\| \leq 1$, by Lemma 2.3

$$\varphi^{(2n)} \begin{pmatrix} 1_n & a \\ a^* & 1_n \end{pmatrix} = \begin{pmatrix} \varphi^{(n)}(1_n) & \varphi^{(n)}(a) \\ \varphi^{(n)}(a)^* & \varphi^{(n)}(1_n) \end{pmatrix} \geq 0.$$

Hence $\|\varphi^{(n)}(a)\| \leq \|\varphi^{(n)}(1_n)\| = \|\varphi(1)\|$. □

Unital complete order isomorphisms are isometric $*$-isomorphisms

Lemma 2.6. *Let A, B be unital C^*-algebras. If $\varphi: A \to B$ is a unital complete order isomorphism, then φ is an isometric $*$-isomorphism.*

Proof.

By the Cauchy-Schwarz inequality for 2-positive maps, $\varphi(a)^*\varphi(a) \leq \varphi(a^*a)$. Hence

$$a^*a = \varphi^{-1}(\varphi(a)^*)\varphi^{-1}(\varphi(a)) \leq \varphi^{-1}(\varphi(a)^*\varphi(a)) \leq \varphi^{-1}(\varphi(a^*a)) = a^*a,$$

and so we have equality throughout. Thus $a^*a = \varphi^{-1}(\varphi(a)^*\varphi(a))$, and so $\varphi(a^*a) = \varphi(a)^*\varphi(a)$.

Now let $c = \begin{pmatrix} b^* & a \\ a^* & 0 \end{pmatrix}$, $a, b \in A$. Since $\varphi^{(2)}$ is 2-positive, we have $\varphi^{(2)}(c^*c) - \varphi^{(2)}(c)^*\varphi^{(2)}(c) \geq 0$, which becomes

$$\begin{pmatrix} \varphi(bb^*) + \varphi(aa^*) & \varphi(ba) \\ \varphi(a^*b^*) & \varphi(a^*a) \end{pmatrix} - \begin{pmatrix} \varphi(b)\varphi(b^*) + \varphi(a)\varphi(a^*) & \varphi(b)\varphi(a) \\ \varphi(a^*)\varphi(b^*) & \varphi(a^*)\varphi(a) \end{pmatrix} = \begin{pmatrix} * & \varphi(ba) - \varphi(b)\varphi(a) \\ * & 0 \end{pmatrix}.$$

Note that if $\begin{pmatrix} x & y \\ y^* & 0 \end{pmatrix} \geq 0$, then for all $t > 0$, $\begin{pmatrix} 1 & -ty \end{pmatrix}\begin{pmatrix} x & y \\ y^* & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -ty^* \end{pmatrix} = x - 2tyy^* \geq 0$, which implies $y = 0$. Hence $\varphi(ba) = \varphi(b)\varphi(a)$, and so φ is a $*$-isomorphism, which by Lemma 1.13 is isometric. $\square$

Example of a positive map which is not completely positive

- Let $A = M_2(\mathbb{C})$, and define $\varphi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ as the transpose map.

★ - This map is a positive unital map.
- Let $e_{j,k}$ be the so-called *matrix units*; that is $e_{j,k}$ is the matrix which is equal to 1 in the (j,k)-entry and zero elsewhere.
- In $M_2(M_2(\mathbb{C})) = M_4(\mathbb{C})$, define

$$a = \begin{pmatrix} e_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

  which is easily seen to be positive.
- Then

$$\varphi^{(2)}(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

  and since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a nonpositive direct summand, $\varphi^{(2)}(a)$ is not positive.

★ - Similar reasoning can be used to show that φ is not completely bounded either.

Lemma 2.7. *Let A and B be C^*-algebras, $S \subseteq A$ an operator system, Ω a locally compact Hausdorff space. If $\varphi : S \to C_0(\Omega)$ or $\varphi : C_0(\Omega) \to B$ and φ is positive, then φ is completely positive.*

Proof.

Fix $n \in \mathbb{N}$ and define $\varphi_\omega : S \to \mathbb{C}$ as in the proof of Lemma 2.1. Then $\varphi \geq 0 \Leftrightarrow \varphi_\omega \geq 0 \ \forall \omega \Leftrightarrow$ for all $0 \leq a = (a_{ij}) \in M_n(S)$, $x = (\alpha_1, \dots \alpha_n)^t$ and ω, $\varphi_\omega(\sum_{ij} \alpha_i a_{ij} \overline{\alpha_j}) = \langle \varphi_\omega x, x \rangle = \geq 0 \Leftrightarrow \varphi_\omega^{(n)}(a) \geq 0 \ \forall \omega$ and $a \geq 0 \Leftrightarrow \varphi^{(n)}(a)(\omega) \geq 0 \ \forall \omega$ and $a \geq 0 \Leftrightarrow \varphi^{(n)}(a) \geq 0 \ \forall a \Leftrightarrow \varphi^{(n)} \geq 0$.

Now suppose that $\varphi : C_0(\Omega) \to B$ is positive, and let $F \geq 0$ in $M_n(C_0(\Omega))$. Then F extends to a bounded continuous function on Ω_1, the one point compactification of Ω (set $F(\infty) = 0$). Fix $\epsilon > 0$ and define an open cover $\mathscr{U} = \{U_\omega\}$ of Ω_1 by $U_\omega = \{x \in \Omega_1 : \|F(x) - F(\omega)\| < \epsilon\}$. Extract a finite open subcover $\{U_{\omega_1}, \dots, U_{\omega_n}\}$. Let $\{p(1), \dots, p(n)\}$ be a partition of unity subordinate to the cover; that is, $p(k) \geq 0$ for all k, $\sum_k p(k) = 1$ for all i, j, and $p(k)(x) = 0$ for $x \notin U_k$. Then $\|F - \sum_k F(\omega_k) p(k)\| < \epsilon$. Observe that

$$\varphi^{(n)}(F(\omega_k)p(k)) = (\varphi(F(\omega_k)_{ij} p(k))) = (F(\omega_k)_{ij} \varphi(p(k))) = F(\omega_k)\varphi(p(k)) \geq 0.$$

Hence $\varphi^{(n)}(\sum_k F(\omega_k) p(k)) \geq 0$, and since ϵ is arbitrary, it follows that $\varphi^{(n)}(F) \geq 0$. □

Lemma 2.8. *Let A and B be C^*-algebras, $S \subseteq A$ an operator system. If $\varphi : S \to M_n(\mathbb{C})$ or $\varphi : M_n(\mathbb{C}) \to B$ and φ is n-positive (that is, $\varphi^{(n)}$ is positive), then φ is completely positive.*

Proof.

Suppose $m \geq n$, and let $f \in \mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{C}^{mn}$. Claim that there is an isometry $V : \mathbb{C}^n \to \mathbb{C}^m$ and a vector $g \in \mathbb{C}^n \otimes \mathbb{C}^n \cong \mathbb{C}^{n^2}$ such that $(V \otimes 1_n)(g) = f$. To see this, let $\{e_k\}$ be a basis for $\mathbb{C}^n$. Then we can express $f = \sum_{k=1}^n f_k \otimes e_k$, $f_k \in \mathbb{C}^m$. Set $F = \operatorname{span}\{f_k\}$. Note that $\dim F \leq n \leq m$, and so there is an isometry $V : \mathbb{C}^n \to \mathbb{C}^m$ such that $\operatorname{ran} V \supseteq F$. Define $g_j = V^* f_j$ (so $V f_j = g_j$). Then $g = \sum_{k=1}^n g_k \otimes e_k$ is the required vector.

Now suppose $\varphi : S \to M_n(\mathbb{C})$ is n-positive and let $a \in M_m(S)^+$. Show $\varphi^{(m)}(a) \geq 0$. Given $f \in \mathbb{C}^{mn}$, find V and g as above. Then

$$\left\langle \varphi^{(m)}(a) f, f \right\rangle = \left\langle \varphi^{(m)}(a)(V \otimes 1) g, (V \otimes 1) g \right\rangle = \left\langle \varphi^{(n)}(V^* a V) g, g \right\rangle \geq 0,$$

and so $\varphi^{(m)}$ is positive.

