CS6462

Probabilistic and Explainable AI

Lesson 2

Conditional Probability

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January 24, 2022

Probability and Conditions

Probability of event A occured and event B occured

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Probability of event A given B has occured

Probability of event B

Conditional Probability

- simplify the specification of probabilities by describing them in terms of conditional probabilities
- defined as the "*likelihood of an event occurring, based on the occurrence of another event*":

  the *conditional probability* ***P(E|F)*** of an event ***E*** given an event ***F***, is the probability that ***E*** occurs given that ***F*** has occurred

Example:

X = tossing a die; determine the probability that a 2 was rolled, GIVEN an even number has been rolled

S = {1, 2, 3, 4, 5, 6}

E = {2} - an event "the number is 2"

F = {2, 4, 6} - an event "the number is even"

P(E|F) = ?

Mathematical Definition

$$P\,(E|F) = \frac{P\,(E \cap F)}{P(F)} \quad , \text{ iff } P(F) > 0$$

Example 1: ***X*** = tossing a die; determine the probability that a 2 was rolled, GIVEN an even number has been rolled

$S = \{1, 2, 3, 4, 5, 6\}$

$P(S) = 1, P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$

$E = \{2\}$ - an event "the number is 2“, $P(E) = 1/6$

$F = \{2, 4, 6\}$ - an event "the number is even“, $P(F) = 3/6$

$E \cap F = \{2\}, P\,(E \cap F) = 1/6$

$P(E|F) = P\,(E \cap F) / P(F) = (1/6)/(3/6) = 1/3 \approx 33.33\ \%$

Composite Probability

Composite probability ***P(E∩F)*** can be determined based on the *conditional probability*

General multiplication:

P (E∩F) = P(F) * P(E|F)

Example 2: ***X*** = tossing a die; the probability that a 2 was rolled, GIVEN an even number has been rolled is 1/3; find the composite probability

S = {1, 2, 3, 4, 5, 6}

P(S) = 1, P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6

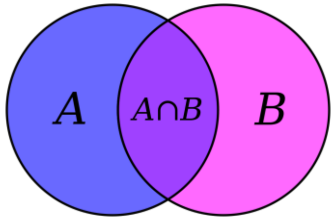
E = {2} - an event "the number is 2“, ***P(E) = 1/6***

F = {2, 4, 6} - an event "the number is even“, ***P(F) = 3/6***

P(E|F) = 1/3

P (E∩F) = P(F) * P(E|F) = (3/6) * (1/3) **≈ 1/6**

Independent Events



Definition:

the occurrence of ***F*** *has no effect upon the specification of the probability of* ***E***

Properties of independent events :

- events ***E*** and ***F*** in ***S*** are *independent* if:

  P(E) = P(E|F)

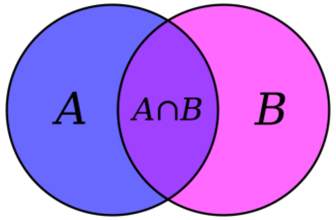
- if events ***E*** and ***F*** in ***S*** are independent then:

  1) $P(E|F) = P(E|F^c)$

  2) $P(E \cap F) = P(E) * P(F)$ *(multiplication)*

Independent Events are not Mutually Exclusive Events

Independent Events (cont.)



Example 3: ***X*** *= Tossing a die;* **P(*E*|*F*) ?**

- ***S = {1, 2, 3, 4, 5, 6}***
- ***P(S) = 1, P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6***
- ***E = {1, 3, 5}*** "an odd number is thrown"
- ***F = {1, 2}*** "the number thrown is less than 3"
- ***E*** **= {1} ∪ {3} ∪ {5},** ***F*** **= {1} ∪ {2}**
- ***E*** **∩** ***F*** **= {1}**
- recall *additivity rule:* $P(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i)$

  P(E) = P(1) + P(3) + P(5) = 3/6 = ½

  P(F) = P(1) + P(2) = 2/6

  P(E∩*F*) = *P(1)* = 1/6

- conditional probability

  P(*E*|*F*) = (1/6)/(2/6) = *½*

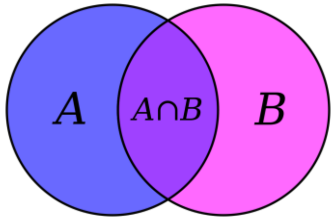
E and F are independent events

$$P\,(E|F) = \frac{P\,(E \cap F)}{P(F)}$$

Independent events

$$P(E) = P(E|F)$$

Conditional Probability – Examples



Example 4: ***X****:* a family has two children; assuming that boys and girls are equally likely, determine the probability that the family has:

a) 1 boy and 1 girl GIVEN the first child is a boy
b) 2 girls GIVEN that at least one is a girl
c) 2 girls GIVEN that the older one is a girl

