CS6462

Probabilistic and Explainable AI

Lesson 5

Random Variables and Probability Distribution

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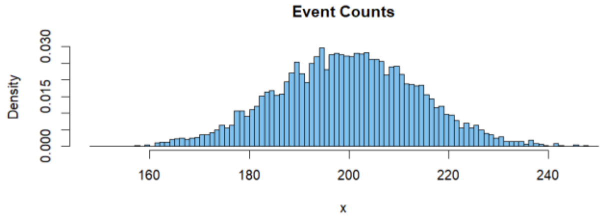
Random Variables and Probability



Definition: Random variables are represented by:

- a function that can take on either a finite number of values, each with an associated probability, or an infinite number of values, whose probabilities are summarized by a density function
- a function that assigns a numerical value to each outcome in ***S***, i.e., a real-valued function defined on ***S***

Example:

- If a coin is tossed three times, the sample space might be described by a list of 8 three-letter words,

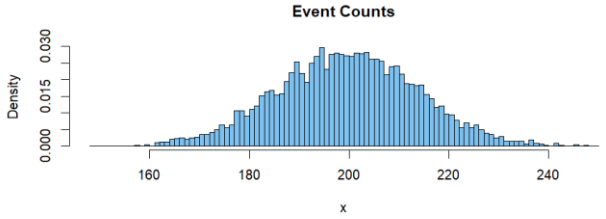
  S = {TTT, TTH, THT, HTT, HHT, HTH, THH, HHH}

- possible random variables:

  $\boldsymbol{F_X(S) = (\#H\text{'s in the word}) = \{0, 1, 2, 3\}, X}$ = #H's in the word

  $\boldsymbol{F_Y(S) = (\#T\text{'s in the word}) = \{0, 1, 2, 3\}, Y}$ = #T's in the word

Discrete Random Variable



Definition:

- a random variable ***X*** is said to be discrete if it can assume only a finite or countable infinite number of distinct values
- a discrete random variable can be defined on both a countable or uncountable sample space

Probability:

- the probability that ***X*** takes on the value ***x***, ***P(X=x)***, is defined as the sum of the probabilities of all sample points in ***S*** that are assigned the value ***x***
- ***P(X=x) = p(x) -*** function that assigns probabilities to each possible value x

  p(x) - the probability function for ***X***

Probability Distribution

Definition:

- The probability distribution for a random variable describes how the probabilities are distributed over the values of the random variable.
- For a discrete random variable $\boldsymbol{X}$ the probability distribution is defined by a *probability mass function*, denoted by $\boldsymbol{F_X(X)}$.
- $\boldsymbol{F_X(x)}$ - provides the probability for each value of the random variable

  $\boldsymbol{F_X(x) = p(x) = P(X=x)}$ for each $\boldsymbol{x}$ within the range of $\boldsymbol{X}$

Rules for mass function $\boldsymbol{F_X(x)}$ *for a discrete random variable* $\boldsymbol{X}$*:*

(1) $\boldsymbol{F_X(x) \geq 0}$ - must be nonnegative for each value of the random variable **X**

(2) $\boldsymbol{\sum_x F_X(x) = 1}$, sum of probabilities for each value of the random variable must = 1

Probability Distribution - Example

$P\,(\text{heads}) = \frac{1}{2} = 0.5$

Example:

Find a formula for the probability distribution of the total number of heads obtained in four tosses of a balanced coin.

S = see table

X= #H’s in the word

P(X=x):

- ***P(X = 0) = 1/16***
- ***P(X = 1) = 4/16***
- ***P(X = 2) = 6/16***
- ***P(X = 3) = 4/16***
- ***P(X = 4) = 1/16***

sample space, probabilities and random variable

Element of sample space	Probability	Value of random variable X (x)
HHHH	1/16	4
HHHT	1/16	3
HHTH	1/16	3
HTHH	1/16	3
THHH	1/16	3
HHTT	1/16	2
HTHT	1/16	2
HTTH	1/16	2
THHT	1/16	2
THTH	1/16	2
TTHH	1/16	2
HTTT	1/16	1
THTT	1/16	1
TTHT	1/16	1
TTTH	1/16	1
TTTT	1/16	0

Probability Distribution – Example (cont.)

