Contents

Chapter 1

Introduction

Many reviews and books exist on most of the topics which I'll discuss here. The unavoidable reference is Mehta's book [1], as well as the physics report of Guhr, Muller-Goering, Weidenmuller [2] for applications of random matrices in condensed matter physics (it contains a wealth of references), and Di Francesco, Ginsparg, Zinn-Justin [3] for random matrices in random surfaces and quantum gravity. There exists also a good introductory book in French from H. Kunz [4]. Let me mention that many aspects of this course were presented (in French) in my thesis [5], as well as in G. Bonnet's thesis [6].

1.1 Origin of Random Matrices

Originally, the idea of studying random matrices in physics was introduced by Wigner [7,8] in 1951. The energy levels of heavy nuclei, probed by neutron beams, are so dense that it is impossible to describe exactly and individually each level. Only a statistical approach is possible: measuring the average density of states, the average level spacing, the average correlation functions, etc... Wigner's claim was that for heavy nuclei, the average energy level distribution is the same as the level distribution of a random matrix. He calculated the level distribution of random Hamiltonians (with Gaussian weight) for various ensembles (real symmetric Hamiltonians if one focuses on time reversal symmetric atoms, hermitian Hamiltonians if the time symmetry is broken,...), and a remarkable agreement with observations was observed later [17, 18, 19]. The theory of random matrices was born [7, 9, 1, 10, 11, 12, 13], and then .

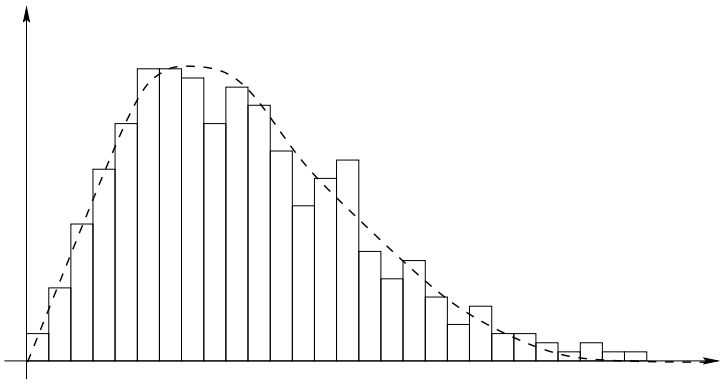


Figure 1.1: experimental energy level spacings histogram for nuclear resonances [18]. In dotted line, level spacing distribution of a random real symmetric Hamiltonian (Gaussian Orthogonal Ensemble).

Since then, this simple idea has found many other applications, in systems with disorder [2,14,15,16] or complexity (chaotic systems [17,20]). Again, a good qualitative description of the level distribution can be obtained by averaging over an ensemble of matrices (either the Hamiltonian or some transmission matrix, or scattering matrix,...).

The success of random matrices lies in the universality of the level distribution. It seems that there is a "central limit theorem" for large random matrices: **when the size of the matrix becomes large** (the number of random variables), **the level distribution tends towards a universal limit distribution, independently of the details of the initial probability law, but dependent only on the symmetry properties**. This conjecture was never proven in a general context, and the most general hypothesis which lead to it are still under investigation. However, it is possible to prove it for a wide range of cases.

Another big step in random matrix theory, came from QCD, and an idea of 't Hooft

[21]. In QCD, the interacting gluons are $N_c \times N_c$ matrix fields ($N_c = 3$ is the number of colors). One has to integrate out the gluon fluctuations, i.e. average them with some probability amplitude. In fact, this is nothing but a particular theory of random matrices (with a chiral symmetry).

Since perturbation theory in the coupling constant was unable to explain quarks confinement, 't Hooft suggested an expansion in N_c. He observed that the diagrammatic expansion greatly simplifies in the limit $N_c \to \infty$: only planar diagrams survive. And at each order in $1/N_c$, only diagrams of a given topology contribute. The $1/N_c$ expansion is a topological expansion.

This remark was not going to reach the goal of explaining confinement, instead, it was going to find another field of application: random surfaces [22, 23, 24, 25, 26, 27].

As in QCD, the diagrammatic expansion of any random matrix integral generates discrete lattices (made of vertices linked with propagators), of given topology, the large N limit giving the planar lattices only.

That observation provides a practical way of counting discrete lattices (triangulated surfaces for instance), and summing over ensembles of lattices. Random matrix integrals can modelize 2D statistical physics systems on a random lattice (for example, Ising model, Potts model, loop gas model,...).

Said differently, random matrices represent some 2D statistical physics model coupled to gravity (fluctuation of the lattice geometry), i.e.: 2D quantum gravity [22, 28, 23, 29, 30, 31, 32, 33]. A theory of Quantum Gravity is still lacking in our understanding of the universe, and any approach to probe it, is welcome. Random matrices provide a well stated definition of quantum gravity. Though it is only in 2D Euclidean space-time instead of 4D Minkowskian space-time, it is a toy model to understand the importance of topology in gravity, and a check for other theories.

Beside quantum gravity, where random matrices have eventually not been so successful, the application to 2D statistical physics if very important. Indeed, since KPZ [34], we know how to relate the critical exponents of a 2D model on a flat surface to the same 2D model in presence of gravity. Random matrices can then help to solve some yet unsolved problems in 2D statistical physics [23].

Another theory where there are fields living on a fluctuating random surface is string theory [35, 36]. String theory too, can be described by random matrices [22, 23, 32]. In string theory, it is necessary to take into account all possible topologies (otherwise there is no interaction between strings), while in random matrices, higher topologies are suppressed by powers of $1/N$, and calculations are possible only in the limit $N \to \infty$. Random matrix physicists have fortunately found a procedure to overcome that difficulty, called the double scaling limit [3]. Random matrices have brought some new exact results in string theories, but their use is limited by the fact that random matrices were able to modelize string theories only in dimension $d \leq 1$. That $d = 1$ barrier is not well understood. Is it only a technical difficulty that we have not been smart enough to overcome, or is there a fundamental obstruction, and no random matrix model can have dimension $d > 1$? The debate is still open.

Eventually, after the discovery that coinciding D-branes behave as matrices, it

was conjectured that a particular super Yang Mills random matrix model could be a non-perturbative definition of string theories, and even M-Theory, the promised Theory of Everything [37, 38]. Somehow, it means that the points of space-time are non-commutative, their coordinates are the eigenvalues of some matrices. The fluctuations around diagonal matrices (the non-diagonal terms) generate the distance between points, as well as all the gauge fields,... In such a picture, space, time, matter, fields are all unified. However, this is a conjecture far for being proven and even accepted at the present time.

This was only a short and non exhaustive introduction of where random matrices are met.

We will now review some of these fields, and describe how random matrices arise, and what answers they bring.

1.2 Random matrices in condensed matter physics

1.2.1 Quantum Chaos

We know that some very simple mechanical systems with only a few degrees of freedom can show (classically) a chaotic behavior: the trajectories become ergodic, the system cannot be parameterized simply by action-angle variables because there are not enough conserved quantities. When the system is not chaotic, the trajectories are confined, they don't visit the whole phase space, whereas the trajectories of a chaotic system are ergodic, and explore the whole phase space.

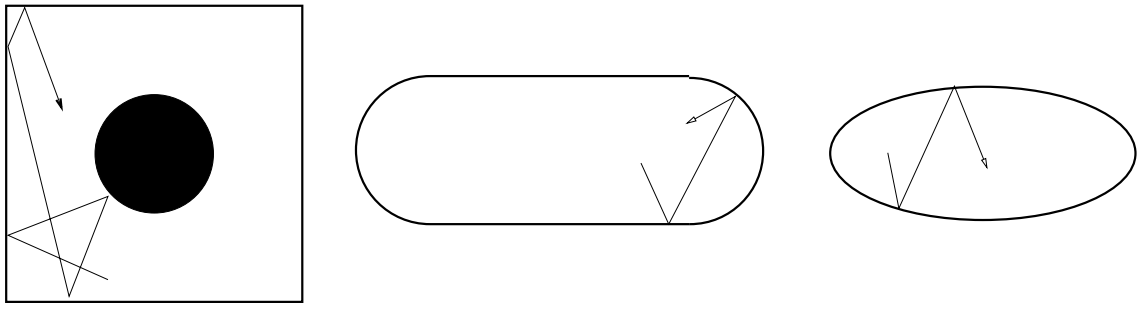
An example of simple systems with either integrable or chaotic behavior is provided by billiards: a particle moves freely within a box, and is reflected on the boundaries. Depending on their shape, some billiards are integrable (the ellipse, the rectangle) or chaotic (the stadium, Sinai's billiard) (fig. 1.2).

Recently it has become possible to build such billiards at microscopic scale: the particle is a single electron moving in a cavity of submicron size, at very low temperature, so that the mean free path is much larger than the size of the cavity. The electron, once it has entered the cavity, has a ballistic motion and bounces on the boundaries many times before exiting.

The question is then: what does chaos mean in quantum mechanics ? Can we see the chaotic behavior in the level distribution ? The answer seems to be Yes.

When some parameter of the system is varied (for instance the radius of the circle in the Sinai's billiard, or an external magnetic field), the energy levels vary but seem to avoid each other, they never cross, they seem to repulse. On the contrary, if the system is classically integrable, the levels ignore each other, there are many crossings (fig. 1.3).

The level repulsion seems to be a characteristic of quantum chaos, it is also a characteristic property of the level distribution of random matrices. Actually, the fact that the distribution is a random matrix distribution is considered as an experimental signature of chaos (fig. 1.4). This is the conjecture of [39], which is not proven yet (see [40] or [2] for a review).



(a) The Sinai's billard (b) The stadium (c) The ellipse

Figure 1.2: (a) The sinai's billiard, and (b) the stadium billiard are both chaotic, while (c) the ellipse is integrable.

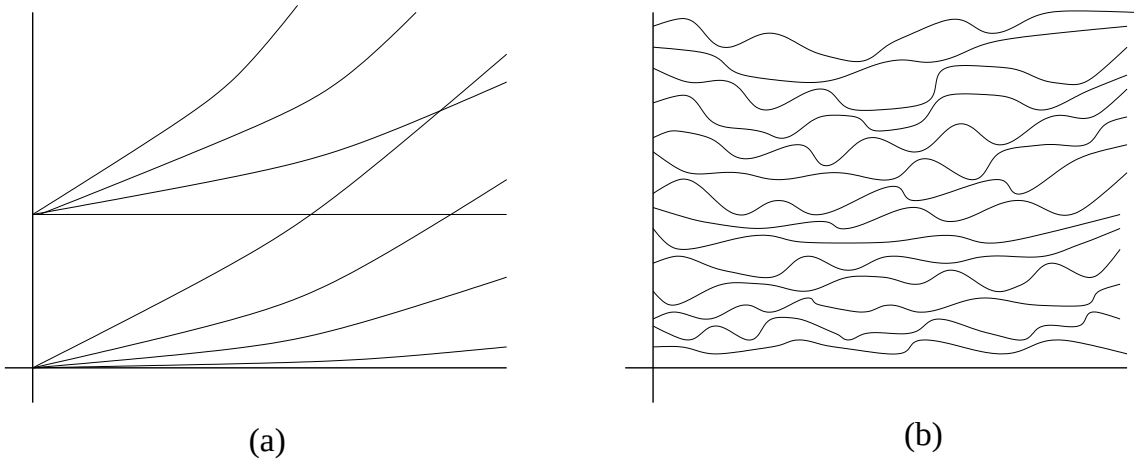


Figure 1.3: Typical behavior of the energy levels of an integrable system (a), and a chaotic system (b), as functions of a parameter.

A possible explanation for the level repulsion (or in other words the interaction or correlation between levels), could be the following: the quantum eigen wave functions tend to localize near the classical periodic trajectories. In an integrable system, the periodic orbits are well localized, and don't cross each other very often, the overlap between wave functions is small, the corresponding eigenvalues are thus almost uncorrelated. On the other hand, for a chaotic system, the classical orbits are ergodic, the wave functions fill up the entire phase space and have a large overlap. Each wave function is very sensitive to the others, the eigenvalues are strongly correlated, they repulse each other.

This hand waving explanation does not constitute a solid justification of the success of random matrix theory. It is still not well understood why a given (even quite simple) Hamiltonian has the same spectrum as a random matrix.

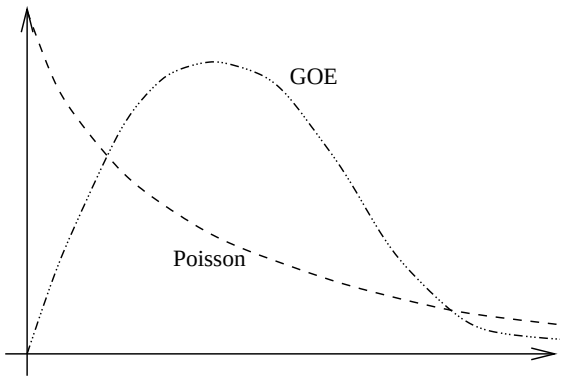


Figure 1.4: Typical distribution of energy level spacings, of an integrable system (Poisson law), and a chaotic system (random matrix law).

The random matrix approach assumes that the Hamiltonian H is a random matrix, chosen randomly in some ensemble, with some probability weight. The universality property of random matrices, ensures that the statistical properties of the spectrum are independent on the probability law, therefore, the simplest choice is to assume a Gaussian law for H:

$$\mathcal{P}(H) = \mathrm{e}^{-\frac{N}{2}\,\mathrm{Tr}\,H^2}$$

The statistical properties of the spectrum depend only on the symmetry of the ensemble to which H belongs.

- Normally, a Hamiltonian H is a hermitian matrix, and unless we have any additional symmetry, the ensemble is the set of hermitian matrices, also called **the Gaussian Unitary Ensemble (GUE)**, which is left invariant under unitary transformations.

- In case the system is time-inversion invariant, H must be a real symmetric matrix. The ensemble is then the set of real symmetric matrices, also called **the Gaussian Orthogonal Ensemble (GOE)**, which is left invariant under orthogonal transformations.

- In case the system is time-inversion invariant, and has a total half integer spin, and if the rotational symmetry is broken, then H is a real self-dual quaternionic matrix (see section 6.7 for definitions). The ensemble is then the set of real self-dual quaternionic matrices, also called **the Gaussian Symplectic Ensemble (GSE)**, which is left invariant under symplectic transformations.

Those three ensembles are the most studied, though it is possible to consider systems with other extra symmetries, and which would lead to other Gaussian ensembles.

Another approach consists in studying the evolution operator U, and assume it is a random matrix. Then, U belongs to other ensembles of matrices, which have the

property to be compact (the eigenvalues belong to the unit circle in the complex plane), and instead of a Gaussian probability law, one can assume a constant probability law. Such ensembles are called **Circular Ensembles**. Without extra symmetries, U is a Unitary matrix, and belongs to the **Circular Unitary Ensemble (CUE)**. If there is time-reversibility, then U is a Orthogonal matrix, and belongs to the **Circular Orthogonal Ensemble (COE)**. If the system has time-reversibility as well as a total half integer spin, and broken rotational symmetry, then U is a Symplectic matrix, and belongs to the **Circular Symplectic Ensemble (CSE)**.

The goal of random matrix theory is to describe the statistical properties of the spectra of matrices in such ensembles.

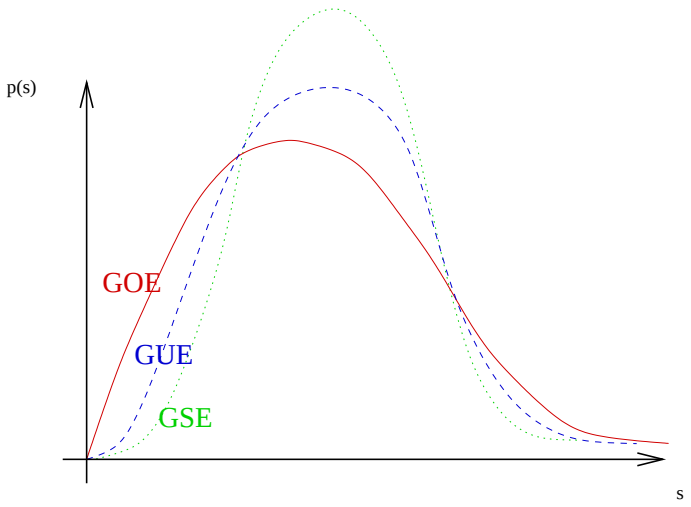


Figure 1.5: Wigner's surmise: the probability law for the spacing distribution (proba that s is the distance between two neighboring eigenvalues), for the three ensembles GOE, GUE and GSE.

Remark: In this course, I will mainly focus the attention to the spectrum, the set of eigenvalues of the matrix, and I will not consider the universal properties of the eigenvectors as well (which are the wave functions in quantum chaotic systems), though this has been done too.

1.2.2 Mesoscopic conductors

Mesoscopic means between microscopic and macroscopic. Macroscopic because we consider conductors made of a large number of atoms (basically that will imply that N, the size of the matrices, will be large: $N \to \infty$), and Microscopic, because the device, which is very small (about micron size) is maintained at a very low temperature (a few K) so that the quantum phase coherence length of the particle wavefunctions is larger than the conductor's size (phase coherence is destroyed by inelastic collisions against

phonons and not by elastic collisions against fixed impurities). In such conductors, quantum effects are essentials. For instance, Ohm's law of additivity for resistances in series, is not satisfied, because there is an interference term:

$$R_{1+2} = |r_1 + r_2|^2 = R_1 + R_2 + 2|r_1||r_2| \cos \delta\phi$$

in usual macroscopic conductors, there is no phase coherence at the scale of the conductor's size, and the cosine of the phase difference averages to zero. In mesoscopic conductors, the phase remains coherent across the whole sample, and the interference term does not vanish. This is just one example of the peculiar behavior of mesoscopic conductors.

When the phase difference, does not average to zero, that means the conductivity is extremely sensitive to the exact shape of the sample, to the presence and exact position of impurities, and to boundary conditions. Two different samples are not alike to have the same conductivity, and the conductivity varies widely when some parameter is varied (for instance an external magnetic field). It is hopeless to determine the exact conductivity of one sample, and again, the only approach is a statistical study, over a large number of samples (or over one sample and varying a parameter). It is then observed that the conductance correlation function is universal, it does not depend on the type of material, or on the parameter which is varied, it is the same for all mesoscopic conductors: it is called the **universal conductance fluctuation**. And that universal correlation function can be derived from a random matrix model: Indeed, the conductivity is linked to the spectrum of the transmission matrix (which tells how electrons are transmitted or reflected across the device, in each mode):

$$G = 2\frac{e^2}{h} \operatorname{tr} t^\dagger t$$

G is the conductance, and t is the transmission matrix (of size $N \times N$ where N is the number of channels available at the Fermi energy, it is proportional to the section of the conductor). t is related to S the scattering matrix, which is a unitary matrix. Due to disorder (position of impurities, exact shape,...) the scattering matrix is a random matrix. Therefore, S belongs to some Circular ensemble:

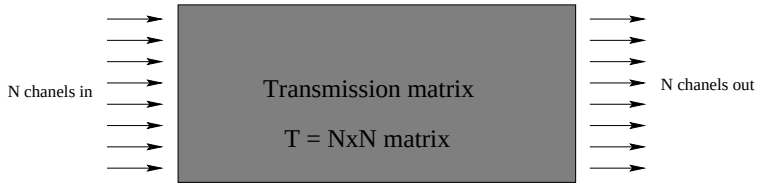


Figure 1.6: The transmission matrix through a conductor

- For electrons moving in a normal metal with broken time reversibility (for instance by an external magnetic field), the S-matrix is unitary, it belongs to the Circular Unitary Ensemble **CUE**.

- If in addition we have time reversibility, S belongs to the Circular Orthogonal Ensemble **COE**.
- And if in addition the system has a half integer total spin with broken rotational invariance, S belongs to the Circular Symplectic Ensemble **CSE**.
- In hybrid conductors (Normal - Supraconductor junction), with or without time-reversibility or rotational symmetry (in case of half integer spin), the limit distributions are given by four possible circular ensembles, sometimes called NS circular ensembles.

Random matrix theory shows that the correlation functions of eigenvalues of random matrices have the property of universality: the correlations depend only on the ensemble, not on the details of the interactions. That explains the *universal conductance fluctuations*.

The possible random matrix ensembles have been classified. They are linked to Cartan's classification of **Symmetric Spaces** (Lie groups with a constant curvature). There are 10 Symmetric Spaces[1], which can be either circular (positive curvature), Gaussian (flat), or hyperbolic (negative curvature).

So far, we have met Gaussian ensembles (the Hamiltonian for instance) and circular ensembles (S-matrix for instance). Hyperbolic ensembles play a very important role as well in many physical systems. These are sometime called "Brownian motion systems" (in Mehta's book), or **transfer matrix ensembles** (Wigner's ones are called TUE, TOE and TSE).

Transfer matrix ensembles

The scattering matrix S, relates the outgoing electrons to the ingoing electrons in the device:

$$S \begin{pmatrix} I \\ I' \end{pmatrix} = \begin{pmatrix} O \\ O' \end{pmatrix}$$

The conservation law

$$|O|^2 + |O'|^2 = |I|^2 + |I'|^2$$

implies that S is a $2N \times 2N$ unitary matrix ($S^\dagger S = 1$). It is usually written in the form

$$S = \begin{pmatrix} r & t \\ t' & r' \end{pmatrix}$$

where r, r' are the reflection $N \times N$ matrices on the left and right side of the conductor, and t, t' are the transmission matrices from right to left and left to right. The conductance is related to the spectrum of $t^\dagger t$:

$$G = 2\frac{e^2}{h} \operatorname{Tr} t^\dagger t$$

that is why we are interested in computing the eigenvalues of $t^\dagger t$ instead of S.

[1] 10 of them can have a rank N arbitrarily large. There exist as well a few exceptional symmetric spaces with $N \leq 8$.

Note that the S-matrix can be block-diagonalized in the following form:

$$S = \begin{pmatrix} u_4^\dagger & 0 \\ 0 & u_1 \end{pmatrix} \begin{pmatrix} \frac{\sinh\phi}{\cosh\phi} & -\frac{1}{\cosh\phi} \\ \frac{1}{\cosh\phi} & \frac{\sinh\phi}{\cosh\phi} \end{pmatrix} \begin{pmatrix} u_3 & 0 \\ 0 & u_2^\dagger \end{pmatrix}$$

where all the u_i are $U(N)$ matrices, and $\phi = \text{diag}\,(\phi_1, \ldots, \phi_N)$ is a diagonal real matrix.

It is not easy to relate the spectrum of S to the spectrum of $t^\dagger t$, that is why it is usual to introduce the transfer matrix M, which relates the left side to the right side of the conductor. By definition:

$$M \begin{pmatrix} I \\ O \end{pmatrix} = \begin{pmatrix} O' \\ I' \end{pmatrix}$$

Indeed one has:

$$\text{Tr}\, t^\dagger t = \text{Tr}\, \frac{2M^\dagger M}{(M^\dagger M + 1)^2}$$

and the eigenvalues of $t^\dagger t$ are directly related to the eigenvalues of $M^\dagger M$.

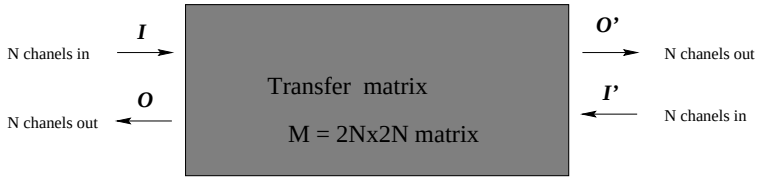


Figure 1.7: The transfer matrix through a conductor

Remark: another interesting property of the transfer matrix is that when you have two conductors in series, the total transfer matrix is the product of the transfer matrices of both conductors.

Now, let us identify the ensemble of matrices to which the transfer matrix belongs.

The symmetry of the transfer matrix is related to the conservation of a non-positive norm:

$$|O|^2 - |I|^2 = |I'|^2 - |O'|^2$$

Thus M belongs to SU(N,N), which is not a compact group, it has a hyperbolic geometry. That fact leads to very important differences with the circular ensembles (to which the matrix S belongs). The eigenvalues of a matrix in a circular ensemble are of the form $e^{i\phi}$, they belong to the compact radius 1 circle. As opposed to it, the eigenvalues of a $SU(N,N)$ matrix are of the form e^{ϕ}. Because the ensemble is not compact, one cannot define a normalizable uniform measure on it (as opposed to circular ensembles). Instead, one can write a Fokker-Planck equation, describing a matrix M following a random walk in the ensemble. The probability law is described by a diffusion equation, like the heat kernel equation:

$$\frac{\partial}{\partial t}\mathcal{P}(M) = \hat{D}\mathcal{P}(M)$$

where $\hat{D}$ is the Laplacian on the ensemble. Here, the "time" t may be any external parameter, usually, it is the length of the conductor, or a magnetic field. knowing the transfer matrix of a conductor of length zero and the diffusion equation, one can derive how the conductivity changes as the length t increases. This equation is known in random matrix theory as the **DMPK equation** (Dorokhov, Mello, Pereyra, Kumar).

A DMPK equation might be written for a circular ensemble as well. In that case, the probability diffuses and because the ensemble is compact, at large time, it tends towards the uniform probability. In other words, for circular ensembles, the DMPK equations has a large t limit, which does not exist for the hyperbolic ensembles.

Symmetric Spaces

The matrix M belongs to the non compact group $SU(N,N)$, but actually, we are interested in the eigenvalues of $t^\dagger t$, or equivalently in the eigenvalues of $M^\dagger M$. To what kind of ensemble does that matrix belong ?

Note that the matrix M can be block-diagonalized in the following form:

$$M = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} \cosh\phi & -\sinh\phi \\ \sinh\phi & -\cosh\phi \end{pmatrix} \begin{pmatrix} u_3 & 0 \\ 0 & u_4 \end{pmatrix}$$

where all the u_i are $U(N)$ matrices, and $\phi = \mathrm{diag}\,(\phi_1, \ldots, \phi_N)$ is a diagonal real matrix. Thus:

$$M^\dagger M = \begin{pmatrix} u_3^\dagger & 0 \\ 0 & u_4^\dagger \end{pmatrix} \begin{pmatrix} \cosh 2\phi & -\sinh 2\phi \\ -\sinh 2\phi & \cosh 2\phi \end{pmatrix} \begin{pmatrix} u_3 & 0 \\ 0 & u_4 \end{pmatrix}$$

which does not depend anymore on u_1 and u_2 as well as on a global phase.

Therefore the matrix $M^\dagger M$ belongs to the quotient space:

$$SU(N,N)/(SU(N) \times SU(N) \times U(1))$$

which is a "symmetric space", it has a constant negative curvature.

It may happen that the system has extra symmetries:

- if the system has time-reversal symmetry, then the u_i matrices have an orthogonal symmetry, they belong to $SO(N)$.
- if the system has a total half integer spin with broken rotational invariance, the u_i matrices are symplectic, they belong to $Sp(N)$.
- and there are other possible symmetries, if the system describes a NS junction (Normal metal - Supraconductor).

In each case, the matrix $M^\dagger M$ belongs to a quotient space which is a symmetric group. Each one has a Laplacian called "Beltrami" operator, and each one has a DMPK equation.

The three cases described here: unitary, orthogonal or symplectic, lead to three hyperbolic ensembles described by a DMPK equation, they are sometimes noted TOE, TUE and TSE (T stands for transfer-matrix).

Summary:
Random matrix theory provides a good tool for this type of problem with disorder, and more generally, in disordered systems, random matrix models provide a good qualitative description of the spectrum. They allow to understand the localization transition. In a localized regime, the wave functions are localized, and the corresponding spectrum is discrete, leading to a Poisson distribution for energy levels, while in a conducting state, the wave functions are spread over the whole sample and the distribution of energy levels is GOE or GUE.

1.3 Applications to Mathematics

1.3.1 Riemann's conjecture

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

can be extended to the complex plane. Riemann conjectured that it's non-trivial zeroes all lie on the line Re $s = 1/2$. If true, such a result would have very important consequences in many areas of maths and physics. For instance in number theory, it would give an estimate of the statistical distribution of prime numbers. Riemann's conjecture has been probed numerically. The striking observation is that the zeroes's distribution seems to agree with the GUE statistics, with an extremely good accuracy. Where does that come from ? Is the Riemann conjecture related to random matrix theory, and how ?

1.3.2 Other fields of mathematics

Random matrices have many other applications in mathematics. As we shall see later, random matrices are very useful in combinatorics, they allow to enumerate planar graphs, triangulated surfaces, knots,...

Random matrices are also studied for themselves, in what is sometimes called "non commutative" probabilities.

Random matrices are also related to integrable hierarchies of partial differential equations [41].

1.4 QCD

Originally, 't Hooft suggested to explore the large N_c (number of colors) expansion of QCD, and he discovered that the Feynmann diagrams are weighted by their topology. That was the first attempt to link random matrices (the matrix is the gluon gauge-field) to surfaces and topology.

However, there are other random matrix approaches to QCD, which have seen a lot of activity and results recently. There is a standard review on that subject from Verbaarschot [42].

Basically, the idea is to study the spectrum of the Dirac operator

$$\not{D} = \not{\partial} + ig\not{A}$$

The properties of the spectrum provide non-perturbative informations and understanding about QCD.

Because of the property of universality of random matrix ensembles, it is expected that the statistical properties of the spectrum don't depend on the details of the Lagrangian, they should depend only on the symmetries. Therefore, some properties of the Dirac operator spectrum can be derived from a simpler model than QCD (the action is chosen quadratic), which obeys the same symmetries.

This approach leads to three classes of ensembles called **chiral Gaussian ensembles**: **chGOE**, **chGUE**, **chGSE**.

The Gaussian-chiral-ensemble actions are:

$$Z_\nu^\beta = \int \mathrm{d}W \prod_{f=1}^{N_f} \det\left(\not{D} + m_f\right) \mathrm{e}^{-N\frac{\Sigma^2\beta}{4}\mathrm{Tr}\, W^\dagger W}$$

where W is a rectangular $n \times m$ matrix, related to the Dirac operator by:

$$\not{D} = \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}$$

$N_c = n+m$ is the total number of colors, $\nu = |n-m|$ is the number of zero modes of the Dirac operator, it selects in which topological sector of QCD we are. $\Sigma = |<\overline{q}q>|$ is the expectation value of the quark fields. And $\beta = 1, 2$ or 4 is related to the symmetry group, chGOE, chGUE or chGSE respectively.

That matrix integral reproduces all the symmetries of QCD, and thus should have the same universal properties as the spectrum of the actual Dirac operator in QCD. That was checked numerically in lattice QCD [43].

1.5 Random matrices and discretized surfaces

Quantum ChromoDynamics states that the quarks have a charge, called the color, of dimension $N_c = 3$. A basis for the charge are the vectors red, green, blue. The strong interaction between quarks is transmitted by some vector bosons, the gluons, which are color gauge fields. These exchange the colors of the quarks. Therefore, there are $N_c^2 - 1 = 8$ gluons (there are 8 and not 9 gluons, because the one which leaves the colors unchanged is not a generator of the $SU(N_c)$ algebra). A convenient way to represent the Feynmann diagrams in QCD, is the following: draw simple lines (with $N_c = 3$ possible colors) for the quark propagators, and double lines (with $N_c^2 = 9$ possible pairs of colors) for the gluons.

More generally, matrix fields are represented by double line propagators, and the corresponding Feynmann diagrams are called **fat graphs** [24,25,26,27]. The perturbative expansion of any matrix field theory, generates fatgraphs, which can be interpreted as lattices which can be drawn on a surface.

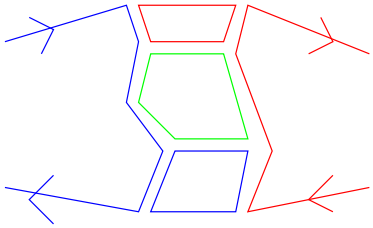


Figure 1.8: A Feynmann diagram contributing to a blue quark-antiquark pair scattered into a red quark-antiquark pair, by exchanging double lines of gluons.

