The Mathematics of Relativity

Johar M. Ashfaque

Einstein's theory of gravity is based on two physical assumptions: gravity is geometry and matter curves space-time. In particular, this means that on the one hand, particles follow geodesics in curved space-time; the resulting motion appears to an observer as the effect of gravity. On the other hand, matter is also a source of space-time curvature and hence of gravity.

Space-time is modelled in terms of differentiable manifolds. This reasonable since physics does not depend on the coordinate system chosen and therefore has to be invariant under general coordinate transformations. We begin by reviewing the basic concepts of manifolds and introduce tensors. Moreover, we will construct covariant derivatives. Einstein's equations of gravity are then formulated while their solutions in the subsequent sections: maximally symmetric space-times and black hole solutions. Finally, the energy-momentum tensor in curved space has to satisfy important conditions, the energy conditions will be discussed.

1 Differential Geometry

1.1 Manifolds, Tangent and Cotangent Spaces

Let us consider a d-dimensional differentiable real manifold M. A coordinate system x is a map from a subset of M to $\mathbb{R}^d$. Of course, for a manifold M, a unique or a preferred coordinate system does not exist. However, the change from one coordinate system x to another coordinate system x' which is defined for the same subset of M for simplicity, has to be smooth. In other words, the transition functions $x \circ x'^{-1}$ and $x' \circ x^{-1}$ have to be differentiable maps from $\mathbb{R}^d$ to $\mathbb{R}^d$. At each point $p \in M$, we may define $T_p(M)$ as the space of tangent vectors. This vector space $T_p(M)$ is d-dimensional and a particular set of basis vectors is given by $\partial_\mu = \partial/\partial x^\mu$ where x^μ are the coordinates. Therefore any vector $V \in T_p(M)$ may be written as

$$V = V^\mu \partial_\mu.$$

Besides the tangent space $T_p(M)$ for every point $p \in M$, we may also define the corresponding cotangent space $T_p^*(M)$ consisting of all linear maps from $T_p(M)$ into $\mathbb{R}$. The basis ∂_μ of the tangent space $T_p(M)$ induces a dual basis dx^ν of the cotangent space $T_p^*(M)$ by virture of

$$dx^\nu(\partial_\mu) = \delta^\nu_\mu.$$

Therefore, as in the case of the tangent space, every vector of the cotangent space $W \in T_p^*(M)$ may be written as

$$W = W_\nu dx^\nu.$$

1.2 Covariance and Tensors

Since physics does not depend on the coordinates which we have chosen, it is essential that all physical statements are independent of the choice of coordinates. A change of coordinates $x \mapsto x'$ induces a change of basis in the tangent space. In particular, basis vectors $\partial_\nu = \partial/\partial x^\nu$ transform into $\partial'_\nu = \partial/\partial x'^\nu$ according to

$$\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu$$

where we have used the chain rule. Under a coordinate transformation $x \mapsto x'$, any vector $V \in T_p(M)$ is invariant

$$V = V^\nu \partial_\nu = V'^\mu \partial'_\mu.$$

However, since the basis vectors transform as

$$\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu$$

we have to impose the following transformation properties on the components of V^μ

$$V'^\mu = V^\nu \frac{\partial x'^\mu}{\partial x^\nu}.$$

In a similar fashion, we can derive the transformation property of cotangent vectors $W \in T^*_p(M)$. Since the basis vectors transform as

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$$

under $x \mapsto x'$, invariance of W implies that the components W_μ have to transform as

$$W = W_\nu dx^\nu = W'_\mu dx'^\mu = W'_\mu \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \Rightarrow W'_\mu = W_\nu \frac{\partial x^\nu}{\partial x'^\mu}.$$

1.3 The Metric

An important role is played by the metric, a particular $(0,2)$ tensor g which at each point $p \in M$ is a non-degenerate symmetric bilinear form g i.e.

$$g : \begin{cases} T_p(M) \times T_p(M) \to \mathbb{R} \\ (u,v) \mapsto g(u,v) \end{cases}$$

with $g(u,v) = g(v,u)$. Non-degeneracy means that for every $u \neq 0$ there exists a vector v such that $g(u,v) \neq 0$. In other words, there is no vector (besides the zero vector) which is orthogonal to every other vector in $T_p(M)$. Note that we do not impose positivity of the metric. This allows us to define space-like, time-like and null vectors.

