### Definition

A *Lie group* $G$ is a space which possesses two structures:
1) structure of a group;
2) structure of a smooth manifold.
These structures are compatible in the sense that the group operations ("multiplication" $a, b \to a \cdot b$ and "taking the inverse element" $a \to a^{-1}$) are both smooth.

Main examples of Lie groups are matrix groups:

- General linear group $GL(n, \mathbb{R})$;
- Special linear group $SL(n, \mathbb{R})$;
- Orthogonal groups $O(n, \mathbb{R})$, $SO(n, \mathbb{R})$;
- Symplectic group $Sp(2n, \mathbb{R})$;
- Unitary group $U(n)$, special unitary group $SU(n)$;
- Triangular group, Upper triangular group;
- Any intersection of the above groups

# Simplest examples: $SO(2)$ and $O(2)$

*Special orthogonal group:*

$$SO(2) = \{\ 2 \times 2 \text{ matrix } A \text{ s.t. } \ AA^{\top} = Id, \det A = 1\}$$

It is not hard to verify that

$$SO(2) = \left\{ \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},\ \phi \in \mathbb{R}\ \ (\text{or, equivalently, } [0, 2\pi]) \right\}$$

As a smooth manifold, $SO(2)$ is a circle.

Similarly, the *orthogonal group:*

$$O(2) = \{\ 2 \times 2 \text{ matrix } A \text{ s.t. } \ AA^{\top} = Id\} =$$

$$\left\{ \begin{pmatrix} \cos\phi & \mp\sin\phi \\ \sin\phi & \pm\cos\phi \end{pmatrix},\ \phi \in \mathbb{R}\ \ (\text{or, equivalently, } [0, 2\pi]) \right\}$$

This is a disjoint union of two circles.

# Simplest examples: $SL(2,\mathbb{R})$

Special linear group (in dimension 2):

$$SL(2,\mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det A = ad - bc = 1 \right\}$$

As a manifold, $SL(2,\mathbb{R})$ represents a second order hypersurface in $\mathbb{R}^4$.

## Proposition

$SL(2,\mathbb{R})$ is three-dimensional and diffeomorphic to the direct product $S^1 \times \mathbb{R}^2$.

Sketch of proof:
Changing variables $a = x + y$, $d = x - y$, $b = u + v$, $c = u - v$ gives

$$ad - bc = 1 \quad \longrightarrow \quad x^2 - y^2 - u^2 + v^2 = 1$$

Or,

$$x^2 + v^2 = 1 + y^2 + u^2$$

If $(y,u) \in \mathbb{R}^2$, then $(x,v)$ belongs to the circle $S^1$ (of variable radius) defined by the above equation.

# Simplest examples: $SU(2)$ and $U(2)$

Unitary group (in dimension 2):

$$U(2) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad a, b, c, d \in \mathbb{C} \mid A\bar{A}^{\top} = Id \right\}$$

Special unitary group (in dimension 2):

$$SU(2) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad a, b, c, d \in \mathbb{C} \mid A\bar{A}^{\top} = Id, \quad \det A = 1 \right\}$$

### Proposition

$SU(2)$ is three-dimensional and diffeomorphic to the 3-dim sphere $S^3$. $U(2)$ is four-dimensional and diffeomorphic to the direct product $S^3 \times S^1$.

## Sketch of proof

In terms of $a, b, c, d$, the condition $A\bar{A}^\top = Id$ is equivalent to 3 equations.
The first one, $a\bar{a} + b\bar{b} = 1$, means that the pair of complex numbers $(a = a_1 + ia_2, b = b_1 + ib_2) \in \mathbb{C}^2 = \mathbb{R}^4$ belongs to the sphere of radius 1 in the sense that $a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1$.
The second equation $c\bar{a} + d\bar{b} = 0$ means that $(c, d)$ is proportional to $(-\bar{b}, \bar{a})$.
In other words, $c = -\lambda\bar{b}$, $d = \lambda\bar{a}$.
Substituting to the third equation $c\bar{c} + d\bar{d} = 1$ gives $|\lambda| = 1$, or equivalently, $\lambda = e^{i\phi}$.
Thus, unitary matrices $A \in U(2)$ are of the form

$$A = \begin{pmatrix} a & b \\ -\lambda\bar{b} & \lambda\bar{a} \end{pmatrix}$$

Notice that the determinant of such a matrix is exactly $\lambda$, so the special unitary group $SU(2)$ is "simpler" and consists of matrices of the form

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

Thus, there is a natural bijection between $SU(2)$ and the 3-sphere $S^3 = \{(a, b) \in \mathbb{C}^2 \mid |a|^2 + |b|^2 = 1\}$.
In the case of $U(2)$ we have one additional (and independent!) parameter $\lambda = e^{i\phi}$, which is simply a point on the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. This basically means that $U(2)$ is (diffeomorphic to) the direct product $S^3 \times S^1$.

# Abelian Lie groups

## Definition

A group is called Abelian (or commutative) if the binary operation is commutative, that is, $a \cdot b = b \cdot a$ for any $a, b \in G$.

Two basic examples:

- Any (finite-dimensional) vector space $V$ over $\mathbb{R}$ carries a natural structure of an Abelian Lie group. The binary operation is the addition: $(\bar{u}, \bar{v}) \mapsto \bar{u} + \bar{v}$.
- A torus $T^n = S^1 \times S^1 \times \cdots \times S^1$ is also an Abelian Lie group. A point on $T^n$ is represented by an $n$-tuple $(\phi_1, \phi_2, \ldots, \phi_n)$ where each $\phi_k$ is understood as an "angle" defined modulo $2\pi$. The binary operation is simply the addition modulo $2\pi$:

$$(\phi_1, \ldots, \phi_n) + (\psi_1, \ldots, \psi_n) = ((\phi_1 + \psi_1) \mathrm{mod} 2\pi, \ldots, (\phi_n + \psi_n) \mathrm{mod} 2\pi)$$

Notice that $T^n$ can easily be presented as a matrix Lie group. We simply assign to each $n$-tuple $(\phi_1, \ldots, \phi_n)$ the following diagonal matrix:

$$\begin{pmatrix} e^{i\phi_1} & & \\ & \ddots & \\ & & e^{i\phi_n} \end{pmatrix}$$

# Remark

Notice that on a vector space, one can define many other structures of Lie groups (not necessarily commutative). For example, on $\mathbb{R}^3$ we can define a (non-commutative) binary operation as follows:

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, \quad x_2 + y_2, \quad x_3 + y_3 + x_1 y_2)$$

The smoothness of this operation is evident. All other axioms are straightforward.
This Lie group also has a natural matrix representation (i.e., matrix group which is isomorphic to it): This is just the group of upper-triangular matrices:

$$G = \left\{ A = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

Although these two Lie groups $(\mathbb{R}^3, +)$ and $(\mathbb{R}^3, \cdot)$ are diffeomorphic as smooth manifolds, they are quite different as Lie groups.

$$GL(n,\mathbb{R}) = \left\{ A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} : \quad \det A \neq 0 \right\}$$

It is a smooth manifold by a very simple reason: $GL(n.\mathbb{R})$ is an open subset in the vector space $\mathbb{R}^{n^2}$ of all $n \times n$-matrices.

Notice also that $GL(n,\mathbb{R})$ is not connected (as a topological space): it consists of two components $\{A \ : \ \det A > 0\}$ and $\{A \ : \ \det A < 0\}$.

### Proposition

In the simplest case $GL(2,\mathbb{R})$, each of these components is diffeomorphic to $S^1 \times \mathbb{R}^3$.

Indeed, any matrix $A \in GL(2,\mathbb{R})$ with $\det A > 0$ can uniquely be presented in the form $A = cA'$, where $c$ is a certain positive constant and $A' \in SL(2,\mathbb{R})$ (it is easy to see that $c = \sqrt{\det A}$).
Thus, comparing to $SL(2,\mathbb{R})$, the group $GL^+(2,\mathbb{R})$ has an additional parameter $c \in \mathbb{R}^+$. In other words, $GL^+(2,\mathbb{R}) \simeq SL(2,\mathbb{R}) \times \mathbb{R}^+$. Since $SL(2,\mathbb{R}) \simeq S^1 \times \mathbb{R}^2$ and $\mathbb{R}^+ \simeq \mathbb{R}$, we conclude $GL^+(2,\mathbb{R}) \simeq S^1 \times \mathbb{R}^3$.

### Definition

A (topological) manifold of dimension $m$ is a topological space $M$ such that each point $P \in M$ admits a neighborhood $U$ homeomorphic to an open subset $V \subset \mathbb{R}^m$.

$U$, together with a homeomorphism $\phi : U \to V \subset \mathbb{R}^m$, is called a *chart*.
A collection of charts covering $M$ is called an *atlas*.
Fixing a chart $U$ is equivalent to defining local coordinates $(x^1, \ldots, x^m)$ on $U$:

$$\phi(P) = (x^1, \ldots, x^m) \in V \subset \mathbb{R}^m$$

Thus, basically we may say that $M$ is a manifold if in a neighborhood of any point $P$ we can introduce local coordinates. Of course, this correspondence

$$P \longleftrightarrow (x^1, \ldots, x^m) = \text{ coordinates of } P$$

must be a bijection continuous in both directions (i.e., a homeomorphism).
We shall always assume two additional conditions:

- $M$ is a Hausdorff space, i.e., any two points $P, Q \in M$, $P \neq Q$, have disjoint neighborhoods.
- $M$ admits a countable atlas

# Smooth manifolds

Terminology: *Smooth* $=$ of class $C^\infty$

Motivation: Having local coordinates $(x^1, \dots, x^m)$ we may work with our manifold (at least locally, i.e., in $U$) in the same way as we do it in $\mathbb{R}^m$. However, the following "problem" appears: some points $P$ belong to several charts $U_1, U_2, \dots$. In other words, we have several choices for local coordinates. Clearly, we want them to be equivalent and we want to have the possibility to change them freely. In differential geometry, one of the main principles is that "properties of the objects we work with should not depend on the choice of local coordinates". For example, if we need to verify the smoothness of a certain object (function, vector field, etc.), we would like to be able to do this in *any local coordinate system* we wish.
This idea can be conceptualised in the following definition:

## Definition

A manifold $M$ is called *smooth* if the transition functions between any two local coordinate systems are smooth.

More precisely: Let $U$ and $U'$ be two intersecting charts, then on their intersection $U \cap U'$ we have two different coordinate systems $(x^1, \ldots, x^m)$ and $(x^{1'}, \ldots, x^{m'})$ and we can naturally define the transition functions between them

$$x^{1'} = h_1(x^1, \ldots, x^m), \ldots, x^{m'} = h_m(x^1, \ldots, x^m)$$

Formally, these functions can be defined as (the components of) the map

$$\phi' \circ \phi^{-1} : \phi(U \cap U') \to \phi'(U \cap U').$$

The smoothness of $M$ means that the functions

$$h_1(x^1, \ldots, x^m), \ldots, h_m(x^1, \ldots, x^m)$$

are smooth in usual sense (and it is so for any two intersecting charts).

# Examples of manifolds

- Vector space $\mathbb{R}^n$
- Any open subset in $\mathbb{R}^n$
- Graph of a smooth function (map)
- Two-dimensional surfaces
- Spheres $S^n$ and Projective spaces $\mathbb{R}P^n$
- Lie Groups
- Homogeneous spaces of Lie groups
- Subsets in $\mathbb{R}^n$ given by a system of equations (with certain "regularity" condition)

In $\mathbb{R}^n$, consider a system of equations:

$$\begin{cases} f_1(x_1, \dots, x_n) = 0, \\ \vdots \\ f_k(x_1, \dots, x_n) = 0. \end{cases}$$

where $f_i$ are smooth and $k \leq n$. Let $M \subset \mathbb{R}^n$ be the set of solutions.
Regularity condition:
The rank of the Jacoby matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \dots & \frac{\partial f_k}{\partial x_n} \end{pmatrix}$$

is maximal (that is, $= k$) at any point $P \in M$.

**Theorem**

*If the regularity condition holds, then $M$ carries the natural structure of a smooth manifold of dimension $n - k$.*

# Implicit function theorem: Example

## Example

Consider the group $O(3)$ as a subset in $\mathbb{R}^9$ (the space of $3 \times 3$-matrices). The condition $A \cdot A^\top = Id$ is a matrix equation which is equivalent to the system of 6 usual equations with 9 unknowns:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longrightarrow \left\{ \begin{array}{ll} f_1 : & a_{11}^2 + a_{12}^2 + a_{13}^2 = 1 \\ f_2 : & a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0 \\ f_3 : & a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0 \\ f_4 : & a_{21}^2 + a_{22}^2 + a_{23}^2 = 1 \\ f_5 : & a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0 \\ f_6 : & a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \end{array} \right.$$

Jacobi matrix $J$:

$$\begin{array}{cc} & \begin{array}{ccccccccc} a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} & a_{31} & a_{32} & a_{33} \end{array} \\ \begin{array}{c} df_1 \\ df_2 \\ df_3 \\ df_4 \\ df_5 \\ df_6 \end{array} & \begin{pmatrix} 2a_{11} & 2a_{12} & 2a_{13} & * & * & * & * & * & * \\ a_{21} & a_{22} & a_{23} & * & * & * & * & * & * \\ a_{31} & a_{32} & a_{33} & * & * & * & * & * & * \\ 0 & 0 & 0 & 2a_{21} & 2a_{22} & 2a_{23} & * & * & * \\ 0 & 0 & 0 & a_{31} & a_{32} & a_{33} & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 2a_{31} & 2a_{32} & 2a_{33} \end{pmatrix} \end{array}$$

Since the rows of $A$ are linearly independent (recall that $\det A = \pm 1$), the same is true for the Jacoby matrix $J$. So the rank of $J$ is maximal and equal to 6. According to the implicit function theorem, the equation $AA^{\top} = Id$ define a smooth manifold of dimension $9 - 6 = 3$ in $\mathbb{R}^9$.

**Example**

Consider the surface $V$, a cone, in $\mathbb{R}^3$ given by $x^2 + y^2 - z^2 = 0$. Compute the differential:

$$df = (2x, 2y, -2z).$$

The differential is nowhere zero except the point $O = (0, 0, 0)$. The implicit function theorem guarantees that locally $V$ has a structure of a smooth manifold at any $P \in V$ different from $O$. It is easy to see that $O = (0, 0, 0)$ is indeed singular in the sense that there is no neighborhood of $O$ in $V$ homeomorphic to a 2-disc.

Conclusion: $V$, as a whole, is not a smooth manifold because of the singular point $O$, this is exactly the point where the regularity condition of the implicit function theorem fails.

# Maps between manifolds

### Definition

Let $F : M \to N$ be a (continuous) map between two smooth manifolds. This map is called *smooth* (or $C^\infty$), if it is so in local coordinates. More precisely, this means the following. Let $P \in M$ and $Q = F(P) \in N$ be its image. Consider local coordinates $(x^1, \ldots, x^m)$ and $(y^1, \ldots, y^n)$ in neighborhoods of $P$ and $Q$ respectively. Then locally $F$ can be written in these local coordinates as:

$$(y^1, \ldots, y^n) = F(x^1, \ldots, x^m), \text{ i.e., } F = \begin{cases} y_1 = f_1(x^1, \ldots, x^m) \\ y_2 = f_2(x^1, \ldots, x^m) \\ \vdots \\ y_n = f_n(x^1, \ldots, x^m) \end{cases}$$

The smoothness of $F$ means that all the functions $f_1, \ldots, f_n$ are smooth.

### Remark

Since the transition functions between charts are smooth, the above definition does not depend of the choice of local coordinates $(x^1, \ldots, x^m)$ and $(y^1, \ldots, y^n)$ in neighborhoods of $P$ and $Q$. We may verify the smoothness condition in any local coordinates we wish.

### Definition

A *diffeomorphism* $F : M \to N$ is a smooth bijective map such that its inverse $F^{-1} : N \to M$ is also smooth.

