

Vector Geometry  
A Supplement for Math 1410

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This text is based on the chapter on vectors from the APEX Calculus textbook by Hartman et al ([www.apexcalculus.com](http://www.apexcalculus.com)) and compiled in this form by Sean Fitzpatrick.

Since this material originated in a Calculus textbook, some of the emphasis and notation will differ from your Linear Algebra textbook; however, most of the material here is covered in Math 1410, and it shouldn't be too difficult to make the connections between this text and the course textbook.

If you read this text using Adobe Reader, you should be able to interact with all of the 3D graphics in the text. This may be of help as you attempt to visualize lines and planes in three dimensions.



# 1: VECTORS

## 1.1 An Introduction to Vectors

Many quantities we think about daily can be described by a single number: temperature, speed, cost, weight and height. There are also many other concepts we encounter daily that cannot be described with just one number. For instance, a weather forecaster often describes wind with its speed and its direction (“... with winds from the southeast gusting up to 30 mph ...”). When applying a force, we are concerned with both the magnitude and direction of that force. In both of these examples, *direction* is important. Because of this, we study *vectors*, mathematical objects that convey both magnitude and direction information.

One “bare-bones” definition of a vector is based on what we wrote above: “a vector is a mathematical object with magnitude and direction parameters.” This definition leaves much to be desired, as it gives no indication as to how such an object is to be used. Several other definitions exist; we choose here a definition rooted in a geometric visualization of vectors. It is very simplistic but readily permits further investigation.

### Definition 1 Vector

A **vector** is a directed line segment.

Given points  $P$  and  $Q$  (either in the plane or in space), we denote with  $\vec{PQ}$  the vector from  $P$  to  $Q$ . The point  $P$  is said to be the **initial point** of the vector, and the point  $Q$  is the **terminal point**.

The **magnitude, length or norm** of  $\vec{PQ}$  is the length of the line segment  $\overline{PQ}$ :  $\|\vec{PQ}\| = \|\overline{PQ}\|$ .

Two vectors are **equal** if they have the same magnitude and direction.

Figure 1.1 shows multiple instances of the same vector. Each directed line segment has the same direction and length (magnitude), hence each is the same vector.

We use  $\mathbb{R}^2$  (pronounced “r two”) to represent all the vectors in the plane, and use  $\mathbb{R}^3$  (pronounced “r three”) to represent all the vectors in space.

Consider the vectors  $\vec{PQ}$  and  $\vec{RS}$  as shown in Figure 1.2. The vectors look to be equal; that is, they seem to have the same length and direction. Indeed, they are. Both vectors move 2 units to the right and 1 unit up from the initial point to reach the terminal point. One can analyze this movement to measure the magnitude of the vector, and the movement itself gives direction information (one could also measure the slope of the line passing through  $P$  and  $Q$  or  $R$  and  $S$ ). Since they have the same length and direction, these two vectors are equal.

This demonstrates that inherently all we care about is *displacement*; that is, how far in the  $x$ ,  $y$  and possibly  $z$  directions the terminal point is from the initial point. Both the vectors  $\vec{PQ}$  and  $\vec{RS}$  in Figure 1.2 have an  $x$ -displacement of 2 and a  $y$ -displacement of 1. This suggests a standard way of describing vectors in the plane. A vector whose  $x$ -displacement is  $a$  and whose  $y$ -displacement is  $b$  will have terminal point  $(a, b)$  when the initial point is the origin,  $(0, 0)$ . This leads

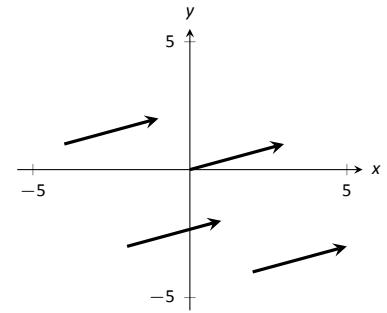


Figure 1.1: Drawing the same vector with different initial points.

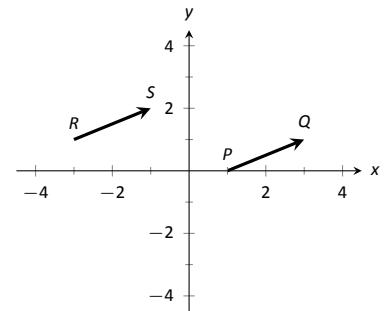


Figure 1.2: Illustrating how equal vectors have the same displacement.

us to a definition of a standard and concise way of referring to vectors.

