COMP9020: Foundations of Computer Science

Assignment 2

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Problem 1:

(a): According to the definition, there has:

$R_1;R_2 = \{(a,c)$: there is a b with $(a,b) \in R_1$ and $(b,c) \in R_2$ $\}$

$(R_1;R_2);R_3 = \{(a,d)$: there are b, d with $(a,b) \in R_1$, $(b,c) \in R_2$ and $(c,d) \in R_3$ $\}$

$R_2;R_3 = \{(b,d)$: there is a c with $(b,c) \in R_2$ and $(c,d) \in R_3\}$

$R_1;(R_2;R_3) = \{(a,d)$: there are b, c with $(a,b) \in R_1$ and $(b,c) \in R_2$, $(c,d) \in R_3$ $\}$

$\therefore$ $(a,b) \in R_1$, $(b,c) \in R_2$, $(c,d) \in R_3$.

Then $(a,c) \in R_1;R_2$, $(b,d) \in R_2;R_3$, $(a,d) \in R_1;(R_2;R_3)$, also, $(a,d) \in (R_1;R_2);R_3$.

Above, $(R_1;R_2);R_3 = R_1;(R_2;R_3)$ is proved.

(b): From the definition, we know $(a,b) \in R_1$.

Cause I = $\{$ (x,x) $x \in S\}$, $\therefore$ (a,a), $(b,b) \in I$.

There has: $I;R_1 = \{$ (a,b): there is a a with $(a,a) \in I$ and $(a,b) \in R_1\}$

$R_1;I = \{(a,b)$: there is a b with $(a,b) \in R_1$ and $(b,b) \in I$ $\}$

So, $(a,b) \in I;R_1$, also, $(a,b) \in R_1;I$.

$I;R_1 = R_1;I = R_1$ where I = $\{(x,x)$: $x \in S\}$ is proved.

(c): We can give a counterexample to disprove $(R_1;R_2)^{\leftarrow} = R_1^{\leftarrow};R_2^{\leftarrow}$.

When $R_1 = \{(1,5)\}$, $R_2 = \{(5,7)\}$, $(1,7) \in R_1;R_2$,

but $R_1^{\leftarrow} = \{(5,1)\}$, $R_2^{\leftarrow} = \{(7,5)\}$, $R_1^{\leftarrow};R_2^{\leftarrow} = \varnothing$,

$\therefore$ $(R_1;R_2)^{\leftarrow} \neq R_1^{\leftarrow};R_2^{\leftarrow}$

(d): $(R_1 \cup R_2);R_3$ **= {** (a,c) : there is a b with $(a,b) \in R_1 \cup R_2$ and $(b,c) \in R_3$ }

There are 3 situations for $(a,b) \in R_1 \cup R_2$:

(1) $(a,b) \in R_1$, $(a,b) \notin R_2$:

$R_1;R_3 = \{\,(a,c)$: there is a b with $(a,b) \in R_1$ and $(b,c) \in R_3\}$

$R_2;R_3 = \varnothing$.

$\therefore$ $(a,c) \in (R_1;R_3) \cup (R_2;R_3)$, also, $(a,c) \in (R_1 \cup R_2);R_3$.

So, $(R_1 \cup R_2);R_3 = (R_1;R_3) \cup (R_2;R_3)$.

(2) $(a,b) \notin R_1$, $(a,b) \in R_2$:

$R_1;R_3 = \varnothing$

$R_2;R_3 = \{\,(a,c)$: there is a b with $(a,b) \in R_2$ and $(b,c) \in R_3\}$

$\therefore$ $(a,c) \in (R_1;R_3) \cup (R_2;R_3)$, also, $(a,c) \in (R_1 \cup R_2);R_3$.

So, $(R_1 \cup R_2);R_3 = (R_1;R_3) \cup (R_2;R_3)$.

(3) $(a,b) \in R_1$ **,** $(a,b) \in R_2$

$R_1;R_3 = \{(a,c)$: there is a b with $(a,b) \in R_1$ and $(b,c) \in R_3\}$

$R_2;R_3 = \{(a,c)$: there is a b with $(a,b) \in R_2$ and $(b,c) \in R_3\}$

$\therefore$ $(a,c) \in (R_1;R_3) \cup (R_2;R_3)$, also, $(a,c) \in (R_1 \cup R_2);R_3$.