The are several different ways to introduce the notion of a tangent vector. We shall use the following simple idea. If we deal with a manifold $M$ embedded in $\mathbb{R}^n$, then a tangent vector at a point $P \in M$ can be defined as just a tangent vector to a certain smooth curve $\gamma(t) \subset M$ passing through $P$. The set of all possible tangent vectors at $P$ is then the tangent space to $M$ at $P$.
Let $\gamma(t)$ be a smooth curve in $M$, i.e., a smooth map $\gamma : (-\epsilon, \epsilon) \to M$. In local coordinates, this map is given as $\gamma(t) = (x^1(t), \ldots, x^m(t))$. Then the tangent vector to $\gamma$ at point $P = \gamma(0)$ is defined to be simply

$$\frac{d\gamma}{dt}(0) = \left( \frac{dx^1}{dt}(0), \frac{dx^2}{dt}(0), \ldots, \frac{dx^m}{dt}(0) \right)$$

Thus, in local coordinates, any tangent vector $\vec{\xi}$ is given as an $m$-tuple $\xi = (\xi^1, \ldots, \xi^m)$, here $m = \dim M$. The problem, however, is that this definition depends on the choice of local coordinates. What happens to $(\xi^1, \ldots, \xi^m)$ if we change local coordinates $(x^1, \ldots, x^m) \to (x^{1'}, \ldots, x^{m'})$?

Standard computation: the tangent vector to $\gamma(t)$ in the new coordinates is

$$\left(\frac{dx^{1'}}{dt}, \ldots, \frac{dx^{m'}}{dt}\right) = \left(\sum_{i=1}^{m} \frac{\partial x^{1'}}{\partial x^i}\frac{dx^i}{dt}, \ldots, \sum_{i=1}^{m} \frac{\partial x^{m'}}{\partial x^i}\frac{dx^i}{dt}\right)$$

In other words, the "new" coordinates of the same tangent vector $\xi$ are:

$$(\xi^{1'}, \ldots, \xi^{m'}) = \left(\sum_{i=1}^{m} \frac{\partial x^{1'}}{\partial x^i}\xi^i, \ldots, \sum_{i=1}^{m} \frac{\partial x^{m'}}{\partial x^i}\xi^i\right) \tag{1}$$

## Definition

A *tangent vector* $\xi$ at a point $P$ is defined in any local coordinate system $(x^1, \ldots, x^m)$ as an $m$-tuple $(\xi^1, \ldots, \xi^m)$; in addition, it is required that the transformation law for the components of $\xi$ under coordinate change $(x^1, \ldots, x^m) \to (x^{1'}, \ldots, x^{m'})$ is given by (1).

## Remark

Notice that the transformation $(\xi^1, \ldots, \xi^m) \to (\xi^{1'}, \ldots, \xi^{m'})$ (at a fixed point $P$) is linear whereas the transformation $(x^1, \ldots, x^m) \to (x^{1'}, \ldots, x^{m'})$ is not.

# Another version

## Definition

A linear mapping $A : C^\infty(M) \to \mathbb{R}$ is called a *derivation* (first order differential operator) at point $P \in M$ if it satisfies the Leibnitz rule:

$$A(f \cdot h) = A(f) \cdot h(P) + A(h) \cdot f(P).$$

The relationship between tangent vectors and derivations is very simple and natural: Each tangent vector $\xi$ defines a derivation of this kind (called *directional derivative along* $\xi$) by:

$$A_\xi(f) = \frac{d}{dt}_{|t=0} f(\gamma(t)) = \sum_i \frac{\partial f}{\partial x^i} \xi^i,$$

where $\gamma(t)$ is a smooth curve such that $\gamma(0) = P$, $\dfrac{d\gamma}{dt}(0) = \xi$.
It is easy to see that the correspondence $\xi \mapsto A_\xi$ is a natural bijection between tangent vectors and derivations. Hence:

## Definition

A *tangent vector* to $M$ at point $P$ is defined to be a derivation at $P$. Usually, we shall denote $A_\xi$ simply by $\xi$ (since we identify them): $\xi(f) = \sum_i \xi^i \dfrac{\partial f}{\partial x^i}$.

Notation: $\xi = \sum_i \xi^i \dfrac{\partial}{\partial x^i}$.

# Tangent vectors

Three points of view:

- A tangent vector $\xi$ to $M$ at a point $P \in M$ is the velocity vector of a smooth curve $\gamma(t) \subset M$ passing through $P$, i.e.,

$$\xi = \left.\frac{d\gamma}{dt}\right|_{t=0}, \quad \text{where } \gamma(0) = P.$$

- If $(x^1, \ldots, x^m)$ is a local coordinate system in a neighbourhood of $P$, then a tangent vector $\xi$ is an $n$-tuple $(\xi^1, \xi^2, \ldots, \xi^m) \in \mathbb{R}^m$.
  Important: if we change local coordinates $(x) \longrightarrow (x')$, then the components of $\xi$ change according to the following rule:

$$\xi^{i'} = \sum_{i+1}^{m} \frac{\partial x^{i'}}{\partial x^i}(P)\,\xi^i, \quad \text{or equivalently} \quad \xi' = J\,\xi,$$

  where $J$ is the Jacobi matrix of the transition map $(x) \longrightarrow (x')$.
- A tangent vector is a derivation (at the point $P$), i.e., a linear map

$$\xi : C^\infty(M) \to \mathbb{R}, \quad \text{such that} \quad \xi(f\,g) = \xi(f)\cdot g(P) + \xi(g)\cdot f(P).$$

# Differential of a smooth map

Let $F : M \to N$ be a smooth map, $P \in M$ and $Q = F(P) \in N$.

## Definition

The *differential* $dF$ (of $F$ at the point $P \in M$) is a linear map between the tangent spaces $T_P M$ and $T_Q N$:

$$dF_{|P} : T_P M \longrightarrow T_Q N$$

which is defined in one of the following equivalent ways:

- take $\xi \in T_P(M)$ and chose any curve $\gamma(t)$ through $P$ such that $\gamma'(0) = \xi$, then $\eta = dF(\xi)$ is the tangent vector to the image of $\gamma$ at the point $Q$:

$$\eta = dF(\xi) = \frac{d}{dt}_{|t=0} F(\gamma(t)) \quad \in T_Q N;$$

- consider $\xi \in T_P(M)$ as a derivation, then $\eta = dF(\xi)$ is the derivation at $Q \in N$ which acts on an arbitrary function $g : N \to \mathbb{R}$ as follows:

$$\eta(g) = \xi(g \circ F);$$

- choose local coordinates $(x^1, \dots, x^m)$ and $(y^1, \dots, y^n)$ in some neighborhoods of $P$ and $Q$ respectively; let $\xi = (\xi^1, \dots, \xi^m)$, then the components of $\eta = dF(\xi)$ are

$$\eta^j = \sum_{i=1}^{m} \frac{\partial f_j}{\partial x^i} \xi^i, \qquad j = 1, \dots, n,$$

In matrix form:

$$\eta = dF(\xi) = J \cdot \xi, \qquad \text{where } J \text{ is the Jacobi matrix of } F \text{ at the point } P$$

That is,

$$dF(\xi) = \begin{pmatrix} \eta^1 \\ \vdots \\ \eta^n \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial x^2} & \cdots & \frac{\partial f_1}{\partial x^m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x^1} & \frac{\partial f_n}{\partial x^2} & \cdots & \frac{\partial f_n}{\partial x^m} \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \vdots \\ \xi^m \end{pmatrix}$$

Here by $f_i$, we denote the functions which (locally) define $F$ in the chosen coordinate systems:

$$y = F(x) \quad \simeq \quad \begin{cases} y^1 = f_1(x^1, \ldots, x^m), \\ \vdots \\ y^n = f_n(x^1, \ldots, x^m) \end{cases}$$

In other words, the matrix of the linear map $dF$ is $J$, the Jacobi matrix of $F$.

# Lie groups (local aspect)

## Definition

A *Lie group* $G$ is a group which is a smooth manifold such that the group operations, i.e., multiplication and inversion, are smooth.

Standard notation for the identity: $e \in G$.

Multiplication in local coordinates: Let $U$ be a neighborhood of $e \in G$ with some local coordinates. Assume that the coordinates of $e$ are $(0, 0, \ldots, 0)$. Take an element $x \in U$ with coordinates $(x^1, \ldots, x^n)$ and another element $y$ with coordinates $(y^1, \ldots, y^n)$. If we multiply them $x, y \mapsto z = x \cdot y$, then the components of $z$ are smooth functions of $x$ and $y$:

$$z^k = z^k(x^1, \ldots, x^n; y^1, \ldots, y^n)$$

It is interesting and useful to look at the Taylor expansion of $z^k$.

## Proposition

$$z^k = x^k + y^k + \sum_{i,j} b^k_{ij} x^i y^j + \ldots \text{(terms of degree} \geq 3) \tag{1}$$

Similarly, if we consider the inversion map $x \mapsto u = x^{-1}$ then the first terms for the Taylor expansion of the components of $u$ are (check it!):

$$u^k = -x^k + \sum_{i,j} b^k_{ij} x^i x^j + \ldots \text{(terms of degree} \geq 3)$$

# Proof of the formula (1)

In general, the Taylor expansion of an arbitrary smooth function $z^k = z^k(x^1, \dots, x^n; y^1, \dots, y^n)$ can be written as:

$$z^k = a_0 + l_j x^j + m_j y^j + a_{ij} x^i x^j + b_{ij} x^i y^j + c_{ij} y^i y^j + \dots$$

(with summation over $i$ and $j$). Substituting $x = e = (0, 0, \dots, 0)$, we have $z = e \cdot y = y$, i.e.,

$$y^k = a_0 + m_j y^j + c_{ij} y^i y^j + \dots$$

Hence,
$a_0 = 0$,
$m_j = 0$ for all $j$ except for $j = k$ and $m_k = 1$,
$c_{ij} = 0$.

Similarly, substituting $y = e = (0, 0, \dots, 0)$ gives:
$l_j = 0$ for all $j$ except for $j = k$ and $l_k = 1$,
$a_{ij} = 0$.

Thus,

$$z^k = x^k + y^k + \sum_{i,j} b_{ij} x^i y^j + \dots,$$

as needed. (The coefficients $b_{ij}$ are different for different $z^k$'s, that is why we denote them by $b^k_{ij}$ in (1)).

# Component of the identity

Any smooth manifold $M$ can be presented as a disjoint union of connected (equivalently, path-connected) manifolds called *connected components* of $M$:

$$M = M_1 \cup M_2 \cup \ldots, \quad M_i \neq \emptyset, \quad M_i \cap M_j = \emptyset$$

Let $G$ be a Lie group, $G = G_0 \cup G_1 \cup \ldots$ be the decomposition into connected components, and $G_0 \subset G$ be the connected component that contains the identity element $e \in G$. In other words, $G_0$ is a connected open-closed subset which contains $e$. Notice that such a subset is unique and can be characterized as the set of those points which can be joined with $e$ by a continuous path.

### Theorem

- *$G_0$ is a normal subgroup in $G$;*
- *$G_0$ itself is a Lie group;*
- *any other connected component $G_i$ of $G$ is diffeomorphic to $G_0$.*

## Proof of the first statement

Why a subgroup? As usual, we need to verify two conditions:
(i) $G_0$ is closed under multiplication, i.e. for any $a, b \in G_0$, their product $a \cdot b$ also belongs to $G_0$.
(ii) $G_0$ is closed under the inversion, i.e. for any $a \in G_0$, its inverse $a^{-1}$ also belongs to $G_0$.

Proof of (i): Since $a, b \in G_0$ and $G_0$ is (path)-connected, there are continuous curves $a(t)$ and $b(t)$ which connect $e$ with $a$ and $b$ respectively, i.e. $a(0) = e$, $a(1) = a$ and $b(0) = e$, $b(1) = b$. Then $c(t) = a(t) \cdot b(t)$ is a continuous curve connecting $e$ with $a \cdot b$. Thus $a \cdot b$ belongs to the connected component that contains $e$, i.e. $G_0$.

Proof of (ii) is similar: It suffices to consider the continuous curves $a(t)$ and $c(t) = (a(t))^{-1}$.

Why normal? We need just to verify that for any $g \in G$ we have $gG_0g^{-1} = G_0$. Since the left multiplication by $g$ and right multiplication by $g^{-1}$ are both diffeomorphisms, then $gG_0g^{-1}$ is obviously a certain connected component of $G$. But the set $gG_0g^{-1}$ contains the identity $e$ (because $geg^{-1} = e$), so this connected component coincides with $G_0$. Thus, $gG_0g^{-1} = G_0$, as needed. Hence $G_0$ is a normal subgroup.

**Theorem**

*Let $U$ be any connected neighborhood of $e \in G$. Then $G_0$ is generated by $U$ in the sense that any element $x \in G_0$ can be represented as $x = u_1 u_2 \dots u_k$ with $u_i \in U$.*

**Remark**

1) If $U$ is a neighborhood of $e$, then $a \cdot U$ and $U \cdot a$ are both neighborhoods of $a \in G$. These neighborhoods are naturally homeomorphic (and diffeomorphic) to $U$, but in general do not coincide.
2) The set $U^{-1} = \{u^{-1} \mid u \in U\}$ is a neighborhood of the identity $e$.

Proof. Consider the subsets $U^1 = U$, $U^2 = U \cdot U = \{u_1 \cdot u_2 \mid u_1, u_2 \in U\}, \dots,$
$U^k = \underbrace{U \cdot U \cdot \ldots \cdot U}_{k \text{ times}} = \{u_1 \cdot u_2 \cdot \ldots \cdot u_k \mid u_1, \dots, u_k \in U\}$
and take $L = \cup_{k=1}^{\infty} U^k$. We need to prove that $L = G_0$.
1) Clearly, $L$ is an open set (as union of open subsets).
2) $L$ is closed, i.e., the complement $G \setminus L$ is open. Indeed, assume $a \in G \setminus L$ and consider its neighborhood $a \cdot U^{-1}$. Check that this neighborhood does not intersect with $L$. By contradiction: if $a \cdot U^{-1} \cap L \neq \emptyset$ then there exist $u$ and $u_1, \dots, u_k$ such that $a \cdot u^{-1} = u_1 \cdot \ldots \cdot u_k$ or equivalently, $a = u_1 \cdot \ldots \cdot u_k \cdot u$. That is $a \in U^{k+1} \subset L$. Contradiction.

3) $L$ is connected.

We need to show that for any product $x = u_1 u_2 \dots u_k \in L$ there exists a continuous path $\gamma(t) \subset L$ which connects $x$ with the identity element $e$. But this is obvious: we simply can take $\gamma(t) = \gamma_1(t) \cdot \gamma_2(t) \cdot \dots \cdot \gamma_k(t)$ where $\gamma_i(t)$ is a continuous path in $U$ that connects $e$ and $u_i$, i.e. $\gamma_i(0) = e$, $\gamma_i(1) = u_i$. Then obviously

$$\gamma(0) = e, \quad \gamma(1) = x, \quad \text{and} \quad \gamma(t) \in L \text{ for } t \in [0,1],$$

as needed.

Thus, $L$ is a connected open-closed subset which contain the identity $e \in G$. Hence (by definition!) $L$ is the connected component of the identity, i.e., $G_0$.

**Definition**

An *algebraic linear group* $G$ is a subgroup in $GL(n,\mathbb{R})$ which is defined by a system of polynomial equations

$$\begin{cases} p_1(a_{11}, a_{12}, \dots, a_{nn}) = 0, \\ \vdots \\ p_m(a_{11}, a_{12}, \dots, a_{nn}) = 0, \end{cases}$$

where $a_{ij}$ are matrix components of $A \in GL(n,\mathbb{R})$.

**Example**

$SL(n,\mathbb{R})$ is given just by one equation $\det A = 1$.

**Example**

$O(n)$ is given by one matrix equation $A \cdot A^\top = Id$ which amounts to $\dfrac{n(n+1)}{2}$ usual equations (see Example in Lecture 2). In the case of $SO(n)$, one additional equation appears: $\det A = 1$.