### Definition 2 Component Form of a Vector

1. The **component form** of a vector  $\vec{v}$  in  $\mathbb{R}^2$ , whose terminal point is  $(a, b)$  when its initial point is  $(0, 0)$ , is  $\langle a, b \rangle$ .
2. The **component form** of a vector  $\vec{v}$  in  $\mathbb{R}^3$ , whose terminal point is  $(a, b, c)$  when its initial point is  $(0, 0, 0)$ , is  $\langle a, b, c \rangle$ .

The numbers  $a, b$  (and  $c$ , respectively) are the **components** of  $\vec{v}$ .

It follows from the definition that the component form of the vector  $\overrightarrow{PQ}$ , where  $P = (x_1, y_1)$  and  $Q = (x_2, y_2)$  is

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle;$$

in space, where  $P = (x_1, y_1, z_1)$  and  $Q = (x_2, y_2, z_2)$ , the component form of  $\overrightarrow{PQ}$  is

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

We practice using this notation in the following example.

### Example 1 Using component form notation for vectors

1. Sketch the vector  $\vec{v} = \langle 2, -1 \rangle$  starting at  $P = (3, 2)$  and find its magnitude.
2. Find the component form of the vector  $\vec{w}$  whose initial point is  $R = (-3, -2)$  and whose terminal point is  $S = (-1, 2)$ .
3. Sketch the vector  $\vec{u} = \langle 2, -1, 3 \rangle$  starting at the point  $Q = (1, 1, 1)$  and find its magnitude.

#### SOLUTION

1. Using  $P$  as the initial point, we move 2 units in the positive  $x$ -direction and  $-1$  units in the positive  $y$ -direction to arrive at the terminal point  $P' = (5, 1)$ , as drawn in Figure 1.3(a).

The magnitude of  $\vec{v}$  is determined directly from the component form:

$$\|\vec{v}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}.$$

2. Using the note following Definition 2, we have

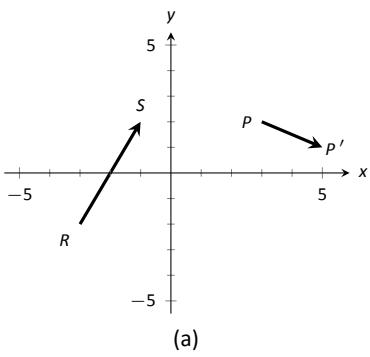
$$\overrightarrow{RS} = \langle -1 - (-3), 2 - (-2) \rangle = \langle 2, 4 \rangle.$$

One can readily see from Figure 1.3(a) that the  $x$ - and  $y$ -displacement of  $\overrightarrow{RS}$  is 2 and 4, respectively, as the component form suggests.

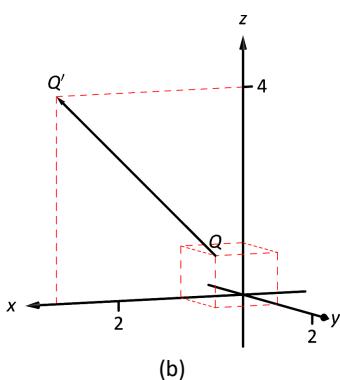
3. Using  $Q$  as the initial point, we move 2 units in the positive  $x$ -direction,  $-1$  unit in the positive  $y$ -direction, and 3 units in the positive  $z$ -direction to arrive at the terminal point  $Q' = (3, 0, 4)$ , illustrated in Figure 1.3(b).

The magnitude of  $\vec{u}$  is:

$$\|\vec{u}\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}.$$



(a)



(b)

Figure 1.3: Graphing vectors in Example 1.



Now that we have defined vectors, and have created a nice notation by which to describe them, we start considering how vectors interact with each other. That is, we define an *algebra* on vectors.

### Definition 3 Vector Algebra

- Let  $\vec{u} = \langle u_1, u_2 \rangle$  and  $\vec{v} = \langle v_1, v_2 \rangle$  be vectors in  $\mathbb{R}^2$ , and let  $c$  be a scalar.

- The addition, or sum, of the vectors  $\vec{u}$  and  $\vec{v}$  is the vector

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle.$$

- The scalar product of  $c$  and  $\vec{v}$  is the vector

$$c\vec{v} = c \langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle.$$

- Let  $\vec{u} = \langle u_1, u_2, u_3 \rangle$  and  $\vec{v} = \langle v_1, v_2, v_3 \rangle$  be vectors in  $\mathbb{R}^3$ , and let  $c$  be a scalar.