$P\,(\text{heads}) = \frac{1}{2} = 0.5$

$$P(X=0) = \frac{1}{16}\,,\ P(X=1) = \frac{4}{16}\,,\ P(X=2) = \frac{6}{16}\,,\ P(X=3) = \frac{4}{16}\,,\ P(X=4) = \frac{1}{16}$$

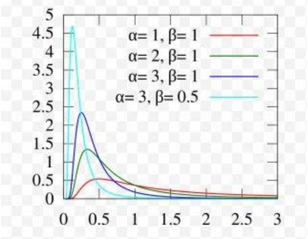
- denominators of the five fractions are the same and the numerators of the five fractions are 1,4,6,4,1
- the numbers in the numerators form a set of ***binomial coefficients*** $\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$

  n = 4, k = x

$$\frac{1}{16} = \binom{4}{0} \quad \frac{4}{16} = \binom{4}{1} \quad \frac{6}{16} = \binom{4}{2} \quad \frac{4}{16} = \binom{4}{3} \quad \frac{1}{16} = \binom{4}{4}$$

- ***$F_X(x)$*** *– the probability mass function can be written as:*

$$f(x) = \frac{\binom{4}{x}}{16}\ for\ x = 0\,,\ 1\,,\ 2\,,\ 3\,,\ 4$$



Cumulative Distribution Function

*Definition: cumulative distribution function of **X***

If ***X*** is a discrete random variable, the function given by:

$F_X(x) = P(x \leq X) = \Sigma_{t \leq x} f(t)$, for all $x \in R$

where $f(t)$ is the value of the probability distribution of ***X*** at ***t***

Rules for cumulative distribution function $F_X(x)$ of a discrete random variable ***X***:

1) $f(-\infty) = 0$ and $f(\infty) = 1$

2) if $a < b$, then $f(a) \leq f(b)$ for any real numbers ***a*** and ***b*** within the range of ***X***

$P\,(\text{heads}) = \frac{1}{2} = 0.5$

Cumulative Distribution Function - Example

Example: tossing a coin four times

determine the cumulative distribution function

$F_X(x) = P(x \leq X) = \Sigma_{t \leq x}\, f(t)$, for all ***$x \in R$***

- cumulative distribution function:

$$F(0) = f(0) = \frac{1}{16}$$

$$F(1) = f(0) + f(1) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

$$F(2) = f(0) + f(1) + f(2) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} = \frac{11}{16}$$

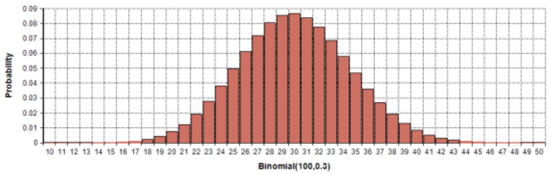
$$F(3) = f(0) + f(1) + f(2) + f(3) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{6} = \frac{15}{16}$$

$$F(4) = f(0) + f(1) + f(2) + f(3) + f(4) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{6} + \frac{1}{16} = \frac{16}{16}$$

$$F(x) = \begin{cases} 0 & for\ x < 0 \\ \frac{1}{16} & for\ 0 \leq x < 1 \\ \frac{5}{16} & for\ 1 \leq x < 2 \\ \frac{11}{16} & for\ 2 \leq x < 3 \\ \frac{15}{16} & for\ 3 \leq x < 4 \\ 1 & for\ x \geq 4 \end{cases}$$

sample space, probabilities and random variable

Element of sample space	Probability	Value of random variable X (x)
HHHH	1/16	4
HHHT	1/16	3
HHTH	1/16	3
HTHH	1/16	3
THHH	1/16	3
HHTT	1/16	2
HTHT	1/16	2
HTTH	1/16	2
THHT	1/16	2
THTH	1/16	2
TTHH	1/16	2
HTTT	1/16	1
THTT	1/16	1
TTHT	1/16	1
TTTH	1/16	1
TTTT	1/16	0



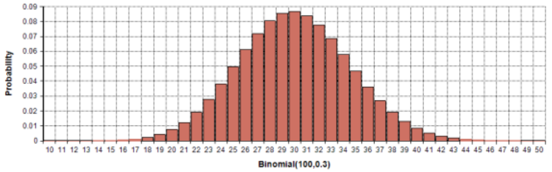
Binomial Distribution

Definition:

- a finite discrete distribution of the discrete random variable ***X***
- arises in situations where a sequence of what is known as *Bernoulli trials* is observed
- a *Bernoulli trial* is an experiment which has exactly two possible outcomes: *success* and *failure*
- *probability of success* is a fixed number **θ** which does not change with the number of experiments ***n***

The probability mass function for a binomial distribution:

$$b(x; n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad \text{for } x = 0, 1, 2, \ldots, n$$



Binomial Distribution - Example

Example: tossing a coin twice

What is the probability of getting one or more heads?

X= #H’s in the word

The probability mass function for a binomial distribution:

The binomial distribution consists of the probabilities of each of the possible numbers of successes on ***n*** trials for independent events that each have a probability of **θ** of occurring.

$$b(x; n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad \text{for } x = 0, 1, 2, \ldots, n$$

For the coin flip example, n = 2:

X=0, θ = ¼ , b(x; n, θ) = 0.25

X=1, θ = ½ , b(x; n, θ) = 0.5

X=2, θ = ½ , b(x; n, θ) = 0.25

sample space

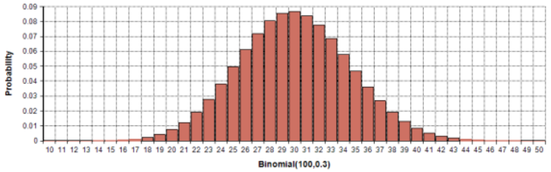
Outcome	First Flip	Second Flip
1	Heads	Heads
2	Heads	Tails
3	Tails	Heads
4	Tails	Tails

probabilities

Number of Heads	Probability
0	1/4
1	1/2
2	1/4

The probability of getting one or more heads is **0.5 + 0.25 = 0.75**

Poisson Distribution



Definition:

- used to model a number of events ***x*** occurring within a given time interval of an experiment.

- the formula for the Poisson *probability mass function* is:

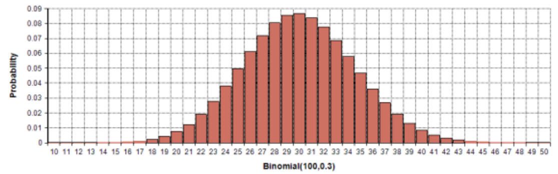
$$P\,(x;\lambda) = \frac{e^{-\lambda} * \lambda^{x}}{x!}$$

, for ***x*** = 0, 1, 2, ⋯ ; **λ** does not have to be an integer

λ is a shape parameter that indicates the average number of events in the given time interval

- *Poisson cumulative probability distribution function:*

$$F\,(x;\lambda) = \sum_{i}^{x} \frac{e^{-\lambda} * \lambda^{i}}{i!}$$



Poisson Distribution - Example

Example:

Radioactivity example with an average of 2 decays/sec.

a) What's the probability of zero decays in one second?

b) What's the probability of more than one decay in one second?

> Radioactive decay is the process by which an unstable atomic nucleus loses energy by radiation.

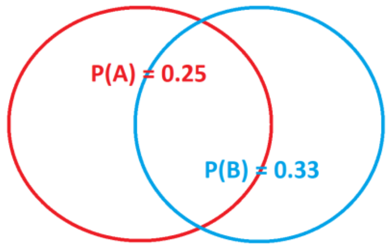
Solution a):

$$p(0,2) = \frac{e^{-2}2^0}{0!} = \frac{e^{-2}\cdot 1}{1} = e^{-2} = 0.135 \rightarrow 13.5\%$$

Solution b):

$$p(>1,2) = 1 - p(0,2) - p(1,2) = 1 - \frac{e^{-2}2^0}{0!} - \frac{e^{-2}2^1}{1!} = 1 - e^{-2} - 2e^{-2} = 0.594 \rightarrow 59.4\%$$

Summary



Random Variable:

- a function that assigns a numerical value to each outcome in ***S***
- discrete random variable

Probability Distribution for Random Variables

- probability mass function
- cumulative distribution function
- binomial distribution
- Poisson distribution

Next Lesson – Bayes’ Theorem

Thank You!

Questions?