**Example**

The upper triangular group is given by $\dfrac{n(n-1)}{2}$ equations of the form $a_{ij} = 0$, $1 \leq j < i \leq n$.

The following theorem gives us a lot of examples of Lie groups.

**Theorem**

*Any algebraic linear group $G \subset GL(n, \mathbb{R})$ is a Lie subgroup in $GL(n, \mathbb{R})$. In particular, $G$ is a Lie group.*

**Sketch of proof and discussion.**

The main statement of the theorem is that $G$ is a smooth submanifold in $GL(n, \mathbb{R})$ (the rest is obvious).

As we know, algebraic submanifolds may have singular points (recall Example in Lecture 2: the cone in $\mathbb{R}^3$ given by $x^2 + y^2 - z^2 = 0$). Thus, we need to explain why they do not appear if, in addition, we assume that our algebraic submanifold is a subgroup?

The reason can be formulated as: HOMOGENEITY

In this context this simply means that any subgroup $G \in GL(n, \mathbb{R})$ has the same topological (and differential) structure at each of its points.

More precisely: Take a neighborhood of $e$ in $G$ as $U_e = G \cap V \subset G$ where $V$ is a certain neighborhood of $e = Id$ in $GL(n, \mathbb{R})$.

Then for any other point $x \in G$, we can consider its neighborhood $U_x$ obtained from $U_e$ by left translation: $U_x = x \cdot U_e = x \cdot (G \cap V) = G \cap (x \cdot V)$. But the left translation by $x$ is a diffeomorphism of $GL(n, \mathbb{R})$, so $U_x$ and $U_e$ are absolutely "isomorphic" from both topological and differential viewpoints.

Conclusion: If locally (in a neighborhood of $e$) $G$ has a structure of a smooth submanifold of $GL(n, \mathbb{R})$, then this condition will hold at any other point $x \in G$. Moreover, if there is at least one "good" point $x_0 \in G$ (not necessarily the identity $e$), then all points $x \in G$ are "good" and, consequently, $G$ has (globally!) the structure of a smooth submanifold in $GL(n, \mathbb{R})$.
Now it remain to notice that although an algebraic submanifold $V$ may have singular points, almost all of its points are still non-singular (i.e., such that locally $V$ is smooth).
Since $G$ is algebraic, non-singular points in $G$ exist and therefore, according to the homogeneity principle, all of them are non-singular.

Two results in the same spirit (each of which immediately implies the above theorem):

### Theorem (Cartan)

*Any closed subgroup of a Lie group is a smooth submanifold and, therefore, a Lie subgroup.*

### Theorem (A. Skopenkov)

*Let $M \subset \mathbb{R}^n$ be a closed subset satisfying the following homogeneity condition. For any two points $a, b \in M$ there are neighborhoods $U_a, U_b \subset \mathbb{R}^n$ and a diffeomorphism $F$ sending $U_a$ onto $U_b$ such that $F(U_a \cap M) = U_b \cap M$ and $F(a) = b$. Then $M$ is a smooth submanifold in $\mathbb{R}^n$.*

# Preliminaries

Consider a smooth manifold $M$ with a smooth vector field $\xi$ on it. This vector field defines a system of autonomous ODE on $M$:

$$\frac{dx}{dt} = \xi(x) \tag{1}$$

A smooth curve $\gamma(t)$, $t \in (-\epsilon, \epsilon)$ is an *integral curve* of $\xi$, if $\frac{d\gamma}{dt}(t) = \xi(\gamma(t))$ (in other words, $\gamma$ is a solution of (1)).

Existence and uniqueness theorem. For any $x \in M$ there is a unique integral curve $\gamma_x(t)$ passing through it (i.e. such that $\gamma_x(0) = x$). In general, such an integral curve is defined only for $t$ from a "small" interval around zero.

Flow $\Phi^t$: To each vector field $\xi$, we can assign (at least locally) a diffeomorphism $\Phi^t$ which shifts each point $x$ along $\xi$ by time $t$, in other words

$$\Phi^t(x) = \gamma_x(t).$$

"Locally" means that, in general, for a given $x \in M$, the flow $\Phi^t$ is defined only for sufficiently small $t$ (depending on $x$!).

Completeness: If each integral curve $\gamma_x(t)$ can be extended (in the sense of $t$) to $\mathbb{R} = (-\infty, \infty)$, then $\xi$ is called complete. Equivalently, completeness of $\xi$ means that the flow $\Phi^t : M \to M$ is globally defined for all $t \in \mathbb{R}$.

Important property: $\Phi^t \circ \Phi^s = \Phi^{t+s}$, i.e., the flow $\Phi^t$ can be understood as a one-parameter group of diffeomorphisms.

The differential of a smooth map $F : M \to N$ is the map $dF : TM \to TN$ defined by

$$dF\left(\frac{d}{dt}\gamma(t)\right) = \frac{d}{dt}F(\gamma(t)).$$

Lie bracket of vector fields: Given two smooth vector fields $\xi$ and $\eta$ on $M$, we can introduce a new vector field $[\xi, \eta]$:

- $[\xi, \eta\,]^k = \xi^i \frac{\partial \eta^k}{\partial x^i} - \eta^i \frac{\partial \xi^k}{\partial x^i}$ (in local coordinates)
- $[\xi, \eta\,](f) = \xi\big(\eta(f)\big) - \eta\big(\xi(f)\big)$ (in terms of derivations)

Important (but obvious) property: If $F : M \to N$ is a diffeomorphism, $dF : TM \to TN$ is its differential, then

$$dF([\xi, \eta]) = [dF(\xi), dF(\eta)]. \tag{2}$$

Another important property: Vector fields $\xi$, $\eta$ commute, i.e., $[\xi, \eta] = 0$, if and only if the corresponding flows commute, i.e. $\Phi^t_\xi \circ \Phi^s_\eta = \Phi^s_\eta \circ \Phi^t_\xi$.

# Left and right translations

Throughout the course we shall use the following notation:

$$
\begin{array}{lll}
L_a : G \to G, & x \mapsto a \cdot x & (\text{left translation by } a \in G) \\
R_a : G \to G, & x \mapsto x \cdot a & (\text{right translation by } a \in G)
\end{array}
$$

Clearly, $L_a$ and $R_a$ are diffeomorphisms of $G$ onto itself.
The corresponding differentials will be denoted by $dL_a$ and $dR_a$:

$$dL_a : TG \to TG \quad \text{and} \quad dR_a : TG \to TG$$

We shall use the same notation for the differential at a fixed point $x \in G$:

$$dL_a : T_xG \to T_{ax}G \quad \text{and} \quad dR_a : T_xG \to T_{xa}G$$

Left and right translations commute:

$$L_a \circ R_b(x) = a \cdot x \cdot b = R_b \circ L_a(x) \text{ for any } a, b \in G.$$

But, in general,

$$L_a \circ L_b \neq L_b \circ L_a, \quad R_a \circ R_b \neq R_b \circ R_a.$$

Also notice that

$$L_a \circ L_b = L_{ab}, \quad R_a \circ R_b = R_{ba}.$$

Similarly, for the differentials:

$$dL_a \circ dL_b = dL_{ab}, \quad dR_a \circ dR_b = dR_{ba}.$$

### Definition

A vector field $\xi$ is called *left invariant*, if it is preserved under left translations, i.e., for any $a \in G$:

$$dL_a(\xi) = \xi.$$

Similarly, a vector field $\eta$ is called *right invariant*, if $dR_a(\eta) = \eta$ for any $a \in G$.

In other words, if we consider the values $\xi(x)$ and $\xi(y)$ of our vector field $\xi$ at two distinct points $x \in G$ and $y = ax \in G$, then they must be related by the linear operator $dL_a : T_xG \to T_yG$, i.e., $dL_a(\xi(x)) = \xi(ax)$. (Similarly for right invariant vector fields: $dR_a(\xi(x)) = \xi(xa)$).

Construction: Take an arbitrary tangent vector $\xi_0 = \xi(e) \in T_eG$ at the identity $e \in G$ and then define a tangent vector $\xi(a) \in T_aG$ at any other point $a \in G$ by putting

$$\xi(a) = dL_a(\xi_0).$$

As a result, we obtain a certain tangent vector $\xi(a)$ for any $a \in G$, i.e., a vector field $\xi$ on $G$. Clearly, $\xi$ is smooth (since $L_a$ depends on $a$ smoothly).

**Proposition**

The vector field $\xi$ on $G$ so obtained is left invariant.

*Proof.* We only need to verify the condition $dL_a(\xi(x)) = \xi(ax)$ for any $x, a \in G$. Notice that for $x = e$, this condition holds by construction. For any other $x \in G$ we have

$$dL_a(\xi(x)) = dL_a\left(dL_x(\xi_0)\right) = dL_a \circ dL_x(\xi_0) = dL_{ax}(\xi_0) = \xi(ax).$$

**Corollary**

*A left invariant vector field $\xi$ is uniquely defined by its "initial" value $\xi_0 = \xi(e)$ at the identity $e \in G$. Moreover, $\xi_0$ can be chosen arbitrarily.*

**Corollary**

*The set of left invariant vector fields is a vector space of dimension $n = \dim G$, which is naturally isomorphic to the tangent space $T_eG$ to $G$ at the identity $e$. This isomorphism is established by the above construction:*

$$\xi_0 \mapsto \xi, \ \textit{where}\ \xi(a) = dL_a(\xi_0).$$

Let $\xi$ be a left invariant vector field on $G$. (All statements below can naturally be reformulated for right invariant vector fields.)

### Proposition

Let $\gamma_e(t)$ be the integral curve of $\xi$ passing through the identity $e$ (i.e., $\gamma_e(0) = e$). Then the integral curve of $\xi$ passing through $x$ is $\gamma_x(t) = x \cdot \gamma_e(t) = L_x(\gamma_e(t))$.

*Proof.*

$$\frac{d}{dt}\gamma_x(t) = \frac{d}{dt}(L_x(\gamma_e(t))) = dL_x(\frac{d}{dt}\gamma_e(t)) = dL_x(\xi(\gamma_e(t))) = \xi(x \cdot \gamma_e(t)) = \xi(\gamma_x(t)).$$

### Corollary

*The left translation of any integral curve of $\xi$ is again an integral curve.*

### Corollary

*The flow $\Phi_\xi^t : G \to G$ of $\xi$ is defined by*

$$\Phi_\xi^t(x) = x \cdot \gamma_e(t), \tag{3}$$

*where $\gamma_e(t)$ is the integral curve of $\xi$ through the identity.*

# Properties of left (right) invariant vector fields

## Proposition

$\xi$ is complete, i.e., the flow $\Phi_\xi^t : G \to G$ of $\xi$ is well defined for all $t \in \mathbb{R}$.

Proof. Assume that the integral curve $\gamma_e(t)$ is defined for $t \in (-\epsilon, \epsilon)$. Formula $\Phi_\xi^t(x) = x \cdot \gamma_e(t)$ shows that the flow $\Phi_\xi^t$ is well defined on the whole group $G$ for $t \in (-\epsilon, \epsilon)$. Then $\Phi^t$ can naturally be defined for all $t \in (-\infty, \infty)$ just by iterating:

$$\Phi_\xi^t = \underbrace{\Phi_\xi^{t/k} \circ \cdots \circ \Phi_\xi^{t/k}}_{k \text{ times}},$$

where $k$ is sufficiently large so that $t/k \in (-\epsilon, \epsilon)$.

## Definition

A smooth map $f : \mathbb{R} \to G$ is called a *one-parameter subgroup* of $G$, if $f(t+s) = f(t) \cdot f(s)$, for any $t, s \in \mathbb{R}$.

## Proposition

$\gamma_e(t)$ is a one-parameter subgroup.

Proof.

$$\gamma_e(t+s) = \Phi_\xi^{t+s}(e) = \Phi_\xi^{s+t}(e) = \Phi_\xi^s \circ \Phi_\xi^t(e) = \Phi_\xi^s(\gamma_e(t)) = \gamma_e(t) \cdot \gamma_e(s).$$

## Proposition (converse)

Let $f : \mathbb{R} \to G$ be a one-parameter subgroup such that $\frac{df}{dt}(0) = \xi_0$, then $f(t)$ is exactly the integral curve (through $e$) of the left invariant vector field $\xi$ on $G$ associated with the "initial" vector $\xi_0 \in T_eG$.

Proof.

$$\frac{df}{dt}(t) = \frac{d}{ds}_{|s=0} f(t+s) = \frac{d}{ds}_{|s=0} f(t)\cdot f(s) =$$

$$\frac{d}{ds}_{|s=0} L_{f(t)} f(s) = dL_{f(t)}\left(\frac{d}{ds}_{|s=0} f(s)\right) = dL_{f(t)}(\xi_0) = \xi(f(t)).$$

Conclusion. For each initial tangent vector $\xi_0 \in T_eG$, there is a unique one-parameter subgroup $f : \mathbb{R} \to G$ such that $\frac{df}{dt}(0) = \xi_0$. This subgroup is exactly the integral curve of the left (and right!) invariant vector field associated with $\xi_0$. In particular, integral curves of left and right invariant vector fields passing through the identity are the same.

Important notation. The one-parameter subgroup $f(t)$ with the initial vector $\xi_0 = f'(0) \in T_eG$ is denoted by $\exp(t\xi_0)$ (see next lecture for comments).

## Proposition

Any left invariant vector field $\xi$ commute with any right invariant vector field $\eta$, that is, $[\xi, \eta] = 0$.

Proof. It follows immediately from the fact that the corresponding flows $\Phi_\xi^t$ and $\Phi_\eta^s$ commute. Indeed, from (3) we have: $\Phi_\xi^t(x) = x \cdot \exp(t\xi_0)$ and, similarly, $\Phi_\eta^s(x) = \exp(s\eta_0) \cdot x$ for the right invariant field $\eta$. Hence:

$$\Phi_\xi^t \circ \Phi_\eta^s(x) = \exp(s\eta_0) \cdot x \cdot \exp(t\xi_0) = \Phi_\eta^s \circ \Phi_\xi^t(x).$$

## Proposition

The Lie bracket of two left invariant vector fields $\xi_1$ and $\xi_2$ is again a left invariant vector field.

Proof. It follows immediately from the fact that "the Lie bracket is preserved under diffeomorphisms". Indeed, from (2) for any $a \in G$ we have:

$$dL_a([\xi_1, \xi_2]) = [dL_a(\xi_1), dL_a(\xi_2)] = [\xi_1, \xi_2],$$

i.e., $[\xi_1, \xi_2]$ is left invariant.

# Summary

- The space of left (right) invariant vector fields is naturally isomorphic (as a vector space) to the tangent space $T_e G$ at the identity.
- Left (right) invariant vector fields are complete
- Integral curves of left (right) invariant vector fields through the identity $e \in G$ are exactly one-parameter subgroups
- The flow on $G$ generated by a left invariant vector field $\xi$ can be written in the form $\Phi_\xi^t(a) = a \cdot \exp(t\xi_0)$, where $\xi_0 = \xi(e)$
- Any left invariant vector field commute with any right invariant vector field
- The space of left invariant vector fields is closed under the Lie bracket

# Lie bracket for left and right invariant vector fields

### Proposition

Any left invariant vector field $\xi$ commute with any right invariant vector field $\eta$, that is, $[\xi, \eta] = 0$.

Proof. It follows immediately from the fact that the corresponding flows $\Phi_\xi^t$ and $\Phi_\eta^s$ commute. Indeed, from (**??**) we have: $\Phi_\xi^t(x) = x \cdot \exp(t\xi_0)$ and, similarly, $\Phi_\eta^s(x) = \exp(s\eta_0) \cdot x$ for the right invariant field $\eta$. Hence:

$$\Phi_\xi^t \circ \Phi_\eta^s(x) = \exp(s\eta_0) \cdot x \cdot \exp(t\xi_0) = \Phi_\eta^s \circ \Phi_\xi^t(x).$$

### Proposition

The Lie bracket of two left invariant vector fields $\xi_1$ and $\xi_2$ is again a left invariant vector field.