★ The other case (when $\varphi : M_n(\mathbb{C}) \to B$) is left as an exercise (if you get stuck, see Paulsen, pp. 34–35). □

Theorem 2.9. *Let A be a unital C^*-algebra, $\mathscr{H}$ a Hilbert space and $\varphi : A \to \mathscr{B}(\mathscr{H})$ a completely positive map. Then there exists a Hilbert space $\mathscr{K}$, a nondegenerate unital $*$-representation $\pi : A \to \mathscr{B}(\mathscr{K})$ and an operator $V \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ with $\|V\|^2 = \|\varphi(1)\|$ such that for all $a \in A$,*

$$\varphi(a) = V^* \pi(a) V.$$

Moreover, if $\phi(1) = 1$, then V is an isometry.

Proof.

The proof is much like that of the Gel'fand-Naĭmark theorem (Theorem 1.18). On the algebraic tensor product $A \otimes \mathscr{H}$, define the scalar product

$$\left\langle \sum_i a_i \otimes f_i, \sum_j b_j \otimes g_j \right\rangle = \sum_{i,j} \left\langle \varphi(b_j^* a_i) f_i, g_j \right\rangle = \left\langle \varphi^{(n)}((b_j^* a_i)) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} \right\rangle,$$

Since φ is assumed to be completely positive, by the last equality it follows that the scalar product is positive semidefinite. Such scalar products satisfy the Cauchy-Schwarz inequality, and so

$$\mathscr{N} := \{u \in A \otimes \mathscr{H} : \langle u, u \rangle = 0\} = \{u \in A \otimes \mathscr{H} : \langle u, v \rangle = 0 \text{ for all } v \in A \otimes \mathscr{H}\}$$

is a subspace of $A \otimes \mathscr{H}$. Hence we get an inner product on $(A \otimes \mathscr{H})/\mathscr{N}$ defined by $\langle u + \mathscr{N}, v + \mathscr{N} \rangle = \langle u, v \rangle$. Set $\tilde{\mathscr{K}}$ to be the Hilbert space completion of $(A \otimes \mathscr{H})/\mathscr{N}$.

Proof of Theorem 2.9, continued

Define a linear map $\tilde{\pi}: A \to \mathscr{B}(A \otimes \mathscr{H})$ by $\tilde{\pi}(a)\sum_i a_i \otimes f_i = \sum_i (aa_i) \otimes f_i$. By complete positivity and Lemma 2.2,

$$\left\langle \tilde{\pi}(a)\sum_i a_i \otimes f_i, \tilde{\pi}(a)\sum_j a_j \otimes f_j \right\rangle = \sum_{i,j}\left\langle \varphi(a_j^* a^* a a_i) f_i, f_j \right\rangle = \left\langle \varphi^{(n)}((a_j^* a^* a a_i)) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle$$

$$\leq \|a\|^2 \left\langle \varphi^{(n)}((a_j^* a_i)) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle = \|a\|^2 \left\langle \sum_i a_i \otimes f_i, \sum_j a_j \otimes f_j \right\rangle.$$

Thus $\tilde{\pi}(a)\mathscr{N} \subseteq \mathscr{N}$, and so $\tilde{\pi}$ gives rise to a well defined map on the quotient space. Also, $\|\tilde{\pi}(a)\| \leq \|a\|$, hence $\tilde{\pi}(a)$ extends to a bounded linear operator on $\tilde{\mathscr{K}}$, which
★ we also denote by $\tilde{\pi}(a)$. Verify that $\tilde{\pi}$ is a unital $*$-homomorphism!

Define $V \in \mathscr{B}(\mathscr{H}, \tilde{\mathscr{K}})$ by $Vf = 1 \otimes f + \mathscr{N}$. Then $\|Vf\|^2 = \langle 1 \otimes f, 1 \otimes f \rangle = \langle \varphi(1)f, f \rangle$, which, upon taking the supremum over f with $\|f\| = 1$ gives $\|V\|^2 = \|\varphi(1)\|$. Clearly,
★ V is an isometry if if $\varphi(1) = 1$. The space $\mathscr{K} := \tilde{\pi}(A)V\mathscr{H} \supset V\mathscr{H}$ reduces $\tilde{\pi}(A)$ and so $\pi := P_{\mathscr{K}}\tilde{\pi}|_{\mathscr{K}}$ is a nondegenerate unital $*$-representation. Finally, for all $f, g \in \mathscr{H}$

$$\langle V^*\pi(a)Vf, g \rangle = \langle \pi(a)(1 \otimes f), 1 \otimes g \rangle = \langle \varphi(a)f, g \rangle,$$

implies $\varphi(a) = V^*\pi(a)V$. □

Application: The Sz.-Nagy dilation theorem

- Let $S \subset C(\mathbb{T})$ be the operator system defined by $S = \{p(z) + \overline{q(z)} : p,\ q \text{ polynomials}\}$.
- Let $T \in \mathscr{B}(\mathscr{H})$ with $\|T\| \leq 1$. Define a map $\varphi : S \to \mathscr{B}(\mathscr{H})$ by $\varphi(p + \overline{q}) = p(T) + q(T)^*$.
- Note that $\varphi(z) = T$, $\varphi(\overline{z}) = T^*$ and $\varphi(1) = 1$. By the Fejér-Riesz theorem, if $f \in S$ and $f \geq 0$, then there is a polynomial g such that $f = \overline{g}g$, and so $\varphi(f) = g(T)^* g(T) \geq 0$.
- Furthermore, since S is norm dense in $C(\mathbb{T})$, φ extends to a positive map on $C(\mathbb{T})$, which we also call φ.
- It follows from Lemma 2.6 that φ is a completely positive map.

Theorem 2.10 (The Sz.-Nagy dilation theorem). *Let $T \in \mathscr{B}(\mathscr{H})$ with $\|T\| \leq 1$. Then there exists a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$ and unitary $U \in \mathscr{B}(\mathscr{K})$ such that $T^n = P_{\mathscr{H}} U^n|_{\mathscr{H}}$ for all $n \in \mathbb{N}$. Furthermore, $\mathscr{K}$ can be chosen so that U is irreducible.*

Proof.

Apply the Stinespring dilation theorem (Theorem 2.9) to write the map φ above as $\varphi(f) = V^* \pi(f) V$. Since φ is unital, V is isometric, and so we can identify $\mathscr{H}$ with a subspace of $\mathscr{K}$. Set $U = \pi(z)$. This is unitary since $\overline{z}z = z\overline{z} = 1$. That ★ $T^n = P_{\mathscr{H}} U^n|_{\mathscr{H}}$ for all $n \in \mathbb{N}$ follows immediately. The last statement is left as an exercise, and follows from the nondegeneracy of π. □

Theorem 2.11. *Let A be a unital C^*-algebra, $S \subseteq A$ an operator system. Then every completely positive map $\varphi : S \to M_n(\mathbb{C})$ can be extended to a completely positive map $\tilde{\varphi} : A \to M_n(\mathbb{C})$ with the same cb-norm.*

Proof.

The goal is to express φ as $\varphi(a) = V^*\pi(a)V$ for all $a \in S$, where $\pi : A \to \mathscr{B}(\mathscr{H})$ is a unital $*$-representation, since by the Stinespring dilation theorem, $\varphi(a) = V^*\pi(a)V$, $a \in A$, defines a cp map on A.

Let $\{e_k\}$ be the standard basis for $\mathbb{C}^n$ and define a linear functional on $M_n(S)$ by

$$\tau_\varphi((s_{ij})) := \sum_{i,j=1}^{n} \langle \varphi(s_{ij})e_j, e_i \rangle = \langle \varphi^{(n)}(s_{ij})e, e \rangle,$$

where e is the vector with n entries e_1 to e_n. Since by assumption φ is cp, it follows that τ_φ is a positive linear functional on $M_n(S)$. By the Krein-Rutman theorem (see the slide *A digression: the Hahn-Banach theorem and its allies*), τ_φ extends to a positive linear functional (also called τ_φ) on $M_n(A)$.

Conversely, suppose τ is a positive linear functional on $M_n(A)$, e_{jk} are the standard matrix units and $a \in A$. Then

$$\varphi_\tau(a) := \left(\tau(a^{jk})\right), \qquad a^{jk} := a \otimes e_{jk},$$

★ defines a linear map from S to $M_n(\mathbb{C})$ with $\varphi_{\tau_\varphi} = \varphi$ and $\tau_{\varphi_\tau} = \tau$. In particular, $\varphi_{\tau_\varphi}(s) = \varphi(s)$ for all s in S.