The metric may be expressed in terms of the basis vectors $dx^\mu \otimes dx^\nu$ of $T_p(M) \times T_p(M)$ using the components $g_{\mu\nu}(x)$ as

$$ds^2 \equiv g_{\mu\nu}(x) dx^\mu \otimes dx^\nu.$$

We will suppress $\otimes$ ad write $g_{\mu\nu}(x)dx^\mu dx^\nu$ as a shorthand notation. This leads to the introduction of the notion of an infinitesimal line element. If $g_{\mu\nu}$ has only positive eigenvalues the manifold is Riemannian while if it has one negative eigenvalue the manifold is Lorentzian. For Lorentzian manifolds, the infinitesimal line element ds^2 determines whether a vector dx^μ, viewed as an infinitesimal distance between points on a manifold, is space-like, time-like or light-like depending on the sign of ds^2.

In the case of a flat d-dimensional Minkowski space, the metric components $g_{\mu\nu}$ are given by $\eta_{\mu\nu}$. Let us consider a few examples of Lorentzian manifolds, restricted to four space-time dimensions for simplicity.

- One example if the Schwarzschild metric for a black hole of mass M

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

  where G is the Newton's constant.

- Another example is the de Sitter metric describing an accelerated expansion of the universe in the inflationary phase

$$ds^2 = -dt^2 + e^{2Ht}((dx_1)^2 + (dx_2)^2 + (dx_3)^2).$$

As in the case of flat space-time, we can use the metric with components $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ to lower and raise indices. In particular, a vector $V \in T_p(M)$ with components $V = V^\mu \partial_\mu$ may be mapped to $W = W_\mu dx^\mu \in T^*_p(M)$ using $W_\mu = g_{\mu\nu}V^\nu$. In other words, the metric induces a natural isomorphism between the tangent space $T_p(M)$ and the cotangent space $T^*_p(M)$.

1.4 Covariant Derivative

With the exception of scalar fields, the partial derivative of any (r, s) tensor is no longer a tensor. For example, let us consider a one-form W i.e. $(0, 1)$ tensor and take the derivative with respect to ∂_μ. From the index structure, we expect that $\partial_\mu W_\nu$ transforms as a $(0, 2)$ tensor. However, under a change of coordinates $\partial_\mu W_\nu$ transforms as

$$\partial'_\mu W'_\nu = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\left(\frac{\partial}{\partial x^\rho}W_\sigma\right) + W_\sigma \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial^2 x^\sigma}{\partial x^\rho \partial x'^\nu}.$$

While the first term is the expected transformation law of a $(0, 2)$ tensor, the second being the inhomogeneous term spoils it. This is always the case when a global symmetry is promoted to a local symmetry and the introduction of a covariant derivative becomes necessary. The covariant derivatives denoted by ∇_μ should satisfy the following properties

- ∇_μ maps (r, s) tensors to $(r, s + 1)$ tensors,
- ∇_μ is linear,
- ∇_μ satisfies the Leibniz rule.

These properties imply that the covariant derivative will be the standard derivative plus a correction term namely the connection. The connection in general relativity consists of a $d \times d$ matrix $\Gamma^\nu_{\mu\lambda}$ for each coordinate labelled by μ.

1.5 Lie Derivative

The covariant derivative is not the only way to define a derivative. Another possibility is given by the Lie derivative $\mathcal{L}_V$ along a vector field $V = V^\mu(x)\partial_\mu$. For example, the Lie derivative acts on the scalar $\phi(x)$ as

$$\mathcal{L}_V \phi(x) = V^\rho(x)\partial_\rho \phi(x).$$

Note that $\mathcal{L}_V \phi(x)$ is again a scalar field. We may extend the definition of a Lie derivative to arbitrary tensor fields. For example, applied to vector fields and one-forms the Lie derivative reads

$$\mathcal{L}_V U^\mu = V^\rho \partial_\rho U^\mu - (\partial_\rho V^\mu)U^\rho,$$

$$\mathcal{L}_V W_\mu = V^\rho \partial_\rho W_\mu + (\partial_\mu V^\rho)W_\rho.$$

2 Einstein's Field Equation

In this section, we introduce Einstein's field equations for gravity. A first guess for the field equations of gravity, relating geometry and matter is $R_{\mu\nu} \propto T_{\mu\nu}$. However, this turns out to be incorrect, since the energy conservation $\nabla^\mu T_{\mu\nu} = 0$ would then imply $\nabla^\mu R_{\mu\nu} = 0$ which is not true in general. However, the Einstein tensor defined by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$$

satisfies $\nabla^\mu G_{\mu\nu} = 0$ and therefore Einstein introduced the field equations

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}$$

which give rise to the correct gravitational law in the Newtonian approximation. The parameter Λ is the cosmological constant and κ is related to the Newton's gravitational constant G by $\kappa^2 = 8\pi G$.