Proof. It follows immediately from the fact that "the Lie bracket is preserved under diffeomorphisms". Indeed, from (**??**) for any $a \in G$ we have:

$$dL_a([\xi_1, \xi_2]) = [dL_a(\xi_1), dL_a(\xi_2)] = [\xi_1, \xi_2],$$

i.e., $[\xi_1, \xi_2]$ is left invariant.

# Summary

- The space of left (right) invariant vector fields is naturally isomorphic (as a vector space) to the tangent space $T_e G$ at the identity.
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- The flow on $G$ generated by a left invariant vector field $\xi$ can be written in the form $\Phi_\xi^t(a) = a \cdot \exp(t\xi_0)$, where $\xi_0 = \xi(e)$
- Any left invariant vector field commute with any right invariant vector field
- The space of left invariant vector fields is closed under the Lie bracket

# The case of matrix groups:

Important remark: Let $G \subset GL(n, \mathbb{R})$ be a matrix Lie group. Then a tangent vector to $G$ at $X_0 \in G$ can (and will) be considered as a certain $n \times n$ matrix. We use our usual idea: tangent vectors to $G$ are tangent vectors to curves lying in $G$. A curve passing through $X_0$ is a family of matrices $X(t) \in G$ (smoothly depending on $t$), $X(0) = X_0$, so its derivative w.r.t. $t$ is again an $n \times n$ matrix:

$$X(t) = \begin{pmatrix} x_{11}(t) & \dots & x_{1n}(t) \\ \vdots & \ddots & \vdots \\ x_{n1}(t) & \dots & x_{nn}(t) \end{pmatrix} \longrightarrow \frac{dX}{dt}(0) = \begin{pmatrix} x'_{11}(0) & \dots & x'_{1n}(0) \\ \vdots & \ddots & \vdots \\ x'_{n1}(0) & \dots & x'_{nn}(0) \end{pmatrix} \in T_{X_0}G$$

The tangent space $T_X G$ is a certain subspace in $M_{n,n}$ (space of all matrices) which depends on $X$. However, if $G = GL(n, \mathbb{R})$, then the tangent space $T_X GL(n, \mathbb{R})$ coincides with $M_{n,n}$ at each point $X$.

## Proposition

Let $A$ be an arbitrary $n \times n$-matrix viewed as a tangent vector to $GL(n, \mathbb{R})$ at $E = Id$. Then the corresponding left invariant vector field on $GL(n, \mathbb{R})$ is $\xi(X) = XA$.

Proof. By construction, $\xi(X) = dL_X(A) = XA$ (left multiplication by $X$ is a linear map, so it coincides with its own differential).

## Proposition

Let $A$ be an arbitrary $n \times n$-matrix viewed as a tangent vector to $GL(n, \mathbb{R})$ at $E = Id$. Then the corresponding one-parameter subgroup is given as

$$\exp(tA) = e^{tA} = E + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \dots$$

Proof. First of all, it is an easy exercise to check that this series converges for any $A$ and $t$ absolutely (and uniformly on any interval $t \in (-T, T)$). Moreover, the resulting matrix function is smooth w.r.t. $t$ and

$$\frac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A. \tag{1}$$

Besides,

$$e^{(t+s)A} = e^{tA} \cdot e^{sA}. \tag{2}$$

Now there are two ways to complete the proof.

- (1) means that $e^{tA}$ is an integral curve of the left invariant vector field $\xi(X) = XA$ through the identity;
- (2) means that $e^{tA}$ is a one-parameter subgroup (with the initial tangent vector $A$).

Natural question: What are analogs of these two statements in the case of an arbitrary matrix Lie group $G \subset GL(n, \mathbb{R})$?

Answer: These statements holds for any $G \subset GL(n, \mathbb{R})$ without any change.

This immediately follows from the uniqueness condition: for any tangent vector $\xi_0 \in T_e G$, there is a unique one-parameter subgroup $f : \mathbb{R} \to G$ such that $\frac{df}{dt}(0) = \xi_0$ and there is a unique left invariant vector field $\xi$ such that $\xi(e) = \xi_0$.

**Corollary**

*Let $\mathfrak{g} = T_E G \subset M_{n,n}$ be the tangent space of a matrix group $G \subset GL(n, \mathbb{R})$ at the identity. Then the tangent space at any other point $X \in G$ is:*

$$T_X G = X \cdot \mathfrak{g} = \mathfrak{g} \cdot X$$

**Corollary**

*In the above notation, consider a system of linear ODE*

$$\frac{dx}{dt} = A(t)x, \quad x \in \mathbb{R}^n.$$

*Let $A(t) \in \mathfrak{g}$ for any $t \in \mathbb{R}$. Then the fundamental solution $X(t)$ belongs to $G$ for any $t \in \mathbb{R}$ (recall that by the fundamental solution we mean $X'(t) = A(t)X(t)$ and $X(0) = E$).*

# Exponential map

## Definition

The exponential map $\exp : T_eG \to G$ is defined by

$$\exp(\xi_0) = \exp(t\xi_0)_{|t=1},$$

where $\xi_0 \in T_eG$ and $\exp(t\xi_0)$, as before, denotes the one-parameter subgroup in $G$ with the initial vector $\xi_0$.

## Remark

Our notation $\exp(t\xi_0)$ can now be understood in two different ways: as the image under "exp" of the tangent vector $t\xi_0$ or as the point on the one-parameter subgroup $\exp(t\xi_0)$ with parameter $t$. In fact, these points coincide (check it!), so the notation causes no confusion.

Properties of the exponential map:

- exp is smooth and globally defined on $T_eG$ as a whole;
- the differential of exp at zero is the identity operator:

$$d\exp : T_eG \to T_eG, \quad d\exp(\xi_0) = \xi_0;$$

- exp is a local diffeomorphism at a neighborhood of zero.

# Parallelizability

Left (or right) translation allows us to identify the tangent space $T_x G$ at an arbitrary point $x \in G$ with $T_e G$. This identification is natural and implies the following topological property of Lie groups.

Theorem

*Any Lie group $G$ is parallelizable, i.e., its tangent bungle $TG$ is trivial:*

$$TG \simeq G \times \mathbb{R}^n, \qquad (n = \dim G).$$

Proof. In general, the triviality of the (tangent) bundle $TM$ means that there is is a smooth map

$$\phi : M \times \mathbb{R}^n \to TM$$

which is linear on each fiber, i.e., $\phi(x, \xi) = (x, A_x(\xi))$, where $A_x : \mathbb{R}^n \to T_x M$ is a linear isomorphism (that identifies $T_x M$ with a fixed vector space $\mathbb{R}^n$).
In our case, such a map is given by

$$\phi : G \times \mathbb{R}^n \to TG, \quad \phi(x, \xi_0) = (x, dL_x(\xi_0)), \quad \xi_0 \in T_e G = \mathbb{R}^n.$$

Corollary

- *Any Lie group $G$ is orientable.*
- *Among closed 2-dim surfaces, only the torus $T^2$ may carry (and indeed, carries) the structure of a Lie group.*

We have not explained any relationship between Lie groups and Lie algebras yet. Here is the first example which demonstrate this relationship explicitly. (This issue will be discussed in detail at the next lecture.)

**Proposition**

*Let $G \in GL(n, \mathbb{R})$ be a matrix Lie group. Let $A, B \in \mathfrak{g} = T_E G$ and $\xi(X) = XA$, $\eta(X) = XB$ be the corresponding left invariant vector fields on $G$. Then the Lie bracket of $\xi$ and $\eta$ is the left invariant vector field of the form $X(AB - BA)$.*

**Corollary**

*The tangent space at the identity $\mathfrak{g} = T_E G$ of any matrix Lie group is closed under the matrix commutator:*

$$A, B \mapsto [A, B] = AB - BA.$$

*Proof.* Notice that it suffices to prove the formula $[\xi, \eta] = X(AB - BA)$ for $GL(n, \mathbb{R})$ only, then it will hold for any Lie subgroup $G \subset GL(n, \mathbb{R})$ automatically. In local coordinates, the proof is straightforward (as local coordinates we just take matrix coefficients $x_{ij}$):

$$[\xi, \eta]_{ij} = \sum_{k,l} \left( \xi_{kl} \frac{\partial \eta_{ij}}{\partial x_{kl}} - \eta_{kl} \frac{\partial \xi_{ij}}{\partial x_{kl}} \right).$$

We have $\xi_{kl} = \sum_\alpha x_{k\alpha} a_{\alpha l}$ and, similarly, $\eta_{kl} = \sum_\alpha x_{k\alpha} b_{\alpha l}$.

Here by $\xi_{ij}, x_{ij}, a_{ij}$, etc. we denote the matrix coefficients of $\xi$, $X$, $A$, etc.

Hence

$$[\xi, \eta]_{ij} = \sum_{k,l} \left( \sum_\alpha x_{k\alpha} a_{\alpha l} \frac{\partial}{\partial x_{kl}} \left( \sum_\beta x_{i\beta} b_{\beta j} \right) \quad - \quad \dots \right) =$$

(It is seen that the partial derivative does not vanish in the only case $(kl) = (i\beta)$ and in this case it equals simply to $b_{\beta j}$. Hence, replacing $k$ by $i$ and $l$ by $\beta$, we have:)

$$= \sum_{\alpha,\beta} \left( x_{i\alpha} a_{\alpha\beta} b_{\beta j} \quad - \quad x_{i\alpha} b_{\alpha\beta} a_{\beta j} \right) =$$

$$= \sum_{\alpha,\beta} x_{i\alpha} \left( a_{\alpha\beta} b_{\beta j} \quad - \quad b_{\alpha\beta} a_{\beta j} \right)$$

But this formula means exactly that $[\xi, \eta](X) = X(AB - BA)$, as stated.

### Definition

A *Lie algebra* $L$ is a vector space endowed with a bilinear operation $[\cdot,\cdot] : L \times L \to L$ (called *commutator* or *Lie bracket*) satisfying two properties:

- $[a,b] = -[b,a]$ for any $a,b \in L$ (skew symmetry);
- $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$ for any $a,b,c \in L$ (Jacobi identity).

### Example

Euclidean space $\mathbb{R}^3$ endowed with the standard vector product.

### Example

The space of smooth vector fields on a given manifold $M$ with the standard Lie bracket of vector fields.

### Example

Any vector space $V$ with the trivial commutator: $[u,v] = 0$ for all $u,v \in V$.

### Example

The space $M_{n,n}$ of all square $n \times n$ matrices with the matrix commutator: $[A,B] = AB - BA$.

# Structure coefficients of a Lie algebra

If a Lie algebra $L$ is finite-dimensional, it is often convenient to define the operation $[\cdot,\cdot]$ by means of structure coefficients.
To do this, consider a basis $e_1, \dots, e_n$ in $L$. Then we have

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k$$

where $c_{ij}^k \in \mathbb{R}$ are certain real numbers called the *structure coefficients* (all together they form the *structural tensor* of $L$).
These coefficients satisfy two natural conditions:

- $c_{ij}^k = -c_{ji}^k$ (skew symmetry)
- $c_{il}^k c_{jm}^l + c_{jl}^k c_{mi}^l + c_{ml}^k c_{ij}^l = 0$ (Jacobi identity)

The above relation can be rewritten in terms of coordinates as follows: if $(a^1, \dots, a^n)$ and $(b^1, \dots, b^n)$ are the coordinates of two elements $a, b \in L$ w.r.t. the basis $e_1, \dots, e_n$, then the coordinates of $[a, b]$ are:

$$[a, b]^k = \sum_{i,j=1}^{n} c_{ij}^k a^i b^j .$$

## The Lie algebra of a Lie group: Definition 1

Let $G$ be a Lie group. Consider the space of all left-invariant vector fields on $G$. We know already that this space is closed w.r.t. the Lie bracket (of vector fields). Thus, this standard bracket give us a natural bilinear operation:

$$\xi, \eta \mapsto [\xi, \eta]$$

on the space of left invariant vector fields. Skew-symmetry is obvious, and the Jacobi identity is just the standard property of this Lie bracket (which holds for any vector fields, not necessarily left-invariant!).

### Definition

The space of left invariant vector fields on $G$ with the standard Lie bracket is called the *Lie algebra* of the Lie group $G$. Notation: $\mathfrak{g}$.

Since the space of left invariant vector fields is naturally isomorphic to $T_eG$, one usually identifies $\mathfrak{g}$ with $T_eG$. This allows us to define the Lie algebra structure on $T_eG$ directly:
Take $\xi_0, \eta_0 \in T_eG$ and consider the corresponding left invariant vector fields $\xi$, $\eta$ (i.e. such that $\xi(e) = \xi_0, \eta(e) = \eta_0$. Then compute the Lie bracket $\zeta = [\xi, \eta]$ and take its value at the identity: $\zeta_0 = \zeta(e) \in T_eG$. As a result, we obtain a bilinear operation $[\cdot, \cdot]$ on $T_eG$:

$$\xi_0, \eta_0 \mapsto \zeta_0 = [\xi, \eta](e).$$

If we have a local coordinate system $(x_1, \dots, x_n)$ in a neighborhood of the identity $e \in G$, then we get a natural basis $e_1 = \partial/\partial x^1, \dots, e_n = \partial/\partial x^n$ in the tangent space $\mathfrak{g} = T_e G$. (As before, we assume that the coordinates of $e$ are $(0, \dots, 0)$.)

It is natural to ask what the structure coefficients of $\mathfrak{g}$ are with respect to this basis? It turns out that in terms of local coordinates the Lie bracket just measures the difference between $x \cdot y$ and $y \cdot x$.

More precisely, let the multiplication $x, y \mapsto z = x \cdot y$ in the local coordinates be given by (we use the notation from Lecture 4):

$$z^k = x^k + y^k + \sum_{ij} b^k_{ij} x^i y^j + \dots$$

Then

$$c^k_{ij} = b^k_{ij} - b^k_{ji}$$

are the structure coefficients of $\mathfrak{g}$ (w.r.t. the chosen coordinate system).

Here is another way to define $[\cdot,\cdot]$ on the tangent space $T_eG = \mathfrak{g}$.
First for any element $a \in G$, we define the map:

$$A_a : G \to G, \quad x \mapsto A_a(x) = axa^{-1}$$

(this map, in fact, defines the so-called *adjoint action* of $G$ on itself).
Obviously, $A_a(e) = e$ and we can consider the differential of $A_a$ at the identity:

$$dA_a : T_eG \to T_eG$$

This differential has special notation $dA_a = \mathrm{Ad}_a$ (here Ad refers to "adjoint").

Let $\xi_0, \eta_0 \in T_eG$. Consider the one-parameter subgroup $\exp(t\xi_0)$ and take $\mathrm{Ad}_{\exp(t\xi_0)}(\eta_0)$. This is a curve lying in $\mathfrak{g} = T_eG$. Its tangent vector is exactly $[\xi_0, \eta_0]$:

$$[\xi_0, \eta_0] = \frac{d}{dt}_{|t=0} \mathrm{Ad}_{\exp(t\xi_0)}(\eta_0) \tag{1}$$

Equivalently (recall the definition of Ad) this can be rewritten as

$$[\xi_0, \eta_0] = \frac{d}{dt}_{|t=0} \frac{d}{ds}_{|s=0} \exp(t\xi_0)\exp(s\eta_0)\exp(-t\xi_0) \tag{2}$$

Let $G \subset GL(n, \mathbb{R})$ be a matrix Lie group, and $\mathfrak{g} = T_E G$ be the tangent space at the identity.

### Proposition

The tangent space $\mathfrak{g} = T_E G$ is closed under the standard matrix commutator, i.e.,

$$\text{if } A, B \in \mathfrak{g}, \text{ then } AB - BA \in \mathfrak{g},$$

and this operation is exactly the Lie bracket on $\mathfrak{g}$ introduced in Definition 3.