- The addition, or sum, of the vectors  $\vec{u}$  and  $\vec{v}$  is the vector

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle.$$

- The scalar product of  $c$  and  $\vec{v}$  is the vector

$$c\vec{v} = c \langle v_1, v_2, v_3 \rangle = \langle cv_1, cv_2, cv_3 \rangle.$$

In short, we say addition and scalar multiplication are computed “component-wise.”

### Example 2 Adding vectors

Sketch the vectors  $\vec{u} = \langle 1, 3 \rangle$ ,  $\vec{v} = \langle 2, 1 \rangle$  and  $\vec{u} + \vec{v}$  all with initial point at the origin.

**SOLUTION** We first compute  $\vec{u} + \vec{v}$ .

$$\begin{aligned}\vec{u} + \vec{v} &= \langle 1, 3 \rangle + \langle 2, 1 \rangle \\ &= \langle 3, 4 \rangle.\end{aligned}$$

These are all sketched in Figure 1.4.

As vectors convey magnitude and direction information, the sum of vectors also convey length and magnitude information. Adding  $\vec{u} + \vec{v}$  suggests the following idea:

“Starting at an initial point, go out  $\vec{u}$ , then go out  $\vec{v}$ .”

This idea is sketched in Figure 1.5, where the initial point of  $\vec{v}$  is the terminal point of  $\vec{u}$ . This is known as the “Head to Tail Rule” of adding vectors. Vector addition is very important. For instance, if the vectors  $\vec{u}$  and  $\vec{v}$  represent forces acting on a body, the sum  $\vec{u} + \vec{v}$  gives the resulting force. Because of various physical applications of vector addition, the sum  $\vec{u} + \vec{v}$  is often referred to as the **resultant vector**, or just the “resultant.”

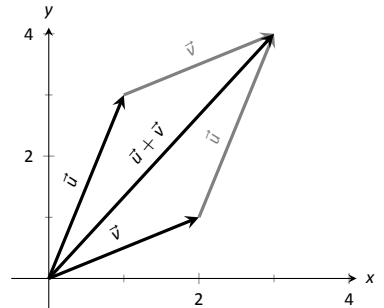


Figure 1.5: Illustrating how to add vectors using the Head to Tail Rule and Parallelogram Law.

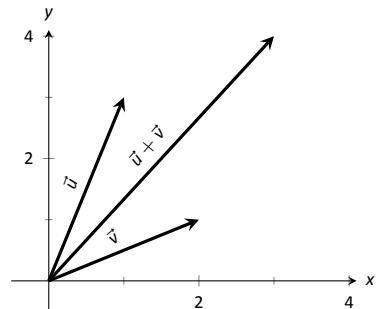


Figure 1.4: Graphing the sum of vectors in Example 2.

Analytically, it is easy to see that  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ . Figure 1.5 also gives a graphical representation of this, using gray vectors. Note that the vectors  $\vec{u}$  and  $\vec{v}$ , when arranged as in the figure, form a parallelogram. Because of this, the Head to Tail Rule is also known as the Parallelogram Law: the vector  $\vec{u} + \vec{v}$  is defined by forming the parallelogram defined by the vectors  $\vec{u}$  and  $\vec{v}$ ; the initial point of  $\vec{u} + \vec{v}$  is the common initial point of parallelogram, and the terminal point of the sum is the common terminal point of the parallelogram.

While not illustrated here, the Head to Tail Rule and Parallelogram Law hold for vectors in  $\mathbb{R}^3$  as well.

It follows from the properties of the real numbers and Definition 3 that

$$\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}.$$

The Parallelogram Law gives us a good way to visualize this subtraction. We demonstrate this in the following example.

### Example 3 Vector Subtraction

Let  $\vec{u} = \langle 3, 1 \rangle$  and  $\vec{v} = \langle 1, 2 \rangle$ . Compute and sketch  $\vec{u} - \vec{v}$ .

**SOLUTION** The computation of  $\vec{u} - \vec{v}$  is straightforward, and we show all steps below. Usually the formal step of multiplying by  $(-1)$  is omitted and we “just subtract.”

$$\begin{aligned}\vec{u} - \vec{v} &= \vec{u} + (-1)\vec{v} \\ &= \langle 3, 1 \rangle + \langle -1, -2 \rangle \\ &= \langle 2, -1 \rangle.\end{aligned}$$

Figure 1.6 illustrates, using the Head to Tail Rule, how the subtraction can be viewed as the sum  $\vec{u} + (-\vec{v})$ . The figure also illustrates how  $\vec{u} - \vec{v}$  can be obtained by looking only at the terminal points of  $\vec{u}$  and  $\vec{v}$  (when their initial points are the same).