Proof of Theorem 2.11, continued

We conclude the proof by showing that φ_{τ_φ} extends to A and has a Stinespring decomposition there, so is cp on A.

Let $(\tilde{\pi}, \mathscr{B}(\tilde{\mathscr{K}}))$ be the associated Gel'fand-Naimark representation to τ_φ, with cyclic vector f, so that $\tilde{\tau}(x) = \left\langle \tilde{\pi}(x) f, f \right\rangle$, $x \in M_n(A)$. Note that $\tilde{\pi}(e_{ij} e_{k\ell}) = \tilde{\pi}(e_{i\ell})$ if $j = k$ and 0 otherwise. Also, if a^{jk} is a matrix with all entries 0 except the (j,k)th, which ★ is a, then $\tilde{\pi}(e_{ij} a^{jk} e_{k\ell}) = \tilde{\pi}(a^{i\ell})$. From this we can see that $\tilde{\pi}((x_{ij})) = (\pi(x_{ij}))$ for some representation π of A on a Hilbert space $\mathscr{K}$ such that $\mathscr{K}^n = \tilde{\mathscr{K}}$.

We write $f = (f_k)$ with respect to this decomposition. Define $V : \mathbb{C}^n \to \mathscr{K}$ by $Ve_k = f_k$. Then $\tau_\varphi(a^{jk}) = \left\langle V^* \pi(a) V e_k, e_j \right\rangle$. Thus, for $a \in A$,

$$\varphi_\tau(a) = \left(\tau(a^{ij})\right) = \left(\left\langle V^* \tilde{\pi}(a^{jk}) V e_k, e_j \right\rangle\right) = V^* \pi(a) V.$$

By the Stinespring dilation theorem, Theorem 2.9, $\varphi_\tau : A \to M_n(\mathbb{C})$ is cp. Since $\varphi(s) = \varphi_\tau(s)$ for $s \in S$, φ_τ is a cp extension of φ from S to A. By definition, $1 \in S$, hence $\|\varphi\| = \|\varphi(1)\| = \|\tilde{\varphi}(1)\| = \|\tilde{\varphi}\|$, so the extension preserves the norm. □

The bounded weak topology

- Let X, Y be Banach spaces. As usual, $\mathscr{B}(X,Y^*)$ denotes the bounded linear maps from X to the dual of Y.
- Can we find a predual for $\mathscr{B}(X,Y^*)$? That is, can we find a Banach space Z such that $Z^* = \mathscr{B}(X,Y^*)$?
- Such a space would allow us to endow $\mathscr{B}(X,Y^*)$ with a weak-$*$ topology, which in this context is called the *bounded weak* or *BW topology*.
- We can define a linear functional on $\mathscr{B}(X,Y^*)$ as follows: let $x \in X$, $y \in Y$ and define $(x\otimes y)(L) = L(x)(y) \in \mathbb{C}$ for $L \in \mathscr{B}(X,Y^*)$. Since $|(x\otimes y)(L)| \leq \|L\|\|x\|\|y\|$, we find that $x\otimes y \in \mathscr{B}(X,Y^*)^*$ and $\|x\otimes y\| \leq \|x\|\|y\|$ (in fact this can be shown to be an equality).
- ★ The map $x\otimes y$ is linear in both terms.
- Let Z be the closed span of finite linear combinations of such elementary tensors in $\mathscr{B}(X,Y^*)^*$.
- Note that $L \in \mathscr{B}(X,Y^*)$ defines a bounded linear functional on Z by linearly ★ extending $\langle L, x\otimes y\rangle = (x\otimes y)(L)$, and in fact, $\mathscr{B}(X,Y^*)$ is isometrically isomorphic to a subspace of Z^* in this way.

The bounded weak topology, continued

- On the other hand, if $T \in Z^*$, define $T_x : Y \to \mathbb{C}$ by $T_x(y) = T(x \otimes y)$. This is easily seen to be bounded, and so $T_x \in Y^*$.
- Now define $L \in \mathscr{B}(X, Y^*)$ by $L(x) = T_x$. Then L is a bounded linear map with $\|L\| \leq \|T\|$, so $\mathscr{B}(X, Y^*)$ is isometrically isomorphic to Z^*.
- Let $(L_\alpha)_\alpha$ be a bounded net in $\mathscr{B}(X, Y^*)$. If $L_\alpha \to L \in \mathscr{B}(X, Y^*)$, then $L_\alpha(x)(y) = \left\langle L_\alpha, x \otimes y \right\rangle \to \left\langle L, x \otimes y \right\rangle = L(x)(y)$, so for all x, $L_\alpha(x)$ converges weakly to $L(x)$.
- On the other hand, if $L_\alpha(x)$ converges weakly to $L(x)$ for all x, then $\left\langle L_\alpha, x \otimes y \right\rangle \to \left\langle L, x \otimes y \right\rangle$ on elementary tensor products, and hence on the linear span of these. Since $(L_\alpha)_\alpha$ is bounded, it therefore converges weak-$*$ to L. Hence the name "bounded weak".

- In what follows, we write B^+ for set of cp maps $\varphi : S \to \mathscr{B}(\mathscr{H})$ with $\|\varphi\| \leq 1$.
- By Alaoglu's theorem, B, the norm closed unit ball of bounded maps from S to $\mathscr{B}(\mathscr{H})$, is compact in the BW-topology. Since B^+ is a norm closed subset of B, it too is compact in the BW-topology.
- ★ We will also make use of the fact that unital completely positive maps are completely contractive (and hence contractive).

The next theorem is an operator system version of the Krein-Rutman theorem.

Theorem 2.12. *Let S be an operator system contained in a C^*-algebra A, $\varphi : S \to \mathscr{B}(\mathscr{H})$ a completely positive map. Then there exists a completely positive map $\tilde{\varphi} : A \to \mathscr{B}(\mathscr{H})$ extending φ and with the same cb-norm.*

Proof.

For each finite dimensional subspace $\mathscr{F} \subseteq \mathscr{H}$, define $\varphi_{\mathscr{F}}$ to be the compression of φ to $\mathscr{F}$, $P_{\mathscr{F}}\varphi(\cdot)|_{\mathscr{F}}$. Note that this is a cp map from S to $\mathscr{B}(\mathscr{F}) \cong M_n(\mathbb{C})$ and $\|\varphi_{\mathscr{F}}\| = \|P_{\mathscr{F}}\varphi(1)|_{\mathscr{F}}\| \leq \|\varphi(1)\| = \|\varphi\|$. By Theorem 2.11, this extends to a cp map $\tilde{\varphi}_{\mathscr{F}}$ from A to $\mathscr{B}(\mathscr{F}) \cong M_n(\mathbb{C})$. We make this into a cp map from A to $\mathscr{B}(\mathscr{H})$ by setting $\tilde{\varphi}_{\mathscr{F}}|_{\mathscr{F}^\perp} = 0$. Order the finite dimensional spaces by inclusion to get a directed set. Then $(\tilde{\varphi}_{\mathscr{F}})_{\mathscr{F}}$ is a bounded net in B^+, and so there is a convergent subnet with limit $\tilde{\varphi} \in B^+$. Note that if $a \in S$ and $f, g \in \mathscr{H}$, by cofinality there is some $\mathscr{F}$ such that $\tilde{\varphi}_{\mathscr{F}}$ is in the subnet and contains f and g. Then for all $\mathscr{G} \supset \mathscr{F}$,

$$\langle \varphi(a)f, g \rangle = \langle \varphi_{\mathscr{F}}(a)f, g \rangle = \langle \tilde{\varphi}_{\mathscr{F}}(a)f, g \rangle = \langle \tilde{\varphi}_{\mathscr{G}}(a)f, g \rangle = \langle \tilde{\varphi}(a)f, g \rangle.$$

□

An application to unital operator spaces

- As we noted earlier, unital completely positive maps are completely contractive. As it happens there is something of a converse to this.
- Let $X \subset A$ be a unital operator space in a unital C^*-algebra A. Define $S := A + A^*$. Then S is an operator system.
- Suppose that $\varphi : X \to \mathscr{B}(\mathscr{H})$ is a unital completely contractive map. Extend φ to S by setting $\varphi(a^*) = \varphi(a)^*$. Then $\varphi : S \to \mathscr{B}(\mathscr{H})$ is well defined and completely positive.
- As a corollary to the Arveson extension theorem (Theorem 2.12), we find that any unital completely contractive map $\varphi : X \to \mathscr{B}(\mathscr{H})$ has an extension to a unital completely positive (and so completely contractive) map $\tilde{\varphi} : A \to \mathscr{B}(\mathscr{H})$.
- If we then apply the Stinespring dilation theorem, we see that there is a unital $*$-homomorphism $\pi : A \to \mathscr{B}(\mathscr{K})$, $\mathscr{K} \supset \mathscr{H}$ a Hilbert space, such that

$$\varphi(a) = P_{\mathscr{H}} \pi(a)|_{\mathscr{H}}.$$

- Note that if X is an operator algebra (so closed under multiplication), this is still valid. In addition, if φ is an algebra homomorphism, then its extension to S is a completely positive map, and so the result applies.