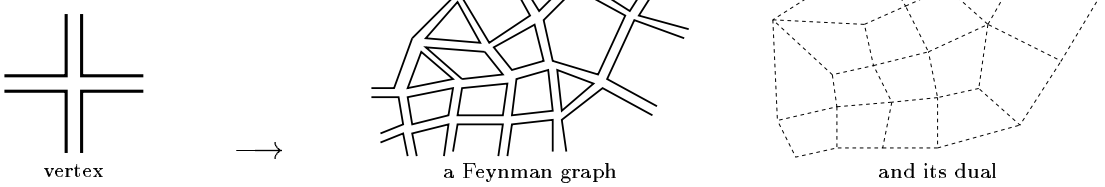
't Hooft has observed that the power of N_c associated to each graph is its Euler-Poincarré characteristic [21]. By varying N we can control the topology of the lattices. For instance when N is sent to infinity, only the maximal Euler-Poincarré characteristic contributes, i.e. the planar graphs. The next order in the $1/N^2$ expansion would give lattices (graphs) which can be drawn on a torus, and so on...

Figure 1.9: two fat Feynmann diagrams contributing to the quark self energy. The left one has characteristic $\chi = 2$, while the right one which is not planar, has $\chi = 0$, it is suppressed by a $1/N^2$ factor compared to the left one.

- As an example, let us consider the following matrix integral:

$$Z = \int \mathrm{d}M \, \mathrm{e}^{-N \,\mathrm{tr}\, \frac{g}{2} M^2 - \frac{g_4}{4} M^4}$$

the inverse of the quadratic term corresponds to the propagator, whose weight is $1/Ng$. The quartic term corresponds to four-legs vertices with weight Ng_4. The summation over matrix indices brings a factor N for each closed loop made of single lines.



A general Feynmann diagram is obtained by gluing together vertices and propagators, i.e. in this case, gluing some square pieces together. We obtain a surface made of elementary square pieces.

If one wants a unitary theory, it is necessary to have some real positive Boltzmann weights, and thus g and g_4 real positive, which means that the potential is not bounded from below and the integral is obviously divergent. However, it is possible for such a divergent integral to make sense, through its perturbative expansion, order by order in powers of N. **The large N limit is necessary**.

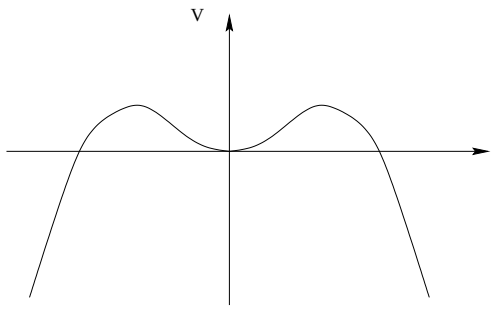


Figure 1.10: A potential not bounded from below.

The potential well is unstable because if the matrix lies in the well near zero, it will eventually escape by tunnel effect and run to infinity. The tunnel effect is exponentially small in the potential, which in this case is accompanied with a factor of N. Therefore, in the large N limit the tunnel effect is suppressed as e^{-N}. That is why the matrix integral makes sense perturbatively in the large N limit. Though, it is important to keep in mind that the matrix integral contains some non-perturbative contribution which can make it unstable.

To summarize, here is the procedure to construct a random-matrix-model for some 2D statistical physics system on a random lattice.
- find a matrix model whose interactions generate the type of elementary pieces you need to build your lattices with. For example quartic interactions to build squares, cubic to build triangles, ...
- compute the matrix integral in the large N limit by any of the methods which will be presented further in this course.
- expand the result as a power series of the coupling constants: you get the number (or the entropy) of the random lattices.

Of course you can incorporate more complicated elementary pieces, carrying decorations, i.e. with some extra degrees of freedom living on the random lattice. We will see later how to formulate the Ising model on a random lattice, or Potts model and others.

1.5.1 Continuous surfaces

In the preceding section, we have been able to generate random discretized surfaces. It would be interesting to be able to go to the continuum limit of continuous surfaces. A continuous surface is the limit of discretized surfaces with a very large number of very small elementary pieces. In the matrix model frame, this limit is reached by fine-tuning some coupling constant $g \to g_c$ so that the average number of elementary pieces diverges.

$$n(g) \to \infty \qquad \text{when} \qquad g \to g_c$$

That means to place oneself within the vicinity of some divergence, or in other words some **critical point** of the model.

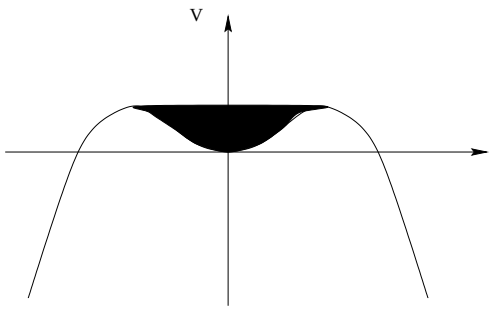


Figure 1.11: At the critical point $g \to g_c$, the eigenvalues of the matrix start to reach the top of the barrier, and the model approaches instability. The model is ill-defined for $g > g_c$ because even in the large N limit, the matrix cannot be confined in the well.

Typically, in the case of the quartic potential example, the critical point occurs when the spectrum of the matrix approaches the top of the barrier, and some of the eigenvalues are about to escape the potential.

The critical behavior is characterized by some critical exponents (power law divergences). For instance, the mean number of elementary pieces diverges with some exponent

$$n(g) \sim n_*(g_c - g)^{-\alpha}$$

the length of the boundary of the surface diverges with some other exponent

$$l(g) \sim l_*(g_c - g)^{-\beta}$$

So that the continuous surfaces will have fractal Hausdorff dimension $\frac{\alpha}{\beta}$. α and β can be derived directly from the matrix integral, and thus we see how random matrix theory can give a very efficient way to compute some geometric properties of random surfaces.

1.5.2 Conformal field theories

Of course, if one fine-tunes more than one coupling constant to some critical value, more complicated critical points can be reached.

For instance consider the Ising model on a random lattice: That model has one coupling constant g coupling to the size of the lattice (the number of squares) and in addition, the usual Ising coupling constant T measuring the strength of the spin interactions, i.e. the temperature. A magnetic field h could also be added. In order to reach the Ising ferromagnetic transition on a continuous surface, one has to fine-tune both coupling constants: $g \to g_c$ so that $n(g)$ diverges and the lattice becomes a continuous surface, and $T \to T_c$ so that the magnetization's correlation length diverges. Of course the critical exponents are different from the ones you would get by fine tuning only one coupling constant.

The set of all possible critical points and critical exponents is closely related to conformal theories [44]. Indeed, the summation over random lattices correspond to vary the geometry of the surface where the statistical physics model lives. Such random surfaces are naturally reparametrization invariant, and therefore are conserved by the conformal group, once the continuous limit is taken. The possible models living on them, can be classified according to the irreducible representations of the conformal group.

Moreover, KPZ [34] have shown how to relate the scaling exponents of a 2D model on a flat space (a regular lattice in the discrete version), to its fluctuating lattice counterpart. There is a relationship, because the flat model is invariant under conformal reparametrization of the surface, i.e. under local fluctuations of the geometry. However, KPZ tells how to relate the exponents of bulk operators only. At the present time it is not yet completely understood how to extend KPZ's formula to boundary operators.

Anyway, the classification of critical points of random matrices, provides some representations of conformal theories. All the so called minimal models are obtained by very simple matrix models, where calculations are quite easy.

For instance, the high-low temperature duality of the Ising model corresponds to a simple linear change of variable in the matrix model.

1.5.3 String theories

Field theories have been very successful for describing elementary particles (standard model), though they are ill-defined due to the fact that we assume elementary particle to be point-like. The simplest alternative theory which can be invented with extended particles is String Theory [36].

A String is a 1-dimensional particle, whose time evolution sweeps a 2D surface across space-time. In quantum mechanics, it is necessary to take into account all possible time evolutions (i.e. all possible surfaces), with some probability amplitude. In other words, string theory is a theory of random surfaces, and one is tempted to apply random matrix technics to it.

The first difficulty, is to include string interactions within the framework of random matrices. Indeed, string interactions correspond to a change of topology, for instance a closed string (a circle) splitting into two strings which merge again, sweeps a torus-like surface (fig. 1.12). In the large N limit of random matrices, such a diagram is suppressed by a $1/N^2$ coefficient. How could one take at the same time the limit

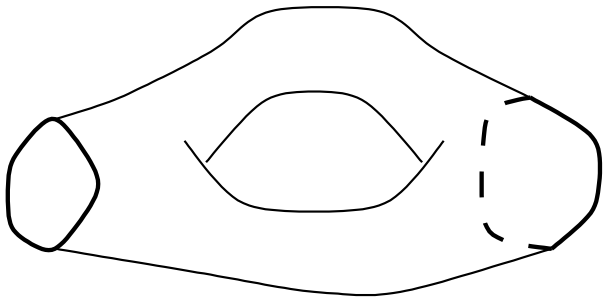


Figure 1.12: A string interaction: a closed string (a circle) splits into two strings which merge again later.

$N \to \infty$ in order for the matrix integral to make sense, and keep track of the higher topologies ? This can be achieved by the double-scaling-limit trick.

Remember that, in order to have continuous surfaces, we have chosen the coupling constant(s) in the vicinity of a critical point $g \to g_c$, so that the mean number of elementary pieces diverges. The matrix integral has a perturbative expansion of the form:

$$Z \sim Z_{\rm reg} \ + \ \sum_h N^{2-2h} \ (g_c - g)^{\gamma_h} \ \zeta_h$$

where γ_h is the critical exponent for the divergence of surfaces of genus h. It can be proven that γ_h is linear in h:

$$\gamma_h = (1-h)(2-\gamma)$$

(it can be roughly explained by some handwaving: the fact that the surface becomes continuous, is independent of its global topology, so g_c does not depend on h and the scaling exponent depends very simply on h).

Now it is easy to see how to proceed: consider the double scaling limit, where $N \to \infty$ and $g \to g_c$ simultaneously, and with the scaling condition that $N^2(g_c - g)^{2-\gamma}$ remains finite. The variable

$$\kappa = N(g_c - g)^{1-\gamma/2}$$

is the string coupling constant. Indeed, each continuous surface is now weighted by κ^{2-2h}:

$$Z \sim \sum_h \kappa^{2-2h} \ \zeta_h$$

In this peculiar limit, many of the terms which were suppressed by the large N limit reappear. The large N limit alone (without double scaling limit) is equivalent to $\hbar \to 0$ limit in quantum mechanics, it is a classical limit, the operators reduce to numbers, and Schroedinger differential equation reduces to an algebraic equation involving no derivative. The same phenomenon happens in matrix models. When the large N limit is taken alone, one gets only algebraic equations, but in the double scaling limit, differential equations reappear (differentiation with respect to κ).

For instance in the simplest matrix model leading to a string theory of central charge $c = 0$, one gets a Painleve equation:

$$u(x)^2 + u''(x) = x$$

(the classical counterpart would be only $u^2(x) = x$, i.e. a typical $1/2$ scaling exponent).

More generally, if we have more than one coupling constant and we are at a higher critical point, we can get a full hierarchy of differential equations corresponding to the variation of each relevant operator. Those hierarchies are integrable hierarchies, like KDV or KP hierarchies. Therefore, there is a link between matrix models and integrable hierarchies, which might be a route to explore that field too...

Let us mention that unfortunately, matrix models have failed to represent the 10-dimensional supersymmetric string, which is the only one which makes sense, and which is expected (or hoped) to be the unified theory of everything in the universe... The matrix models are limited to dimension less than 1. At least, nobody has ever found a way to cross that barrier, and it is not known whether it is a real barrier corresponding to some deep physical or mathematical reason, or it is just a technical one, and we have not been smart enough so far to cross it...

1.6 M(atrix) Theory

Three Random matrix models have been introduced in the context of M-Theory [38]. They are supposed to provide a non-perturbative definition of M-theory (and some of its compactifications, as type IIa and type IIb string theories), in the infinite momentum frame [45, 46, 47].

In all those theories, the action is the Super-Yang-Mills model (SYM) normally defined in 10 dimensions, and reduced to $d = 0$, 1 or 2 dimensions. In each case, the partition function is of the form:

$$Z = \int \mathrm{d}[X^\mu]\mathrm{d}[A_i]\,\mathrm{d}[\theta^\alpha]\ \mathrm{e}^{-Tr\,S[X,\theta,A]}$$

$X^\mu(x_i)$, $\mu = 1 \ldots 10$, $i = 0 \ldots d-1$ are $N \times N$ hermitian matrices, which depend on d space-time dimensions. A_i are d gauge fields (present only if $d = 2$). $\theta^\alpha(x_i)$, $\alpha = 1 \ldots 32$ are $N \times N$ matrices whose entries are grasmann anticommuting variables, they are the superpartners of the X^μ. Let us note Γ_μ the 10-dimensional 32×32 gamma-matrices, and for $d = 2$, γ_i are the d-dimensional gamma-matrices.

The actions S are the following:

- BFSS [46]: M-theory, SYM reduced to d=1:

$$S = \int \mathrm{d}x_0\ \frac{1}{2}\sum_\mu \dot{X}^\mu \dot{X}^\mu + \sum_\alpha \overline{\theta}^\alpha \dot{\theta}^\alpha - \frac{1}{4}\sum_{\mu,\nu}[X^\mu, X^\nu]^2 - \frac{1}{2}\sum_\mu \overline{\theta}\Gamma_\mu[X^\mu, \theta]$$

- DVV [45]: Type IIa string, SYM reduced to d=2:

$$S = \int\int \mathrm{d}x^2 \ \frac{1}{2}(D_i X^\mu)^2 + \overline{\theta}\gamma_i D_i \theta + \frac{1}{2}F_{ij}^2 - \frac{1}{4}[X^\mu, X^\nu]^2 - \frac{1}{2}\overline{\theta}\Gamma_\mu[X^\mu, \theta]$$

  where $D_i = \partial_i + A_i$ are the covariant derivatives, and $F_{ij} = \partial_i A_j - \partial_j A_i$ is the stress tensor of the 2-d gauge field A_i.

- IKKT [47]: Type IIb string, SYM reduced to d=0:

$$S = -\frac{1}{4}[X^\mu, X^\nu]^2 - \frac{1}{2}\overline{\theta}\Gamma_\mu[X^\mu, \theta]$$

1.7 Summary

We have seen that random matrices are involved in many different fields of physics, let us summarize here what kind of matrices play a role for what kind of application. In the next part, we will present some tools for calculating with random matrices, independently of the underlying physical problem, and we will consider only abstract matrices. But for intuition it is important to have in mind some physical examples.

- Quantum chaos: the matrix is the Hamiltonian. Though the Hamiltonian is not random (it is given), it is observed and conjectured (but not understood) that (when the corresponding classical mechanics problem is chaotic), its spectrum has the same distribution as the spectrum of a random matrix. This is why Quantum Chaos people consider random Hamiltonians. Random Hamiltonians belong to Gaussian ensembles, like GUE, GOE or GSE depending on their symmetries. One can also consider random evolution operators, those belong to circular ensembles, like CUE, COE or CSE.

- Disordered systems: the matrix is the Hamiltonian. It is naturally a random matrix due to disorder (impurities for instance). We average over samples or over variations of an external parameter such as a magnetic field. Again, the Hamiltonian can belong to GOE, GUE or GSE ensembles depending on its symmetry properties.

- Mesoscopic conductors: the matrix is the transfer matrix M, or the scattering S matrix. These are naturally random due to disorder. The S matrix is unitary or orthogonal or symplectic. The corresponding ensembles are the circular ensembles. The transfer matrix M belongs to some hyperbolic ensembles (like TUE, TOE or TSE), which are non compact. An equilibrium universal normalizable probability law cannot be found for them. One can only write a "time"-evolution equation, which is a diffusion equation: the DMPK equation.

- QCD: the matrix is the Dirac operator. It is random because we integrate out the gauge field fluctuations. The exact probability law of the true QCD Dirac operator is very complicated. Universality allows to simplify the problem: we

study a matrix ensemble with the same symmetries than QCD, and with a simpler probability law, a Gaussian law. This leads to the three chiral ensembles: chGOE, chGUE, chGSE.

- Statistical mechanics on a random lattice: the matrix here is abstract, it does not have a physical meaning. It is a field whose fluctuations generate by perturbative expansion, some Feynmann's diagrams. The symmetry of the matrix corresponds to the orientability properties of the diagrams: GUE generates orientable graphs, GOE non-orientable graphs. Introducing several random matrices coupled together, one can generate lattices with some fields living on the sites of the lattice, for instance Ising model, Potts model, etc...
  Let me emphasize that this procedure is ill defined beyond the perturbative large N limit. It is necessary to take $N \to \infty$.

- Conformal field theories: same as previous, but now we are interested in the limit where the discrete lattices become continuous surfaces. One has to stand near a critical point where the average number of sites diverges. That generates some scaling exponents, which are in close relationship with conformal theories. By fine-tuning more than one coupling constant, one can reach different critical points, where the lattice becomes continuous and at the same time the fields living on it become critical (correlation length diverges). The KPZ equation relates the critical exponents for those models on a random surface to the ones on a fixed flat surface.

- String theories: one wants to generate continuous surfaces, of all genera, and wants to keep track of the topology. This can be achieved by a double scaling limit: at the same time the size of the matrix tends to infinity (this generates the topological expansion, and is necessary for the number of degrees of freedom to become infinite):
  $$N \to \infty$$
  and the coupling constants are fine-tuned to some critical point
  $$g \to g_c$$
  with a scaling law
  $$N(g_c - g)^{1-\gamma/2} = \kappa \qquad \text{finite}$$
  κ is the string coupling constant, which is coupled to the topology of the surfaces. In this limit one has continuous surfaces of all topologies. The free energy and correlation functions obey an integrable hierarchy of differential equations (KDV or KP hierarchy).

- M(atrix) Theory: the matrices are D0 brane fields. The action is 10D Super Yang Mills reduced to a smaller dimension (0, 1 or 2). Due to supersymmetry, the large N limit or coupling constant fine tunings are not necessary for the integral to make sense.

Chapter 2

Ensembles of random matrices

In the first part of this course, we have met a manifold of applications of random matrices to physics. In Nuclear physics, quantum chaos, disordered systems, QCD,... the main goal is to understand the Universality of random matrices: prove if possible this universality, i.e. that the exact model is irrelevant, only the symmetries are, and therefore one can study a Gaussian model. Then to study the Gaussian model, compute its correlations, spacing statistics,... And of course classify the possible symmetries, the possible universality classes. That will be the topic of this second lecture.

2.1 Random matrices

A random matrix model is defined by:

- A symmetry group G
- An ensemble E, along with its measure $\mathrm{d}M$ (invariant under G)
- and a probability law $\mathcal{P}(M)$ of the form of a Boltzmann weight $\mathrm{e}^{-N^2\mathcal{V}(M)}$ with a potential $\mathcal{V}$.

One is interested in the induced statistics of eigenvalues.

The claim of universality, is that for large N, the statistics of eigenvalues at short range, depends only on G, and not on $\mathcal{P}(M)$. The limit statistical distribution of eigenvalues can therefore be obtained from any law $\mathcal{P}(M)$, and in particular by its simplest version, the Gaussian law[1] $\mathcal{P}(M) = \frac{1}{Z}\,\mathrm{e}^{-N\,\mathrm{tr}\,\frac{g}{2}M^2}$, or in case E is compact, by the constant law $\mathcal{P}(M) = 1$.

For instance, in case the probability law is of the form (2.1.1), the limit distribution should not depend on the potential V.

We will now examine each of the elements of a random matrix model: The ensemble E, with its measure $\mathrm{d}M$, and the probability law $\mathcal{P}(M)$. Then, we will study the different universalities encountered.

[1] Of course, that is not true if the model is near a critical point. A higher degree potential is then needed. Critical points too have some universal properties.

2.1.1 The Ensemble E

So far, we have seen many examples of ensembles, which can be separated into three types: circular, gaussian (flat), or hyperbolic.

- **Gaussian ensembles (flat ensembles).**
  The matrix is for instance a Hamiltonian, or the Dirac operator in QCD. The eigenvalues lie on the real axis (which is not compact). We have met at least 6 possible symmetries, depending on the time-reversibility property and the rotational invariance. These are the GOE, GUE, GSE Wigner Dyson ensembles and their three chiral analogous for QCD: chGOE, chGUE, chGSE. The Bogoliubov-de Gennes Hamiltonians which describe the Andreev reflections in Normal-Supraconductor junctions (NS) belong to four other ensembles (fig.2.1).

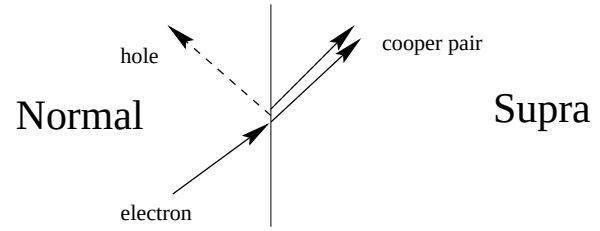


Figure 2.1: Andreev reflection between a normal metal and a supraconductor. The incident electron is reflected as a hole and a Cooper pair is transmitted.

  In total that amounts to 10 ensembles.

- **Circular ensembles.**
  The matrix is for instance a Scattering matrix (S-Matrix). It is a unitary matrix (unitarity comes from the conservation of some positive norm: conservation of the number of particles). The eigenvalues belong to the unit circle in the complex plane. Because of the compacity of these ensembles, the universal distribution can be obtained with a uniform probability. The Wigner Dyson ensembles are COE, CUE, CSE, and again it is possible to define some chiral ones and NS ones.

- **Hyperbolic ensembles.**
  The matrix is for instance a transfer matrix. It belongs to a group of the form $SU(N,N)$, because it conserves a non-positive norm (number of ingoing particles minus outgoing particles should be the same on each side of a conductor). Such ensembles are not compact. They have a constant negative curvature, they are hyperbolic symmetric spaces. It is not possible to define a universal normalizable probability. Instead, is is possible to write a diffusion equation for the probability law, as a function of a "time" t (any external parameter can be the "time", for instance the length of the conductor). The diffusion equation describes the probability of presence of a matrix which performs a random walk on the ensemble

E, it takes the form of a heat diffusion, it is called the DMPK equation:

$$\frac{\partial \mathcal{P}(M,t)}{\partial t} = \Delta \mathcal{P}(M,t)$$

Where Δ is the Laplacian on the ensemble E. For symmetric spaces, Δ is called the Beltrami operator.

The same equation could be written for the circular and gaussian ensembles, for which a large t limit exists. In the hyperbolic cases, there is no large t limit, that's why one can only write down a DMPK equation, and it is not possible to define a universal large t limit.

Again, depending on the time-reversibility and rotational symmetries, many of these hyperbolic ensembles have been studied along with their corresponding DMPK equations. They are sometimes called TOE, TUE, TSE (T is for Transfer-matrix), and their chiral and NS versions.

The common property of all those ensembles, is that they are symmetric spaces, i.e. Lie groups with a constant curvature. The circular ensembles have a positive curvature, the gaussian ones have a zero curvature (tangent spaces of the circular ones, indeed, the real axis is the limit of a large radius circle), and the hyperbolic ensembles have a negative curvature.

Symmetric spaces

A symmetric space is a Lie Group with a constant curvature. It is like a sphere, it can be described by radial and angular coordinates. The radial coordinates are the eigenvalues of the matrix.

There is a classification of all possible symmetric spaces, found by Cartan. There are only 10 symmetric spaces of arbitrarily large rank N, plus a few exceptional ones with rank $N \leq 8$ which we will ignore in the following. Each of them can come with positive, negative or null curvature.

Symmetric spaces are in one to one correspondence with some diagrams, called "root lattices" (derived from the Dynkin diagrams of the corresponding Lie Algebras). The root lattice contains all the information about the symmetric space attached to it. The root lattice is a finite set of points in $\mathbb{R}^N$ (each one is called a root). Among them, half of them are called positive roots, the others are called negative roots, that means we have chosen an ordering on $\mathbb{R}^N$. Among the positive roots, a subset of them form a basis, these are the simple roots. The set of simple roots contains all the information about the algebra. The roots have a multiplicity, which is an integer.

We will see later that the knowledge of the set roots is sufficient to find the Beltrami operator, or the measure for the corresponding matrix ensembles.

Here is a table of Symmetric spaces, their root lattices, in correspondence with Random Matrix ensembles [48, 49].

Sym. Space	Roots	Circular	Flat	Hyperbolic
$\frac{SL(N,\mathbb{R})}{SO(N)}$	A_{N-1}	COE	GOE	
$\frac{SL(N,\mathbb{C})}{SU(N)}$	A_{N-1}	CUE	GUE	
$\frac{SU(2N)}{Sp(2N)}$	A_{N-1}	CSE	GSE	
$\frac{SO(2N+1,\mathbb{C})}{SO(2N+1)}$	B_N			
$\frac{SO(2N,\mathbb{C})}{SO(2N)}$	D_N	NS-D	NS-D	
$\frac{Sp(2N,\mathbb{C})}{Sp(2N)}$	C_N	NS-C	NS-C	
$\frac{Sp(2N,\mathbb{R})}{U(N)}$	C_N	NS-CI	NS-CI	TOE
$\frac{SO(4N)}{U(2N)}$	C_N	NS-DIII	NS-DIII	TSE
$\frac{SO(4N+2)}{U(2N+1)}$	BC_N			
$\frac{SO(p,q)}{SO(p)\times SO(q)}$	$B_q D_q$		chGOE	
$\frac{SU(p,q)}{SU(p)\times SU(q)}$	BC_q		chGUE	(p=q) TUE
$\frac{Sp(2p,2q)}{Sp(2p)\times Sp(2q)}$	BC_q		chGSE	

2.1.2 The probability law $\mathcal{P}(M)$

Let us consider a random matrix M of size $N \times N$ from an ensemble E and a symmetry group G, together with a probability law

$$\mathcal{P}(M)$$

That probability law should depend only on the invariants of the matrix (under G), i.e. the traces of the matrix and its powers:

$$T_k = \frac{1}{N} \mathrm{Tr}\ M^k \qquad k = 1 \dots N$$

(which we have normalized by $1/N$ so that T_k is of order $O(1)$ for N large). The set of T_k, $k = 1..N$ is a complete set of invariants of the matrix under a change of coordinate systems, indeed, any other $\mathrm{Tr}\ M^k$ with $k > N$ could be written as a function of the N first T_k's, the determinant too,...

Usually the probability law is written as a Boltzmann weight:

$$\mathcal{P}(M) = \mathrm{e}^{-N^2 \mathcal{V}(M)} = \mathrm{e}^{-N^2 \mathcal{V}[T_k]}$$

Let us explain why there is a N^2 in front of the potential $\mathcal{V}$:
The total number of degrees of freedom is of order N^2, thus the measure term $\mathrm{d}M$ is a product of N^2 terms, i.e. of order e^{N^2}. The action term has to be of the same order. If it is larger, that means the measure term is irrelevant, the matrix is not random but trapped at the bottom of the potential. If it is smaller, the potential is irrelevant, and the matrix is free, it escapes at infinity.

example 1

The simplest example, which we will consider, and which is of practical use is the following:

$$\mathcal{P}(M) = \frac{1}{Z} \mathrm{e}^{-N \operatorname{tr} V(M)} \tag{2.1.1}$$

where $V(x)$ is a polynomial potential, generally bounded from below (except when one is concerned with applications to 2D statistical physics on a random lattice as we have seen).

$$V(x) = \frac{g}{2}x^2 + \frac{g_3}{3}x^3 + \frac{g_4}{4}x^4 + \ldots + \frac{g_n}{n}x^n + \ldots$$

Z is the partition function, which contains almost all the information about the problem, most of our efforts will aim at calculating Z.

$$Z = \int \mathrm{d}M \ \mathrm{e}^{-N \operatorname{tr} V(M)}$$

Differentiating Z with respect to the g_n's gives expectation values and correlations.

example 2

In particular, the Gaussian ensembles correspond to a quadratic potential:

$$\mathcal{P}(M) = \frac{1}{Z} \ \mathrm{e}^{-N \operatorname{tr} \frac{g}{2} M^2}$$

example 3

The circular ensembles ($E = G = U(N), O(N)$ or $sp(2N)$) are compact. The limit distribution is of the form:

$$\mathcal{P}(M) = 1/\mathrm{vol}(E)$$

2.1.3 The measure

Now, we are going to study the measure which enters the definition of random matrix ensembles.

For simplicity, let us begin with the ensemble of Hermitian matrices (i.e. GUE). $\mathrm{d}M$ is the Haar measure on this ensemble:

$$\mathrm{d}[M] = \prod_{i=1}^{N} \mathrm{d}M_{ii} \ \prod_{i<j} \mathrm{dRe}M_{ij} \, \mathrm{dIm}M_{ij}$$

$\mathrm{d}[M]$ is invariant under changes of basis, it is invariant under unitary transformations of M:

$$M \to U^\dagger M U \qquad \text{where} \quad U^\dagger U = 1$$

Any hermitian matrix can be diagonalized by a unitary transformation, i.e. for each M, there exists a unitary matrix U and a diagonal matrix Λ such that:

$$M = U^\dagger \Lambda U \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \quad , \quad U^\dagger U = 1 \tag{2.1.2}$$

(of course, this decomposition is not unique).

M has N^2 independent coefficients, Λ has N coefficients and U has N^2-N. We can perform the change of variables $M \to (\Lambda, U)$, and we have to compute the Jacobian:

$$\mathrm{d}[M] = J \mathrm{d}[\Lambda] \mathrm{d}[U]$$

Because the Haar measure is invariant under a unitary transformation, the Jacobian J depends only on Λ, and it is sufficient to compute it in the vicinity of $U = 1$. The differential of 2.1.2 reads:

$$\mathrm{d}M = \mathrm{d}U^\dagger \Lambda U + U^\dagger \mathrm{d}\Lambda U + U^\dagger \Lambda \mathrm{d}U \qquad , \qquad \mathrm{d}U^\dagger U + U^\dagger \mathrm{d}U = 0 \tag{2.1.3}$$

when $U = 1$ we have (we write the coordinates):
for $i = j$: $\mathrm{d}M_{ii} = \mathrm{d}\lambda_i$
for $i < j$: $\mathrm{dRe}M_{ij} = (\lambda_i - \lambda_j)\mathrm{dRe}U_{ij}$, $\mathrm{dIm}M_{ij} = (\lambda_i - \lambda_j)\mathrm{dIm}U_{ij}$
The change of variables $M \to (\Lambda, U)$ is diagonal, its Jacobian is simply the product:

$$J = \prod_{i<j} (\lambda_i - \lambda_j)^2 \qquad , \qquad \text{GUE}$$

Other ensembles

The same calculation can be performed for other ensembles.

- for the ensemble of real symmetric matrices (GOE) one would have the Haar measure:

  $$\mathrm{d}[M] = \prod_{i=1}^{N} \mathrm{d}M_{ii} \prod_{i<j} \mathrm{d}M_{ij}$$

  and one can diagonalize M with a orthogonal transformation:

  $$M = U^t \Lambda U \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \quad , \quad U^t U = 1 \tag{2.1.4}$$

  and the Jacobian of the change of variables $M \to (\Lambda, U)$ is

  $$J = \prod_{i<j} |\lambda_i - \lambda_j| \qquad , \qquad \text{GOE}$$

- For the Symplectic ensemble GSE:

  $$J = \prod_{i<j} |\lambda_i - \lambda_j|^4 \qquad , \qquad \text{GSE}$$

- For circular ensembles:

  $$J = \prod_{k<l} |\mathrm{e}^{i\theta_k} - \mathrm{e}^{i\theta_l}|^\beta \qquad , \qquad \beta = 1, 2, 4 \text{ for COE, CUE, CSE}$$

where the eigenvalues are now the $e^{i\theta_k}$ on the unit circle, with measure $d\theta_k$, and β =1,2,4 for COE, CUE and CSE. Note that the Jacobian can also be written

$$J = \prod_{k<l} |\sin \frac{1}{2}(\theta_k - \theta_l)|^\beta \qquad , \qquad \beta = 1, 2, 4$$

If the θ_k are rescaled by a radius r:

$$\theta_k = \frac{\lambda_k}{r} \qquad , \qquad \lambda_k \in [0, 2\pi r]$$

the Jacobian is

$$J = \prod_{k<l} |\sin \frac{1}{2r}(\lambda_k - \lambda_l)|^\beta$$

in the large r limit, the circular measure tends towards the gaussian one. Somehow, the gaussian ensembles are the limits of the circular ensembles when the curvature (inverse of the radius) tends towards zero.

- Other symmetric spaces:

$$J = \prod_{\vec{\alpha}>0} \sin\left(\sqrt{R}(\vec{\lambda}.\vec{\alpha})\right)^{m_{\vec{\alpha}}}$$

where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N)$, $\vec{\alpha}$ are the positive roots with multiplicity $m_{\vec{\alpha}}$, and R is the curvature (positive, negative or zero).

