APEX CALCULUS *FOR* MA 301

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14: VECTOR ANALYSIS

In previous chapters we have explored a relationship between vectors and integration. Our most tangible result: if $\vec{v}(t)$ is a vector–valued velocity function of a moving object, then integrating $\vec{v}(t)$ from $t = a$ to $t = b$ gives the displacement of that object over that time interval.

This chapter explores completely different relationships between vectors and integration. These relationships will enable us to compute the work done by a magnetic field in moving an object along a path and how much air moves through an oddly–shaped screen in space, among other things.

Our upcoming work with integration will benefit from a review. We are not concerned here with techniques of integration, but rather what an integral "does" and how that relates to the notation we use to describe it.

Integration Review

Recall from Section 13.1 that when R is a region in the x-y plane, $\iint_R dA$ gives the area of the region R. The integral symbols are "elongated esses" meaning "sum" and dA represents "a small amount of area." Taken together, $\iint_R dA$ means "sum up, over R, small amounts of area." This sum then gives the total area of R. We use two integral symbols since R is a two–dimensional region.

Now let $z = f(x,y)$ represent a surface. The double integral $\iint_R f(x,y)\ dA$ means "sum up, over R, function values (heights) given by f times small amounts of area." Since "height $\times$ area = volume," we are summing small amounts of volume over R, giving the total signed volume under the surface $z = f(x,y)$ and above the x-y plane.

This notation does not directly inform us *how* to evaluate the double integrals to find an area or a volume. With additional work, we recognize that a small amount of area dA can be measured as the area of a small rectangle, with one side length a small change in x and the other side length a small change in y. That is, $dA = dx\,dy$ or $dA = dy\,dx$. We could also compute a small amount of area by thinking in terms of polar coordinates, where $dA = r\,dr\,d\theta$. These understandings lead us to the iterated integrals we used in Chapter 13.

Let us back our review up farther. Note that $\int_1^3\ dx = x\Big|_1^3 = 3 - 1 = 2$. We have simply measured the length of the interval $[1,3]$. We could rewrite the above integral using syntax similar to the double integral syntax above:

$$\int_1^3\ dx = \int_I dx, \quad \text{where } I = [1,3].$$

We interpret "$\int_I dx$" as meaning "sum up, over the interval I, small changes in x." A change in x is a length along the x-axis, so we are adding up along I small lengths, giving the total length of I.

We could also write $\int_1^3 f(x)\,dx$ as $\int_I f(x)\,dx$, interpreted as "sum up, over I, heights given by $y = f(x)$ times small changes in x." Since "height$\times$length = area," we are summing up areas and finding the total signed area between $y = f(x)$ and the x-axis.

This method of referring to the process of integration can be very powerful. It is the core of our notion of the Riemann Sum. When faced with a quantity to compute, if one can think of a way to approximate its value through a sum, the one is well on their way to constructing an integral (or, double or triple integral) that computes the desired quantity. We will demonstrate this process throughout this chapter, starting with the next section.

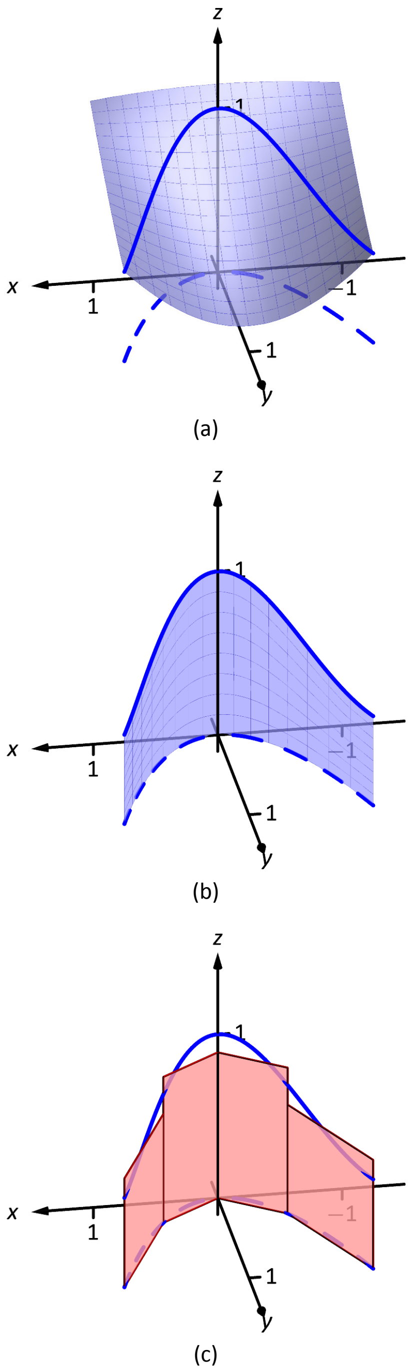


Figure 14.1: Finding area under a curve in space.

14.1 Introduction to Line Integrals

We first used integration to find "area under a curve." In this section, we learn to do this (again), but in a different context.

Consider the surface and curve shown in Figure 14.1(a). The surface is given by $f(x, y) = 1 - \cos(x)\sin(y)$. The dashed curve lies in the x-y plane and is the familiar $y = x^2$ parabola from $-1 \le x \le 1$; we'll call this curve C. The curve drawn with a solid line in the graph is the curve in space that lies on our surface with x and y values that lie on C.

The question we want to answer is this: what is the area that lies below the curve drawn with the solid line? In other words, what is the area of the region above C and under the the surface f? This region is shown in Figure 14.1(b).

We suspect the answer can be found using an integral, but before trying to figure out what that integral is, let us first try to approximate its value.

In Figure 14.1(c), four rectangles have been drawn over the curve C. The bottom corners of each rectangle lie on C, and each rectangle has a height given by the function $f(x, y)$ for some (x, y) pair along C between the rectangle's bottom corners.

As we know how to find the area of each rectangle, we are able to approximate the area above C and under f. Clearly, our approximation will be *an approximation*. The heights of the rectangles do not match exactly with the surface f, nor does the base of each rectangle follow perfectly the path of C.

In typical calculus fashion, our approximation can be improved by using more rectangles. The sum of the areas of these rectangles gives an approximate value of the true area above C and under f. As the area of each rectangle is "height$\times$width", we assert that the

$$\text{area above } C \approx \sum (\text{heights} \times \text{widths}).$$

When first learning of the integral, and approximating areas with "heights$\times$

Notes:

widths", the width was a small change in x: dx. That will not suffice in this context. Rather, each width of a rectangle is actually approximating the arc length of a small portion of C. In Section 11.5, we used s to represent the arc–length parameter of a curve. A small amount of arc length will thus be represented by ds.

The height of each rectangle will be determined in some way by the surface f. If we parametrize C by s, an s-value corresponds to an (x, y) pair that lies on the parabola C. Since f is a function of x and y, and x and y are functions of s, we can say that f is a function of s. Given a value s, we can compute $f(s)$ and find a height. Thus

$$\begin{aligned} \text{area under } f \text{ and above } C &\approx \sum(\text{heights} \times \text{widths}); \\ \text{area under } f \text{ and above } C &= \lim_{||\Delta s||\to 0} \sum f(c_i)\Delta s_i \\ &= \int_C f(s)\, ds. \end{aligned} \tag{14.1}$$

Here we have introduce a new notation, the integral symbol with a subscript of C. It is reminiscent of our usage of $\iint_R$. Using the train of thought found in the Integration Review preceding this section, we interpret "$\int_C f(s)\, ds$" as meaning "sum up, along a curve C, function values $f(s)\times$small arc lengths." It is understood here that s represents the arc–length parameter.

All this leads us to a definition. The integral found in Equation 14.1 is called a **line integral**. We formally define it below, but note that the definition is very abstract. On one hand, one is apt to say "the defintion makes sense," while on the other, one is equally apt to say "but I don't know what I'm supposed to do with this definition." We'll address that after the definition, and actually find an answer to the area problem we posed at the beginning of this section.

Definition 91 Line Integral Over Scalar Field

Let C be a smooth curve parametrized by s, the arc–length parameter, and let f be a continuous function of s. A **line integral** is an integral of the form

$$\int_C f(s)\, ds = \lim_{||\Delta s||\to 0} \sum_{i=1}^{n} f(c_i)\Delta s_i,$$

where $s_1 < s_2 < \ldots < s_n$ is any partition of the s-interval over which C is defined, c_i is any value in the i^{th} subinterval, Δs_i is the width of the i^{th} subinterval, and $||\Delta s||$ is the length of the longest subinterval in the partition.

Note: Definition 91 uses the term **scalar field** which has not yet been defined. Its meaning is discussed in the paragraph preceding Definition 96 when it is compared to a **vector field**.

Notes:

When C is a **closed** curve, i.e., a curve that ends at the same point at which it starts, we use

$$\oint_C f(s)\ ds \quad \text{instead of} \quad \int_C f(s)\ ds.$$

The definition of the line integral does not specify whether C is a curve in the plane or space (or hyperspace), as the definition holds regardless. For now, we'll assume C lies in the x-y plane.

This definition of the line integral doesn't really say anything new. If C is a curve and s is the arc–length parameter of C on $a \leq s \leq b$, then

$$\int_C f(s)\ ds = \int_a^b f(s)\ ds.$$

The real difference with this integral from the standard "$\int_a^b f(x)\ dx$" we used in the past is that of context. Our previous integrals naturally summed up values over an interval on the x-axis, whereas now we are summing up values over a curve. If we can parametrize the curve with the arc–length parameter, we can evaluate the line integral just as before.

The trouble here is that we have generally avoided direct use of the arc–length parameter s in the past as it is usually difficult to use. We continue that methodology here.

Given a curve C, find a parametrization of C: $x = g(t)$ and $y = h(t)$, for continuous functions g and h, where $a \leq t \leq b$. We can represent this parametrization with a vector–valued function, $\vec{r}(t) = \langle g(t), h(t)\rangle$.

In Section 11.5, we defined the arc–length parameter in Equation 11.1 as

$$s(t) = \int_0^t ||\ \vec{r}\,'(u)\ ||\ du.$$

By the Fundamental Theorem of Calculus, $ds = ||\ \vec{r}\,'(t)\ ||\ dt$. We can substitute the right hand side of this equation for ds in the line integral definition.

We can view f as being a function of x and y since it is a function of s. Thus $f(s) = f(x, y) = f\big(g(t), h(t)\big)$. This gives us a concrete way to evaluate a line integral:

$$\int_C f(s)\ ds = \int_a^b f\big(g(t), h(t)\big)||\ \vec{r}\,'(t)\ ||\ dt.$$

We write this as a Key Idea, along with its three–dimensional analogue, followed by an example where we finally evaluate an integral and find an area.

Notes:

Key Idea 50 Evaluating a Line Integral

- Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t)\rangle$, $a \leq t \leq b$, where g and h are continuously differentiable, and let $z = f(x, y)$. Then
$$\int_C f(s)\, ds = \int_a^b f\big(g(t), h(t)\big)\, ||\, \vec{r}\,'(t)\, ||\, dt.$$
- Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t), k(t)\rangle$, $a \leq t \leq b$, where g, h and k are continuously differentiable, and let $w = f(x, y, z)$. Then
$$\int_C f(s)\, ds = \int_a^b f\big(g(t), h(t), k(t)\big)\, ||\, \vec{r}\,'(t)\, ||\, dt.$$

To be clear, the first point of Key Idea 50 can be used to find the area under a surface $z = f(x, y)$ and above a curve C.

Let's finally do an example where we actually compute an area.

Example 374 Evaluating a line integral: area under a surface over a curve.

Find the area under the surface $f(x, y) = \cos(x) + \sin(y) + 2$ over the curve C, which is the segment of the line $y = 2x + 1$ on $-1 \leq x \leq 1$, as shown in Figure 14.2.

Solution Our first step is to represent C with a vector–valued function. Since C is a simple line, and we have a explicit relationship between y and x (namely, that y is $2x+1$), we can let $x = t$, $y = 2t+1$, and write $\vec{r}(t) = \langle t, 2t+1\rangle$ for $-1 \leq t \leq 1$.

We find the values of f over C as $f(x, y) = f(t, 2t+1) = \cos(t)+\sin(2t+1)+2$. We also need $||\, \vec{r}\,'(t)\, ||$; with $\vec{r}\,'(t) = \langle 1, 2\rangle$, we have $||\, \vec{r}\,'(t)\, || = \sqrt{5}$. Thus $ds = \sqrt{5}\, dt$.

The area we seek is

$$\begin{aligned}\int_C f(s)\, ds &= \int_{-1}^{1} \big(\cos(t) + \sin(2t+1) + 2\big)\sqrt{5}\, dt\\ &= \sqrt{5}\Big(\sin(t) - \frac{1}{2}\cos(2t+1) + 2t\Big)\Big|_{-1}^{1}\\ &\approx 14.418 \text{ units}^2.\end{aligned}$$

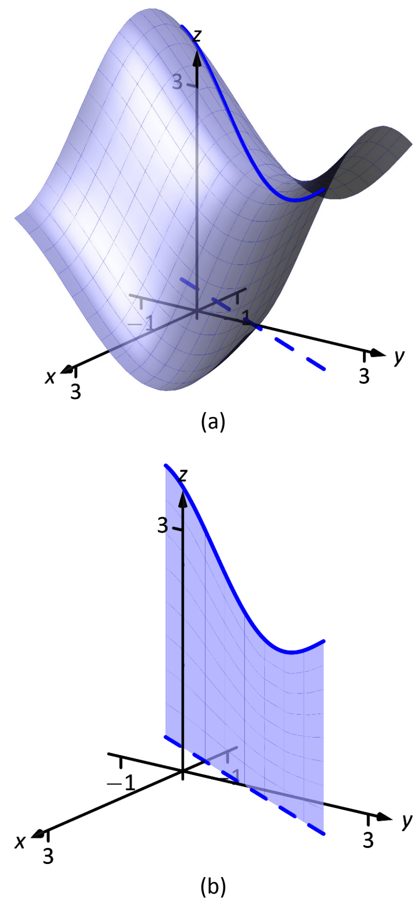


Figure 14.2: Finding area under a curve in Example 374.

We will practice setting up and evaluating a line integral in another example,

Notes:

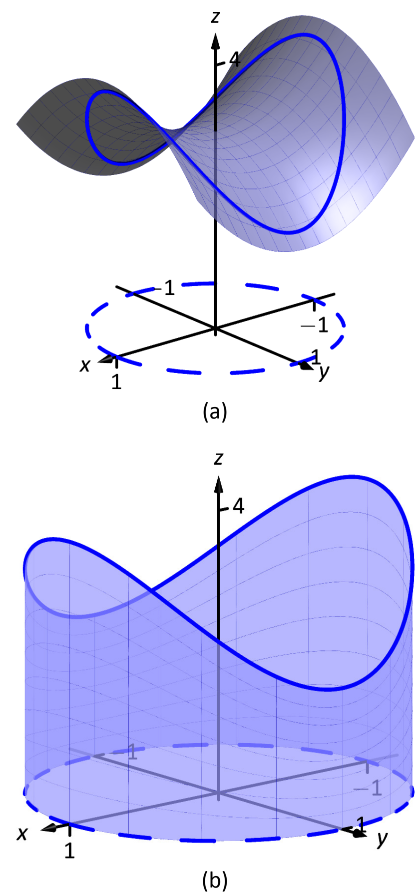


Figure 14.3: Finding area under a curve in Example 375.

then find the area described at the beginning of this section.

Example 375 Evaluating a line integral: area under a surface over a curve.

Find the area over the unit circle in the x-y plane and under the surface $f(x,y) = x^2 - y^2 + 3$, shown in Figure 14.3.

Solution The curve C is the unit circle, which we will describe with the parametrization $\vec{r}(t) = \langle \cos t, \sin t\rangle$ for $0 \le t \le 2\pi$. We find $||\,\vec{r}\,'(t)\,|| = 1$, so $ds = 1dt$.

We find the values of f over C as $f(x,y) = f(\cos t, \sin t) = \cos^2 t - \sin^2 t + 3$. Thus the area we seek is (note the use of the $\oint f(s)ds$ notation):

$$\begin{aligned}\oint_C f(s)\, ds &= \int_0^{2\pi} \left(\cos^2 t - \sin^2 t + 3\right) dt \\ &= 6\pi.\end{aligned}$$

(Note: we may have approximated this answer from the start. The unit circle has a circumference of 2π, and we may have guessed that due to the apparent symmetry of our surface, the average height of the surface is 3.)

We now consider the example that introduced this section.

Example 376 Evaluating a line integral: area under a surface over a curve.

Find the area under $f(x,y) = 1 - \cos(x)\sin(y)$ and over the parabola $y = x^2$, from $-1 \le x \le 1$.

Solution We parametrize our curve C as $\vec{r}(t) = \langle t, t^2\rangle$ for $-1 \le t \le 1$; we find $||\,\vec{r}\,'(t)\,|| = \sqrt{1+4t^2}$, so $ds = \sqrt{1+4t^2}\,dt$.

Replacing x and y with their respective functions of t, we have $f(x,y) = f(t,t^2) = 1 - \cos(t)\sin(t^2)$. Thus the area under f and over C is found to be

$$\int_C f(s)\, ds = \int_{-1}^{1} \Big(1 - \cos(t)\sin\left(t^2\right)\Big)\sqrt{1+t^2}\, dt.$$

This integral is impossible to evaluate using the techniques developed in this text. We resort to a numerical approximation; accurate to two places after the decimal, we find the area is

$$= 2.17.$$

Notes:

We give one more example of finding area.

Example 377 **Evaluating a line integral: area under a curve in space.**
Find the area above the x-y plane and below the helix parametrized by $\vec{r}(t) = \langle \cos t, 2\sin t, t/\pi\rangle$, for $0 \le t \le 2\pi$, as shown in Figure 14.4.

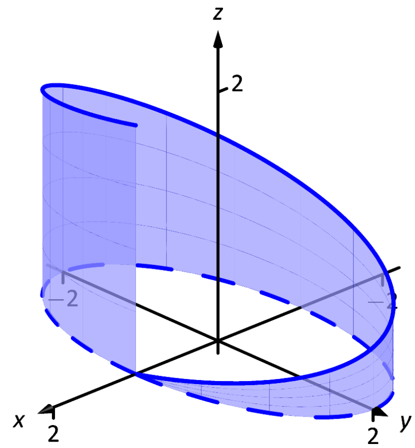


Figure 14.4: Finding area under a curve in Example 377.

Solution Note how this is problem is different than the previous examples: here, the height is not given by a surface, but by the curve itself.

We use the given vector-valued function $\vec{r}(t)$ to determine the curve C in the x-y plane by simply using the first two components of $\vec{r}(t)$: $\vec{c}(t) = \langle \cos t, 2\sin t\rangle$. Thus $ds = ||\,\vec{c}\,'(t)\,||\,dt = \sqrt{\sin^2 t + 4\cos^2 t}\,dt$.

The height is not found by evaluating a surface over C, but rather it is given directly by the third component of $\vec{r}(t)$: t/π. Thus

$$\oint_C f(s)\,ds = \int_0^{2\pi} \frac{t}{\pi}\sqrt{\sin^2 t + 4\cos^2 t}\,dt \approx 9.69,$$

where the approximation was obtained using numerical methods.

Properties of Line Integrals

Many properties of line integrals can be inferred from general integration properties. For instance, if k is a scalar, then $\int_C k f(s)ds = k\int_C f(s)ds$.

One property in particular of line integrals is worth noting. If C is a curve composed of subcurves C_1 and C_2, where they share only one point in common (see Figure 14.5(a)), then the line integral over C is the sum of the line integrals over C_1 and C_2:

$$\int_C f(s)\,ds = \int_{C_1} f(s)\,ds + \int_{C_2} f(s)\,ds.$$

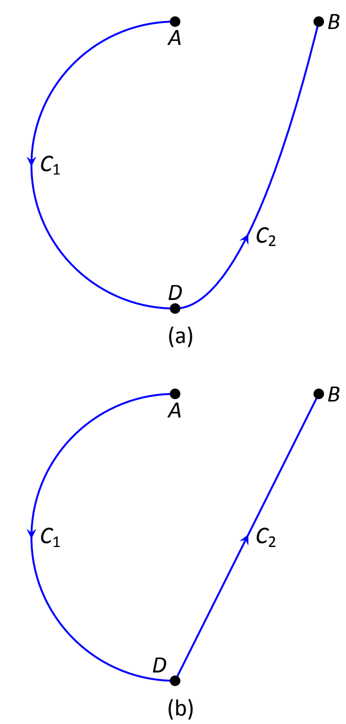


Figure 14.5: Illustrating properties of line integrals.

This property allows us to evaluate line integrals over some curves C that are not smooth. Note how in Figure 14.5(b) the curve is not smooth at D, so by our definition of the line integral we cannot evaluate $\int_C f(s)ds$. However, one can evaluate line integrals over C_1 and C_2 and their sum will be the desired quantity.

A curve C that is composed of two or more smooth curves is said to be **piecewise smooth**. In this chapter, any statement that is made about smooth curves also holds for piecewise smooth curves.

We state these properties as a theorem.

Notes:

Theorem 94 Properties of Line Integrals Over Scalar Fields

Let C be a smooth curve parametrized by the arc–length parameter s, let f and g be continuous functions of s, and let k_1 and k_2 be scalars. Then:

1. $\displaystyle\int_C \big(k_1 f(s) + k_2 g(s)\big)\, ds = k_1 \int_C f(s)\, ds + k_2 \int_C g(s)\, ds.$
2. Let C be piecewise smooth, composed of smooth components C_1 and C_2. Then
$$\int_C f(s)\, ds = \int_{C_1} f(s)\, ds + \int_{C_2} f(s)\, ds.$$

Mass and Center of Mass

We first learned integration as a method to find area under a curve, then later used integration to compute a variety of other quantities, such as arc length, volume, force, etc. In this section, we also introduced line integrals as a method to find area under a curve, and now we explore one more application.

Let a curve C (either in the plane or in space) represent a thin wire with variable density $\delta(s)$. We can approximate the mass of the wire by dividing the wire (i.e., the curve) into small segments and assume the density is constant across these small segments. The mass of each segment is density of the segment $\times$ its length; by summing up the approximate mass of each segment we can approximate the total mass:

$$\text{Total Mass of Wire } = \sum \delta(s_i)\Delta s_i.$$

By taking the limit as the length of the segments approaches 0, we have the definition of the line integral as seen in Definition 91. When learning of the line integral, we let $f(s)$ represent a height; now we let $f(s) = \delta(s)$ represent a density.

We can extend this understanding of computing mass to also compute the center of mass of a thin wire. (As a reminder, the center of mass can be a useful piece of information as objects rotate about that center.) We give the relevant formulas in the next definition, followed by an example. Note the similarities between this definition and Definition 90, which gives similar properties of solids in space.

Notes:

Definition 92 Mass, Center of Mass of Thin Wire

Let a thin wire lie along a smooth curve C with continuous density function $\delta(s)$, where s is the arc length parameter.

1. The **mass** of the thin wire is $M = \displaystyle\int_C \delta(s)\,ds$.
2. The **moment about the y-z plane** is $M_{yz} = \displaystyle\int_C x\delta(s)\,ds$.
3. The **moment about the x-z plane** is $M_{xz} = \displaystyle\int_C y\delta(s)\,ds$.
4. The **moment about the x-y plane** is $M_{xy} = \displaystyle\int_C z\delta(s)\,ds$.
5. The **center of mass** of the wire is

$$(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right).$$

Example 378 Evaluating a line integral: calculating mass.

A thin wire follows the path $\vec{r}(t) = \langle 1+\cos t, 1+\sin t, 1+\sin(2t)\rangle$, $0 \leq t \leq 2\pi$. The density of the wire is determined by its position in space: $\delta(x, y, z) = y + z$ gm/cm. The wire is shown in Figure 14.6, where a light color indicates low density and a dark color represents high density. Find the mass and center of mass of the wire.

SOLUTION We compute the density of the wire as

$$\delta(x, y, z) = \delta\big(1+\cos t, 1+\sin t, 1+\sin(2t)\big) = 2+\sin t+\sin(2t).$$

We compute ds as

$$ds = ||\,\vec{r}\,'(t)\,||\,dt = \sqrt{\sin^2 t+\cos^2 t+4\cos^2(2t)}\,dt = \sqrt{1+4\cos^2(2t)}\,dt.$$

Thus the mass is

$$M = \oint_C \delta(s)\,ds = \int_0^{2\pi} \big(2+\sin t+\sin(2t)\big)\sqrt{1+4\cos^2(2t)}\,dt \approx 21.08\text{gm}.$$

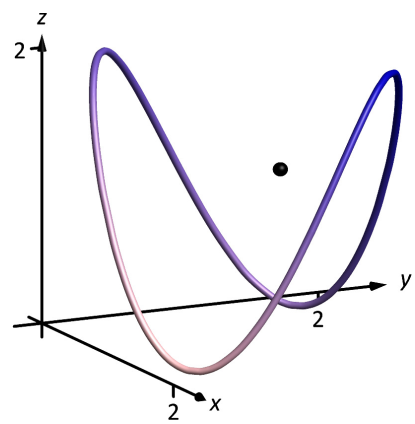


Figure 14.6: Finding the mass of a thin wire in Example 378.