Lemma 2.12. *Let A, B be unital C^*-algebras, $X \subseteq A$ an operator space, $\varphi : X \to B$. Define an operator system*

$$S_X := \left\{ \begin{pmatrix} \lambda 1 & a \\ b^* & \mu 1 \end{pmatrix} : \lambda, \mu \in \mathbb{C},\ a, b \in X \right\} \subseteq M_2(A),$$

and a map $\varphi' : S_X \to M_2(B)$ by

$$\varphi'\left(\begin{pmatrix} \lambda 1 & a \\ b^* & \mu 1 \end{pmatrix} \right) = \begin{pmatrix} \lambda 1 & \varphi(a) \\ \varphi(b)^* & \mu 1 \end{pmatrix}.$$

Then φ is completely contractive iff φ' is completely positive.

Proof.

Let $S \in M_n(S_X)$, where $S_{ij} = \begin{pmatrix} \lambda_{ij} 1 & a_{ij} \\ b_{ij}^* & \mu_{ij} 1 \end{pmatrix}$. Performing the so-called canonical shuffle, we have that S is unitarily equivalent to a matrix of the form $S' = \begin{pmatrix} L & C \\ D^* & M \end{pmatrix}$, where $L = (\lambda_{ij} 1)$, $M = (\mu_{ij} 1)$, $C = (a_{ij})$ and $D = (b_{ij})$. Clearly $S \geq 0$ iff $S' \geq 0$, which implies that $L, M \geq 0$ and $D = C$. Furthermore, $\varphi'^{(n)}(S)$ becomes, after the canonical shuffle, $\begin{pmatrix} L & \varphi^{(n)}(C) \\ \varphi^{(n)}(D)^* & M \end{pmatrix}$, so it suffices to show that this is positive when $S' \geq 0$.

For $\epsilon > 0$, let $L_\epsilon = L + \epsilon 1$, $M_\epsilon = M + \epsilon 1$. These are positive and invertible, and so $S' \geq 0$ iff $\begin{pmatrix} 1 & G_\epsilon \\ G_\epsilon^* & 1 \end{pmatrix} \geq 0$ for all $\epsilon > 0$, where $G_\epsilon = L_\epsilon^{-1/2} C M_\epsilon^{-1/2}$. We therefore have by Lemma 2.3 that $S' \geq 0$ iff $\|G_\epsilon\| \leq 1$ for all $\epsilon > 0$. By assumption, φ is completely contractive, so $\|\varphi^{(n)}(G_\epsilon)\| \leq 1$ for all ϵ. Since $\varphi(\alpha 1 \cdot a \cdot \beta 1) = \alpha \varphi(a) \beta$ for all $\alpha, \beta \in \mathbb{C}$ and $a \in X$, we have that $\varphi^{(n)}(G_\epsilon) = L_\epsilon^{-1/2} \varphi^{(n)}(C) M_\epsilon^{-1/2}$. Consequently, since $\varphi^{(n)}$ is contractive, $\begin{pmatrix} 1 & L_\epsilon^{-1/2} \varphi^{(n)}(C) M_\epsilon^{-1/2} \\ (L_\epsilon^{-1/2} \varphi^{(n)}(C) M_\epsilon^{-1/2})^* & 1 \end{pmatrix} \geq 0$; equivalently, $\begin{pmatrix} L_\epsilon & \varphi^{(n)}(C) \\ \varphi^{(n)}(C)^* & M_\epsilon \end{pmatrix} \geq 0$; equivalently $\varphi'^{(n)}(S) \geq 0$.

On the other hand, if φ' is completely positive, then after the canonical shuffle, we have that $\begin{pmatrix} 1 & \varphi^{(n)}(C) \\ \varphi^{(n)}(C)^* & 1 \end{pmatrix} \geq 0$ for C any contractive matrix in $M_n(X)$, and so by Lemma 2.3, $\varphi^{(n)}$ is contractive, and hence φ is completely contractive. $\square$

Viewing operator spaces as an off-diagonal entry of a 2×2 operator system is referred to as *the off-diagonal technique.* We next use it to prove an operator space version of the Hahn-Banach theorem.

Theorem 2.13. *Let X be a subspace of a unital C^*-algebra A, $\varphi : X \to \mathscr{B}(\mathscr{H})$ completely bounded. Then there exists a completely bounded map $\tilde{\varphi} : A \to \mathscr{B}(\mathscr{H})$ extending φ and such that $\|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb}$.*

Proof.

We assume, without loss of generality, that $\|\varphi\|_{cb} = 1$. By Lemma 2.12, the associated map $\varphi' : S_X \to M_2(\mathscr{B}(\mathscr{H}))$ is unital and completely positive, and so by the Arveson extension theorem (Theorem 2.11), φ' extends to a completely positive map $\tilde{\varphi}' : M_2(A) \to M_2(\mathscr{B}(\mathscr{H}))$. Now for all $x \in X$, define $\tilde{\varphi}(x)$ to be the upper right entry of $\varphi'\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right)$. Clearly, this is a linear map extending φ. By the Stinespring dilation theorem, there is representation π such that $\tilde{\varphi}'(a) = P_{\mathscr{B}(\mathscr{H})}\pi(a)|_{\mathscr{B}(\mathscr{H})}$ for all

★ $a \in M_2(A)$. Since unital $*$-representations of C^*-algebras are completely contractive,

$$\|\tilde{\varphi}^{(n)}(x)\| \leq \left\|\tilde{\varphi}'^{(n)}\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right)\right\| \leq \left\|\pi^{(n)}\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right)\right\| \leq \left\|\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right\| = \|x\|, \quad \text{for all } x \in M_n(X),$$

it follows that $\tilde{\varphi}$ is completely contractive. □

★ One can use this theorem to prove the Arveson extension theorem, showing that the two theorems are equivalent.

Lemma 2.14. *Let A be a unital C^*-algebra, $\varphi : A \to \mathscr{B}(\mathscr{H})$ a completely bounded map. There exist completely positive maps $\varphi_i : A \to \mathscr{B}(\mathscr{H})$, $\|\varphi_i\|_{cb} = \|\varphi\|_{cb}$, $i = 1, 2$, such that*

$$\tilde{\varphi}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{pmatrix} \varphi_1(a) & \varphi(b) \\ \varphi^*(c) & \varphi_2(d) \end{pmatrix}$$

is completely positive (here $\varphi^*(c) = \varphi(c^*)^*$). Furthermore, if φ is completely contractive, we may take $\varphi_1(1) = \varphi_2(1) = 1$.

Proof.

Without loss of generality, take $\|\varphi\|_{cb} = 1$. By Lemma 2.12, we extend φ to $\varphi' : S_A \to \mathscr{B}(\mathscr{H}^2)$. Then by the Arveson extension theorem (Theorem 2.11) we extend φ' to a cp map $\tilde{\varphi}' : M_2(A) \to \mathscr{B}(\mathscr{H}^2)$. The matrix $D = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in S_A$, so

$\tilde{\varphi}'(D) = \varphi'(D) = \begin{pmatrix} 0 & \varphi(b) \\ \varphi^*(c) & 0 \end{pmatrix}$. Likewise, for $a \leq 1$,

$$0 \leq \tilde{\varphi}'\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) \leq \tilde{\varphi}'\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \text{ implying } \tilde{\varphi}'\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof of Lemma 2.14, continued

So since any element of A is a linear combination of positive elements, there exists a completely positive φ_1 such that $\tilde{\varphi}'\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} \varphi_1(a) & 0 \\ 0 & 0 \end{pmatrix}$. The existence of φ_2 is analogously proved. By construction, $\varphi_1(1) = \varphi_2(1) = 1$, and so $\|\varphi_1\|_{cb} = \|\varphi_2\|_{cb} = \|\varphi\|_{cb}$. $\square$

We next prove a generalisation of the Stinespring dilation theorem due to Haagerup, Paulsen and Wittstock.