Proof. Indeed, according to Definition 3,

$$[A, B] = \tfrac{d}{dt}_{|t=0} \mathrm{Ad}_{\exp(tA)} B = \tfrac{d}{dt}_{|t=0} \left( \exp(tA) B \exp(-tA) \right) =$$
$$\left( \tfrac{d}{dt}_{|t=0} \exp(tA) \right) B \exp(-0 \cdot A) + \exp(0 \cdot A) B \left( \tfrac{d}{dt}_{|t=0} \exp(-tA) \right) =$$
$$ABE - EBA = AB - BA.$$

# Example: $SO(n)$

$$SO(n) = \{X \in GL(n, \mathbb{R}) \;:\; X^\top = X^{-1}, \;\; \det X = 1\}$$

### Proposition

The tangent space $T_E SO(n)$ consists of skew-symmetric matrices:

$$so(n) = T_E SO(n) = \{A \in M_{n,n} \mid A^\top = -A\},$$

so(n) is closed under the standard matrix commutator and, therefore, carries the structure of a Lie algebra.

Proof. Let $A \in T_E SO(n)$. Consider any curve $X(t) \subset SO(n)$ such that $X(0) = E$ and $X'(0) = A$. For this curve, we have $X^\top(t)X(t) = E$ for any $t \in \mathbb{R}$. Let us differentiate this identity w.r.t. $t$ at the point $t = 0$ (the derivative is obviously zero, since the right hand side does not depend on $t$):

$$0 = \frac{d}{dt}|_{t=0} X^\top(t)X(t) = (X^\top)'(0)X(0) + X^\top(0)X'(0) = A^\top E + EA = A^\top + A$$

that is, $A^\top = -A$, as needed. In other words, any tangent vector to $SO(n)$ at the identity is a skew-symmetric matrix.

Conversely, take an arbitrary skew symmetric matrix $A$.
Consider $X(t) = \exp(tA)$. Obviously, $X(0) = E$ and $X'(0) = A$. Besides,

$$X^\top(t) = (\exp(tA))^\top = \exp\left(tA^\top\right) = \exp(-tA) = (\exp(tA))^{-1} = X^{-1}(t),$$

and

$$\det X(t) = \det\exp(tA) = e^{\operatorname{tr}(tA)} = e^0 = 1,$$

i.e., $X(t) \subset SO(n)$ and, therefore, $A \in T_E SO(n)$ as the tangent vector to a curve lying in $SO(n)$.
The second part of Proposition follows, of course, from the general construction, but can be verified directly:
Let $A, B$ be skew-symmetric, then

$$[A, B]^\top = (AB - BA)^\top = B^\top A^\top - A^\top B^\top =$$

$$(-B)(-A) - (-A)(-B) = BA - AB = -[A, B],$$

i.e., $[A, B]$ is skew-symmetric too.

# Equivalence of 1 and 3

We use the following standard fact in differential geometry:
Let $\xi$ and $\eta$ be smooth vector fields and $\Phi^t$ be the flow of $\xi$. Then the Lie bracket $[\xi, \eta]$ at a fixed point $x \in M$ can be defined as:

$$[\xi, \eta](x) = \frac{d}{dt}|_{t=0} d\Phi^{-t}(\eta(\Phi^t(x))) \tag{3}$$

Here $d\Phi^t$ is the differential of $\Phi^t$ viewed as a linear map from $T_x M$ to $T_{\Phi^t(x)}M$, and $d\Phi^{-t} = (d\Phi^t)^{-1} : T_{\Phi^t(x)}M \to T_x M$ is the inverse map.
The right hand side of (3) is just the definition of the Lie derivative $\mathcal{L}_\xi \eta$ of $\eta$ along $\xi$ (which, as we know, coincides with $[\xi, \eta]$).
In our case of Lie groups:

- $\xi$ and $\eta$ are left invariant;
- $x$ is the identity $e$;
- $\Phi^t(x) = \Phi^t(e) = \exp(t\xi_0)$;
- $\eta(\Phi^t(x)) = \eta(\exp(t\xi_0)) = dL_{\exp(t\xi_0)}\eta_0$ (since $\eta$ is left invariant);
- $\Phi^t$ is just the right multiplication by $\exp(t\xi_0)$, thus $d\Phi^{-t} = dR_{\exp(-t\xi_0)}$.

Conclusion: $d\Phi^{-t}(\eta(\Phi^t(x)) = dR_{\exp(-t\xi_0)} \circ dL_{\exp(t\xi_0)}\eta_0 = \mathrm{Ad}_{\exp(t\xi_0)}\eta_0$

But this means exactly that Def 1 via left invariant vector fields and Def 3 via the adjoint action Ad are absolutely the same.

# Equivalence of 2 and 3

As we noticed (see (2)), Definition 3 is equivalent to

$$[\xi_0, \eta_0] = \frac{d}{dt}_{|t=0} \frac{d}{ds}_{|s=0} \exp(t\xi_0)\exp(s\eta_0)\exp(-t\xi_0)$$

Let us rewrite this in local coordinates. Clearly, $\exp(t\xi_0)^k = t\xi_0^k + \dots$ and $\exp(s\eta_0)^k = s\eta_0^k + \dots$ (where $\dots$ denotes higher order terms).
Now using the coordinate representation for multiplication (see Lecture 4)

$$z^k = x^k + y^k + \sum_{ij} b_{ij}^k x^i y^j + \dots$$

we get (in the double power series below we only need the coefficient for $st$):

$$\exp(t\xi_0)\exp(s\eta_0)\exp(-t\xi_0) = (t\xi_0^k + \dots)(s\eta_0^k + \dots)(-t\xi_0^k + \dots) =$$

$$(t\xi_0^k + \dots)(s\eta_0^k - t\xi_0^k - st \sum b_{ij}^k \eta_0^i \xi_0^j + \dots) =$$

$$(t\xi_0^k + s\eta_0^k - t\xi_0^k - st \sum b_{ij}^k \eta_0^i \xi_0^j + \sum \left( b_{ij}^k (t\xi_0^i)(s\eta_0^j - t\xi_0^j - st \sum b_{\alpha\beta}^j \eta_0^\alpha \xi_0^\beta) \right) + \dots =$$

$$s\eta_0^k + st \sum_{ij} (b_{ij}^k - b_{ji}^k) \xi_0^i \eta_0^j + \dots$$

Now it remains to notice that $[\xi, \eta]^k$ (in the sense of Definition 3) is

$$\frac{d}{dt}_{|t=0} \frac{d}{ds}_{|s=0} \exp(t\xi_0)\exp(s\eta_0)\exp(-t\xi_0) =$$

$$\frac{d}{dt}_{|t=0} \frac{d}{ds}_{|s=0} (s\eta_0^k + st \sum_{ij} (b_{ij}^k - b_{ji}^k)\xi_0^i \eta_0^j + \dots) = \sum_{ij} (b_{ij}^k - b_{ji}^k)\xi_0^i \eta_0^j$$

That is,

$$c_{ij}^k = b_{ij}^k - b_{ji}^k,$$

as was to be proved.

## General linear group $GL(n, \mathbb{R})$

$$GL(n, \mathbb{R}) = \{X \quad n \times n \text{ matrix} \quad : \quad \det X \neq 0\}$$

Properties of $GL(n, \mathbb{R})$:

- $\dim GL(n, \mathbb{R}) = n^2$;
- $GL(n, \mathbb{R})$ is not compact;
- $GL(n, \mathbb{R})$ is not connected but consists of two connected components $GL_+(n, \mathbb{R}) = \{X \ : \ \det X > 0\}$ and $GL_-(n, \mathbb{R}) = \{X \ : \ \det X < 0\}$;
- the Lie algebra $gl(n, \mathbb{R})$ consists of all $n \times n$ matrices.

# Special linear group $SL(n,\mathbb{R})$

$$SL(n,\mathbb{R}) = \{X \quad n\times n \text{ matrix} \quad : \quad \det X = 1\}$$

Properties of $SL(n,\mathbb{R})$:

- $\dim SL(n,\mathbb{R}) = n^2 - 1$;
- $SL(n,\mathbb{R})$ is not compact;
- $SL(n,\mathbb{R})$ is connected;
- the Lie algebra $sl(n,\mathbb{R})$ consists of $n\times n$ matrices with zero trace;
$$sl(n,\mathbb{R}) = \{A \ : \ \mathrm{tr}\, A = 0\};$$
- $SL(n,\mathbb{R})$ is a normal subgroup in $GL(n,\mathbb{R})$ and the quotient group $GL(n,\mathbb{R})/SL(n,\mathbb{R})$ is isomorphic to $\mathbb{R}^*$ (the multiplicative subgroup in $\mathbb{R}$, i.e., $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$ under multiplication).
- $SL(n,\mathbb{R})$ is not simply connected, $\pi_1(SL(n,\mathbb{R})) = \mathbb{Z}_2$ for $n>2$, and $\pi_1(SL(2,\mathbb{R})) = \mathbb{Z}$.

# The Lie group associated with a bilinear form

Let $\mathcal{B}$ be a bilinear form on $\mathbb{R}^n$:

$$\mathcal{B} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad u, v \mapsto \mathcal{B}(u, v), \quad \text{where } u, v \in \mathbb{R}^n, \mathcal{B}(u, v) \in \mathbb{R}.$$

Standard examples: inner product, symplectic form.
Each bilinear form $\mathcal{B}$ can naturally be given by its matrix $B = (b_{ij})$:

$$\mathcal{B}(u, v) = \sum_{i,j} b_{ij} u_i v_j = \begin{pmatrix} u_1 & \dots & u_n \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Consider the set of all invertible linear transformations $X : \mathbb{R}^n \to \mathbb{R}^n$ that preserve the form $\mathcal{B}$.

$$G_{\mathcal{B}} = \{X \in GL(n, \mathbb{R}) \ : \ \mathcal{B}(Xu, Xv) = \mathcal{B}(u, v) \text{ for all } u, v \in \mathbb{R}^n\}$$

Here we identify non-degenerate matrices $X \in GL(n, \mathbb{R})$ with invertible linear transformations $X : \mathbb{R}^n \to \mathbb{R}^n$.
The fact that $G_{\mathcal{B}}$ is a group is standard: the set of invertible transformations preserving "something" is always a group.

## Proposition

$G_{\mathcal{B}}$ is an algebraic linear group (and, therefore, a Lie subgroup in $GL(n,\mathbb{R})$). In matrix notation, $G_{\mathcal{B}}$ is defined by one matrix equation (quadratic in $X$) :

$$G_{\mathcal{B}} = \{X \in GL(n,\mathbb{R}) \;:\; X^\top B X = B\}.$$

The corresponding Lie algebra $\mathfrak{g}_{\mathcal{B}} = T_E G_{\mathcal{B}}$ is

$$\mathfrak{g}_{\mathcal{B}} = \{A \in gl(n,\mathbb{R}) \;:\; A^\top B + BA = 0\}.$$

Proof. The equation $X^\top B X = B$ is just a reformulation of $\mathcal{B}(Xu, Xv) = \mathcal{B}(u, v)$ in terms of matrices. Thus, $G_{\mathcal{B}}$ is algebraic.
To describe its Lie algebra $\mathfrak{g}_{\mathcal{B}}$ we need to verify the following property:

$$A^\top B + BA = 0 \quad \text{if and only if} \quad (\exp(tA))^\top B \exp(tA) = B$$

$\Leftarrow$ follows just from differentiating $(\exp(tA))^\top B \exp(tA) = B$ at $t = 0$.
$\Rightarrow$ can be obtained from the following argument:
$A^\top B + BA = 0$ implies $BA = (-A^\top)B$, hence by induction $BA^n = (-A^\top)^n B$ and therefore:

$$B\exp(tA) = B\sum_n \frac{t^n A^n}{n!} = \sum_n \frac{t^n(-A^\top)^n}{n!} B = \exp(-tA^\top)B$$

and finally $(\exp(tA))^\top B \exp(tA) = \exp(tA^\top)\exp(-tA^\top)B = B$, as required.

In general, there is no restriction on $\mathcal{B}$: this form may be symmetric or skew-symmetric, neither symmetric nor skew-symmetric, degenerate or non-degenerate.
The properties of $G_{\mathcal{B}}$ essentially depend on the properties of $\mathcal{B}$, as we shall see below.
However, the following statement holds:

### Proposition

If bilinear forms $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent in the sense that $B_1 = C^\top B_2 C$ for a certain invertible matrix $C$ (this means that these two forms are related by a suitable change of coordinates), then the corresponding groups $G_{\mathcal{B}_1}$ and $G_{\mathcal{B}_2}$ are isomorphic and the isomorphism is given by conjugation:

$$\Phi : G_{\mathcal{B}_1} \to G_{\mathcal{B}_2}, \qquad X \mapsto \Phi(X) = CXC^{-1}.$$

*Proof.* The conjugation $X \mapsto CXC^{-1}$ is an isomorphism between any Lie group $G$ and its image $\Phi(G)$. Thus, we only need to prove that $\Phi(G_{\mathcal{B}_1}) = G_{\mathcal{B}_2}$. Let $X \in G_{\mathcal{B}_1}$, then

$$(CXC^{-1})^\top B_2 CXC^{-1} = (C^{-1})^\top X^\top C^\top B_2 CXC^{-1} =$$

$$(C^{-1})^\top X^\top B_1 X C^{-1} = (C^{-1})^\top B_1 C^{-1} = B_2.$$

Thus, $CXC^{-1} \in G_{\mathcal{B}_2}$, i.e. $\Phi(G_{\mathcal{B}_1}) \subset G_{\mathcal{B}_2}$. The proof that $G_{\mathcal{B}_2} \subset \Phi(G_{\mathcal{B}_2})$ is similar.

# Orthogonal groups $O(n)$ and $SO(n)$

$O(n)$ is a particular case of $G_{\mathcal{B}}$, namely, $B = E$ (identity matrix).

$$O(n) = \{X \in GL(n, \mathbb{R}) \ : \ X^\top X = E\}$$

Properties of O(n):

- $\dim O(n) = \frac{n(n-1)}{2}$;
- $O(n)$ is compact;
- $O(n)$ consists of two connected components; the connected component of the identity is $SO(n) = \{X \in O(n) \ : \ \det X = 1\}$;
- $SO(n)$ is not simply connected, $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n > 2$, and $\pi_1(SO(2)) = \mathbb{Z}$;
- the Lie algebra $so(n)$ consists of skew-symmetric matrices;
- each one-parameter subgroup in $SO(n)$ is conjugate to

$$X(t) = \begin{pmatrix} X_1 & & & \\ & X_2 & & \\ & & \ddots & \\ & & & X_k \end{pmatrix} \text{ where } X_i \text{ is either } \begin{pmatrix} \cos \varphi_i t & -\sin \varphi_i t \\ \sin \varphi_i t & \cos \varphi_i t \end{pmatrix} \text{ or } 1$$

# Pseudo-orthogonal groups $O(p,q)$ and $SO(p,q)$

$O(p,q)$ is a particular case of $G_{\mathcal{B}}$ for

$$\mathcal{B}(u,v) = u_1v_1 + \cdots + u_pv_p - u_{p+1}v_{p+1} - \cdots - u_{p+q}v_{p+q},$$

or, equivalently,

$$B = E_{p,q} = \mathrm{diag}(\underbrace{1,\ldots,1}_{p},\underbrace{-1,\ldots,-1}_{q})$$

We assume that $p+q=n$ so that $\mathcal{B}$ is non-degenerate. In addition, neither $p$ nor $q$ is 0 (if $p$ or $q$ is zero, then $O(p,q) = O(n)$).

Important case: $O(1,3)$ known as the *Lorentz group*.

Properties of O(p,q):

- $\dim O(p,q) = \frac{n(n-1)}{2}$, $n = p+q$;
- $O(p,q)$ is not compact;
- $O(p,q)$ is not connected and consists of 4 connected components; $SO(p,q) = O(p,q) \cap SL(n,\mathbb{R})$ consists of 2 components;
- $\det X = \pm 1$ for $X \in O(p,q)$;
- the Lie algebra $so(p,q)$ consists of the matrices $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ where $A_1, A_2, A_3, A_4$ are submatrices of dimension $p\times p$, $p\times q$, $q\times p$ and $q\times q$ respectively, such that $A_1$ and $A_4$ are skew-symmetric and $A_3^\top = A_2$. Equivalently, this simply means that $E_{p,q}A$ is skew-symmetric.