Notes:

We compute the moments about the coordinate planes:

$$M_{yz} = \oint_C x\delta(s)\,ds = \int_0^{2\pi} (1+\cos t)\big(2+\sin t+\sin(2t)\big)\sqrt{1+4\cos^2(2t)}\,dt \approx 21.08.$$
$$M_{xz} = \oint_C y\delta(s)\,ds = \int_0^{2\pi} (1+\sin t)\big(2+\sin t+\sin(2t)\big)\sqrt{1+4\cos^2(2t)}\,dt \approx 26.35$$
$$M_{xy} = \oint_C z\delta(s)\,ds = \int_0^{2\pi} \big(1+\sin(2t)\big)\big(2+\sin t+\sin(2t)\big)\sqrt{1+4\cos^2(2t)}\,dt \approx 25.40$$

Thus the center of mass of the wire is located at

$$(\overline{x},\overline{y},\overline{z}) = \left(\frac{M_{yz}}{M},\frac{M_{xz}}{M},\frac{M_{xy}}{M}\right) \approx (1, 1.25, 1.20),$$

as indicated by the dot in Figure 14.6. Note how in this example, the curve C is "centered" about the point $(1,1,1)$, though the variable density of the wire pulls the center of mass out along the y and z axes.

We end this section with a callback to the Integration Review that preceded this section. A line integral looks like: $\int_C f(s)\,ds$. As stated before the definition of the line integral, this means "sum up, along a curve C, function values $f(s) \times$ small arc lengths." When $f(s)$ represents a height, we have "height $\times$ length = area." When $f(s)$ is a density (and we use $\delta(s)$ by convention), we have "density (mass per unit length) $\times$ length = mass."

We'll expand our uses for line integrals after the next section. First, we'll investigate a new mathematical object, called a vector field.

Notes:

Exercises 14.1

Terms and Concepts

1. Explain how a line integral can be used to find the area under a curve.

2. How does the evaluation of a line integral given as $\int_C f(s)\,ds$ differ from a line integral given as $\oint_C f(s)\,ds$?

3. Why are most line integrals evaluated using Key Idea 50 instead of "directly" as $\int_C f(s)\,ds$?

4. Sketch a closed, piecewise smooth curve composed of three subcurves.

Problems

In Exercises 5 – 10, a planar curve C is given along with a surface f that is defined over C. Evaluate the line integral $\displaystyle\int_C f(s)\,ds$.

5. C is the line segment joining the points $(-2,-1)$ and $(1,2)$; the surface is $f(x,y)=x^2+y^2+2$.

6. C is the segment of $y=3x+2$ on $[1,2]$; the surface is $f(x,y)=5x+2y$.

7. C is the circle with radius 2 centered at the point $(4,2)$; the surface is $f(x,y)=3x-y$.

8. C is the curve given by $\vec{r}(t)=\langle \cos t+t\sin t, \sin t-t\cos t\rangle$ on $[0,2\pi]$; the surface is $f(x,y)=5$.

9. C is the piecewise curve composed of the line segments that connect $(0,1)$ to $(1,1)$, then connect $(1,1)$ to $(1,0)$; the surface is $f(x,y)=x+y^2$.

10. C is the piecewise curve composed of the line segment joining the points $(0,0)$ and $(1,1)$, along with the quarter–circle parametrized by $\langle \cos t, -\sin t+1\rangle$ on $[0,\pi/2]$(which starts at the point $(1,1)$ and ends at $(0,0)$; the surface is $f(x,y)=x^2+y^2$.

In Exercises 11 – 14, a planar curve C is given along with a surface f that is defined over C. Set up the line integral $\displaystyle\int_C f(s)\,ds$, then approximate its value using technology.

11. C is the portion of the parabola $y=2x^2+x+1$ on $[0,1]$; the surface is $f(x,y)=x^2+2y$.

12. C is the portion of the curve $y=\sin x$ on $[0,\pi]$; the surface is $f(x,y)=x$.

13. C is the ellipse given by $\vec{r}(t)=\langle 2\cos t, \sin t\rangle$ on $[0,2\pi]$; the surface is $f(x,y)=10-x^2-y^2$.

14. C is the portion of $y=x^3$ on $[-1,1]$; the surface is $f(x,y)=2x+3y+5$.

In Exercises 15 – 18, a parametrized curve C in space is given. Find the area above the x-y plane that is under C.

15. C: $\vec{r}(t)=\langle 5t, t, t^2\rangle$ for $1\leq t\leq 2$.

16. C: $\vec{r}(t)=\langle \cos t, \sin t, \sin(2t)+1\rangle$ for $0\leq t\leq 2\pi$.

17. C: $\vec{r}(t)=\langle 3\cos t, 3\sin t, t^2\rangle$ for $0\leq t\leq 2\pi$.

18. C: $\vec{r}(t)=\langle 3t, 4t, t\rangle$ for $0\leq t\leq 1$.

In Exercises 19 – 20, a parametrized curve C is given that represents a thin wire with density δ. Find the mass and center of mass of the thin wire.

19. C: $\vec{r}(t)=\langle \cos t, \sin t, t\rangle$ for $0\leq t\leq 4\pi$; $\delta(x,y,z)=z$.

20. C: $\vec{r}(t)=\langle t-t^2, t^2-t^3, t^3-t^4\rangle$ for $0\leq t\leq 1$; $\delta(x,y,z)=x+2y+2z$. Use technology to approximate the value of each integral.

14.2 Vector Fields

We have studied functions of two or three variables, where the input of such functions is a point (either a point in the plane or in space) and the output is a number.

We could also create functions where the input is a point (again, either in the plane or in space), but the output is a *vector*. For instance, we could create the following function: $\vec{F}(x,y) = \langle x+y, x-y\rangle$, where $\vec{F}(2,3) = \langle 5,-1\rangle$. We are to think of $\vec{F}$ assigning the vector $\langle 5,-1\rangle$ to the point $(2,3)$; in some sense, the vector $\langle 5,-1\rangle$ lies at the point $(2,3)$.

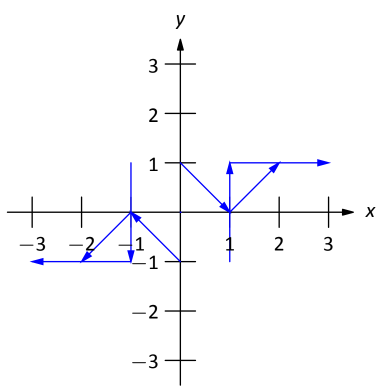
Such functions are extremely useful in any context where magnitude and direction are important. For instance, we could create a function $\vec{F}$ that represents the electromagnetic force exerted at a point by a electromagnetic field, or the velocity of air as it moves across an airfoil.

Because these functions are so important, we need to formally define them.

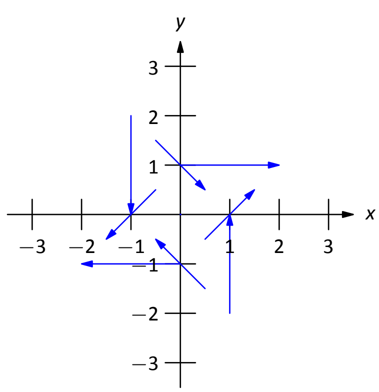
Definition 93 Vector Field

1. A **vector field in the plane** is a function $\vec{F}(x,y)$ whose domain is a subset of $\mathbb{R}^2$ and whose output is a two–dimensional vector:
$$\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle.$$

2. A **vector field in space** is a function $\vec{F}(x,y,z)$ whose domain is a subset of $\mathbb{R}^3$ and whose output is a three–dimensional vector:
$$\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle.$$



(a)



(b)

Figure 14.7: Demonstrating methods of graphing vector fields.

This definition may seem odd at first, as a special type of function is called a "field." However, as the function determines a "field of vectors", we can say the field is *defined by* the function, and thus the field *is* a function.

Visualizing vector fields helps cement this connection. When graphing a vector field in the plane, the general idea is to draw the vector $\vec{F}(x,y)$ at the point (x,y). For instance, using $\vec{F}(x,y) = \langle x+y, x-y\rangle$ as before, at $(1,1)$ we would draw $\langle 2,0\rangle$.

In Figure 14.7(a), one can see that the vector $\langle 2,0\rangle$ is drawn *starting from* the point $(1,1)$. A total of 8 vectors are drawn, with the x- and y-values of $-1,0,1$. In many ways, the resulting graph is a mess; it is hard to tell what this field "looks like."

In Figure 14.7(b), the same field is redrawn with each vector $\vec{F}(x,y)$ drawn *centered on* the point (x,y). This makes for a better looking image, though the

Notes:

long vectors can cause confusion: when one vector intersects another, the image looks cluttered.

A common way to address this problem is limit the length of each arrow, and represent long vectors with thick arrows, as done in Figure 14.8(a). Usually we do not use a graph of a vector field to determine exactly the magnitude of a particular vector. Rather, we are more concerned with the relative magnitudes of vectors: which are bigger than others? Thus limiting the length of the vectors is not problematic.

Drawing arrows with variable thickness is best done with technology; search the documentation of your favorite graphing program for terms like "vector fields" or "slope fields" to learn how. Technology obviously allows us to plot many vectors in a vector field nicely; in Figure 14.8(b), we see the same vector field drawn with many vectors, and finally get a clear picture of how this vector field behaves. (If this vector field represented the velocity of air moving across a flat surface, we could see that the air tends to move either to the upper–right or lower–left, and moves very slowly near the origin.)

We can similarly plot vector fields in space, as shown in Figure 14.9, though it is not often done. The plots get very busy very quickly, as there are lots of arrows drawn in a small amount of space. In Figure 14.9 the field $\vec{F} = \langle -y, x, z\rangle$ is graphed. If one could view the graph from above, one could see the arrows point in a cirlce about the z-axis. One should also note how the arrows far from the origin are larger than those close to the origin.

It is good practice to try to visualize certain vector fields in one's head. For instance, consider a point mass at the origin and the vector field that represents the gravitational force exerted by the mass at any point in the room. The field would consist of arrows pointing toward the origin, increasing in size as they near the origin (as the gravitational pull is strongest near the point mass).

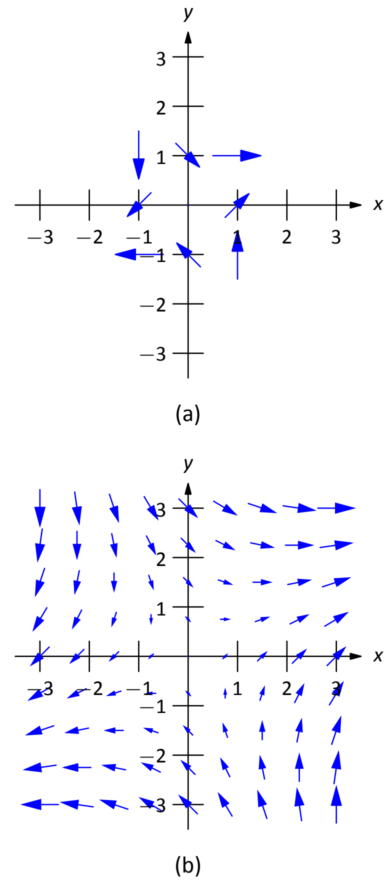


Figure 14.8: Demonstrating methods of graphing vector fields.

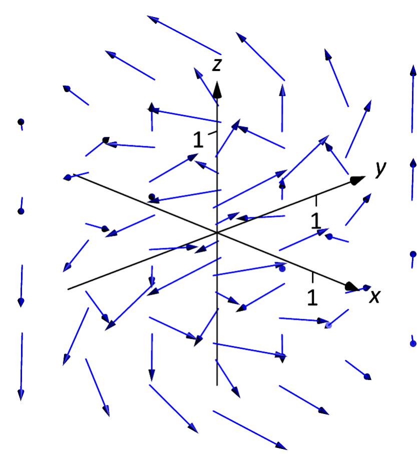


Figure 14.9: Graphing a vector field in space.

Vector Field Notation and Del Operator

Definition 93 defines a vector field $\vec{F}$ using the notation

$$\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle \quad \text{and} \quad \vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle.$$

That is, the components of $\vec{F}$ are each functions of x and y (and also z in space). As done in other contexts, we will drop the "of x, y and z" portions of the notation and refer to vector fields in the plane and in space as

$$\vec{F} = \langle M, N\rangle \quad \text{and} \quad \vec{F} = \langle M, N, P\rangle,$$

respectively, as this shorthand is quite convenient.

Another item of notation will become useful: the "del operator." Recall in Section 12.6 how we used the symbol ∇ (pronounced "del") to represent the

Notes:

gradient of a function of two variables. That is, if $z = f(x,y)$, then "del f" $= \nabla f = \langle f_x, f_y\rangle$.

We now define ∇ to be the "del operator." It is a vector whose components are partial derivative operations.

In the plane, $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right\rangle$; in space, $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right\rangle$.

Let $\vec F = \langle x+\sin y, y^2+z, x^2\rangle$. We can use vector operations and find the dot product of ∇ and $\vec F$:

$$\begin{aligned}\nabla\cdot\vec F &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle x+\sin y, y^2+z, x^2\rangle\\ &= \frac{\partial}{\partial x}(x+\sin y) + \frac{\partial}{\partial y}(y^2+z) + \frac{\partial}{\partial z}(x^2)\\ &= 1+2y.\end{aligned}$$

We can also compute their cross products:

$$\begin{aligned}\nabla\times\vec F &= \left\langle \frac{\partial}{\partial y}\big(x^2\big) - \frac{\partial}{\partial z}\big(y^2+z\big), \frac{\partial}{\partial z}\big(x+\sin y\big) - \frac{\partial}{\partial x}\big(x^2\big), \frac{\partial}{\partial x}\big(y^2+z\big) - \frac{\partial}{\partial y}\big(x+\sin y\big)\right\rangle\\ &= \langle -1, -2x, -\cos y\rangle.\end{aligned}$$

We do not yet know why we would want to compute the above. However, as we next learn about properties of vector fields, we will see how these dot and cross products with the del operator are quite useful.

Divergence and Curl

Two properties of vector fields will prove themselves to be very important: divergence and curl.

If the vector field represents the velocity of a fluid or gas, then the divergence of the field is a measure of the "compressibility" of the fluid. If the divergence is negative at a point, it means that the fluid is compressing: more fluid is going into the point than is going out. If the divergence is positive, is means the fluid is expanding: more fluid is going out at that point than going in. A divergence of zero means the same amount of fluid is going in as is going out. If the divergence is zero at all points, we say the field is **incompressible**.

It turns out that the proper measure of divergence is simply $\nabla\cdot\vec F$, as stated in the following definition.

Notes:

Definition 94 Divergence of a Vector Field

The **divergence** of a vector field $\vec{F}$ is

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F}.$$

- In the plane, with $\vec{F} = \langle M, N \rangle$, $\text{div}\,\vec{F} = M_x + N_y$.
- In space, with $\vec{F} = \langle M, N, P \rangle$, $\text{div}\,\vec{F} = M_x + N_y + P_z$.

Curl is a measure of the spinning action of the field. Let $\vec{F}$ represent the flow of water over a flat surface. If a small round cork were held in place at a point in the water, would the water cause the cork to spin? No spin corresponds to zero curl; counterclockwise spin corresponds to positive curl and clockwise spin corresponds to negative curl.

In space, things are a bit more complicated. Again let $\vec{F}$ represent the flow of water, and imagine suspending a tennis ball in one location in this flow. The water may cause the ball to spin along an axis. If so, the curl of the vector field is a *vector* (not a *scalar*, as before), parallel to the axis of rotation, following a right hand rule: when the thumb of one's right hand points in the direction of the curl, the ball will spin in the direction of the curling fingers of the hand.

In space, it turns out the proper measure of curl is $\nabla \times \vec{F}$, as stated in the following definition. To find the curl of a planar vector field $\vec{F} = \langle M, N \rangle$, embed it into space as $\vec{F} = \langle M, N, 0 \rangle$ and apply the cross product definition. Since M and N are functions of just x and y (and not z), all partial derivatives with respect to z become 0 and the result is simply $\langle 0, 0, N_x - M_y \rangle$. The third component is the measure of curl.

Definition 95 Curl of a Vector Field

- Let $\vec{F} = \langle M, N \rangle$ be a vector field in the plane. The **curl** of $\vec{F}$ is $\text{curl}\,\vec{F} = N_x - M_y$.
- Let $\vec{F} = \langle M, N, P \rangle$ be a vector field in space. The **curl** of $\vec{F}$ is $\text{curl}\,\vec{F} = \nabla \times \vec{F} = \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle$.

We adopt the convention of referring to curl as $\nabla \times \vec{F}$, regardless of whether $\vec{F}$ is a vector field in two or three dimensions.

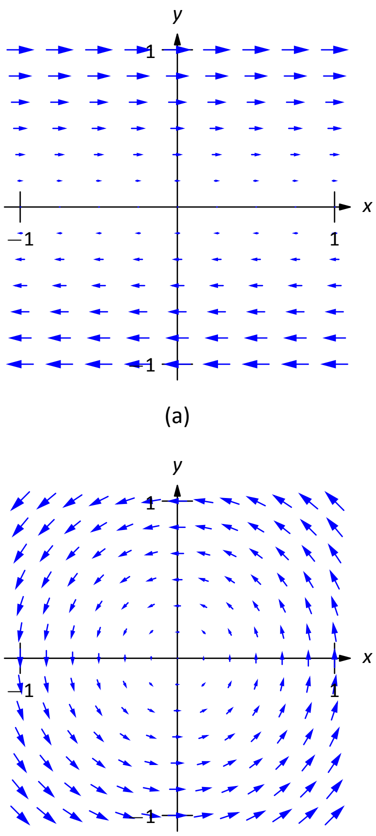
We now practice computing these quantities.

Notes:

Example 379 Computing divergence and curl of planar vector fields

For each of the planar vector fields given below, view its graph and try to visually determine if its divergence and curl are 0. Then compute the divergence and curl.

1. $\vec{F} = \langle y, 0\rangle$ (see Figure 14.10(a))
2. $\vec{F} = \langle -y, x\rangle$ (see Figure 14.10(b))
3. $\vec{F} = \langle x, y\rangle$ (see Figure 14.11(a))
4. $\vec{F} = \langle \cos y, \sin x\rangle$ (see Figure 14.11(b))



(a)

(b)

Figure 14.10: The vector fields in parts (a) and (b) in Example 379.

Solution

1. The arrow sizes are constant along any horizontal line, so if one were to draw a small box anywhere on the graph, it would seem that the same amount of fluid would enter the box as exit. Therefore it seems the divergence is zero; it is, as

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = M_x + N_y = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(0) = 0.$$

At any point on the x-axis, arrows above it move to the right and arrows below it move to the left, indicating that a cork placed on the axis would spin clockwise. A cork placed anywhere above the x-axis would have water above it moving to the right faster than the water below it, also creating a clockwise spin. A clockwise spin also appears to be created at points below the x-axis. Thus it seems the curl should be negative (and not zero). Indeed, it is:

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = N_x - M_y = \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(y) = -1.$$

2. It appears that all vectors that lie on a circle of radius r, centered at the origin, have the same length (and indeed this is true). That implies that the divergence should be zero: draw any box on the graph, and any fluid coming in will lie along a circle that takes the same amount of fluid out. Indeed, the divergence is zero, as

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = M_x + N_y = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0.$$

Clearly this field moves objects in a circle, but would it induce a cork to spin? It appears that yes, it would: place a cork anywhere in the flow, and

Notes:

the point of the cork closest to the origin would feel less flow than the point on the cork farthest from the origin, which would induce a counterclockwise flow. Indeed, the curl is positive:

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = N_x - M_y = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 1 - (-1) = 2.$$

Since the curl is constant, we conclude the induced spin is the same no matter where one is in this field.

3. At the origin, there are many arrows pointing out but no arrows pointing in. We conclude that at the origin, the divergence must be positive (and not zero). If one were to draw a box anywhere in the field, the edges farther from the origin would have larger arrows passing through them than the edges close to the origin, indicating that more is going from a point than going in. This indicates a positive (and not zero) divergence. This is correct:

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = M_x + N_y = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 1 + 1 = 2.$$

One may find this curl to be harder to determine visually than previous examples. One might note that any arrow that induces a clockwise spin on a cork will have an equally sized arrow inducing a counterclockwise spin on the other side, indicating no spin and no curl. This is correct, as

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = N_x - M_y = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) = 0.$$

4. One might find this divergence hard to determine visually as large arrows appear in close proximity to small arrows, each pointing in different directions. Instead of trying to rationalize a guess, we compute the divergence:

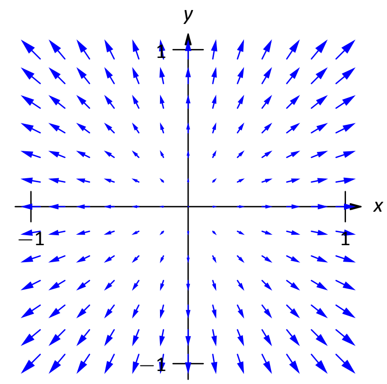
$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = M_x + N_y = \frac{\partial}{\partial x}(\cos y) + \frac{\partial}{\partial y}(\sin x) = 0.$$

Perhaps surprisingly, the divergence is 0.

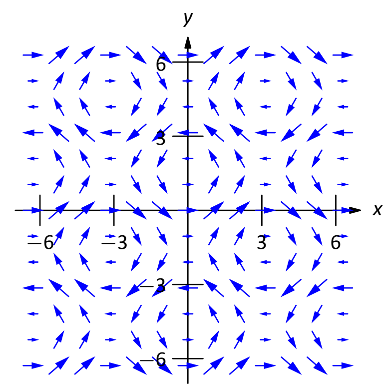
Will all the loops of different directions in the field, one is apt to reason the curl is variable. Indeed, it is:

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = N_x - M_y = \frac{\partial}{\partial x}(\sin x) - \frac{\partial}{\partial y}(\cos y) = \cos x + \sin y.$$

Depending on the values of x and y, the curl may be positive, negative, or zero.



(a)



(b)

Figure 14.11: The vector fields in parts (c) and (d) in Example 379.

Notes:

Example 380 **Computing divergence and curl of vector fields in space**
Compute the divergence and curl of each of the following vector fields.

1. $\vec{F} = \langle x^2 + y + z, -x - z, x + y \rangle$
2. $\vec{F} = \langle e^{xy}, \sin(x + z), x^2 + y \rangle$

Solution We compute the divergence and curl of each field following the definitions.

1. $\text{div}\,\vec{F} = \nabla \cdot \vec{F} = M_x + N_y + P_z = 2x + 0 + 0 = 2x.$
   $$\begin{aligned}\text{curl}\,\vec{F} = \nabla \times \vec{F} &= \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle \\ &= \langle 1 - (-1), 1 - 1, -1 - (1) \rangle = \langle 2, 0, -2 \rangle.\end{aligned}$$
   For this particular field, no matter the location in space, a spin is induced with axis parallel to $\langle 2, 0, -2 \rangle$.
2. $\text{div}\,\vec{F} = \nabla \cdot \vec{F} = M_x + N_y + P_z = ye^{xy} + 0 + 0 = ye^{xy}.$
   $$\begin{aligned}\text{curl}\,\vec{F} = \nabla \times \vec{F} &= \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle \\ &= \langle 1 - \cos(x + z), -2x, \cos(x + z) - xe^{xy} \rangle.\end{aligned}$$

Example 381 **Creating a field representing gravitational force**
The force of gravity between two objects is inversely proportional to the square of the distance between the objects. Locate a point mass at the origin. Create a vector field $\vec{F}$ that represents the gravitational pull of the point mass at any point (x, y, z). Find the divergence and curl of this field.

Solution The point mass pulls toward the origin, so at (x, y, z), the force will pull in the direction of $\langle -x, -y, -z \rangle$. To get the proper magnitude, it will be useful to find the unit vector in this direction. Dividing by its magnitude, we have

$$\vec{u} = \left\langle \frac{-x}{\sqrt{x^2 + y^2 + z^2}}, \frac{-y}{\sqrt{x^2 + y^2 + z^2}}, \frac{-z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle.$$

The magnitude of the force is inversely proportional to the square of the distance between the two points. Letting k be the constant of proportionality, we have the magnitude as $\dfrac{k}{x^2 + y^2 + z^2}$. Multiplying this magnitude by the unit vector above, we have the desired vector field:

$$\vec{F} = \left\langle \frac{-kx}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-ky}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-kz}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle.$$

Notes:

We leave it to the reader to confirm that div $\vec{F} = 0$ and curl $\vec{F} = \vec{0}$.

The analogous planar vector field is given in Figure 14.12. Note how all arrows point to the origin, and the magnitude gets very small when "far" from the origin.

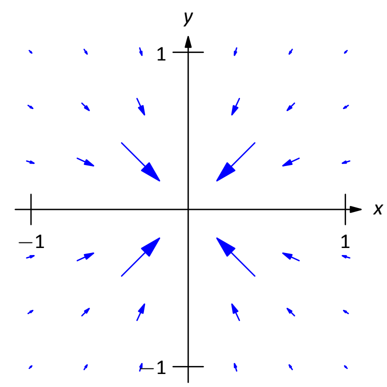


Figure 14.12: A vector field representing gravitational force.

A function $z = f(x, y)$ naturally induces a vector field, $\vec{F} = \nabla f = \langle f_x, f_y \rangle$. Given what we learned of the gradient in Section 12.6, we know that the vectors of $\vec{F}$ point in the direction of greatest increase of f. Because of this, f is said to be the **potential function** of $\vec{F}$. Vector fields that are the gradient of potential functions will play an important role in the next section.

Example 382 A vector field that is the gradient of a potential function

Let $f(x, y) = 3 - x^2 - 2y^2$ and let $\vec{F} = \nabla f$. Graph $\vec{F}$, and find the divergence and curl of $\vec{F}$.

Solution Given f, we find $\vec{F} = \nabla f = \langle -2x, -4y \rangle$. A graph of $\vec{F}$ is given in Figure 14.13(a). In part (b) of the figure, the vector field is given along with a graph of the surface itself; one can see how each vector is pointing in the direction of "steepest uphill", which, in this case, is not simply just "toward the origin."

We leave it to the reader to confirm that div $\vec{F} = -6$ and curl $\vec{F} = 0$.