Theorem 2.15. *Let A be a unital C^*-algebra, $\varphi : A \to \mathscr{B}(\mathscr{H})$ completely bounded. Then there exists a Hilbert space $\mathscr{K}$, a unital $*$-homomorphism $\pi : A \to \mathscr{B}(\mathscr{K})$ and operators V_1, $V_2 \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ with $\|V_1\|\|V_2\| = \|\varphi\|_{cb}$ such that for all $a \in A$,*

$$\varphi(a) = V_1^* \pi(a) V_2.$$

Moreover, if $\|\varphi\|_{cb} = 1$, then V_1 and V_2 may be taken to be isometries.

Proof.

Without loss of generality take $\|\varphi\|_{cb} = 1$. Define φ_1, φ_2 and $\tilde{\varphi}'$ as in the last lemma. Let $(\pi', V', \mathscr{K}')$ be the minimal (ie, irreducible) Stinespring representation for $\tilde{\varphi}'$, and note that since $\tilde{\varphi}'$ is unital, we can take V' to be isometric.

★ Arguing as in the proof of Theorem 2.10, we can decompose $\mathscr{K}' = \mathscr{K} \oplus \mathscr{K}$ and we have a unital $*$-homomorphism $\pi : A \to \mathscr{B}(\mathscr{H})$ such that

$$\pi'\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \pi(a) & \pi(b) \\ \pi(c) & \pi(d) \end{pmatrix}.$$

Proof of Theorem 2.15, continued

Since φ_1 is unital and V' is isometric, we see that for $h \in \mathscr{H}$

$$\begin{pmatrix} h \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_1(1) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h \\ 0 \end{pmatrix} = V^* \begin{pmatrix} \pi(1) & 0 \\ 0 & 0 \end{pmatrix} V \begin{pmatrix} h \\ 0 \end{pmatrix} = V^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V \begin{pmatrix} h \\ 0 \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} 0 \\ h \end{pmatrix} = V^* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} V \begin{pmatrix} 0 \\ h \end{pmatrix},$$

and so V has the form $V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$, where $V_1, V_2 \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ are isometries.

Reading off the upper right corner of $\tilde{\varphi}'$, we conclude that $\varphi(a) = V_1^* \pi(a) V_2$ for all $a \in A$. $\square$

Theorem 2.16. *Let A be a unital C^*-algebra, $\varphi : A \to \mathscr{B}(\mathscr{H})$ completely bounded. Then there exists a completely positive map $\psi : A \to \mathscr{B}(\mathscr{H})$ such that $\|\varphi\|_{cb} \geq \|\psi\|_{cb}$ and the maps $\psi \pm \mathrm{Re}\,\varphi$, $\psi \pm \mathrm{Im}\,\varphi$ are completely positive. Consequently, every completely bounded map is a linear combination of at most four completely positive maps.*

Proof.

Use the HPW decomposition theorem (Theorem 2.15) to write $\varphi(a) = V_1^* \pi(a) V_2$ with $\|V_1\|^2 = \|V_2\|^2 = \|\varphi\|_{cb}$. Define

$$\psi(a) := \tfrac{1}{2}(V_1^* \pi(a) V_1 + V_2^* \pi(a) V_2).$$

Then ψ is cp, $\|\psi\|_{cb} = \|\psi(1)\| \leq \|\varphi\|_{cb}$. Since $\varphi^*(a) = V_2^* \pi(a) V_1$, we have the following, which are cp:

$$\psi(a) \pm \mathrm{Re}\,\varphi(a) = \tfrac{1}{2}(V_1 \pm V_2)^* \pi(a)(V_1 \pm V_2), \quad \text{and}$$
$$\psi(a) \pm \mathrm{Im}\,\varphi(a) = \tfrac{1}{2}(V_1 \mp iV_2)^* \pi(a)(V_1 \mp iV_2).$$

Finally, the last statement holds since

$$\psi = \tfrac{1}{2}[((\psi + \mathrm{Re}\,\varphi) - (\psi - \mathrm{Re}\,\varphi)) + i((\psi + \mathrm{Im}\,\varphi) - (\psi - \mathrm{Im}\,\varphi))]. \qquad \square$$

- We wish to abstractly characterise operator systems, spaces and algebras.
- As we noted, the morphisms in these categories are "complete" maps.
- If we are dealing with an **operator system** S, the appropriate morphisms are the completely positive maps. The notion of positivity requires a cone of positive elements. Complete positivity requires such cones be specified for $M_n(S)$ for all $n \in \mathbb{N}$. Note too that conjugation of $a \in M_n(S)$ by a scalar matrix $c \in M_{mn}(\mathbb{C})$ gives an element of $M_m(S)$. Furthermore, if $a \in M_n(S)^+$, then we should have $c^* ac \in M_m(S)^+$. Finally, there should be a "unit" in S, and the cones should in some sense be closed and generate each $M_n(S)_h$.

- If we are dealing with an **operator space** X, the morphisms should be completely bounded maps. Consequently, we need not only a norm on the original space, but also norms on each $M_n(X)$. These norms should fit together in the right way. For example, if a is a direct sum of two smaller matrices, then the norm of a should be the maximum of the norms of the smaller matrices. Likewise, if $a \in M_n(X)$ and b, c are scalar matrices of the appropriate sizes, then $\|bac\| \leq \|b\|\|a\|\|c\|$. We call such a system of norms an *ℓ^∞ norm structure*.
- If we are dealing with an **operator algebra** A, then it should be an operator space. Furthermore, for each n the norms should be submultiplicative. That is, we require multiplication to be completely contractive bilinear map.
- We will find that these properties will be enough to ensure that any abstract object which satisfies them are completely isometrically isomorphic to a concrete object of the same sort.

Archimedean matrix orders

- Let S be a complex vector space with involution (ie, a conjugate linear map $*$ such that $(a^*)^* = a$). Denote by S_h the set of all elements $a \in S$ for which $a^* = a$. Note that by the usual construction of real and imaginary parts, we can express $a \in S$ as $a = a_r + ia_i$, where $a_r, a_i \in S_h$.
- For each n we are given a full cone in $C_n \in M_n(S)$. In particular, $C_n \cap -C_n = \{0\}$ and $C_n + -C_n = S_h$.
- We require that for every $n, m \in \mathbb{N}$, $c \in M_{n,m}(\mathbb{C})$, $c^* C_n c \subseteq C_m$. We refer to $\{C_n\}_n$ as a matrix order on S.
- The cones C_n should be *archimedean*, in the sense that there is a *complete order unit* e. So e is an order unit for C_1, and for all $n \in \mathbb{N}$, 1_n, $n \times n$ diagonal matrix with e in every entry on the main diagonal is an order unit for C_n. (so for all $x \in M_n(S)_h$, there exists real $t > 0$ such that $t1_n \pm x \in C_n$)
- The cone C_n should be *algebraically closed*; that is, for all n, $t1_n + x \in C_n$ for all $t > 0$ implies $x \in C_n$.
- We call a $*$-vector space S with a full archimedean algebraically closed matrix order as above an *abstract operator system*.
- Let S, S' be two (abstract) operator systems with matrix orders $\{C_n\}_n$ and $\{C'_n\}_n$, respectively. A linear map $\varphi : S \to S'$ is *completely positive* if for each n and all $x \in C_n$, $\varphi^{(n)}(x) \in C'_n$. We say that $\varphi : S \to S'$ is a *complete order isomorphism* if φ is invertible and both φ and φ^{-1} are completely positive.

Proposition 2.18. *Let S be an abstract operator system, and for $a \in M_n(S)$, define*

$$\|a\|_n := \inf\left\{ t \in \mathbb{R} : \begin{pmatrix} t1_n & a \\ a^* & t1_n \end{pmatrix} \in C_{2n} \right\}.$$

Then $\|\cdot\|_n$ is a norm on $M_n(S)$ and C_n is closed in this norm.