So, $(R_1 \cup R_2);R_3 = (R_1;R_3) \cup (R_2;R_3)$.

Above the 3 situations, $(R_1 \cup R_2);R_3 = (R_1;R_3) \cup (R_2;R_3)$ is proved.

(e): We can give a counterexample to disprove $R_1;(R_2 \cap R_3) = (R_1;R_2) \cap (R_1;R_3)$.

When $R_1 = \{(1,5)\ ,\ (1,6)\}$, $R_2 = \{(5,7)\}$, $R_3 = \{(6,7)\}$:

$R_1;(R_2 \cap R_3) = \varnothing$, $(R_1;R_2) \cap (R_1;R_3) = \{(1,7)\}$,

which means $R_1;(R_2 \cap R_3) \neq (R_1;R_2) \cap (R_1;R_3)$.

So, this relation is disproved.

Problem 2:

(a): We can assume P(j) has the proportion like $R^i = R^j$.

Approach to prove when $R^i = R^{i+1}$, then $R^j = R^i$ for all $j \geq i$:

[B]: P(i) is true

[I]: Suppose there is a P(k) coincidently for some $k \geq i$,

then $R^k = R^i$, for R^{k+1}:

$$
\begin{aligned}
R^{k+1} &:= R^k \cup (R^k;R) && (\text{ definition of } R^{k+1}) \\
&= R^i \cup (R^i;R) && (\text{IH}) \\
&= R^{i+1} && (\text{ definition of } R^{i+1}) \\
&= R^i && (\text{ definition of } R^i)
\end{aligned}
$$

So, $R^k = R^i$, also, $R^{k+1} = R^i$, in this way, P(k) implies P(k+1).

Above, it's proved if there is an i such that $R^i = R^{i+1}$, then $R^j = R^i$ for all $j \geq i$.

(b): Cause we assume P(j), then $R^{j+1} := R^i \cup (R;R^i)$, there has a relation that $R^j \subseteq R^{j+1}$ for all $j \geq 0$, according to the definition of transitivity, for all $k \leq i$, $R^k \subseteq R^i$.

We've showed that $R^k = R^i$ for all $k \geq i$ in (a), so $R^k \subseteq R^i$ when $k \geq i$.

Therefore, $R^k \subseteq R^i$ for all $k \geq 0$.

(c): Approach to show P(n) holds for all $n \in \mathbb{N}$:

We know for all $m \in \mathbb{N}$: $R^n;R^m = R^{n+m}$, $R^0 := I = \{(x,x) : x \in S\}$, and $I;R_1 = R_1;I = R_1$ from Problem 1 (b),

when $n = 0$: there has:

$$
\begin{aligned}
R^0;R^m &= I;R^m \\
&= R^m;I \\
&= R^m \\
&= R^{0+m}
\end{aligned}
$$

P(n) holds.

When $n \neq 0$:

From $R^{i+1} := R^i \cup (R;R^i)$ for $i \geq 0$, we know $R^{n+1} := R^n \cup (R;R^n)$ for $n \in \mathbb{N}$:

$$
\begin{aligned}
R^n;R^m &= R^{n+m} \\
R^{n+1};R^m &= (R^n \cup (R;R^n));R^m \\
&= (R^n;R^m) \cup ((R;R^n);R^m) \\
&= R^{n+m} \cup (R;(R^n;R^m)) \\
&= R^{n+m} \cup (R;R^{n+m}) \\
&= R^{n+m} \cup (R^{n^0};R^{n+m}) \\
&= R^{n+m} \cup R^{n^0+n+m} \\
&= R^{n+m} \cup R^{1+n+m} \\
&= R^{n+1+m}
\end{aligned}
$$

P(n) holds.

Above, P(n) holds for all $n \in \mathbb{N}$.

(d): From the above questions in problem 2, we know that $R^{k+1} \subseteq R^k$.

If $(a,c) \in R^{k+1}$, then there exists $b_0, b_1, b_2, ..., b_{k-1}, b_k, b_{k+1} \in S$, so that $a = b_0,\ c = b_{k+1}$, also, $(b_i, b_{i+1}) \in R$ for $0 \le i \le k$.