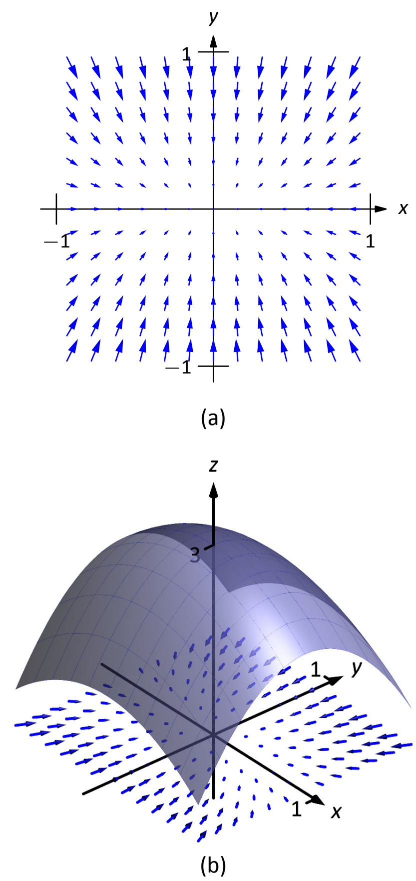


Figure 14.13: A graph of a function $z = f(x, y)$ and the vector field $\vec{F} = \nabla f$ in Example 382.

This section introduces the concept of a vector field. The next section "applies calculus" to vector fields. A common application is this: let $\vec{F}$ be a vector field representing a force (hence it is called a "force field," though this name has a decidedly comic-book feel) and let a particle move along a curve C under the influence of this force. What work is performed by the field on this particle? The solution lies in correctly applying the concepts of line integrals in the context of vector fields.

Notes:

Exercises 14.2

Terms and Concepts

1. Give two quantities that can be represented by a vector field in the plane or in space.

2. In your own words, describe what it means for a vector field to have a negative divergence at a point.

3. In your own words, describe what it means for a vector field to have a negative curl at a point.

4. The divergence of a vector field $\vec{F}$ at a particular point is 0. Does this mean that $\vec{F}$ is incompressible? Why/why not?

Problems

In Exercises 5 – 8, sketch the given vector field over the rectangle with opposing corners $(-2,-2)$ and $(2,2)$, sketching one vector for every point with integer coordinates (i.e., at $(0,0)$, $(1,2)$, etc.).

5. $\vec{F} = \langle x, 0\rangle$

6. $\vec{F} = \langle 0, x\rangle$

7. $\vec{F} = \langle 1, -1\rangle$

8. $\vec{F} = \langle y^2, 1\rangle$

In Exercises 9 – 18, find the divergence and curl of the given vector field.

9. $\vec{F} = \langle x, y^2\rangle$

10. $\vec{F} = \langle -y^2, x\rangle$

11. $\vec{F} = \langle \cos(xy), \sin(xy)\rangle$

12. $\vec{F} = \left\langle \dfrac{-2x}{(x^2+y^2)^2}, \dfrac{-2y}{(x^2+y^2)^2} \right\rangle$

13. $\vec{F} = \langle x+y, y+z, x+z\rangle$

14. $\vec{F} = \langle x^2+z^2, x^2+y^2, y^2+z^2\rangle$

15. $\vec{F} = \nabla f$, where $f(x,y) = \frac{1}{2}x^2 + \frac{1}{3}y^3$.

16. $\vec{F} = \nabla f$, where $f(x,y) = x^2y$.

17. $\vec{F} = \nabla f$, where $f(x,y,z) = x^2y + \sin z$.

18. $\vec{F} = \nabla f$, where $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$.

14.3 Line Integrals over Vector Fields

Suppose a particle moves along a curve C under the influence of an electromagnetic force described by a vector field $\vec{F}$. Since a force is inducing motion, work is performed. How can we calculate how much work is performed?

Recall that when moving in a straight line, if $\vec{F}$ represents a constant force and $\vec{d}$ represents the direction and length of travel, then work is simply $W = \vec{F} \cdot \vec{d}$. However, we generally want to be able to calculate work even if $\vec{F}$ is not constant and C is not a straight line.

As we have practiced many times before, we can calculate work by first approximating, then refining our approximation through a limit that leads to integration.

Assume as we did in Section 14.1 that C can be parametrized by the arc length parameter s. Over a short piece of the curve with length ds, the curve is approximately straight and our force is approximately constant. The straight–line direction of this short length of curve is given by $\vec{T}$, the unit tangent vector; let $\vec{d} = \vec{T}\,ds$, which gives the direction and magnitude of a small section of C. Thus work over this small section of C is $\vec{F} \cdot \vec{d} = \vec{F} \cdot \vec{T}\,ds$.

Summing up all the work over these small segments gives an approximation of the work performed. By taking the limit as ds goes to zero, and hence the number of segments approaches infinity, we can obtain the exact amount of work. Following the logic presented at the beginning of this chapter in the Integration Review, we see that

$$W = \int_C \vec{F} \cdot \vec{T}\,ds,$$

a line integral.

This line integral is beautiful in its simplicity, yet is not so useful in making actual computations (largely because the arc length parameter is so difficult to work with). To compute actual work, we need to parametrize C with another parameter t via a vector–valued function $\vec{r}(t)$. As stated in Section 14.1, $ds = \|\,\vec{r}\,'(t)\,\|\,dt$, and recall that $\vec{T} = \vec{r}\,'(t)/\|\,\vec{r}\,'(t)\,\|$. Thus

$$W = \int_C \vec{F} \cdot \vec{T}\,ds = \int_C \vec{F} \cdot \frac{\vec{r}\,'(t)}{\|\,\vec{r}\,'(t)\,\|}\|\,\vec{r}\,'(t)\,\|\,dt = \int_C \vec{F} \cdot \vec{r}\,'(t)\,dt = \int_C \vec{F} \cdot d\vec{r}, \quad (14.2)$$

where the final integral uses the differential $d\vec{r}$ for $\vec{r}\,'(t)\,dt$.

These integrals are known as **line integrals over vector fields**. By contrast, the line integrals we dealt with in Section 14.1 are sometimes referred to as **line integrals over scalar fields**. Just as a vector field is defined by a function that returns a vector, a scalar field is a function that returns a scalar, such as $z = f(x, y)$. We waited until now to introduce this terminology so we could contrast the concept with vector fields.

Notes:

We formally define this line integral, then give examples and applications.

Definition 96 Line Integral Over Vector Field

Let $\vec{F}$ be a vector field with continuous components defined on a smooth curve C, parametrized by $\vec{r}(t)$, and let $\vec{T}$ be the unit tangent vector of $\vec{r}(t)$. The **line integral over $\vec{F}$ along** C is

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds.$$

In Definition 96, note how the dot product $\vec{F}\cdot\vec{T}$ is just a scalar. Therefore, this new line integral is really just a special kind of line integral found in Section 14.1; letting $f(s) = \vec{F}(s)\cdot\vec{T}(s)$, the right–hand side simply becomes $\int_C f(s)\,ds$, and we can use the techniques of that section to evaluate the integral. We combine those techniques, along with parts of Equation (14.2), to clearly state how to evaluate a line integral over a vector field in the following Key Idea.

Key Idea 51 Evaluating a Line Integral over a Vector Field

Let $\vec{F}$ be a vector field with continuous components defined on a smooth curve C, parametrized by $\vec{r}(t)$, $a \leq t \leq b$, where $\vec{r}$ is continuously differentiable. Then

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}\big(\vec{r}(t)\big)\cdot\vec{r}\,'(t)\,dt.$$

An important concept implicit in this Key Idea: we can use any continuously differentiable parametrization $\vec{r}(t)$ of C that preserves the orientation of C: there isn't a "right" one. In practice, choose one that seems easy to work with.

Notation note: the above Definition and Key Idea implicitly evaluate $\vec{F}$ along the curve C, which is parametrized by $\vec{r}(t)$. For instance, if $\vec{F} = \langle x+y, x-y\rangle$ and $\vec{r}(t) = \langle t^2, \cos t\rangle$, then evaluating $\vec{F}$ along C means substituting the x- and y-components of $\vec{r}(t)$ in for x and y, respectively, in $\vec{F}$. Therefore, along C, $\vec{F} = \langle x+y, x-y\rangle = \big\langle t^2+\cos t, t^2-\cos t\big\rangle$. Since we are substituting the *output* of $\vec{r}(t)$ for the *input* of $\vec{F}$, we write this as $\vec{F}\big(\vec{r}(t)\big)$. This is a slight abuse of notation as technically the input of $\vec{F}$ is to be a *point*, not a *vector*, but this shorthand is useful.

We use an example to practice evaluating line integrals over vector fields.

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Example 383 Evaluating a line integral over a vector field: computing work

Two particles move from $(0,0)$ to $(1,1)$ under the influence of the force field $\vec{F} = \langle x, x+y\rangle$. One particle follows C_1, the line $y = x$; the other follows C_2, the curve $y = x^4$, as shown in Figure 14.14. Force is measured in newtons and distance is measured in meters. Find the work performed by each particle.

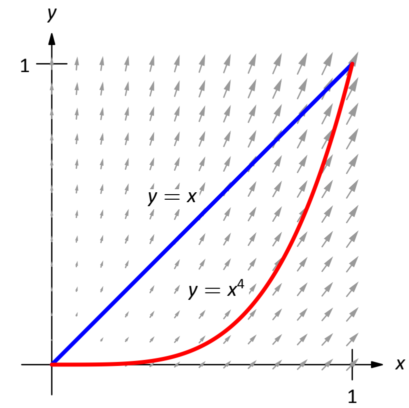

Figure 14.14: Paths through a vector field in Example 383.

Solution To compute work, we need parametrize each path. We use $\vec{r}_1(t) = \langle t, t\rangle$ to parametrize $y = x$, and let $\vec{r}_2(t) = \langle t, t^4\rangle$ parametrize $y = x^4$; for each, $0 \le t \le 1$.

Along the straight–line path, $\vec{F}\big(\vec{r}_1(t)\big) = \langle x, x+y\rangle = \langle t, t+t\rangle = \langle t, 2t\rangle$. We find $\vec{r}\,'_1(t) = \langle 1, 1\rangle$. The integral that computes work is:

$$\begin{aligned}\int_{C_1} \vec{F}\cdot d\vec{r} &= \int_0^1 \langle t, 2t\rangle \cdot \langle 1, 1\rangle\ dt\\ &= \int_0^1 3t\,dt\\ &= \frac{3}{2}t^2\Big|_0^1 = 1.5 \text{ joules.}\end{aligned}$$

Along the curve $y = x^4$, $\vec{F}\big(\vec{r}_2(t)\big) = \langle x, x+y\rangle = \big\langle t, t+t^4\big\rangle$. We find $\vec{r}\,'_2(t) = \big\langle 1, 4t^3\big\rangle$. The work performed along this path is

$$\begin{aligned}\int_{C_2} \vec{F}\cdot d\vec{r} &= \int_0^1 \big\langle t, t+t^4\big\rangle \cdot \big\langle 1, 4t^3\big\rangle\ dt\\ &= \int_0^1 \big(t + 4t^4 + 4t^7\big)\,dt\\ &= \big(\frac{1}{2}t^2 + \frac{4}{5}t^5 + \frac{1}{2}t^8\big)\Big|_0^1 = 1.8 \text{ joules.}\end{aligned}$$

Note how differing amounts of work are performed along the different paths. This should not be too surprising: the force is variable, one path is longer than the other, etc.

Example 384 Evaluating a line integral over a vector field: computing work

Two particles move from $(-1,1)$ to $(1,1)$ under the influence of a force field $\vec{F} = \langle y, x\rangle$. One moves along the curve C_1, the parabola defined by $y = 2x^2 - 1$. The other particle moves along the curve C_2, the bottom half of the circle defined by $x^2+(y-1)^2 = 1$, as shown in Figure 14.15. Force is measured in pounds

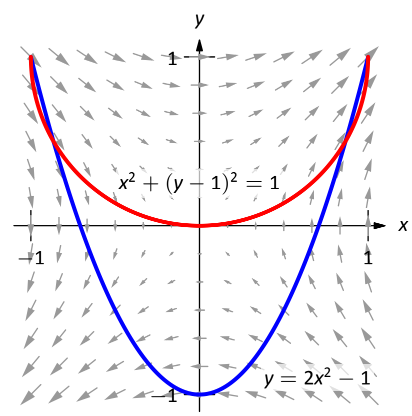

Figure 14.15: Paths through a vector field in Example 384.

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and distances are measured in feet. Find the work performed by moving each particle along its path.

Solution We start by parametrizing C_1: the parametrization $\vec{r}_1(t) = \left\langle t, 2t^2-1\right\rangle$ is straightforward, giving $\vec{r}\,'_1 = \langle 1, 4t\rangle$. On C_1, $\vec{F}\big(\vec{r}_1(t)\big) = \langle y, x\rangle = \left\langle 2t^2-1, t\right\rangle$.

Computing the work along C_1, we have:

$$\begin{aligned}\int_{C_1} \vec{F}\cdot d\vec{r}_1 &= \int_{-1}^{1} \left\langle 2t^2-1, t\right\rangle \cdot \langle 1, 4t\rangle\ dt\\ &= \int_{-1}^{1} \left(2t^2-1+4t^2\right) dt = 2 \text{ ft-lbs.}\end{aligned}$$

For C_2, it is probably simplest to parametrize the half circle using sine and cosine. Recall that $\vec{r}(t) = \langle \cos t, \sin t\rangle$ is a parametrization of the unit circle on $0 \leq t \leq 2\pi$; we add 1 to the second component to shift the circle up one unit, then restrict the domain to $\pi \leq t \leq 2\pi$ to obtain only the lower half, giving $\vec{r}_2(t) = \langle \cos t, \sin t + 1\rangle$, $\pi \leq t \leq 2\pi$, and hence $\vec{r}\,'_2(t) = \langle -\sin t, \cos t\rangle$ and $\vec{F}\big(\vec{r}_2(t)\big) = \langle y, x\rangle = \langle \sin t + 1, \cos t\rangle$.

Computing the work along C_2, we have:

$$\begin{aligned}\int_{C_2} \vec{F}\cdot d\vec{r}_2 &= \int_{\pi}^{2\pi} \langle \sin t + 1, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\ dt\\ &= \int_{\pi}^{2\pi} \big(-\sin^2 t - \sin t + \cos^2 t\big)\, dt = 2 \text{ ft-lbs.}\end{aligned}$$

Note how the work along C_1 and C_2 in this example is the same. We'll address why later in this section when *conservative fields* and *path independence* are discussed.

Properties of Line Integrals Over Vector Fields

Line integrals over vector fields share the same properties as line integrals over scalar fields, with one important distinction. The orientation of the curve C matters with line integrals over vector fields, whereas it did not matter with line integrals over scalar fields.

It is relatively easy to see why. Let C be the unit circle. The area under a surface over C is the same whether we traverse the circle in a clockwise or counterclockwise fashion, hence the line integral over a scalar field on C is the same irrespective of orientation. On the other hand, if we are computing work done by a force field, direction of travel definitely matters. Opposite directions create

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opposite signs when computing dot products, so traversing the circle in opposite directions will create line integrals that differ by a factor of -1.

Theorem 95 Properties of Line Integrals Over Vector Fields

Let $\vec{F}$ and $\vec{G}$ be vector fields with continuous components defined on a smooth curve C, parametrized by $\vec{r}(t)$, and let k_1 and k_2 be scalars. Then:

1. $\displaystyle\int_C \big(k_1\vec{F} + k_2\vec{G}\big)\cdot d\vec{r} = k_1\int_C \vec{F}\cdot d\vec{r} + k_2\int_C \vec{G}\cdot d\vec{r}.$
2. Let C be piecewise smooth, composed of smooth components C_1 and C_2. Then
$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r}.$$
3. Let C^* be the curve C with opposite orientation, parametrized by $\vec{r}^{\,*}$. Then
$$\int_C \vec{F}\cdot d\vec{r} = -\int_{C^*}\vec{F}\cdot d\vec{r}^{\,*}.$$

We demonstrate using these properties in the following example.

Example 385 Using properties of line integrals over vector fields

Let $\vec{F} = \langle 3(y-1/2), 1\rangle$ and let C be the path that starts at $(0,0)$, goes to $(1,1)$ along the curve $y = x^3$, then returns to $(0,0)$ along the line $y = x$, as shown in Figure 14.16. Evaluate $\oint_C \vec{F}\cdot d\vec{r}$.

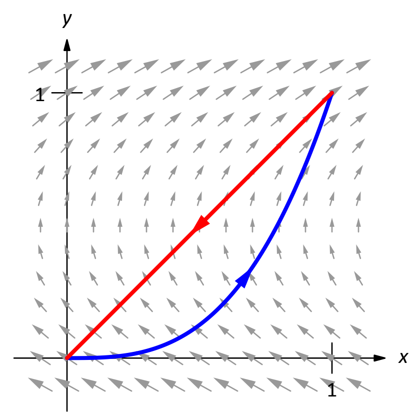


Figure 14.16: The vector field and curve in Example 385.

Solution As C is piecewise smooth, we break it into two components C_1 and C_2, where C_1 follows the curve $y = x^3$ and C_2 follows the curve $y = x$.

We parametrize C_1 with $\vec{r}_1(t) = \big\langle t, t^3\big\rangle$ on $0 \le t \le 1$, with $\vec{r}\,'_1(t) = \big\langle 1, 3t^2\big\rangle$. We will use $\vec{F}\big(\vec{r}_1(t)\big) = \big\langle 3(t^3 - 1/2), 1\big\rangle$.

While we always have unlimited ways in which to parametrize a curve, there are 2 "direct" methods to choose from when parametrizing C_2. The parametrization $\vec{r}_2(t) = \langle t, t\rangle$, $0 \le t \le 1$ traces the correct line segment but with the wrong orientation. Using Property 3 of Theorem 95, we can use this parametrization and negate the result.

Another choice is to use the techniques of Section 10.5 to create the line with the orientation we desire. We wish to start at $(1,1)$ and travel in the $\vec{d} = \langle -1,-1\rangle$ direction for one length of $\vec{d}$, giving equation $\vec{\ell}(t) = \langle 1,1\rangle + t\,\langle -1,-1\rangle = \langle 1-t, 1-t\rangle$ on $0 \le t \le 1$.

Notes:

Either choice is fine; we choose $\vec{r}_2(t)$ to practice using line integral properties. We find $\vec{r}\,'_2(t) = \langle 1, 1\rangle$ and $\vec{F}\big(\vec{r}_2(t)\big) = \langle 3(t-1/2), 1\rangle$.

Evaluating the line integral (note how we subtract the integral over C_2 as the orientation of $\vec{r}_2(t)$ is opposite):

$$\begin{aligned}\oint_C \vec{F}\cdot d\vec{r} &= \int_{C_1}\vec{F}\cdot d\vec{r}_1 - \int_{C_2}\vec{F}\cdot d\vec{r}_2\\ &= \int_0^1 \big\langle 3(t^3-1/2), 1\big\rangle\cdot\big\langle 1, 3t^2\big\rangle\,dt - \int_0^1 \langle 3(t-1/2), 1\rangle\cdot\langle 1, 1\rangle\,dt\\ &= \int_0^1 \Big(3t^3+3t^2-3/2\Big)\,dt - \int_0^1\Big(3t-1/2\Big)\,dt\\ &= (1/4)-(1)\\ &= -3/4.\end{aligned}$$

If we interpret this integral as computing work, the negative work implies that the motion is mostly *against* the direction of the force, which seems plausible when we look at Figure 14.16.

Example 386 Evaluating a line integral over a vector field in space

Let $\vec{F} = \langle -y, x, 1\rangle$, and let C be the portion of the helix given by $\vec{r}(t) = \langle \cos t, \sin t, t/(2\pi)\rangle$ on $[0, 2\pi]$, as shown in Figure 14.17. Evaluate $\int_C \vec{F}\cdot d\vec{r}$.

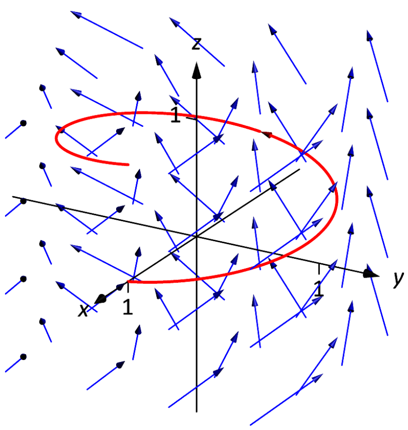


Figure 14.17: The graph of $\vec{r}(t)$ in Example 386.

Solution A parametrization is already given for C, so we just need to find $\vec{F}\big(\vec{r}(t)\big)$ and $\vec{r}\,'(t)$.

We have $\vec{F}\big(\vec{r}(t)\big) = \langle -\sin t, \cos t, 1\rangle$ and $\vec{r}\,'(t) = \langle -\sin t, \cos t, 1/(2\pi)\rangle$. Thus

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} \langle -\sin t, \cos t, 1\rangle\cdot\langle -\sin t, \cos t, 1/(2\pi)\rangle\,dt\\ &= \int_0^{2\pi}\Big(\sin^2 t + \cos^2 t + \frac{1}{2\pi}\Big)dt\\ &= 2\pi + 1 \approx 7.28.\end{aligned}$$

The Fundamental Theorem of Line Integrals

We are preparing to make important statements about the value of certain line integrals over special vector fields. Before we can do that, we need to define some terms that describe the domains over which a vector field is defined.

A region in the plane is **connected** if any two points in the region can be joined by a piecewise smooth curve that lies entirely in the region. In Figure

Notes:

14.18, sets R_1 and R_2 are connected; set R_3 is not connected, though it is composed of two connected subregions.

A region is **simply connected** if every simple closed curve that lies entirely in the region can be continously deformed (shrunk) to a single point without leaving the region. (A curve is **simple** if it does not cross itself.) In Figure 14.18, only set R_1 is simply connected. Region R_2 is not simple as any closed curve that goes around the "hole" in R_2 cannot be continously shrunk to a single point. As R_3 is not even connected, it cannot be simply connected, though again it consists of two simply connected subregions.

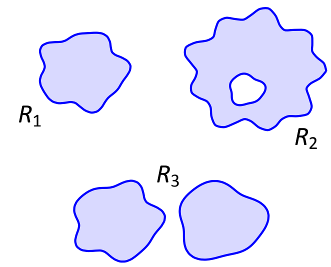
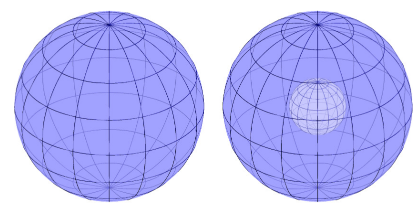
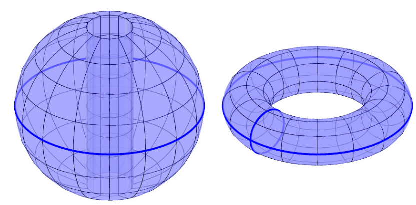


Figure 14.18: R_1 is simply connected; R_2 is connected, but not simply connected; R_3 is not connected.

We have applied these terms to regions of the plane, but they can be extended intuitively to regions in space (and hyperspace). In Figure 14.19(a), the sphere (at left) and the sphere with a subsphere removed (at right) are both simply connected. Any simple closed path that lies entirely within these domains can be continuously deformed into a single point. In Figure 14.19(b), neither domain is simply connected. A left, the sphere has a hole that extends its length and the pictured closed path cannot be deformed to a point. At right, two paths are illustrated on the torus that cannot be shrunk to a point.



(a)



(b)

Figure 14.19: The domains in (a) are simply connected, while the domains in (b) are not.

We will use the terms connected and simply connected in subsequent definitions and theorems.

Recall how in Example 384 particles moved from $A = (-1, 1)$ to $B = (1, 1)$ along two different paths, wherein the same amount of work was performed along each path. It turns out that regardless of the choice of path from A to B, the amount of work performed under the field $\vec{F} = \langle y, x\rangle$ is the same. Since our expectation is that differing amounts of work are performed along different paths, we give such special fields a name.

Definition 97 Conservative Field, Path Independent

Let $\vec{F}$ be a vector field defined on an open, connected domain D in the plane or in space containing points A and B. If the line integral $\int_C \vec{F}\cdot d\vec{r}$ has the same value for all choices of paths C starting at A and ending at B, then

- $\vec{F}$ is a **conservative field** and
- The line integral $\int_C \vec{F}\cdot d\vec{r}$ is **path independent** and can be written as
$$\int_C \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot\ d\vec{r}.$$

When $\vec{F}$ is a conservative field, the line integral from points A to B is sometimes written as $\int_A^B \vec{F}\cdot d\vec{r}$ to emphasize the independence of its value from the

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choice of path; all that matters are the beginning and ending points of the path.

How can we tell if a field is conservative? To show a field $\vec{F}$ is conservative using the definition, we need to show that *all* line integrals from points A to B have the same value. It is equivalent to show that *all* line integrals over closed paths C are 0. Each of these tasks are generally nontrivial.

There is a simpler method. Consider the surface defined by $z = f(x, y) = xy$. We can compute the gradient of this function: $\nabla f = \langle f_x, f_y \rangle = \langle y, x \rangle$. Note that this is the field from Example 384, which we have claimed is conservative. We will soon give a theorem that states that a field $\vec{F}$ is conservative if, and only if, it is the gradient of some scalar function f. To show $\vec{F}$ is conservative, we need to determine whether or not $\vec{F} = \nabla f$ for some function f. (We'll later see that there is a yet simpler method). To recognize the special relationship between $\vec{F}$ and f in this situation, f is given a name.

Definition 98 Potential Function

Let f be a function defined on a open domain D in the plane or in space (i.e., $z = f(x, y)$ or $w = f(x, y, z)$) and let $\vec{F} = \nabla f$, the gradient of f. Then f is a **potential function** of $\vec{F}$.