3 Maximally Symmetric Space-Times

Let us discuss the solutions of Einstein's equations with $T_{\mu\nu} = 0$ i.e. vacuum solutions without any matter. In particular, we are interested in those space-times with maximal symmetry. The symmetries of space-time are given by Killing vector fields X which by definition satisfy $\mathcal{L}_X g_{\mu\nu} = 0$. A Killing vector field is linear dependent if it can be written as a linearly combination of other Killing vector fields with constant coefficients.

The question arises how many linearly independent Killing fields i.e. isometries, may a manifold have. For example, the isometries of d-dimensional Minkowski space are given by Lorentz transformations and translations in space and time. Therefore, we have d translational isometries and $d(d-1)/2$ rotational isometries including boosts i.e. in total $d(d+1)/2$ isometries. It can be shown that a manifold of dimension d can only have most $d(d+1)/2$ linearly independent Killing vector fields. The space-times which satisfy this bound are referred to as maximally symmetric space-times. Minkowski space-time is therefore an example of a maximally symmetric space-time.

For maximally symmetric space-time, the curvature has to be the same everywhere. If we think in terms of translational and rotational isometries, it is obvious that the curvature has to be the same at each point of the manifold and in each direction. Therefore we should be able to express the Riemann tensor in term of the Ricci scalar as

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)}(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}).$$

Therefore we see that we may classify maximally symmetric space-times according to their dimension, the value of the Ricci scalar as well as whether the space-time manifold of Riemannian or Lorentzian.

Let us first consider the case of Riemannian manifolds in which the maximally symmetric space-times are locally Euclidean, spherical or hyperbolic. The line element of these spaces may be written in a compact way

$$ds^2 = \frac{d\chi^2}{1-k\chi^2} + \chi^2 d\Omega_{d-1}^2 \equiv dK_d^2$$

where $k \in \{0, \pm 1\}$ and $d\Omega_{d-1}^2$ is the line element of the unit sphere S^{d-1}. For $k = 0$, we obtain the Euclidean space in spherical coordinates where χ is the radial coordinate. For $k = 1$ after the coordinate transformation $\chi = \sin\phi$ where $\phi \in [0, \pi[$, the line element reads

$$ds^2 = d\phi^2 + \sin^2\phi d\Omega_{d-1}^2$$

which corresponds to a unit sphere. In the case of $k = -1$, we can use $\chi = \sinh\psi$ with $\psi \in [0, \infty[$ to get the line element of a hyperboloid

$$ds^2 = d\psi^2 + \sinh^2\psi d\Omega_{d-1}^2.$$

Also in the case of the Lorentzian manifold, we find three maximally symmetric space-times depending on the sign of the Ricci scalar R. For $R = 0$, the maximally symmetric space-time is Minkowski space-time. For $R < 0$, the maximally symmetric space-time is AdS space and for $R > 0$, the maximally symmetric space-time is the dS space.

3.1 Minkowski Space-Time

The d-dimensional Minkowski space-time is a solution to the vacuum Einstein equations with $\Lambda = 0$. We use coordinates such that the line element ds is given by

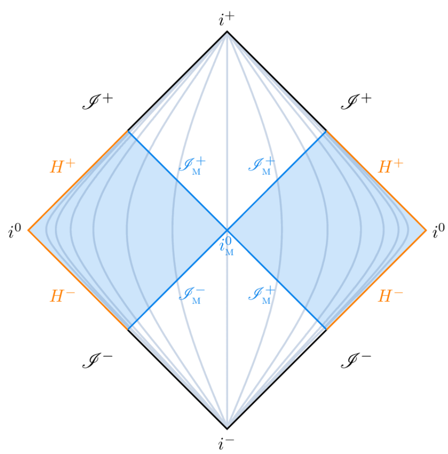
$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu.$$

Let us study the causal structure of this space-time. In general, the causal structure may be visualized by a conformal diagram, which is also referred to as the Penrose diagram. A conformal diagram is defined by the following two properties. To study the causal structure of space-time, we have to use coordinates that vary in a finite range only. Furthermore, null geodesics should always remain straight lines at angles of $\pm 45°$.