Because S has k elements, there exists i, j with $0 \le i \le j \le k+1$, like $b_i = b_j$, which means $(b_i, b_j) \in R$, also, $(a,c) \in R^{k+1-(j-i)}$. As $j > i,\ k+1-(j-1) \le k$, then $(a,c) \in R^n$ for some $n \le k$.

We can know $R^n \subseteq R^k$ in (b), then $(a,c) \in R^k$.

So, that's why $R^k = R^{k+1}$ when $|S| = k$.

(e): We can assume $(a,b) \in R^k$, $(b,c) \in R^k$. Then there has $b_1, b_2, ..., b_{n-1}, b_n$, when $n \le k$, and $C_1, C_2, ..., C_{r-1}, C_r$ when $r \le k$.

Then: $(a,b_1),\ (b_i, b_{i+1})$ for $1 \le i \le n$, $(b_n, b),\ (b, c_1),\ (c_i, c_{i+1})$ for $1 \le i \le r$, (c_r, c), which means $(a,c) \in R^{n+r+1}$. In (b) and (d), we know $R^{n+r+1} \subseteq R^k$, then $(a,c) \in R^k$.

Therefore, R^k is transitive.

(f): From (e), we can know that $(R \cup R^{\leftarrow})^K$ is transitive.

$\because\ (a,b) \in R,\ (b,a) \in R^{\leftarrow}, \quad \therefore \{(a,b),\ (b,a)\} \subseteq R \cup R^{\leftarrow}$,

Because of the content of the problem, then $(R \cup R^{\leftarrow})^0 = I$.

$(a,a) \in (R \cup R^{\leftarrow})^0$ and $(R \cup R^{\leftarrow})^0 \subseteq (R \cup R^{\leftarrow})^k$

$\therefore\ (a,a) \in (R \cup R^{\leftarrow})^0 \subseteq (R \cup R^{\leftarrow})^k$ then, $(R \cup R^{\leftarrow})^k$ is reflexive.

So, $R \cup R^{\leftarrow}$ is symmetric.

Also, $(R \cup R^{\leftarrow})^1 = I \cup (R \cup R^{\leftarrow})$, $(R \cup R^{\leftarrow})^1 = (R \cup R^{\leftarrow})^0 \cup ((R \cup R^{\leftarrow});(R \cup R^{\leftarrow})^0)$

So, $(R \cup R^{\leftarrow})^1$ is summetric.

We can assume that $(R \cup R^{\leftarrow})^n$ is symmetric, where $n \ge 1$.

Known from (c) that $(R \cup R^{\leftarrow})^{n+1} = (R \cup R^{\leftarrow})^1;(R \cup R^{\leftarrow})^n$, $(R \cup R^{\leftarrow})^{n+1} = (R \cup R^{\leftarrow})^n;(R \cup R^{\leftarrow})^1$

there has $(a,c) \in (R \cup R^{\leftarrow})^{n+1}$, then $(a,b) \in (R \cup R^{\leftarrow})^1,\ (b,c) \in (R \cup R^{\leftarrow})^n$,

so, $(b,a) \in (R \cup R^{\leftarrow})^1,\quad (c,b) \in (R \cup R^{\leftarrow})^n,\ (c,a) \in (R \cup R^{\leftarrow})^{n+1}$.

Then, $(R \cup R^{\leftarrow})^{n+1}$ is symmetric, so, $(R \cup R^{\leftarrow})^k$ is symmetric.

Above, it's proved that $(R \cup R^{\leftarrow})^k$ is an equivalence relation.

Problem 3:

(a): Binary tree data structure can be defined as either:

(B) an empty tree

(R) a node followed by two binary trees

(b): The count function:

count(T):

(B): if (T = 0): (where T = 0 means the binary tree is empty)

return 0

(R): else:

return count(T_1) + count(T_2) +1

(where T_1 is the left child binary tree, and T_2 is the right child binary tree)

(c): The leaves function:

leaves(T):

(B): if (T = 0): (where T = 0 means the binary tree is empty)

return $\frac{1}{2}$

(R): else:

return $\lfloor$leaves(T_1) + leaves(T_2)$\rfloor$

(where T_1 is the left child binary tree, and T_2 is the right child binary tree)

(d): The internal function:

internal(T):

(B): if (T = 0): (where T = 0 means the binary tree is empty)

return $-\frac{1}{2}$

(R): else:

return $\lfloor$internal(T_1) + internal(T_2)$\rfloor$ +1

(where T_1 is the left child binary tree, and T_2 is the right child binary tree)

(e): There are two situations:

(1) There is one node in the binary tree, according to the definition we showed in (b) and (c), internal(T) = 0, leaves(T) = 1, which fits leaves(T) = 1 + internal(T) P(T) holds.