For instance, consider the A root lattice, all the roots have the same multiplicity $m_{\vec{\alpha}} = \beta$, the positive roots are $e_i - e_j$ with $i < j$ (the e_i are the basis vectors of $\mathbb{R}^N$. In the zero curvature case J reduces to

$$J = \prod_{i<j} |\lambda_i - \lambda_j|^\beta$$

2.2 Universality

So far, I have said that "some" statistical properties of the spectrum of large random matrices are universal. In order to talk about universality, it is necessary to be able to compare the spectrum of different matrices.

2.2.1 The level density $\rho(\lambda)$

The level density is defined as

$$\rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} < \delta(\lambda - \lambda_i) >= \frac{1}{N} \langle \text{Tr } \delta(\lambda - M) \rangle$$

where the average <> has been performed with the probability law $\mathcal{P}(M)$.

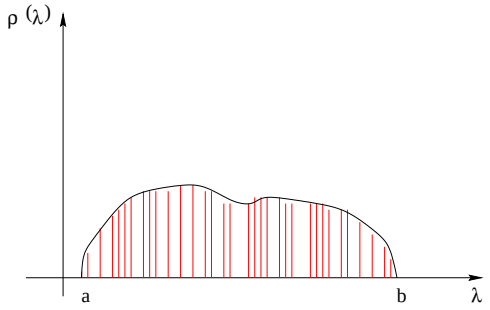


Figure 2.2: Typical spectrum of a matrix. In the large N limit, the spectrum is well approximated by a continuous density of levels $\rho(\lambda)$.

The density is not universal, it depends on the precise form of the probability law $\mathcal{P}(M)$.

The mean spacing between consecutive eigenvalues is then:

$$\Delta = \frac{1}{N\rho(\lambda)}$$

it is not constant along the spectrum, it depends on λ.

In order to compare the statistical properties of spectra of different matrices, it is necessary to "rescale" the spectrum so that the two spectra have the same average level spacing. This procedure is called "unfolding" of the spectrum.

2.2.2 Unfolded spectrum

Consider the counting function

$$D(\lambda) = \frac{1}{N}\left\langle \sum_i \theta(\lambda - \lambda_i) \right\rangle = \int_a^\lambda \rho(l)\mathrm{d}l$$

where θ is the heavyside step function. $D(\lambda)$ increases by $\frac{1}{N}$ each time λ meets one λ_i, therefore, the $\mu_i = D(\lambda_i)$ have a constant mean-spacing equal to

$$\tilde{\Delta} = \frac{1}{N}$$

And it is easy to see that the density of the μ_i is constant:

$$\tilde{\rho}(\mu) = \frac{1}{N}\sum_{i=1}^{N} < \delta(\mu - \mu_i) >= 1$$

The $\mu_i = D(\lambda_i)$ are called the unfolded variables.

The procedure to unfold the spectrum is the following:

- first compute the average density $\rho(\lambda)$ and its primitive $D(\lambda)$.

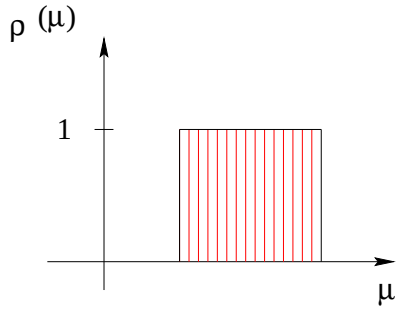


Figure 2.3: The unfolded spectrum has density 1.

- then make a change of variable $\lambda_i \to \mu_i = D(\lambda_i)$
- then compute all the statistical properties by averaging over the μ_i's instead of the λ_i's.
That is how you can find some universal properties.

Here are the observables most often described:

2.2.3 Level spacing distribution

The level spacing distribution $p(s)$ is the property that $s\tilde{\Delta}$ is the distance between two consecutive eigenvalues, i.e. the probability that $s\tilde{\Delta} = \mu_{i+1} - \mu_i$ (if we assume the eigenvalues ordered), i.e. $p(s)\mathrm{d}s$ measures the number of μ_i such that $|\mu_{i+1} - \mu_i|/\tilde{\Delta} \in [s, s + \mathrm{d}s]$.

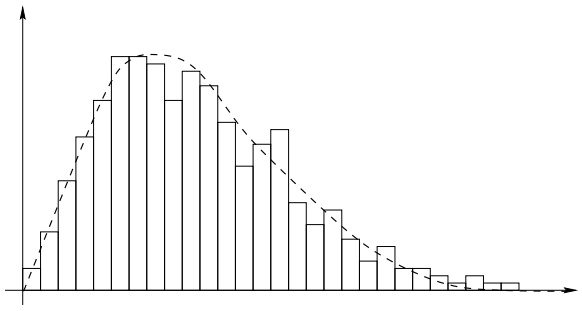


Figure 2.4: Level spacing distribution of the energy levels of nuclei.

Let us consider the function $E(\mu, \nu)$, defined as the probability that the interval $[\mu, \nu]$ contains no eigenvalue.

We have:

$$p(|\mu - \nu|/\tilde{\Delta}) = -\frac{1}{N(N-1)} \frac{\mathrm{d}^2 E(\mu, \nu)}{\mathrm{d}\mu \mathrm{d}\nu}$$

Indeed,

$$E(\mu,\nu) = \left\langle \prod_{k=1}^{N} (1 - \chi(\mu_k)) \right\rangle$$

where the average is taken with the matrix probability law, the μ_k are the (unordered) unfolded eigenvalues, and $\chi(\mu)$ is the charcteristic function of the interval $[\mu,\nu]$ (its derivative is a δ function).

Therefore:

$$\frac{\mathrm{d}E(\mu,\nu)}{\mathrm{d}\mu} = \sum_{i=1}^{N} \left\langle \delta(\mu_i - \mu) \prod_{k\neq i} (1 - \chi(\mu_k)) \right\rangle$$

and

$$\frac{\mathrm{d}^2E(\mu,\nu)}{\mathrm{d}\mu\mathrm{d}\nu} = -\sum_{i\neq j} \left\langle \delta(\mu_i - \mu)\delta(\mu_j - \nu) \prod_{k\neq i,j} (1 - \chi(\mu_k)) \right\rangle$$

The right hand side is the probability that there is no eignevalue in $[\mu,\nu]$, and that there is one eigenvalue at μ and one at ν, it is thus exactly the probability that μ and ν are consecutive eigenvalues. If we are talking about the unfolded spectrum, or if we can assume that the density of eigenvalues is constant, we expect the spacing probability to depend only on the distance $s = |\nu - \mu|/\tilde{\Delta}$. (see reference [1] for a proper derivation).

Example: the Poisson distribution

Consider the case where all the μ_i are independent variables with constant density. Then $E(\mu.\nu)$ is simply:

$$E(\mu,\nu) = \prod_i \int_{\mu_i \notin [\mu,\nu]} \mathrm{d}\mu_i = \left(\int_{\mu}^{\nu} \mathrm{d}\mu_1 \right)^N = (1 - (\mu - \nu))^N = (1 - s/N)^N$$

where $s = N|\mu - \nu|$. In the large N limit, we have $E(s) \sim \mathrm{e}^{-s}$, and by taking the second derivative, we get the Poisson law:

$$\bullet \qquad \text{Poisson} \quad : \quad p(s) = \mathrm{e}^{-s} \tag{2.2.5}$$

The Wigner surmise

When the energy levels are independent random variables, the spacing distribution is a Poisson law. On the contrary, when the energy levels are the eigenvalues of a random matrix, they are not independent, and the spacing distribution does not look at all like a Poisson law. Indeed Wigner suggested the following spacing distributions:

$$\bullet \qquad \text{GUE} \quad : \quad \frac{32}{\pi^2} s^2 \, \mathrm{e}^{-\frac{4}{\pi}s^2} \tag{2.2.6}$$

$$\bullet \qquad \text{GOE} \quad : \quad \frac{\pi}{2} s \, \mathrm{e}^{-\frac{\pi}{4}s^2} \tag{2.2.7}$$

$$\bullet \qquad \text{GSE} \quad : \quad \frac{2^{18}}{3^6\pi^3} s^4 \, \mathrm{e}^{-\frac{64}{9\pi}s^2} \tag{2.2.8}$$

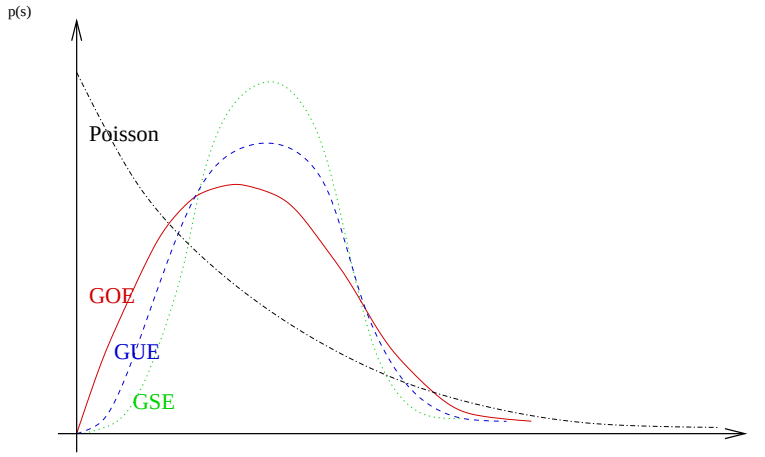


Figure 2.5: The three Wigner laws.

2.2.4 The correlation function

The other usual observable is the 2-point eigenvalues connected correlation function. $\rho(\lambda,\mu)\mathrm{d}\lambda\,\mathrm{d}\mu$ is the probability that there is one eigenvalue in $[\lambda,\lambda+\mathrm{d}\lambda]$ and one eigenvalue in $[\mu,\mu+\mathrm{d}\mu]$. In other words:

$$\rho(\lambda,\mu)=\frac{1}{N^2}\left\langle \mathrm{Tr}\;\delta(\lambda-M)\,\mathrm{Tr}\;\delta(\mu-M)\right\rangle=\frac{1}{N^2}\sum_{i,j=1}^{N}\left\langle\delta(\lambda-\lambda_i)\delta(\mu-\lambda_j)\right\rangle$$

and the connected correlation function is:

$$\rho_c(\lambda,\mu)=\rho(\lambda,\mu)-\rho(\lambda)\rho(\mu)$$

The function $\rho_c(\lambda,\mu)$ has the property to be universal. It does not depend on the details of the system, only on its symmetries. There are two regimes in which that function shows some universality:

- short range regime

  where $|\lambda-\mu|\sim O(1/N)$ is of the same order as the mean spacing $\Delta=1/N\rho(\lambda)$. For the three Dyson's ensembles, one finds the three universal laws in the form:

  $$\rho_c(\lambda,\mu)\sim-\rho(\lambda)\rho(\mu)\;Y\left(\frac{|\lambda-\mu|}{\Delta}\right)$$

  $Y(r)$ is the unfolded correlation function, one finds the following universal functions for the three Wigner-Dyson ensembles:

  – GUE:

  $$Y(r)=\left(\frac{\sin\pi r}{\pi r}\right)^2 \tag{2.2.9}$$

– GOE:

$$Y(r) = \left(\frac{\sin \pi r}{\pi r}\right)^2 - \left(\mathrm{si}\, \pi r - \frac{\pi}{2}\mathrm{sgn}\, r\right) \left(\frac{\cos \pi r}{\pi r} - \frac{\sin \pi r}{(\pi r)^2}\right)$$

– GSE:

$$Y(r) = \left(\frac{\sin 2\pi r}{2\pi r}\right)^2 - \mathrm{si}\, 2\pi r \left(\frac{\cos 2\pi r}{2\pi r} - \frac{\sin 2\pi r}{(2\pi r)^2}\right)$$

where

$$\mathrm{si}(r) = \int_0^r \frac{\sin x}{x} \mathrm{d}x$$

- long range regime

  where $|\lambda - \mu| \sim O(1)$. The correlation function has high frequency oscillations of frequency of order N (see the short range regime above), and one has to smooth out those oscillations. The smoothed correlation function is of order $1/N^2$, and again, it is universal. It depends only on the endpoints of the spectrum, not on the details of the model. For instance, for GUE with endpoints a and b, one finds:

$$\rho_{c\,\mathrm{smoothed}}(\lambda, \mu) = -\frac{1}{4\pi^2 N^2} \frac{(\lambda-\mu)^2 - (\lambda-a)(\lambda-b) - (\mu-a)(\mu-b)}{(\lambda-\mu)^2 \sqrt{(\lambda-a)(b-\lambda)(\mu-a)(b-\mu)}} \quad (2.2.10)$$

2.2.5 Critical points and other universal regimes

So far, we have considered the generic properties of the spectrum, i.e. in the bulk (far from the endpoints), and away from any critical point. However, there exists many other universal regimes (see [50] for phase transitions in random matrices). Here we will consider only hermitian matrices.

- Spectrum near the edges.

  For a generic potential (away from criticality), the average density of levels drops as a square root near the edges (fig (2.6)), while the asymptotic correlation function of eq.2.2.10 seems to diverge. Zooming in into the edge, one observe an exponential tail to the density of levels, and a Airy Kernel law for the correlation function:

$$\rho_c(a + N^{-2/3}x, a + N^{-2/3}y) \sim \text{ cte } \left(\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y}\right)^2 \quad (2.2.11)$$

  That behavior is universal, it occurs for any potential V (away from criticality), it can therefore be obtained from a gaussian law.

- Critical points.

  In case the potential is critical (2nd order phase transition), the behavior at the edges changes (fig (2.7)). The density of levels has a power law singularity

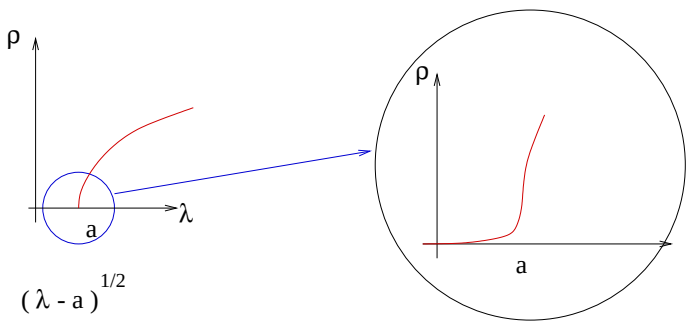


Figure 2.6: The density of eigenvalues has square-root singularities at the edges. An enlargement of the edge's vicinity shows a universal tail related to the Airy function.

(which is no more a square-root). In particular, the critical points corresponding to (p, q) conformal minimal models in conformal field theory, have a p/q power law exponent:

$$\rho(\lambda) \sim (\lambda - a)^{p/q}$$

The tail and correlation function don't correspond anymore to the Airy kernel. They are described by functions solutions of differential equations like Painlevé equation. They obey a hierarchy of differential equations, like KdV or KP hierarchy.

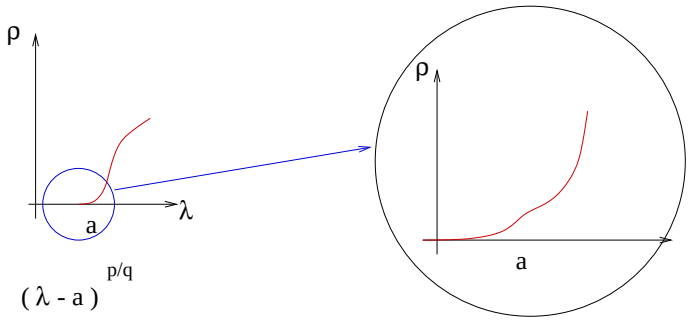


Figure 2.7: At a critical point, the density of eigenvalues has power law singularities at the edges. An enlargement of the edge's vicinity shows a universal tail related to some integrable hierarchy of differential equations.

- Merging of edges.

  When the support of ρ is not connected, it may occur that when a coupling constant is varied, two connected parts come into contact (fig (2.8)). That generally leads to a first order phase transition. The universal behavior is related to a Painlevé II equation.

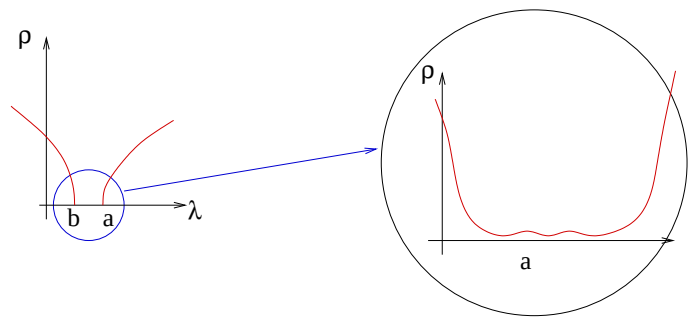


Figure 2.8: When two edges merge, an enlargement of the merging area shows a universal behavior related to a Painlevé II equation.

The classification of all possible behaviors has not been completed at the present time, and it has hardly been explored for non hermitian ensembles. That classification, and its relationship with integrability and differential equations is a very active area of mathematics and physics.

I will not derive those universal behaviors in this introductory course.

2.3 Technics for studying random matrices

A large number of approaches has been invented in order to compute things in a random matrix model. I will give here a short survey of the main technics used, and in the other parts of this course, I will detail some of them.

- The saddle point method: it gives the large N leading contribution to matrix integrals. There is an analogy with a Coulomb gaz of eigenvalues in a potential well. One usually gets linear or algebraic equations. That method is available only when one is able to integrate out the angular degrees of freedom, and write the integral in terms of the eigenvalues only, therefore it is often inapplicable to multi-matrix integrals.

- The Orthogonal Polynomials method: that method is exact for any N, it merely simplifies in the large N limit. There is an analogy with a gaz of fermions. Instead of algebraic equations, it leads to recurrence equations, which can also be written as operator equations like $[P,Q] = 1/N$. Again, such a method is not available for many multi-matrix models.

- Equations of motion: the equations of motion (invariance of an integral under a change of variable) can be written for any model. It is only in the large N limit and for some particular models that the equations close. They have a geometrical interpretation, in terms of splitting boundaries of random surfaces. They allow a $1/N^2$ expansion.

- Supersymetric Efetov's method: it exists only for gaussian potentials, and for any number (even infinite: matrix field theory) of matrices in any ensemble. It mainly amounts to writing average values of determinants with the help of fermions (and bosons for negative powers of determinants), and integrate out the original matrices. It is very efficient for disordered systems where one can assume the potentials gaussian. I will not present that method in this course.

- Character expansion method: it can be considered as an extension of the saddle point method. One decomposes the integrand onto the basis of characters, and uses the orthogonality of characters in order to perform the matrix integral. One is then left with a sum over representations (Young diagrams) instead of a matrix integral. The large N limit amounts to approximate that sum by its largest term, the contribution of only one representation (one gets a saddle point equation for the length of the Young diagram lines).

- Renormalization: one changes the size of the matrix $N \to N+1$, and integrates out the last row and column of the matrix, in order to rewrite it as another $N \times N$ matrix model with a new action $S + \frac{1}{N}\delta S$. That gives β-functions, critical points, critical exponents and the phase diagram. That method could in principle work for any model, and can be used approximately.

- Many other methods have probably been invented...

Let me mention that unfortunately, we have so far met two types of models: those which can be solved by nearly every of the previous methods, and those which can be solved by none of the previous: solvable / non solvable (or at least non solved up to now). All those methods either work exactly, or not at all. We are lacking an approximate method.

Chapter 3

Random Matrices and discretized surfaces

3.1 Perturbative expansion: feynmann graphs and discretized surfaces

Consider the following matrix integral (M is a $N \times N$ hermitian matrix)

$$\tilde{Z} = \int \mathrm{d}M \mathrm{e}^{-N \operatorname{tr} V(M)} \tag{3.1.1}$$

$$\text{with} \qquad V(M) = V_0 + \frac{g}{2}M^2 + \frac{g_3}{3}M^3 + \cdots + \frac{g_k}{k}M^k = V_0 + \frac{g}{2}M^2 + \delta V \tag{3.1.2}$$

We will expand it perturbatively. The idea of the diagrammatic expansion, is to consider the higher degree terms as perturbations of a Gaussian integral, and Taylor-expand $\exp -N \operatorname{tr} \delta V$:

$$\tilde{Z} = \mathrm{e}^{-N^2 V_0} \int \mathrm{d}M \mathrm{e}^{-\frac{1}{2}Ng \operatorname{tr} M^2} (1 - N \operatorname{tr} \delta V(M) + \frac{N^2}{2} \operatorname{tr}^2 \delta V(M) + \ldots) \tag{3.1.3}$$

in other words:

$$\tilde{Z} = \tilde{Z}_0 \left\langle \mathrm{e}^{-N \operatorname{tr} \delta V(M)} \right\rangle_0 = 1 - N \left\langle \operatorname{tr} \delta V \right\rangle_0 + \frac{N^2}{2} \left\langle (\operatorname{tr} \delta V)^2 \right\rangle_0 + \ldots \tag{3.1.4}$$

where index 0 means Gaussian average. Wick's theorem tells us how to compute Gaussian averages: One has to group the variables in pairs, in all possible ways, and replace each pair by the 2-variables average. For instance, if x, y, z, t are Gaussian variables, we have:

$$\langle xyzt \rangle_0 = \langle xy \rangle_0 \langle zt \rangle_0 + \langle xz \rangle_0 \langle yt \rangle_0 + \langle xt \rangle_0 \langle yz \rangle_0$$

That can be represented diagrammatically by linking each pair by a line called "propagator". Each term of the expansion of Eq. (3.1.4) can thus be drawn as a diagram whose vertices represent all the variables of a monome of δV, and are linked by propagators. Feynmann rules tell how to draw such diagrams, and what is their numerical value.

3.1.1 Feynmann rules

Propagator

The propagator is the expectation value of one pair of Gaussian variables, it is the inverse of the quadratic part of the potential:

$$= \left\langle M_i^j M_k^l \right\rangle_0 = \frac{1}{Ng}\, \delta_l^i \delta_j^k \tag{3.1.5}$$

Each propagator links two ordered pairs of indices $(i,j) \to (k,l)$ (with the convention that we follow indices down to up). Because of this ordering (specific to hermitian matrices), the propagators are oriented double lines.

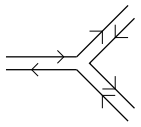
Vertices

Each monome $\left(N(g_k/k)\ \mathrm{Tr}\ M^k\right)$ of $(N\,\mathrm{tr}\,\delta V)$ (cf eq.3.1.2) is represented by a diagram with k external legs, since it contains k matrices M free to be paired by propagators. Here again, the index structure gives an orientation to the double lines.

For instance, the cubic term in the potential

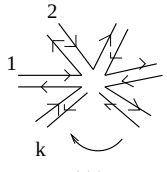
$$N\frac{g_3}{3}\ \mathrm{tr}\ M^3 \ = \ N\frac{g_3}{3} \sum_{i,j,k,l,m,n} M_i^j M_k^l M_m^n\, \delta_j^k \delta_l^m \delta_n^i$$

is represented by:



$$= N\frac{g_3}{3}\, \delta_j^k \delta_l^m \delta_n^i \tag{3.1.6}$$

and more generally, the degree k term is:



$$= N\frac{g_k}{k}\, \delta_{j_1}^{i_2} \delta_{j_2}^{i_3} \ldots \delta_{j_{k-1}}^{i_k} \delta_{j_k}^{i_1} \tag{3.1.7}$$

Diagrams with indices

Now, we have got all the tools to compute the perturbative expansion of Eq. (3.1.4) :

$$\tilde{Z} = \sum_{\{n_k\}} \left\langle \prod_k \frac{1}{n_k!} \left(-N\frac{g_k}{k}\,\mathrm{tr}\ M^k\right)^{n_k} \right\rangle_0 \tag{3.1.8}$$

Each term of the form

$$\left\langle \prod_k \frac{1}{n_k!} \left(-N\frac{g_k}{k}\,\mathrm{tr}\ M^k\right)^{n_k} \right\rangle_0 \tag{3.1.9}$$

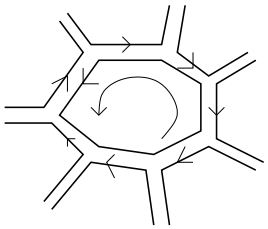


Figure 3.1: Contribution of a loop (a closed single line)

is expanded according to Wick's theorem, as a sum over all possible diagrams with n_k k legs vertices, linked by propagators. Each diagram worths:

$$\frac{1}{\Omega}\left(\frac{1}{gN}\right)^{n_v}\prod_k(-Ng_k)^{n_k}\sum_{indices}\prod\left(\delta^{\text{indices}}_{\text{indices}}\right) \tag{3.1.10}$$

where:

- Ω is the symmetry factor of the diagram, it is a combinatoric factor related to the number of vertices permutations keeping the total diagram unchanged, it contains in particular the $(n_k!)k^{n_k}$ factors.
- n_k is the number of k legs vertices of the diagram.
- $n_v = \sum_k n_k$ is the total number of vertices.
- n_p is the total number of propagators. Note that $n_p = \sum_k kn_k/2$.
- the product of δ functions tells how to glue the double lines. The structure of propagator and vertices (3.1.5,3.1.7) oblige the indices to be constant along single lines.

Index summation: loops

It is possible to compute the sum over all indices in Eq. (3.1.10) . Indeed, indices appear only in the δ functions, and are constant along each single line. Each closed single line carrying an index i, contributes as a factor $\delta^i_i = 1$. The sum over i, thus gives a factor N for each closed single line (loop) (cf fig.3.1).

Thus, Eq. (3.1.10) reduces to:

$$\frac{1}{\Omega}\left(\frac{1}{gN}\right)^{n_v}\prod_k(-Ng_k)^{n_k}\,N^l \tag{3.1.11}$$

where l is the number of loops of the diagram.

Diagrams without indices

Eventually, the expansion of Eq. (3.1.4) can be represented as a sum over all possible diagrams without indices.

Let us summarize the Feynmann rules here:

- draw all possible diagrams made of oriented double lines vertices and propagators glued together consistently with the orientation of lines.
- to each propagator (double line) is associated a factor $1/(Ng)$
- to each k legs vertex is associated a factor $(-Ng_k)$
- to each closed single line is associated a factor N
- then divide the value of a given diagram (calculated by multiplying all the propagator, vertices and loop factors), by the symmetry factor Ω of the graph. This means that equivalent diagrams (which can be obtained by relabeling the vertices or indices) are not counted twice.
- then sum over all possible diagrams.

A diagram with n_k k-legs vertices, n_p propagators, l loops, and with symmetry factor Ω contributes to the partition function $\tilde{Z}$ as:

$$\frac{1}{\Omega} N^{n_k - n_p + l} g^{-n_p} \prod_k (-g_k)^{n_k} \tag{3.1.12}$$

Note that there is no really simple method for computing Ω, but anyway, we won't need to know it in what follows. Only the topological properties of the diagrams will matter. Let us illustrate the Feynmann rules with an example.

3.1.2 Example

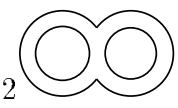
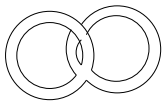
Let us compute $\tilde{Z} = \int \mathrm{d}M \mathrm{e}^{-N \operatorname{tr}(\frac{g}{2}M^2 + \frac{g_4}{4}M^4)}$ at order 1 in g_4 power series expansion. At that order we have:

$$\tilde{Z} = 1 - N\frac{g_4}{4} \left\langle \operatorname{tr} M^4 \right\rangle_0 + O(g_4^2) = 1 - N\frac{g_4}{4} \sum_{ijkl} \left\langle M_i^j M_j^k M_k^l M_l^i \right\rangle_0 + O(g_4^2) \tag{3.1.13}$$

The Gaussian average of the product of 4 matrices is given by Wick's theorem: one has to take all pairs of matrices:

$$\left\langle M_i^j M_j^k M_k^l M_l^i \right\rangle_0 = \left\langle M_i^j M_j^k \right\rangle_0 \left\langle M_k^l M_l^i \right\rangle_0 + \left\langle M_i^j M_l^i \right\rangle_0 \left\langle M_j^k M_k^l \right\rangle_0 + \left\langle M_i^j M_k^l \right\rangle_0 \left\langle M_j^k M_l^i \right\rangle_0$$

That equality can be represented diagrammatically (note that the 2 first terms are identical up to a permutation of indices):

$\left\langle M_i^j M_j^k M_k^l M_l^i \right\rangle_0 = 2$  $+$ 

Which gives (with propagator Eq. (3.1.5)):

$$= 2\frac{1}{(Ng)^2} \delta_i^k \delta_j^j \delta_k^i \delta_l^l + \frac{1}{(Ng)^2} \delta_i^l \delta_j^k \delta_j^i \delta_k^l$$

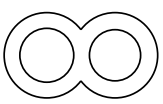
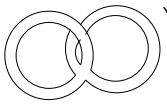
and:

$$\tilde{Z} = 1 - N\frac{g_4}{4}\frac{1}{(Ng)^2}\sum_{ijkl}\left(2\delta_i^k\delta_j^j\delta_k^i\delta_l^l + \delta_i^l\delta_j^k\delta_j^i\delta_k^l\right) + \dots$$

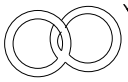
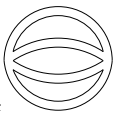
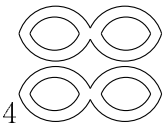
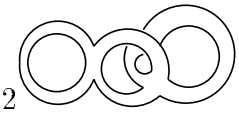
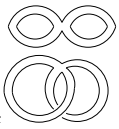
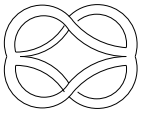
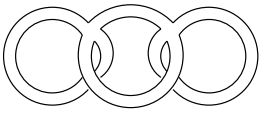
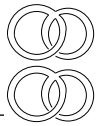
The summation over indices can be carried out thanks to the δ's:

$$\tilde{Z} = 1 - N\frac{g_4}{4}\frac{1}{(Ng)^2}\left(2N^3 + N\right) + O(g_4^2)$$

That equality is represented diagrammatically by

$$\tilde{Z} = 1 - N\frac{g_4}{4}\frac{1}{N^2g^2}\left(2\ \text{  }\ +\ \text{  }\right)$$

which could have been obtained directly from the Feynmann rules. As an example, we write the expansion at order 2 in g_4:

$$\begin{aligned}\tilde{Z} = \ & 1 - N\tfrac{g_4}{4}\tfrac{1}{N^2g^2}\left(2\ \text{  }\ +\ \text{  }\right) \\ & + \tfrac{1}{2!}N^2\left(\tfrac{g_4}{4}\right)^2\tfrac{1}{N^4g^4}\left(\begin{array}{lll} 4\ \text{  } & +32\ \text{  } & +4\ \text{  } \\ +16\ \text{  } & +32\ \text{  } & +4\ \text{  } \\ +4\ \text{  } & +8\ \text{  } & +\ \text{  } \end{array}\right)\end{aligned} \tag{3.1.14}$$

Or numerically:

$$\begin{aligned}\tilde{Z} = \ & 1 - N\tfrac{g_4}{4}\tfrac{1}{(Ng)^2}\left(2N^3 + N\right) \\ & + \tfrac{N^2}{2}\left(\tfrac{g_4}{4}\right)^2\tfrac{1}{N^4g^4}\left(4N^4 + 16N^2 + 4N^2 + 32N^4 + 32N^2 + 8N^2 + 4N^6 + 4N^4 + N^2\right) \\ & + O(g_4^3)\end{aligned} \tag{3.1.15}$$

3.1.3 Duality: Diagrams $\longleftrightarrow$ Surfaces

The diagrams involved in the perturbation expansion of matrix integrals can be viewed as polygonal surfaces. Indeed, propagators and vertices are made of double lines, which can be viewed as small ribbons. They have a 2D structure, the ribbons can be drawn on surfaces, they possess two faces. Each k-legs vertex can be drawn as a polygonal

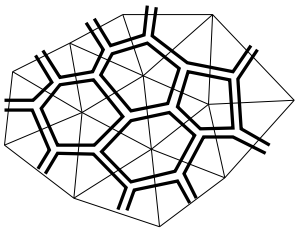
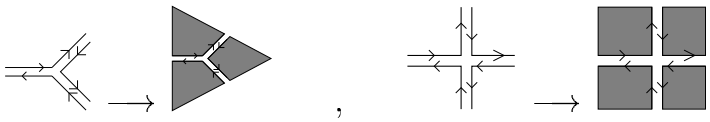


Figure 3.2: A Feynmann graph with 3-legs vertices, and its dual: a surface made of triangles.

with k sides (a k-gone). And a diagram is a collection of polygons glued side by side by propagators.