Consider first two-dimensional Minkowski space-time with metric $ds^2 = -dt^2 + dx^2$ where $-\infty < t, x < \infty$. Introducing light-cone coordinates of the form $u = t - x$ and $v = t + x$ and mapping these to finite interval throigh $\tilde{u} = \arctan(u)$, $\tilde{v} = \arctan(v)$, the metric reads

$$ds^2 = -\frac{1}{\cos^2\tilde{u}\cos^2\tilde{v}} d\tilde{u}\tilde{v} = \frac{1}{4\cos^2\tilde{u}\cos^2\tilde{v}}(-d\tilde{t}^2 + d\tilde{x}^2)$$

where we have introduced $\tilde{t} = \frac{1}{2}(\tilde{u} + \tilde{v})$ and $\tilde{x} = \frac{1}{2}(\tilde{v} - \tilde{u})$. In this way, we have mapped two-dimensional Minkowski space-time into a finite region given by $-\pi < \tilde{t} + \tilde{x} < \pi$ and $-\pi < \tilde{t} - \tilde{x} < \pi$. Note that this metric is conformal to Minkowski space-time. Since null geodesics are invariant under conformal transformations, the null geodesics are still straight lines at $\pm 45°$. Therefore, we may use the coordinates $\tilde{t}$ and $\tilde{x}$ to draw the conformal diagram of Minkowski space-time. The conformal diagram below there are various infinities present.



First of all, there are three special points which are referred to as

$$i^+$$

which are the future time-like infinities

$$i^-$$

which are the past time-like infinities and

$$i^0$$

which are the space-like infinities. Time-like geodesics begin at $i^-$and end at i^+ whereas space-like geodesics begin and end at i^0. Furthermore, we may define

$$\mathcal{J}^+$$

which are the future null infinities connecting i^0 and i^+ as well as

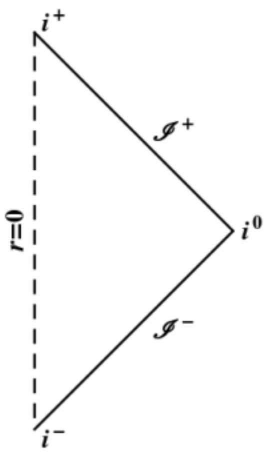
$$\mathcal{J}^-$$

which are the past null infinities connecting i^0 and i^+. All null geodesics originate from $\mathcal{J}^-$ and reach $\mathcal{J}^+$.

For higher-dimensional Minkowski space-time, the conformal diagram looks slightly different. We start from the metric

$$ds^2 = -dt^2 + d\vec{x}^2 = -dt^2 + dr^2 + r^2 d\Omega^2_{d-2}$$

where r is the radial direction $r^2 = \vec{x}^2$. Note that now r is restricted to values $r \geq 0$. Repeating the same analysis as in the two-dimensional case, the conformal diagram of d-dimensional Minkowski space-time is given by the diagram below.



To draw this figure we have suppressed a sphere S^{d-2} which we have to attach at each point of the conformal diagram.

4 Black Holes

An interesting class of solutions of Einstein's equations of motion are black holes which by definition have at least one event horizon. An event horizon is a boundary in space-time beyond which events cannot influence an outside observer. For example, Minkowski space-time has no event horizon since all inextendable null curves start at $\mathcal{J}^-$ and terminate at $\mathcal{J}^+$.

4.1 Schwarzschild Black Hole

The simplest, spherically symmetric vacuum solution to Einstein's equations with $\Lambda = 0$ in d-dimensions is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2$$

where

$$f(r) = 1 - \frac{2\mu}{r^{d-3}}$$

which is referred to as the Schwarzschild metric. Historically, for $d = 4$ where $f(r) = 1 - 2\mu/r$, this was the first non-trivial solution to Einstein's equations found in 1916. $d\Omega_{d-2}$ is the infinitesimal angular element in $d-2$ dimensions. μ is related to the mass of a black hole

$$\mu = \frac{8\pi GM}{(d-2)Vol(S^{d-2})}, \qquad Vol(S^{d-2}) = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$$