(2) When there are nodes more than one, suppose leaves(T) = 1 + internal(T) holds, according to the definition we show in (b) and (c), then:

$$\begin{aligned} \text{leaves(T)} &= \lfloor \text{leaves}(T_1) + \text{leaves}(T_2)\rfloor \\ &= \lfloor 1+ \text{internal(T)} \rfloor \\ &= 1 + \text{internal(T)} \end{aligned}$$

P(T) hods.

Above, it's proved P(T) holds for all binary tree T.

Problem 4:

(a): Defining the requirements as formulas of propositional logic firstly:

Ap: Alpha uses channel hi; Aq: Alpha uses channel lo;

Bp: Bravo uses channel hi; Bq: Bravo uses channel lo;

Cp: Charlie uses channel hi; Cq: Charlie uses channel lo;

Dp: Delta uses channel hi; Dq: Delta uses channel lo.

According to the definition of propositional formulas:

(i) $\varphi_1 = ((Ap \vee Aq) \wedge (Bp \vee Bq) \wedge (Cp \vee Cq) \wedge (Dp \vee Dq))$

(ii) $\varphi_2 = ((\neg Ap \vee \neg Aq) \wedge (\neg Bp \vee \neg Bq) \wedge (\neg Cp \vee \neg Cq) \wedge (\neg Dp \vee \neg Dq))$

(iii) $\varphi_3 = (\neg((Ap \wedge Bp) \vee (Aq \wedge Bq))) \wedge (\neg((Bp \wedge Cp) \vee (Bq \wedge Cq))) \wedge (\neg((Cp \wedge Dp) \vee (Cq \wedge Dq)))$

(b): (i) Following is the satisfying truth assignment:

$\varphi_1 = ((Ap \vee Aq) \wedge (Bp \vee Bq) \wedge (Cp \vee Cq) \wedge (Dp \vee Dq))$

φ_1	$Ap \vee Aq$	$Bp \vee Bq$	$Cp \vee Cq$	$Dp \vee Dq$
T	T	T	T	T

$\varphi_2 = ((\neg Ap \vee \neg Aq) \wedge (\neg Bp \vee \neg Bq) \wedge (\neg Cp \vee \neg Cq) \wedge (\neg Dp \vee \neg Dq))$

φ_2	$Ap \wedge Aq$	$Bp \wedge Bq$	$Cp \wedge Cq$	$Dp \wedge Dq$
T	F	F	F	F

$\varphi_3 = (\neg((Ap \wedge Bp) \vee (Aq \wedge Bq))) \wedge (\neg((Bp \wedge Cp) \vee (Bq \wedge Cq))) \wedge (\neg((Cp \wedge Dp) \vee (Cq \wedge Dq)))$

φ_3	$Ap \wedge Bp$	$Aq \wedge Bq$	$Bp \wedge Cp$	$Bq \wedge Cq$	$Cp \wedge Dp$	$Cq \wedge Dq$
T	F	F	F	F	F	F

Recapitulative truth assignment:

Ap	Aq	Bp	Bq	Cp	Cq	Dp	Dq	φ_1	φ_2	φ_3	$\varphi_1 \wedge \varphi_2 \wedge \varphi_3$
F	T	F	T	F	T	F	T	T	T	T	T
T	F	T	F	T	F	T	F	T	T	T	T

So, for these two situation in recapitulative truth assignment, all requirements are met.

(ii) To avoid interference, the near networks cannot use the same channel, so there are 2 situations:

(1) When Alpha uses channel lo, Bravo uses channel hi, Charlie uses the channel lo, Delta uses the channel hi;
(2) When Alpha uses channel hi, Bravo uses channel lo, Charlie uses the channel hi, Delta uses the channel lo.