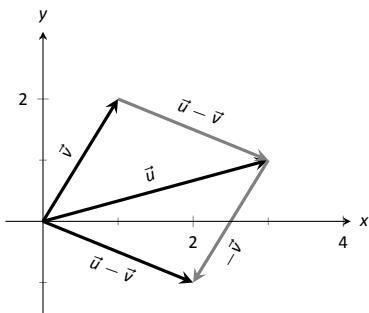


Figure 1.6: Illustrating how to subtract vectors graphically.

### Example 4 Scaling vectors

1. Sketch the vectors  $\vec{v} = \langle 2, 1 \rangle$  and  $2\vec{v}$  with initial point at the origin.
2. Compute the magnitudes of  $\vec{v}$  and  $2\vec{v}$ .

#### SOLUTION

1. We compute  $2\vec{v}$ :

$$\begin{aligned}2\vec{v} &= 2 \langle 2, 1 \rangle \\ &= \langle 4, 2 \rangle.\end{aligned}$$

Both  $\vec{v}$  and  $2\vec{v}$  are sketched in Figure 1.7. Make note that  $2\vec{v}$  does not start at the terminal point of  $\vec{v}$ ; rather, its initial point is also the origin.

2. The figure suggests that  $2\vec{v}$  is twice as long as  $\vec{v}$ . We compute their magnitudes to confirm this.

$$\begin{aligned}||\vec{v}|| &= \sqrt{2^2 + 1^2} \\ &= \sqrt{5}. \\ ||2\vec{v}|| &= \sqrt{4^2 + 2^2} \\ &= \sqrt{20} \\ &= \sqrt{4 \cdot 5} = 2\sqrt{5}.\end{aligned}$$

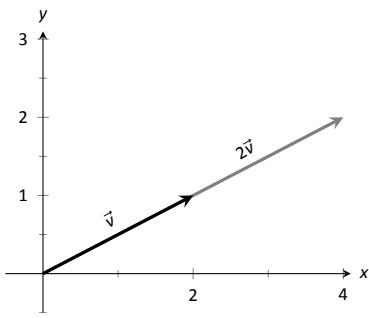


Figure 1.7: Graphing vectors  $\vec{v}$  and  $2\vec{v}$  in Example 4.

As we suspected,  $2\vec{v}$  is twice as long as  $\vec{v}$ .

The **zero vector** is the vector whose initial point is also its terminal point. It is denoted by  $\vec{0}$ . Its component form, in  $\mathbb{R}^2$ , is  $\langle 0, 0 \rangle$ ; in  $\mathbb{R}^3$ , it is  $\langle 0, 0, 0 \rangle$ . Usually the context makes it clear whether  $\vec{0}$  is referring to a vector in the plane or in space.

Our examples have illustrated key principles in vector algebra: how to add and subtract vectors and how to multiply vectors by a scalar. The following theorem states formally the properties of these operations.

### Theorem 1 Properties of Vector Operations

The following are true for all scalars  $c$  and  $d$ , and for all vectors  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$ , where  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  are all in  $\mathbb{R}^2$  or where  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  are all in  $\mathbb{R}^3$ :

1.  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$  Commutative Property
2.  $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$  Associative Property
3.  $\vec{v} + \vec{0} = \vec{v}$  Additive Identity
4.  $(cd)\vec{v} = c(d\vec{v})$
5.  $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$  Distributive Property
6.  $(c + d)\vec{v} = c\vec{v} + d\vec{v}$  Distributive Property
7.  $0\vec{v} = \vec{0}$
8.  $\|\vec{cv}\| = |c| \cdot \|\vec{v}\|$
9.  $\|\vec{u}\| = 0$  if, and only if,  $\vec{u} = \vec{0}$ .

As stated before, each vector  $\vec{v}$  conveys magnitude and direction information. We have a method of extracting the magnitude, which we write as  $\|\vec{v}\|$ . *Unit vectors* are a way of extracting just the direction information from a vector.

### Definition 4 Unit Vector

A **unit vector** is a vector  $\vec{v}$  with a magnitude of 1; that is,

$$\|\vec{v}\| = 1.$$

Consider this scenario: you are given a vector  $\vec{v}$  and are told to create a vector of length 10 in the direction of  $\vec{v}$ . How does one do that? If we knew that  $\vec{u}$  was the unit vector in the direction of  $\vec{v}$ , the answer would be easy:  $10\vec{u}$ . So how do we find  $\vec{u}$ ?