Proof.

Suppose $\begin{pmatrix} t1_n & a \\ a^* & t1_n \end{pmatrix} \in C_{2n}$ (t exists since C_{2n} is archimedean). Conjugating with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ gives $\begin{pmatrix} t1_n & -a \\ -a^* & t1_n \end{pmatrix} \in C_{2n}$, and so by summing, $2t1_{2n} \in C_{2n}$. Since $t'1_{2n} \in C_{2n}$ for any $t' \geq 0$ and $C_{2n} \cap -C_{2n} = \{0\}$, it follows that $t \geq 0$, and so $\|a\|_n \geq 0$. If $\|a\|_n = 0$, then since C_{2n} is algebraically closed, $\tilde{a} = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix} \in C_{2n}$. Conjugating as above

★ gives $-\tilde{a} \in C_{2n}$, and so $a = 0$. By similar manipuations, one also finds for all $s \in \mathbb{C}$, $a, b \in M_n(S)$ that $\|sa\|_n = |s|\|a\|_n$, $\|a+b\|_n \leq \|a\|_n + \|b\|_n$ and $\|a^*\|_n = \|a\|_n$. Finally, suppose $(a_k)_k \subset C_n$ with $\|a - a_k\|_n \to 0$. Note that $a_k = a_k^*$, and so $a = a^*$. Then for any $t > 0$, there is some M so that for all $m \geq M$, $0 \leq \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} t1_n & a - a_m \\ a - a_m & t1_n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $= 2(t1_n + a - a_m)$, and so $t1_n + a > a_M$ is in C_n. Thus since C_n is algebraically closed,

★ $a \in C_n$. By the way, these norms form an ℓ^∞ norm structure. □

Linear functionals associated to matrix maps

- Let S be an abstract operator system. Recall that we use the notation e_j for the jth standard basis element in $\mathbb{C}^n$ and e_{ij} for the (i,j)th matrix unit in $M_n(\mathbb{C})$. Note then that if $a \in S$, $a \otimes e_{ij} \in M_n(S)$ with a in the (i,j)th entry and 0 elsewhere.
- Given a linear functional $\lambda : M_n(S) \to \mathbb{C}$ and $a \in S$, define $\lambda_{ij}(a) = \lambda(a \otimes e_{ij})$.
- Recall the following construction from the proof of Theorem 2.10: To $\lambda : M_n(S) \to \mathbb{C}$ we associate a linear map $\varphi_\lambda : S \to M_n(\mathbb{C})$ by $\varphi_\lambda(a) = (\lambda_{ij}(a))$.
- Likewise, to a linear map $\varphi : S \to M_n(\mathbb{C})$, we associate a linear functional $\lambda_\varphi : M_n(S) \to \mathbb{C}$ by $\lambda_\varphi(x) = \sum_{i,j=1}^n \langle \varphi(x_{ij})e_j, e_i \rangle$.
- We have $\lambda_{(\varphi_\lambda)} = \lambda$ and $\varphi_{(\lambda_\varphi)} = \varphi$.
- We proved the following while proving Theorem 2.10:
-

Proposition 2.19. *Let S be an abstract operator system, $\lambda : M_n(S) \to \mathbb{C}$ a linear functional, and define a linear map $\varphi_\lambda : S \to M_n(\mathbb{C})$ as above. If $\lambda \geq 0$, then φ_λ is completely positive.*

Theorem 2.20 (Choi and Effros). *Let S be an abstract operator system with order unit e. Then there exists a Hilbert space $\mathscr{H}$, an operator system $\tilde{S} \subseteq \mathscr{B}(\mathscr{H})$ and a complete order isomorphism $\varphi : S \to \tilde{S}$ with $\varphi(e) = 1_{\mathscr{H}}$. Conversely, if $\tilde{S}$ is an operator system on $\mathscr{B}(\mathscr{H})$, then $\tilde{S}$ is an abstract operator system with order unit $e = 1_{\mathscr{H}}$.*

Recall that we showed after Lemma 2.5 that in the concrete case, unital complete order isomorphisms of operator systems are isometric $*$-isomorphisms. ★ The same holds in the abstract case as well.

Proof.

Assume that S is an abstract operator system with order unit e. We also assume that S is endowed with the matrix norms from Proposition 2.18. Define $\mathscr{P}_n = \{\varphi : S \to M_n(\mathbb{C}) : \varphi$ is cp and $\varphi(e) = 1\}$. Claim that for each n, $\mathscr{P}_n \neq \emptyset$. Note that because C_1 is a cone, there exists $x \in M_1(S)_h \setminus C_1$. By the Separation Theorem (see slide *A digression: the Hahn-Banach theorem and its allies*), there is a nonzero linear functional λ with $\lambda(x) < 0$ and $\lambda(C_1) \geq 0$. Furthermore, since C_1 is archimedean and there is some $t \in \mathbb{R}^+$ such that $te + x \in C_1$, it follows that $\lambda(e) = -\frac{1}{t}\lambda(x) > 0$, and we can scale this so that $\lambda(e) = 1$. Hence $\mathscr{P}_1 \neq \emptyset$. The direct sum of cp maps is cp, and so for all n, $\mathscr{P}_n \neq \emptyset$.

Proof of Theorem 2.20, continued

Define $\mathscr{H} = \bigoplus_n (\bigoplus_{\varphi \in \mathscr{P}_n} M_n)$ and $\Phi : S \to \mathscr{H}$ by $\Phi(a) = \bigoplus_n (\bigoplus_{\varphi \in \mathscr{P}_n} \varphi(a))$ (in the ℓ^∞ sense). By definition, Φ is completely positive. To show that Φ is a complete order isomorphism, it suffices to show that Φ^{-1} is completely positive (by the same proof as in Lemma 2.5, we will then have Φ^{-1} bounded). We prove the contrapositive; namely, that if $x \in M_n(S) \backslash C_n$, then there is some m and $\varphi \in \mathscr{P}_m$ such that $\varphi^{(n)}(x) \ngeq 0$. So assume we have such an x and let λ be a positive linear functional on $M_n(S)$ with $\lambda(x) < 0$ and $\lambda(1_n) > 0$. Let $\varphi = \varphi_\lambda : S \to M_n(\mathbb{C})$, $\{e_k\}$ the standard basis vectors for $\mathbb{C}^n$ and set $f \in \mathbb{C}^{n^2}$ the column vector formed from the e_ks. Then $\varphi^{(n)}(x) = (\varphi(x_{k\ell})) = ((\lambda(x_{k\ell} \otimes e_{ij})))$, and so

$$\left\langle \varphi^{(n)}(x) f, f \right\rangle = \sum_{k,\ell} \left\langle (\lambda(x_{k\ell} \otimes e_{ij})) e_\ell, e_k \right\rangle = \sum_{k,\ell} \lambda(x_{k\ell} \otimes e_{k\ell}) = \lambda(x) < 0.$$

Problem: It might happen that $\varphi(e) \neq 1_n \in M_n(\mathbb{C})$. Note that for $x \in S$, $\begin{pmatrix} \|x\| e & x \\ x^* & \|x\| e \end{pmatrix} \geq 0$ and φ is 2-positive, so $\ker \varphi(e) \subseteq \ker \varphi(x)$. Likewise, $\ker \varphi(e) \subseteq \ker \varphi(x)^*$. Let p be the projection onto $\operatorname{ran} \varphi(e)$, and let $m = \dim \operatorname{ran} p$. Note $p\varphi(x)p = \varphi(x)$. Let $a \in M_{n,m}(\mathbb{C})$ with $a^* \varphi(e) a = 1_m$, and $b \in M_{m,n}(\mathbb{C})$ with $ab = p$. Set $\varphi' = a^* \varphi(\cdot) a : S \to M_m(\mathbb{C})$. Then $\varphi' \in \mathscr{P}_m$ and

$$\left\langle \varphi'^{(n)}(x) bf, bf \right\rangle = \sum_{k,\ell} \left\langle \varphi'(x) be_\ell, be_k \right\rangle = \sum_{k,\ell} \left\langle p\varphi(x) pe_\ell, e_k \right\rangle = \sum_{k,\ell} \left\langle \varphi(x) e_\ell, e_k \right\rangle = \left\langle \varphi^{(n)}(x) f, f \right\rangle < 0. \quad \square$$