This is the duality transformation of the diagram:

diagram	$\leftrightarrow$ polygonal surface
k-legs vertex	$\leftrightarrow$ k-gone
propagator	$\leftrightarrow$ edge
loop	$\leftrightarrow$ vertex
n_v (number of vertices)	$\leftrightarrow$ number of faces
n_p (number of propagators)	$\leftrightarrow$ number of edges
l (number of loops)	$\leftrightarrow$ number of vertices
Ω symmetry factor	$\leftrightarrow$ # automorphism group of the surface

In this duality, every vertex becomes the center of a polygon, every propagator is orthogonal to an edge, every loop surrounds one vertex of the polygonal surface: One nearly has:

$$\tilde{Z} = \sum_{\text{surfaces}} \frac{1}{\Omega} N^{n_k - n_p + l} g^{-n_p} \prod_k (-g_k)^{n_k} \tag{3.1.16}$$

Unfortunately it is not completely true, because the duality makes sense only for connected diagrams, and in general, the expansion of $\tilde{Z}$ involves non connected diagrams (cf Eq. (3.1.14) with 3 non connected diagrams at order g_4^2).

Connected diagrammer

It is possible to get rid of non-connected diagrams, by introducing a new partition function $Z = \ln \tilde{Z}$.
Z has the same perturbative expansion as $\tilde{Z}$, except that it contains only connected diagrams.

$$Z = \ln \tilde{Z} = \sum_{\text{connected diagrams}} \frac{1}{\Omega} N^{n_k - n_p + l} g^{-n_p} \prod_k (-g_k)^{n_k} \tag{3.1.17}$$

Duality now allows to write Eq. (3.1.16) for Z:

$$Z = \ln \tilde{Z} = \sum_{\text{surfaces}} \frac{1}{\Omega} N^{n_k - n_p + l} g^{-n_p} \prod_k (-g_k)^{n_k} \tag{3.1.18}$$

geometrical interpretation

Z is the partition function of a set of random polygonal surfaces with a Boltzmann weight Eq. (3.1.18) .

- g_k is a fugacity for $k-$gones,
- g is the binding energy between neighboring polygons.
- N couples to the number $\chi = n_v - n_p + l$, which is the Euler-Poincare characteristic, which characterizes the topology of the surface.

Remark: Note that the fugacity of polygons is not g_k but $(-g_k)$. For the Boltzmann weight is positive, we need the g_k's to be negative. But then, the potential is not bounded from below, and the matrix integral is ill-defined. One can show that, in the large N limit (or at least order by order in $1/N$ expansion), the integral Z which is well defined for positive g_k's can be extended analytically to negative g_k's, and it makes sense to consider Z as the partition function of polygonal surfaces, order by order in $1/N$.

Continuous surfaces limit

By tuning the g_k's, we can reach a limit where very large surfaces (made of a very large number of tiny polygons) will dominate the sum. In this limit, Z can be interpreted as a partition function for continuous 2D surfaces. As we have seen above, that limit requires to continue analytically the matrix integral to negative g_k's. We will discuss it with more attention below.

Before, let us comment the χ factor, and the topology of polygonal surfaces.

3.1.4 $1/N^2$ topological expansion

So far, we have considered the diagrammatic expansion as a power series expansion in powers of g_k's. It can be viewed as well as a power series expansion in $1/N$.

Indeed, each term of the expansion of Eq. (3.1.18) is weighted by a power of N, with exponent N^χ, where

$$\chi = n_v - n_p + l \tag{3.1.19}$$

n_v = total number of faces, n_P = number of edges, l = number of vertices of the surface.

χ is called Euler Poincaré characteristic of the surface. It is a topological invariant. For a closed surface of genus h (number of holes), one has:

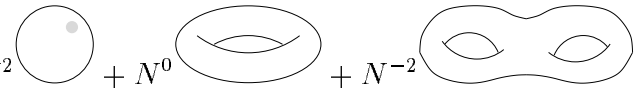
$$\chi = 2 - 2h \tag{3.1.20}$$

(a sphere has no hole $h = 0$, a torus has one hole $h = 1$, ...etc).

the value of χ tells on which surface the diagram can be drawn so that its lines don't cross. Only the diagrams with $\chi = 2$ can be drawn on the plane without crossing.

The perturbative expansion of Z, viewed as a power series in $1/N$ is called topological expansion:

$$Z = \sum_{h=0}^{\infty} N^{2-2h} Z_h = N^2 + N^0 + N^{-2} + \dots \tag{3.1.21}$$



Z_h is the partition function of polygonal surfaces including only surfaces of genus h.

Spherical limit: $N \to \infty$

In the large N limit, only $h = 0$ surfaces dominate. This is the spherical limit, because we keep only surfaces which have the topology of the sphere $\chi = 2$.

$$\lim_{N\to\infty} \frac{1}{N^2} Z = Z_0 \tag{3.1.22}$$

The next orders in $1/N$ for N large allow to compute the partition function Z_h with surfaces of fixed genus h.

3.2 Multimatrix integrals

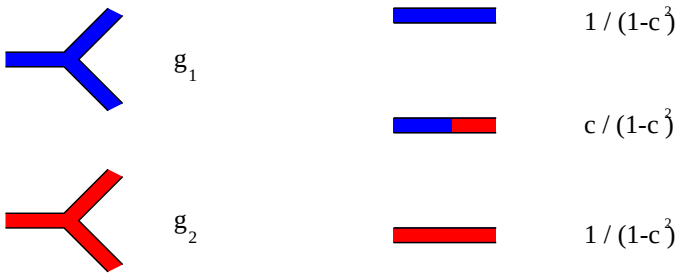
3.2.1 The Ising model

Consider the following 2-matrices integral:

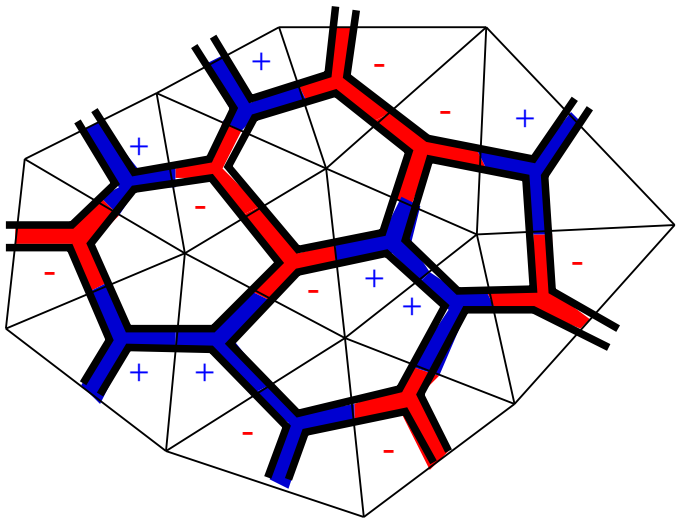
$$Z = \int \mathrm{d}M_1 \mathrm{d}M_2 \; \mathrm{e}^{-N \,\mathrm{Tr}\left[\frac{g_1}{3}M_1^3 + \frac{g_2}{3}M_2^3 + \frac{1}{2}M_1^2 + \frac{1}{2}M_2^2 - cM_1M_2\right]} \tag{3.2.23}$$

where M_1 and M_2 are two $N \times N$ hermitian matrices.

The two cubic terms give two 3-legs vertices, and the quadratic form $\begin{pmatrix} 1 & -c \\ -c & 1 \end{pmatrix}^{-1}$ gives rise to 3 propagators:



The corresponding Feynmann diagrams are triangulated surfaces, with two types of triangles: blue or red. In other words, we can say that each triangle carries an index (or spin) $\sigma = +1$ or -1.



The weight associated to a given diagram is:

$$\frac{1}{\Omega} g_1^{n_+} g_2^{n_-} c^{n_{+-}} (1-c^2)^{-n_p}$$

where n_σ is the number of triangles which have spin σ, n_p is the total number of propagators ($2n_p = 3(n_1+n_2)$), and n_{+-} is the number of propagators linking triangles of different spins:

$$2n_{+-} = n_p - \sum_{<ij>} \sigma_i \sigma_j$$

If we note $g = \sqrt{g_1 g_2}(\sqrt{c}/(1-c^2))^{3/2}$ and $\mathrm{e}^{-2h} = g_1/g_2$, and $c = \mathrm{e}^{2J}$, we see that the weight of each Feynmann diagram is:

$$\frac{1}{\Omega} g^n \mathrm{e}^{-J\sum_{<ij>}\sigma_i\sigma_j - h\sum_i \sigma_i}$$

i.e. we have a Ising model on a random triangular lattice. g is the "cosmological constant" coupled to n the total number of triangles of the lattice, J is the Ising coupling (the temperature), and h is an external magnetic field.

The study of the matrix integral, allows to find the phase diagram of this Ising model coupled to gravity. The ferromagnetic transition corresponds to a $(4,3)$ conformal minimal model.

3.2.2 The $O(n)$ model

Another interesting model is the $O(n)$ model:

$$Z = \int \mathrm{d}M \prod_{\alpha=1}^{n} \mathrm{d}A_\alpha \; \mathrm{e}^{-N \,\mathrm{Tr}\left[V(M) + M(A_1^2 + \ldots + A_n^2)\right]}$$

where M as well as $A_1, \ldots, A_n$ are $N \times N$ hermitian matrices.

One can easily integrate out the A_α's and define the model for any real value of n.

It turns out that only $n \in [-2, 2]$ lead to an interesting critical behavior. In particular for $n = -2\cos(p/q\,\pi)$, the critical points are those of a (p, q) conformal minimal model. The $n \to +2$ limit describes a Kosterlitz-Thouless transition.

The Feynmann diagrams of the $O(n)$ model generate the configurations of a gaz of self avoiding loops on a random surface.

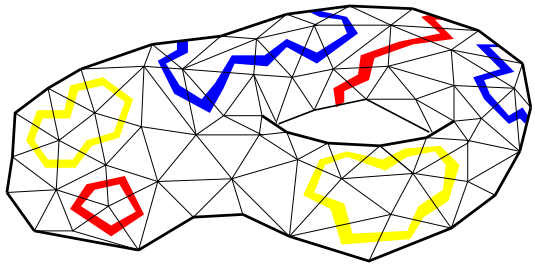


Figure 3.3: $O(n)$ model: a configuration of self avoiding loops on a random triangulated surface.

3.2.3 The Q-states Potts model

The Potts model's configurations on a random lattice are generated by the following matrix integral.

$$Z = \int \prod_{\alpha=1}^{Q} \mathrm{d}M_\alpha \; \mathrm{e}^{-N \,\mathrm{Tr}\left[\sum_{\alpha=1}^{Q} V(M_\alpha) + \sum_{\alpha,\beta=1}^{Q} M_\alpha M_\beta\right]}$$

This model was solved for cubic potentials.

many other matrix integrals corresponding to a statistical physics model on a random surface were introduced, for instance the 6-vertex model, the 3-color model, the meanders model (yet unsolved), ... Many of them are yet unsolved.

3.3 Non hermitian matrices

So far, we have dealt with hermitian matrices only, and the propagators and vertices were made of oriented double lines.

3.3.1 Real symmetric matrices

Consider the following matrix integral

$$\tilde{Z} = \int \mathrm{d}M \mathrm{e}^{-N \,\mathrm{tr}\, V(M)} \tag{3.3.24}$$

$$\text{with} \qquad V(M) = V_0 + \frac{g}{2}M^2 + \frac{g_3}{3}M^3 + \cdots + \frac{g_k}{k}M^k = V_0 + \frac{g}{2}M^2 + \delta V \tag{3.3.25}$$

where M is a $N \times N$ real symmetric matrix:

$$M_{ij} = M_{ji}$$

The independent variables are the M_{ij} with $i \leq j$.

The propagator (quadratic part of the potential $\mathrm{Tr}\, M^2$), is now:

$$\langle M_{ij} M_{kl} \rangle_0 = \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} \qquad i \leq j, k \leq l$$

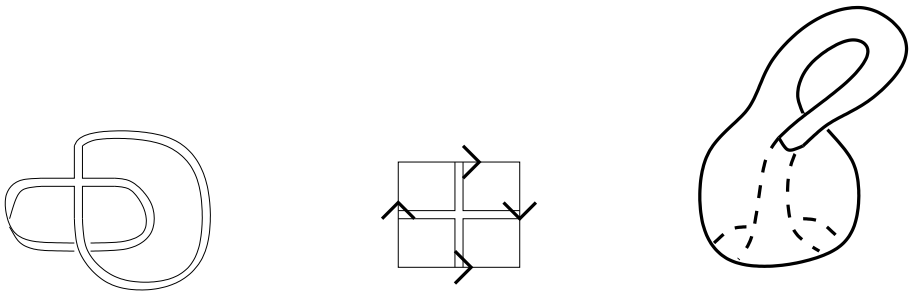
The orientation of single lines does not matter anymore. Everything else is unchanged compared to the hermitian case. The Feynmann diagrams are obtained by gluing together vertices with propagators. The dual diagrams are discrete surfaces made of polygonal pieces. The power of N associated to each diagram is its Euler-Poincarre characteristic.

what changes is that:

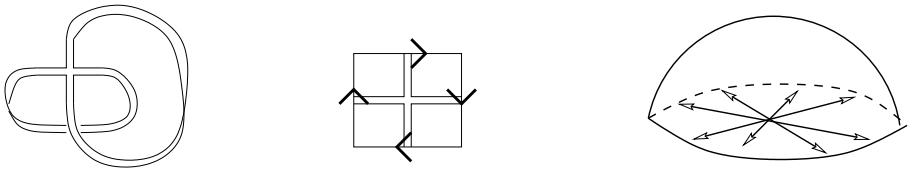
- the surfaces may be non-oriantable surfaces like Klein's bottle or the projective plane, or others.
- some non orientable surfaces may have an odd Euler characteristic. Therefore, $\ln \tilde{Z}$ has a power series expansion in powers of $1/N$ instead of powers of $1/N^2$.

Examples:

- Klein's bottle ($\chi = 0$):



- Projective plane ($\chi = 1$):



3.3.2 Real self-dual Quaternionic matrices

One gets the same diagrams as in the real-symmetric matrix case, except that one has to change $N \to -N/2$.

3.3.3 Complex matrices

Non hermitian complex matrices give rise to oriented surfaces, and the double lines are oriented as well.

Chapter 4

The Saddle Point Method

We will try to compute a matrix integral of the form

$$\int \mathrm{d}M \mathrm{e}^{-N \operatorname{tr} V(M)}$$

in the large N limit (N is the size of the matrix as well as the coefficient in front of the potential), by a saddle point method.

We will see that the saddle point method for matrix integrals is a little more difficult than the usual saddle point method for simple integrals.

We will introduce a very important notion: the Resolvent function (or Green function):

$$W(z) = \frac{1}{N} \left\langle \operatorname{tr} \frac{1}{z - M} \right\rangle$$

in term of which the saddle point equation is either a
- quadratic equation
- or a linear equation.

We will illustrate the method on two examples: the quadratic potential (which gives the famous semi-circle law for the density of states), and the quartic potential.

Then, the saddle point method will allow us to find some universal properties (the 2-point correlation function), and the critical points.

Eventually, we will show how to extend the method to more complicated matrix integrals.

4.1 The saddle point method for the 1-hermitian matrix model

We will try to compute the following matrix integral:

$$Z = \mathrm{e}^{-N^2 F} = \int \mathrm{d}M \ \mathrm{e}^{-N \,\mathrm{tr}\, V(M)} \tag{4.1.1}$$

for a hermitian $N \times N$ matrix M, with a polynomial potential V, in the large N limit.

4.1.1 Why the naive saddle point method does not work ?

Consider the naive "col" $M = M_0$ such that $\mathrm{tr}\, V'(M_0) = 0$:

$$Z \sim \mathrm{e}^{-N \,\mathrm{tr}\, V(M_0)} \ \frac{1}{\sqrt{\det(S''^*(M_0))}} \ \times \ (\text{zero modes}) \tag{4.1.2}$$

In the usual saddle-point method, $\mathrm{e}^{-N \,\mathrm{tr}\, V(M_0)}$ is the value of the action at the col, it is the leading large N contribution, while the other terms coming from $S''^*(M_0)$ (the functional second derivative of the action at M_0, with the zero modes excluded), and the zero-modes contributions, are supposed to be subdominant.

Obviously, here, there are a lot of zero modes, because any $M = UM_0U^\dagger$ ($U =$ unitary) is a col. The number of zero-modes is thus $N^2 - N$ which is of order N^2, and a product of N^2 terms is of the same order than the leading term e^{N^2} ! The subdominant terms are as large as the leading term, which means that the saddle-point approximation is not valid.

We will see below how to use the saddle-point method properly: with the eigenvalues.

4.1.2 Coulomb gaz analogy

Let's go to the eigenvalues representation of 4.1.1 (we use the change of variable from section 2.1.3):

$$\begin{aligned} Z &= \int \mathrm{d}M \ \mathrm{e}^{-N \,\mathrm{tr}\, V(M)} = \mathrm{e}^{-N^2 F} && (4.1.3)\\ &= \mathrm{cte} \int \mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_N \ \prod_{i<j} (\lambda_i - \lambda_j)^2 \ \prod_i \mathrm{e}^{-N V(\lambda_i)} && (4.1.4)\\ &= \mathrm{cte} \int \mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_N \ \mathrm{e}^{-N^2 S[\lambda_1, \dots, \lambda_N]} && (4.1.5) \end{aligned}$$

$$\text{action } S = \frac{1}{N} \sum_i V(\lambda_i) \ - \frac{2}{N^2} \sum_{i<j} \ln |\lambda_i - \lambda_j| \tag{4.1.6}$$

We recognize a gaz of eigenvalues in dimension 1, with a 2-body coulomb repulsion, and a bulk potential V.
The saddle point is the equilibrium position for the gaz $\lambda_i = \tilde{\lambda}_i$:

$$\frac{\partial S}{\partial \lambda_i} = 0 = V'(\tilde{\lambda}_i) - \frac{2}{N} \sum_{j \neq i} \frac{1}{\tilde{\lambda}_i - \tilde{\lambda}_j} \tag{4.1.7}$$

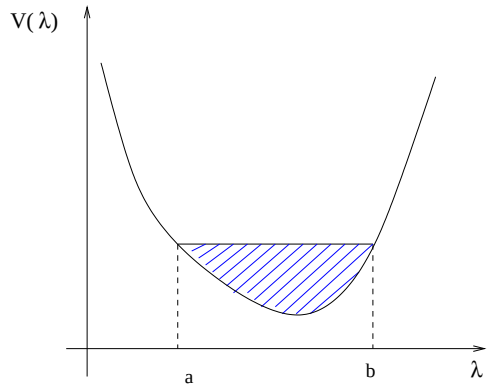


Figure 4.1: The gaz of eigenvalues in a bulk potential V.

4.1.3 Solution of the saddle point equation

In order to solve equation 4.1.7, it is convenient to introduce the following function:

$$\tilde{W}(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z - \tilde{\lambda}_i} \tag{4.1.8}$$

Let us compute $\tilde{W}^2$:

$$\begin{aligned}
\tilde{W}^2(z) &= \frac{1}{N^2} \sum_{i,j} \frac{1}{(z - \tilde{\lambda}_i)(z - \tilde{\lambda}_j)} \\
&= \frac{1}{N^2} \sum_{i} \frac{1}{(z - \tilde{\lambda}_i)^2} + \frac{1}{N^2} \sum_{i \neq j} \frac{1}{(z - \tilde{\lambda}_i)(z - \tilde{\lambda}_j)} \\
&= -\frac{1}{N} W'(z) + \frac{1}{N^2} \sum_{i \neq j} \frac{1}{\tilde{\lambda}_i - \tilde{\lambda}_j} \left(\frac{1}{z - \tilde{\lambda}_i} - \frac{1}{z - \tilde{\lambda}_j} \right) \\
&= -\frac{1}{N} W'(z) + \frac{2}{N^2} \sum_{i \neq j} \frac{1}{\tilde{\lambda}_i - \tilde{\lambda}_j} \frac{1}{z - \tilde{\lambda}_i}
\end{aligned}$$

neglecting the $O(1/N)$ term, and using 4.1.7, we get:

$$\begin{aligned}\tilde{W}^2(z) &= \frac{1}{N}\sum_i \frac{V'(\tilde{\lambda}_i)}{z-\tilde{\lambda}_i} \\ &= \frac{1}{N}\sum_i \frac{V'(z)-(V'(z)-V'(\tilde{\lambda}_i))}{z-\tilde{\lambda}_i} \\ &= V'(z)\tilde{W}(z) - P(z)\end{aligned}$$

where

$$P(z) = \frac{1}{N}\sum_i \frac{V'(z)-V'(\tilde{\lambda}_i)}{z-\tilde{\lambda}_i} \tag{4.1.9}$$

is a polynomial in z of degree less than the degree of $V'(z)$. Therefore, we have a second degree equation for the resolvent:

$$\boxed{\tilde{W}^2(z) = V'(z)\tilde{W}(z) - P(z)} \tag{4.1.10}$$

whose solution is:

$$\tilde{W}(z) = \frac{1}{2}\left(V'(z) - \sqrt{V'^2(z) - 4P(z)}\right) \tag{4.1.11}$$

So far, the polynomial $P(z)$ is not known, but 4.1.11 is much simpler than the initial equation 4.1.7. Indeed, we now have to determine only a small number of unknowns, the coefficients of $P(z)$. Note that the leading coefficient of $P(z)$ is known, it is the same as V'.

Now that we have determined the function $\tilde{W}(z)$, let us go back to the determination of the $\tilde{\lambda}_i$, and their statistical distribution.

4.1.4 The resolvent function

In spectral theory and in particular in random matrix models, it is usual to consider the "resolvent function", also called "Green function" $W(z)$:

$$W(z) = \frac{1}{N}\left\langle \mathrm{Tr}\,\frac{1}{z-M}\right\rangle = \frac{1}{N}\sum_{k=0}^{\infty}\frac{\left\langle \mathrm{Tr}\,M^k\right\rangle}{z^{k+1}} = \frac{1}{N}\sum_i \left\langle \frac{1}{z-\lambda_i}\right\rangle \tag{4.1.12}$$

The function $W(z)$ encodes all the statistical properties of the spectrum. For instance all the moments $<\mathrm{Tr}\,M^k>$ are the coefficients of the large z expansion of $W(z)$:

$$\frac{1}{N}<\mathrm{Tr}\,M^k> = -\frac{1}{2i\pi}\oint W(z)z^k \mathrm{d}z \tag{4.1.13}$$

where the contour is a large circle around $z\to\infty$.

Note that $W(z)$ is defined for z complex and is initially defined for large z only. We shall see below, that $W(z)$ can be analytically extended to the whole complex plane

except along a cut located on the real axis. Indeed, $W(z)$ is singular when z is close to the (average) spectrum of M. In other words, $W(z)$ is singular when $z \to < \lambda_i >$. In average, the eigenvalues are real, and fall at the bottom of the potential, they tend to concentrate along an interval $[a, b]$ on the real axis[1].

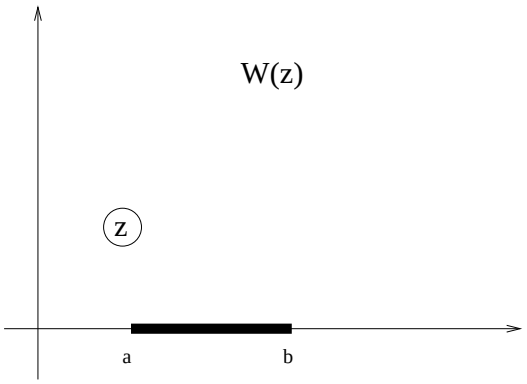


Figure 4.2: The resolvent has a cut on the real axis.

If one introduces the average density of eigenvalues $\rho(\lambda)$:

$$\rho(\lambda) = \frac{1}{N} \left\langle \sum_i \delta(\lambda - \lambda_i) \right\rangle = \frac{1}{N} \langle \text{ Tr } \delta(\lambda - M) \rangle \tag{4.1.14}$$

then:

$$W(z) = \int_a^b \rho(\lambda) \mathrm{d}\lambda \ \frac{1}{z - \lambda} \tag{4.1.15}$$

and the moments $<$ Tr $M^k >$ are:

$$\frac{1}{N} < \text{ Tr } M^k > = \int_a^b \rho(\lambda) \, \lambda^k \, \mathrm{d}\lambda = -\frac{1}{2i\pi} \oint W(z) z^k \mathrm{d}z \tag{4.1.16}$$

where the integration contour now encloses the interval $[a, b]$. When the contour gets very close to the interval $[a, b]$, one sees that:

$$\rho(\lambda) = -\frac{1}{2i\pi}(W(\lambda + i0) - W(\lambda - i0)) \tag{4.1.17}$$

Knowing $W(z)$ is equivalent to knowing $\rho(\lambda)$, the relationship between both is either 4.1.15 or 4.1.17.

In the large N limit, in the saddle point approximation, we have (see eq. 4.1.11:

$$W(z) \sim \tilde{W}(z) + O(1/N) \tag{4.1.18}$$

[1] Actually, the support of the average density is not necessarily connected, it can be a collection of disconnected intervals $[a, b] \cup [c, d] \cup \ldots$. That will be discussed in section 4.4.

therefore

$$W(z) \sim \frac{1}{2}\left(V'(z) - \sqrt{V'^2(z) - 4P(z)}\right) \tag{4.1.19}$$

which can have many cuts, the endpoints of the cuts are where the polynomial in the square-root vanishes. The number of cuts is at most equal to the degree of V'.

4.1.5 One-cut assumption.

When the potential $V(z)$ has only one well (or at least one deepest well), the eigenvalues all fall down to the bottom of this well, and the support of $\rho(\lambda)$ should have only one connected component $[a,b]$. In this case, $W(z)$ should have only one cut in the complex plane, along the interval $[a,b]$. This one-cut condition is sufficient to determine the unknown polynomial $P(z)$ and the endpoints a and b. Indeed, the polynomial under the square root in 4.1.19 should have only 2 single roots, and all the others should be double roots.

We must have:

$$W(z) \sim \frac{1}{2}\left(V'(z) - M(z)\sqrt{(z-a)(z-b)}\right) \tag{4.1.20}$$

where $M(z)$ is an unknown polynomial such that:

$$4P(z) = V'^2(z) - M^2(z)(z-a)(z-b) \tag{4.1.21}$$

One can check that there are as many unknowns as equations, and all the unknown coefficients of P, of M as well as a and b can be determined.

A simpler method is the following: Note that at large z we have

$$W(z) \sim \frac{1}{z} \tag{4.1.22}$$

Divide 4.1.20 by $\sqrt{(z-a)(z-b)}$, and take the large z limit:

$$M(z) = \mathrm{Pol}\ \frac{V'(z)}{\sqrt{(z-a)(z-b)}} = \frac{V'(z)}{\sqrt{(z-a)(z-b)}} - \frac{1}{2i\pi}\oint \frac{1}{z-x}\frac{V'(x)}{\sqrt{(x-a)(x-b)}}\mathrm{d}x \tag{4.1.23}$$

which you insert back into 4.1.20:

$$\boxed{W(z) = \frac{1}{4i\pi}\oint \frac{V'(x)}{z-x}\frac{\sqrt{(z-a)(z-b)}}{\sqrt{(x-a)(x-b)}}\mathrm{d}x} \tag{4.1.24}$$

where the contour encloses the cut $[a,b]$.

Now, the large z condition 4.1.22 implies the two equations:

$$\frac{1}{2i\pi}\oint V'(x)\frac{\sqrt{(z-a)(z-b)}}{\sqrt{(x-a)(x-b)}}\mathrm{d}x = 0 \tag{4.1.25}$$

$$\frac{1}{2i\pi}\oint xV'(x)\frac{\sqrt{(z-a)(z-b)}}{\sqrt{(x-a)(x-b)}}\mathrm{d}x = 2 \tag{4.1.26}$$

which determine a and b as functions of the potential V.

With equations 4.1.24 (or 4.1.20) and these two conditions, we have a full solution of the saddle point equation. The average density of eigenvalues is:

$$\rho(\lambda) = \frac{1}{2\pi} M(\lambda) \sqrt{(\lambda - a)(b - \lambda)} \tag{4.1.27}$$

Remark 1: note that this solution makes sense only if $M(\lambda)$ is positive for $\lambda \in [a, b]$, otherwise it means that the one-cut asumption was not valid.
Remark 2: the density $\rho(\lambda)$ has generically square-root singularities at its end-points, except when $M(\lambda)$ has a zero (multiple-zero) at a or b, in which case we have a different singular behavior: $\rho(\lambda) \sim (\lambda - a)^{m+1/2}$. It is exactly the condition to have a critical point, and $m + 1/2$ is one of the universal critical exponents.

4.2 Examples

4.2.1 Example: the Gaussian potential, Wigner's semi-circle law

As an example consider a quadratic potential:

$$V(z) = \frac{g}{2} z^2 \tag{4.2.28}$$

i.e. $V'(z) = gz$.

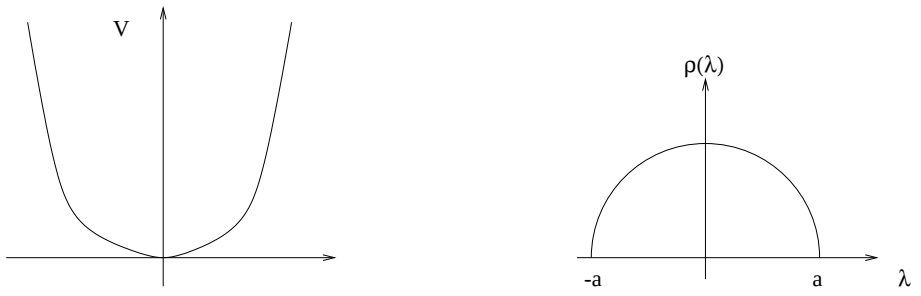


Figure 4.3: The quadratic potential leads to the semi-circle law for the density.

The polynomial $P(z)$ of 4.1.9 has degree 0, it is a constant $P(z) = g$. Therefore:

$$W(z) = \frac{1}{2}\left(gz - \sqrt{g^2 z^2 - 4g}\right) = \frac{1}{2}\left(gz - g\sqrt{(z - a)(z + a)}\right) \tag{4.2.29}$$

with $a^2 = 4/g$. The density of eigenvalues is

$$\rho(\lambda) = \frac{g}{2\pi} \sqrt{a^2 - \lambda^2}$$

If you write $\lambda = a \cos\phi$, then $\rho(\lambda) = \frac{2}{a} \sin\phi$. This is the famous semi-circle law.

4.2.2 Example: The quartic potential

Consider the potential

$$V(z) = \frac{g}{4}z^4 - \frac{m}{2}z^2$$

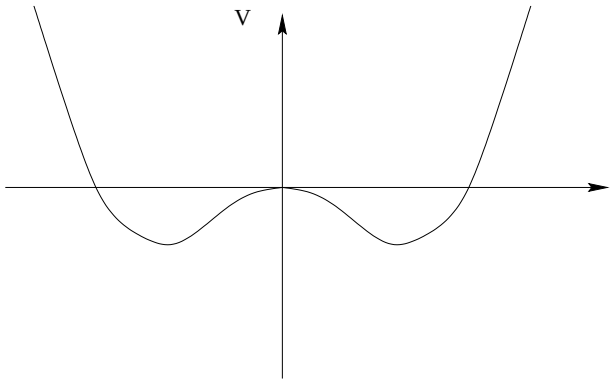


Figure 4.4: Quartic potential.

Because the potential is even, we expect the cut to be symmetrical $a = -b$, and the polynomials P and M even of degree 2:

$$M(z) = gz^2 + c$$

Equation 4.1.23 implies:

$$M(z) = \text{Pol}\ (gz^3 - mz)(z^2 - a^2)^{-1/2} = \text{Pol}\ (gz^2 - m)(1 + \frac{a^2}{2z^2} + \frac{3a^4}{8z^4} + \ldots) = gz^2 + \frac{ga^2}{2} - m$$

and

$$W(z) = \frac{1}{2}\left(gz^3 - mz - (gz^2 + \frac{ga^2}{2} - m)\sqrt{z^2 - a^2}\right)$$

The condition $W(z) \sim 1/z$ implies:

$$g\frac{a^4}{8} + (g\frac{a^2}{2} - m)\frac{a^2}{2} = 2$$

i.e.

$$a^2 = \frac{2}{3g}(m + \sqrt{m^2 + 12g})$$

The density of eigenvalues is:

$$\rho(\lambda) = \frac{1}{2i\pi}(g\lambda^2 + \frac{ga^2}{2} - m)\sqrt{a^2 - \lambda^2}$$

Remark 1: That solution makes sense only when $\rho(\lambda)$ is positive on $[-a, a]$, i.e.

$$m^2 < 4g$$

If $m^2 > 4g$, then the double wells are so deep that the eigenvalue distribution breaks into two connected components, one in each well. The one-cut asumption is not valid anymore. One should then make a 2-cuts asumption. There is 1^{st} order a phase transition.

