Redundant and quasi-redundant calibration with point sources

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1 Visibilities due to a Poisson distribution of dim and unresolved point sources

The visibility is given without explicit dependence on frequency, as

$$V(\boldsymbol{u}) = \int A(\boldsymbol{\theta}) I(\boldsymbol{\theta}) e^{-2\pi i \boldsymbol{u}.\boldsymbol{\theta}} d^2\boldsymbol{\theta}, \tag{1}$$

where $\boldsymbol{u} = \frac{\boldsymbol{b}}{\lambda}$, $A(\boldsymbol{\theta})$ is the antenna beam and the sky intensity is $I(\boldsymbol{\theta})$.

For point sources, we can use the equation for the sky brightness given as [1],

$$I = \int_0^{S_{max}} S \frac{dN}{dS} dS \quad [\text{Jy sr}^{-1}], \tag{2}$$

where S is the source flux expressed as a simple power law in frequency [1],

$$S_\nu = S_{\nu_0} \left(\frac{\nu}{\nu_0} \right)^{-\gamma} \quad [\text{Jy}] \tag{3}$$

and the source count distribution is [2]

$$\frac{dN}{dS}(\nu) = \alpha \left(\frac{S}{1\,\text{Jy}} \right)^{-\beta} \quad [\text{Jy}^{-1}\text{sr}^{-1}]. \tag{4}$$

The variables in equations 3 and 4 are obtained from point source surveys conducted at a particular frequency. The values used are: $\nu_0 \sim 150$ MHz, $\gamma = 0.8$, $\alpha_{\nu_0} = 4000$ [Jy^{-1} sr^{-1}] and $\beta = 1.75$. In equation 2 we set an upper limit, S_{max} on the flux density of sources such that sources above this limit must be omitted and treated individually.

The value chosen for S_{max} depends on the sensitivity of HIRAX which is $\sim 12\ \mu$Jy for the full 1024 element HIRAX array in a single day [3]. Assuming a 5σ detection of point sources is required to treat these bright sources individually, we can set the $S_{max} = 5 \times 12\ \mu$Jy. For a scaled down version of the full array, we can multiply this value by $\sqrt{\frac{1024}{\mathrm{N}_{dish}}}$.

We can rewrite the visibility in equation 1 by substituting equation 2 such that

$$V(\boldsymbol{u}) = \int_{S_{min}}^{S_{max}} S\frac{dN}{dS}dS \int A(\boldsymbol{\theta})e^{-2\pi i\boldsymbol{u}.\boldsymbol{\theta}}d^2\boldsymbol{\theta} \quad [\mathrm{Jy}], \tag{5}$$

from which we can see that the two integrals can be computed separately. We can evaluate the dS integral by substituting equation 4 and then simplifying to obtain a constant value of $\frac{\alpha}{2-\beta}(S_{max}^{2-\beta} - S_{min}^{2-\beta})$.

To evaluate the second integral, let us write the expression for the Gaussian beam of antenna i as, $A_i(\boldsymbol{\theta}) = e^{-\frac{\theta_{sep}^2}{2\sigma^2}}$, where θ_{sep} is a 1D vector in radians, calculated as the separation of each pixel in a HEALPix map, with 2D angular vector, $\boldsymbol{\theta}$, from the zenith which has a 2D angular vector, $\boldsymbol{\theta}^{zenith}$. The quantity, σ, is obtained using the FWHM given as λ/D_{dish}. Assuming equal beams for all antennas, we use $A(\boldsymbol{\theta}) = A_i(\boldsymbol{\theta})A_j(\boldsymbol{\theta})$. Using this, we can write the visibility as a summation over the number of pixels, n_{pix}, as

$$V(\boldsymbol{u}) = \frac{\alpha}{2-\beta}(S_{max}^{2-\beta} - S_{min}^{2-\beta})\left(\frac{4\pi}{n_{pix}}\sum_{n=1}^{n_{pix}} e^{-\frac{\theta_{sep,n}^2}{\sigma^2}} e^{-2\pi i\boldsymbol{u}.\boldsymbol{\theta}_n}\right). \tag{6}$$

2 Visibility covariance

We can now compute the visibility covariance for baselines a and b.