ℓ^∞ matrix norms and Ruan's axioms

- Recall that a *concrete operator space* X is simply a norm closed subspace of $\mathscr{B}(\mathscr{H},\mathscr{K})$, $\mathscr{H}$ and $\mathscr{K}$ Hilbert spaces, and of course we can identify $\mathscr{B}(\mathscr{H},\mathscr{K})$ with a closed subspace of $\mathscr{B}(\mathscr{H}\oplus\mathscr{K})$.
- For all $m, n \in \mathbb{N}$, the $m \times n$ matrices over X, $M_{m,n}(X)$, inherits a norm structure from $M_{m,n}(\mathscr{B}(\mathscr{H}\oplus\mathscr{K})) \cong \mathscr{B}(\mathscr{H}^m \oplus \mathscr{K}^n))$.
- Note too, that we can pad any operator in $M_{m,n}(\mathscr{B}(\mathscr{H}\oplus\mathscr{K}))$ to make it an operator on $M_p(\mathscr{B}(\mathscr{H}\oplus\mathscr{K}))$ for some p without increasing the norm.
- Because interchanging rows and columns does not alter the norm of an operator in $M_p(\mathscr{B}(\mathscr{H}\oplus\mathscr{K}))$, we can identify $M_n(M_m(X)) \cong M_m(M_n(X)) \cong M_{mn}(X)$ (these are complete isomorphisms!). We also have the following, called Ruan's axioms:

R1 For all $n \in \mathbb{N}$, $\alpha, \beta \in M_n(\mathbb{C})$, $x \in M_n(X)$,

$$\|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\|, \quad \text{and}$$

R2 For all $x \in M_m(X)$, $y \in M_n(X)$,

$$\left\| \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}.$$

- We refer to (semi-)norms satisfying Ruan's axioms as an *ℓ^∞ matrix (semi-)norm* structure.
- We call a vector space X with an ℓ^∞ matrix norm structure an *abstract operator space*.

Theorem 2.21. *A vector space X with set of norms $\{\|\cdot\|_n\}_n$ defined on $M_n(X)$ is an abstract operator space iff it is completely isometrically isomorphic to a a concrete operator space.*

Proof.

It is obvious from the earlier discussion that a concrete operator space is an abstract operator space. So we assume X is an abstract operator space and construct a complete isometry from X onto a concrete operator space. We do this by constructing an abstract operator system containing X as a corner, and then applying Theorem 2.20.

We formally define $X^* = \{v^* : v \in X\}$ with $f^* + g^* := (f+g)^*$ and $\alpha f^* := (\overline{\alpha} f)^*$ for all $f, g \in X$ and $\alpha \in \mathbb{C}$. We take $S = \mathbb{C} \oplus \mathbb{C} \oplus X \oplus X^*$, with the elements of S viewed as matrices of the form $\begin{pmatrix} \alpha & f \\ g^* & \beta \end{pmatrix}$, and introduce an involution on S by requiring $\begin{pmatrix} \alpha & f \\ g^* & \beta \end{pmatrix}^* = \begin{pmatrix} \overline{\alpha} & g \\ f^* & \overline{\beta} \end{pmatrix}$. This matrix is in S_h iff $\alpha, \beta \in \mathbb{R}$ and $g = f$. With a canonical shuffle, elements of $M_n(S)$ have the form $\begin{pmatrix} \alpha & f \\ g^* & \beta \end{pmatrix}$ where $\alpha, \beta \in M_n(\mathbb{C})$, $f \in M_n(X)$, $g^* \in M_n(X^*)$, and those in $M_n(S)_h$ satisfying the further requirement that α, β are selfadjoint and $g = f$. Observe that elements of $M_n(S)$ are $2n \times 2n$ matrices!

Proof of Theorem 2.21, continued

We need to define an order structure on S. Motivated by the concrete case presented in Lemma 2.13, we define C_n to be the subset of $M_n(S)_h$ as above with $\|(\alpha+\epsilon 1)^{-1/2} f(\beta+\epsilon 1)^{-1/2}\| \leq 1$ for all $\epsilon > 0$. If $a = (a_{ij}) = \begin{pmatrix} \alpha_{ij} & f_{ij} \\ g_{ij}^* & \beta_{ij} \end{pmatrix} \in M_n(S)$ and $\gamma = (\gamma_{ij}) \in M_{m,n}(\mathbb{C})$, then $\gamma a \gamma^* = \sum \gamma_{ij} a_{jk} \overline{\gamma_{\ell k}}$, and so after the canonical shuffle, has the form $\begin{pmatrix} \gamma\alpha\gamma^* & \gamma f \gamma^* \\ \gamma g^* \gamma^* & \gamma\beta\gamma^* \end{pmatrix}$. This will be in C_m if the original matrix is in C_n (with $\gamma f^* \gamma^* = (\gamma f \gamma^*)^*$). To see this, observe that

$$
\begin{aligned}
&(\gamma\alpha\gamma^* + \epsilon 1)^{-1/2} \gamma f \gamma^* (\gamma\beta\gamma^* + \epsilon 1)^{-1/2} \\
&= \left[(\gamma\alpha\gamma^* + \epsilon 1)^{-1/2} \gamma(\alpha + \delta 1)^{1/2}\right]\left[(\alpha+\delta 1)^{-1/2} f (\beta + \delta 1)^{-1/2}\right]\left[(\beta+\delta 1)^{1/2} f^* (\gamma\beta\gamma^* + \epsilon 1)^{-1/2}\right].
\end{aligned}
$$

The middle term has norm less than or equal to 1 by assumption when $\delta > 0$. The same will be true for the first term if $\gamma(\alpha+\delta 1)\gamma^* \leq \gamma\alpha\gamma^* + \epsilon 1$, which holds for $\delta \leq \epsilon / \|\gamma\gamma^*\|$. Similarly for the third term, proving that $\gamma a \gamma^* \in C_m$, as claimed.

We need to show that the C_ns are cones. It is clear that $tC_n \subseteq C_n$, and by the second Ruan axiom that if $a, b \in C_n$, then $a \oplus b \in C_{2n}$. Letting $\alpha^* = \begin{pmatrix} 1 & 1 \end{pmatrix} \in M_{2n,n}(\mathbb{C})$, we have $\alpha^*(a \oplus b)\alpha = a + b \in C_n$, showing that C_n is closed under addition.

Proof of Theorem 2.21, continued

Note that $\begin{pmatrix} \alpha_n & x_n \\ x_n^* & \beta_n \end{pmatrix} \in C_n$ converges to $\begin{pmatrix} \alpha & x \\ y^* & \beta \end{pmatrix}$ means that $\alpha_n \to \alpha$, $\beta_n \to \beta$, $x_n \to x$, $x_n^* \to y$, and so $y = x^*$. Furthermore, $(\alpha_n + \epsilon 1)^{-1/2} \to (\alpha + \epsilon 1)^{-1/2}$ and $(\beta_n + \epsilon 1)^{-1/2} \to (\beta + \epsilon 1)^{-1/2}$, so we see that $\begin{pmatrix} \alpha & x \\ x^* & \beta \end{pmatrix} \in C_n$, and so C_n is norm closed. It is clearly then algebraically closed. It is also easily verified that $M_n(S)_h = C_n - C_n$.

We claim that $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a matrix order unit. Since $M_n(S)_h = C_n - C_n$, it suffices to show that if $\begin{pmatrix} \alpha & x \\ x^* & \beta \end{pmatrix} \in C_n$, then for some $t > 0$, $t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \alpha & x \\ x^* & \beta \end{pmatrix} \in C_n$.

This will follow if we can find t so that $\|(t1 - \alpha + \epsilon 1)^{-1/2}(\alpha + \epsilon 1)^{1/2}\| \leq 1$ for $\epsilon > 0$, with a similar statement for β. Equivalently, we want t so that $\alpha + \epsilon 1 \leq t1 - \alpha + \epsilon 1$, which follows if $t \geq 2\|\alpha\|$. Thus $t \geq 2\max\{\|\alpha\|, \|\beta\|\}$ will do.