We now state the Fundamental Theorem of Line Integrals, which connects conservative fields and path independence to fields with potential functions.

Theorem 96 Fundamental Theorem of Line Integrals

Let $\vec{F}$ be a vector field whose components are continuous on an open, connected domain D in the plane or in space.

1. $\vec{F}$ is conservative if and only if there exists a differentiable function f such that $\vec{F} = \nabla f$.

2. If $\vec{F}$ is conservative, then given any points A and B in D,

$$\int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A).$$

Once again considering Example 384, we have $A = (-1, 1)$ and $B = (1, 1)$ and $\vec{F} = \langle y, x \rangle$. In that example, we evaluated two line integrals from A to B and found the value of each was 2. Note that $f(x, y) = xy$ is a potential function for

Notes:

$\vec{F}$. Following the Fundamental Theorem of Line Integrals, consider $f(B) - f(A)$:

$$f(B) - f(A) = f(1,1) - f(-1,1) = 1 - (-1) = 2,$$

the same value given by the line integrals.

We practice using this theorem again in the next example.

Example 387 Using the Fundamental Theorem of Line Integrals

Let $\vec{F} = \left\langle 3x^2y + 2x, x^3 + 1 \right\rangle$, $A = (0,1)$ and $B = (1,4)$. Use the first part of the Fundamental Theorem of Line Integrals to show that $\vec{F}$ is conservative, then choose any path from A to B and confirm the second part of the theorem.

Solution To show $\vec{F}$ is conservative, we need to find $z = f(x,y)$ such that $\vec{F} = \nabla f = \langle f_x, f_y \rangle$. That is, we need to find f such that $f_x = 3x^2y + 2x$ and $f_y = x^3 + 1$. As all we know about f are its partial derivatives, we recover f by integration:

$$\int \frac{\partial f}{\partial x}\, dx = f(x,y) + C(y).$$

Note how the constant of integration is more than "just a constant": it is anything that acts as a constant when taking a derivative with respect to x. Any function that is a function of y (containing no x's) acts as a constant when deriving with respect to x.

Integrating f_x in this example gives:

$$\int \frac{\partial f}{\partial x}\, dx = \int (3x^2y + 2x)\, dx = x^3y + x^2 + C(y).$$

Likewise, integrating f_y with respect to y gives:

$$\int \frac{\partial f}{\partial y}\, dy = \int (x^3 + 1)\, dy = x^3y + y + C(x).$$

These two results should be equal with appropriate choices of $C(x)$ and $C(y)$:

$$x^3y + x^2 + C(y) = x^3y + y + C(x) \quad \Rightarrow \quad C(x) = x^2 \quad \text{and} \quad C(y) = y.$$

We find $f(x,y) = x^3y + x^2 + y$, a potential function of $\vec{F}$. (If $\vec{F}$ were not conservative, no choice of $C(x)$ and $C(y)$ would give equality.)

By the Fundamental Theorem of Line Integrals, regardless of the path from A to B,

$$\begin{aligned}\int_A^B \vec{F} \cdot d\vec{r} &= f(B) - f(A)\\ &= f(1,4) - f(0,1)\\ &= 9 - 1 = 8.\end{aligned}$$

Notes:

To illustrate the validity of the Fundamental Theorem, we pick a path from A to B. The line between these two points would be simple to construct; we choose a slightly more complicated path by choosing the parabola $y = x^2 + 2x + 1$. This leads to the parametrization $\vec{r}(t) = \langle t, t^2 + 2t + 1\rangle$, $0 \leq t \leq 1$, with $\vec{r}\,'(t) = \langle t, 2t + 2\rangle$. Thus

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_C \vec{F}\big(\vec{r}(t)\big) \cdot \vec{r}\,'(t)\, dt \\
&= \int_0^1 \big\langle 3(t)(t^2 + 2t + 1) + 2t, t^3 + 1\big\rangle \cdot \langle t, 2t + 2\rangle\ dt \\
&= \int_0^1 \big(5t^4 + 8t^3 + 3t^2 + 4t + 2\big)\, dt \\
&= \big(t^5 + 2t^4 + t^3 + 2t^2 + 2t\big)\Big|_0^1 \\
&= 8,
\end{aligned}$$

which matches our previous result.

The Fundamental Theorem of Line Integrals states that we can determine whether or not $\vec{F}$ is conservative by determining whether or not $\vec{F}$ has a potential function. This can be difficult. A simpler method exists if the domain of $\vec{F}$ is simply connected (not just connected as needed in the Fundamental Theorem of Line Integrals), which is a reasonable requirement. We state this simpler method as a theorem.

Theorem 97 Curl of Conservative Fields

Let $\vec{F}$ be a vector field whose components are continuous on an open, simply connected domain D in the plane or in space. Then $\vec{F}$ is conservative if and only if curl $\vec{F} = 0$ or $\vec{0}$, respectively.

In Example 387, we showed that $\vec{F} = \langle 3x^2y + 2x, x^3 + 1\rangle$ is conservative by finding a potential function for $\vec{F}$. Using the above theorem, we can show that $\vec{F}$ is conservative much more easily by computing its curl:

$$\text{curl}\,\vec{F} = N_x - M_y = 3x^2 - 3x^2 = 0.$$

Notes:

Exercises 14.3

Terms and Concepts

1. T/F: In practice, the evaluation of line integrals over vector fields involves computing the magnitude of a vector–valued function.

2. Let $\vec{F}(x, y)$ be a vector field in the plane and let $\vec{r}(t)$ be a two–dimensional vector–valued function. Why is "$\vec{F}\big(\vec{r}(t)\big)$" an "abuse of notation"?

3. T/F: The orientation of a curve C matters when computing a line integral over a vector field.

4. T/F: The orientation of a curve C matters when computing a line integral over a scalar field.

5. Under "reasonable conditions," if $\text{curl}\,\vec{F} = \vec{0}$, what can we conclude about the vector field $\vec{F}$?

6. Let $\vec{F}$ be a conservative field and let C be a closed curve. Why are we able to conclude that $\oint_C \vec{F} \cdot d\vec{r} = 0$?

Problems

In Exercises 7 – 12, a vector field $\vec{F}$ and a curve C are given. Evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

7. $\vec{F} = \langle y, y^2 \rangle$; C is the line segment from $(0, 0)$ to $(3, 1)$.

8. $\vec{F} = \langle x, x + y \rangle$; C is the portion of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

9. $\vec{F} = \langle y, x \rangle$; C is the top half of the unit circle, beginning at $(1, 0)$ and ending at $(-1, 0)$.

10. $\vec{F} = \langle xy, x \rangle$; C is the portion of the curve $y = x^3$ on $-1 \le x \le 1$.

11. $\vec{F} = \langle z, x^2, y \rangle$; C is the line segment from $(1, 2, 3)$ to $(4, 3, 2)$.

12. $\vec{F} = \langle y + z, x + z, x + y \rangle$; C is the helix $\vec{r}(t) = \langle \cos t, \sin t, t/(2\pi) \rangle$ on $0 \le t \le 2\pi$.

In Exercises 13 – 16, find the work performed by the force field $\vec{F}$ moving a particle along the path C.

13. $\vec{F} = \langle y, x^2 \rangle$ N; C is the segment of the line $y = x$ from $(0, 0)$ to $(1, 1)$, where distances are measured in meters.

14. $\vec{F} = \langle y, x^2 \rangle$ N; C is the portion of $y = \sqrt{x}$ from $(0, 0)$ to $(1, 1)$, where distances are measured in meters.

15. $\vec{F} = \langle 2xy, x^2, 1 \rangle$ lbs; C is the path from $(0, 0, 0)$ to $(2, 4, 8)$ via $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ on $0 \le t \le 2$, where distance are measured in feet.

16. $\vec{F} = \langle 2xy, x^2, 1 \rangle$ lbs; C is the path from $(0, 0, 0)$ to $(2, 4, 8)$ via $\vec{r}(t) = \langle t, 2t, 4t \rangle$ on $0 \le t \le 2$, where distance are measured in feet.

In Exercises 17 – 20, a conservative vector field $\vec{F}$ and a curve C are given.

1. **Find a potential function f for $\vec{F}$.**
2. **Compute** $\text{curl}\,\vec{F}$**.**
3. **Evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ directly, i.e., using Key Idea 51.**
4. **Evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ using the Fundamental Theorem of Line Integrals.**

17. $\vec{F} = \langle y + 1, x \rangle$, C is the line segment from $(0, 1)$ to $(1, 0)$.

18. $\vec{F} = \langle 2x + y, 2y + x \rangle$, C is curve parametrized by $\vec{r}(t) = \langle t^2 - t, t^3 - t \rangle$ on $0 \le t \le 1$.

19. $\vec{F} = \langle 2xyz, x^2z, x^2y \rangle$, C is curve parametrized by $\vec{r}(t) = \langle 2t + 1, 3t - 1, t \rangle$ on $0 \le t \le 2$.

20. $\vec{F} = \langle 2x, 2y, 2z \rangle$, C is curve parametrized by $\vec{r}(t) = \langle \cos t, \sin t, \sin(2t) \rangle$ on $0 \le t \le 2\pi$.

21. Prove part of Theorem 97: let $\vec{F} = \langle M, N, P \rangle$ be a conservative vector field. Show that $\text{curl}\,\vec{F} = 0$.

14.4 Flow, Flux, Green's Theorem and the Divergence Theorem

Flow and Flux

Line integrals over vector fields have the natural interpretation of computing work when $\vec{F}$ represents a force field. It is also common to use vector fields to represent velocities. In these cases, the line integral $\int_C \vec{F} \cdot d\vec{r}$ is said to represent **flow**.

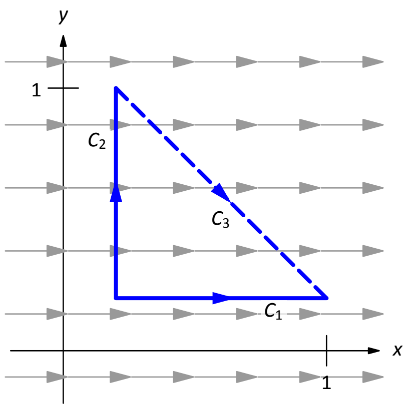


Figure 14.20: Illustrating the principles of flow and flux.

Let the vector field $\vec{F} = \langle 1, 0 \rangle$ represent the velocity of water as it moves across a smooth surface, depicted in Figure 14.20. A line integral over C will compute "how much water is moving *along* the path C."

In the figure, "all" of the water above C_1 is moving along that curve, whereas "none" of the water above C_2 is moving along that curve (the curve and the flow of water are at right angles to each other). Because C_3 has nonzero horizontal and vertical components, "some" of the water above that curve is moving along the curve.

When C is a closed curve, we call flow **circulation**, represented by $\oint_C \vec{F} \cdot d\vec{r}$.

The "opposite" of flow is **flux**, a measure of "how much water is moving *across* the path C." If a curve represents a filter in flowing water, flux measures how much water will pass through the filter. Considering again Figure 14.20, we see that a screen along C_1 will not filter any water as no water passes across that curve. Because of the nature of this field, C_2 and C_3 each filter the same amount of water per second.

The terms "flow" and "flux" are used apart from velocity fields, too. Flow is measured by $\int_C \vec{F} \cdot d\vec{r}$, which is the same as $\int_C \vec{F} \cdot \vec{T}\, ds$ by Definition 96. That is, flow is a summation of the amount of $\vec{F}$ that is *tangent* to the curve C.

By contrast, flux is a summation of the amount of $\vec{F}$ that is *orthogonal* to the direction of travel. To capture this orthogonal amount of $\vec{F}$, we use $\int_C \vec{F} \cdot \vec{n}\, ds$ to measure flux, where $\vec{n}$ is a unit vector orthogonal to the curve C. (Later, we'll measure flux across surfaces, too. For example, in physics it is useful to measure the amount of a magnetic field that passes through a surface.)

How is $\vec{n}$ determined? We'll later see that if C is a closed curve, we'll want $\vec{n}$ to point to the outside of the curve (measuring how much is "going out"). We'll also adopt the convention that closed curves should be traversed counterclockwise.

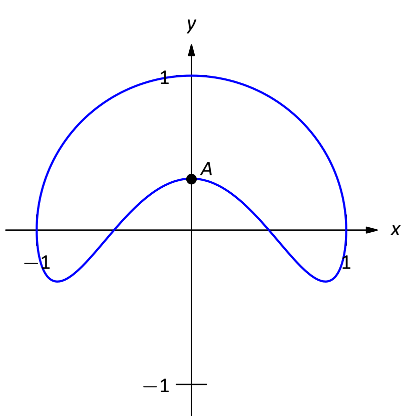


Figure 14.21: Determining "counterclockwise" is not always simple without a good definition.

(If C is a complicated closed curve, it can be difficult to determine what "counterclockwise" means. Consider Figure 14.21. Seeing the curve as a whole, we know which way "counterclockwise" is. If we zoom in on point A, one might incorrectly choose to traverse the path in the wrong direction. So we offer this definition: *a closed curve is being traversed counterclockwise if the outside is to the right of the path and the inside is to the left.*)

Notes:

When a curve C is traversed counterclockwise by $\vec{r}(t) = \langle f(t), g(t)\rangle$, we rotate $\vec{T}$ clockwise 90° to obtain $\vec{n}$:

$$\vec{T} = \frac{\langle f'(t), g'(t)\rangle}{||\,\vec{r}\,'(t)\,||} \quad \Rightarrow \quad \vec{n} = \frac{\langle g'(t), -f'(t)\rangle}{||\,\vec{r}\,'(t)\,||}.$$

Letting $\vec{F} = \langle M, N\rangle$, we calculate flux as:

$$\begin{aligned}\int_C \vec{F}\cdot\vec{n}\,ds &= \int_C \vec{F}\cdot\frac{\langle g'(t), -f'(t)\rangle}{||\,\vec{r}\,'(t)\,||}||\,\vec{r}\,'(t)\,||\,dt\\ &= \int_C \langle M, N\rangle\cdot\langle g'(t), -f'(t)\rangle\;dt\\ &= \int_C \Big(M\,g'(t) - N f'(t)\Big)dt\\ &= \int_C M\,g'(t)\,dt - \int_C N f'(t)\,dt.\end{aligned}$$

As the x and y components of $\vec{r}(t)$ are $f(t)$ and $g(t)$ respectively, the differentials of x and y are $dx = f'(t)dt$ and $dy = g'(t)dt$. We can then write the above integrals as:

$$= \int_C M\,dy - \int_C N\,dx.$$

This is often written as one integral (not incorrectly, though somewhat confusingly, as this one integral has two "d's"):

$$= \int_C M\,dy - N\,dx.$$

We summarize the above in the following definition.

Notes:

Definition 99 Flow, Flux

Let $\vec{F} = \langle M, N \rangle$ be a vector field with continuous components defined on a smooth curve C, parametrized by $\vec{r}(t) = \langle f(t), g(t) \rangle$, let $\vec{T}$ be the unit tangent vector of $\vec{r}(t)$, and let $\vec{n}$ be the clockwise 90°degree rotation of $\vec{T}$.

- The **flow** of $\vec{F}$ over C is

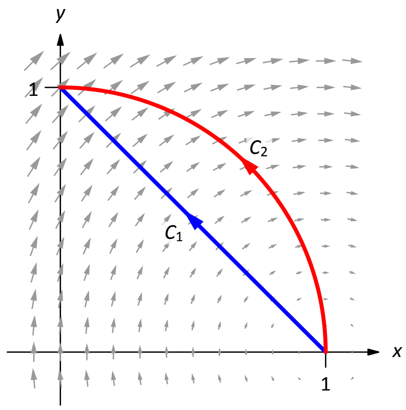
$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot d\vec{r}.$$

- The **flux** of $\vec{F}$ over C is

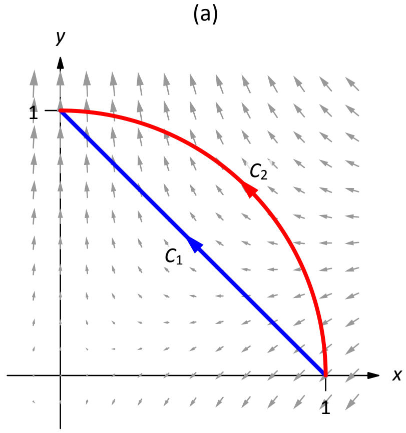
$$\int_C \vec{F} \cdot \vec{n}\, ds = \int_C M\, dy - N\, dx = \int_C \Big(M\, g'(t) - N\, f'(t) \Big)\, dt.$$

This definition of flow also holds for curves in space, though it does not make sense to measure "flux across a curve" in space.

Measuring flow is essentially the same as finding work performed by a force as done in the previous examples. Therefore we practice finding only flux in the following example.



(a)



(b)

Figure 14.22: Illustrating the curves and vector fields in Example 388. In (a) the vector field is $\vec{F}_1$, and in (b) the vector field is $\vec{F}_2$.

Example 388 Finding flux across curves in the plane

Curves C_1 and C_2 each start at $(1, 0)$ and end at $(0, 1)$, where C_1 follows the line $y = 1 - x$ and C_2 follows the unit circle, as shown in Figure 14.22. Find the flux across both curves for the vector fields $\vec{F}_1 = \langle y, -x + 1 \rangle$ and $\vec{F}_2 = \langle -x, 2y - x \rangle$.

Solution We begin by finding parametrizations of C_1 and C_2. As done in Example 385, parametrize C_1 by creating the line that starts at $(1, 0)$ and moves in the $\langle -1, 1 \rangle$ direction: $\vec{r}_1(t) = \langle 1, 0 \rangle + t\langle -1, 1 \rangle = \langle 1 - t, t \rangle$, for $0 \le t \le 1$. We parametrize C_2 with the familiar $\vec{r}_2(t) = \langle \cos t, \sin t \rangle$ on $0 \le t \le \pi/2$. For reference later, we give each function and its derivative below:

$$\vec{r}_1(t) = \langle 1 - t, t \rangle\,, \quad \vec{r}\,'_1(t) = \langle -1, 1 \rangle\,.$$

$$\vec{r}_2(t) = \langle \cos t, \sin t \rangle\,, \quad \vec{r}\,'_2(t) = \langle -\sin t, \cos t \rangle\,.$$

When $\vec{F} = \vec{F}_1 = \langle y, -x + 1 \rangle$ (as shown in Figure 14.22(a)), over C_1 we have $M = y = t$ and $N = -x + 1 = -(1 - t) + 1 = t$. Using Definition 99, we

Notes:

compute the flux:

$$\begin{aligned}\int_{C_1} \vec{F}\cdot\vec{n}\,ds &= \int_{C_1} \Big(M\,g'(t) - N f'(t)\Big)\,dt\\ &= \int_0^1 \Big(t(1) - t(-1)\Big)\,dt\\ &= \int_0^1 2t\,dt\\ &= 1.\end{aligned}$$

Over C_2, we have $M = y = \sin t$ and $N = -x + 1 = 1 - \cos t$. Thus the flux across C_2 is:

$$\begin{aligned}\int_{C_1} \vec{F}\cdot\vec{n}\,ds &= \int_{C_1} \Big(M\,g'(t) - N f'(t)\Big)\,dt\\ &= \int_0^{\pi/2} \Big((\sin t)(\cos t) - (1-\cos t)(-\sin t)\Big)\,dt\\ &= \int_0^{\pi/2} \sin t\,dt\\ &= 1.\end{aligned}$$

Notice how the flux was the same across both curves. This won't hold true when we change the vector field.

When $\vec{F} = \vec{F}_2 = \langle -x, 2y - x\rangle$ (as shown in Figure 14.22(b)), over C_1 we have $M = -x = t - 1$ and $N = 2y - x = 2t - (1-t) = 3t - 1$. Computing the flux across C_1:

$$\begin{aligned}\int_{C_1} \vec{F}\cdot\vec{n}\,ds &= \int_{C_1} \Big(M\,g'(t) - N f'(t)\Big)\,dt\\ &= \int_0^1 \Big((t-1)(1) - (3t-1)(-1)\Big)\,dt\\ &= \int_0^1 (4t-2)\,dt\\ &= 0.\end{aligned}$$

Over C_2, we have $M = -x = -\cos t$ and $N = 2y - x = 2\sin t - \cos t$. Thus the

Notes:

flux across C_2 is:

$$\begin{aligned}\int_{C_1} \vec{F}\cdot\vec{n}\,ds &= \int_{C_1}\Big(M\,g'(t) - N\,f'(t)\Big)\,dt\\ &= \int_0^{\pi/2}\Big((-\cos t)(\cos t) - (2\sin t - \cos t)(-\sin t)\Big)\,dt\\ &= \int_0^{\pi/2}\big(2\sin^2 t - \sin t\cos t - \cos^2 t\big)\,dt\\ &= \pi/4 - 1/2 \approx 0.285.\end{aligned}$$

We analyze the results of this example below.

In Example 388, we saw that the flux across the two curves was the same when the vector field was $\vec{F}_1 = \langle y, -x+1\rangle$. This is not a coincidence. We show why they are equal in Example 393. In short, the reason is this: the divergence of $\vec{F}_1$ is 0, and when $\operatorname{div}\vec{F} = 0$, the flux across any two paths with common beginning and ending points will be the same.

We also saw in the example that the flux across C_1 was 0 when the field was $\vec{F}_2 = \langle -x, 2y-x\rangle$. Flux measures "how much" of the field crosses the path from left to right (following the conventions established before). Positive flux means most of the field is crossing from left to right; negative flux means most of the field is crossing from right to left; zero flux means the same amount crosses from left to right as from right to left. When we consider Figure 14.22(b), it seems plausible that the same amount of $\vec{F}_2$ was crossing C_1 from left to right as from right to left.

Green's Theorem

There is an important connection between the circulation around a closed region R and the curl of the vector field inside of R, as well as a connection between the flux across the boundary of R and the divergence of the field inside R. These connections are described by Green's Theorem and the Divergence Theorem, respectively. We'll explore each in turn.

Green's Theorem states "the counterclockwise circulation around a closed region R is equal to the sum of the curls over R."

Notes:

Theorem 98 Green's Theorem

Let R be a closed, bounded region of the plane whose boundary C is composed of finitely many smooth curves, let $\vec{r}(t)$ be a counterclockwise parametrization of C, and let $\vec{F} = \langle M, N \rangle$ where N_x and M_y are continuous over R. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\,dA.$$

We'll explore Green's Theorem through an example.

Example 389 Confirming Green's Theorem

Let $\vec{F} = \left\langle -y, x^2+1 \right\rangle$ and let R be the region of the plane bounded by the triangle with vertices $(-1,0)$, $(1,0)$ and $(0,2)$, shown in Figure 14.23. Verify Green's Theorem; that is, find the circulation of $\vec{F}$ around the boundary of R and show that is equal to the double integral of curl $\vec{F}$ over R.

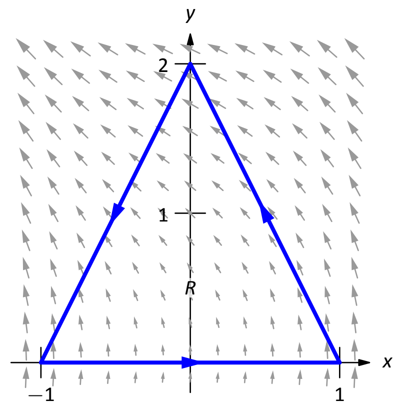


Figure 14.23: The vector field and planar region used in Example 389.

Solution The curve C that bounds R is composed of 3 lines. While we need to traverse the boundary of R in a counterclockwise fashion, we may start anywhere we choose. We arbitrarily choose to start at $(-1,0)$, move to $(1,0)$, etc., with each line parametrized by $\vec{r}_1(t)$, $\vec{r}_2(t)$ and $\vec{r}_3(t)$, respectively.

We leave it to the reader to confirm that the following parametrizations of the three lines are accurate:

$\vec{r}_1(t) = \langle 2t-1, 0\rangle$, for $0 \leq t \leq 1$, with $\vec{r}_1\,'(t) = \langle 2, 0\rangle$,
$\vec{r}_2(t) = \langle 1-t, 2t\rangle$, for $0 \leq t \leq 1$, with $\vec{r}_2\,'(t) = \langle -1, 2\rangle$, and
$\vec{r}_3(t) = \langle -t, 2-2t\rangle$, for $0 \leq t \leq 1$, with $\vec{r}_3\,'(t) = \langle -1, -2\rangle$.

The circulation around C is found by summing the flow along each of the sides of the triangle. We again leave it to the reader to confirm the following computations:

$$\int_{C_1} \vec{F} \cdot d\vec{r}_1 = \int_0^1 \left\langle 0, (2t-1)^2+1 \right\rangle \cdot \langle 2, 0\rangle\,dt = 0,$$
$$\int_{C_2} \vec{F} \cdot d\vec{r}_2 = \int_0^1 \left\langle -2t, (1-t)^2+1 \right\rangle \cdot \langle -1, 2\rangle\,dt = 11/3, \text{ and}$$
$$\int_{C_3} \vec{F} \cdot d\vec{r}_3 = \int_0^1 \left\langle 2t-2, t^2+1 \right\rangle \cdot \langle -1, -2\rangle\,dt = -5/3.$$

The circulation is the sum of the flows: 2.

We confirm Green's Theorem by computing $\iint_R \text{curl}\,\vec{F}\,dA$. We find curl $\vec{F} = 2x+1$. The region R is bounded by the lines $y = 2x+2$, $y = -2x+2$ and $y = 0$.

Notes:

Integrating with the order $dx\,dy$ is most straightforward, leading to

$$\int_0^2 \int_{y/2-1}^{1-y/2} (2x+1)\,dx\,dy = \int_0^2 (2-y)\,dy = 2,$$

which matches our previous measurement of circulation.