Property 8 of Theorem 1 holds the key. If we divide  $\vec{v}$  by its magnitude, it becomes a vector of length 1. Consider:

$$\begin{aligned} \left\| \frac{1}{\|\vec{v}\|} \vec{v} \right\| &= \frac{1}{\|\vec{v}\|} \|\vec{v}\| && \text{(we can pull out } \frac{1}{\|\vec{v}\|} \text{ as it is a scalar)} \\ &= 1. \end{aligned}$$

So the vector of length 10 in the direction of  $\vec{v}$  is  $10 \frac{1}{\|\vec{v}\|} \vec{v}$ . An example will make this more clear.

### Example 5 Using Unit Vectors

Let  $\vec{v} = \langle 3, 1 \rangle$  and let  $\vec{w} = \langle 1, 2, 2 \rangle$ .

1. Find the unit vector in the direction of  $\vec{v}$ .
2. Find the unit vector in the direction of  $\vec{w}$ .
3. Find the vector in the direction of  $\vec{v}$  with magnitude 5.

#### SOLUTION

1. We find  $\|\vec{v}\| = \sqrt{10}$ . So the unit vector  $\vec{u}$  in the direction of  $\vec{v}$  is

$$\vec{u} = \frac{1}{\sqrt{10}} \vec{v} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle.$$

2. We find  $\|\vec{w}\| = 3$ , so the unit vector  $\vec{z}$  in the direction of  $\vec{w}$  is

$$\vec{z} = \frac{1}{3} \vec{w} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle.$$

3. To create a vector with magnitude 5 in the direction of  $\vec{v}$ , we multiply the unit vector  $\vec{u}$  by 5. Thus  $5\vec{u} = \langle 15/\sqrt{10}, 5/\sqrt{10} \rangle$  is the vector we seek. This is sketched in Figure 1.8.

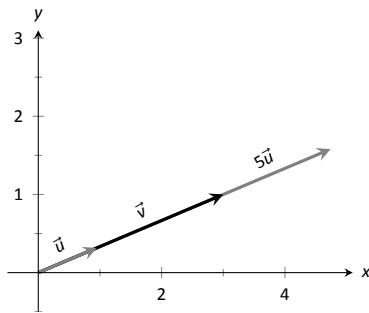


Figure 1.8: Graphing vectors in Example 5. All vectors shown have their initial point at the origin.

The basic formation of the unit vector  $\vec{u}$  in the direction of a vector  $\vec{v}$  leads to a interesting equation. It is:

$$\vec{v} = \|\vec{v}\| \frac{1}{\|\vec{v}\|} \vec{v}.$$

We rewrite the equation with parentheses to make a point:

$$\vec{v} = \underbrace{\|\vec{v}\|}_{\text{magnitude}} \cdot \underbrace{\left( \frac{1}{\|\vec{v}\|} \vec{v} \right)}_{\text{direction}}.$$

This equation illustrates the fact that a vector has both magnitude and direction, where we view a unit vector as supplying *only* direction information. Identifying unit vectors with direction allows us to define **parallel vectors**.

### Definition 5 Parallel Vectors

1. Unit vectors  $\vec{u}_1$  and  $\vec{u}_2$  are **parallel** if  $\vec{u}_1 = \pm \vec{u}_2$ .
2. Nonzero vectors  $\vec{v}_1$  and  $\vec{v}_2$  are **parallel** if their respective unit vectors are parallel.

It is equivalent to say that vectors  $\vec{v}_1$  and  $\vec{v}_2$  are parallel if there is a scalar  $c \neq 0$  such that  $\vec{v}_1 = c\vec{v}_2$  (see marginal note).

If one graphed all unit vectors in  $\mathbb{R}^2$  with the initial point at the origin, then the terminal points would all lie on the unit circle. Based on what we know from trigonometry, we can then say that the component form of all unit vectors in  $\mathbb{R}^2$  is  $\langle \cos \theta, \sin \theta \rangle$  for some angle  $\theta$ .

A similar construction in  $\mathbb{R}^3$  shows that the terminal points all lie on the unit sphere. These vectors also have a particular component form, but its derivation is not as straightforward as the one for unit vectors in  $\mathbb{R}^2$ . Important concepts about unit vectors are given in the following Key Idea.

### Key Idea 1 Unit Vectors

1. The unit vector in the direction of  $\vec{v}$  is

$$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v}.$$

2. A vector  $\vec{u}$  in  $\mathbb{R}^2$  is a unit vector if, and only if, its component form is  $\langle \cos \theta, \sin \theta \rangle$  for some angle  $\theta$ .
3. A vector  $\vec{u}$  in  $\mathbb{R}^3$  is a unit vector if, and only if, its component form is  $\langle \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta \rangle$  for some angles  $\theta$  and  $\varphi$ .