Remark 2: One can see that a critical point occurs at $g_c = -\frac{m^2}{12}$. Indeed, we will see that the large N limit of the matrix integral can be extended analytically to $g < 0$. When $g = g_c$, we have $M(\lambda) = g(\lambda^2 - a^2)$, the density has no more a square root edge, at g_c you have $\rho(\lambda) \sim (\lambda - a)^{3/2}$. One can show that it correspond to a second order phase transition, associated to a $(3, 2)$ conformal minimal model, with central charge $c = 0$, which is called "pure gravity" critical point.

4.3 A linear equation for the resolvent

So far, we have found the resolvent from the saddle-point equation, thanks to a second degree equation 4.1.10.

We will see now that it is possible to rewrite the saddle point equation 4.1.7 as a linear equation. Indeed, when z is one of the $\tilde{\lambda}_i$, eq.4.1.7 reads:

$$\boxed{W(z + i0) + W(z - i0) = V'(z) \qquad \text{for} \qquad z \in [a, b]} \tag{4.3.30}$$

This is a linear equation for $W(z)$.

4.3.1 Solution of the linear equation

The solution of a linear equation is the sum of one particular solution (here $W(z) = 1/2\ V'(z)$ is a solution) and the general solution of the homogeneous equation:

$$W(z) = \frac{1}{2}V'(z) + f(z) \qquad , \qquad f(z + i0) + f(z - i0) = 0$$

If you write $f(z) = h(z)\sqrt{(z - a)(z - b)}$, the equation for f becomes:

$$h(z + i0) = h(z - i0)$$

i.e. $h(z)$ has no discontinuity along the cut $[a, b]$, it is an analytical function. The large z behavior implies that $h(z)$ is a polynomial, and we prefer to write it $h(z) = -1/2\ M(z)$:

$$W(z) = \frac{1}{2}\left(V'(z) - M(z)\sqrt{(z - a)(z - b)}\right)$$

which is identical to the solution 4.1.20 found from the quadratic equation.

4.4 The multicut case

In case the potential has several wells, the support of the density may be disconnected. Just to illustrate, let's say that it has two connected components $[a,b] \cup [c,d]$, and $a < b < c < d$. Starting either from the linear equation 4.3.30 or from the quadratic equation 4.1.10, we find that $W(z)$ must have the form:

$$W(z) = \frac{1}{2}\left(V'(z) - M(z)\sqrt{(z-a)(z-b)(z-c)(z-d)}\right) = \frac{1}{2}\left(V'(z) - M(z)\sqrt{\sigma(z)}\right) \tag{4.4.31}$$

For short we note $\sigma(z) = (z-a)(z-b)(z-c)(z-d)$. Again, we have:

$$M(z) = \text{Pol}\ \frac{V'(z)}{\sqrt{\sigma(z)}} = \frac{V'(z)}{\sqrt{\sigma(z)}} - \frac{m_1}{z} - \frac{m_2}{z^2} - \frac{m_3}{z^3} - O(\frac{1}{z^4}) \tag{4.4.32}$$

M is of degree $degV'-2$, and the coefficients m_i are functions of a,b,c,d. The condition that $W(z) \sim 1/z$ determines m_1, m_2 and m_3, it gives only three equations, while we have four unknowns a,b,c,d ! In general, when there are s connected components, we have $2s$ unknowns, and the large z condition $W(z) \sim 1/z$ provides only $s+1$ equations.

Therefore, in the multicut case, the saddle point equation 4.1.7 alone is not sufficient to determine $W(z)$ completely.

The filling fraction x

Consider the filling fraction x of eigenvalues in the first cut $[a,b]$:

$$x = \int_a^b \rho(\lambda) = \frac{1}{2i\pi}\int_a^b M(\lambda)\sqrt{\sigma(\lambda)}\mathrm{d}\lambda \qquad , \qquad 1-x = \int_c^d \rho(\lambda)\mathrm{d}\lambda \tag{4.4.33}$$

At given x, all the endpoints a,b,c,d can be determined by the large z condition as functions of x. $W(z)$ can be determined as a function of x, and therefore the free energy can be determined as a function $F(x)$ of x. How to determine x ?

x must be chosen such that the free energy $F(x)$ is minimal :

$$x = x_0 \qquad , \qquad \frac{\partial F}{\partial x} = 0$$

In other words, a small variation of x should not change the free energy at first order. A variation of x means varying the number of eigenvalues in each cut, i.e. send one eigenvalue from one cut to another one. For instance let us compute the amount of energy needed to move one eigenvalue from b to c. The force experienced by one eigenvalue λ_i is (from 4.1.6):

$$f(\lambda_i) = -V'(\lambda_i) + \frac{2}{N}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j} \tag{4.4.34}$$

in the large N limit, it becomes:

$$f(\lambda) = -V'(\lambda) + 2W(\lambda) \qquad \text{when} \quad \lambda \notin [a,b]\cup[c,d]$$

The amount of energy needed to move it from b to c is thus:

$$\Gamma = \int_b^c 2W(z) - V'(z)\mathrm{d}z = -\int_b^c M(z)\sqrt{\sigma(z)}$$

Therefore, at equilibrium we have the extra equation

$$\int_b^c M(z)\sqrt{\sigma(z)} = 0 \tag{4.4.35}$$

This allows to determine all the unknowns a, b, c, d. Note that the filling fraction x_0 is given by:

$$\int_a^b M(z)\sqrt{\sigma(z)} = 2i\pi x_0 \tag{4.4.36}$$

In general, when there are s cuts, we have $s-1$ such conditions (the integral of $M\sqrt{\sigma}$ between two successive cuts vanishes), which means that the number of equations matches exactly the number of unknowns.

The effective potential

The primitive of the effective force 4.4.34

$$V_{\text{eff}}(z) = V(z) - 2\int_a^z \mathrm{Re}W(x)\mathrm{d}x = 2\mathrm{Re}\int_a^z M(x)\sqrt{\sigma(x)} \tag{4.4.37}$$

is the effective potential experienced by one eigenvalue in the field of the others. The saddlepoint equation reads:

$$V'_{\text{eff}}(z) = 0 \qquad \text{for} \qquad z \in \text{supp}\,\rho \tag{4.4.38}$$

which means that the effective potential is constant in each cut, and equation 4.4.35 means that the constant is the same for each cut.

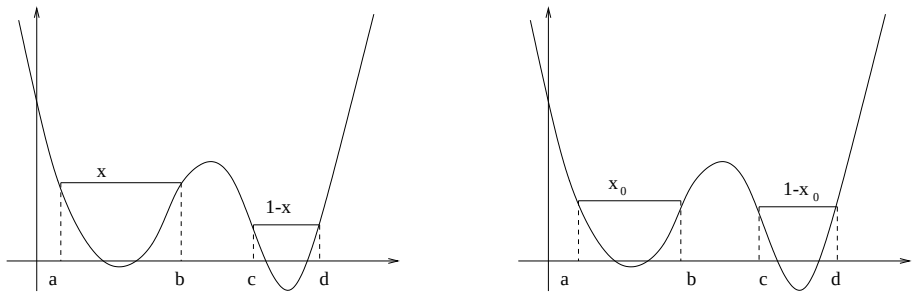


Figure 4.5: We have a solution for each value of x. The equilibrium condition gives x_0 such that the effective potential is the same in each cut.

An intuitive picture is the following: imagine a liquid of eigenvalues. The saddlepoint equations tells you that the level of the liquid is horizontal in each well, and 4.4.35 tells you that by tunnel effect, the level should be the same in all wells.

4.5 Universality

Let us come back to the one cut case.

Let us introduce a temperature T, i.e. just replace the potential V by $\frac{1}{T}V$, and we try to compute the free energy $F(T)$, and every observable as a function of the temperature.

In particular, the density of eigenvalues is a function of T, as well as the resolvent $W(z)$.

Let us take the derivative of the linear saddle point equation 4.3.30:

$$W(z+i0)+W(z-i0)=\frac{1}{T}V'(z) \tag{4.5.39}$$

with respect to T:

$$\Omega(z)=\frac{\partial}{\partial T}TW(z) \qquad , \qquad \Omega(z+i0)+\Omega(z-i0)=0 \tag{4.5.40}$$

we see that the equation for $\Omega(z)$ does not depend on V. The large z condition $W(z)\sim 1/z$ implies $\Omega(z)\sim 1/z$ and determines $\Omega(z)$ completely:

$$\Omega(z)=\frac{\partial}{\partial T}TW(z)=\frac{1}{\sqrt{(z-a)(z-b)}} \tag{4.5.41}$$

This is an example of a universal property. The derivative of the density of eigenvalues

$$\frac{\partial}{\partial T}T\rho(\lambda)=\frac{1}{2\pi\sqrt{(\lambda-a)(b-\lambda)}} \tag{4.5.42}$$

is independent on the potential, it is the same for every random matrix model (under the one-cut asumption).

remark: This trick of taking the derivative of the linear equation with respect to a parameter of the potential is very useful.

The specific heat $u(T)$

Another interesting universal result is the following: Let us introduce the specific heat

$$u(T)=\frac{\mathrm{d}^2}{\mathrm{d}T^2}T^2F(T) \tag{4.5.43}$$

It can be proven that:

$$\boxed{u(T)=2\ln(b-a)} \tag{4.5.44}$$

4.5.1 The two-points function

We will try to determine the connected 2-points correlation function

$$\rho_c(\lambda,\mu) =< \sum_{i,j} \delta(\lambda-\lambda_i)\delta(\mu-\lambda_j) >_c =< \text{ Tr } \delta(\lambda-M)\delta(\mu-M) >_c \tag{4.5.45}$$

Similarly to the average density $\rho(\lambda)$, the 2-point function can be related to a 2-point resolvent $W(z,z')$:

$$W(z,z') =< \text{ Tr } \frac{1}{z-M}\frac{1}{z'-M} >_c = \int\int \mathrm{d}\lambda \mathrm{d}\mu \ \rho_c(\lambda,\mu) \ \frac{1}{z-\lambda}\ \frac{1}{z'-\mu} \tag{4.5.46}$$

and conversely:

$$-4\pi^2\rho_c(\lambda,\mu) = W(\lambda+i0,\mu+i0)-W(\lambda-i0,\mu+i0)-W(\lambda+i0,\mu-i0)+W(\lambda-i0,\mu-i0) \tag{4.5.47}$$

The 2-point function can be obtained by differentiating the partition function:

$$Z = \int \mathrm{d}M \mathrm{e}^{-N \,\mathrm{tr}\, V(M)} = \mathrm{e}^{-N^2 F} \tag{4.5.48}$$

with respect to the potential $V(z) = \sum_k \frac{g_k}{k} z^k$ in the following way:

$$\frac{\mathrm{d}}{\mathrm{d}V(z)} := \sum_{k=0}^{\infty} \frac{k}{z^{k+1}} \frac{\mathrm{d}}{\mathrm{d}g_k} \tag{4.5.49}$$

Indeed:

$$\frac{\mathrm{d}}{\mathrm{d}V(z)} V(z') = \frac{1}{z-z'} \tag{4.5.50}$$

so that

$$W(z) = \frac{1}{Z}\frac{\mathrm{d}}{\mathrm{d}V(z)} Z = \frac{\mathrm{d}F}{\mathrm{d}V(z)} \tag{4.5.51}$$

$$W(z,z') = \frac{\mathrm{d}^2 F}{\mathrm{d}V(z)\mathrm{d}V(z')} = \frac{\mathrm{d}W(z)}{\mathrm{d}V(z')} \tag{4.5.52}$$

Let us apply the operator $\mathrm{d}/\mathrm{d}V(z)$ to the linear saddle point equation 4.3.30, we get:

$$W(z+i0,z') + W(z-i0,z') = -\frac{1}{(z-z')^2} \qquad , \qquad z \in [a,b] \tag{4.5.53}$$

A particular solution is simply $-\frac{1}{2}\frac{1}{(z-z')^2}$, and we have to add the solution of the homogeneous equation:

$$W(z,z') = \frac{-1}{2(z-z')^2}\left(1 - \frac{f(z,z')}{\sqrt{(z-a)(z-b)}\sqrt{(z'-a)(z'-b)}}\right) \tag{4.5.54}$$

$f(z,z')$ must have no cut along $[a,b]$, it must be symmetrical in z and z', the large z behavior implies that f is a polynomial, of degree at most 1 in each variable, and

the condition that $W(z,z')$ is regular when $z = z'$ determines $W(z,z')$ completely. Eventually, we have:

$$W(z,z') = \frac{-1}{4\sqrt{\sigma(z)\sigma(z')}} \left(1 - \left(\frac{\sqrt{\sigma(z)} - \sqrt{\sigma(z')}}{z - z'} \right)^2 \right) \tag{4.5.55}$$

which again is independent on the potential V. It depends on the potential only through a and b.

4.6 The saddle point method for Multimatrix integrals

Consider a two hermitian $N \times N$ matrices integral:

$$Z = \int \mathrm{d}M_1 \mathrm{d}M_2 \mathrm{e}^{-N \operatorname{Tr} V_1(M_1) + V_2(M_2) - M_1 M_2} \tag{4.6.56}$$

And make the change of variables to the eigenvalues a_i of M_1 and b_j of M_2:

$$M_1 = UAU^\dagger \qquad , \qquad M_2 = VBV^\dagger$$

$$A = \operatorname{diag}(a_1, \ldots, a_N) \qquad , \qquad B = \operatorname{diag}(b_1, \ldots, b_N)$$

then:

$$Z = \int \mathrm{d}U \mathrm{d}V \mathrm{d}a_i \mathrm{d}b_i \prod_{i<j} (a_i - a_j)^2 (b_i - b_j)^2 \prod_i \mathrm{e}^{-N(V_1(a_i) + V_2(b_i))} \mathrm{e}^{N \operatorname{Tr} UAU^\dagger VBV^\dagger}$$

U can easily be integrated out just by shifting V to $U^\dagger V$:

$$Z = \left(\int \mathrm{d}U \right) \int \mathrm{d}V \mathrm{d}a_i \mathrm{d}b_i \prod_{i<j} (a_i - a_j)^2 (b_i - b_j)^2 \prod_i \mathrm{e}^{-N(V_1(a_i) + V_2(b_i))} \mathrm{e}^{N \operatorname{Tr} AVBV^\dagger}$$

Integrating V out seems more difficult. It is actually possible and the result is quite simple, it is the famous integral of Harisch-Chandra - Itzykson - Zuber:

$$\int \mathrm{d}V \mathrm{e}^{N \operatorname{Tr} AVBV^\dagger} = \mathrm{cte}_N \frac{\det \mathrm{e}^{N a_i b_j}}{\prod_{i<j} (a_i - a_j)(b_i - b_j)} \tag{4.6.57}$$

so that

$$Z = \mathrm{cte} \int \mathrm{d}a_i \mathrm{d}b_i \prod_{i<j} (a_i - a_j)(b_i - b_j) \prod_i \mathrm{e}^{-N(V_1(a_i) + V_2(b_i))} \det \mathrm{e}^{N a_i b_j} \tag{4.6.58}$$

The naive saddle point method

A naive thing to do is to expand the determinant as a sum over the permutations and relabel the indices:

$$Z = N! \, \mathrm{cte} \int \mathrm{d}a_i \mathrm{d}b_i \prod_{i<j} (a_i - a_j)(b_i - b_j) \prod_i \mathrm{e}^{-N(V_1(a_i)+V_2(b_i))} \prod_i \mathrm{e}^{N a_i b_i}$$

even though that integral looks simpler than 4.6.58, the saddle point method cannot be applied to it, because the integrand is no more positive. If you find a col, $a_i = \tilde{a}_i$, $b_i = \tilde{b}_i$, you will have another col, with almost the same action, just by exchanging a pair of indices for a. That second col has almost the same action, and an opposite sign, so that it cancels the first col almost exactly. The leading behavior of Z at large N is therefore not given by the col !

4.6.1 A saddle point approximation for the angular integral

Consider integral 4.6.57 in the large N limit:

$$\mathrm{e}^{I(a_i,b_j)} = \int \mathrm{d}V \mathrm{e}^{N \, \mathrm{Tr} \, AVBV^\dagger}$$

and assume that the a_i and b_j have continuous densities $\rho_1(a)$ and $\rho_2(b)$, and write $I(a_i, b_j)$ as a functional of those densities:

$$I(a_i, b_j) = I[\rho_1, \rho_2]$$

and take the functional derivatives of I with respect to ρ_1 and ρ_2:

$$\frac{\delta I}{\delta \rho_1(\lambda_1)} = f(\lambda_1) - W_1(\lambda_1) \qquad , \qquad \frac{\delta I}{\delta \rho_2(\lambda_2)} = g(\lambda_2) - W_2(\lambda_2)$$

where $W_i(\lambda)$ are the resolvents associated to the densities $\rho_i(\lambda)$. Then P. Zinn-Justin showed that the functions f and g are functional inverse of one another:

$$f(g(\lambda)) = \lambda \qquad , \qquad g(f(\lambda)) = \lambda$$

Saddle point equation

We now have all the ingredients to find the linear saddle point equation for the 2-matrix integral. For the matrix M_1, we have the saddle point equation:

$$W_1(z_1 + i0) + W_1(z_1 - i0) = V_1'(z_1) - f(z_1 - i0) + W_1(z_1 - i0)$$

i.e.

$$W_1(z_1 + i0) = V_1'(z_1) - f(z_1 - i0)$$

Formally, we will write:

$$W_1(z_1) = V_1'(z_1) - f(z_1) \qquad , \qquad W_2(z_2) = V_2'(z_2) - g(z_2) \qquad , \qquad fog = id \quad (4.6.59)$$

4.6.2 Example: the Ising model.

Let us take the following example:

$$V_1(z) = V_2(z) = \frac{g}{3}z^3 + \frac{c}{2}z^2$$

g is the cosmological constant, coupled to the area of the lattice, and c is the Ising temperature coupled to the energy of neighboring spins. It is possible to introduce a magnetic field by taking different couplings g_1 and g_2 for the two matrices. Just for simplicity, here, let us assume that there is no magnetic field and $g_1 = g_2 = g$.

Then it is expected that the two matrices will have the same equilibrium distribution, $\rho_1 = \rho_2$, and both have the same resolvent $W(z)$, and the functions $f = g$.

The saddle point equation reads:

$$W(z) = gz^2 + cz - f(z) \qquad , \qquad f(f(z)) = z$$

How can one solve such a functional equation ? Consider the behavior at large z, when z belongs to the physical sheet:

$$W(z) \sim 1/z$$

which implies:

$$f(z) \sim f_0(z) = gz^2 + cz - 1/z + O(1/z^2)$$

the condition $f(f(z)) = z$ implies that in two other sheets, the function $f(z)$ has the following large z behavior:

$$f(z) \sim f_\pm(z) = \pm\sqrt{\frac{z}{g}} - \frac{c}{2g} \pm \frac{c^2}{8g}\sqrt{\frac{1}{gz}} \pm \frac{1 - c^4/64g^2}{2z\sqrt{gz}} + O(1/z^2)$$

so that the Riemann surface generated by $f(z)$ has at least 3 sheets. The simplest solution (not the only one) is to assume that it has only three sheets, and that $f(z)$ is solution of a third degree algebraic equation:

$$f^3(z) - A(z)f^2(z) - B(z)f(z) - C(z) = 0$$

The coefficients A, B, C are themselves polynomials in z, they are found from the large z behavior:

$$A(z) = f_0(z) + f_+(z) + f_-(z) = gz^2 + cz - \frac{c}{g}$$

$$B(z) = -f_0(f_+ + f_-) - f_+f_- = cz^2 + \frac{c^2+1}{g}z + \text{cte}_1$$

$$C(z) = f_0 f_+ f_- = -z^3 - \frac{c}{g}z^2 + \text{cte}_1 z + \text{cte}_2$$

Therefore we have:

$$f^3 + z^3 - gz^2f^2 - cfz(f+z) - \frac{c}{g}(f^2+z^2) - \frac{c^2+1}{g}fz - \text{cte}_1(f+z) - \text{cte}_2 = 0 \quad (4.6.60)$$

So far, c_1 and c_2 are undetermined constants, they have to be determined by a one-cut asumption. One should compute the discriminant of equation 4.6.60, which is a polynomial of degree 9 in z, and write that it should have enough double roots, and get c_1 and c_2 as functions of c and g. We shall not do it here, and leave it as an exercise to the reader.

The solution can be written parametrically with a parameter s:

$$gz = Cs + A - D^2 + \frac{1}{2}(s-B)\left(s - \sqrt{s^2 - 4D^2}\right)$$

$$gf(z) = Cs + A - D^2 + \frac{1}{2}(s-B)\left(s + \sqrt{s^2 - 4D^2}\right)$$

where A, B, C and D are constants which can be found as functions of g and c from the large z behavior of $f(z)$:

$$C = 1 \qquad , \qquad B = 1 - c - 2A = \frac{2D^2 - A^2}{2D^2 + A} \qquad , \qquad 2D^2 = A\frac{1-c-A}{c+2A} = A\frac{A+B}{1-B}$$

$$2g^2(c+2A)^2 = A(1-c-A)(A(1-c-A) - (c+2A)(1-(c+2A)^2))$$

The next step is finding the critical points. The condition for a critical point is that the discriminant vanishes at the endpoints of the density.

One condition for criticality is $B = -2C$. Another condition is $B = 2D$.

Both together give: $c_c = 1 - 2\sqrt{7}$ and $g_c = 2$, which is the Ising critical point, it has the critical exponents of the $(4,3)$ rational model.

Chapter 5

Orthogonal Polynomials Method

5.1 Hermitian Ensemble

Consider the matrix integral

$$Z = \int \mathrm{d}M \mathrm{e}^{-N \operatorname{tr} V(M)} = \text{cte} \int \mathrm{d}\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_i \mathrm{e}^{-NV(\lambda_i)} \tag{5.1.1}$$

where M is a $N \times N$ Hermitian matrix with eigenvalues λ_i, $i \in [0, \ldots, N-1]$.

The Jacobian of the change of variables $M \to (\lambda_1, \ldots \lambda_N)$ is the square of a Vandermonde determinant:

$$\Delta(\lambda_i) = \prod_{i<j} (\lambda_i - \lambda_j) = \det\left(\lambda_i^j\right) \tag{5.1.2}$$

Note that one can perform any linear combination of columns without changing the determinant so that:

$$\Delta(\lambda_i) = \det P_j(\lambda_i) \tag{5.1.3}$$

where $P_n(\lambda)$ is an arbitrary family of monic polynomials ($\deg P_n = n$).

The determinants can be expanded as sums of permutations:

$$Z = \sum_{\sigma,\tau} (-1)^\sigma (-1)^\tau \prod_i \int \mathrm{d}\lambda_i P_{\sigma_i}(\lambda_i) P_{\tau_i}(\lambda_i) \mathrm{e}^{-NV(\lambda_i)}$$

The trick is to chose the polynomials P_n orthogonal with respect to the following weight:

$$< P_n . P_m > := \int \mathrm{d}\lambda P_n(\lambda) P_m(\lambda) \mathrm{e}^{-NV(\lambda)} = h_n \delta_{nm}$$

therefore

$$Z = \sum_{\sigma,\tau} (-1)^\sigma (-1)^\tau \prod_i \delta_{\sigma_i,\tau_i} \prod_{i=0}^{N-1} h_i = N! \prod_{i=0}^{N-1} h_i$$

5.1.1 Slatter Determinant

One can think of it as a system of free fermions.

$$Z = \int \mathrm{d}\lambda_i \ |\Psi(\lambda_1, \ldots, \lambda_N)|^2$$
$$\Psi(\lambda_1, \ldots, \lambda_N) = \det \psi_i(\lambda_j)$$
$$\psi_n(\lambda) = \frac{1}{\sqrt{h_n}} \ P_n(\lambda) \ \mathrm{e}^{-NV(\lambda)}$$

Ψ is a Slatter determinant of N fermions occupying the 1-particle states $\psi_0, \ldots, \psi_{N-1}$. The state $N-1$ is the Fermi level. The N first wave functions of the orthogonal polynomial family form the Fermi sea.

Therefore, we can have the insight that almost every large N property will be controlled by the vicinity of the Fermi level:

$$n \sim N$$

5.1.2 Correlation functions

Note that the joint probability density of all N eigenvalues is the square of a determinant:

$$R_N(\lambda_1, \ldots, \lambda_N) = \frac{1}{N!} \ |\det(\psi_i(\lambda_j))|^2 \tag{5.1.4}$$

The square of a determinant can be written as the determinant of the square:

$$|\det A|^2 = \det AA^\dagger$$

Here, we have:

$$|\det(\psi_i(\lambda_j))|^2 = \det\left(\sum_{n=0}^{N-1} \psi_n(\lambda_i)\psi_n(\lambda_j)\right)$$

We introduce the Kernel:

$$K(\lambda, \mu) = \frac{1}{N} \sum_{n=0}^{N-1} \psi_n(\lambda)\psi_n(\mu) \tag{5.1.5}$$

The kernel $K(\lambda, \mu)$ encodes all the information about the statistics of the eigenvalues. The joint probability density is:

$$R_N(\lambda_1, \ldots, \lambda_N) = \frac{N^N}{N!} \ \det_{i,j}(K(\lambda_i, \lambda_j)) \tag{5.1.6}$$

Thanks to the orthogonality properties of the ψ_n, the kernel $K(\lambda, \mu)$ satisfies some remarkable projection relations:

$$\int \mathrm{d}\lambda K(\lambda, \lambda) = 1 \qquad , \qquad \int \mathrm{d}\mu K(\lambda, \mu)K(\mu, \nu) = \frac{1}{N} K(\lambda, \nu) \tag{5.1.7}$$

Dyson's theorem

It is possible to prove that the partially integrated probability densities:

$$R_k(\lambda_1, \ldots, \lambda_k) = \int \mathrm{d}\lambda_{k+1} \ldots \mathrm{d}\lambda_N \; R_N(\lambda_1, \ldots, \lambda_N)$$

take the form of a $k \times k$ determinant:

$$R_k(\lambda_1, \ldots, \lambda_k) = N^k \frac{N-k!}{N!} \det_{k\times k}(K(\lambda_i, \lambda_j)) \tag{5.1.8}$$

In addition, the R_k are related to the k-points correlation functions. They are proportional to the k-points correlation functions when all the λ_i are distinct:

$$\rho_k(\lambda_1, \ldots, \lambda_k) = \frac{N!}{N-k!N^k} R_k(\lambda_1, \ldots, \lambda_k) + \delta \,\text{terms}$$

For instance we have the density of eigenvalues (the 1-point function):

$$\rho(\lambda) = R_1(\lambda) = K(\lambda, \lambda) \tag{5.1.9}$$

The 2-point correlation function:

$$\begin{aligned} \rho_2(\lambda, \mu) &= \frac{N-1}{N} R_2(\lambda, \mu) + \frac{1}{N} R_1(\lambda)\delta(\lambda - \mu) && (5.1.10) \\ &= K(\lambda, \lambda)K(\mu, \mu) - K(\lambda, \mu)K(\mu, \lambda) + \frac{1}{N}\rho(\lambda)\delta(\lambda - \mu) && (5.1.11) \end{aligned}$$

its connected part is

$$\rho_c(\lambda, \mu) = -K(\lambda, \mu)^2 \tag{5.1.12}$$

and so on, all the correlation functions can be found this way.

The main aim of the orthogonal polynomial's approach is to be able to compute the kernel $K(\lambda, \mu)$, in particular in the large N limit.

5.1.3 Level spacing

As an illustration, we shall show how the level spacing distribution is related to the kernel $K(\lambda, \mu)$.

We have already showed before, that the spacing distribution is the second derivative of $E_\mu(s)$: the probability that there is no eigenvalue in the interval $[\mu, \mu + s]$.

$$\begin{aligned} E_\mu(s) &= \int_{\lambda_i \notin [\mu, \mu+s]} \det K(\lambda_i, \lambda_j) && (5.1.13) \\ &= \sum_{\sigma,\tau} (-1)^\tau \int_{\lambda_i \notin [\mu, \mu+s]} \prod_i \psi_i(\lambda_{\sigma_i}) \psi_{\tau_i}(\lambda_{\sigma_i}) \mathrm{d}\lambda_i && (5.1.14) \\ &= \sum_{\sigma,\tau} (-1)^\tau \prod_i \left(\delta_{i,\tau_i} - \int_\mu^{\mu+s} \psi_i(\lambda)\psi_{\tau_i}(\lambda)\mathrm{d}\lambda \right) && (5.1.15) \end{aligned}$$

$$= N! \sum_{\tau} (-1)^{\tau} \prod_{i} \left(\delta_{i,\tau_i} - \int_{\mu}^{\mu+s} \psi_i(\lambda)\psi_{\tau_i}(\lambda)\mathrm{d}\lambda \right) \tag{5.1.16}$$

$$= N! \det_{i,j} \left[\delta_{i,j} - \int_{\mu}^{\mu+s} \psi_i(\lambda)\psi_j(\lambda)\mathrm{d}\lambda \right] \tag{5.1.17}$$

$$= N! \prod_{k} (1-\alpha_k) \tag{5.1.18}$$

where the α's are the eigenvalues of the matrix $\int_{\mu}^{\mu+s} \psi_i(\lambda)\psi_j(\lambda)\mathrm{d}\lambda$. Let $\vec{u} = (u_0, \ldots u_{N-1})$ be an eigenvector with eigenvalue α:

$$\sum_{j=0}^{N-1} u_j \int_{\mu}^{\mu+s} \psi_i(\lambda)\psi_j(\lambda)\mathrm{d}\lambda = \alpha u_i$$

Now consider the function

$$f(\lambda) = \sum_{i=0}^{N-1} u_i \psi_i(\lambda)$$

We have:

$$\int_{\mu}^{\mu+s} K(\lambda,\mu) f(\mu) = \alpha f(\lambda) \tag{5.1.19}$$

The α's are the eigenvalues of the kernel $K(\lambda,\mu)$ acting by convolution in the space of L_2 functions.

Universality tells us that when s is small enough, and in the large N limit, the kernel $K(\lambda,\mu)$ tends towards

$$K(\lambda,\mu) \sim \frac{\sin\left(N\pi\rho(\lambda)(\lambda-\mu)\right)}{N\pi\rho(\lambda)(\lambda-\mu)}$$

And the spacing distribution is the Wigner surmise:

$$p(s) = s^2 \mathrm{e}^{-s^2}$$

5.1.4 How to determine the orthogonal polynomials

Naive recurrence

The first polynomial of degree 0 is:

$$P_0 = 1$$

The normalization h_0 is therefore:

$$h_0 = \int \mathrm{d}\lambda \mathrm{e}^{-NV(\lambda)}$$

The second polynomial of degree 1 is of the form:

$$P_1(\lambda) = \lambda - \text{ cte}$$

The cte is determined by the condition that $< P_0, P_1 >= 0$, i.e.:

$$\text{cte} \int \mathrm{d}\lambda \mathrm{e}^{-NV(\lambda)} = \int \mathrm{d}\lambda \lambda \mathrm{e}^{-NV(\lambda)}$$

once cte is known, we know P_1 and we can compute h_1 And so on,..., by recurrence, one can compute all the $P_n(\lambda)$.

Of course, that method is not satisfactory, because it is inapplicable for large values of n.

Integral expression

The orthogonal polynomials have an exact expression in the form of an integral:

$$P_n(\lambda) = \int \mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_n \prod_{i<j}(\lambda_i - \lambda_j)^2 \prod_{i=1}^{n}(\lambda - \lambda_i) \prod_{i=1}^{n} \mathrm{e}^{-NV(\lambda_i)} \qquad (5.1.20)$$

$$= \frac{\int \mathrm{d}M_{n\times n} \det(\lambda - M)\mathrm{e}^{-N \operatorname{tr} V(M)}}{\int \mathrm{d}M_{n\times n} \mathrm{e}^{-N \operatorname{tr} V(M)}} = \langle \det(\lambda - M) \rangle_{n\times n} \qquad (5.1.21)$$

The orthogonal polynomial of degree n is the average characteristic polynomial of a random $n \times n$ hermitian matrix.