$$\begin{aligned} C_{ab} &= \langle V(\boldsymbol{u}_a)V^*(\boldsymbol{u}_b)\rangle \\ &= \langle I_\nu^2\rangle \int A^2(\boldsymbol{\theta})e^{-2\pi i(\boldsymbol{u}_a-\boldsymbol{u}_b).\boldsymbol{\theta}}d^2\boldsymbol{\theta} \quad [\mathrm{Jy}]^2. \end{aligned} \tag{7}$$

The power spectrum for point sources following a Poisson distribution is written as

$$C^{PS,Poisson} = \langle I_\nu^2\rangle = \int_0^{S_{max}} S^2\frac{dN}{dS}dS \quad [\mathrm{Jy}^2\ \mathrm{sr}^{-1}], \tag{8}$$

which simplifies to a constant value of $\frac{\alpha}{3-\beta}(S_{max}^{3-\beta} - S_{min}^{3-\beta})$. This is substituted into equation 7.

Writing the integral as a summation over n_{pix} again, we have

$$C_{ab} = \frac{\alpha}{3-\beta}\left(S_{max}^{3-\beta} - S_{min}^{3-\beta}\right)\left(\frac{4\pi}{n_{pix}}\sum_{n=1}^{n_{pix}} e^{-\frac{2\theta_{sep,n}^2}{\sigma^2}} e^{-2\pi i(\boldsymbol{u}_a - \boldsymbol{u}_b).\boldsymbol{\theta}_n}\right), \tag{9}$$

for a, b within a redundant block.

2.1 Visibilities due to bright and resolved point sources

We can write the visibility equation for the bright sources with flux density, $S > S_{max}$, that can be resolved by the telescope. We start off with a single source, whose sky brightness can be written as a Dirac delta function as

$$\begin{aligned} V_{singlesource}(\boldsymbol{u}) &= S_{source}\int A(\boldsymbol{\theta})\delta_D^2(\boldsymbol{\theta} - \boldsymbol{\theta}_s)e^{-2\pi i\boldsymbol{u}.\boldsymbol{\theta}}d^2\boldsymbol{\theta} \\ &= S_{source}A(\boldsymbol{\theta}_s)e^{-2\pi i\boldsymbol{u}.\boldsymbol{\theta}_s} \end{aligned} \tag{10}$$

where $\boldsymbol{\theta}_s$ is the position of the source. The absolute magnitude of all visibilities will be equal for a single source. Specifically, if we consider the source to be at the centre of the beam, such that $\boldsymbol{\theta}_s = \boldsymbol{\theta}_{zenith}$, then $A(\boldsymbol{\theta}) = 1$. The absolute magnitude of all visibilities will then be $V_{singlesource}(\boldsymbol{u}) = S_{source}$. Such a source would provide no phase information.

We code this up as

$$V_{brightsources} = \sum_{n=1}^{n_{pix}} I_n e^{-\frac{\theta_{sep,n}^2}{\sigma^2}} e^{-2\pi i\boldsymbol{u}.\boldsymbol{\theta}_n}, \tag{11}$$

where I_n is the sky brightness of each pixel, which depends on whether or not a source is contained in the pixel, such that $I_n = 0$ for pixels with no bright sources. All bright sources are simulated to occupy a single pixel so that each source individually can be approximated as a Dirac delta function, thereby returning the result from the analytic expression for a single source, given in equation 10.

References

[1] S. G. Murray, C. M. Trott, and C. H. Jordan. An improved statistical point-source foreground model for the epoch of reionization. *The Astrophysical Journal*, 845(1):7, aug 2017.

[2] Mario G. Santos, Asantha Cooray, and Lloyd Knox. Multifrequency analysis of 21 centimeter fluctuations from the era of reionization. *The Astrophysical Journal*, 625(2):575–587, Jun 2005.

[3] L. B. Newburgh, K. Bandura, M. A. Bucher, T.-C. Chang, H. C. Chiang, J.F. Cliche, R. Davé, M. Dobbs, C. Clarkson, K. M. Ganga, and et al. Hirax: a probe of dark energy and radio transients. *Ground-based and Airborne Telescopes VI*, Aug 2016.