# Symplectic group $Sp(2n, \mathbb{R})$

$Sp(2n, \mathbb{R})$ is a group $G_{\mathcal{J}}$ of linear transformations that preserve a bilinear non-degenerate skew-symmetric form $\mathcal{J}$. Usually, as the matrix of $\mathcal{J}$ one takes:

$$J = J_{2n} = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

so that

$$Sp(2n, \mathbb{R}) = \{X \in GL(2n, \mathbb{R}) \ : \ X^\top J X = J\}$$

Notice that in this situation, our vector space has to be even-dimensional, i.e., $\mathbb{R}^{2n}$, since odd-dimensional skew-symmetric matrices are necessarily degenerate.

Properties of $Sp(2n, \mathbb{R})$:

- $\dim Sp(2n, \mathbb{R}) = n(2n+1)$;
- $Sp(2n, \mathbb{R})$ is not compact;
- $Sp(2n, \mathbb{R})$ is connected;
- $\det X = 1$ for $X \in Sp(2n, \mathbb{R})$;
- the Lie algebra $sp(2n, \mathbb{R})$ consists of the matrices $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ where $A_1, A_2, A_3, A_4$ are submatrices of dimension $n \times n$ such that: $A_1 = -A_4^\top$ and $A_3$ and $A_2$ are symmetric. Equivalently: $JA$ is symmetric.
- $Sp(2n, \mathbb{R})$ is not simply connected and $\pi_1(Sp(2n, \mathbb{R})) = \mathbb{Z}$.

Throughout this lecture, we identify the Lie algebra $\mathfrak{g}$ of a Lie group $G$ with the tangent space at the identity: $\mathfrak{g} = T_eG$. We are going to discuss the relationship between $G$ and $\mathfrak{g}$. In this context it is natural to assume that $G$ is connected (indeed, the identity component $G_0$ is naturally attached to $T_eG = \mathfrak{g}$ whereas the other components know nothing about $\mathfrak{g}$).

**Proposition**

Let $G$ be a Lie group and $H \subset G$ be a Lie subgroup. Then the tangent space $\mathfrak{h} = T_eH \subset T_eG$ is a Lie subalgebra in $\mathfrak{g} = T_eG$ (which is, of course, the Lie algebra of $H$).

Notice that in the case of matrix groups we already used this statement many times.

Question: Conversely, given some (abstract) subalgebra $\mathfrak{h} \subset \mathfrak{g} = T_eG$, does it correspond to any subgroup $H \subset G$?

The answer is, in fact, yes... But this subgroup is not necessarily a Lie subgroup. Here is an example which explains this phenomenon.

### Example

Let $G = T^2$ be a two-dimensional torus endowed with the standard structure of a commutative Lie group, and let $\phi_1, \phi_2$ be standard angle coordinates on $T^2$ defined modulo $2\pi$. Consider the one-dimensional subspace in $\mathfrak{g} = T_e T^2$ spanned by a vector $\xi_0 = (a, b)$. This is a subalgebra in $\mathfrak{g}$. The corresponding subgroup is just $\exp t\xi_0$. It is easy to see that in local coordinates:
$\exp(t\xi_0) = (at, bt)$
If $a/b$ is rational, then $\exp(t\xi_0)$ is closed and, hence, is a Lie subgroup in $T^2$ (according to the Cartan theorem). If $a/b$ is not rational, then $\exp(t\xi_0)$ represents the so-called irrational winding on $T^2$ which can be considered as an immersion $f : \mathbb{R} \to T^2$, but not an embedding.
Nevertheless, this is a subgroup, moreover locally (i.e. for small $t \in \mathbb{R}$) it is a submanifold.

In general, consider two Lie groups $G$ and $H$ and a smooth monomorphism $f : H \to G$. Then two situations are possible:

- $f(H)$ is closed and then is a Lie subgroup in $G$,
- $f(H)$ is not closed

### Definition

In the latter case, $f(H)$ is called a *virtual Lie subgroup* of $G$.

### Proposition

Let $\mathfrak{h} \subset \mathfrak{g} = T_e G$ be a Lie subalgebra. Then there exist a unique connected Lie subgroup (possibly virtual) $H \subset G$ such that $\mathfrak{h} = T_e H$.

# Normal subgroups and ideals

**Definition**
A subgroup $H \subset G$ is called *normal* if $gHg^{-1} = H$ for any $g \in G$.

**Definition**
A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called an *ideal* if $[\xi, \mathfrak{h}] \subset \mathfrak{h}$ for any $\xi \in \mathfrak{g}$.

**Proposition**
Let $H \subset G$ be a connected Lie subgroup (possibly virtual) and $\mathfrak{h} = T_e H \subset \mathfrak{g} = T_e G$ be the corresponding Lie subalgebra. Then $H$ is normal if and only if $\mathfrak{h}$ is an ideal.

"$H$ is normal $\Rightarrow \mathfrak{h}$ is an ideal" is easy (because the proof is reduced to "differentiation"):

$\exp t\xi \cdot H \cdot \exp(-t\xi) = H$ implies $\mathrm{Ad}_{\exp t\xi}\mathfrak{h} = \mathfrak{h}$ (first differentiation);
Differentiating $\mathrm{Ad}_{\exp t\xi}\mathfrak{h} = \mathfrak{h}$ w.r.t $t$ for $t = 0$ gives $[\xi, \mathfrak{h}] \subset \mathfrak{h}$, as needed.

"$H$ is normal $\Leftarrow \mathfrak{h}$ is an ideal" is more difficult (because the proof consists in "integration").

Fix $\xi \in \mathfrak{g}$ and consider the family of linear operators $\mathrm{Ad}_{\exp t\xi}$ on $\mathfrak{g}$. It is easy to see that this family is in fact a one-parameter subgroup. So it naturally defines a vector field on $\mathfrak{g}$ which is, of course,

$$\zeta(\eta) = \frac{d}{dt}|_{t=0}\mathrm{Ad}_{\exp t\xi}(\eta) = [\xi, \eta], \quad \text{for any } \eta \in \mathfrak{g}$$

In partucular, $\mathrm{Ad}_{\exp t\xi}(\eta)$ is exactly the integral curve of this vector field passing through $\eta \in \mathfrak{g}$.

Now "$\mathfrak{h}$ is an ideal in $\mathfrak{g}$" means exactly, that this vector field $\zeta(\eta) = [\xi, \eta]$ is tangent to $\mathfrak{h}$ (at each point $\eta \in \mathfrak{h}$). Therefore, its integral curves passing through $\eta \in \mathfrak{h}$ cannot leave $\mathfrak{h}$, i.e.

$$\eta \in \mathfrak{h} \quad \Rightarrow \quad \mathrm{Ad}_{\exp t\xi}\eta \subset \mathfrak{h}.$$

In other words, $\mathrm{Ad}_{\exp t\xi}\mathfrak{h} \subset \mathfrak{h}$ and therefore $\mathrm{Ad}_{\exp t\xi}\mathfrak{h} = \mathfrak{h}$ since $\mathrm{Ad}_{\exp t\xi}$ is a linear isomorphism.

Next, using the standard property $g \cdot \exp s\eta \cdot g^{-1} = \exp(s\mathrm{Ad}_g\eta)$, we conclude that $\exp t\xi \cdot \exp s\eta \cdot \exp(-t\eta) = \exp(s\mathrm{Ad}_{\exp t\xi}\eta) \subset H$ for any $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{h}$.

It remains to notice that the elements of the form $\exp t\xi$ generate $G$ and those of the form $\exp s\eta$ generate $H$. Hence, $gHg^{-1} = H$ for any $g \in G$, as required.

## Example

Consider the Lie algebra $\mathfrak{t}$ of upper triangular $n \times n$ matrices:

$$\mathfrak{t} = \left\{ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \right\} \quad \text{and let} \quad \mathfrak{h} = \left\{ \begin{pmatrix} 0 & 0 & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \right\} \subset \mathfrak{t}\,.$$

Clearly, $\mathfrak{h} \subset \mathfrak{t}$ is a subalgebra. The corresponding (connected) group and subgroup are:

$$T = \left\{ \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ 0 & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_{nn} \end{pmatrix} \right\} \quad \text{and} \quad H = \left\{ \begin{pmatrix} 1 & 0 & \dots & x_{1n} \\ 0 & 1 & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_{nn} \end{pmatrix} \right\} \subset T,$$

where $x_{ii} > 0$ (the others are arbitrary).
It is not hard to see that $\mathfrak{h} \subset \mathfrak{t}$ is an ideal, and $H \subset T$ is a normal subgroup.

### Definition

The *commutant of $G$* is a subgroup $G' \subset G$ generated by all commutators, i.e. elements of the form $aba^{-1}b^{-1} \quad (a, b \in G)$.

### Definition

The *commutant of $\mathfrak{g}$* is a subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ generated by all commutators, i.e. elements of the form $[\xi, \eta] \quad (\xi, \eta \in \mathfrak{g})$.

### Proposition

- $G'$ is a normal connected Lie subgroup (possibly virtual) in $G$. The quotient group $G/G'$ is commutative and $G'$ is the smallest normal subgroup with this property.
- $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$. The quotient Lie algebra $\mathfrak{g}/\mathfrak{g}'$ is commutative and $\mathfrak{g}'$ is the smallest ideal with this property.
- $\mathfrak{g}' = T_e G'$, i.e., $\mathfrak{g}'$ is the Lie algebra of $G'$.

### Corollary

*A connected Lie group $G$ is commutative if and only if its Lie algebra $\mathfrak{g}$ is commutative.*

### Definition

A Lie group *homomorphism* from $H$ to $G$ is a smooth map $\Phi : H \to G$ which is a homomorphism, i.e. such that

$$\Phi(h_1 \cdot h_2) = \Phi(h_1) \cdot \Phi(h_2) \quad \text{for any } h_1, h_2 \in H.$$

### Definition

A *homomorphism* of (abstract) Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$ is a linear map $\phi : \mathfrak{h} \to \mathfrak{g}$ which "preserves Lie bracket", i.e.,

$$\phi[\eta_1, \eta_2] = [\phi(\eta_1), \phi(\eta_2)] \quad \text{for any } \eta_1, \eta_2 \in \mathfrak{h}.$$

Consider a homomorphism of Lie groups $\Phi : H \to G$ and its differential $d\Phi : T_e H \to T_e G$ at the identity. Since $\mathfrak{h} = T_e H$ and $\mathfrak{g} = T_e G$, we get a linear map between the corresponding Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$.

### Theorem

$d\Phi : \mathfrak{h} \to \mathfrak{g}$ *is a homomorphism of Lie algebras.*

Important! (see next lecture): The converse is also true: under certain natural restrictions, any homomorphism of Lie algebras $\phi : \mathfrak{h} \to \mathfrak{g}$ induces a unique homomorphism $\Phi : H \to G$ such that $\phi = d\Phi$.

### Lemma

*Let* $\eta \in \mathfrak{h}$ *and* $\exp t\eta$ *be the corresponding one-parameter subgroup in* $H$. *Then*

$$\Phi(\exp t\eta) = \exp(t\, d\Phi(\eta))$$

Proof is simple. It is sufficient to notice that the left hand side and right hand side of this relation both represent one-parameter subgroups in $G$. Moreover, the initial vectors of these two subgroups are the same, namely $d\Phi(\eta)$. Thus, because of the uniqueness, these subgroups coincide.

### Lemma

$$d\Phi(\mathrm{Ad}_h \eta) = \mathrm{Ad}_{\Phi(h)} d\Phi(\eta).$$

Follows from differentiating $\Phi(h \cdot \exp t\eta \cdot h^{-1}) = \Phi(h) \cdot \exp(t\, d\Phi(\eta)) \cdot \Phi(h)^{-1}$.

To complete the proof we differentiate the identity

$$d\Phi(\mathrm{Ad}_{\exp t\eta_1} \eta_2) = \mathrm{Ad}_{\Phi(\exp t\eta_1)} d\Phi(\eta_2) = \mathrm{Ad}_{\exp(t d\Phi(\eta_1))} d\Phi(\eta_2)$$

to get $d\Phi[\eta_1, \eta_2] = [d\Phi(\eta_1), d\Phi(\eta_2)]$, as was to be proved.

Let $H$ and $G$ be connected Lie groups and $\mathfrak{h}$, $\mathfrak{g}$ be the corresponding Lie algebras.

The main goal of this lecture is the following fundamental

**Theorem**
*Let $\phi : \mathfrak{h} \to \mathfrak{g}$ be a Lie algebra homomorphism. If $H$ is simply connected, then there exists a unique homomorphism $\Phi : H \to G$ such that $\phi = d\Phi$.*

The simply connectedness assumption is very important for the existence of $\Phi$. The uniqueness part of the statement holds in general case: if $\Phi$ exists, it is unique.

### Definition

A (path-connected) topological space $X$ is called simply connected if any closed curve $\gamma : S^1 \to X$ can be contracted to a point in the following sense: there exists a continuous map $\Gamma : D^2 \to X$ (where $D^2$ is a disc bounded by $S^1$) such that $\Gamma$ restricted to $S^1 = \partial D^2$ is $\gamma$.

An equivalent formulation: $X$ is simply connected if and only if it is path-connected, and whenever $\gamma_1 : [0, 1] \to X$ and $\gamma_2 : [0, 1] \to X$ are two paths (i.e., continuous maps) with the same endpoints (i.e., $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$), then $\gamma_1$ and $\gamma_2$ are homotopic as paths with fixed endpoints. Intuitively, this means that $\gamma_1$ can be "continuously deformed" to get $\gamma_2$ while keeping the endpoints fixed.

- The Euclidean plane $\mathbb{R}^2$ is simply connected, but $\mathbb{R}^2$ minus the origin $(0, 0)$ is not. If $n > 2$, then both $\mathbb{R}^n$ and $\mathbb{R}^n$ minus the origin are simply connected.
- Analogously: the $n$-dimensional sphere $S^n$ is simply connected if and only if $n \geq 2$.
- The torus $T^2$, the cylinder, the Möbius strip are not simply connected.
- The special orthogonal group $SO(n)$ is not simply connected for $n \geq 2$.

## Example

Let $H$ be a complex unit circle

$$H = S^1 = \{z \in \mathbb{C} \mid z = e^{i\alpha}\},$$

and $G$ be the set of positive real numbers viewed as a group under multiplication

$$G = \mathbb{R}_+ = \{x \in \mathbb{R} \mid x = e^{\beta} > 0\}.$$

$\mathfrak{h} = T_e H$ can be identified with the pure imaginary axis on the complex plane, while $\mathfrak{g}$ can naturally be considered as the real line $\mathbb{R}$.
The exponential maps are obvious:

$$\exp_H : \mathfrak{h} = i\mathbb{R} \to H \text{ is } i\alpha \mapsto e^{i\alpha}$$
$$\exp_G : \mathfrak{g} = \mathbb{R} \to G \text{ is } \beta \mapsto e^{\beta}$$

A homomorphism $\phi : \mathfrak{h} \to \mathfrak{g}$ can naturally be defined as

$$\phi(i\alpha) = \alpha, \qquad \text{where} \quad i\alpha \in \mathfrak{h},\ \alpha \in \mathfrak{g}.$$

It is easy to see that we cannot construct any non-trivial homomorphism $\Phi : S^1 \to \mathbb{R}_+$.
However, locally (for small $\alpha$) such a homomorphism exists: $\Phi(e^{i\alpha}) = e^{\alpha}$; and we indeed have $\Phi(e^{i\alpha_1} e^{i\alpha_2}) = \Phi(e^{i(\alpha_1+\alpha_2)}) = e^{\alpha_1+\alpha_2} = \Phi(e^{i\alpha_1})\Phi(e^{i\alpha_2})$. But trying to extend it from a small neighborhood to the whole circle $S^1$ we get a problem with periodicity: $e^{i\cdot 0} = e^{i\cdot 2\pi}$ whereas $\Phi(e^{i\cdot 0}) = e^0 \neq e^{2\pi} = \Phi(e^{i\cdot 2\pi})$.