We now embed X in S via $a \mapsto \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. By definition, the induced norm on S (Proposition 2.18) agrees with the operator space norm on X, and hence by Theorem 2.20, we see that the embedding is a complete isometry on X, finishing the proof. □

Application of Ruan's theorem: Quotient operator spaces

- Let V be an operator space, W a closed subspace of V, $q: V \to V/W$ the quotient map. Define norms on $M_n(V/W)$ by

$$\|q^{(n)}(v)\|_n = \inf\{\|v+w\|_n : w \in M_n(W)\}.$$

Claim that these norms satisfy Ruan's axioms, and hence make V/W into an operator space.

★
- Fix $n \in \mathbb{N}$, $\alpha, \beta \in M_n(\mathbb{C})$, $v \in M_n(V)$. It is easily checked that $q^{(n)}(\alpha v \beta) = \alpha q^{(n)}(v)\beta$.
- Given $\epsilon > 0$, and $w \in M_n(W)$ such that $\|v+w\|_n \leq \|q^{(n)}(v)\|_n + \epsilon$, we have, since V is an operator space,

$$\|q^{(n)}(\alpha v \beta)\|_n = \|\alpha q^{(n)}(v)\beta\|_n \leq \|\alpha(v+w)\beta\| \leq \|\alpha\|\|v+w\|\|\beta\| \leq \|\alpha\|(\|q^{(n)}(v)\|_n + \epsilon)\|\beta\|.$$

Since ϵ is arbitrary, *R1* follows.
- Again fix $\epsilon > 0$ and now suppose $v_1 \in M_m(V)$, $v_2 \in M_n(V)$. Choose w_1, w_2 such that $\|v_1 + w_1\| \leq \|q^{(m)}(v_1)\|_m + \epsilon$, $\|v_2 + w_2\| \leq \|q^{(n)}(v_2)\|_n + \epsilon$. Then

$$\left\| q^{(m+n)}\left(\begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}\right)\right\| \leq \left\|\begin{pmatrix} v_1 + w_1 & 0 \\ 0 & v_2 + w_2 \end{pmatrix}\right\| = \max\{\|v_1 + w_1\|, \|v_2 + w_2\|\}$$
$$\leq \max\{\|q^{(m)}(v_1)\|, \|q^{(n)}(v_2)\|\} + \epsilon.$$

★ Taking $\epsilon \to 0$, gives one inequality. The other is obtained from *R1*. Hence *R2* holds and V/W is an abstract operator space.
- Note that in this case the quotient map q is completely contractive.

Application of Ruan's theorem: Mapping spaces

- Let V, W be operator spaces. As usual we write $\mathscr{CB}(V,W)$ for the space of completely bounded maps from V to W.
- There is a canonical isomorphism from $M_n(\mathscr{CB}(V,W))$ to $\mathscr{CB}(V,M_n(W))$, and using this we can define matrix norms over $\mathscr{CB}(V,W)$. We claim that these norms make $\mathscr{CB}(V,W)$ into an operator space.
- Let $\Phi=(\varphi_{jk})\in M_n(\mathscr{CB}(V,W))$, $\alpha,\beta\in M_n(\mathbb{C})$, $x=(x_{jk})\in M_r(V)$. Then

$$(\alpha\Phi\beta)^{(r)}(x)=((\alpha\Phi\beta)(x_{jk}))=\bigoplus_1^r\alpha\Phi^{(r)}(x)\bigoplus_1^r\beta,$$

  and so $\|(\alpha\Phi\beta)^{(r)}(x)\|\leq\|\alpha\|\|\Phi^{(r)}(x)\|\|\beta\|$. It follows that

$$\|\alpha\Phi\beta\|_n=\|\alpha\Phi\beta\|_{cb}=\sup_r\|(\alpha\Phi\beta)^{(r)}\|\leq\|\alpha\|\sup_r\|\Phi^{(r)}\|\|\beta\|=\|\alpha\|\|\Phi\|_{cb}\|\beta\|=\|\alpha\|\|\Phi\|_n\|\beta\|,$$

  so *R1* holds.
- Let $\Phi\in M_n(\mathscr{CB}(V,W)),\Psi\in M_m(\mathscr{CB}(V,W))$, x as above. Then $\|(\Phi\oplus\Psi)^{(r)}(x)\|=\|((\Phi\oplus\Psi)(x_{jk}))\|$, which after a canonical shuffle equals $\|\Phi^{(r)}(x)\oplus\Psi^{(r)}(x)\|$. Hence

$$\|\Phi\oplus\Psi\|_{n+m}=\sup_r\max\{\|\Phi^{(r)}\|,\|\Psi^{(r)}\|\}=\max\{\sup_r\|\Phi^{(r)}\|,\sup_r\|\Psi^{(r)}\|\}=\max\{\|\Phi\|_n,\|\Psi\|_m\}.$$

  Hence *R2* holds and $\mathscr{CB}(V,W)$ is an operator space.

Application of Ruan's theorem: Dual of an operator space

- Consider the last example with $W = \mathbb{C}$. We then get an operator space structure on $V^* := \mathscr{C}\mathscr{B}(V,\mathbb{C}) = \mathscr{B}(V,\mathbb{C})$, since as we have noted, continuous linear functionals are automatically completely bounded. We call V^* viewed in this way the *operator space dual* of V.
- With this operator space structure, $M_n(V^*) \cong \mathscr{C}\mathscr{B}(V, M_n(\mathbb{C}))$ completely isometrically.
- Indeed, the complete isometry is implemented through the identification of $M_n(\mathbb{C}) \otimes V^*$ with $\mathscr{B}(V, M_n(\mathbb{C}))$
- Of course V^{**} is an operator space as well. Let $j : V \to V^{**}$ be the canonical embedding. Then j is a complete isometry. To see this, let $x \in M_r(V)$. Then $j^{(r)}(x) \in M_r(V^{**})$, and so can be identified with a map $\hat{x} : V^* \to M_r(\mathbb{C})$ where $(\hat{x})_{jk}(\lambda) = \lambda(x_{jk})$ for $\lambda \in V^*$. Show that $\|\hat{x}\|_{cb} = \|x\|$.
- Let $\lambda = (\lambda_{jk}) \in M_n(V^*)$. Then $\|\hat{x}^{(n)}(\lambda)\| = \|(\lambda_{ij}(x_{k\ell}))\| \leq \|\lambda\|_{cb}\|x\|$, and so $\|\hat{x}\|_{cb} \leq \|x\|$.
- To get the other inequality, use Ruan's theorem (Theorem 2.21) to identify V completely isometrically with a subspace of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. Let $\{\mathscr{H}_m\}$ be a sequence of nested subspaces of dimension m such that $\|x\| = \sup_m \|P_{\mathscr{H}_m} x|_{\mathscr{H}_m}\|$. Let $\{e_k\}$ be an orthonormal set with the property that for each m, the first m vectors span $\mathscr{H}_m$, and use these to define linear functionals $\gamma_{jk}(x) := \langle x e_k, e_j \rangle$. Then $P_{\mathscr{H}_m} x|_{\mathscr{H}_m} = \Gamma_m(x) := (\gamma_{jk}(x))_{j,k=1}^m \in M_m(V^*)$ and $\|\Gamma_m\| \leq 1$. Hence $\|\hat{x}\|_{cb} \geq \sup_m \|\hat{x}^{(m)}(\Gamma_m)\| = \sup_m \|P_{\mathscr{H}_m} x|_{\mathscr{H}_m}\| = \|x\|$.

We say that a unital Banach algebra $\mathscr{A}$ is a *abstract operator algebra* if $\|1\| = 1$, it is an operator space and multiplication is completely contractive; that is, for any n and $A, B \in M_n(\mathscr{A})$, $\|AB\|_n \leq \|A\|_n \|B\|_n$ where AB is the usual matrix product.

Theorem 2.22. *If $\mathscr{A}$ is an abstract operator algebra then there exists a Hilbert space $\mathscr{H}$ and a completely isometric unital homomorphism $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$.*

There is also a non-unital version of this theorem due to Blecher.

Due to a lack of time, we are unable to present a proof of this, though there are several available in the course references which you are now in a position to read.

Here's a sample application.

- Let J be a closed two sided ideal of an operator algebra A.
- Then A/J is a quotient Banach algebra.
- Equip A/J with the quotient ℓ^∞ norm structure as an operator space.
- Then A/J is an operator algebra.