Example 390 Using Green's Theorem

Let $\vec{F} = \langle \sin x, \cos y\rangle$ and let R be the region enclosed by the curve C parametrized by $\vec{r}(t) = \left\langle 2\cos t + \frac{1}{10}\cos(10t), 2\sin t + \frac{1}{10}\sin(10t)\right\rangle$ on $0 \le t \le 2\pi$, as shown in Figure 14.24. Find the circulation around C.

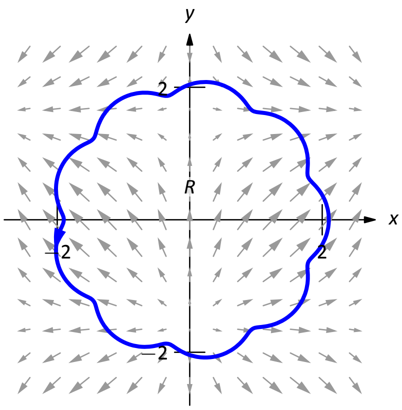


Figure 14.24: The vector field and planar region used in Example 390.

Solution Computing the circulation directly using the line integral looks difficult, as the integrand will include terms like "$\sin\left(2\cos t + \frac{1}{10}\cos(10t)\right)$."

Green's Theorem states that $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\,dA$; since $\text{curl}\,\vec{F} = 0$ in this example, the double integral is simply 0 and hence the circulation is 0.

Since $\text{curl}\,\vec{F} = 0$, we can conclude that the circulation is 0 in two ways. One method is to employ Green's Theorem as done above. The second way is to recognize that $\vec{F}$ is a conservative field, hence there is a function $z = f(x,y)$ wherein $\vec{F} = \nabla f$. Let A be any point on the curve C; since C is closed, we can say that C "begins" and "end" at A. By the Fundamental Theorem of Line Integrals, $\oint_C \vec{F}\,d\vec{r} = f(A) - f(A) = 0$.

One can use Green's Theorem to find the area of an enclosed region by integrating along its boundary. Let C be a closed curve, enclosing the region R, parametrized by $\vec{r}(t) = \langle f(t), g(t)\rangle$. We know the area of R is computed by the double integral $\iint_R dA$, where the integrand is 1. By creating a field $\vec{F}$ where $\text{curl}\,\vec{F} = 1$, we can employ Green's Theorem to compute the area of R as $\oint_C \vec{F}\cdot d\vec{r}$.

One is free to choose any field $\vec{F}$ to use as long as $\text{curl}\,\vec{F} = 1$. Common choices are $\vec{F} = \langle 0, x\rangle$, $\vec{F} = \langle -y, 0\rangle$ and $\vec{F} = \langle -y/2, x/2\rangle$. We demonstrate this below.

Example 391 Using Green's Theorem to find area

Let C be the closed curve parametrized by $\vec{r}(t) = \left\langle t - t^3, t^2\right\rangle$ on $-1 \le t \le 1$, enclosing the region R, as shown in Figure 14.25. Find the area of R.

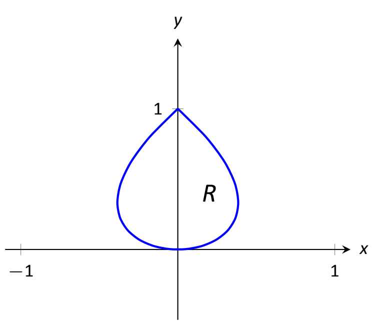


Figure 14.25: The region R, whose area is found in Example 391.

Solution We can choose any field $\vec{F}$, as long as $\text{curl}\,\vec{F} = 1$. We choose $\vec{F} = \langle -y, 0\rangle$. We also confirm (left to the reader) that $\vec{r}(t)$ traverses the region

Notes:

R in a counterclockwise fashion. Thus

$$\begin{aligned}\text{Area of } R &= \iint_R dA \\ &= \oint_C \vec{F}\cdot d\vec{r} \\ &= \int_{-1}^{1} \left\langle -t^2, 0\right\rangle \cdot \left\langle 1-3t^2, 2t\right\rangle \, dt \\ &= \int_{-1}^{1} (-t^2)(1-3t^2)\, dt \\ &= \frac{8}{15}.\end{aligned}$$

The Divergence Theorem

Green's Theorem makes a connection between the circulation around a closed region R and the sum of the curls over R. The Divergence Theorem makes a somewhat "opposite" connection: the total flux across the boundary of R is equal to the sum of the divergences over R.

Theorem 99 The Divergence Theorem (in the plane)

Let R be a closed, bounded region of the plane whose boundary C is composed of finitely many smooth curves, let $\vec{r}(t)$ be a counterclockwise parametrization of C, and let $\vec{F} = \langle M, N\rangle$ where M_x and N_y are continuous over R. Then

$$\oint_C \vec{F}\cdot\vec{n}\, ds = \iint_R \text{div}\,\vec{F}\, dA.$$

Example 392 Confirming the Divergence Theorem

Let $\vec{F} = \langle x-y, x+y\rangle$, let C be the circle of radius 2 centered at the origin and define R to be the interior of that circle, as shown in Figure 14.26. Verify the Divergence Theorem; that is, find the flux across C and show it is equal to the double integral of $\text{div}\,\vec{F}$ over R.

Solution We parametrize the circle in the usual way, with $\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$,

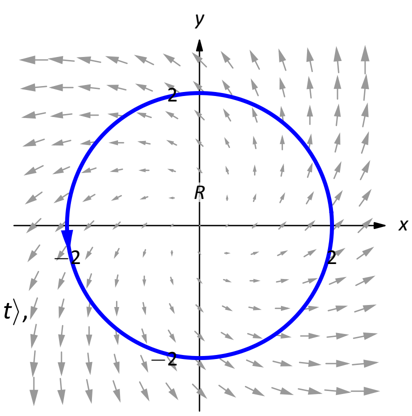


Figure 14.26: The region R used in Example 392.

Notes:

$0 \le t \le 2\pi$. The flux across C is

$$\begin{aligned}\oint_C \vec F\cdot\vec n\,ds &= \oint_C \big(Mg'(t) - Nf'(t)\big)\,dt\\ &= \int_0^{2\pi} \big((2\cos t - 2\sin t)(2\cos t) - (2\cos t + 2\sin t)(-2\sin t)\big)\,dt\\ &= \int_0^{2\pi} 4\,dt = 8\pi.\end{aligned}$$

We compute the divergence of $\vec F$ as $\text{div}\,\vec F = M_x + N_y = 2$. Since the divergence is constant, we can compute the following double integral easily:

$$\iint_R \text{div}\,\vec F\,dA = \iint_R 2\,dA = 2\iint_R dA = 2(\text{area of } R) = 8\pi,$$

which matches our previous result.

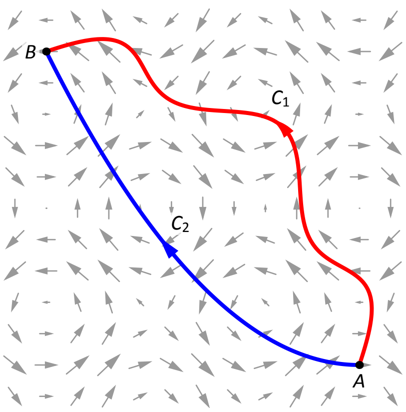


Figure 14.27: As used in Example 393, the vector field has a divergence of 0 and the two paths only intersect at their initial and terminal points.

Example 393 **Flux when** $\text{div}\,\vec F = 0$

Let $\vec F$ be any field where $\text{div}\,\vec F = 0$, and let C_1 and C_2 be any two nonintersecting paths, except that each begin at point A and end at point B (see Figure 14.27). Show why the flux across C_1 and C_2 is the same.

Solution By referencing Figure 14.27, we see we can make a closed path C that combines C_1 with C_2, where C_2 is traversed with its opposite orientation. We label the enclosed region R. Since $\text{div}\,\vec F = 0$, the Divergence Theorem states that

$$\oint_C \vec F\cdot\vec n\,ds = \iint_R \text{div}\,\vec F\,dA = \iint_R 0\,dA = 0.$$

Using the properties and notation given in Theorem 95, consider:

$$\begin{aligned}0 &= \oint_C \vec F\cdot\vec n\,ds\\ &= \int_{C_1} \vec F\cdot\vec n\,ds + \int_{C_2^*} \vec F\cdot\vec n\,ds\end{aligned}$$

(where C_2^* is the path C_2 traversed with opposite orientation)

$$\begin{aligned}&= \int_{C_1} \vec F\cdot\vec n\,ds - \int_{C_2} \vec F\cdot\vec n\,ds.\\ \int_{C_2} \vec F\cdot\vec n\,ds &= \int_{C_1} \vec F\cdot\vec n\,ds.\end{aligned}$$

Thus the flux across each path is equal.

Notes:

Exercises 14.4

Terms and Concepts

1. Let $\vec{F}$ be a vector field and let C be a curve. *Flow* is a measure of the amount of $\vec{F}$ going ________________ C; *flux* is a measure of the amount of $\vec{F}$ going ________________ C.

2. What is circulation?

3. Green's Theorem states, informally, that the circulation around a closed curve that bounds a region R is equal to the sum of ________________ across R.

4. The Divergence Theorem states, informally, that the outward flux across a closed curve that bounds a region R is equal to the sum of ________________ across R.

5. Let $\vec{F}$ be a vector field and let C_1 and C_2 be any nonintersecting paths except that each starts at point A and ends at point B. If ________ $= 0$, then $\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2}\vec{F}\cdot\vec{T}\,ds$.

6. Let $\vec{F}$ be a vector field and let C_1 and C_2 be any nonintersecting paths except that each starts at point A and ends at point B. If ________ $= 0$, then $\int_{C_1} \vec{F}\cdot\vec{n}\,ds = \int_{C_2}\vec{F}\cdot\vec{n}\,ds$.

Problems

In Exercises 7 – 12, a vector field $\vec{F}$ and a curve C are given. Evaluate $\int_C \vec{F}\cdot\vec{n}\,ds$, the flux of $\vec{F}$ over C.

7. $\vec{F} = \langle x+y, x-y\rangle$; C is the curve with initial and terminal points $(3,-2)$ and $(3,2)$, respectively, parametrized by $\vec{r}(t) = \langle 3t^2, 2t\rangle$ on $-1 \le t \le 1$.

8. $\vec{F} = \langle x+y, x-y\rangle$; C is the curve with initial and terminal points $(3,-2)$ and $(3,2)$, respectively, parametrized by $\vec{r}(t) = \langle 3, t\rangle$ on $-2 \le t \le 2$.

9. $\vec{F} = \langle x^2, y+1\rangle$; C is line segment from $(0,0)$ to $(2,4)$.

10. $\vec{F} = \langle x^2, y+1\rangle$; C is the portion of the parabola $y = x^2$ from $(0,0)$ to $(2,4)$.

11. $\vec{F} = \langle y, 0\rangle$; C is the line segment from $(0,0)$ to $(0,1)$.

12. $\vec{F} = \langle y, 0\rangle$; C is the line segment from $(0,0)$ to $(1,1)$.

In Exercises 13 – 16, a vector field $\vec{F}$ and a closed curve C, enclosing a region R, are given. Verify Green's Theorem by evaluating $\oint_C \vec{F}\cdot d\vec{r}$ and $\iint_R \operatorname{curl}\vec{F}\,dA$, showing they are equal.

13. $\vec{F} = \langle x-y, x+y\rangle$; C is the closed curve composed of the parabola $y = x^2$ on $0 \le x \le 2$ followed by the line segment from $(2,4)$ to $(0,0)$.

14. $\vec{F} = \langle -y, x\rangle$; C is the unit circle.

15. $\vec{F} = \langle 0, x^2\rangle$; C the triangle with corners at $(0,0)$, $(2,0)$ and $(1,1)$.

16. $\vec{F} = \langle x+y, 2x\rangle$; C the curve that starts at $(0,1)$, follows the parabola $y = (x-1)^2$ to $(3,4)$, then follows a line back to $(0,1)$.

In Exercises 17 – 20, a closed curve C enclosing a region R is given. Find the area of R by computing $\oint_C \vec{F}\cdot d\vec{r}$ for an appropriate choice of vector field $\vec{F}$.

17. C is the ellipse parametrized by $\vec{r}(t) = \langle 4\cos t, 3\sin t\rangle$ on $0 \le t \le 2\pi$.

18. C is the curve parametrized by $\vec{r}(t) = \langle \cos t, \sin(2t)\rangle$ on $-\pi/2 \le t \le \pi/2$.

19. C is the curve parametrized by $\vec{r}(t) = \langle \cos t, \sin(2t)\rangle$ on $0 \le t \le 2$.

20. C is the curve parametrized by $\vec{r}(t) = \langle 2\cos t + \frac{1}{10}\cos(10t), 2\sin t + \frac{1}{10}\sin(10t)\rangle$ on $0 \le t \le 2\pi$.

In Exercises 21 – 24, a vector field $\vec{F}$ and a closed curve C, enclosing a region R, are given. Verify the Divergence Theorem by evaluating $\oint_C \vec{F}\cdot\vec{n}\,ds$ and $\iint_R \operatorname{div}\vec{F}\,dA$, showing they are equal.

21. $\vec{F} = \langle x-y, x+y\rangle$; C is the closed curve composed of the parabola $y = x^2$ on $0 \le x \le 2$ followed by the line segment from $(2,4)$ to $(0,0)$.

22. $\vec{F} = \langle -y, x\rangle$; C is the unit circle.

23. $\vec{F} = \langle 0, y^2\rangle$; C the triangle with corners at $(0,0)$, $(2,0)$ and $(1,1)$.

24. $\vec{F} = \langle x^2/2, y^2/2\rangle$; C the curve that starts at $(0,1)$, follows the parabola $y = (x-1)^2$ to $(3,4)$, then follows a line back to $(0,1)$.

14.5 Parametrized Surfaces and Surface Area

Thus far we have focused mostly on 2-dimensional vector fields, measuring flow and flux along/across curves in the plane. Both Green's Theorem and the Divergence Theorem make connections between planar regions and their boundaries. We now move our attention to 3-dimensional vector fields, considering both curves and surfaces in space.

Note: We use the letter S to denote Surface Area. This section begins a study into surfaces, and it is natural to label a surface with the letter "S". We distinguish a surface from its surface area by using a calligraphic S to denote a surface: $\mathcal{S}$. When writing this letter by hand, it may be useful to add serifs to the letter, such as: $\mathscr{S}$

We are accustomed to describing surfaces as functions of two variables, usually written as $z = f(x, y)$. For our coming needs, this method of describing surfaces will prove to be insufficient. Instead, we will *parametrize* our surfaces, describing them as the set of terminal points of some vector–valued function $\vec{r}(u, v) = \langle f(u, v), g(u, v), h(u, v)\rangle$. The bulk of this section is spent practicing the skill of describing a surface $\mathcal{S}$ using a vector–valued function. Once this skill is developed, we'll show how to find the surface area S of a parametrically–defined surface $\mathcal{S}$, a skill needed in the remaining sections of this chapter.

Note: A function is *one to one* on its domain if the function never repeats an output value over the domain. In the case of $\vec{r}(u, v)$, $\vec{r}$ is one to one if $\vec{r}(u_1, v_1) \neq \vec{r}(u_2, v_2)$ for all points $(u_1, v_1) \neq (u_2, v_2)$ in the domain of $\vec{r}$.

Definition 100 Parametrized Surface

Let $\vec{r}(u, v) = \langle f(u, v), g(u, v), h(u, v)\rangle$ be a vector–valued function that is continuous and one to one on the interior of its domain R in the u-v plane. The set of all terminal points of $\vec{r}$ (i.e., the *range* of $\vec{r}$) is the **surface** $\mathcal{S}$, and $\vec{r}$ along with its domain R form a **parametrization** of $\mathcal{S}$.

This parametrization is **smooth** on R if $\vec{r}_u$ and $\vec{r}_v$ are continuous and $\vec{r}_u \times \vec{r}_v$ is never $\vec{0}$ on the interior of R.

Given a point (u_0, v_0) in the domain of a vector–valued function $\vec{r}$, the vectors $\vec{r}_u(u_0, v_0)$ and $\vec{r}_v(u_0, v_0)$ are tangent to the surface $\mathcal{S}$ at $\vec{r}(u_0, v_0)$ (a proof of this is developed in the Exercise section). The definition of smoothness dictates that $\vec{r}_u \times \vec{r}_v \neq \vec{0}$; this ensures that neither $\vec{r}_u$ nor $\vec{r}_v$ are $\vec{0}$, nor are they ever parallel. Therefore smoothness guarantees that $\vec{r}_u$ and $\vec{r}_v$ determine a plane that is tangent to $\mathcal{S}$.

A surface $\mathcal{S}$ is said to be **orientable** if a field of normal vectors can be defined on $\mathcal{S}$ that vary continuously along $\mathcal{S}$. This definition may be hard to understand; it may help to know that orientable surfaces are often called "two sided." A sphere is an orientable surface, and one can easily envision an "inside" and "outside" of the sphere. A paraboloid is orientable, where again one can generally envision "inside" and "outside" sides (or "top" and "bottom" sides) to this surface. Just about every surface that one can imagine is orientable, and we'll assume all surfaces we deal with in this text are orientable.

It is enlightening to examine a classic non-orientable surface: the Möbius

Notes:

band, shown in Figure 14.28. Vectors normal to the surface are given, starting at the point indicated in the figure. These normal vectors "vary continuously" as they move along the surface. Letting each vector indicate the "top" side of the band, we can easily see near any vector which side is the "top".

However, if as we progress along the band, we recognize that we are labeling "both sides" of the band as the top; in fact, there are not two "sides" to this band, but one. The Möbius band is a non-orientable surface.

We now practice parameterizing surfaces.

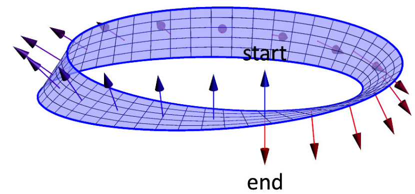


Figure 14.28: A Möbius band, a non-orientable surface.

Example 394 Parameterizing a surface over a rectangle

Parametrize the surface $z = x^2 + 2y^2$ over the rectangular region R defined by $-3 \le x \le 3$, $-1 \le y \le 1$.

Solution parameterizing surfaces, given in the form $z = f(x, y)$ over rectangular regions is straightforward. We can simply let $x = u$ and $y = v$, and let $\vec{r}(u, v) = \langle u, v, f(u, v)\rangle$. In this instance, we have $\vec{r}(u, v) = \langle u, v, u^2 + 2v^2\rangle$, for $-3 \le u \le 3$, $-1 \le v \le 1$. This surface is graphed in Figure 14.29.

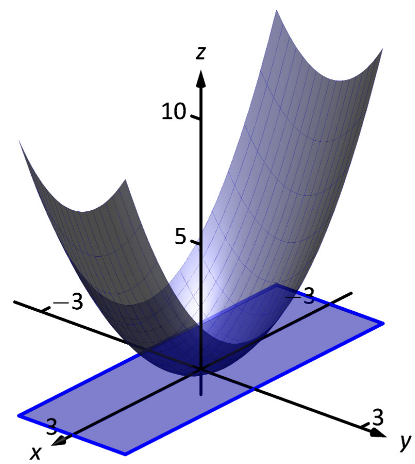


Figure 14.29: The surface parametrized in Example 394.

Example 395 Parameterizing a surface over a circle

Parametrize the surface $z = x^2 + 2y^2$ over the circular region R enclosed by the circle of radius 2 that is centered at the origin.

Solution We can parametrize the circular boundary of R with the vector–valued function $\langle 2\cos u, 2\sin u\rangle$, where $0 \le u \le 2\pi$. We can obtain the interior of R by scaling this function by a variable amount, i.e., by multiplying by v: $\langle 2v\cos u, 2v\sin u\rangle$, where $0 \le v \le 1$.

It is important to understand the role of v in the above function. When $v = 1$, we get the boundary of R, a circle of radius 2. When $v = 0$, we simply get the point $(0, 0)$, the center of R (which can be thought of as a circle with radius of 0). When $v = 1/2$, we get the circle of radius 1 that is centered at the origin, which is the circle *halfway* between the boundary and the center.

Thus far, we have determined the x and y components of our parametrization of the surface: $x = 2v\cos u$ and $y = 2v\sin u$. We find the z component simply by using $z = f(x, y) = x^2 + 2y^2$:

$$z = (2v\cos u)^2 + 2(2v\sin u)^2 = 4v^2\cos^2 u + 8v^2\sin^2 u.$$

Thus $\vec{r}(u, v) = \langle 2v\cos u, 2v\sin u, 4v^2\cos^2 u + 8v^2\sin^2 u\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 1$, which is graphed in Figure 14.30. The way that this graphic was generated highlights how the surface was parametrized. When viewing from above, one can see lines emanating from the origin; the represent different values of u as u sweeps from an angle of 0 up to 2π. One can also see concentric circles, each

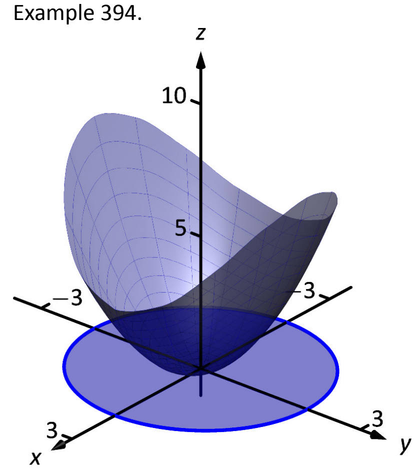


Figure 14.30: The surface parametrized in Example 395.

Notes:

corresponding to a different value of v.

Examples 394 and 395 demonstrate an important principle when parameterizing surfaces given in the form $z = f(x, y)$ over a region R: if one can determine x and y in terms of u and v, then z follows directly as $z = f(x, y)$.

In the following two examples, we parametrize the same surface over triangular regions. Each will use v as a "scaling factor" as done in Example 395.

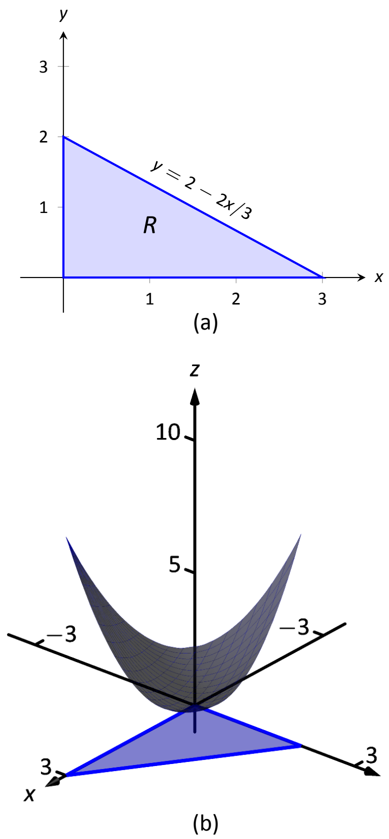


Figure 14.31: Part (a) shows a graph of the region R, and part (b) shows the surface over R, as defined in Example 396.

Example 396 Parameterizing a surface over a triangle

Parametrize the surface $z = x^2 + 2y^2$ over the triangular region R enclosed by the coordinate axes and the line $y = 2 - 2x/3$, as shown in Figure 14.31(a).

Solution We may begin by letting $x = u$, $0 \leq u \leq 3$, and $y = 2 - 2u/3$. This gives only the line on the "upper" side of the triangle. To get all of the region R, we can once again scale y by a variable factor, v.

Still letting $x = u$, $0 \leq u \leq 3$, we let $y = v(2 - 2u/3)$, $0 \leq v \leq 1$. When $v = 0$, all y-values are 0, and we get the portion of the x-axis between $x = 0$ and $x = 3$. When $v = 1$, we get the upper side of the triangle. When $v = 1/2$, we get the line $y = 1/2(2 - 2u/3) = 1 - u/3$, which is the line "halfway up" the triangle.

Letting $z = f(x, y) = x^2 + 2y^2$, we have $\vec{r}(u, v) = \langle u, v(2 - 2u/3), u^2 + 2\big(v(2 - 2u/3)\big)^2\rangle$, $0 \leq u \leq 3$, $0 \leq v \leq 1$. This surface is graphed in Figure 14.31(b). Again, when one looks from above, we can see the scaling effects of v: the series of lines that run to the point $(3, 0)$ each represent a different value of v.

Another common way to parametrize the surface is to begin with $y = u$, $0 \leq u \leq 2$. Solving the equation of the line $y = 2 - 2x/3$ for x, we have $x = 3 - 3y/2$, leading to using $x = v(3 - 3u/2)$, $0 \leq v \leq 1$. With $z = x^2 + 2y^2$, we have $\vec{r}(u, v) = \langle v(3 - 3u/2), u, \big(v(3 - 3u/2)\big)^2 + 2v^2\rangle$, $0 \leq u \leq 2$, $0 \leq v \leq 1$.

Example 397 Parameterizing a surface over a triangle

Parametrize the surface $z = x^2 + 2y^2$ over the triangular region R enclosed by the lines $y = 3 - 2x/3$, $y = 1$ and $x = 0$ as shown in Figure 14.32(a).

Solution While the region R in this example is very similar to the region R in the previous example, and our method of parameterizing the surface is fundamentally the same, it will feel as though our answer is much different than before.

We begin with letting $x = u$, $0 \leq u \leq 3$. We may be tempted to let $y = v(3 - 2u/3)$, $0 \leq v \leq 1$, but this is incorrect. When $v = 1$, we obtain the upper line of the triangle as desired. However, when $v = 0$, the y-value is 0, which

Notes:

does not lie in the region R.