# Uniqueness

For any homomorphism $\Phi : H \to G$ and its differential $d\Phi = \phi : \mathfrak{h} \to \mathfrak{g}$, we have the commutative diagram:

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\phi = d\Phi} & \mathfrak{g} \\ \exp\downarrow & & \downarrow\exp \\ H & \xrightarrow[\Phi]{} & G \end{array}$$

Or equivalently: $\Phi(\exp\xi) = \exp(d\Phi(\xi))$.
Locally, in a small neighborhood $U(e) \subset H$ the exponential map is invertible, i.e., we can define the inverse map $\exp^{-1} : U(e) \to \mathfrak{h}$:

$$\exp^{-1}(h) = \xi, \quad \text{such that} \quad \exp\xi = h.$$

Thus, in this neighborhood we have

$$\Phi(h) = \exp\left(\phi\left(\exp^{-1}(h)\right)\right).$$

This means that locally $\Phi$ can be reconstructed from its differential $\phi = d\Phi$.
The global result follows immediately from the fact that $H$ is generated by $U(e)$ (since $H$ is connected).

# Continuation of $\Phi$ along a path

Let $\Phi : H \to G$ be a homomorphism and $\Phi(x) = y$. We want to describe the differential of $\Phi$ at $x$ by means of $\phi = d\Phi|_e$. As we know, any tangent vector $T_x H$ can be obtained by left translation from a certain vector $\xi \in \mathfrak{h} = T_e G$, i.e. can be represented as

$$dL_x(\xi), \quad \xi \in \mathfrak{h} = T_e H.$$

## Lemma

$$d\Phi|_x(dL_x(\xi)) = dL_y(\phi(\xi)), \quad \text{where} \quad y = \Phi(x).$$

*In other words, we have the commutative diagram:*

$$\begin{array}{ccccc} & \mathfrak{h} = T_e H & \xrightarrow{\phi = d\Phi|_e} & \mathfrak{g} = T_e G & \\ dL_x & \downarrow & & \downarrow & dL_y \\ & T_x H & \xrightarrow[d\Phi|_x]{} & T_y G & \end{array}$$

Proof follows immediately from "differentiating" the commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\Phi} & G \\ L_x \downarrow & & \downarrow L_y \\ H & \xrightarrow[\Phi]{} & G \end{array}$$

which simply means that $\Phi(L_x h) = \Phi(x \cdot h) = \Phi(x) \cdot \Phi(h) = L_y \Phi(h)$.

Now let $\gamma(t)$ be a smooth path in $H$ connecting $e$ with $x$. Let us introduce the vector $\xi(t) \in \mathfrak{h}$ such that $\gamma'(t) = dL_{\gamma(t)}\xi(t)$ (in other words, $\xi(t)$ is just the tangent vector to $\gamma(t)$ if we identify $T_{\gamma(t)}H$ with $\mathfrak{h} = T_eH$ by means of left translation).
Notice that if $\xi(t)$ is given, we can look at the relation

$$\frac{dh}{dt} = dL_h\,\xi(t) \tag{1}$$

as a non-autonomous ODE on $H$. In particular, the (unique!) solution of this equation starting at $e$ is exactly our initial curve $\gamma(t)$ that joints $e$ with $x$. Notice that all the other solutions are of the form $h(t) = a \cdot \gamma(t)$.

### Lemma

*If $\Phi : H \to G$ is a homomorphism, then the image $\Phi(\gamma(t))$ of $\gamma(t)$ is the solution of the following non-autonomous ODE on $G$:*

$$\frac{dg}{dt} = dL_g\,\phi(\xi(t)). \tag{2}$$

Proof follows immediately from the previous Lemma and the standard formula $d\Phi(\gamma'(t)) = \Phi(\gamma(t))'$.

Conclusion: If we know $\phi : \mathfrak{h} \to \mathfrak{g}$, then we can uniquely reconstruct $\Phi(x)$ for any $x \in \mathfrak{h}$ that can be jointed with $e$ by a smooth path. Namely, we consider a path $\gamma(t)$ between $e$ and $x$ (i.e., $e = \gamma(0)$, $x = \gamma(1)$). Then we take the (unique) solution $\tilde{\gamma}(t)$ of (2) on $G$ such that $\tilde{\gamma}(0) = e$ and put, by definition, $\Phi(x) = \tilde{\gamma}(1)$. Since $H$ is assumed to be connected, we can do it for any $x \in H$.
We still have 2 important things to verify:

- $\Phi(x)$ so obtained does not depend on the path between $e$ and $x$;
- $\Phi$ is indeed a homomorphism, i.e., $\Phi(x_1 \cdot x_2) = \Phi(x_1) \cdot \Phi(x_2)$.

The second item is relatively easy.
Let $\gamma_1(t)$ and $\gamma_2(t)$ be some paths connecting $e \in H$ with $x_1$ and $x_2$ respectively. As a path between $e$ and $x_1 \cdot x_2$, we choose the path $\gamma$ that consists of two parts: $\gamma_1(t)$ and $x_1 \cdot \gamma_2(t)$ (after a suitable reparametrization, it is a continuous path).
Denote by $y_1$ and $y_2$ the images of $x_1$ and $x_2$ obtained by "continuation" of $\Phi$ along $\gamma_1$ and $\gamma_2$. By construction, to find $\Phi(x_1 \cdot x_2)$ we need to apply the above "continuation" procedure to $\gamma = \gamma_1 + x_1 \cdot \gamma_2$.
We claim that this procedure gives the curve in $G$ of the form $\tilde{\gamma} = \tilde{\gamma}_1 + y_1 \cdot \tilde{\gamma}_2$. This statement is non-trivial for the second parts of $\gamma$ and $\tilde{\gamma}$ and follows from the fact $y_1 \cdot \tilde{\gamma}_2(t)$ and $x_1 \cdot \gamma_2(t)$ satisfy the equations (2) and (1) respectively with $\xi$ replaced by $\xi_2$ (here we use the left-invariance property of (2) and (1)). Thus, the endpoint of $\tilde{\gamma}$ is $y_1 \cdot$ endpoint of $\gamma_2 = y_1 \cdot y_2$, that is $\Phi(x_1 \cdot x_2) = \Phi(x_1) \cdot \Phi(x_2)$, as required.

The fact that $\Phi(x)$ so obtained does not depend on the path between $e$ and $x$ is not so easy.

**Lemma**

*Let $\gamma_1(t)$ and $\gamma_2(t)$ be two smooth curves connecting $e \in H$ with $x \in H$, $t \in [0,1]$, and $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ be the corresponding "transported curves" as before. If $\gamma_1$ and $\gamma_2$ are homotopic (as curves with fixed ends), then $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$. In other words, the result of the "continuation" procedure does not change if we replace the curve $\gamma_1$ by any other curve homotopic to it.*

The assumption that $\gamma_1$ and $\gamma_2$ are homotopic is crucial. That is why, for this general statement we have to assume that

$H$ is simply connected

Under this condition, $\Phi(x)$ is well defined (i.e., does not depend on the choice of $\gamma(t)$ connecting $x$ with the identity $e$.
The final remark is that the map $\Phi$ obtained from $\phi$ by "continuation" along a curve is smooth. Indeed in a small neighborhood of $e \in H$ it can be written as $\exp \circ \phi \circ \exp^{-1}$ (and therefore is smooth as a superposition of smooth maps). In a neighborhood of any other point $x \in H$, $\Phi$ is smooth by homogeneity principle: $\Phi$ can be presented as $L_{\Phi(x)} \circ \Phi \circ L_{x^{-1}}$ (where $\Phi$ between two left translations is defined in a neighborhood of $e \in H$).

Let $G$ be a Lie group and $M$ a smooth manifold.

**Definition**

An action of $G$ on $M$ is a smooth map $F : G \times M \to M$ satisfying the following properties:

- $F(e, \cdot) : M \to M$ is the identity map, i.e., $F(e, x) = x$ for any $x \in M$,
- $F(ab, x) = F(a, F(b, x))$ for any $x \in M$; in other words, $F(ab, \cdot)$ is the superposition of the maps $F(a, \cdot)$ and $F(b, \cdot)$.

It follows from the definition that $F(a, \cdot) : M \to M$ is a diffeomorphism. Thus, denoting $F(a, \cdot)$ by $\hat{a}$, we may think of this action as a map

$$\hat{} : G \to \mathrm{Diff}(M), \quad a \mapsto \hat{a}.$$

Then the above properties simply mean that this map is a homomorphism of groups. Moreover, if we think of Diff($M$) as an infinite-dimensional Lie group then the action can be understood as a homomorphism of Lie groups.

# Examples

- Left action and adjoint actions of $G$ on itself:

$$\hat{a} = L_a : G \to G, \qquad L_a(x) = a \cdot x,$$

and

$$\hat{a} = A_a : G \to G, \qquad A_a(x) = a \cdot x \cdot a^{-1}$$

- Natural action of $SO(3)$ on $\mathbb{R}^3$
- Action of the circle $S^1 = \{z \in \mathbb{C} \ : \ z = e^{i\alpha}\}$ on $\mathbb{R}^2$:

$$e^{i\alpha} \mapsto \widehat{e^{i\alpha}} : \mathbb{R}^2 \to \mathbb{R}^2, \quad \text{rotation by } \alpha \text{ given by } \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

- Action of $SL(2,\mathbb{R})$ on the complex upper half-plane by fractional linear transformations (Möbius transformations):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \quad \longrightarrow \quad z \mapsto \frac{az+b}{cz+d}$$

- Action of $\mathbb{R}$ (under addition) on the torus $T^2$ by translations:

$$t \in \mathbb{R} \quad \mapsto \qquad \hat{t}(\phi_1, \phi_2) = (\phi_1 + at, \phi_2 + bt).$$

# Induced action of the Lie algebra

If we think of an action of $G$ on $M$ as a homomorphism from $G$ to $\mathrm{Diff}(M)$, then it is natural to ask: What is its differential as a homomorphism between the corresponding Lie algebras?
In fact, as a Lie algebra of $\mathrm{Diff}(M)$, it is very natural to consider the space $\mathrm{Vect}(M)$ of smooth vector fields on $M$.
Thus, the differential of the action is a map from $\mathfrak{g}$ to $\mathrm{Vect}(M)$.

## Definition

Let $\xi \in \mathfrak{g}$. The vector field $\hat{\xi}$ on $M$ associated to $\xi$ is defined as

$$\hat{\xi}(x) = \frac{d}{dt}|_{t=0}\widehat{\exp t\xi}(x).$$

The map $\hat{}: \mathfrak{g} \to \mathrm{Vect}(M)$ can naturally be considered as the *differential of the action* $\hat{}: G \to \mathrm{Diff}(M)$.

## Proposition

$\hat{}: \mathfrak{g} \to \mathrm{Vect}(M)$ is an (anti)homomorphism of Lie algebras, i.e., is linear and satisfies

$$\widehat{[\xi, \eta]} = -[\hat{\xi}, \hat{\eta}].$$

Let $u, v$ be two vector fields on $M$ and $\Phi_u^t$, $\Phi_v^s$ be the corresponding flows viewed as one-parameter groups of diffeomorphisms. Using our interpretation of $\mathrm{Diff}(M)$ and $\mathrm{Vect}(M)$ as an infinite dimensional Lie group and its Lie algebra, we may say that $\Phi_u^t = \exp tu$, $\Phi_v^s = \exp sv$.
Then the standard Lie bracket of $u$ and $v$ can be defined by the formula:

$$[u, v](x) = \frac{d^2}{dtds}|_{t=0,s=0}\Phi_u^{-t}\Phi_v^s\Phi^t{}_u(x),$$

whereas, according to our standard definition of the Lie algebra commutator, we have

$$[\xi, \eta] = \frac{d^2}{dtds}|_{t=0,s=0} \exp t\xi \exp s\eta \exp(-t\xi).$$

It is easy to see that these two definitions differs by sign.

Notice that these two formulas immediately give the proof of the proposition.

**Theorem**

*A smooth action $G \to \mathrm{Diff}(M)$ of a connected Lie group $G$ can be uniquely reconstructed from its differential $\mathfrak{g} \to \mathrm{Vect}(M)$.*

# Orbits

We consider an action of a Lie group $G$ on a smooth manifold $M$.

**Definition**

The orbit $\mathcal{O}(x)$ of a point $x \in M$ is the set of points of $M$ to which $x$ can be moved by the elements of $G$:

$$\mathcal{O}(x) = \{y \in M \mid y = \hat{g}(x),\ g \in G\}$$

It is easy to see that

- $x \in \mathcal{O}(x)$;
- if $y \in \mathcal{O}(x)$, then $\mathcal{O}(x) = \mathcal{O}(y)$.

In particular, $M$ can be presented as the disjoint union of orbits.
In general, $\mathcal{O}(x)$ is not necessarily a submanifold in $M$, but it is always an immersed submanifold.

**Example**

For the action of $SO(3)$ on the Euclidean space $\mathbb{R}^3$, the orbits are of two types: 1) spheres centered at the origin, 2) the origin (exceptional orbit).

# Stabilizers

## Definition

For every $x \in M$, we define the stabilizer subgroup of $x$ (also called the isotropy group or stationary subgroup) as the set of all elements in $G$ that fix $x$:

$$\mathrm{St}(x) = \{g \in G \mid \hat{g}(x) = x\}$$

This is indeed a subgroup of $G$, moreover this subgroup is closed. Also it is easy to see that if $x$ and $y$ belong to the same orbit, then their stabilizers $\mathrm{St}(x)$ and $\mathrm{St}(y)$ are isomorphic. Indeed, if $y \in \mathcal{O}(x)$, then there is $a \in G$ s.t. $y = \hat{a}(x)$. Then

$$g \in \mathrm{St}(x) \quad \text{if and only if} \quad aga^{-1} \in \mathrm{St}(y)$$

In other words, $\mathrm{St}(y) = a\,\mathrm{St}(x)\,a^{-1}$, i.e. the stabilizers are conjugate.

## Example

For the standard action of $SO(3)$ on $\mathbb{R}^3$, the stabilizer $St(P)$, where $P = (0,0,1) \in \mathbb{R}^3$, is

$$St(P) = \left\{ \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \simeq SO(2) \subset SO(3).$$

For $P_0 = (0,0,0)$, the stabilizer $St(P_0)$ is the whole group $SO(3)$.

# Invariants

## Definition

An invariant of an action of $G$ on $M$ is a smooth function $f : M \to \mathbb{R}$ with the property:

$$f(\hat{g}(x)) = f(x) \quad \text{for any } x \in M, g \in G.$$

Sometimes one wants to consider polynomial or rational invariants only.

## Proposition

*Let $G$ be a connected Lie group acting on $M$. Then $f : M \to \mathbb{R}$ is an invariant if and only if $f$ satisfies the following relations:*

$$\hat{\xi}(f) = 0, \quad \textit{for any } \xi \in \mathfrak{g}.$$

The set of invariants is a ring in the sense that it is closed under multiplication and addition. Moreover, if $f_1, \dots, f_k$ are invariants, then any smooth function $F(f_1, \dots, f_k)$ is an invariant too.
The number of "functionally independent" invariants is bounded by the codimension of a generic orbit.

## Example

For the standard action of $SO(3)$ on $\mathbb{R}^3$, there is essentially only one invariant:

$$f(x, y, z) = x^2 + y^2 + z^2.$$

# Example

The action of $\mathbb{R}$ (under addition) on the torus $T^2$ by translations:

$$t \in \mathbb{R} \quad \mapsto \qquad \hat{t}(\phi_1, \phi_2) = (\phi_1 + at, \phi_2 + bt).$$

Assume that $a/b$ is irrational. Then

- the orbit of any point $(\phi_1, \phi_2) \in T^2$ is an irrational winding. It is not a submanifold, however it is the image of $\mathbb{R}$ under immersion.
- the stabilizer of any point $(\phi_1, \phi_2) \in T^2$ is trivial, i.e., $\{0\}$.
- there are no smooth invariants, since orbits are everywhere dense; however a local invariant exists: $f(\phi_1, \phi_2) = b\phi_1 - a\phi_2$.