These formulas can come in handy in a variety of situations, especially the formula for unit vectors in the plane.

### Example 6 Finding Component Forces

Consider a weight of 50lb hanging from two chains, as shown in Figure 1.9. One chain makes an angle of  $30^\circ$  with the vertical, and the other an angle of  $45^\circ$ . Find the force applied to each chain.

**SOLUTION** Knowing that gravity is pulling the 50lb weight straight down, we can create a vector  $\vec{F}$  to represent this force.

$$\vec{F} = 50 \langle 0, -1 \rangle = \langle 0, -50 \rangle.$$

We can view each chain as “pulling” the weight up, preventing it from falling. We can represent the force from each chain with a vector. Let  $\vec{F}_1$  represent the force from the chain making an angle of  $30^\circ$  with the vertical, and let  $\vec{F}_2$  represent the force from the other chain. Convert all angles to be measured from the horizontal (as shown in Figure 1.10), and apply Key Idea 1. As we do not yet know the magnitudes of these vectors, (that is the problem at hand), we use  $m_1$  and  $m_2$  to represent them.

$$\vec{F}_1 = m_1 \langle \cos 120^\circ, \sin 120^\circ \rangle$$

$$\vec{F}_2 = m_2 \langle \cos 45^\circ, \sin 45^\circ \rangle$$

As the weight is not moving, we know the sum of the forces is  $\vec{0}$ . This gives:

$$\vec{F} + \vec{F}_1 + \vec{F}_2 = \vec{0}$$

$$\langle 0, -50 \rangle + m_1 \langle \cos 120^\circ, \sin 120^\circ \rangle + m_2 \langle \cos 45^\circ, \sin 45^\circ \rangle = \vec{0}$$

The sum of the entries in the first component is 0, and the sum of the entries in the second component is also 0. This leads us to the following two equations:

$$m_1 \cos 120^\circ + m_2 \cos 45^\circ = 0$$

$$m_1 \sin 120^\circ + m_2 \sin 45^\circ = 50$$

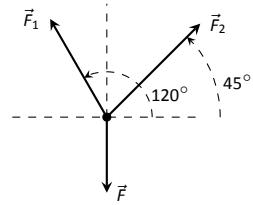


Figure 1.10: A diagram of the force vectors from Example 6.

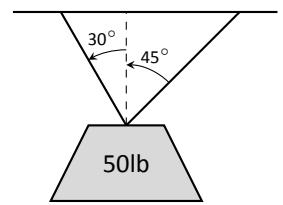


Figure 1.9: A diagram of a weight hanging from 2 chains in Example 6.

This is a simple 2-equation, 2-unkown system of linear equations. We leave it to the reader to verify that the solution is

$$m_1 = 50(\sqrt{3} - 1) \approx 36.6; \quad m_2 = \frac{50\sqrt{2}}{1 + \sqrt{3}} \approx 25.88.$$

It might seem odd that the sum of the forces applied to the chains is more than 50lb. We leave it to a physics class to discuss the full details, but offer this short explanation. Our equations were established so that the *vertical* components of each force sums to 50lb, thus supporting the weight. Since the chains are at an angle, they also pull against each other, creating an “additional” horizontal force while holding the weight in place.

Unit vectors were very important in the previous calculation; they allowed us to define a vector in the proper direction but with an unknown magnitude. Our computations were then computed component-wise. Because such calculations are often necessary, the *standard unit vectors* can be useful.

#### Definition 6 Standard Unit Vectors

1. In  $\mathbb{R}^2$ , the standard unit vectors are

$$\vec{i} = \langle 1, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1 \rangle.$$

2. In  $\mathbb{R}^3$ , the standard unit vectors are

$$\vec{i} = \langle 1, 0, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1, 0 \rangle \quad \text{and} \quad \vec{k} = \langle 0, 0, 1 \rangle.$$

#### Example 7 Using standard unit vectors

1. Rewrite  $\vec{v} = \langle 2, -3 \rangle$  using the standard unit vectors.

2. Rewrite  $\vec{w} = 4\vec{i} - 5\vec{j} + 2\vec{k}$  in component form.

#### SOLUTION

- 1.

$$\begin{aligned}\vec{v} &= \langle 2, -3 \rangle \\ &= \langle 2, 0 \rangle + \langle 0, -3 \rangle \\ &= 2\langle 1, 0 \rangle - 3\langle 0, 1 \rangle \\ &= 2\vec{i} - 3\vec{j}\end{aligned}$$

