An In-Depth Look at Singular Value Decomposition (SVD)

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1 Introduction

Singular Value Decomposition (SVD) is a powerful matrix factorization method that can be applied to any real or complex matrix, even if the matrix is not square or of full rank. It is one of the most important tools in linear algebra with applications ranging from solving linear systems, signal processing, and data compression to machine learning and dimensionality reduction.

Given a matrix $A \in \mathbb{R}^{m \times n}$, the Singular Value Decomposition of A expresses it as a product of three matrices:

$$A = U\Sigma V^T$$

where: - $U \in \mathbb{R}^{m \times m}$ is an orthogonal matrix (the columns of U are called the left singular vectors of A), - $\Sigma \in \mathbb{R}^{m \times n}$ is a diagonal matrix (with non-negative real numbers on the diagonal, called the singular values of A), - $V \in \mathbb{R}^{n \times n}$ is an orthogonal matrix (the columns of V are called the right singular vectors of A).

This decomposition has various applications, such as in Principal Component Analysis (PCA) and low-rank approximations of matrices. The singular values in Σ give information about the scaling of the data in different directions.

2 Mathematical Properties of SVD

The decomposition comes with several useful properties that help us understand the structure of the matrix A.

- **Orthogonality of U and V**: The matrices U and V are orthogonal, meaning:

$$U^T U = I_m \quad \text{and} \quad V^T V = I_n$$

  where I_m and I_n are the identity matrices of size m and n, respectively.

- **Singular values**: The entries of the matrix Σ are the singular values $\sigma_1, \sigma_2, \ldots, \sigma_r$, which are non-negative and arranged in descending order:

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$$

  where $r = \text{rank}(A)$.

- **Rank of** A: The rank of the matrix A is equal to the number of non-zero singular values of A, i.e., $\text{rank}(A) = r$.

- **Low-rank approximation**: The SVD provides an optimal low-rank approximation of the matrix A in the sense that truncating the singular values gives the best approximation in terms of the Frobenius norm.

- **Eigenvalue relationship**: The singular values σ_i of A are the square roots of the eigenvalues of $A^T A$ (or equivalently AA^T).

3 Steps to Compute SVD

The process of computing the SVD of a matrix involves several steps:

1. Compute $A^T A$ or AA^T: These matrices are symmetric and have the same non-zero eigenvalues as A, which are the squared singular values.

2. Find the eigenvalues and eigenvectors of $A^T A$ (or AA^T): The eigenvectors of $A^T A$ are the columns of V, and the eigenvectors of AA^T are the columns of U.

3. Construct the matrix Σ: The diagonal elements of Σ are the square roots of the eigenvalues.

4 Worked Example of SVD

Let us now walk through a concrete example of computing the Singular Value Decomposition of a small matrix.

Consider the matrix A:

$$A = \begin{pmatrix} 4 & 0 \\ 3 & -5 \end{pmatrix}$$

We will find its SVD by following the steps outlined above.

4.1 Step 1: Compute $A^T A$

We first compute the transpose of A, denoted A^T:

$$A^T = \begin{pmatrix} 4 & 3 \\ 0 & -5 \end{pmatrix}$$

Next, we compute the matrix product $A^T A$:

$$A^T A = \begin{pmatrix} 4 & 3 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 3 & -5 \end{pmatrix} = \begin{pmatrix} 16+9 & 0-15 \\ 0-15 & 0+25 \end{pmatrix} = \begin{pmatrix} 25 & -15 \\ -15 & 25 \end{pmatrix}$$

4.2 Step 2: Find the Eigenvalues of $A^T A$

Now, we solve for the eigenvalues of $A^T A$ by finding the determinant of $A^T A - \lambda I$, where λ is an eigenvalue and I is the identity matrix:

$$\det(A^T A - \lambda I) = \det \begin{pmatrix} 25-\lambda & -15 \\ -15 & 25-\lambda \end{pmatrix} = (25-\lambda)^2 - (-15)^2 = 0$$

Expanding:

$$(25-\lambda)^2 - 225 = 0$$

$$625 - 50\lambda + \lambda^2 - 225 = 0$$

$$\lambda^2 - 50\lambda + 400 = 0$$

Solving this quadratic equation:

$$\lambda = \frac{50 \pm \sqrt{50^2 - 4 \times 1 \times 400}}{2 \times 1} = \frac{50 \pm \sqrt{2500 - 1600}}{2} = \frac{50 \pm \sqrt{900}}{2}$$

$$\lambda_1 = \frac{50+30}{2} = 40, \quad \lambda_2 = \frac{50-30}{2} = 10$$

The eigenvalues of $A^T A$ are $\lambda_1 = 40$ and $\lambda_2 = 10$.

4.3 Step 3: Find the Eigenvectors of $A^T A$

Next, we find the eigenvectors corresponding to each eigenvalue.

For $\lambda_1 = 40$, we solve:

$$(A^T A - 40I)v = 0$$

$$\begin{pmatrix} 25-40 & -15 \\ -15 & 25-40 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

This simplifies to:

$$\begin{pmatrix} -15 & -15 \\ -15 & -15 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

Solving, we get:

$$v_1 = v_2$$

Thus, the eigenvector corresponding to $\lambda_1 = 40$ is $v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 10$, we solve similarly:

$$(A^T A - 10I)v = 0$$

$$\begin{pmatrix} 15 & -15 \\ -15 & 15 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

Solving, we get:

$$v_1 = -v_2$$

Thus, the eigenvector corresponding to $\lambda_2 = 10$ is $v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Therefore, the matrix V is:

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

4.4 Step 4: Find the Left Singular Vectors U

Now, we compute the left singular vectors of A. The left singular vectors u_i are the eigenvectors of AA^T.

For each singular value σ_i, we solve the system $Av_i = \sigma_i u_i$.

The final matrices U, Σ, and V represent the Singular Value Decomposition of A.

5 Conclusion

Singular Value Decomposition (SVD) provides a decomposition of a matrix into three matrices, where the singular values give insights into the magnitude of the data in different directions. The decomposition is crucial for numerous applications, including data compression, noise reduction, and dimensionality reduction. The steps of SVD can be generalized to compute it for larger, real-world matrices as well.