We will describe the general method of proceeding following this example. For now, consider $y = 1 + v(2 - 2u/3, 0 \le v \le 1$. Note that when $v = 1$, we have $y = 3 - 2u/3$, the upper line of the boundary of R. Also, when $v = 0$, we have $y = 1$, which is the lower boundary of R. With $z = x^2 + 2y^2$, we determine $\vec{r}(u,v) = \langle u, 1+v(2-2u/3), u^2+2\big(1+v(2-2u/3)\big)^2\rangle, 0 \le u \le 3, 0 \le v \le 1$.

Given a surface of the form $z = f(x,y)$, one can often determine a parametrization of the surface over a region R in a manner similar to determining bounds of integration over a region R. Using the techniques of Section 13.1, suppose a region R can be described by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, i.e., the area of R can be found using the iterated integral

$$\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx.$$

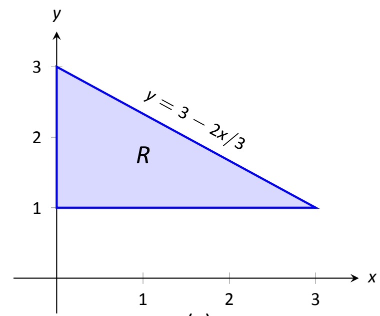
When parameterizing the surface, we can let $x = u$, $a \le u \le b$, and we can let $y = g_1(u) + v\big(g_2(u) - g_1(u)\big)$, $0 \le v \le 1$. The parametrization of x is straightforward, but look closely at how y is determined. When $v = 0$, $y = g_1(u) = g_1(x)$. When $v = 1$, $y = g_2(u) = g_2(x)$.

As a specific example, consider the triangular region R from Example 397, shown in Figure 14.32(a). Using the techniques of Section 13.1, we can find the area of R as

$$\int_0^3 \int_1^{3-2x/3} dy\,dx.$$

Following the above discussion, we can set $x = u$, where $0 \le u \le 3$, and set $y = 1+v\big(3-2u/3-1\big) = 1+v(2-2u/3)$, $0 \le v \le 1$, as used in that example.

One can do a similar thing if R is bounded by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, but for the sake of simplicity we leave it to the reader to flesh out those details. The principles outlined above are given in the following Key Idea for reference.



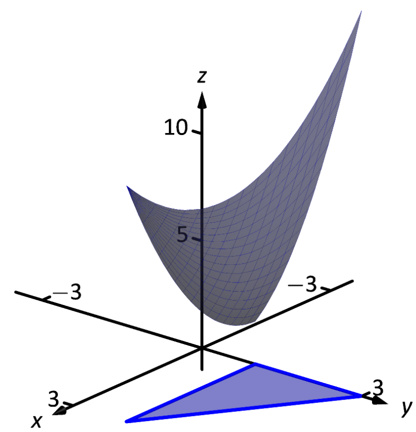


Figure 14.32: Part (a) shows a graph of the region R, and part (b) shows the surface over R, as defined in Example 397.

Key Idea 52 parameterizing Surfaces

Let a surface $\mathcal{S}$ be the portion of the graph of $z = f(x,y)$ that lies above a region R in the x-y plane. Let R be bounded by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, i.e., the area of R can be found using the iterated integral $\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$, and let $h(u,v) = g_1(u) + v\big(g_2(u) - g_1(u)\big)$.
$\mathcal{S}$ can be parametrized as

$$\vec{r}(u,v) = \big\langle u, h(u,v), f\big(u, h(u,v)\big)\big\rangle, \quad a \le u \le b,\ 0 \le v \le 1.$$

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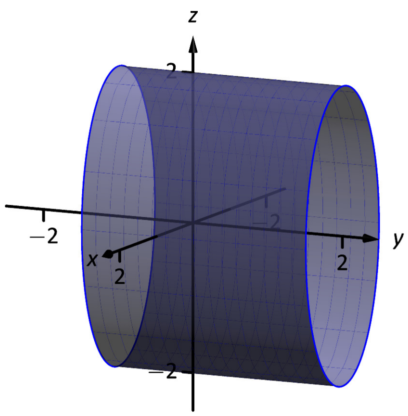


Figure 14.33: The cylinder parametrized in Example 398.

Example 398 Parameterizing a cylinderical surface

Find a parametrization of the cylinder $x^2 + z^2/4 = 1$, where $-1 \le y \le 2$, as shown in Figure 14.33.

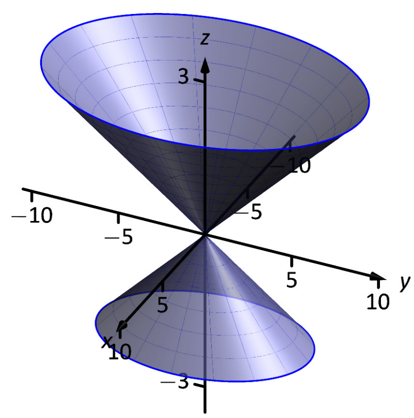
Solution The equation $x^2 + z^2/4 = 1$ describes an ellipse in the x-z plane that extends without bound parallel to the y-axis. This ellipse has a vertical major axis of length 4, a horizontal minor axis of length 2, and is centered at the origin. We can parametrize this ellipse using sines and cosines; our parametrization can begin with

$$\vec{r}(u,v) = \langle \cos u, ???, 2\sin u\rangle\,, \quad 0 \le u \le 2\pi,$$

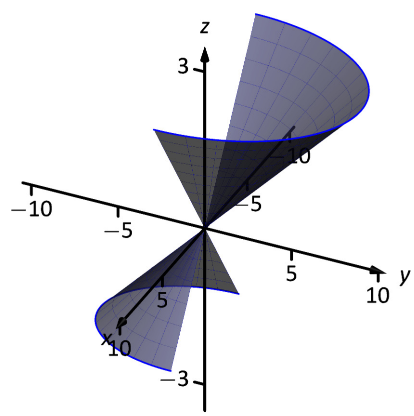
where we still need to determine the y component.

While the cylinder $x^2 + z^2/4 = 1$ is satisfied by any y value, the problem states that all y values are to be between $y = -1$ and $y = 2$. Since the value of y does not depend at all on the values of x or z, we can use another variable, v, to describe y. Our final answer is

$$\vec{r}(u,v) = \langle \cos u, v, 2\sin u\rangle\,, \quad 0 \le u \le 2\pi, \quad -1 \le v \le 2.$$



(a)



(b)

Figure 14.34: An elliptic cone in (a), drawn again in (b) with its domain restricted, as described in Example 399.

Example 399 Parameterizing an elliptic cone

Find a parametrization of the elliptic cone $z^2 = \frac{x^2}{4} + \frac{y^2}{9}$, where $-2 \le z \le 3$, as shown in Figure 14.34(a).

Solution One way to parametrize this cone is to recognize that given a z value, the cross section of the cone at that z value is an ellipse with equation $\frac{x^2}{(2z)^2} + \frac{y^2}{(3z)^2} = 1$. We can let $z = v$, for $-2 \le v \le 3$ and then parametrize the above ellipses using sines, cosines and v.

We can parametrize the x component of our surface with $x = 2z\cos u$ and the y component with $y = 3z\sin u$, where $0 \le u \le 2\pi$. Putting all components together, we have

$$\vec{r}(u,v) = \langle 2v\cos u, 3v\sin u, v\rangle\,, \quad 0 \le u \le 2\pi, \quad -2 \le v \le 3.$$

When v takes on negative values, the radii of the cross–sectional ellipses become "negative," which can lead to some surprising results. Consider Figure 14.34(b), where the cone is graphed for $0 \le u \le \pi$. Because v is negative below the x-y plane, the radii of the cross–sectional ellipses are negative, and the opposite side of the cone is sketched below the x-y plane.

Notes:

Example 400 Parameterizing an ellipsoid
Find a parametrization of the ellipsoid $\frac{x^2}{25}+y^2+\frac{z^2}{4}=1$ as shown in Figure 14.35(a).

Solution Recall Key Idea 40 from Section 10.2, which states that all unit vectors in space have the form $\langle \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\rangle$ for some angles θ and φ. If we choose our angles appropriately, this allows us to draw the unit sphere. To get an ellipsoid, we need only scale each component of the sphere appropriately.

The x-radius of the given ellipsoid is 5, the y-radius is 1 and the z-radius is 2. Substituting u for θ and v for φ, we have $\vec r(u,v)=\langle 5\sin u\cos v, \sin u\sin v, 2\cos u\rangle$, where we still need to determine the ranges of u and v.

Note how the x and y components of $\vec r$ have $\cos v$ and $\sin v$ terms, respectively. This hints at the fact that ellipses are drawn parallel to the x-y plane as v varies, which implies we should have v range from 0 to 2π.

One may be tempted to let $0\le u\le 2\pi$ as well, but note how the z component is $2\cos u$. We only need $\cos u$ to take on values between -1 and 1 once, therefore we can restrict u to $0\le u\le\pi$.

The final parametrization is thus

$$\vec r(u,v)=\langle 5\sin u\cos v, \sin u\sin v, 2\cos u\rangle,\quad 0\le u\le\pi,\quad 0\le v\le 2\pi.$$

In Figure 14.35(b), the ellipsoid is graphed on $\frac{\pi}{4}\le u\le\frac{2\pi}{3}$, $\frac{\pi}{4}\le v\le\frac{3\pi}{2}$ to demonstrate how each variable affects the surface.

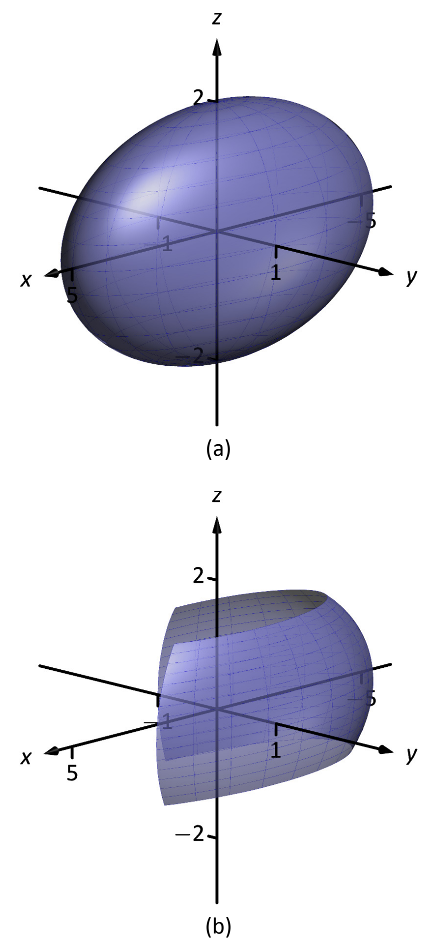


Figure 14.35: An ellipsoid in (a), drawn again in (b) with its domain restricted, as described in Example 400.

Surface Area

It will become important in the following sections to be able to compute the surface area of a surface $\mathcal{S}$ given a smooth parametrization $\vec r(u,v)$, $a\le u\le b$, $c\le v\le d$. Following the principles given in the integration review at the beginning of this chapter, we can say that

$$\text{Surface Area of } \mathcal{S} = S = \iint_{\mathcal{S}} dS,$$

where dS represents a small amount of surface area. That is, to compute total surface area S, add up lots of small amounts of surface area dS across the entire surface $\mathcal{S}$. The key to finding surface area is knowing how to compute dS. We begin by approximating.

In Section 13.5 we used the area of a plane to approximate the surface area of a small portion of a surface. We will do the same here.

Let R be the region of the u-v plane bounded by $a\le u\le b$, $c\le v\le d$ as shown in Figure 14.36(a). Partition R into rectangles of width $\Delta u=\frac{b-a}{n}$ and

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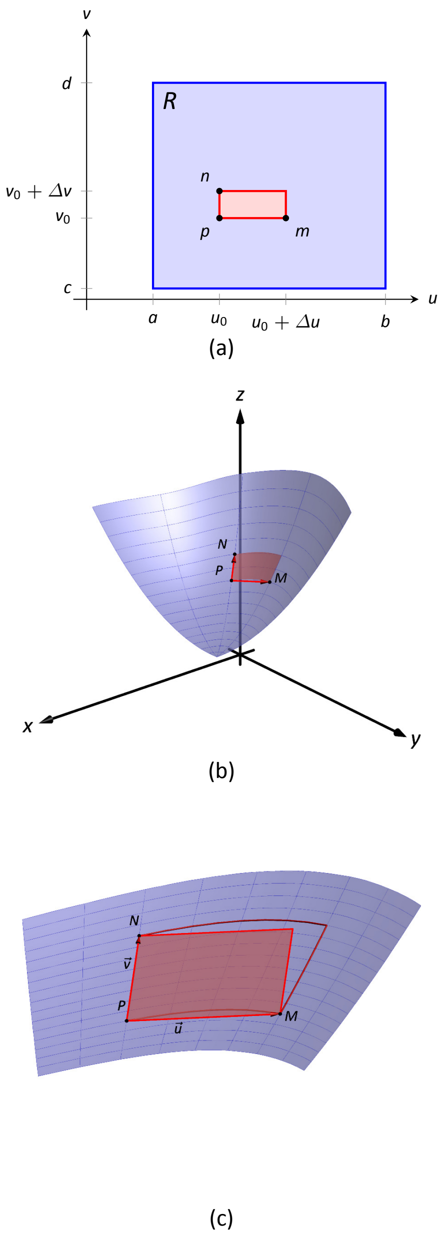


Figure 14.36: Illustrating the process of finding surface area by approximating with planes.

height $\Delta v = \frac{d-c}{n}$, for some n. Let $p = (u_0, v_0)$ be the lower left corner of some rectangle in the partition, and let m and n be neighboring corners as shown.

The point p maps to a point $P = \vec r(u_0, v_0)$ on the surface $\mathcal{S}$, and the rectangle with corners p, m and n maps to some region (probably not rectangular) on the surface as shown in Figure 14.36(b), where $M = \vec r(m)$ and $N = \vec r(n)$. We wish to approximate the surface area of this mapped region.

Let $\vec u = M - P$ and $\vec v = N - P$. These two vectors form a parallelogram, illustrated in Figure 14.36(c), whose area *approximates* the surface area we seek. In this particular illustration, we can see that parallelogram does not particularly match well the region we wish to approximate, but that is acceptable; by increasing the number of partitions of R, Δu and Δv shrink and our approximations will become better.

From Section 10.4 we know the area of this parallelogram is $|| \, \vec u \times \vec v \, ||$. If we repeat this approximation process for each rectangle in the partition of R, we can sum the areas of all the parallelograms to get an approximation of the surface area S:

$$\text{Surface area of } \mathcal{S} = S \approx \sum_{j=1}^{n}\sum_{i=1}^{n} || \, \vec u_{i,j} \times \vec v_{i,j} \, || \, ,$$

where $\vec u_{i,j} = \vec r(u_i + \Delta u, v_j) - \vec r(u_i, v_j)$ and $\vec v_{i,j} = \vec r(u_i, v_j + \Delta v) - \vec r(u_i, v_j)$.

From our previous calculus experience, we expect that taking a limit as $n \to \infty$ will result in the exact surface area. However, the current form of the above double sum makes it difficult to realize what the result of that limit is. The following rewriting of the double summation will be helpful:

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{i=1}^{n} || \, \vec u_{i,j} \times \vec v_{i,j} \, || = \\
&\sum_{j=1}^{n}\sum_{i=1}^{n} \Big|\Big| \, \big(\vec r(u_i + \Delta u, v_j) - \vec r(u_i, v_j)\big) \times \big(\vec r(u_i, v_j + \Delta v) - \vec r(u_i, v_j)\big) \, \Big|\Big| = \\
&\sum_{j=1}^{n}\sum_{i=1}^{n} \left|\left| \, \frac{\vec r(u_i + \Delta u, v_j) - \vec r(u_i, v_j)}{\Delta u} \times \frac{\vec r(u_i, v_j + \Delta v) - \vec r(u_i, v_j)}{\Delta v} \, \right|\right| \Delta u \Delta v.
\end{aligned}$$

We now take the limit as $n \to \infty$, forcing Δu and Δv to 0. As $\Delta u \to 0$,

$$\frac{\vec r(u_i + \Delta u, v_j) - \vec r(u_i, v_j)}{\Delta u} \to \vec r_u(u_i, v_j) \quad \text{and}$$

$$\frac{\vec r(u_i, v_j + \Delta v) - \vec r(u_i, v_j)}{\Delta v} \to \vec r_v(u_i, v_j).$$

Notes:

(This limit process also demonstrates that $\vec{r}_u(u,v)$ and $\vec{r}_v(u,v)$ are tangent to the surface $\mathcal{S}$ at $\vec{r}(u,v)$. We don't need this fact now, but it will be important in the next section.)

Thus, in the limit, the double sum leads to a double integral:

$$\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{n}\|\,\vec{u}_{i,j}\times\vec{v}_{i,j}\,\| = \int_c^d\int_a^b \|\,\vec{r}_u\times\vec{r}_v\,\|\ du\,dv.$$

Theorem 100 Surface Area of Parametrically Defined Surfaces

Let $\vec{r}(u,v)$ be a smooth parametrization of a surface $\mathcal{S}$ over a region R of the u-v plane.

- The surface area differential dS is: $dS = \|\,\vec{r}_u\times\vec{r}_v\,\|\ dA$.
- The surface area S of $\mathcal{S}$ is

$$S = \iint_{\mathcal{S}} dS = \iint_R \|\,\vec{r}_u\times\vec{r}_v\,\|\ dA.$$

Example 401 Finding the surface area of a parametrized surface

Using the parametrization found in Example 395, find the surface area of $z = x^2+2y^2$ over the circle of radius 2, centered at the origin.

SOLUTION In Example 395, we parametrized the surface as $\vec{r}(u,v) = \left\langle 2v\cos u, 2v\sin u, 4v^2\cos^2 u + 8v^2\sin^2 u\right\rangle$, for $0\le u\le 2\pi$, $0\le v\le 1$. To find the surface area using Theorem 100, we need $\vec{r}_u\times\vec{r}_v$. We find:

$$\begin{aligned}
\vec{r}_u &= \left\langle -2v\sin u, 2v\cos u, 8v^2\cos u\sin u\right\rangle\\
\vec{r}_v &= \left\langle 2\cos u, 2\sin v, 8v\cos^2 u + 16v\sin^2 u\right\rangle\\
\vec{r}_u\times\vec{r}_v &= \left\langle 16v^2\cos u, 32v^2\sin u, -4v\right\rangle\\
\|\,\vec{r}_u\times\vec{r}_v\,\| &= \sqrt{256v^4\cos^2 u + 1024v^4\sin^2 u + 16v^2}.
\end{aligned}$$

Thus the surface area is

$$\begin{aligned}
S = \iint_{\mathcal{S}} dS &= \iint_R \|\,\vec{r}_u\times\vec{r}_v\,\|\ dA\\
&= \int_0^1\int_0^{2\pi}\sqrt{256v^4\cos^2 u + 1024v^4\sin^2 u + 16v^2}\,du\,dv \approx 53.59.
\end{aligned}$$

Notes:

There is a lot of tedious work in the above calculations and the final integral is nontrivial. The use of a computer-algebra system is highly recommended.

In Section 14.1, we recalled the arc length differential $ds = \|\vec{r}\,'(t)\|\,dt$. In subsequent sections, we used that differential, but in most applications the "$\|\vec{r}\,'(t)\|$" part of the differential canceled out of the integrand (to our benefit, as integrating the square roots of functions is generally difficult). We will find a similar thing happens when we use the surface area differential *dS* in the following sections. That is, our main goal is not to be able to compute surface area; rather, surface area is tool to obtain other quantities that are more important and useful. In our applications, we will use *dS*, but most of the time the "$\|\vec{r}_u \times \vec{r}_v\|$" part will cancel out of the integrand, making the subsequent integration easier to compute.

Notes:

Exercises 14.5

No problems written.

14.6 Surface Integrals

Consider a smooth surface $\mathcal{S}$ that represents a thin sheet of metal. How could we find the mass of this metallic object?

If the density of this object is constant, then we can find mass via "mass=density×surface area," and we could compute the surface area using the techniques of the previous section.

What if the density were not constant, but variable, described by a function $\delta(x, y, z)$? We can describe the mass using our general integration techniques as

$$\text{mass} = \iint_{\mathcal{S}} dm,$$

where dm represents "a little bit of mass." That is, to find the total mass of the object, sum up lots of little masses over the surface.

How do we find the "little bit of mass" dm? On a small portion of the surface with surface area ΔS, the density is approximately constant, hence $dm \approx \delta(x, y, z)\Delta S$. As we use limits to shrink the size of ΔS to 0, we get $dm = \delta(x, y, z)dS$; that is, a little bit of mass is equal to a density times a small amount of surface area. Thus the total mass of the thin sheet is

$$\text{mass} = \iint_{\mathcal{S}} \delta(x, y, z)\, dS. \tag{14.3}$$

To evaluate the above integral, we would seek a smooth parametrization of $\mathcal{S}$ $\vec{r}(u, v)$ over a region R of the u-v plane. The density would become a function of u and v, and we would integrate $\iint_R \delta(u, v)\, || \vec{r}_u \times \vec{r}_v ||\ dA$.

The integral in Equation (14.3) is a specific example of a more general construction defined below.

Definition 101 Surface Integral

Let $G(x, y, z)$ be a continuous function defined on a surface $\mathcal{S}$. The **surface integral of** G **on** $\mathcal{S}$ is

$$\iint_{\mathcal{S}} G(x, y, z)\, dS.$$

Surface integrals can be used to measure a variety of quantities beyond mass. If $G(x, y, z)$ measures the static charge density at a point, then the surface integral will compute the total static charge of the sheet. If G measures the amount of fluid passing through a screen (represented by $\mathcal{S}$) at a point, then the surface integral gives the total amount of fluid going through the screen.

Notes:

Example 402 Finding the mass of a thin sheet

Find the mass of a thin sheet modeled by the plane $2x + y + z = 3$ over the triangular region of the x-y plane bounded by the coordinate axes and the line $y = 2-2x$, as shown in Figure 14.37, with density function $\delta(x, y, z) = x^2+5y+z$, where all distances are measured in cm and the density is given as gm/cm^2.

Solution We begin by parameterizing the planar surface $\mathcal{S}$. Using the techniques of the previous section, we can let $x = u$ and $y = v(2 - 2u)$, where $0 \leq u \leq 1$ and $0 \leq v \leq 1$. Solving for z in the equation of the plane, we have $z = 3 - 2x - y$, hence $z = 3 - 2u - v(2 - 2u)$, giving the parametrization $\vec{r}(u, v) = \langle u, v(2 - 2u), 3 - 2u - v(2 - 2u)\rangle$.

We need $dS = \|\, \vec{r}_u \times \vec{r}_v \,\| \, dA$, so we need to compute $\vec{r}_u$, $\vec{r}_v$ and the norm of their cross product. We leave it to the reader to confirm the following:

$$\vec{r}_u = \langle 1, -2v, 2v - 2\rangle, \quad \vec{r}_v = \langle 0, 2 - 2u, 2u - 2\rangle,$$

$$\vec{r}_u \times \vec{r}_v = \langle 4 - 4u, 2 - 2u, 2 - 2u\rangle \quad \text{and} \quad \|\, \vec{r}_u \times \vec{r}_v \,\| = 2\sqrt{6}\sqrt{(u - 1)^2}.$$

We need to be careful to not "simplify" $\|\, \vec{r}_u \times \vec{r}_v \,\| = 2\sqrt{6}\sqrt{(u - 1)^2}$ as $2\sqrt{6}(u-1)$; rather, it is $2\sqrt{6}|u - 1|$. In this example, u is bounded by $0 \leq u \leq 1$, and on this interval $|u - 1| = 1 - u$. Thus $dS = 2\sqrt{6}(1 - u)dA$.

The density is given as a function of x, y and z, for which we'll substitute the corresponding components of $\vec{r}$ (with the slight abuse of notation that we used in previous sections):

$$\begin{aligned} \delta(x, y, z) &= \delta\big(\vec{r}(u, v)\big) \\ &= u^2 + 5v(2 - 2u) + 3 - 2u - v(2 - 2u) \\ &= u^2 - 8uv - 2u + 8v + 3. \end{aligned}$$

Thus the mass of the sheet is:

$$\begin{aligned} M &= \iint_{\mathcal{S}} dm \\ &= \iint_R \delta\big(\vec{r}(u, v)\big) \,\|\, \vec{r}_u \times \vec{r}_v \,\|\, dA \\ &= \int_0^1 \int_0^1 \big(u^2 - 8uv - 2u + 8v + 3\big)\big(2\sqrt{6}(1 - u)\big)\, du\, dv \\ &= \frac{31}{\sqrt{6}} \approx 12.66 \text{ gm}. \end{aligned}$$

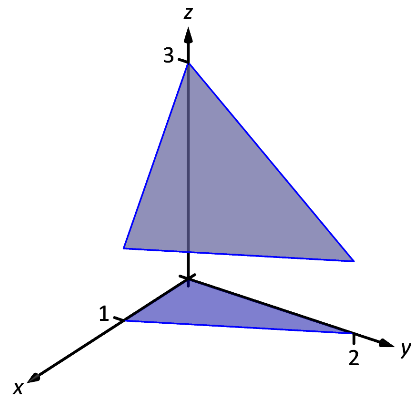


Figure 14.37: The surface whose mass is computed in Example 402.

Notes:

Flux

Let a surface $\mathcal{S}$ lie within a vector field $\vec{F}$. One is often interested in measuring the *flux* of $\vec{F}$ across $\mathcal{S}$; that is, measuring "how much of the vector field passes across $\mathcal{S}$." For instance, if $\vec{F}$ represents the velocity field of moving air and $\mathcal{S}$ represents the shape of an air filter, the flux will measure how much air is passing through the filter per unit time.

As flux measures the amount of $\vec{F}$ passing across $\mathcal{S}$, we need to find the "amount of $\vec{F}$ orthogonal to $\mathcal{S}$." Similar to our measure of flux in the plane, this is equal to $\vec{F}\cdot\vec{n}$, where $\vec{n}$ is a unit vector normal to $\mathcal{S}$ at a point. We now consider how to find $\vec{n}$.