- 2.

$$\begin{aligned}\vec{w} &= 4\vec{i} - 5\vec{j} + 2\vec{k} \\ &= \langle 4, 0, 0 \rangle + \langle 0, -5, 0 \rangle + \langle 0, 0, 2 \rangle \\ &= \langle 4, -5, 2 \rangle\end{aligned}$$

These two examples demonstrate that converting between component form and the standard unit vectors is rather straightforward. Many mathematicians prefer component form, and it is the preferred notation in this text. Many engineers prefer using the standard unit vectors, and many engineering text use that notation.

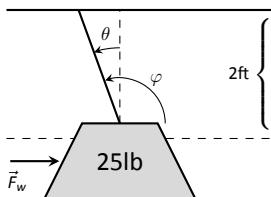


Figure 1.11: A figure of a weight being pushed by the wind in Example 8.

**Example 8 Finding Component Force**

A weight of 25lb is suspended from a chain of length 2ft while a wind pushes the weight to the right with constant force of 5lb as shown in Figure 1.11. What angle will the chain make with the vertical as a result of the wind's pushing? How much higher will the weight be?

**SOLUTION** The force of the wind is represented by the vector  $\vec{F}_w = 5\vec{i}$ . The force of gravity on the weight is represented by  $\vec{F}_g = -25\vec{j}$ . The direction and magnitude of the vector representing the force on the chain are both unknown. We represent this force with

$$\vec{F}_c = m \langle \cos \varphi, \sin \varphi \rangle = m \cos \varphi \vec{i} + m \sin \varphi \vec{j}$$

for some magnitude  $m$  and some angle with the horizontal  $\varphi$ . (Note:  $\theta$  is the angle the chain makes with the *vertical*;  $\varphi$  is the angle with the *horizontal*.)

As the weight is at equilibrium, the sum of the forces is  $\vec{0}$ :

$$\begin{aligned}\vec{F}_c + \vec{F}_w + \vec{F}_g &= \vec{0} \\ m \cos \varphi \vec{i} + m \sin \varphi \vec{j} + 5\vec{i} - 25\vec{j} &= \vec{0}\end{aligned}$$

Thus the sum of the  $\vec{i}$  and  $\vec{j}$  components are 0, leading us to the following system of equations:

$$\begin{aligned}5 + m \cos \varphi &= 0 \\ -25 + m \sin \varphi &= 0\end{aligned}\tag{1.1}$$

This is enough to determine  $\vec{F}_c$  already, as we know  $m \cos \varphi = -5$  and  $m \sin \varphi = 25$ . Thus  $F_c = \langle -5, 25 \rangle$ . We can use this to find the magnitude  $m$ :

$$m = \sqrt{(-5)^2 + 25^2} = 5\sqrt{26} \approx 25.5\text{lb}.$$

We can then use either equality from Equation (1.1) to solve for  $\varphi$ . We choose the first equality as using arccosine will return an angle in the 2<sup>nd</sup> quadrant:

$$5 + 5\sqrt{26} \cos \varphi = 0 \Rightarrow \varphi = \cos^{-1} \left( \frac{-5}{5\sqrt{26}} \right) \approx 1.7682 \approx 101.31^\circ.$$

Subtracting  $90^\circ$  from this angle gives us an angle of  $11.31^\circ$  with the vertical.

We can now use trigonometry to find out how high the weight is lifted. The diagram shows that a right triangle is formed with the 2ft chain as the hypotenuse with an interior angle of  $11.31^\circ$ . The length of the adjacent side (in the diagram, the dashed vertical line) is  $2 \cos 11.31^\circ \approx 1.96\text{ft}$ . Thus the weight is lifted by about 0.04ft, almost 1/2in.

The algebra we have applied to vectors is already demonstrating itself to be very useful. There are two more fundamental operations we can perform with vectors, the *dot product* and the *cross product*. The next two sections explore each in turn.