Proof:

It is sufficient to prove that the above expression satisfies:

$$< P_n . \lambda^m >= 0 \qquad \text{for all} \quad m \leq n-1$$

We have:

$$< P_n . \lambda^m > = \int \mathrm{d}\lambda \, \mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_n \prod_{i<j}(\lambda_i - \lambda_j)^2 \, \lambda^m \prod_{i=1}^{n}(\lambda - \lambda_i) \, \mathrm{e}^{-NV(\lambda)} \prod_{i=1}^{n} \mathrm{e}^{-NV(\lambda_i)}$$

$$= \int \mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_{n+1} \left[\prod_{i=1}^{n+1} \mathrm{e}^{-NV(\lambda_i)} \prod_{i<j\leq n+1} (\lambda_i - \lambda_j)\right] \left[\lambda_{n+1}^m \prod_{i<j\leq n} (\lambda_i - \lambda_j)\right]$$

The first bracket is antisymmetric in the $n+1$ variables, therefore we can antisymmetrize the second bracket too. The result is zero because any non-zero antisymmetric polynomial of $n+1$ variables must be of degree at least n, while here, the second bracket is a polynomial of degree at most $n-1$ in each variable.

That integral expression for the orthogonal polynomials was not used often in the literature. It is useful if one wants to find some WKB asymptotics.

5.1.5 Recurrence equations

The most convenient way to determine the orthogonal polynomials is finding a recurrence equation.

Consider the polynomial $\lambda P_n(\lambda)$, of degree $n+1$. It can be expanded on the basis of the P_m's with $m \leq n+1$:

$$\lambda P_n(\lambda) = P_{n+1} + \beta(n) P_n + r(n) P_{n-1} + r(n,2) P_{n-2} + \dots r(n,k) P_{n-k} + \dots \quad (5.1.22)$$

If you write that:

$$< \lambda P_n . P_m > = < P_n . \lambda P_m > = \int \mathrm{d}\lambda \;\; \lambda \, P_n(\lambda) \, P_m(\lambda) \, \mathrm{e}^{-NV(\lambda)} \quad (5.1.23)$$

you find that $r(n,k) = 0$ for all $k \geq 2$, and $r(n) = h_n / h_{n-1}$:

$$\lambda P_n(\lambda) = P_{n+1} + \beta(n) P_n + \frac{h_n}{h_{n-1}} P_{n-1} \quad (5.1.24)$$

This recurrence equation can be rewritten in terms of the wave functions $\psi_n = \frac{1}{\sqrt{h_n}} P_n \mathrm{e}^{-NV/2}$:

$$\lambda \psi_n(\lambda) = \alpha(n+1) \psi_{n+1} + \beta(n) \psi_n + \alpha(n) \psi_{n-1} \qquad , \qquad \alpha(n) = \sqrt{\frac{h_n}{h_{n-1}}} \quad (5.1.25)$$

In the same line, we can compute P_n' and expand it on the basis of the P_{n-k} with $k \geq 1$:

$$P_n'(\lambda) = n P_{n-1} + s(n,2) P_{n-2} + \dots s(n,k) P_{n-k} + \dots \quad (5.1.26)$$

or equivalently for the wave functions ψ_n:

$$\frac{1}{N} \frac{\partial}{\partial \lambda} \psi_n(\lambda) = -\frac{1}{2} V'(\lambda) \psi_n(\lambda) + \frac{n}{N} \frac{1}{\alpha(n)} \psi_{n-1} + \gamma(n,2) \psi_{n-2} + \dots + \gamma(n,k) \psi_{n-k} + \dots \quad (5.1.27)$$

Note that the multiplication by $V'(\lambda)$ can be written in terms of $\alpha(n)$ and $\beta(n)$. Now, the integration by parts:

$$\int \psi_n' \psi_m = -\int \psi_n \psi_m' \quad (5.1.28)$$

implies that $\gamma_{n,k} = 0$ for all $k > \deg V'$, and imposes some relations between the coefficients γ, α and β. Eventually, it is possible to write some recurrence equations for all these coefficients. One can check that there is enough equations to determine all the unknown sequences.

5.1.6 Example: The Gaussian case

Consider the quadratic potential:

$$V(\lambda) = \frac{g}{2} \lambda^2 \qquad \text{i.e.} \quad V'(\lambda) = g\lambda \qquad , \qquad \deg V' = 1$$

We have:

$$\lambda \psi_n(\lambda) = \alpha(n+1) \psi_{n+1} + \beta(n) \psi_n + \alpha(n) \psi_{n-1} \qquad , \qquad \alpha(n) = \sqrt{\frac{h_n}{h_{n-1}}}$$

$$\begin{aligned}\frac{1}{N}\frac{\partial}{\partial\lambda}\psi_n(\lambda) &= -\frac{g}{2}\lambda\psi_n(\lambda)+\frac{n}{N}\frac{1}{\alpha(n)}\psi_{n-1}\\ &= -\frac{g}{2}\alpha(n+1)\psi_{n+1}-\frac{g}{2}\beta(n)\psi_n+\left(\frac{n}{N}\frac{1}{\alpha(n)}-\frac{g}{2}\alpha(n)\right)\psi_{n-1}\end{aligned}$$

The integration by parts

$$<\psi_n'.\psi_m>=-<\psi_n.\psi_m'>$$

implies that (with $m=n$ and $m=n-1$):

$$\beta(n)=0 \qquad \text{and} \qquad \frac{n}{N}\frac{1}{\alpha(n)}-\frac{g}{2}\alpha(n)=\frac{g}{2}\alpha(n)$$

Therefore

$$\alpha(n)=\sqrt{\frac{n}{Ng}} \qquad \text{and} \qquad h_n=h_0\frac{n!}{(Ng)^n}=\sqrt{\frac{\pi}{Ng}}\frac{n!}{(Ng)^n} \tag{5.1.29}$$

The Gaussian matrix integral is thus:

$$\int \mathrm{d}\lambda_1\ldots\mathrm{d}\lambda_N\ \prod_{i<j}(\lambda_i-\lambda_j)^2\ \prod_i \mathrm{e}^{-\frac{Ng}{2}\lambda_i^2}=\frac{\pi^{N/2}}{(Ng)^{N^2/2}}\prod_{n=0}^{N}n!$$

Of course we could have found this result directly from

$$\int \mathrm{d}M\mathrm{e}^{-\frac{Ng}{2}\,\mathrm{tr}\,M^2}=\left(\frac{\pi}{Ng}\right)^{N^2/2} \qquad \text{and} \qquad \mathrm{Vol}\,U(N)/U(1)^N=\frac{\pi^{N(N-1)/2}}{\prod_{n=0}^{N}n!}$$

The orthogonal polynomials $P_n(\lambda)$ satisfy the following relations:

$$\lambda P_n(\lambda)=P_{n+1}(\lambda)+\frac{n}{Ng}P_{n-1}(\lambda) \qquad , \qquad P_n'(\lambda)=nP_{n-1}(\lambda) \tag{5.1.30}$$

The P_n's are the Hermite polynomials:

$$P_n(\lambda)=\frac{1}{(Ng)^{n/2}}H_n(\sqrt{Ng}\,\lambda) \tag{5.1.31}$$

5.1.7 Example: Quartic potential

Consider the quartic potential:

$$V(\lambda)=\frac{1}{T}(\frac{1}{4}\lambda^4-\frac{1}{2}\lambda^2) \qquad \text{i.e.} \quad V'(\lambda)=\frac{1}{T}(\lambda^3-\lambda) \qquad , \qquad \deg V'=3 \tag{5.1.32}$$

We have:

$$\lambda\psi_n=\alpha_{n+1}\psi_{n+1}+\beta_n\psi_n+\alpha_n\psi_{n-1}$$

$$\frac{1}{N}\psi_n'+\frac{\lambda^3-\lambda}{2T}\psi_n=\frac{n}{N}\frac{1}{\alpha_n}\psi_{n-1}+\gamma_{n,2}\psi_{n-2}+\gamma_{n,3}\psi_{n-3}$$

The integration by parts

$$< \psi'_n . \psi_m >= - < \psi_n . \psi'_m >$$

implies that (with $m = n$, $m = n-1$, $m = n-2$ and $m = n-3$):

$$2T\gamma_{n,3} = \alpha_n \, \alpha_{n-1} \, \alpha_{n-2} \tag{5.1.33}$$

$$2T\gamma_{n,2} = \alpha_n \, \alpha_{n-1} \, (\beta_{n-2} + \beta_{n-1} + \beta_n) \tag{5.1.34}$$

$$\frac{Tn}{N}\frac{1}{\alpha_n^2} = \alpha_{n-1}^2 + \alpha_n^2 + \alpha_{n+1}^2 + \beta_{n-1}^2 + \beta_n\beta_{n-1} + \beta_n^2 - 1 \tag{5.1.35}$$

$$0 = \beta_n^3 - \beta_n + \alpha_{n+1}^2(2\beta_n + \beta_{n+1}) + \alpha_n^2(2\beta_n + \beta_{n-1}) \tag{5.1.36}$$

The initial condition $\beta_0 = 0$ implies that $\beta_n = 0$ for all n, and we have a recurrence equation for α_n:

$$\frac{Tn}{N} = \alpha_n^2 \left(\alpha_{n-1}^2 + \alpha_n^2 + \alpha_{n+1}^2 - 1\right) \tag{5.1.37}$$

remark 1: note that $\alpha_n(T)$ depends only on the variable $t = \frac{Tn}{N}$.
remark 2: In the large n limit, one can make the assumption that α_n depends smoothly on n, so that at leading order in $1/n$ we have

$$\alpha_{n+1} \sim \alpha_n \sim \alpha_{n-1} = \alpha \tag{5.1.38}$$

and in this limit, the recurrence equation for α_n turns into an algebraic equations for $\alpha(t)$:

$$t = 3\alpha^4 - \alpha^2 \tag{5.1.39}$$

whose solution is

$$\alpha^2 = \frac{1}{6}\left(1 + \sqrt{1 + 12t}\right) \tag{5.1.40}$$

remark: this suggests that we have a critical point at $T_c = -\frac{1}{12}$, if we assume that we can analytically extend the model to negative T.

remark 3: The assumption 5.1.38 is not always right. When $t < \frac{1}{4}$, one can check that this assumption is wrong. The correct assumption is that the even and odd sequences α_{2n} and α_{2n+1} behave smoothly with n for large n, and we will then write:

$$\alpha_{2n} \sim \alpha_{2n+2} \sim \alpha \qquad \text{and} \qquad \alpha_{2n-1} \sim \alpha_{2n+1} \sim \tilde{\alpha} \tag{5.1.41}$$

and α and $\tilde{\alpha}$ obey two algebraic equations:

$$t = \alpha^2(\alpha^2 + 2\tilde{\alpha}^2 - 1) \qquad , \qquad t = \tilde{\alpha}^2(\tilde{\alpha}^2 + 2\alpha^2 - 1) \tag{5.1.42}$$

which has solutions:

$$\alpha^2 = \frac{1}{2}\left(1 + \sqrt{1 - 4t}\right) \qquad , \qquad \tilde{\alpha}^2 = \frac{1}{2}\left(1 - \sqrt{1 - 4t}\right) \tag{5.1.43}$$

or:

$$\alpha_n^2 \sim \frac{1}{2}\left(1 + (-1)^n\sqrt{1 - 4\frac{nT}{N}}\right) \tag{5.1.44}$$

That solution exists only when $0 < t < \frac{1}{4}$, and it corresponds to the two-cuts solution.

remark 4: This example is not generic, because our quartic potential is symmetric, it has a Z_2 symmetry. If we consider a general non-symmetric quartic potential:

$$V(\lambda) = \frac{1}{T}\left(\frac{1}{4}\lambda^4 + \frac{h}{3}\lambda^3 - \frac{1}{2}\lambda^2\right)$$

we would find that in the small T regime, neither of the two subsequences α_{2n} or α_{2n+1} behaves smoothly with n in the large n limit. They seem to behave chaotically. Actually, they have a quasiperiodic behavior of the type:

$$\alpha(n) \sim A\mathrm{cn}^2(\pi n x) + B$$

where $x(T)$ is generically some irrational number. It is only in the symmetric case that we have $x = \frac{1}{2}$ and that we have a $(-1)^n$ behavior ($\mathrm{cn}(u)$ is the elliptical cosine function, it is 2π periodic).

In the general case of a higher degree potential with several wells, the multicut regime leads to a multi-quasiperiodic behavior for the sequences α_n and β_n.

5.1.8 Operator notation:

The recurrence equations are much easier to deal with when written with the help of Dirac's notations.

$$|n>= \psi_n \tag{5.1.45}$$

and we define the operators $\hat{Q}$ and $\hat{P}$, the multiplication and derivation with respect to λ:

$$\hat{Q}|n>= \lambda\psi_n(\lambda) \qquad , \qquad \hat{P} = \frac{1}{N}\frac{\partial}{\partial\lambda} \tag{5.1.46}$$

of course we have the canonical commutation equation:

$$[\hat{P}, \hat{Q}] = \frac{1}{N} \tag{5.1.47}$$

$1/N$ plays the role of $\hbar$ and the large N limit corresponds to the classical limit.

In order to write the recurrence equations, we introduce the shift operator x:

$$x|n>= |n+1> \qquad , \qquad x^\dagger|n>= |n-1> \qquad , \qquad x^\dagger x = 1 \tag{5.1.48}$$

Note that we nearly have $x^\dagger = x^{-1}$, actually, x has only a right inverse. We also introduce the level operator $\hat{n}$:

$$\hat{n}|n>= n|n> \qquad , \qquad \hat{n}^\dagger = \hat{n} \tag{5.1.49}$$

and for the sequences $\alpha(n)$, we will need to introduce the operators $\alpha(\hat{n})$ which we will write as $\hat{\alpha}$.

According to 5.1.25, the effect of a multiplication by λ writes:

$$\hat{Q}|n>= \lambda\psi_n(\lambda) = \alpha(n+1)\psi_{n+1} + \beta(n)\psi_n + \alpha(n)\psi_{n-1} = \hat{\alpha}x|n> + \hat{\beta}|n> + x^\dagger\hat{\alpha}|n> \tag{5.1.50}$$

or in operator notations:

$$\hat{Q} = \hat{\alpha} x + \hat{\beta} + x^{\dagger} \hat{\alpha} \tag{5.1.51}$$

Note that 5.1.51 is a mere rewriting of 5.1.25, and this notation makes obvious that $\hat{Q}$ is a hermitian operator (i.e. that $< \lambda P_n . P_m >=< P_n . \lambda P_m >$).

In the same way, we write the derivation operator $\hat{P}$ with the coefficients $\gamma(n,k)$ and the shift operator from 5.1.27:

$$\hat{P} = \sum_{k=0}^{\deg V'} x^{\dagger k} \hat{\gamma_k} - \hat{\gamma_k} x^k \tag{5.1.52}$$

The antihermiticity of $\hat{P}$ (which stems from the integration by parts $\int \psi_n' \psi_m = -\int \psi_n \psi_m'$) appears clearly with these notations.

Now, for any operator $\hat{P}$ which we decompose in power series of x and $x^{\dagger}$, we note P_k the coefficient of x^k and P_{-k} the coefficient of $x^{\dagger k}$, and P_+ the sum of P_k with positive k and P_- the sum with negative k. Looking at positive powers in equation 5.1.27, we have:

$$\hat{P}_+ = -\frac{1}{2} V'(\hat{Q})_+ \tag{5.1.53}$$

and by anti hermiticity of $\hat{P}$, we have $\hat{P}_0 = 0$ and $\hat{P}_- = -\hat{P}_+^{\dagger}$, which implies:

$$\hat{P} = \frac{1}{2} \left(V'(\hat{Q})_- - V'(\hat{Q})_+ \right) \tag{5.1.54}$$

Looking at the term of degree 0 and -1 in x in equation 5.1.27, we have:

$$V_0' = 0 \qquad , \qquad V_{-1}' = x^{\dagger} \frac{\hat{n}}{N} \frac{1}{\hat{\alpha}} \tag{5.1.55}$$

We have thus two equations (operator equations) involving only two unknowns $\hat{\alpha}$ and $\hat{\beta}$, so that we can solve them. Note that if we apply equation 5.1.55 to a state $|n>$, we get two recurrence equations for the sequences $\alpha(n)$ and $\beta(n)$.

Let us illustrate the method on the quadratic and quartic potentials again.

Example: quadratic potential

Again, we consider the potential:

$$V(\lambda) = \frac{g}{2} \lambda^2 \qquad , \qquad V'(\lambda) = g\lambda \qquad , \qquad \deg V' = 1 \tag{5.1.56}$$

The operator $\hat{Q}$ reads:

$$\hat{Q} = \hat{\alpha} x + \hat{\beta} + x^{\dagger} \hat{\alpha} \tag{5.1.57}$$

and $V'(\hat{Q}) = g\hat{Q}$ reads:

$$V_+' = g\hat{\alpha} x \qquad , \qquad V_0' = g\hat{\beta} \qquad , \qquad V_-' = g x^{\dagger} \hat{\alpha} \tag{5.1.58}$$

The operator $\hat{P}$ is thus:

$$\hat{P} = \frac{g}{2} \left(x^{\dagger} \hat{\alpha} - \hat{\alpha} x \right) \tag{5.1.59}$$

And the equations for $\hat{\alpha}$ and $\hat{\beta}$ are:

$$\begin{aligned} V_0' = \quad 0 = \quad & g\hat{\beta} && (5.1.60) \\ V_{-1}' = \quad x^\dagger \tfrac{\hat{n}}{N}\tfrac{1}{\hat{\alpha}} = \quad & gx^\dagger\hat{\alpha} && (5.1.61) \end{aligned}$$

whose solution is

$$\hat{\beta} = 0 \qquad , \qquad \hat{\alpha}^2 = \frac{\hat{n}}{Ng} \tag{5.1.62}$$

which is the same as 5.1.29.

Example: quartic potential

Consider the quartic potential:

$$V(\lambda) = \frac{1}{T}(\frac{1}{4}\lambda^4 - \frac{1}{2}\lambda^2) \tag{5.1.63}$$

and write

$$\hat{Q} = \hat{\alpha}x + \hat{\beta} + x^\dagger\hat{\alpha} \tag{5.1.64}$$

and we have $V'(\hat{Q}) = \frac{1}{T}(\hat{Q}^3 - \hat{Q})$:

$$\begin{aligned} TV'(\hat{Q}) = \quad & \hat{\alpha}x\hat{\alpha}x\hat{\alpha}x && (5.1.65) \\ + \quad & \hat{\alpha}x\hat{\alpha}x\hat{\beta} + \hat{\alpha}x\hat{\beta}\hat{\alpha}x + \hat{\beta}\hat{\alpha}x\hat{\alpha}x \\ + \quad & \hat{\alpha}x\hat{\alpha}xx^\dagger\hat{\alpha} + \hat{\alpha}xx^\dagger\hat{\alpha}\hat{\alpha}x + x^\dagger\hat{\alpha}\hat{\alpha}x\hat{\alpha}x + \hat{\alpha}x\hat{\beta}\hat{\beta} + \hat{\beta}\hat{\alpha}x\hat{\beta} + \hat{\beta}\hat{\beta}\hat{\alpha}x - \hat{\alpha}x \\ + \quad & \hat{\alpha}x\hat{\beta}x^\dagger\hat{\alpha} + \hat{\alpha}xx^\dagger\hat{\alpha}\hat{\beta} + x^\dagger\hat{\alpha}\hat{\alpha}x\hat{\beta} + x^\dagger\hat{\alpha}\hat{\beta}\hat{\alpha}x + \hat{\beta}x^\dagger\hat{\alpha}\hat{\alpha}x + \hat{\beta}\hat{\alpha}xx^\dagger\hat{\alpha} + \hat{\beta}^3 - \hat{\beta} \\ + \quad & x^\dagger\hat{\alpha}x^\dagger\hat{\alpha}\hat{\alpha}x + x^\dagger\hat{\alpha}\hat{\alpha}xx^\dagger\hat{\alpha} + \hat{\alpha}xx^\dagger\hat{\alpha}x^\dagger\hat{\alpha} + x^\dagger\hat{\alpha}\hat{\beta}\hat{\beta} + \hat{\beta}x^\dagger\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\beta}x^\dagger\hat{\alpha} - x^\dagger\hat{\alpha} \\ + \quad & x^\dagger\hat{\alpha}x^\dagger\hat{\alpha}\hat{\beta} + x^\dagger\hat{\alpha}\hat{\beta}x^\dagger\hat{\alpha} + \hat{\beta}x^\dagger\hat{\alpha}x^\dagger\hat{\alpha} \\ + \quad & x^\dagger\hat{\alpha}x^\dagger\hat{\alpha}x^\dagger\hat{\alpha} \end{aligned}$$

on which we read the equations $V_0' = 0$ and $V_{-1}' = x^\dagger\frac{\hat{n}}{N\hat{\alpha}}$:

$$\begin{aligned} 0 = \quad & \hat{\alpha}x\hat{\beta}x^\dagger\hat{\alpha} + \hat{\alpha}xx^\dagger\hat{\alpha}\hat{\beta} + x^\dagger\hat{\alpha}\hat{\alpha}x\hat{\beta} + x^\dagger\hat{\alpha}\hat{\beta}\hat{\alpha}x + \hat{\beta}x^\dagger\hat{\alpha}\hat{\alpha}x + \hat{\beta}\hat{\alpha}xx^\dagger\hat{\alpha} + \hat{\beta}^3 - \hat{\beta} \\ x^\dagger\frac{\hat{n}T}{N}\frac{1}{\hat{\alpha}} = \quad & x^\dagger\hat{\alpha}x^\dagger\hat{\alpha}\hat{\alpha}x + x^\dagger\hat{\alpha}\hat{\alpha}xx^\dagger\hat{\alpha} + \hat{\alpha}xx^\dagger\hat{\alpha}x^\dagger\hat{\alpha} + x^\dagger\hat{\alpha}\hat{\beta}\hat{\beta} + \hat{\beta}x^\dagger\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\beta}x^\dagger\hat{\alpha} - x^\dagger\hat{\alpha} \end{aligned} \tag{5.1.66}$$

One solution is $\hat{\beta} = 0$, and then:

$$x^\dagger\frac{\hat{n}T}{N}\frac{1}{\hat{\alpha}} = x^\dagger\hat{\alpha}x^\dagger\hat{\alpha}\hat{\alpha}x + x^\dagger\hat{\alpha}\hat{\alpha}xx^\dagger\hat{\alpha} + \hat{\alpha}xx^\dagger\hat{\alpha}x^\dagger\hat{\alpha} - x^\dagger\hat{\alpha} \tag{5.1.67}$$

When we apply this operator equation to a state $|n>$, we recover the recurrence equation 5.1.37 for the sequence α_n.

5.1.9 Large n limit

As we have seen in the example of the quartic potential, when n is large, the recurrence equations reduce to algebraic equations. The simplest assumption (which is valid only in the one-cut case) is that $\alpha(n)$ behaves smoothly with n for large n. As we have seen in that example too, and that is obvious from equation 5.1.55, if we introduce a temperature T as a multiplicative constant in front of the potential $\frac{1}{T}V$, the sequence $\alpha_n(T)$ depends on n and T through the variable $t = Tn/N$ only.

$$\alpha_n(T) \sim \alpha(t) \tag{5.1.68}$$

and in the large n limit, $\alpha(t)$ obeys an algebraic equation of the type:

$$t = f(\alpha(t)) \tag{5.1.69}$$

The free energy

We have already shown that the partition function is simply:

$$Z(N,T) = \mathrm{e}^{N^2 F(N,T)} = N! \prod_{n=0}^{N-1} h_n \qquad \text{and} \qquad \alpha_n^2(T) = \frac{h_n}{h_{n-1}} \tag{5.1.70}$$

so that:

$$(N+1)^2 F(N+1,T) + (N-1)^2 F(N-1,T) - 2N^2 F(N,T) = 2\log \alpha_N(T) \tag{5.1.71}$$

$F(n,T)$ depends on n and T only through the variable $t = nT/N$, and we have in the large N limit:

$$\frac{\partial^2}{\partial t^2} t^2 F(t) = 2\log \alpha(t) \tag{5.1.72}$$

and we are interested in $n = N$, i.e. $t = T$. The knowledge of the function $\alpha(T)$ is sufficient to compute the free energy $F(T)$, and therefore all the correlation functions.

The classical limit

In this large N limit, the canonical commutation relation $[P,Q] = \frac{1}{N}$ reduce to $[P,Q] = 0$, so that P can be written as a function $P(Q)$. This limit is equivalent to replace all operators by numbers, for instance the operator x becomes a number x (indeed, the assumption $\alpha_n \sim \alpha_{n+1}$ means that x commutes with $\hat{n}$). The operators $\hat{Q}$ and $\hat{P}$ thus read:

$$Q \sim \beta + \alpha\left(x + \frac{1}{x}\right) \tag{5.1.73}$$

and

$$P \sim \sum_{k=1}^{\deg V'} \gamma_k (x^{-k} - x^k) \tag{5.1.74}$$

The latter can be factorized as $(x - 1/x)$ times a polynomial in $(x + 1/x)$, or in other words a polynomial in Q (we write $Q = \lambda$):

$$\lambda = \beta + \alpha(x + 1/x) \qquad , \qquad P(\lambda) = -\frac{1}{2}\alpha(x - 1/x)\ M(\lambda) \tag{5.1.75}$$

where $M(\lambda)$ is a polynomial. Note that the relation λ as a function of x can be inverted:

$$x = \frac{1}{2\alpha}\left(-\beta \pm \sqrt{(\lambda - \beta)^2 - 4\alpha^2}\right) \tag{5.1.76}$$

and

$$\alpha(x - 1/x) = \sqrt{(\lambda - a)(\lambda - b)} \qquad \text{with} \qquad a = \beta - 2\alpha \qquad , \qquad b = \beta + 2\alpha \tag{5.1.77}$$

The function $P(\lambda)$ can then be written:

$$P(\lambda) = -\frac{1}{2}M(\lambda)\sqrt{(\lambda - a)(\lambda - b)} \tag{5.1.78}$$

If you compare with the saddle-point method, notice that this is exactly the expression of the density of eigenvalues:

$$-i\pi\rho(\lambda) = P(\lambda) = -\frac{1}{2}M(\lambda)\sqrt{(\lambda - a)(\lambda - b)} \tag{5.1.79}$$

a and b are the endpoints of the distribution of eigenvalues in the large n limit, this gives a physical meaning to α and β:

- $\beta = \frac{a+b}{2}$ is the center of the distribution
- $\alpha = \frac{b-a}{4}$ is the width of the distribution

2-point correlation function

Remember that the connected 2-point correlation function is related to the orthogonal polynomials by the kernel:

$$K(\lambda, \mu) = \frac{1}{N}\sum_{n=0}^{N-1} \psi_n(\lambda)\psi_n(\mu) \qquad , \qquad \rho_c(\lambda, \mu) = -K(\lambda, \mu)^2 \tag{5.1.80}$$

In order to find the 2-point correlation function, we need to find the kernel $K(\lambda, \mu)$ in the large N limit.

The first step is the Darboux-Christoffel theorem.

The Darboux-Christoffel theorem

which states that:

$$(\lambda - \mu)K(\lambda, \mu) = \frac{\alpha_N}{N}\left(\psi_N(\lambda)\psi_{N-1}(\mu) - \psi_{N-1}(\lambda)\psi_N(\mu)\right) \tag{5.1.81}$$

It can be proven easily from 5.1.25: all the terms of the sum collapse except at the boundaries.

Thus, we need to have asymptotics of the orthogonal polynomials $P_n(\lambda)$ only for $n \sim N$, which is in agreement with the fact that $n = N$ is the Fermi level of our system of free fermions, and that all the physical properties are related to the vicinity of the Fermi level.

WKB Asymptotics for the orthogonal polynomials

Remember that the operators $\hat{Q}$ and $\hat{P}$ are defined as:

$$\hat{Q}\psi_n(\lambda) = \lambda\psi_n(\lambda) \qquad \text{and} \qquad \hat{P}\psi_n(\lambda) = \frac{1}{N}\psi_n'(\lambda) \tag{5.1.82}$$

therefore, in the large N limit, where $\hat{P}$ becomes a function of λ (and $P(\lambda) = -i\pi\rho(\lambda)$), we have:

$$\psi_n'(\lambda) \sim -Ni\pi\rho(\lambda)\psi_n(\lambda) \tag{5.1.83}$$

which can be integrated to give the leading large N asymptotics for ψ_n:

$$\psi_n(\lambda) \sim \mathrm{e}^{-Ni\pi\int^\lambda \rho(\mu)\mathrm{d}\mu} \tag{5.1.84}$$

A next to leading order calculation is more tricky, and can be performed from the integral expression of $P_n(\lambda)$, it would give:

$$\psi_n(\lambda) \sim \frac{1}{\sqrt{2\pi}}\sqrt{-\frac{\partial x}{\partial\lambda}}\; x^{n-N-\frac{1}{2}}\; \mathrm{e}^{-Ni\pi\int^\lambda \rho(\mu)\mathrm{d}\mu} + \text{c.c.} \qquad , \qquad \lambda = \beta + \alpha(x + 1/x) \tag{5.1.85}$$

One can have a hint for the x^{n-N} term, indeed:

$$\psi_n = |n> = x^{n-N}|N> \tag{5.1.86}$$

and the $\frac{1}{\sqrt{2\pi}}\sqrt{-\frac{\partial x}{\partial\lambda}}$ term is exactly what is needed for the ψ_n to be orthogonal.

Remark 1: when $\lambda \notin [a,b]$, the complex conjugate c.c. is not present. $P(\lambda)$ is real, and $\mathrm{e}^{N\int^\lambda P(\lambda)\mathrm{d}\lambda}$ decreases for large λ. When $\lambda \in [a,b]$, both determinations of $x(\lambda)$ should be taken into account, and that's why there is the c.c. term. $\psi_n(\lambda)$ is indeed real. This phenomenon is quite general with WKB approximations, we have two solutions, and we should chose the solution which decreases at infinity outside the potential well, and an oscillatory solution within the well.

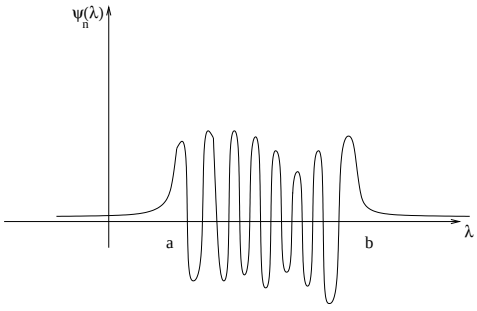


Figure 5.1: The WKB approximation: exponentially decreasing outside $[a,b]$, oscillatory within $[a,b]$.

Remark 2:

If we are in a multicut situation, i.e. when the support of ρ is disconnected with $s+1$ connected intervals, the asymptotic form of the orthogonal polynomials can be computed too, it involves hyper elliptical Riemann theta functions:

$$\Theta(\tau|\vec{u}) = \sum_{\vec{n}} \mathrm{e}^{i\pi\vec{n}.\tau\vec{n}} \mathrm{e}^{2i\pi\vec{u}.\vec{n}} \tag{5.1.87}$$

where $\vec{u} = (u_1, \ldots, u_s)$ is a s dimensional vector, and $\vec{n}$ denotes a vector whose coordinates are integers varying from $-\infty$ to $+\infty$. And $\tau = \tau_{ij}$ is a $s \times s$ real positive symmetric invertible matrix. When τ is well specified, we often omit it, and write $\Theta(\vec{u})$. The Theta function obeys some periodicity relations:

$$\Theta(\vec{u}+\vec{n}) = \Theta(\vec{u}) \qquad , \qquad \Theta(\vec{u}+\tau\vec{n}) = \mathrm{e}^{-i\pi\vec{n}.\tau\vec{n}} \mathrm{e}^{-2i\pi\vec{n}.\vec{u}}\, \Theta(\vec{u}) \tag{5.1.88}$$

The orthogonal polynomial then takes the form, in the large n limit:

$$P_n(\lambda) \sim \ \mathrm{cte}_{\,n} \sqrt{\left|\frac{\partial \vec{u}}{\partial \lambda}\right|}\ \frac{\Theta(N\vec{x}+\vec{u}+(2N-2n-1)\vec{u}_\infty)}{\Theta(N\vec{x}+(2N-2n)\vec{u}_\infty)}\ \left(\frac{\Theta_1(\vec{u}+\vec{u}_\infty)}{\Theta_1(\vec{u}-\vec{u}_\infty)}\right)^{n-N}\ \mathrm{e}^{-iN\pi\int^\lambda \rho} \tag{5.1.89}$$

see references [51] or [52,53] for more details.