Given a smooth parametrization $\vec{r}(u,v)$ of $\mathcal{S}$, the work in the previous section showing the development of our method of computing surface area also shows that $\vec{r}_u(u,v)$ and $\vec{r}_v(u,v)$ are tangent to $\mathcal{S}$ at $\vec{r}(u,v)$. Thus $\vec{r}_u\times\vec{r}_v$ is orthogonal to $\mathcal{S}$, and we let

$$\vec{n} = \frac{\vec{r}_u\times\vec{r}_v}{||\,\vec{r}_u\times\vec{r}_v\,||},$$

which is a unit vector normal to $\mathcal{S}$ at $\vec{r}(u,v)$.

The measurement of flux across a surface is a surface integral; that is, to measure total flux we sum the product of $\vec{F}\cdot\vec{n}$ times a small amount of surface area: $\vec{F}\cdot\vec{n}\,dS$.

A nice thing happens with the actual computation of flux: the $||\,\vec{r}_u\times\vec{r}_v\,||$ terms go away. Consider:

$$\begin{aligned}\text{Flux} &= \iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,dS\\ &= \iint_R \vec{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{||\,\vec{r}_u\times\vec{r}_v\,||}\,||\,\vec{r}_u\times\vec{r}_v\,||\ dA\\ &= \iint_R \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA.\end{aligned}$$

The above only makes sense if $\mathcal{S}$ is orientable; the normal vectors $\vec{n}$ must vary continuously across $\mathcal{S}$. We assume that $\vec{n}$ does vary continuously. (If the parametrization $\vec{r}$ of $\mathcal{S}$ is smooth, then our above definition of $\vec{n}$ will vary continuously.)

Notes:

Definition 102 Flux over a surface

Let $\vec{F}$ be a vector field with continuous components defined on an orientable surface $\mathcal{S}$ with normal vector $\vec{n}$. The **flux** of $\vec{F}$ across $\mathcal{S}$ is

$$\text{Flux} = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS.$$

If $\mathcal{S}$ is parametrized by $\vec{r}(u, v)$, which is smooth on its domain R, then

$$\text{Flux} = \iint_{R} \vec{F}\big(\vec{r}(u, v)\big) \cdot (\vec{r}_u \times \vec{r}_v)\, dA.$$

Since $\mathcal{S}$ is orientable, we adopt the convention of saying one passes from the "back" side of $\mathcal{S}$ to the "front" side when moving across the surface parallel to the direction of $\vec{n}$. Also, when $\mathcal{S}$ is closed, it is natural to speak of the regions of space "inside" and "outside" $\mathcal{S}$. We also adopt the convention that when $\mathcal{S}$ is a closed surface, $\vec{n}$ should point to the outside of $\mathcal{S}$. If $\vec{n} = \vec{r}_u \times \vec{r}_v$ points inside $\mathcal{S}$, use $\vec{n} = \vec{r}_v \times \vec{r}_u$ instead.

When the computation of flux is positive, it means that the field is moving from the back side of $\mathcal{S}$ to the front side; when flux is negative, it means the field is moving opposite the direction of $\vec{n}$, and is moving from the front of $\mathcal{S}$ to the back. When $\mathcal{S}$ is not closed, there is not a "right" and "wrong" direction in which $\vec{n}$ should point, but one should be mindful of its direction to make full sense of the flux computation.

We demonstrate the computation of flux, and its interpretation, in the following examples.

Example 403 Finding flux across a surface

Let $\mathcal{S}$ be the surface given in Example 402; that is, let $\mathcal{S}$ be the surface paramtetrized by $\vec{r}(u, v) = \langle u, v(2-2u), 3-2u-v(2-2u)\rangle$ on $0 \leq u \leq 1$, $0 \leq v \leq 1$, and let $\vec{F} = \langle 1, x, -y\rangle$, as shown in Figure 14.38.

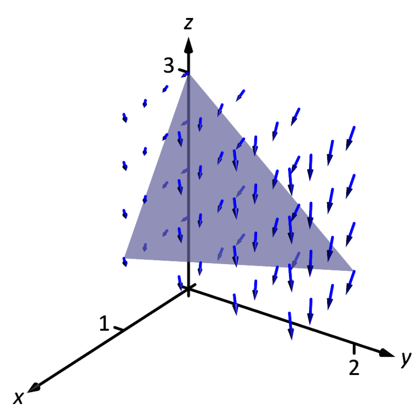


Figure 14.38: The surface and vector field used in Example 403.

Solution Using our work from the previous example, we have $\vec{n} = \vec{r}_u \times \vec{r}_v = \langle 4-4u, 2-2u, 2-2u\rangle$. We also need $\vec{F}\big(\vec{r}(u, v)\big) = \langle 1, u, -v(2-2u)\rangle$.

Notes:

Thus the flux of $\vec{F}$ across $\mathcal{S}$ is:

$$\begin{aligned}\text{Flux} &= \iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,dS\\ &= \iint_R \langle 1, u, -v(2-2u)\rangle \cdot \langle 4-4u, 2-2u, 2-2u\rangle\,dA\\ &= \int_0^1\int_0^1 \big(-4u^2v - 2u^2 + 8uv - 2u - 4v + 4\big)\,du\,dv\\ &= 5/3.\end{aligned}$$

To make full use of this numeric answer, we need to know the direction in which the field is passing across $\mathcal{S}$. The graph in Figure 14.38 helps, but we need a method that is not dependent on a graph.

Pick a point (u,v) in the interior of R and consider $\vec{n}(u,v)$. For instance, choose $(1/2,1/2)$ and look at $\vec{n}(1/2,1/2) = \langle 2,1,1\rangle$. This vector has positive x, y and z components. Generally speaking, one has *some* idea of what the surface $\mathcal{S}$ looks like, as that surface is for some reason important. In our case, we know $\mathcal{S}$ is a plane with z-intercept of $z=3$. Knowing $\vec{n}$ and the flux measurement of positive $5/3$, we know that the field must be passing from "behind" $\mathcal{S}$, i.e., the side the origin is on, to the "front" of $\mathcal{S}$.

Example 404 Flux across surfaces with shared boundaries

Let $\mathcal{S}_1$ be the unit disk in the x-y plane, and let $\mathcal{S}_2$ be the paraboloid $z = 1 - x^2 - y^2$, for $z \geq 0$, as graphed in Figure 14.39. Note how these two surfaces each have the unit circle as a boundary.

Let $\vec{F}_1 = \langle 0,0,1\rangle$ and $\vec{F}_2 = \langle 0,0,z\rangle$. Find the flux of each field across each surface.

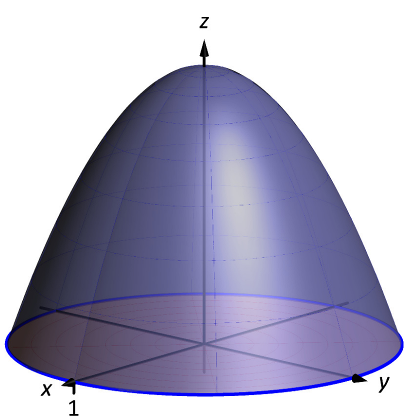


Figure 14.39: The surfaces used in Example 404.

Solution We begin by parameterizing each surface.

The boundary of the unit disk in the x-y plane is the unit circle, which can be described with $\langle \cos u, \sin u, 0\rangle$, $0 \leq u \leq 2\pi$. To obtain the interior of the circle as well, we can scale by v, giving

$$\vec{r}_1(u,v) = \langle v\cos u, v\sin u, 0\rangle, \quad 0\leq u\leq 2\pi \quad 0\leq v\leq 1.$$

As the boundary of $\mathcal{S}_2$ is also the unit circle, the x and y components of $\vec{r}_2$ will be the same as those of $\vec{r}_1$; we just need a different z component. With $z = 1-x^2-y^2$, we have

$$\vec{r}_2(u,v) = \langle v\cos u, v\sin u, 1 - v^2\cos^2 u - v^2\sin^2 u\rangle = \langle v\cos u, v\sin u, 1-v^2\rangle,$$

where $0\leq u\leq 2\pi$ and $0\leq v\leq 1$.

We now compute the normal vectors $\vec{n}_1$ and $\vec{n}_2$.

Notes:

$\vec{n}_1$: $\vec{r}_{1u} = \langle -v\sin u, v\cos u, 0\rangle$, $\vec{r}_{1v} = \langle \cos u, \sin u, 0\rangle$, so $\vec{n}_1 = \vec{r}_{1u} \times \vec{r}_{1v} = \langle 0, 0, -v\rangle$ (after simplifying).

Similarly, $\vec{n}_2$: $\vec{r}_{2u} = \langle -v\sin u, v\cos u, 0\rangle$, $\vec{r}_{2v} = \langle \cos u, \sin u, -2v\rangle$, so $\vec{n}_2 = \vec{r}_{2u} \times \vec{r}_{2v} = \langle -2v^2\cos u, -2v^2\sin u, -v\rangle$ (after simplifying).

Note that both of these normal vectors point *down*, i.e., in the negative z-direction.

We are now set to compute flux. Over field $\vec{F}_1 = \langle 0, 0, 1\rangle$:

$$\begin{aligned}\text{Flux across } \mathcal{S}_1 &= \iint_{\mathcal{S}_1} \vec{F}_1 \cdot \vec{n}_1 \, dS \\ &= \iint_R \langle 0, 0, 1\rangle \cdot \langle 0, 0, -v\rangle \, dA \\ &= \int_0^1 \int_0^{2\pi} (-v)\, du\, dv \\ &= -\pi.\end{aligned}$$

$$\begin{aligned}\text{Flux across } \mathcal{S}_2 &= \iint_{\mathcal{S}_2} \vec{F}_1 \cdot \vec{n}_2 \, dS \\ &= \iint_R \langle 0, 0, 1\rangle \cdot \langle -2v^2\cos u, -2v^2\sin u, -v\rangle \, dA \\ &= \int_0^1 \int_0^{2\pi} (-v)\, du\, dv \\ &= -\pi.\end{aligned}$$

These two results are equal and negative. Each are negative because the normal vector is pointing in the negative z-directions, whereas $\vec{F}_1$ points constantly in the positive z-direction. As the flux is opposite the direction of $\vec{n}_1$, the flux is measured as negative.

We can also intuitively understand why the results are equal. Consider $\vec{F}_1$ to represent the flow of air, and let each surface represent a filter. Since $\vec{F}_1$ is constant, and moving "straight up," it makes sense that all air passing through $\mathcal{S}_1$ also passes through $\mathcal{S}_2$, and vice–versa.

We now compute the flux across each surface with $\vec{F}_2 = \langle 0, 0, z\rangle$:

$$\text{Flux across } \mathcal{S}_1 = \iint_{\mathcal{S}_1} \vec{F}_2 \cdot \vec{n}_1 \, dS.$$

Notes:

Over $\mathcal{S}_1$, $\vec{F}_2 = \vec{F}_2\big(\vec{r}_2(u,v)\big) = \langle 0,0,0\rangle$. Therefore,

$$\begin{aligned}
&= \iint_R \langle 0,0,0\rangle \cdot \langle 0,0,-v\rangle \, dA\\
&= \int_0^1 \int_0^{2\pi} (0)\, du\, dv\\
&= 0.
\end{aligned}$$

$$\text{Flux across } \mathcal{S}_2 = \iint_{\mathcal{S}_2} \vec{F}_2 \cdot \vec{n}_2 \, dS.$$

Over $\mathcal{S}_2$, $\vec{F}_2 = \vec{F}_2\big(\vec{r}_2(u,v)\big) = \langle 0,0,1-v^2\rangle$. Therefore,

$$\begin{aligned}
&= \iint_R \langle 0,0,1-v^2\rangle \cdot \langle -2v^2\cos u, -2v^2\sin u, -v\rangle \, dA\\
&= \int_0^1 \int_0^{2\pi} (v^3 - v)\, du\, dv\\
&= -\pi/2.
\end{aligned}$$

This time the measurements of flux differ. Over $\mathcal{S}_1$, the field $\vec{F}_2$ is just $\vec{0}$, hence there is no flux. Over $\mathcal{S}_2$, the flux is again negative as $\vec{F}_2$ points in the positive z direction over $\mathcal{S}_2$ while $\vec{n}_2$ points in the negative z direction.

In the previous example, the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ form a closed surface that is piecewise smooth. That the measurement of flux across each surface was the same for some fields (and not for others) is reminiscent of a result from Section 14.4, where we measured flux across curves. The quick answer to why the flux was the same when considering $\vec{F}_1$ is that $\text{div}\,\vec{F}_1 = 0$. In the next section, we'll see the second part of the Divergence Theorem which will more fully explain this occurrence.

Notes:

Exercises 14.6

No problems written.

14.7 The Divergence Theorem and Stokes' Theorem

The Divergence Theorem

Theorem 99 gives the Divergence Theorem in the plane, which states that the flux of a vector field across a closed *curve* equals the sum of the divergences over the region enclosed by the curve. Recall that the flux was measured via a line integral, and the sum of the divergences was measured through a double integral.

We now consider the three-dimensional version of the Divergence Theorem. It states, in words, that the flux across a closed *surface* equals the sum of the divergences over the region enclosed by the surface. Since we are in space (versus the plane), we measure flux via a surface integral, and the sums of divergences will be measured through a triple integral.

Note: the term "outer unit normal vector" used in Theorem 101 means $\vec{n}$ points to the outside of $\mathcal{S}$.

Theorem 101 The Divergence Theorem (in space)

Let D be a closed domain in space whose boundary is an orientable, piecewise smooth surface $\mathcal{S}$ with outer unit normal vector $\vec{n}$, and let $\vec{F}$ be a vector field whose components are differentiable on D. Then

$$\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\, dS = \iiint_D \operatorname{div}\vec{F}\, dV.$$

Example 405 Using the Divergence Theorem in space

Let D be the domain in space bounded by the planes $z = 0$ and $z = 2x$, along with the cylinder $x = 1 - y^2$, as graphed in Figure 14.40, let $\mathcal{S}$ be the boundary of D, and let $\vec{F} = \langle x + y, y^2, 2z\rangle$.

Verify the Divergence Theorem by finding the total outward flux of $\vec{F}$ across $\mathcal{S}$, and show this is equal to $\iiint_D \operatorname{div}\vec{F}\, dV$.

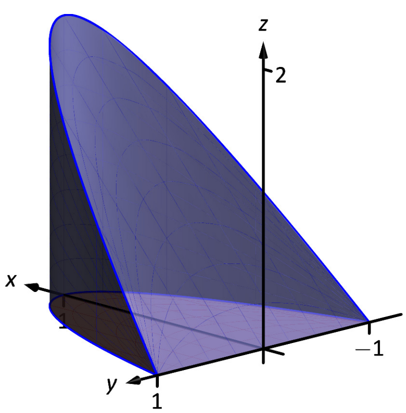


Figure 14.40: The surfaces used in Example 405.

Solution The surface $\mathcal{S}$ is piecewise smooth, comprising surfaces $\mathcal{S}_1$, which is part of the plane $z = 2x$, surface $\mathcal{S}_2$, which is part of the cylinder $x = 1 - y^2$, and surface $\mathcal{S}_3$, which is part of the plane $z = 0$. To find the total outward flux across $\mathcal{S}$, we need to compute the outward flux across each of these three surfaces.

We leave it to the reader to confirm that surfaces $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ can be pa-

Notes:

rameterized by $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ respectively as

$$\begin{aligned}
\vec{r}_1(u,v) &= \left\langle v(1-u^2), u, 2v(1-u^2) \right\rangle, \\
\vec{r}_2(u,v) &= \left\langle (1-u^2), u, 2v(1-u^2) \right\rangle, \\
\vec{r}_3(u,v) &= \left\langle v(1-u^2), u, 0 \right\rangle,
\end{aligned}$$

where $-1 \leq u \leq 1$ and $0 \leq v \leq 1$ for all three functions.

We compute a unit normal vector $\vec{n}$ for each as $\frac{\vec{r}_u \times \vec{r}_v}{||\ \vec{r}_u \times \vec{r}_v\ ||}$, though recall that as we are integrating $\vec{F} \cdot \vec{n}\ dS$, we actually only use $\vec{r}_u \times \vec{r}_v$. Finally, in previous flux computations, it did not matter which direction $\vec{n}$ pointed as long as we made note of its direction. When using the Divergence Theorem, we need $\vec{n}$ to point to the outside of the closed surface, so in practice this means we'll either use $\vec{r}_u \times \vec{r}_v$ or $\vec{r}_v \times \vec{r}_u$, depending on which points outside of the closed surface $\mathcal{S}$.

We leave it to the reader to confirm the following cross products and integrations are correct.

For $\mathcal{S}_1$, we need to use $\vec{r}_{1v} \times \vec{r}_{1u} = \langle 2(u^2-1), 0, 1-u^2 \rangle$. (Note the z-component is nonnegative as $u \leq 1$, therefore this vector always points up, meaning *to the outside*, of $\mathcal{S}$.) The flux across $\mathcal{S}_1$ is:

$$\begin{aligned}
\text{Flux across } \mathcal{S}_1 \colon &= \iint_{\mathcal{S}_1} \vec{F} \cdot \vec{n}_1\ dS \\
&= \int_0^1 \int_{-1}^1 \vec{F}\big(\vec{r}_1(u,v)\big) \cdot \big(\vec{r}_{1v} \times \vec{r}_{1u}\big)\ du\ dv \\
&= \int_0^1 \int_{-1}^1 \left\langle v(1-u^2)+u, u^2, 4v(1-u^2) \right\rangle \cdot \left\langle 2(u^2-1), 0, 1-u^2 \right\rangle\ du\ dv \\
&= \int_0^1 \int_{-1}^1 \big(2u^4v + 2u^3 - 4u^2v - 2u + 2v\big)\ du\ dv \\
&= \frac{16}{15}.
\end{aligned}$$

For $\mathcal{S}_2$, we use $\vec{r}_{2u} \times \vec{r}_{2v} = \langle 2(1-u^2), 4u(1-u^2), 0 \rangle$. (Note the x-component

Notes:

is always positive, meaning this vector points outside $\mathcal{S}$.) The flux across $\mathcal{S}_2$ is:

$$\begin{aligned}
\text{Flux across } \mathcal{S}_2\text{:} &= \iint_{\mathcal{S}_2} \vec{F}\cdot\vec{n}_2\,dS \\
&= \int_0^1\int_{-1}^1 \vec{F}\big(\vec{r}_2(u,v)\big)\cdot\big(\vec{r}_{2u}\times\vec{r}_{2v}\big)\,du\,dv \\
&= \int_0^1\int_{-1}^1 \big\langle 1-u^2+u, u^2, 4v(1-u^2)\big\rangle\cdot\big\langle 2(1-u^2), 4u(1-u^2), 0\big\rangle\,du\,dv \\
&= \int_0^1\int_{-1}^1 \big(4u^5-2u^4-2u^3+4u^2-2u-2\big)\,du\,dv \\
&= \frac{32}{15}.
\end{aligned}$$

For $\mathcal{S}_3$, we use $\vec{r}_{3u}\times\vec{r}_{3v} = \langle 0,0,u^2-1\rangle$. (Note the z-component is always negative, meaning this vector points down, outside of $\mathcal{S}$.) The flux across $\mathcal{S}_3$ is:

$$\begin{aligned}
\text{Flux across } \mathcal{S}_3\text{:} &= \iint_{\mathcal{S}_3} \vec{F}\cdot\vec{n}_3\,dS \\
&= \int_0^1\int_{-1}^1 \vec{F}\big(\vec{r}_3(u,v)\big)\cdot\big(\vec{r}_{3u}\times\vec{r}_{3v}\big)\,du\,dv \\
&= \int_0^1\int_{-1}^1 \big\langle v(1-u^2)+u, u^2, 0\big\rangle\cdot\big\langle 0,0,u^2-1\big\rangle\,du\,dv \\
&= \int_0^1\int_{-1}^1 0\,du\,dv \\
&= 0.
\end{aligned}$$

Thus the total outward flux, measured by surface integrals across all three component surfaces of $\mathcal{S}$, is $16/15+32/15+0=48/15=16/5=3.2$. We now find the total outward flux by integrating $\text{div}\,\vec{F}$ over D.

Following the steps outlined in Section 13.6, we see the bounds of x, y and z can be set as (thinking "surface to surface, curve to curve, point to point"):

$$0\le z\le 2x;\quad 0\le x\le 1-y^2;\quad -1\le y\le 1.$$

With $\text{div}\,\vec{F}=1+2y+2=2y+3$, we find the total outward flux of $\vec{F}$ over $\mathcal{S}$ as:

$$\text{Flux} = \iiint_D \text{div}\,\vec{F}\,dV = \int_{-1}^1\int_0^{1-y^2}\int_0^{2x}\big(2y+3\big)\,dz\,dx\,dy = 16/5,$$

the same result we obtained previously.

Notes:

In Example 405 we see that the total outward flux of a vector field across a closed surface can be found two different ways because of the Divergence Theorem. One computation took far less work to obtain. In that particular case, since $\mathcal{S}$ was comprised of three separate surfaces, it was far simpler to compute one triple integral than three surface integrals (each of which required partial derivatives and a cross product). In practice, if outward flux needs to be measured, one would choose only one method. We will use both methods in this section simply to reinforce the truth of the Divergence Theorem.

We practice again in the following example.

Example 406 Using the Divergence Theorem in space

Let $\mathcal{S}$ be the surface formed by the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$, and the unit disk centered at the origin in the x-y plane, graphed in Figure 14.42, and let $\vec{F} = \langle 0, 0, z\rangle$. (This surface and vector field were used in Example 404.)

Verify the Divergence Theorem; find the total outward flux across $\mathcal{S}$ and evaluate the triple integral of $\operatorname{div} \vec{F}$, showing that these two quantities are equal.

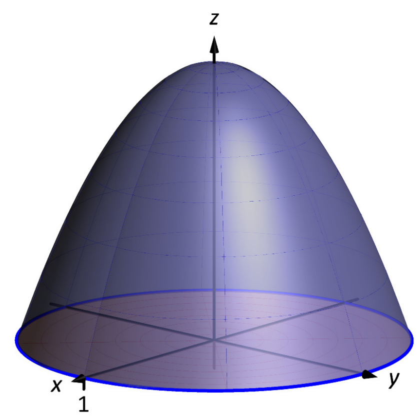


Figure 14.41: The surfaces used in Example 406.

Solution We find the flux across $\mathcal{S}$ first. As $\mathcal{S}$ is piecewise–smooth, we decompose it into smooth components $\mathcal{S}_1$, the disk, and $\mathcal{S}_2$, the paraboloid, and find the flux across each.

In Example 404, we found the flux across $\mathcal{S}_1$ is 0. We also found that the flux across $\mathcal{S}_2$ was $-\pi/2$, but we need to look closer at how this was computed. The normal vector used in that example was $\vec{n} = \langle -2v^2\cos u, -2v^2\sin u, -v\rangle$. Note how $\vec{n}$ points inside $\mathcal{S}$; there are several ways to determine this, though the easiest in this case is to consider the z-component of $\vec{n}$, which is $-v$. Any normal vector pointing to the outside of $\mathcal{S}$ on $\mathcal{S}_2$ should point "up," i.e., have a positive z-component. Our flux computation should use $-\vec{n}$ instead of $\vec{n}$, giving an outward flux of $\pi/2$. Thus the total outward flux is $0 + \pi/2 = \pi/2$.

We now compute $\iiint_D \operatorname{div} \vec{F}\, dV$. We can describe D as the domain bounded by (think "surface to surface, curve to curve, point to point"):

$$0 \leq z \leq 1 - x^2 - y^2, \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, \quad -1 \leq x \leq 1.$$

This description of D is not very easy to integrate. With polar, we can do better. Let R represent the unit disk, which can be described in polar simply as r, where $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. With $x = r\cos\theta$ and $y = r\sin\theta$, the surface $\mathcal{S}_2$ becomes

$$z = 1 - x^2 - y^2 \Rightarrow 1 - (r\cos\theta)^2 - (r\sin\theta)^2 \Rightarrow 1 - r^2.$$

Thus D can be described as the domain bounded by:

$$0 \leq z \leq 1 - r^2, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

Notes:

With div $\vec{F} = 1$, we can integrate, recalling that $dV = r\,dz\,dr\,d\theta$:

$$\iiint_D \operatorname{div} \vec{F}\,dV = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} 1\,dz\,dr\,d\theta = \frac{\pi}{2},$$

which matches our flux computation above.

Example 407 **A "paradox" of the Divergence Theorem and Gauss's Law**

The magnitude of many physical quantities (such as light intensity or electromagnetic and gravitational forces) follow an "inverse square law": the magnitude of the quantity at a point is inversely proportional to the square of the distance to the source of the quantity.

Let a point light source be placed at the origin and let $\vec{F}$ be the vector field which describes the intensity and direction of the emanating light. At a point (x, y, z), the unit vector describing the direction of the light passing through that point is $\langle x, y, z\rangle/\sqrt{x^2+y^2+z^2}$. As the intensity of light follows the inverse square law, the magnitude of $\vec{F}$ at (x, y, z) is $k/(x^2+y^2+z^2)$ for some constant k. Taken together,

$$\vec{F}(x, y, z) = \frac{k}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z\rangle.$$

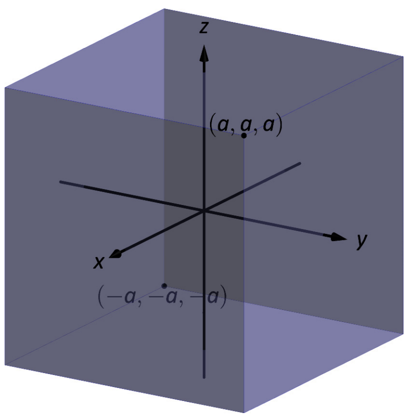


Figure 14.42: The cube used in Example 407.