with G being the Newton constant and $Vol(S^{d-2})$ being the volume of the sphere S^{d-2}. The parameter M represents the mass of the black hole centred at the spatial origin. According to Birkhoff's theorem, this is the unique time independent spherically symmetric solution to Einstein's equations in the vacuum. Obviously, there are two special values $r = 0$ and $r = r_h$ for the radial coordinate. The origin $r = 0$ is a singularity since the curvature becomes infinite. This is a curvature singularity and not just a coordinate singularity since this divergence occurs in any coordinate system. The second special value $r = r_h$ is given by $f(r_h) = 0$. This condition gives

$$r_h = (2\mu)^{\frac{1}{d-3}}$$

which is referred to as the Schwarzschild radius. In the special case of $d = 4$ dimensions, we have $r_h = 2GM$. For any number of dimensions, the curvature is finite at r_h.

4.2 Black Hole Thermodynamics

The thermal properties of black holes may be summarized in four laws which have analogues in the corresponding four laws of standard thermodynamics. The four laws of black hole thermodynamics read as follows.

The zeroth law of black hole thermodynamics states that the surface gravity κ is constant over the horizon. This implies thermal equilibrium. Due to the zeroth law, the surface gravity corresponds to the temperature. The same applies to the electrostatic potential Φ and the angular velocity Ω of a charged or rotating black hole.

The first law states energy conservation: the change in the mass M of the black hole is related to the change in its area A, angular momentum J and charge Q by

$$dM = \frac{\kappa}{8\pi G}\delta A + \Omega\delta J + \Phi\delta Q.$$

For $J = Q = 0$ and relating κ to the Hawking temperature of the Schwarzschild black hole $T_H = \kappa/(2\pi)$ we obtain

$$dM = \frac{1}{4G}T_H\delta A \equiv T_H\delta S_{BH} \Rightarrow S_{BH} = \frac{A}{4G}$$

with Bekenstein-Hawking entropy S_{BH}.

The second law states that the total entropy of a system consisting of a black hole and matter contributions never decreases i.e.

$$dS_{tot} = dS_{matter} + dS_{BH} \geq 0.$$

The third law corresponds to Nernst's law: it is impossible to reduce the surface gravity κ to zero by a finite sequence of operations.

These four laws are analogous to the four laws of standard thermodynamics.

5 Energy Conditions

In addition to the vacuum solutions to Einstein's equations considered so far, there are also solutions corresponding to a specific matter distribution which enters in the Einstein equations by virtue of the energy-momentum tensor $T_{\mu\nu}$. From the matter field action, we may determine $T_{\mu\nu}$. However, sometimes it is not desirable to specify a particular matter system in the form of a Lagrangian and an associated energy-momentum tensor since a general theory of gravity and its phenomena should be maximally independent of any assumptions concerning non-gravitational physics. For instance, this applies to the proof of important theorems for black holes such as no-hair theorems and black hole thermodynamics.

However, in order to obtain sensible results we have to impose certain criteria on the form of the energy-momentum tensor which are met by relevant matter theories realized in nature. Such criteria are given by energy conditions. Let us list these conditions for a d-dimensional gravitational system.

- Null Energy Condition: the null energy condition holds if for any arbitrary null vector ζ^μ

  $$T_{\mu\nu}\zeta^\mu\zeta^\nu \geq 0.$$

- Weak Energy Condition: the weak energy condition holds if for any arbitrary time-like vector ξ^μ

  $$T_{\mu\nu}\xi^\mu\xi^\nu \geq 0.$$

  Note that in the case of a future-directed time-like vector ξ^μ, $T_{\mu\nu}\xi^\mu\xi^\nu$ is the energy density of the matter as measured by an observer whose relativistic velocity is given by ξ. According to the weak energy condition, this energy density should be non-negative.

- Strong Energy Condition: for $d > 2$, the strong energy condition holds if for any time-like vector ξ^μ

  $$\left(T_{\mu\nu} - \frac{1}{d-2}g_{\mu\nu}T\right)\xi^\mu\xi^\nu \geq 0.$$

- Dominant Energy Condition: the dominant energy condition is satisfied if for any null vector ζ^μ

$$T_{\mu\nu}\zeta^\mu\zeta^\nu \geq 0$$

and

$$T^{\mu\nu}\zeta_\mu$$

is non-space-like vector.