The correlation functions

substituting 5.1.85 into 5.1.81 we have the kernel $K(\lambda,\mu)$. We will compute it in two different regimes:

- short distance behavior $|\lambda-\mu| \sim O(1/N)$. We then find that:

$$K(\lambda,\mu) \sim \frac{\sin\left(N\pi\int_\mu^\lambda \rho\right)}{N\pi(\lambda-\mu)} \sim \frac{\sin\left(N\pi(\lambda-\mu)\rho(\lambda)\right)}{N\pi(\lambda-\mu)} \tag{5.1.90}$$

  and the connected two point correlation function is the universal cardinal-sinus square:

$$\rho_c(\lambda,\mu) \sim -\rho(\lambda)\rho(\mu)\left(\frac{\sin\left(N\pi(\lambda-\mu)\right)}{N\pi(\lambda-\mu)}\right)^2 \tag{5.1.91}$$

- long distance behavior $|\lambda-\mu| \sim O(1)$, and we smooth out the oscillations of high frequency $1/N$:

$$\rho_c^{\mathrm{smooth}}(\lambda,\mu) \sim \frac{1}{4\pi^2}\frac{1}{\sqrt{(\lambda-a)(\lambda-b)(\mu-a)(\mu-b)}}\frac{2\lambda\mu-(a+b)(\lambda+\mu)+2ab}{(\lambda-\mu)^2} \tag{5.1.92}$$

  which is a universal function too, indeed, it does not depend on the potential $V(\lambda)$.

5.2 Orthogonal case

So far, we have studied only the case of hermitian matrices, $\beta = 2$. The orthogonal polynomials method can be adapted to other matrix ensembles as well. Let us start with the Orthogonal ensemble, i.e. $\beta = 1$.

Consider the integral:

$$Z_N = \int \mathrm{d}\lambda_0 \dots \mathrm{d}\lambda_{N-1} \ |\Delta(\lambda_i)|^\beta \prod_{i=1}^{N} \mathrm{e}^{-NV(\lambda_i)} \tag{5.2.93}$$

where $\beta = 1$ and $\Delta(\lambda_i)$ is the Vandermonde determinant:

$$\Delta(\lambda_i) = \prod_{i<j} (\lambda_i - \lambda_j) = \det\left(\lambda_i^j\right) = \det\left(P_j(\lambda_i)\right) \tag{5.2.94}$$

the last equality holds for any family of monic polynomials $\deg P_j = j$.

We have:

$$Z_N = \int \mathrm{d}\lambda_0 \dots \mathrm{d}\lambda_{N-1} \ \det\left(P_j(\lambda_i)\right) \prod_{i<j} \mathrm{sgn}(\lambda_i - \lambda_j) \ \prod_{i=0}^{N-1} \mathrm{e}^{-NV(\lambda_i)} \tag{5.2.95}$$

Like in the unitary case, the trick is to choose a suitable family which makes the calculation of integral 5.2.93 simple, i.e. decouple the λ_i, and transform the multiple integral into product of simple integrals. Unfortunately, it does not seem possible here, because of the absolute value.

However, note the following identities, which allow to transform products of sgn into sums:

- even case

$$\prod_{j=0}^{2n-1} \mathrm{sgn}(\lambda - \lambda_j) = 1 + \sum_{k=0}^{2n-1} \mathrm{sgn}(\lambda - \lambda_k) \prod_{j \neq k} \mathrm{sgn}(\lambda_k - \lambda_j) \tag{5.2.96}$$

- odd case

$$\prod_{j=0}^{2n} \mathrm{sgn}(\lambda - \lambda_j) = \sum_{k=0}^{2n} \mathrm{sgn}(\lambda - \lambda_k) \prod_{j \neq k} \mathrm{sgn}(\lambda_k - \lambda_j) \tag{5.2.97}$$

(proof: both sides of these equalities are functions of λ, equal to ± 1 on intervals between the λ_j's, and change their sign at each λ_j). and more generally:

- even case

$$\prod_{0 \leq i < j \leq 2n-1} \mathrm{sgn}(\lambda_i - \lambda_j) = \sum_{\text{pairings}} (-1)^{\text{pairs crossing}} \prod_{\text{pairs}(i,j)} \mathrm{sgn}(\lambda_i - \lambda_j) \tag{5.2.98}$$

- odd case

$$\prod_{0\le i<j\le 2n} \mathrm{sgn}(\lambda_i-\lambda_j) = \sum_{k=0}^{2n} \sum_{\text{pairings}} (-1)^{\text{pairs crossing}} \prod_{\text{pairs}(i,j)\neq k} \mathrm{sgn}(\lambda_i-\lambda_j) \quad (5.2.99)$$

that means that integral 5.2.93 can be decomposed into a sum of products of two-variables integrals of the type:

$$\int \mathrm{d}\lambda \mathrm{d}\mu \; P_n(\lambda) P_m(\mu) \,\mathrm{sgn}(\lambda-\mu)\; \mathrm{e}^{-NV(\lambda)}\, \mathrm{e}^{-NV(\mu)} \quad (5.2.100)$$

and our choice of orthogonal polynomials will be the following:

The scalar product

we consider the following skew scalar-product:

$$< f.g >= - < g.f >:= \int \mathrm{d}\lambda \mathrm{d}\mu \; f(\lambda) g(\mu) \,\mathrm{sgn}(\lambda-\mu)\; \mathrm{e}^{-NV(\lambda)}\, \mathrm{e}^{-NV(\mu)} \quad (5.2.101)$$

and we chose our polynomials P_n, such that:

$$< P_{2n}.P_{2m} >= \; < P_{2n+1}.P_{2m+1} >= \; 0 \quad (5.2.102)$$

$$< P_{2n+1}.P_{2m} >= \qquad h_n \delta_{nm} \quad (5.2.103)$$

The partition function

- when N is even, we write it $2N$.

  After relabeling the dum indices, the partition function 5.2.93 reduces to:

$$Z_{2N} = \frac{2N!}{2^N N!} \sum_{\sigma} (-1)^{\sigma} \prod_{i=0}^{N-1} < P_{\sigma_{2i}}.P_{\sigma_{2i+1}} > \quad (5.2.104)$$

  which is non vanishing only if σ_{2i} and σ_{2i+1} have different parities. Up to another relabeling (which adds a 2^N factor), we can assume that $\sigma_{2i} = 2\tau_i + 1$ and $\sigma_{2i+1} = 2\pi_i$, and:

$$Z_{2N} = \frac{2N!}{N!} \sum_{\pi,\tau} (-1)^{\pi} (-1)^{\tau} \prod_{i=0}^{N-1} < P_{2\tau_i+1}.P_{2\pi_i} > \quad (5.2.105)$$

  5.2.102 then implies that:

$$Z_{2N} = 2N! \prod_{n=0}^{N-1} h_n \quad (5.2.106)$$

- when N is odd, we write it $2N+1$.

  After relabeling the dum indices, the partition function 5.2.93 reduces to:

$$Z_{2N+1} = \frac{2N!}{2^N N!} \sum_{k=0}^{N} \sum_{\sigma} (-1)^{\sigma} \prod_{i=0}^{k-1} < P_{\sigma_{2i}} . P_{\sigma_{2i+1}} > \left(\int \mathrm{d}\lambda P_{\sigma_{2k}}(\lambda) \mathrm{e}^{-NV(\lambda)} \right) \prod_{i=k+1}^{N} < P_{\sigma_{2i-1}} . P_{\sigma_{2i}} >$$

  which is non zero when $\sigma_{2k} = 2N$. And then we must have, up to a relabeling: $\sigma_{2i} = 2\tau_i$, $\sigma_{2i+1} = 2\tau_i + 1$:

$$Z_{2N+1} = 2N! \prod_{n=0}^{N-1} h_n \left(\int \mathrm{d}\lambda P_{2N}(\lambda) \mathrm{e}^{-NV(\lambda)} \right) \tag{5.2.107}$$

Correlation functions

Instead of the partition function, let us compute the correlation functions, using the same method. As an illustration, we will consider here that N is even, and we rewrite it $2N$. The case N odd is not more difficult.

For instance, the average density of the λ_i's is:

$$\rho(\lambda_0) = \frac{1}{Z_N} \int \mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_N \prod_{0 \le i < j \le N} |\lambda_i - \lambda_j| \prod_{i=0}^{N} \mathrm{e}^{-NV(\lambda_i)} \tag{5.2.108}$$

and the two point correlation function is:

$$\rho(\lambda_0, \lambda_1) = \frac{1}{Z_N} \int \mathrm{d}\lambda_2 \dots \mathrm{d}\lambda_N \prod_{0 \le i < j \le N} |\lambda_i - \lambda_j| \prod_{i=0}^{N} \mathrm{e}^{-NV(\lambda_i)} \tag{5.2.109}$$

Using the orthogonal polynomials in the Vandermonde determinant, we can rewrite all correlation functions in term of a kernel:

$$K(\lambda, \mu) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{2h_n} \left(P_{2n+1}(\lambda) P_{2n}(\mu) - P_{2n}(\lambda) P_{2n+1}(\mu) \right) \mathrm{e}^{-NV(\lambda)} \mathrm{e}^{-NV(\mu)} \tag{5.2.110}$$

and we write:

$$R(\lambda, \mu) = \int \mathrm{d}\nu \ \mathrm{sgn}(\mu - \nu) \, K(\lambda, \nu) \tag{5.2.111}$$

$$H(\lambda, \mu) = \int \mathrm{d}\nu \ \mathrm{sgn}(\lambda - \nu) \, R(\nu, \mu) \tag{5.2.112}$$

Then we have:

$$\rho(\lambda) = R(\lambda, \lambda) \tag{5.2.113}$$

and

$$\rho(\lambda, \mu) = K(\lambda, \mu)\mathrm{sgn}(\lambda - \mu) + 2\rho(\lambda)\rho(\mu) + 2R(\lambda, \mu)R(\mu, \lambda) + 2K(\lambda, \mu)H(\mu, \lambda) \tag{5.2.114}$$

Determination of the skew-orthogonal-polynomials

Our monic polynomials $P_n(\lambda)$ are determined by the orthogonality condition 5.2.102. Note that the orthogonality condition is not sufficient to define the P_n's uniquely. There is an ambiguity for the odd degree polynomials P_{2n+1}, because $P_{2n+1} + \mathrm{cte}_n P_{2n}$ would be another suitable family of orthogonal polynomials, which satisfy 5.2.102.

One can show that the even degree polynomial P_{2n} is unique, and the odd degree polynomial P_{2n+1} is defined up to an arbitrary linear combination with P_{2n}. This ambiguity does not affect the value of h_n.

In order to compute the orthogonal polynomials, one should impose an extra condition, for instance that:

- the coefficient of λ^{2n} in P_{2n+1} is zero,
- or $\int \mathrm{d}\lambda P_{2n+1}(\lambda)\mathrm{e}^{-NV(\lambda)} = 0$,
- ... etc.

However, there exists a remarkable integral expression for the orthogonal polynomial:

$$\begin{aligned} P_{2n}(\lambda) &= \frac{1}{Z_{2n}} \int \mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_{2n} \prod_{i<j} |\lambda_i - \lambda_j| \prod_{i=1}^{2n} (\lambda - \lambda_i) \prod_{i=1}^{2n} \mathrm{e}^{-NV(\lambda_i)} \\ &= \int \mathrm{d}M_{2n\times 2n} \det(\lambda - M)\mathrm{e}^{-N \,\mathrm{tr}\, V(M)} / \int \mathrm{d}M_{2n\times 2n}\mathrm{e}^{-N \,\mathrm{tr}\, V(M)} \\ &= \langle \det(\lambda - M)\rangle_{2n\times 2n} \end{aligned} \tag{5.2.115}$$

where M is a $2n\times 2n$ real symmetric matrix. So, the even degree orthogonal polynomial is nothing but the average characteristic polynomial.

The odd degree polynomial has a similar expression:

$$\begin{aligned} P_{2n+1}(\lambda) &= \frac{1}{Z_{2n}} \int \mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_{2n} \prod_{i<j} |\lambda_i - \lambda_j| (\lambda + \sum_{i=1}^{2n} \lambda_i + c_n) \prod_{i=1}^{2n} (\lambda - \lambda_i) \prod_{i=1}^{2n} \mathrm{e}^{-NV(\lambda_i)} \\ &= \int \mathrm{d}M_{2n\times 2n} (\lambda + \mathrm{tr}\, M + c_n) \det(\lambda - M)\mathrm{e}^{-N \,\mathrm{tr}\, V(M)} / \int \mathrm{d}M_{2n\times 2n}\mathrm{e}^{-N \,\mathrm{tr}\, V(M)} \\ &= \langle (\lambda + \mathrm{tr}\, M + c_n) \det(\lambda - M)\rangle_{2n\times 2n} \end{aligned} \tag{5.2.116}$$

where c_n is arbitrary. The choice $c_n = 0$ is the one such that the term in λ^{2n} vanishes.

5.3 Symplectic case

The orthogonal polynomials method can be applied also to the Symplectic ensemble, i.e. $\beta = 4$.

Consider the integral:

$$Z_N = \int \mathrm{d}\lambda_0 \dots \mathrm{d}\lambda_{N-1} \ |\Delta(\lambda_i)|^\beta \prod_{i=1}^{N} \mathrm{e}^{-NV(\lambda_i)} \tag{5.3.117}$$

where $\beta = 4$ and $\Delta(\lambda_i)$ is the Vandermonde determinant:

$$\Delta_N(\lambda_i) = \prod_{i<j}(\lambda_i - \lambda_j) = \det\left(\lambda_i^j\right) = \det\left(P_j(\lambda_i)\right) \tag{5.3.118}$$

Again, the trick is to choose a suitable family of polynomials which makes the calculation of integral 5.3.117 simple, i.e. decouple the λ_i, and transform the multiple integral into product of simple integrals.

In this purpose, we will rewrite the Δ^4 as a single Determinant. Introduce n extra variables $(\mu_0, \dots, \mu_{N-1})$, and consider the Vandermonde determinant of size $2N$, with the λ_i's and μ_i's. Divide it by $\prod_i(\lambda_i - \mu_i)$ and take the limit $\mu_i \to \lambda_i$, you get:

$$\lim_{\mu_i \to \lambda_i} \frac{\Delta_{2N}(\lambda_i, \mu_i)}{\prod_i(\lambda_i - \mu_i)} = \Delta_N^4 \tag{5.3.119}$$

and note that the left hand side can be written as $\lim \det(P_j(\lambda_i)P_j(\mu_i))/\prod(\lambda_i - \mu_i) = \det(P_j(\lambda_i)P_j'(\lambda_i))$:

$$\Delta_N^4(\lambda_i) = \det \begin{pmatrix} P_0(\lambda_0) & \dots & P_{2N-1}(\lambda_0) \\ \vdots & & \vdots \\ P_0(\lambda_{N-1}) & \dots & P_{2N-1}(\lambda_{N-1}) \\ P_0'(\lambda_0) & \dots & P_{2N-1}'(\lambda_0) \\ \vdots & & \vdots \\ P_0'(\lambda_{N-1}) & \dots & P_{2N-1}'(\lambda_{N-1}) \end{pmatrix} \tag{5.3.120}$$

That means that integral 5.3.117 can be decomposed into a sum of products of single integrals of the type:

$$\int \mathrm{d}\lambda \ P_n(\lambda)P_m'(\lambda) \ \mathrm{e}^{-NV(\lambda)} \tag{5.3.121}$$

and our choice of orthogonal polynomials will be the following:

The scalar product

we consider the following skew scalar-product:

$$< f.g >= - < g.f >:= \int \mathrm{d}\lambda \ (f(\lambda)g'(\lambda) - f'(\lambda)g(\lambda)) \ \mathrm{e}^{-NV(\lambda)} \tag{5.3.122}$$

and we chose our polynomials P_n, such that:

$$< P_{2n}.P_{2m} >= \ < P_{2n+1}.P_{2m+1} >= \ 0 \tag{5.3.123}$$

$$< P_{2n+1}.P_{2m} >= \qquad h_n \delta_{nm} \tag{5.3.124}$$

The partition function and correlation functions

With the orthogonal polynomials defined above, it is now easy to compute the partition function:

$$Z_N = 2N! \prod_{n=0}^{N-1} h_n \tag{5.3.125}$$

and we can also compute the density of eigenvalues and the 2-point correlation function, in terms of the following kernel:

$$K(\lambda,\mu) = \frac{1}{N} \sum_{n=0}^{N-1} (P_{2n+1}(\lambda)P_{2n}(\mu) - P_{2n}(\lambda)P_{2n+1}(\mu))\mathrm{e}^{-NV(\lambda)/2}\mathrm{e}^{-NV(\mu)/2} \tag{5.3.126}$$

and

$$R(\lambda,\mu) = \frac{\partial}{\partial\mu}K(\lambda,\mu) \qquad , \qquad H(\lambda,\mu) = \frac{\partial^2}{\partial\lambda\partial\mu}K(\lambda,\mu) \tag{5.3.127}$$

We have then:

$$\rho(\lambda) = R(\lambda,\lambda) \tag{5.3.128}$$

and

$$\rho(\lambda,\mu) = K(\lambda,\mu) + 2\rho(\lambda)\rho(\mu) + 2K(\lambda,\mu)H(\lambda,\mu) + 2R(\lambda,\mu)R(\mu,\lambda) \tag{5.3.129}$$

which is very similar to the $\beta = 1$ case.

Determination of the skew-orthogonal-polynomials

Our monic polynomials $P_n(\lambda)$ are determined by the orthogonality condition 5.3.123. Notice that, like in the $\beta = 1$ case, there is an ambiguity for the odd degree polynomials P_{2n+1}: the even degree polynomial P_{2n} is unique, and the odd degree polynomial P_{2n+1} is defined up to an arbitrary linear combination with P_{2n}.

There exists a remarkable integral expression for the orthogonal polynomial:

$$\begin{aligned} P_{2n}(\lambda) &= \frac{1}{Z_n} \int \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_n \prod_{i<j} (\lambda_i - \lambda_j)^4 \prod_{i=1}^{n} (\lambda - \lambda_i)^2 \prod_{i=1}^{n} \mathrm{e}^{-NV(\lambda_i)} \\ &= \int \mathrm{d}M_{n\times n} \det(\lambda - M)\mathrm{e}^{-\frac{N}{2}\mathrm{tr}\,V(M)} / \int \mathrm{d}M_{n\times n}\mathrm{e}^{-\frac{N}{2}\mathrm{tr}\,V(M)} \\ &= \langle \det(\lambda - M) \rangle_{n\times n} \end{aligned} \tag{5.3.130}$$

where M is a $n \times n$ self-dual matrix whose entries are real quaternions (cf section.6.7). So, the even degree orthogonal polynomial is nothing but the average characteristic polynomial.

The odd degree polynomial has a similar expression:

$$\begin{aligned} P_{2n+1}(\lambda) &= \frac{1}{Z_n} \int \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_n \prod_{i<j} (\lambda_i - \lambda_j)^4 (\lambda + 2\sum_{i=1}^{n} \lambda_i + c_n) \prod_{i=1}^{n} (\lambda - \lambda_i)^2 \prod_{i=1}^{n} \mathrm{e}^{-NV(\lambda_i)} \\ &= \int \mathrm{d}M_{n\times n} (\lambda + \mathrm{tr}\, M + c_n)\, \det(\lambda - M) \mathrm{e}^{-\frac{N}{2}\mathrm{tr}\, V(M)} / \int \mathrm{d}M_{n\times n} \mathrm{e}^{-\frac{N}{2}\mathrm{tr}\, V(M)} \\ &= \langle (\lambda + \mathrm{tr}\, M + c_n) \det(\lambda - M) \rangle_{n\times n} \end{aligned} \tag{5.3.131}$$

where c_n is arbitrary. The choice $c_n = 0$ is the one such that the term in λ^{2n} vanishes.

Chapter 6

Equations of Motion's Method

One way of computing matrix integrals is the equation of motions, which proceed from the invariance of an integral under change of variable.

6.1 The Hermitian one matrix model

Consider the following hermitian-matrix integral:

$$Z = \mathrm{e}^{N^2 F} = \int \mathrm{d}M \ \mathrm{e}^{-N \,\mathrm{tr}\, V(M)} \tag{6.1.1}$$

where M is a $N \times N$ hermitian matrix, and $\mathrm{d}M$ is the Haar measure:

$$\mathrm{d}M = \prod_{i=1}^{N} \mathrm{d}M_{ii} \ \prod_{i<j} \mathrm{dRe}M_{ij} \, \mathrm{dIm}M_{ij} \tag{6.1.2}$$

and consider the change of variable M:

$$M \to M + \epsilon M^l \tag{6.1.3}$$

The integrand is changed to:

$$\mathrm{e}^{-N \,\mathrm{tr}\, V(M)} \to (1 - \epsilon N \,\mathrm{tr}\, M^l V'(M)) \mathrm{e}^{-N \,\mathrm{tr}\, V(M)} + O(\epsilon^2) \tag{6.1.4}$$

and the measure is changed to $\mathrm{d}M \to J(M)\mathrm{d}M$

Calculation of the Jacobian $J(M)$

we have:

$$J(M) = \det \frac{\partial \left(M + \epsilon M^l\right)_{ij}}{\partial M_{i'j'}} \sim \left(1 + \epsilon \,\mathrm{Tr}\, \frac{\partial {M^l}_{ij}}{\partial M_{i'j'}}\right) \tag{6.1.5}$$

where Tr is understood as a trace over the $N^2 \times N^2$ matrix, i.e. $i = i'$ and $j = j'$.

Write the differential:

$$\mathrm{d}M^l_{ii} = \sum_{k=0}^{l-1} M^k_{im} \mathrm{d}M_{mn} M^{l-k-1}_{ni} \longrightarrow \frac{\partial {M^l}_{ii}}{\partial M_{ii}} = \sum_{k=0}^{l-1} M^k_{ii} M^{l-k-1}_{ii} \tag{6.1.6}$$

$$\mathrm{d}\mathrm{Re}M^l_{ij} = \sum_{k=0}^{l-1} \mathrm{Re}M^k_{im}\mathrm{d}M_{mn}M^{l-k-1}_{nj} \longrightarrow \frac{\partial \mathrm{Re}M^l{}_{ij}}{\partial \mathrm{Re}M_{ij}} = \sum_{k=0}^{l-1} \mathrm{Re}M^k_{ii}M^{l-k-1}_{jj} \tag{6.1.7}$$

$$\mathrm{d}\mathrm{Im}M^l_{ij} = \sum_{k=0}^{l-1} \mathrm{Im}M^k_{im}\mathrm{d}M_{mn}M^{l-k-1}_{nj} \longrightarrow \frac{\partial \mathrm{Im}M^l{}_{ij}}{\partial \mathrm{Im}M_{ij}} = \sum_{k=0}^{l-1} \mathrm{Re}M^k_{ii}M^{l-k-1}_{jj} \tag{6.1.8}$$

Eventually you get:

$$J(M) = 1 + \epsilon \mathrm{Re} \sum_{k=0}^{l-1} \left(\sum_i M^k_{ii}M^{l-k-1}_{ii} + 2\sum_{i<j} M^k_{ii}M^{l-k-1}_{jj} \right) = 1 + \epsilon \sum_{k=0}^{l-1} \mathrm{tr}\, M^k \,\mathrm{tr}\, M^{l-k-1} \tag{6.1.9}$$

6.1.1 The equations of motion

$$\begin{aligned} Z &= \int \mathrm{d}M \mathrm{e}^{-N\,\mathrm{tr}\,V(M)} \\ &= \int \mathrm{d}M \mathrm{e}^{-N\,\mathrm{tr}\,V(M)} \left(1 + \epsilon \sum_{k=0}^{l-1} \mathrm{tr}\, M^k \,\mathrm{tr}\, M^{l-1-k} - N\,\mathrm{tr}\, M^l V'(M) \right) + O(\epsilon^2) \end{aligned}$$

At order 1 in ϵ we have:

$$\boxed{\left\langle \sum_{k=0}^{l-1} \mathrm{tr}\, M^k \,\mathrm{tr}\, M^{l-1-k} - N\,\mathrm{tr}\, M^l V'(M) \right\rangle = 0} \tag{6.1.10}$$

If one introduces the following notations:

$$T_k = \frac{1}{N} \left\langle \mathrm{tr}\, M^k \right\rangle \qquad , \qquad T_{k,j} = \left\langle \mathrm{tr}\, M^k \,\mathrm{tr}\, M^j \right\rangle - \left\langle \mathrm{tr}\, M^k \right\rangle \left\langle \mathrm{tr}\, M^j \right\rangle \tag{6.1.11}$$

and

$$V(M) = \sum_{k=1}^{d} \frac{g_k}{k} M^k \tag{6.1.12}$$

the equations of motion reduce to

$$\boxed{\sum_{k=0}^{l-1} T_k T_{l-1-k} + \frac{1}{N^2} T_{k,l-1-k} = \sum_{k=1}^{d} g_k T_{l+k-1}} \tag{6.1.13}$$

Virasoro constraints

Note that:

$$N^2 T_k = -k \frac{1}{Z} \frac{\partial Z}{\partial g_k} \qquad , \qquad N^4 T_k T_j + N^2 T_{k,j} = kj \frac{1}{Z} \frac{\partial^2 Z}{\partial g_k \partial g_j} \tag{6.1.14}$$

so that:

$$L_l Z = 0 \qquad \text{for} \qquad l \geq -1 \tag{6.1.15}$$

with

$$L_l = \frac{1}{N^2} 2l \frac{\partial}{\partial g_l} + \frac{1}{N^2} \sum_{k=1}^{l-1} k(l-k) \frac{\partial^2}{\partial g_k \partial g_{l-k}} + \sum_{k=1}^{d} (k+l) g_k \frac{\partial}{\partial g_{k+l}} \tag{6.1.16}$$

and the L_l's satisfy a Virasoro algebra:

$$[L_l, L_j] = (l-j) L_{l+j} \tag{6.1.17}$$

6.2 Equations of motion for the resolvent

Once again, the best way to deal with the equations of motion involves the resolvent function:

$$W(z) = \frac{1}{N} \left\langle \operatorname{tr} \frac{1}{z - M} \right\rangle = \sum_{l=0}^{\infty} \frac{T_l}{z^{l+1}} \tag{6.2.18}$$

We introduce the "bi-resolvent" as well:

$$W(z, z') = \frac{1}{N} \left\langle \operatorname{tr} \frac{1}{z - M} \operatorname{tr} \frac{1}{z' - M} \right\rangle_c = \sum_{l,j=0}^{\infty} \frac{T_{lj}}{z^{l+1} z'^{j+1}} \tag{6.2.19}$$

If one multiplies Eq. (6.1.13) by $1/z^{l+1}$ and sums over l, the equations of motion become:

$$\boxed{W^2(z) + \frac{1}{N^2} W(z,z) = V'(z) W(z) - P(z)} \tag{6.2.20}$$

where $P(z)$ is a polynomial of degree $\deg V - 2$:

$$P(z) = \mathrm{Pol} V'(z) W(z) = \sum_{k=0}^{d} \sum_{j=0}^{k-2} g_k T_{k-2-j} z^j \tag{6.2.21}$$

6.3 Large N limit

In the large N limit, one can neglect the biresolvent term, and the equation of motion reduces to the quadratic equation we have already met in the saddle point method Eq. (4.1.10):

$$W^2(z) = V'(z) W(z) - P(z) \tag{6.3.22}$$

whose solution is:

$$W(z) = \frac{1}{2}\left(V'(z) - \sqrt{V'^2(z) - 4P(z)}\right) \tag{6.3.23}$$

and the one cut solution takes the form:

$$W(z) = \frac{1}{2}\left(V'(z) - M(z)\sqrt{(z-a)(z-b)}\right) \tag{6.3.24}$$

6.4 $1/N$ expansion

Note that:

$$W(z) = \frac{\partial}{\partial V(z)} F \tag{6.4.25}$$

$$W(z,z') = \frac{\partial^2}{\partial V(z)\partial V(z')} F = \frac{\partial}{\partial V(z')} W(z) \tag{6.4.26}$$

where we have defined:

$$\frac{\partial}{\partial V(z)} := \sum_{k=0}^{\infty} \frac{k}{z^{k+1}} \frac{\partial}{\partial g_k} \tag{6.4.27}$$

This is enough to perform a systematic $1/N$ expansion of the equation of motion.

We will write:

$$F = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} F_{[h]} \tag{6.4.28}$$

$$W(z) = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} W_{[h]}(z) \tag{6.4.29}$$

and

$$W(z,z') = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} W_{[h]}(z,z') \qquad , \qquad W_{[h]}(z,z') = \frac{\partial}{\partial V(z')} W_{[h]}(z) \tag{6.4.30}$$

We can compute $W_{[0]}$ from Eq. (6.3.22). Once $W_{[0]}(z)$ is known, we can compute $W_{[0]}(z,z')$, and in particular chose $z' = z$, so that we have $W_{[0]}(z,z)$. $W_{[1]}(z)$ now satisfies the following equation:

$$(V'(z) - 2W_{[0]}(z))W_{[1]}(z) = W_{[0]}(z,z) + P_{[1]}(z) \tag{6.4.31}$$

and by recursion, on can in principle compute any $W_{[h]}(z)$, and any $F_{[h]}$.

As an example, let us write the expression of the Torus Free Energy $F_{[1]}$, for the one-cut solution:

$$F_{[1]} = \frac{1}{12} \ln\left((a-b)^4 M(a) M(b)\right) \tag{6.4.32}$$

6.5 Diagrammatic interpretation: Loop Equations

Remember that matrix integrals Feynmann expansions generate discrete polygonal random surfaces:

$$Z = \ln \int \mathrm{d}M \mathrm{e}^{-N \,\mathrm{tr}\, V(M)} = \sum_{\text{surfaces}} \frac{1}{\Omega} N^{\chi} g_2^{-n_2} \quad (-g_3)^{n_3} \dots (-g_k)^{n_k} \tag{6.5.33}$$

where the potential is:

$$V(z) = \sum_{k=2}^{\infty} \frac{g_k}{k} z^k \tag{6.5.34}$$

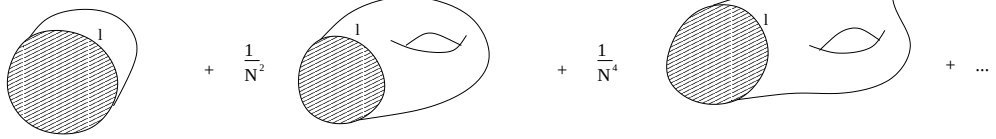
and where $\chi = 2 - 2h$ is the Euler-Poincarre characteristic of the surface, n_2 is the number of edges, n_3 the number of triangles, ... n_k the number of k-gones,... and Ω is the symmetry factor of the surface (the number of automorphisms of the surface).

Similarly, any vacuum expectation value, represents surfaces with boundaries:

- T_l is the generating function for surfaces with one boundary (a disc) of length l:

$$T_l = \sum_{\#\partial S = l} \frac{1}{\Omega} N^{\chi-1} g_2^{-n_2} \quad (-g_3)^{n_3} \dots (-g_k)^{n_k} \tag{6.5.35}$$

  where $\chi = 2 - 2h - 1$.

$T_l =$ 

- $W(z) = \sum_l T_l / z^{l+1}$ is the generating function for surfaces with one boundary of any length. The length is now weighted by a fugacity z:

$$W(z) = \sum_{1\text{boundary}} \frac{1}{\Omega} N^{\chi-1} g_2^{-n_2} \quad (-g_3)^{n_3} \dots (-g_k)^{n_k} \; z^{-(l+1)} \tag{6.5.36}$$

- $T_{l,j}$ is the generating function for surfaces with two boundaries (cylinders) of length l and j:

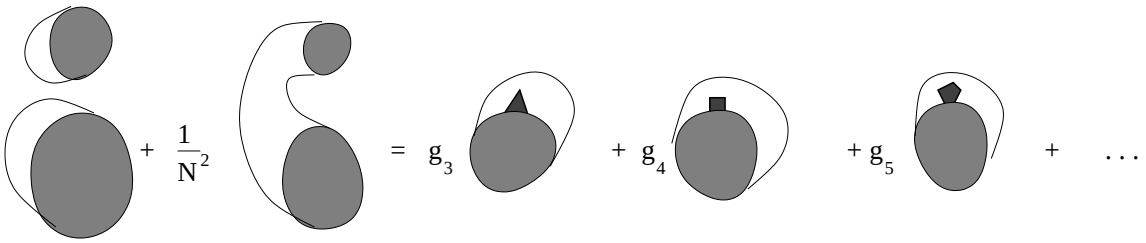
$$T_{lj} = \sum_{\#\partial_1 S = l, \#\partial_2 S = j} \frac{1}{\Omega} N^{\chi} g_2^{-n_2} \quad (-g_3)^{n_3} \dots (-g_k)^{n_k} \tag{6.5.37}$$

  where $\chi = 2 - 2h - 2$.