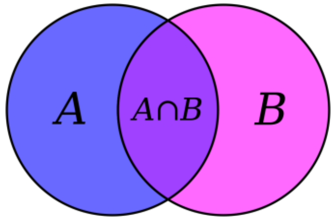
$$P\,(E|F) = \frac{P\,(E \cap F)}{P(F)}$$

Solution a) 1 boy and 1 girl GIVEN the first child is a boy

- $S = \{bb, gg, bg, gb\}$
- $P(S) = 1$, $P(bb) = P(gg) = P(bg) = P(gb) = 1/4$
- $E = \{bg, gb\}$ "1 boy and 1 girl",
- $F = \{bb, bg\}$ "the first child is a boy", $P(F) = 1/2$
- $E \cap F = \{bg\}$, $P(E \cap F) = 1/4$

$P(E|F) = (1/4)/(1/2) = 1/2$

Conditional Probability – Examples (cont.)



Example 4 (cont.):

S = {bb, gg, bg, gb}

P(S) = 1, P(bb) = P(gg) = P(bg) = P(gb) = 1/4

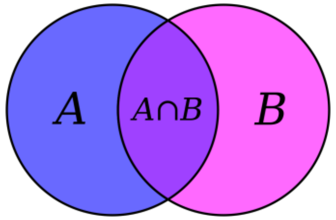
Solution b) 2 girls GIVEN that at least one is a girl

- ***E = {gg}*** "2 girls"
- ***F = {gg, bg, gb}*** "at least one is a girl", ***P(F) = 3/4***
- ***E∩F = {gg}, P(E∩F) = 1/4***

P(E|F) = (1/4)/(3/4) = 1/3

$$P\,(E|F) = \frac{P\,(E\cap F)}{P(F)}$$

Conditional Probability – Examples (cont.)



Example 4 (cont.):

S = {bb, gg, bg, gb}

P(S) = 1, P(bb) = P(gg) = P(bg) = P(gb) = ¼

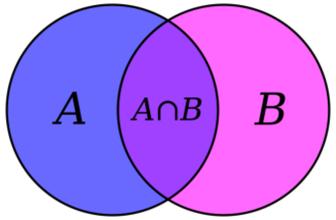
Solution c) 2 girls GIVEN that the older one is a girl

- ***E = {gg}*** "2 girls"
- ***F = {gg, gb}*** "older one is a girl", ***P(F) = ½***
- ***E∩F = {gg}, P(E∩F) = ¼***

P(E|F) = (1/4)/(1/2) = ½

$$P\,(E|F) = \frac{P\,(E\cap F)}{P(F)}$$

Conditional Probability – Examples (cont.)

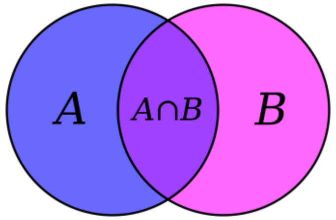


Example 5: ***X***: 200 patients who had either hip surgery or knee surgery were asked whether they were satisfied or dissatisfied regarding the result of their surgery. The following table summarizes their response.

Surgery	Satisfied	Dissatisfied	Total
Knee	70	25	95
Hip	90	15	105
Total	160	40	200

If one person from the 200 patients is randomly selected, determine the probability that the person was satisfied GIVEN that the person had knee surgery.

Conditional Probability – Examples (cont.)



Example 5 (cont.):

S = {1, 2, … , 200}

P(S) = 1, P(1) = P(2) = … = P(200) = 1/200

E = {1, 2, … , 160 } "satisfied"

F = {1, 2, …., 95} "had knee surgery"

E∩F = 70 "satisfied with knee surgery"

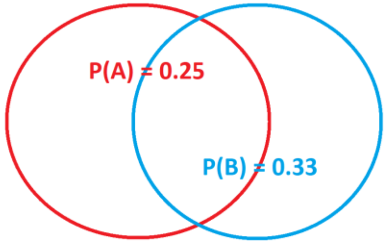
P(E∩F) = 70/200

P(F) = 95/200

P(E|F) = (70/200)/(95/200) = 70/95 = 0.7368 = 73.68%

$$P\,(E|F) = \frac{P\,(E \cap F)}{P(F)}$$

Summary



Conditional Probability

- *Probability – likelihood of an event occurring, based on the occurrence of another event*
- *Mathematical definition:*

$$\boldsymbol{P\,(E|F) = \frac{P\,(E \cap F)}{P(F)}} \quad , \text{ iff } \boldsymbol{P(F) > 0}$$

- *General multiplication:*

$$\boldsymbol{P\,(E \cap F) = P(F) * P(E|F)}$$

- *Independent events:*

$$\boldsymbol{P(E) = P(E|F)}$$

- *Next Lesson* – Probability Multiplication

Thank You!

Questions?