Assume that $a/b = n/m$ is rational (we take $m > 0$ minimal). Then

- the orbit of any point $(\phi_1, \phi_2) \in T^2$ is closed and diffeomorphic to $S^1$,
- the stabilizer of any point $(\phi_1, \phi_2) \in T^2$ is of the form $\alpha\mathbb{Z} = \langle \alpha \rangle$, where $\alpha = 2\pi n/a = 2\pi m/b$,
- there is one invariant function $f(\phi_1, \phi_2) = m\phi_1 - n\phi_2$.

# Example

The adjoint action of $GL(n, \mathbb{R})$ on itself:

$$\text{For } C \in GL(n, \mathbb{R}), \text{ we consider } \hat{C}(X) = A_C(X) = CXC^{-1}.$$

- The orbit of $X \in GL(n, \mathbb{R})$ consists of all the matrices similar to $X$. Two matrices belong to the same orbit if and if they have the same Jordan normal form. In particular, if $X$ and $Y$ are diagonalizable, then $\mathcal{O}(X) = \mathcal{O}(Y)$ if and only if $X$ and $Y$ have the same eigenvalues.
- The stabilizer of $X$ consists of those non-degenerate matrices $C$ that commute with $X$. In particular, if $X$ is diagonal with distinct diagonal elements, then $\mathrm{St}(X)$ is the subgroup of diagonal matrices in $GL(n, \mathbb{R})$. If $X = \lambda \cdot Id$, then $\mathrm{St}(X) = GL(n, \mathbb{R})$.
- The (basis) invariants of this action are the coefficients $f_0(X), \dots, f_{n-1}(X)$ of the characteristic polynomial of $X$:

  $$P_t(X) = \det(X - t \cdot Id) = f_0(X) + f_1(X)t + f_2(X)t^2 + \dots + f_{n-1}(X)t^{n-1} + (-t)^n.$$

  In partucular, $\operatorname{tr} X$ and $\det X$ are invariants of the adjoint action.

### Definition

An action of $G$ on $M$ is called transitive, if the only orbit is $M$ itself.

Construction: Let $G$ act on $M$ transitively. Fix a point $x \in M$ and for any $y \in M$, consider the subset

$$G_y = \{g \in G \mid \hat{g}(x) = y\}$$

Equivalently, we can consider the natural projection $\pi : G \to M$ defined by $\pi(g) = \hat{g}(x)$. Then $G_y$ can be characterized as the preimage $\pi^{-1}(y)$ (i.e., fiber of this projection over the point $y \in M$).
It is easy to see that $g_1, g_2 \in G_y$ if and only if $g_1^{-1} g_2 \in \mathrm{St}(x)$. This means, in fact, that $g_2$ and $g_1$ belongs to the same left coset of the stabilizer subgroup $\mathrm{St}(x)$.
Conclusion: there is a natural bijection between $M$ and the space $G/\mathrm{St}(x)$ of left cosets w.r.t. $\mathrm{St}(x)$.
Conversely: if $H \subset G$ is a closed subgroup, then we can consider the canonical projection $\pi : G \to G/H$, $g \mapsto gH$ (where $G/H$ is considered as the (abstract) space of left cosets of $H$). Obviously, we can define the natural action of $G$ on $G/H$ (in algebraic sense)

$$\hat{a}(gH) = (ag)H.$$

Notice that the stabilizer of this action for the subgroup $H$ (viewed as a left coset) is this subgroup itself.

### Theorem

*Let $H \subset G$ be a Lie subgroup. Then the space of left cosets $G/H$ can be endowed with the structure of a smooth manifold in such a way that:*

- *the canonical projection $\pi : G \to G/H$ is smooth and, moreover, is a locally trivial fibration;*
- *the canonical action of $G$ on $G/H$ is smooth;*
- *if $H$ is normal, then $G/H$ is a Lie group and $\pi$ is a Lie group homomorphism.*

It is important that the above smooth structure on $G/H$ is unique.

### Definition

A smooth manifold $M$ endowed with a transitive action of a Lie group $G$ is called a homogeneous space of $G$.

According to this theorem, every homogeneous space $M$ of a Lie group $G$ is isomorphic to $G/H$ for a certain Lie subgroup $H \subset M$.

### Example

$SO(3)$ acts transitively on $S^2$. Hence $S^2 = SO(3)/SO(2)$.

### Example

$SL(2,\mathbb{R})$ acts transitively on the upper half plane $L \subset \mathbb{C}$. The stabiliser of $i \in L$ consists of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \quad \text{satisfying} \quad \frac{ai+b}{ci+d} = i$$

It is easy to see that $a = d$, $b = -c$ and $a^2 + b^2 = 1$, thus $\mathrm{St}(i) = SO(2)$. Hence $L = SL(2,\mathbb{R})/SO(2)$.

### Example

Consider the action of $GL(n,\mathbb{R})$ on the space $\mathcal{P}$ of positive definite bilinear forms. We may think of such forms as symmetric positive definite matrices. Then the action is given by:

$$\hat{X}(B) = XBX^\top, \qquad X \in GL(n,\mathbb{R}), \quad B \ \text{ symmetric positive definite}$$

This action is transitive because each form can be reduced to the canonical form which is simply $E_n = Id$. The stabilizer of this form is:

$$\mathrm{St}(E_n) = \{X \in GL(n,\mathbb{R}) \mid XE_nX^\top = XX^\top = E_n\} = O(n)$$

Hence we conclude $\mathcal{P} = GL(n,\mathbb{R})/O(n)$.

We consider a smooth action of a Lie group $G$ on a smooth manifold $M$.

### Definition

An action of $G$ on $M$ is called *transitive*, if the only orbit is $M$ itself.

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Hence we conclude $\mathcal{P} = GL(n,\mathbb{R})/O(n)$.

# Linear representations of a Lie group

A linear representation of $G$ is a particular case of action: now $M$ is a vector space $V$, and each diffeomorphism $\hat{a} : V \to V$ is linear.

## Definition

A *linear representation* of a Lie group $G$ on a vector space $V$ is a homomorphism $\Phi : G \to GL(V)$.

Here by $GL(V)$ we mean the group of all invertible linear transformations of $V$:

$$GL(V) = \{X : V \to V \mid X \text{ is linear and invertible}\}.$$

If $V = \mathbb{R}^n$, then $GL(V) = GL(n, \mathbb{R})$.
Each linear representation $\Phi : G \to GL(V)$ induces the linear representation of the corresponding Lie algebra $\mathfrak{g}$ on $V$:

$$\phi = d\Phi : \mathfrak{g} \to \mathrm{End}(V).$$

We use standard notation $\mathrm{End}(V)$ for the Lie algebra of *all* linear transformations on $V$. Alternatively, we can say $\mathrm{End}(V) = gl(V)$.

We shall consider only finite dimensional case: $\dim V < \infty$, however infinite-dimensional representations are also very important.

### Theorem

*Let $G$ be connected and simply connected. Then for any linear representation $\phi$ of the Lie algebra $\mathfrak{g}$ on $V$ there is a unique linear representation of the Lie group $G$ on $V$ such that $\phi = d\Phi$.*

All basic notions related to actions:

- orbits
- stabilizers
- invariants

are naturally transferred to the case of representations.

### Definition

A *faithful representation* $\Phi$ of $G$ on a vector space $V$ is a linear representation in which different elements $g$ of $G$ are represented by distinct linear mappings $\Phi(g)$, in other words, $\Phi : G \to GL(V)$ is injective.
Similarly for Lie algebras.

(In the case of actions, in a similar situation we say that the action is *effective*.)

# Examples

- All classical matrix groups $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $O(n)$, $SO(n)$, upper triangular group $T(n)$ have the standard (natural) representation on $\mathbb{R}^n$.
- *Adjoint representation* of $G$ on its Lie algebra $\mathfrak{g}$

$$\mathrm{Ad} : G \to GL(\mathfrak{g}), \quad \mathrm{Ad}_a(\eta) = \frac{d}{dt}|_{t=0} a \cdot \exp t\eta \cdot a^{-1}.$$

- *Adjoint representation* of $\mathfrak{g}$ on itself (as the differential of $d\Phi$):

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad \mathrm{ad}_\xi \eta = [\xi, \eta] = \frac{d}{dt}|_{t=0} \mathrm{Ad}_{\exp t\xi}(\eta).$$

  Notice that the fact that $\xi \mapsto \mathrm{ad}_\xi$ is a representation is equivalent to the Jacobi identity, so that this representation can be introduced for any abstract Lie algebra (without referring to Lie groups).
- For any matrix Lie group $G \subset GL(n,\mathbb{R})$ one can introduce two natural actions on the space of $n \times n$ matrices:

$$X \in G \quad \mapsto \quad \mathrm{Ad}_X(B) = XBX^{-1},$$

  and

$$X \in G \quad \mapsto \quad \mathrm{Sq}_X(B) = XBX^{\top}.$$

  In the first case the space of $n \times n$-matrices is naturally identified with $\mathrm{End}(\mathbb{R}^n)$, whereas in the second case, this space is viewed as the space of bilinear forms.

# Reducible and irreducible representations

Consider a representation $\Phi : G \to GL(V)$ of a (connected) Lie group $G$ on a vector space $V$ (and resp. $\phi : \mathfrak{g} \to \mathrm{End}(V)$ for its Lie algebra).

### Definition

A subspace $L \subset V$ is called *$\Phi$–invariant*, if it is invariant under any operator $\Phi(g)$, $g \in G$. In other words, $\Phi(g)v \in L$ for any $v \in L$, $g \in G$. (The same definition for $\phi$-invariance).

### Definition

The representation $\Phi$ is called *irreducible*, if it does not admits any non-trivial invariant subspaces. Otherwise, it is called *reducible*.

Notice that "$L$ is $\Phi$–invariant" is equivalent to "$L$ is $\phi$–invariant" so that $\Phi$ and $\phi$ are irreducible or reducible simultaneously.

### Example

- The natural representation of $SO(n)$ on $\mathbb{R}^n$ is irreducible.
- The natural representation of the upper triangular group $T(n)$ on $\mathbb{R}^n$ is reducible. As an invariant subspace, we can take, for instance, the one-dimensional subspace generated by the first basis vector $e_1$.
- The action Sq of $GL(n, \mathbb{R})$ on the space of bilinear forms (see above) is reducible. There are two invariant subspaces: symmetric forms and skew-symmetric forms.

If $L \subset V$ is an invariant subspace for $\Phi$, then we can "restrict" our representation onto $L$:

$$\Phi|_L : G \to GL(L), \qquad \Phi|_L(v) = \Phi(v) \in L \quad \text{for } v \in L,$$

and, similarly, $\phi|_L : G \to \mathrm{End}(L)$.

**Definition**

$\Phi|_L$ and $\phi|_L$ are *restrictions* of $\Phi$ and $\phi$ respectively onto the invariant subspace $L$.

Let $L$ be invariant, consider the quotient space $V/L$. It turns out that $\Phi$ induces a natural representation $\tilde{\Phi}$ on $V/L$ by:

$$\tilde{\Phi}(g)(v + L) = \Phi(g)v + L.$$

**Definition**

$\tilde{\Phi} : G \to GL(V/L)$ is a representation called *quotient representation* of $G$ on $V/L$. Similarly, the representation $\tilde{\phi} : \mathfrak{g} \to \mathrm{End}(V/L)$ defined by the same formula

$$\tilde{\phi}(\xi)(v + L) = \phi(\xi)v + L$$

is called the *quotient representation* of $\mathfrak{g}$ on $V/L$.

# Example

The natural representation of the upper triangular group $T(n)$ on $\mathbb{R}^n$ has the invariant subspace $L$ spanned by the $k$ first basis vectors $e_1, \dots, e_k$:

$$L = \{(x_1, \dots, x_k, 0, \dots, 0)\}.$$

Thus we can restrict this representation onto $L \simeq \mathbb{R}^k$ and consider the quotient representation on $\mathbb{R}^n / L \simeq \mathbb{R}^{n-k}$. If

$$X = \Phi(X) = \begin{pmatrix} \boxed{\begin{matrix} t_{11} & * & * \\ 0 & \ddots & * \\ 0 & 0 & t_{kk} \end{matrix}} & \begin{matrix} * & * & * \\ * & * & * \\ * & * & * \end{matrix} \\ \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} & \boxed{\begin{matrix} t_{k+1,k+1} & * & * \\ 0 & \ddots & * \\ 0 & 0 & t_{nn} \end{matrix}} \end{pmatrix} \in T(n)$$

The the matrices giving the restriction and the quotient are respectively:

$$\Phi|_L(X) = \begin{pmatrix} t_{11} & * & * \\ 0 & \ddots & * \\ 0 & 0 & t_{kk} \end{pmatrix} \quad \text{and} \quad \tilde{\Phi}(X) = \begin{pmatrix} t_{k+1,k+1} & * & * \\ 0 & \ddots & * \\ 0 & 0 & t_{nn} \end{pmatrix}$$

Costruction: Let $\Phi_1$ and $\Phi_2$ be representations of $G$ on vector spaces $V_1$ and $V_2$ respectively. Then we can construct a natural representation of $G$ on the direct sum $V_1 \oplus V_2$ by assigning to each $g \in G$ the operator on $V_1 \oplus V_2$ acting as follows:

$$(v_1, v_2) \quad \mapsto \quad (\Phi_1(g)v_1, \Phi_2(g)v_2)$$

(Here we consider elements of $V_1 \oplus V_2$ as pairs $(v_1, v_2)$, $v_i \in V_i$.) This representation is called the *sum* $\Phi_1 + \Phi_2$ of $\Phi_1$ and $\Phi_2$.
If we think of $\Phi_i(g)$ as a square matrix (of dimension $k_i = \dim V_i$), $i = 1, 2$, then $(\Phi_1 + \Phi_2)(g)$ is the square matrix of dimension $k_1 + k_2$ of the form:

$$(\Phi_1 + \Phi_2)(g) = \begin{pmatrix} \Phi_1(g) & 0 \\ 0 & \Phi_2(g) \end{pmatrix}.$$

Obviously, the representation $\Phi_1 + \Phi_2$ is reducible and $V_1, V_2 \subset V_1 \oplus V_2$ are its invariant subspaces.

### Definition

A represenatation $\Phi$ of $G$ on $V$ is called *completely reducible* (or *semisimple*) if it can be presented as the sum of irreducible representations. In other words, $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ is the direct sum of invariant subspaces, and each restriction $\Phi|_{V_i}$ is irreducible.

# Examples

- The (adjoint) action of $SO(n)$ on the vector space of all $n \times n$ matrices is completely reducible. There are three invariant subspaces:
  - skew-symmetric matrices;
  - scalar matrices $\lambda \cdot Id$, $\lambda \in \mathbb{R}$;
  - symmetric matrices with zero trace.

  On each of these subspaces the representation is irreducible (except for one particular case $SO(4)$, where the space of skew-symmetric matrices splits into two 3-dimensional invariant subspaces; but the representation is still completely reducible).
- The natural action of the upper triangular group $T(n)$ on $\mathbb{R}^n$ is reducible, but not completely reducible. Invariant subspaces $L \subset \mathbb{R}^n$ exist but they do not admit any invariant complement (i.e., invariant $L' \subset \mathbb{R}^n$ such that $\mathbb{R}^n = L \oplus L'$).
- Any (reducible) linear representation of a compact Lie group $G$ is completely reducible.
  This follows immediately from the following statement: For any linear representation $\Phi$ of a compact Lie group $G$ on $V$ there exists a positive definite $G$-invariant bilinear form $B$ (that is $B(\Phi(g)v_1, \Phi(g)v_2) = B(v_1, v_2)$ for any $v_1, v_2 \in V$, $g \in G$). Hence, "$L$ is invariant " implies "$L^\perp$ is invariant" so that we may proceed by induction.