Consider the cube, centered at the origin, with sides of length $2r$ for some $r > 0$ (hence corners of the cube lie at (a, a, a), $(-a, -a, -a)$, etc., as shown in Figure **??**). Find the flux across the six faces of the cube and compare this to $\iint_D \operatorname{div} \vec{F}\,dV$.

Solution Let $\mathcal{S}_1$ be the "top" face of the cube, which can be parametrized by $\vec{r}(u, v) = \langle u, v, a\rangle$ for $-a \le u \le a$, $-a \le v \le a$. We leave it to the reader to confirm that $\vec{r}_u \times \vec{r}_v = \langle 0, 0, 1\rangle$, which points outside of the cube.

The flux across this face is:

$$\begin{aligned}\text{Flux} &= \iint_{\mathcal{S}_1} \vec{F}\cdot\vec{n}\,dS\\ &= \int_{-a}^{a}\int_{-a}^{a} \vec{F}\big(\vec{r}(u, v)\big)\cdot\big(\vec{r}_u \times \vec{r}_v\big)\,du\,dv\\ &= \int_{-a}^{a}\int_{-a}^{a} \frac{k\,a}{(u^2+v^2+a^2)^{3/2}}\,du\,dv.\end{aligned}$$

This double integral is *not* trivial to compute, requiring multiple trigonometric substitutions. This example is not meant to stress integration techniques, so we leave it to the reader to confirm the result is

$$= \frac{2k\pi}{3}.$$

Notes:

Note how the result is independent of a; no matter the size of the cube, the flux through the top surface is always $2k\pi/3$.

An argument of symmetry shows that the flux through each of the six faces is $2\pi/3$, thus the total flux through the faces of the cube is $6 \times 2k\pi/3 = 4k\pi$.

It takes a bit of algebra, but we can show that $\text{div}\,\vec{F} = 0$. Thus the Divergence Theorem would seem to imply that the total flux through the faces of the cube should be

$$\text{Flux } = \iiint_D \text{div}\,\vec{F}\,dV = \iiint_D 0\,dV = 0,$$

but clearly this does not match the result from above. What went wrong?

Revisit the statement of the Divergence Theorem. One of the conditions is that the components of $\vec{F}$ must be differentiable on the domain enclosed by the surface. In our case, $\vec{F}$ is *not* differentiable at the origin – it is not even defined! As $\vec{F}$ does not satisfy the conditions of the Divergence Theorem, it does not apply, and we cannot expect $\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,dA = \iiint_D \text{div}\,\vec{F}\,dV$.

Since $\vec{F}$ is differentiable everywhere except the origin, the Divergence Theorem does apply over any domain that does not include the origin. Let $\mathcal{S}_2$ be any surface that encloses the cube used before, and let $\hat{D}$ be the domain *between* the cube and $\mathcal{S}_2$; note how $\hat{D}$ does not include the origin and so the Divergence Theorem does apply over this domain. The total outward flux over $\hat{D}$ is thus $\iint_{\hat{D}} \text{div}\,\vec{F}\,dV = 0$, which means the amount of flux coming out of $\mathcal{S}_2$ is the same as the amount of flux coming out of the cube. The conclusion: the flux across *any* surface enclosing the origin will be $4k\pi$.

This has an important consequence in electrodynamics. Let q be a point charge at the origin. The electric field generated by this point charge is

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}},$$

i.e., it is $\vec{F}$ with $k = q/(4\pi\varepsilon_0)$, where ε_0 is a physical constant (the "permittivity of free space"). Gauss's Law states that the outward flux of $\vec{E}$ across any surface enclosing the origin is q/ε_0.

Stokes' Theorem

Just as the spatial Divergence Theorem of this section is an extension of the planar Divergence Theorem, Stokes' Theorem is the spatial extension of Green's Theorem. Recall that Green's Theorem states that the circulation of a vector field around a closed curve in the plane is equal to the sum of the curls of the field over the region enclosed by the curve. Stokes' Theorem effectively makes the same statement: given a closed curve that lies on a surface $\mathcal{S}$, the circulation

Notes:

of a vector field around that curve is the same as the sum of "the curls of the field" across the enclosed surface. We use quotes around "the curls of the field" to signify that this statement is not quite correct, as we do not sum curl $\vec{F}$, but curl $\vec{F}\cdot\vec{n}$, where $\vec{n}$ is a unit vector normal to $\mathcal{S}$. That is, we sum the portion of curl $\vec{F}$ that is orthogonal to $\mathcal{S}$ at a point.

Green's Theorem dictated that the curve was to be traversed counterclockwise when measuring circulation. Stokes' Theorem will follow a right hand rule: when the thumb of one's right hand points in the direction of $\vec{n}$, the path C will be traversed in the direction of the curling fingers of the hand.

Theorem 102 Stokes' Theorem

Let $\mathcal{S}$ be a piecewise smooth, orientable surface whose boundary is a piecewise smooth curve C, let $\vec{n}$ be a unit vector normal to $\mathcal{S}$, let C be traversed with respect to $\vec{n}$ according to the right hand rule, and let the components of $\vec{F}$ have continuous first partial derivatives over $\mathcal{S}$. Then

$$\oint_C \vec{F}\cdot\,d\vec{r} = \iint_{\mathcal{S}} (\text{curl}\,\vec{F})\cdot\vec{n}\,dS.$$

In general, the best approach to evaluating the surface integral in Stokes' Theorem is to parametrize the surface $\mathcal{S}$ with a function $\vec{r}(u,v)$. We can find a unit normal vector $\vec{n}$ as

$$\vec{n} = \frac{\vec{r}_u\times\vec{r}_v}{||\,\vec{r}_u\times\vec{r}_v\,||}.$$

Since $dS = ||\,\vec{r}_u\times\vec{r}_v\,||\ dA$, the surface integral in practice is evaluated as

$$\iint_{\mathcal{S}} (\text{curl}\,\vec{F})\cdot(\vec{r}_u\times\vec{r}_v)\,dA,$$

where $\vec{r}_u\times\vec{r}_v$ may be replaced by $\vec{r}_v\times\vec{r}_u$ to properly match the direction of this vector with the orientation of the parameterization of C.

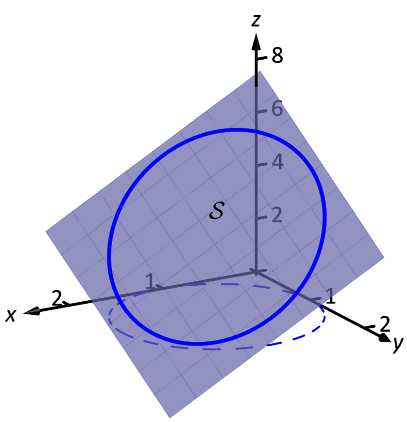


Figure 14.43: As given in Example 408, the surface $\mathcal{S}$ is the portion of the plane bounded by the curve.

Example 408 Verifying Stokes' Theorem

Consider the planar surface $f(x,y) = 7-2x-2y$, let C be the curve in space that lies on this surface above the circle of radius 1 and centered at $(1,1)$ in the x-y plane, let $\mathcal{S}$ be the planar region enclosed by C, as illustrated in Figure 14.43, and let $\vec{F} = \langle x+y, 2y, y^2\rangle$. Verify Stoke's Theorem by showing $\oint_C \vec{F}\cdot\,d\vec{r} = \iint_{\mathcal{S}}(\text{curl}\,\vec{F})\cdot\vec{n}\,dS$.

Solution We begin by parameterizing C and then find the circulation. A unit circle centered at $(1,1)$ can be parametrized with $x = \cos t+1$, $y =$

Notes:

$\sin t+1$ on $0 \le t \le 2\pi$; to put this curve on the surface f, make the z component equal $f(x,y)$: $z = 7-2(\cos t+1)-2(\sin t+1) = 3-2\cos t-2\sin t$. All together, we parametrize C with $\vec r(t) = \langle \cos t+1, \sin t+1, 3-2\cos t-2\sin t\rangle$.

The circulation of $\vec F$ around C is

$$\begin{aligned}\oint_C \vec F\cdot\, d\vec r &= \int_0^{2\pi} \vec F\big(\vec r(t)\big)\cdot \vec r\,'(t)\, dt\\ &= \int_0^{2\pi} \big(2\sin^3 t - 2\cos t\sin^2 t + 3\sin^2 t - 3\cos t\sin t\big)\, dt\\ &= 3\pi.\end{aligned}$$

We now parametrize $\mathcal{S}$. (We re-use the letter "r" for our surface as this is our custom.) Based on the parametrization of C above, we describe $\mathcal{S}$ with $\vec r(u,v) = \langle v\cos u+1, v\sin u+1, 3-2v\cos u-2v\sin u\rangle$, where $0 \le u \le 2\pi$ and $0 \le v \le 1$.

We leave it to the reader to confirm that $\vec r_u \times \vec r_v = \langle 2v, 2v, v\rangle$. As $0 \le v \le 1$, this vector always has a non-negative z-component, which the right–hand rule requires given the orientation of C used above. We also leave it to the reader to confirm $\operatorname{curl}\vec F = \langle 2y, 0, -1\rangle$.

The surface integral of Stokes' Theorem is thus

$$\begin{aligned}\iint_{\mathcal{S}} (\operatorname{curl}\vec F)\cdot\vec n\, dS &= \iint_{\mathcal{S}} (\operatorname{curl}\vec F)\cdot(\vec r_u\times\vec r_v)\, dA\\ &= \int_0^1\int_0^{2\pi} \langle 2v\sin u+2, 0, -1\rangle\cdot\langle 2v, 2v, v\rangle\, du\, dv\\ &= 3\pi,\end{aligned}$$

which matches our previous result.

One of the interesting results of Stokes' Theorem is that if two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ share the same boundary, then $\iint_{\mathcal{S}_1}(\operatorname{curl}\vec F)\cdot\vec n\, dS = \iint_{\mathcal{S}_2}(\operatorname{curl}\vec F)\cdot\vec n\, dS$. That is, the value of these two surface integrals is somehow independent of the interior of the surface. We demonstrate this principle in the next example.

Example 409 Stokes' Theorem and surfaces that share a boundary

Let C be the curve given in Example 408 and note that it lies on the surface $z = 6-x^2-y^2$. Let $\mathcal{S}$ be the region of this surface bounded by C, and let $\vec F = \langle x+y, 2y, y^2\rangle$ as in the previous example. Compute $\iint_{\mathcal{S}}(\operatorname{curl}\vec F)\cdot\vec n\, dS$ to show it equals the result found in the previous example.

Solution We begin by demonstrating that C lies on the surface $z = 6-x^2-y^2$. We can parametrize the x and y components of C with $x = \cos t+1$,

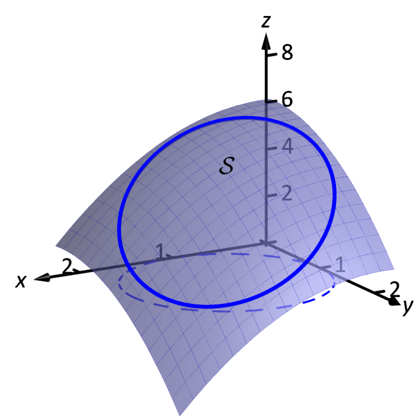


Figure 14.44: As given in Example 409, the surface $\mathcal{S}$ is the portion of the plane bounded by the curve.

Notes:

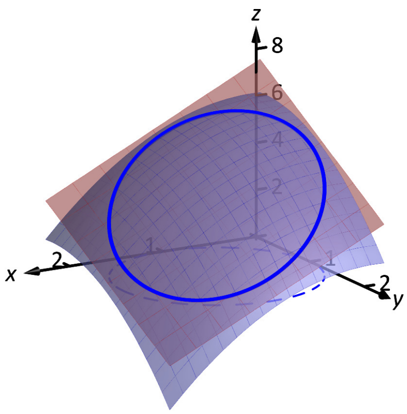


Figure 45: Illustrating how the surfaces in Examples 408 and 409 have the same boundary.

$y = \sin t + 1$ as before. Lifting these components to the surface f gives the z component as $z = 6-x^2-y^2 = 6-(\cos t+1)^2-(\sin t+1)^2 = 3-2\cos t-2\sin t$, which is the same z component as found in Example 408. Thus the curve C lies on the surface $z = 6 - x^2 - y^2$, as illustrated in Figure 14.44.

Since C and $\vec{F}$ are the same as in the previous example, we already know that $\oint_C \vec{F}\cdot\, d\vec{r} = 3\pi$. We confirm that this is also the value of $\iint_{\mathcal{S}}(\text{curl}\,\vec{F})\cdot\vec{n}\,dS$.

We parametrize $\mathcal{S}$ with

$$\vec{r}(u,v) = \langle v\cos u + 1, v\sin u + 1, 6 - (v\cos u+1)^2 - (v\sin u + 1)^2\rangle,$$

where $0 \le u \le 2\pi$ and $0 \le v \le 1$, and leave it to the reader to confirm that

$$\vec{r}_u \times \vec{r}_v = \big\langle 2v\big(v\cos u + 1\big), 2v\big(v\sin u + 1\big), v\big\rangle\,,$$

which also conforms to the right–hand rule with regard to the orientation of C. With $\text{curl}\,\vec{F} = \langle 2y, 0, -1\rangle$ as before, we have

$$\iint_{\mathcal{S}}(\text{curl}\,\vec{F})\cdot\vec{n}\,dS =$$
$$\int_0^1\int_0^{2\pi}\langle 2v\sin u + 2, 0, -1\rangle\cdot\big\langle 2v\big(v\cos u+1\big), 2v\big(v\sin u + 1\big), v\big\rangle\;du\,dv =$$
$$3\pi.$$

Even though the surfaces used in this example and in Example 408 are very different, because they share the same boundary, Stokes' Theorem guarantees they have equal "sum of curls" across their respective surfaces.

Notes:

A: Solutions To Selected Problems

Chapter 14

Section 14.1

1. When C is a curve in the plane and f is a surface defined over C, then $\int_C f(s)\,ds$ describes the area under the spatial curve that lies on f, over C.

2. The evaluation is the same. The $\oint$ notation signifies that the curve C is a closed curve, though the evaluation is the same.

3. The variable s denotes the arc-length parameter, which is generally difficult to use. The Key Idea allows one to parametrize a curve using another, ideally easier-to-use, parameter.

4. Answers will vary.

5. $12\sqrt{2}$

6. $41\sqrt{10}/2$

7. 40π

8. $10\pi^2$

9. Over the first subcurve of C, the line integral has a value of $3/2$; over the second subcurve, the line integral has a value of $4/3$. The total value of the line integral is thus $17/6$.

10. Over the first subcurve of C, the line integral has a value of $2\sqrt{2}/3$; over the second subcurve, the line integral has a value of $\pi - 2$. The total value of the line integral is thus $\pi + 2\sqrt{2}/3 - 2$.

11. $\int_0^1 (5t^2 +_2 t + 2)\sqrt{(4t+1)^2 + 1}\,dt \approx 17.071$

12. $\int_0^\pi t\sqrt{1+\cos^2 t}\,dt \approx 6.001$

13. $\oint_0^{2\pi} \left(10 - 4\cos^2 t - \sin^2 t\right)\sqrt{\cos^2 t + 4\sin^2 t}\,dt \approx 74.986$

14. $\int_{-1}^1 \left(3t^3 + 2t + 5\right)\sqrt{9t^4+1}\,dt \approx 15.479$

15. $7\sqrt{26}/3$

16. 2π

17. $8\pi^3$

18. $5/2$

19. $M = 8\sqrt{2}\pi^2$; center of mass is $(0, -1/(2\pi), 8\pi/3)$.

20. $M \approx 0.237$; center of mass is approximately $(0.173, 0.099, 0.065)$.

Section 14.2

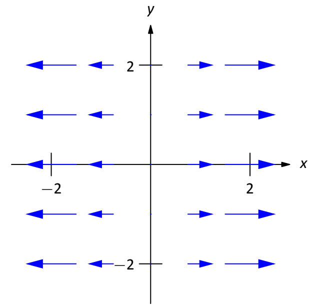
1. Answers will vary. Appropriate answers include velocities of moving particles (air, water, etc.); gravitational or electromagnetic forces.

2. Specific answers will vary, though should relate to the idea that "more of the vector field is moving into that point than out of that point."

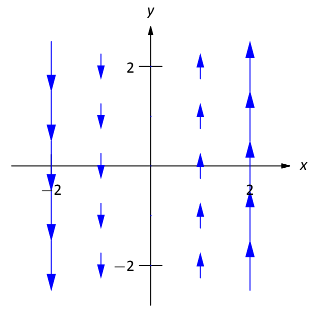
3. Specific answers will vary, though should relate to the idea that the vector field is spinning clockwise at that point.

4. No; to be incompressible, the divergence needs to be 0 everywhere, not just at one point.

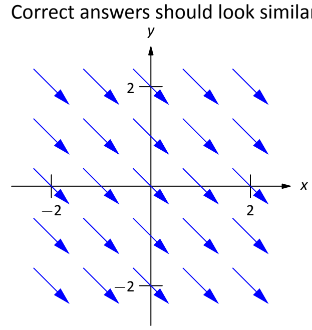
5. Correct answers should look similar to



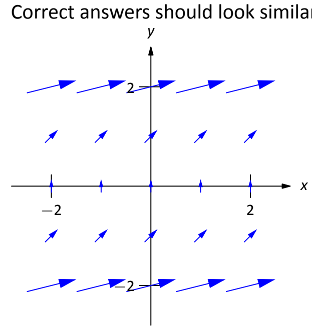
6. Correct answers should look similar to



7. Correct answers should look similar to



8. Correct answers should look similar to



9. $\text{div}\,\vec{F} = 1 + 2y$
   $\text{curl}\,\vec{F} = 0$

10. $\text{div}\,\vec{F} = 0$
    $\text{curl}\,\vec{F} = 1 + 2y$

11. $\text{div}\,\vec{F} = x\cos(xy) - y\sin(xy)$
    $\text{curl}\,\vec{F} = y\cos(xy) + x\sin(xy)$

12. $\text{div}\,\vec{F} = \frac{4}{(x^2+y^2)^2}$
    $\text{curl}\,\vec{F} = 0$

13. $\text{div}\,\vec{F} = 3$
    $\text{curl}\,\vec{F} = \langle -1, -1, -1\rangle$

14. div $\vec{F} = 2x + 2y + 2z$
curl $\vec{F} = \langle 2y, 2z, 2x\rangle$

15. div $\vec{F} = 1 + 2y$
curl $\vec{F} = 0$

16. div $\vec{F} = 2y$
curl $\vec{F} = 0$

17. div $\vec{F} = 2y - \sin z$
curl $\vec{F} = \vec{0}$

18. div $\vec{F} = \frac{2}{(x^2+y^2+z^2)^2}$
curl $\vec{F} = \vec{0}$

Section 14.3

1. False. It is true for line integrals over scalar fields, though.
2. The input of $\vec{F}$ should be a point in the plane, not a two dimensional vector.
3. True.
4. False.
5. We can conclude that $\vec{F}$ is conservative.
6. By the Fundamental Theorem of Line Integrals, since $\vec{F}$ is conservative, $\oint_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$, where f is a potential function for $\vec{F}$ and A and B are the initial and terminal points of C, respectively. Since C is a closed curve, $A = B$, and hence $f(B) - f(A) = 0$.
7. $11/6$. (One parametrization for C is $\vec{r}(t) = \langle 3t, t\rangle$ on $0 \le t \le 1$.)
8. $5/3$. (One parametrization for C is $\vec{r}(t) = \langle t, t^2\rangle$ on $0 \le t \le 1$.)
9. 0. (One parametrization for C is $\vec{r}(t) = \langle \cos t, \sin t\rangle$ on $0 \le t \le \pi$.)
10. $2/5$. (One parametrization for C is $\vec{r}(t) = \langle t, t^3\rangle$ on $-1 \le t \le 1$.)
11. 12. (One parametrization for C is $\vec{r}(t) = \langle 1, 2, 3\rangle + t\langle 3, 1, -1\rangle$ on $0 \le t \le 1$.)
12. 1.
13. $5/6$ joules. (One parametrization for C is $\vec{r}(t) = \langle t, t\rangle$ on $0 \le t \le 1$.)
14. $13/15$ joules. (One parametrization for C is $\vec{r}(t) = \langle t, \sqrt{t}\rangle$ on $0 \le t \le 1$.)
15. 24 ft-lbs.
16. 24 ft-lbs.
17. (a) $f(x, y) = xy + x$
    (b) curl $\vec{F} = 0$.
    (c) 1. (One parametrization for C is $\vec{r}(t) = \langle t, -1t\rangle$ on $0 \le t \le 1$.)
    (d) 1 (with $A = (0, 1)$ and $B = (1, 0)$, $f(B) - f(A) = 1$.)
18. (a) $f(x, y) = x^2 + xy + y^2$
    (b) curl $\vec{F} = 0$.
    (c) 0.
    (d) 0 (with $A = (0, 0)$ and $B = (0, 0)$, $f(B) - f(A) = 0$.)
19. (a) $f(x, y) = x^2yz$
    (b) curl $\vec{F} = \vec{0}$.
    (c) 250.
    (d) 250 (with $A = (1, -1, 0)$ and $B = (5, 5, 2)$, $f(B) - f(A) = 250$.)
20. (a) $f(x, y) = x^2 + y^2 + z^2$
    (b) curl $\vec{F} = \vec{0}$.
    (c) 0.
    (d) 0 (with $A = (1, 0, 0)$ and $B = (1, 0, 0)$, $f(B) - f(A) = 250$.)
21. Since $\vec{F}$ is conservative, it is the gradient of some potential function. That is, $\nabla f = \langle f_x, f_y, f_z\rangle = \vec{F} = \langle M, N, P\rangle$. In particular, $M = f_x$, $N = f_y$ and $P = f_z$.
    Note that curl $\vec{F} = \langle P_y - N_z, M_z - P_x, N_x - M_y\rangle = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy}\rangle$, which, by Theorem 69, is $\langle 0, 0, 0\rangle$.

Section 14.4

1. along, across
2. It is the measure of flow around the entirety of a closed curve C.
3. the curl of $\vec{F}$, or curl $\vec{F}$
4. the divergence of $\vec{F}$, or div $\vec{F}$
5. curl $\vec{F}$
6. div $\vec{F}$
7. 12
8. 12
9. $-2/3$
10. $10/3$
11. $1/2$
12. $1/2$
13. The line integral $\oint_C \vec{F}\cdot d\vec{r}$, over the parabola, is $38/3$; over the line, it is -10. The total line integral is thus $38/3 - 10 = 8/3$. The double integral of curl $\vec{F} = 2$ over R also has value $8/3$.
14. Both the line integral and double integral have value of 2π.
15. Three line integrals need to be computed to compute $\oint_C \vec{F}\cdot d\vec{r}$. It does not matter which corner one starts from first, but be sure to proceed around the triangle in a counterclockwise fashion.
    From $(0, 0)$ to $(2, 0)$, the line integral has a value of 0. From $(2, 0)$ to $(1, 1)$ the integral has a value of $7/3$. From $(1, 1)$ to $(0, 0)$ the line integral has a value of $-1/3$. Total value is 2.
    The double integral of curl $\vec{F}$ over R also has value 2.
16. Two line integrals need to be computed to compute $\oint_C \vec{F}\cdot d\vec{r}$. Along the parabola, the line integral has value 25.5. Along the line, the line integral has value -21. Together, the total value is 4.5
    The double integral of curl $\vec{F}$ over R also has value 4.5.
17. Any choice of $\vec{F}$ is appropriate as long as curl $\vec{F} = 1$. When $\vec{F} = \langle -y/2, x/2\rangle$, the integrand of the line integral is simply 6. The area of R is 12π.
18. Any choice of $\vec{F}$ is appropriate as long as curl $\vec{F} = 1$. The choices of $\vec{F} = \langle -y, 0\rangle$ and $\langle 0, x\rangle$ each lead to reasonable integrands. The area of R is $4/3$.
19. Any choice of $\vec{F}$ is appropriate as long as curl $\vec{F} = 1$. The choices of $\vec{F} = \langle -y, 0\rangle$, $\langle 0, x\rangle$ and $\langle -y/2, x/2\rangle$ each lead to reasonable integrands. The area of R is $16/15$.
20. Any choice of $\vec{F}$ is appropriate as long as curl $\vec{F} = 1$. The choice of $\vec{F} = \langle -y/2, x/2\rangle$ leads to a reasonable integrand after simplification. The area of R is $41\pi/10$.
21. The line integral $\oint_C \vec{F}\cdot\vec{n}\, ds$, over the parabola, is $-22/3$; over the line, it is 10. The total line integral is thus $-22/3 + 10 = 8/3$. The double integral of div $\vec{F} = 2$ over R also has value $8/3$.
22. Both the line integral and double integral have value of 0.

23. Three line integrals need to be computed to compute $\oint_C \vec{F}\cdot\vec{n}\,ds$. It does not matter which corner one starts from first, but be sure to proceed around the triangle in a counterclockwise fashion.

    From $(0,0)$ to $(2,0)$, the line integral has a value of 0. From $(2,0)$ to $(1,1)$ the integral has a value of $1/3$. From $(1,1)$ to $(0,0)$ the line integral has a value of $1/3$. Total value is $2/3$.

    The double integral of div $\vec{F}$ over R also has value $2/3$.

24. Two line integrals need to be computed to compute $\oint_C \vec{F}\cdot\vec{n}\,ds$. Along the parabola, the line integral has value $159/20$. Along the line, the line integral has value 6. Together, the total value is $279/20$.

    The double integral of div $\vec{F}$ over R also has value $279/20$.

Section 14.5

No problems written.

Section 14.6

No problems written.