- $W(z, z')$ is the generating function for surfaces with two boundaries of any length. The lengths are weighted by fugacities z and z':

$$W(z, z') = \sum_{2\text{boundaries}} \frac{1}{\Omega} N^{\chi-1} g_2^{-n_2} \quad (-g_3)^{n_3} \dots (-g_k)^{n_k} \; z^{-(l+1)} z'^{-(l'+1)} \tag{6.5.38}$$

The equations of motion can be represented diagrammatically:



Somehow: adding polygonal pieces to the boundary is equivalent to split the boundary in two.

6.6 Real symmetric matrices $\beta = 1$

The same method works for real symmetric matrices instead of hermitian matrices. Consider a matrix integral

$$Z = \int \mathrm{d}M \mathrm{e}^{-N \operatorname{tr} V(M)} \tag{6.6.39}$$

where M is a real symmetric matrix of size $N \times N$, with Haar measure:

$$\mathrm{d}M = \prod_i M_{ii} \prod_{i<j} M_{ij} \tag{6.6.40}$$

Now write that the matrix integral is left unchanged by $M \to M + \epsilon M^l$. After calculating the Jacobian, one finds the following equation of motion:

$$\frac{1}{2} \sum_{k=0}^{l-1} \left\langle \operatorname{tr} M^k \operatorname{tr} M^{l-k-1} \right\rangle + \frac{l}{2} \left\langle \operatorname{tr} M^{l-1} \right\rangle = N \left\langle \operatorname{tr} M^l V'(M) \right\rangle \tag{6.6.41}$$

or in terms of the T_k's:

$$\boxed{\frac{1}{2} \sum_{k=0}^{l-1} \left(T_k T_{l-k-1} + \frac{1}{N^2} T_{k,l-k-1} \right) + \frac{1}{N} \frac{l}{2} T_{l-1} = \sum_{k=0}^{d} g_k T_{k+l-1}} \tag{6.6.42}$$

or in terms of the resolvent:

$$\boxed{\frac{1}{2} \left(W^2(z) - \frac{1}{N} W'(z) + \frac{1}{N^2} W(z,z) \right) = W(z) V'(z) - P(z)} \tag{6.6.43}$$

Note that we now have odd powers of $1/N$. Again, $W(z)$ can be computed recursively order by order in powers of $1/N$.

6.7 Real quaternionic self dual matrices $\beta = 4$

The $\beta = 4$ case can be treated in a similar way.

Consider the matrix integral:

$$Z = \int \mathrm{d}M \mathrm{e}^{-\frac{N}{2}\,\mathrm{tr}\,V(M)} \tag{6.7.44}$$

where M is a real quaternionic self-dual matrix of size $N \times N$, with Haar measure:

$$\mathrm{d}M = \prod_i M_{ii}^{(0)} \prod_{i<j} \prod_{\alpha=0}^{3} M_{ij}^{(\alpha)} \tag{6.7.45}$$

where the α index is the quaternion component index.

Quaternions

A quaternion q is a 2×2 matrix decomposed on the basis of Id and the three matrices e_1, e_2, e_3:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad , \quad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{6.7.46}$$

we write:

$$q = q^{(0)} + q^{(1)} e_1 + q^{(2)} e_2 + q^{(3)} e_3$$

A real quaternion is such that each $q^{(\alpha)}$ is a real number. (Note that a non zero real quaternion is invertible because $\det q = \sum_\alpha {q^{(\alpha)}}^2 > 0$. The set of quaternions is a non-commutative field).

We introduce the conjugate:

$$\overline{q} = q^{(0)} - q^{(1)} e_1 - q^{(2)} e_2 - q^{(3)} e_3$$

Self-duality means that:

$$\overline{M_{ij}} = M_{ji}$$

The trace of a quaternionic $N \times N$ matrix is the trace of the corresponding $2N \times 2N$ complex matrix.

All the eigenvalues λ_i's of a real self-dual quaternionic matrix are degenerate twice. We have for instance:

$$\mathrm{tr}\, M = 2 \sum_i \lambda_i \qquad , \qquad \det M = \prod_i \lambda_i^2$$

We will thus write:

$$T_k = \frac{1}{2N} \left\langle \mathrm{tr}\, M^k \right\rangle = \frac{1}{N} \left\langle \sum_{i=1}^{N} \lambda_i^k \right\rangle$$

$$W(z) = \sum_{k=0}^{\infty} \frac{T_k}{z^{k+1}} = \frac{1}{2N} \left\langle \mathrm{tr}\, \frac{1}{z - M} \right\rangle = \frac{1}{N} \left\langle \sum_{i=1}^{N} \frac{1}{z - \lambda_i} \right\rangle$$

Equations of motion

Now write that the matrix integral

$$Z = \int \mathrm{d}M \mathrm{e}^{-\frac{N}{2}\,\mathrm{tr}\,V(M)} \tag{6.7.47}$$

is left unchanged by $M \to M + \epsilon M^l$. After calculating the Jacobian, one finds the following equation of motion:

$$\frac{1}{2}\sum_{k=0}^{l-1} \left\langle \mathrm{tr}\, M^k \,\mathrm{tr}\, M^{l-k-1} \right\rangle - \frac{l}{2}\left\langle \mathrm{tr}\, M^{l-1} \right\rangle = N \left\langle \mathrm{tr}\, M^l V'(M) \right\rangle \tag{6.7.48}$$

or in terms of the T_k's:

$$\boxed{2\sum_{k=0}^{l-1}\left(T_k T_{l-k-1} + \frac{1}{N^2} T_{k,l-k-1}\right) - \frac{1}{N} l T_{l-1} = \sum_{k=0}^{d} g_k T_{k+l-1}} \tag{6.7.49}$$

or in terms of the resolvent:

$$\boxed{2W^2(z) + \frac{1}{N}W'(z) + \frac{2}{N^2}W(z,z) = W(z)V'(z) - P(z)} \tag{6.7.50}$$

Note that we have odd powers of $1/N$. Again, $W(z)$ can be computed recursively order by order in powers of $1/N$.

Chapter 7

Double scaling limit, strings and integrable hierarchies

We have seen (part.3) that random matrices can describe random (polygonal) surfaces. The identification is based on the large N perturbative expansion and makes sense only in the $N \to \infty$ limit, i.e. for fixed genus surfaces. Near a critical point $g \to g_c$, surfaces with a very large number of polygons dominate the sum, and we have a model for continuous surfaces, of given topology.

A string's time evolution sweeps a continuous surface, and we hope to describe string theory from a matrix model. Unfortunately, string interactions require change of genus. Apparently, the string coupling constant is $1/N = 0$.

The simple limit, where we first take $N \to \infty$, and then $g \to g_c$, cannot describe string theory. We have to chose a double scaling limit where at the same time $N \to \infty$ and $g \to g_c$ with a scaling $N^{-1} \sim \kappa(g - g_c)^\alpha$; the string coupling constant is then κ.

We will see that in this regime, the free energy and correlation functions obey some differential equations, which belong to a integrable hierarchy.

7.1 String theory

In Field theories, the particles are pointlike, and their time evolution sweeps lines in space-time. Their interactions can be represented by Feynmann diagrams, made of lines and vertices.

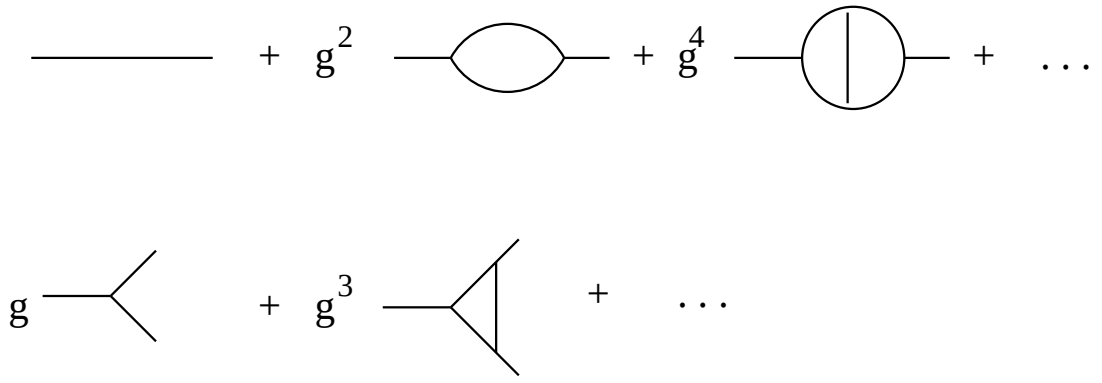


Figure 7.1: Diagrams contributing to the self energy and vertex of a point-particle.

In string theory, the particles are lines, and their time-evolution sweeps surfaces in space-time. Their interactions can be represented as surfaces.

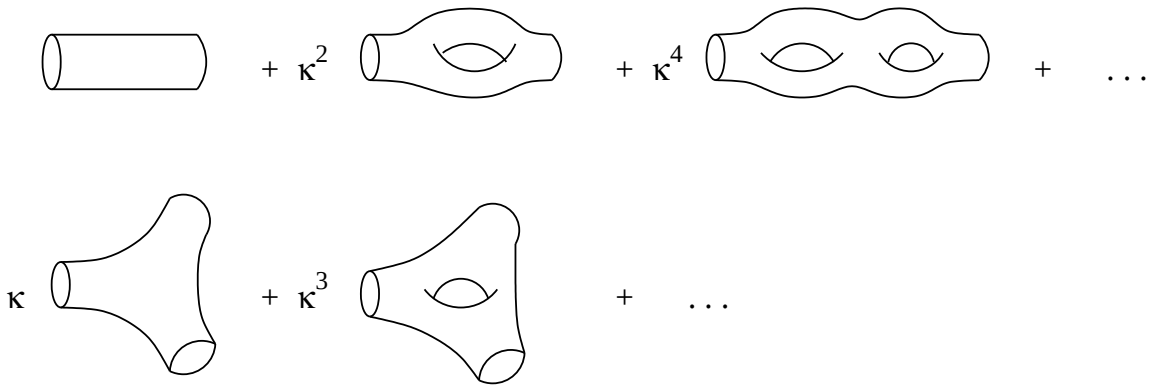


Figure 7.2: Diagrams contributing to the self energy and vertex of a string, κ is the string coupling constant.

The string coupling constant couples to the topology of the surface, the weight of a diagram is:

$$\kappa^{-\chi} \quad , \quad \chi = 2 - 2 \times \text{genus} - \#\text{boundaries} \tag{7.1.1}$$

In a random matrix model, where N is the size of the matrix, each diagram is weighted by a power of N of the form:

$$N^{\chi} \tag{7.1.2}$$

It is tempting to do the identification $\kappa = 1/N$! Unfortunately, the matrix model's diagrammatic expansion is defined only for $N \to \infty$.

7.2 Double scaling limit

Remember that the diagrammatic expansion of the free energy of a matrix model is (cf part.3):

$$F_{\text{matrix}} = N^2 F_0 + F_1 + \ldots + N^{2-2h} F_h \tag{7.2.3}$$

where F_h is the generating function for surfaces of genus h.

Near a critical point $g \to g_c$ (where g is a coupling constant of the matrix model, "the cosmological constant"), we have a singularity:

$$F_h \sim f_h (g_c - g)^{2\alpha_h} \quad + F_{\text{reg } h} \tag{7.2.4}$$

Notice that $F_{\text{reg } h}$ does not contribute to the large surfaces (the limit of continuous surfaces).

Is is possible to prove that α_h depends linearly on h:

$$\alpha_h = (h-1)(\gamma - 2) \tag{7.2.5}$$

γ is called the "string susceptibility exponent".

Therefore, if we set

$$\kappa = \frac{1}{N}(g_c - g)^{\alpha_1} \qquad \text{finite} \tag{7.2.6}$$

we have a generating function for continuous surfaces with contribution of all genus, with a string coupling constant κ.

We will now study the possible critical points and the exponents γ.

7.3 Critical Points

Consider the following matrix integral (M is a $N \times N$ hermitian matrix):

$$Z = \mathrm{e}^F = \int \mathrm{d}M \mathrm{e}^{-N \operatorname{tr} V(M)} \tag{7.3.7}$$

$$= \int \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_N \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_i \mathrm{e}^{-NV(\lambda_i)} \tag{7.3.8}$$

The eigenvalues are in an effective potential:

$$V_{\text{eff}}(\lambda) = V(\lambda) - \frac{2}{N} \sum_j \ln(\lambda - \lambda_j) \tag{7.3.9}$$

Consider the example of the inverted quartic potential:

$$V(M) = -\frac{g}{4} M^4 + \frac{1}{2} M^2 \tag{7.3.10}$$

The matrix integral Z is well defined only for $g < 0$, though it is possible to prove that when $N \to \infty$, $Z(g)$ can be continued analytically to $g > 0$. Indeed, the tunnel effect for eigenvalues escaping the potential well is of order e^{-N} and is suppressed for large N faster than any power of $1/N$, it is not perturbative, the instability does not affect the perturbative expansion.

The large N limit of this model for $g < g_c$ is a classical limit, the quantum effects (tunnel effect) are neglectable. We have seen (cf part.4) that we have algebraic equations for the observables (examples: $a(g)$, $u = F''(g)$)

Note that the density of eigenvalues has square root endpoints:

$$\rho(\lambda) \underset{\lambda \to a}{\sim} (\lambda - a)^{\frac{1}{2}} \tag{7.3.11}$$

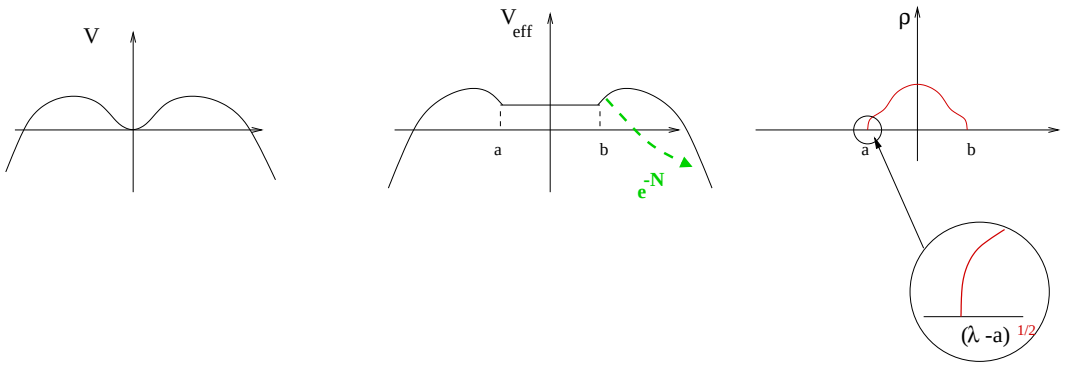


Figure 7.3: Quartic potential, the corresponding effective potential (flat between the two endpoints of the spectrum), and the eigenvalues density, for $g < g_c$. The tunnel effect is suppressed as e^{-N}.

Actually, the tunnel effect is of order $e^{-N \times \text{height of the barrier}}$. And when we approach the critical point $g \to g_c = \frac{1}{12}$, the height of the barrier vanishes. This is why there is a critical point. In the vicinity of the critical point, the tunnel effect is no more neglectable and has to be taken into account, that is why a classical approximation is not sufficient, we will need a quantum approach. Instead of algebraic equations, we will have differential equations in this limit (analogous in mechanics: classical gives algebraic relations between energy and position, quantum gives Schrödinger equation).

At the critical point, the edges of the density of eigenvalues don't behave as a square roots anymore, they behave with another power law:

$$\rho(\lambda) \underset{\lambda \to a}{\sim} (\lambda - a)^{\mu} \tag{7.3.12}$$

(in this example $\mu = 3/2$).

7.3.1 The one-matrix model

We have seen that in the one matrix model, in the $N \to \infty$ limit, the density of eigenvalues obeys a second degree equation, and it has the form:

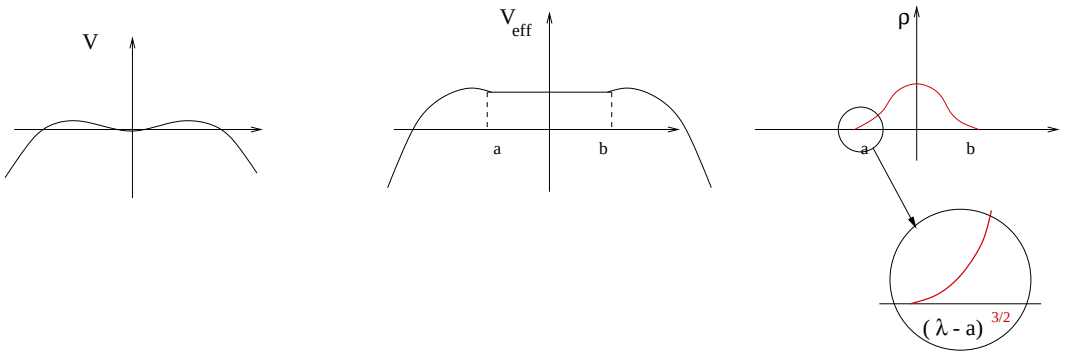


Figure 7.4: When $g \to g_c$, the height of the barrier of the effective potential vanishes. The edges of the density are no more square roots.

$$\rho(\lambda) = \frac{1}{2\pi} M(\lambda) \sqrt{(\lambda - a)(b - \lambda)} \tag{7.3.13}$$

$M(\lambda)$ is a polynomial (related to the potential by $M(\lambda) = \mathrm{Pol}\, V'(\lambda)/\sqrt{(\lambda - a)(\lambda - b)}$). Generically, $M(a)$ and $M(b)$ are non zero, and ρ has square root singularities at its edges.

At a critical point, $M(a)$ can vanish at order m ($M(\lambda) \sim (\lambda - a)^m$), and one can have:

$$\rho(\lambda) \underset{\lambda \to a}{\sim} (\lambda - a)^{m + \frac{1}{2}} \tag{7.3.14}$$

In case M vanishes at both a and b, with exponents m_a and m_b, only the largest one will be important. In case $m_a = m_b$, a degeneracy factor 2 will appear in the calculations.

7.3.2 Multi-matrix model

In more general matrix models, the density of eigenvalues obeys a higher degree algebraic equation (or even in some model, some non algebraic equations). For the rational models, we have a behavior:

$$\rho(\lambda) \underset{\lambda \to a}{\sim} (\lambda - a)^{\frac{p}{q}} \tag{7.3.15}$$

which realizes the critical exponents of a (p, q) rational conformal model, of central charge:

$$c = 13 - 6 \left(\frac{p}{q} + \frac{q}{p} \right)$$

The one matrix model always has $q = 2$, and $p = 2m + 1$. The chain of $q - 1$ matrices coupled linearly, has a p/q exponent.

7.4 The orthogonal polynomial's method in the vicinity of the critical point

In order to study the vicinity of the critical point, the algebraic equations of the saddle point method are not sufficient. One needs a quantum approach. We will consider the method of orthogonal polynomials (cf part.5) instead.

We take the notations of part.5.

7.4.1 Reminder of the orthogonal polynomials method

In order to compute a matrix integral like

$$Z = \mathrm{e}^F = \int \mathrm{d}M \mathrm{e}^{-\frac{N}{g}\mathrm{tr}\, V(M)} \tag{7.4.16}$$

one introduces a family of orthogonal polynomials $P_n(\lambda)$:

$$\int_{-\infty}^{\infty} \mathrm{d}\lambda\ P_n(\lambda)\, P_m(\lambda)\ \mathrm{e}^{-\frac{N}{g}V(\lambda)} = h_n \delta_{nm}$$

and the family of wave functions:

$$\psi_n(\lambda) = \frac{1}{\sqrt{h_n}} P_n(\lambda) \mathrm{e}^{-\frac{N}{2g}V(\lambda)}$$

in terms of which the partition function is simply:

$$Z = N! \prod_{n=0}^{N-1} h_n$$

The orthogonal polynomials and the wave functions are computed by recurrence. We decompose:

$$\lambda\psi_n(\lambda) = \alpha(n+1)\psi_{n+1}(\lambda) + \beta(n)\psi_n(\lambda) + \alpha(n)\psi_{n-1}(\lambda) \tag{7.4.17}$$

where $\alpha(n) = \sqrt{h_n/h_{n-1}}$, and $\beta(n)$ are so far unknown sequences.

and

$$\frac{1}{N}\frac{\partial}{\partial\lambda}\psi_n(\lambda) + \frac{1}{2g}V'(\lambda)\psi_n(\lambda) = \frac{1}{\alpha(n)}\frac{n}{N}\psi_{n-1}(\lambda) + \dots \tag{7.4.18}$$

If we define the operators Q = multiplication by λ and $P = \frac{1}{N}\partial/\partial\lambda$ derivation w.r.t λ, we have the canonical equation:

$$[P, Q] = \frac{1}{N} \tag{7.4.19}$$

The classical limit

In the classical limit $N \to \infty$, P and Q commute. P can be written as a function of Q, we write $Q = \lambda$ and $P = -i\pi\rho(\lambda)$ where $\rho(\lambda)$ is the density of eigenvalues. We find that

$$\rho(\lambda) = \frac{1}{2\pi} M(\lambda)\sqrt{(\lambda - a)(b - \lambda)}$$

Double scaling limit

In the basis of the ψ_n, the operators Q and P have non-zero coefficients only at finite distance off the diagonal.

$$Q\psi_n = \sum_m Q_{nm}\psi_m \qquad , \qquad Q_{nm} = 0 \text{ if } |m-n| > 1 \tag{7.4.20}$$

$$P\psi_n = \sum_m P_{nm}\psi_m \qquad , \qquad P_{nm} = 0 \text{ if } |m-n| > \deg V' \tag{7.4.21}$$

In the large n limit, we will approximate Q and P by differential operators:

$$Q \sim \mathrm{d}^2 + 2u \qquad , \qquad P \sim \mathrm{d}^p + v\mathrm{d}^{p-2} + \dots \tag{7.4.22}$$

and eq.7.4.19 will allow to compute those operators.

7.5 Example: the quartic potential

Consider the potential:

$$V(\lambda) = -\frac{1}{4}\lambda^4 + \frac{1}{2}\lambda^2 \tag{7.5.23}$$

We write the operators Q and P with the help of the shift operator x:

$$x\psi_n = \psi_{n+1} \qquad x = \begin{pmatrix} 0 & 1 & 0 & \\ 0 & 0 & 1 & \ddots \\ & 0 & \ddots & 1 \end{pmatrix} \tag{7.5.24}$$

We have:

$$Q = \alpha x + x^\dagger \alpha$$

where $\alpha = \mathrm{diag}(\alpha(0), \alpha(1), \dots, \alpha(n), \dots)$ is an infinite diagonal matrix.

Similarly, we have

$$P = x^{\dagger 3}\gamma + x^\dagger \epsilon - \epsilon x - \gamma x^3$$

where γ and ϵ are infinite diagonal matrices.

We have seen in part.5 that:

$$\frac{1}{g}V'_{-1} = x^\dagger \frac{1}{\alpha}\frac{\hat{n}}{N}$$

where V'_{-1} means that we have to take only the terms just below the diagonal in $V'(Q)$. One finds the equation:

$$-\alpha^2(n)(\alpha^2(n-1) + \alpha^2(n) + \alpha^2(n+1)) + \alpha^2(n) = t = \frac{n}{N}g \tag{7.5.25}$$

Note that $\alpha(n, g)$ depends only on the reduced variable $t = ng/N$. We note it $\alpha(t)$. All the observables of the system correspond to $n \sim N$, and therefore to $t \sim g$. We will often write g instead of t.

In the large N classical limit, this equation reduces to an algebraic equation:

$$g = f(\alpha^2) = -3\alpha^4 + \alpha^2 \qquad \longrightarrow \quad \alpha^2 = \frac{1}{6}\left(1 - \sqrt{1-12g}\right) \tag{7.5.26}$$

which allows to find the critical point:

$$g_c = \frac{1}{12} \qquad , \qquad \alpha_c^2 = \frac{1}{6} \tag{7.5.27}$$

In the vicinity of g_c, we rescale:

$$g = g_c(1 - N^{-\delta})\tau) \qquad , \qquad \alpha^2 = \alpha_c^2(1 - N^{-\eta}u(\tau)) \tag{7.5.28}$$

In this limit, we have:

$$\alpha^2(n+1) \sim \alpha^2(t) + \frac{g}{N}\frac{\partial \alpha^2(t)}{\partial t} + \frac{g^2}{2N^2}\frac{\partial^2 \alpha^2(t)}{\partial t^2} + \dots$$

or in terms of $u(\tau)$:

$$\alpha^2(n+1) \sim \alpha_c^2(1 - N^{-\eta}u + N^{\delta-\eta-1}u' + \frac{1}{2}N^{2\delta-\eta-2}u'' + \dots)$$

inserting that into eq.7.5.25 we get at leading order for each term:

$$3N^{-2\eta}u^2 - N^{2\delta-2\eta-2}u'' = 3N^{-\delta}\tau$$

We can read the scalings:

$$\eta = \frac{2}{5} \qquad \text{and} \qquad \delta = \frac{4}{5} \tag{7.5.29}$$

and $u(\tau)$ obeys the Painlevé I equation:

$$\frac{3}{4}u^2 - \frac{1}{4}u'' = \frac{3}{4}\tau \tag{7.5.30}$$

from the knowledge of $u(\tau)$, we can recover $\alpha(t)$ and h_n and all the statistical properties...

The double scaling limit is such that:

$$N^2(1 - g/g_c)^{2-\gamma} = \kappa^2 \text{ finite}$$

here we have:

$$N^2 N^{-(2-\gamma)\delta}\tau^{2-\gamma} = \kappa^2$$

therefore

$$2 - \gamma = \frac{2}{\delta}$$

here $\gamma = -\frac{1}{2}$.

7.6 General case, method of commutation relations

7.6.1 scalings, exponent γ

We have just seen an explicit example, but we could have found the result (the equation for $u(\tau)$) much faster. The scaling approximation for $\alpha(n)$ amounts to write a scaling for the operator Q:

$$Q = Q_c + N^{-\eta}(\mathrm{d}^2 - 2u) \qquad , \qquad \frac{ng}{N} = g_c(1 - N^{-\delta}\tau) \tag{7.6.31}$$

where $\mathrm{d} = \partial/\partial\tau$ and u is a function of τ. The d^2 term is related to the fact that Q has off diagonal terms, which allow to jump to $n-1$ and $n+1$, i.e. formally $N^{-\eta}\mathrm{d}^2 = \frac{\partial^2}{\partial n^2} \sim \frac{1}{N^2}$. This implies that:

$$\eta + 2\delta = 2 \tag{7.6.32}$$

The operator P has the same form:

$$P = P_c + N^{-\zeta}(\mathrm{d}^p - v\mathrm{d}^{p-2} - \ldots) \tag{7.6.33}$$

note that P must be anti-hermitic. In the classical limit, we have $Q_c = a$ and $P_c(Q_c) = 2i\pi\rho(a) = 0$, therefore $P_c = 0$. Moreover, we have at the critical point:

$$\rho(\lambda) \sim (\lambda - a)^{m+\frac{1}{2}}$$

so that

$$p = 2m+1 \qquad \text{and} \qquad \zeta = (m + \frac{1}{2})\eta$$

The relationship 7.4.19 implies $1 = (m + \frac{3}{2})\eta$, i.e.:

$$\eta = \frac{2}{2m+3} \qquad \text{and} \qquad \delta = 1 - \frac{1}{2m+3} \tag{7.6.34}$$

The string susceptibility exponent is then $\gamma = 2 - \frac{2}{\delta}$:

$$\boxed{\gamma = -\frac{1}{m+1}} \tag{7.6.35}$$

7.6.2 The string equation

We will now use eq.7.4.19 to find a differential equation for $u(\tau)$. We rewrite:

$$Q = Q_c + N^{-\eta}\mathbf{Q} \qquad , \qquad P = N^{-p\eta/2}\mathbf{P} \tag{7.6.36}$$

where

$$\mathbf{Q} = \mathrm{d}^2 - 2u \qquad , \qquad \mathbf{P} = \mathrm{d}^p - \sum_{k=2}^{p} v_k \mathrm{d}^{p-k} \tag{7.6.37}$$

are differential operators ($\mathbf{P}$ must be anti-hermitic, $p = 2m+1$). They must obey:

$$[\mathbf{P}, \mathbf{Q}] = 1 \tag{7.6.38}$$

Example

Let us start with the example $m = 1$:

$$\mathbf{Q} = \mathrm{d}^2 - 2u \qquad , \qquad \mathbf{P} = \mathrm{d}^3 - (v\mathrm{d} + \mathrm{d}v)$$

The commutation equation

$$[\mathbf{P}, \mathbf{Q}] = 1$$

implies:

$$1 = -(6u' - 4v')\mathrm{d}^2 - (6u'' - 4v'')\mathrm{d} + (-2u''' + v''' + 4u'v)$$

i.e.

$$v = \frac{3}{2}u + \mathrm{cte}_1 \qquad \text{and} \qquad -\frac{1}{2}u'' + 3u^2 + 4\,\mathrm{cte}_1 u + \mathrm{cte}_2 = \tau \tag{7.6.39}$$

up to the two integration constants (which can be absorbed by a suitable reparametrization), we recover the Painlevé 1 equation:

$$3u^2 - \frac{1}{2}u'' = \tau$$

Note that in the quartic potential, we had the same equation for $u/2$ instead of u, this is an effect of degeneracy due to the symmetry of the quartic potential.

7.6.3 The Gelfand-Dikii differential polynomials

It is possible to find the general solution of the equation $[\mathbf{P}, \mathbf{Q}] = 1$. Indeed, consider the following operators:

$$P_k = \left(\mathbf{Q}^{k-\frac{1}{2}}\right)_+ \tag{7.6.40}$$

where the subscript $+$ means that we expand formally the power series $\mathbf{Q}^{k-\frac{1}{2}} = \mathrm{d}^{2k-1}(1 - 2\mathrm{d}^{-1}u)^{k-\frac{1}{2}}$ and keep only the positive powers of d. In the same way, we define $A_k = \left(\mathbf{Q}^{k-1/2}\right)_- = \mathbf{Q}^{k-1/2} - P_k$.

It is easy to see that the commutator:

$$[P_k, \mathbf{Q}] = -[A_k, \mathbf{Q}]$$

contains no power of d (indeed, the left hand side contains only positive powers, and the right hand side negative powers), it must be a function of τ. Actually, it is a derivative, and we write:

$$[P_k, \mathbf{Q}] = R_k'(u) \tag{7.6.41}$$

$R_k(u)$ is the Gelfand-Dikii differential polynomial. The first of them are:

$$R_1 = -2u \qquad , \qquad R_2 = 3u^2 - \frac{1}{2}u'' \qquad , \qquad R_3 = -5u^3 + \frac{5}{4}u'^2 + \frac{5}{2}uu'' - \frac{1}{8}u^{(4)} \qquad , \qquad \dots \tag{7.6.42}$$

they satisfy a recurrence:

$$R_{k+1}' = \frac{1}{4}R_k''' - 2uR_k' - u'R_k \tag{7.6.43}$$

The general solution of the commutation equation is then:

$$\mathbf{P} = \sum_{k=1}^{m+1} t_k P_k \tag{7.6.44}$$

where the t_k are integration constants, and the string equation is:

$$\sum_{k=1}^{m+1} t_k R_k(u) = \tau \tag{7.6.45}$$

The critical point we were interested in was $t_k = 0$ for $k \leq m$ and $t_{m+1} = 1$:

$$R_{m+1}(u) = \tau \tag{7.6.46}$$

for $m = 1$ we recover the Painlevé I equation.

Remark: In multimatrix models, one can have higher order differential operators for Q:

$$\mathbf{Q} = \mathrm{d}^q - \sum_{k=1}^{q} u_k \mathrm{d}^{q-k} + \dots \qquad , \qquad \mathbf{P} = \mathrm{d}^p - \sum_{k=1}^{p} v_k \mathrm{d}^{p-k}$$

The same methods work.

See reference [3] for more details.

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