# Exercises 1.1

## Terms and Concepts

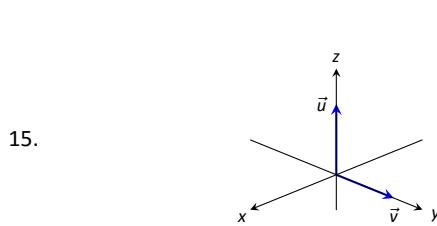
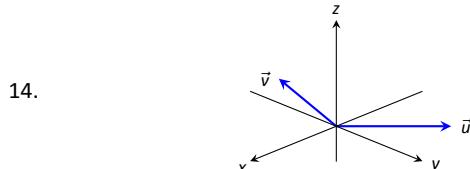
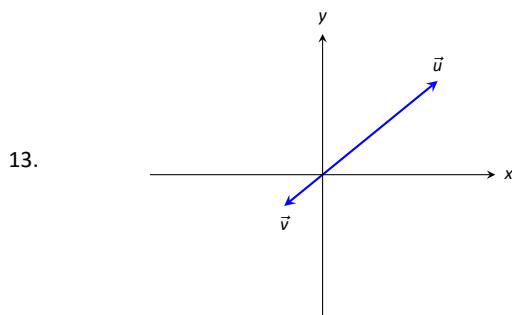
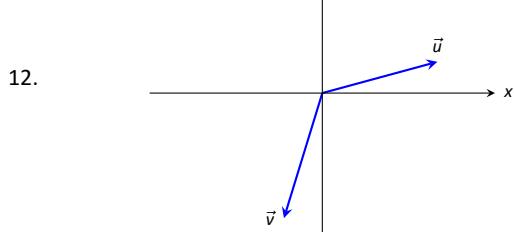
1. Name two different things that cannot be described with just one number, but rather need 2 or more numbers to fully describe them.
2. What is the difference between  $(1, 2)$  and  $\langle 1, 2 \rangle$ ?
3. What is a unit vector?
4. What does it mean for two vectors to be parallel?
5. What effect does multiplying a vector by  $-2$  have?

## Problems

In Exercises 6 – 9, points  $P$  and  $Q$  are given. Write the vector  $\vec{PQ}$  in component form and using the standard unit vectors.

6.  $P = (2, -1)$ ,  $Q = (3, 5)$
7.  $P = (3, 2)$ ,  $Q = (7, -2)$
8.  $P = (0, 3, -1)$ ,  $Q = (6, 2, 5)$
9.  $P = (2, 1, 2)$ ,  $Q = (4, 3, 2)$
10. Let  $\vec{u} = \langle 1, -2 \rangle$  and  $\vec{v} = \langle 1, 1 \rangle$ .
  - (a) Find  $\vec{u} + \vec{v}$ ,  $\vec{u} - \vec{v}$ ,  $2\vec{u} - 3\vec{v}$ .
  - (b) Sketch the above vectors on the same axes, along with  $\vec{u}$  and  $\vec{v}$ .
  - (c) Find  $\vec{x}$  where  $\vec{u} + \vec{x} = 2\vec{v} - \vec{x}$ .
11. Let  $\vec{u} = \langle 1, 1, -1 \rangle$  and  $\vec{v} = \langle 2, 1, 2 \rangle$ .
  - (a) Find  $\vec{u} + \vec{v}$ ,  $\vec{u} - \vec{v}$ ,  $\pi\vec{u} - \sqrt{2}\vec{v}$ .
  - (b) Sketch the above vectors on the same axes, along with  $\vec{u}$  and  $\vec{v}$ .
  - (c) Find  $\vec{x}$  where  $\vec{u} + \vec{x} = \vec{v} + 2\vec{x}$ .

In Exercises 12 – 15, sketch  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{u} + \vec{v}$  and  $\vec{u} - \vec{v}$  on the same axes.



In Exercises 16 – 19, find  $\|\vec{u}\|$ ,  $\|\vec{v}\|$ ,  $\|\vec{u} + \vec{v}\|$  and  $\|\vec{u} - \vec{v}\|$ .

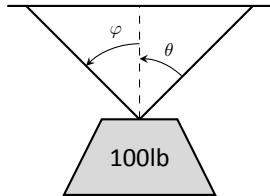
16.  $\vec{u} = \langle 2, 1 \rangle$ ,  $\vec{v} = \langle 3, -2 \rangle$
17.  $\vec{u} = \langle -3, 2, 2 \rangle$ ,  $\vec{v} = \langle 1, -1, 1 \rangle$
18.  $\vec{u} = \langle 1, 2 \rangle$ ,  $\vec{v} = \langle -3, -6 \rangle$
19.  $\vec{u} = \langle 2, -3, 6 \rangle$ ,  $\vec{v} = \langle 10, -15, 30 \rangle$
20. Under what conditions is  $\|\vec{u}\| + \|\vec{v}\| = \|\vec{u} + \vec{v}\|$ ?

In Exercises 21 – 24, find the unit vector  $\vec{u}$  in the direction of  $\vec{v}$ .

21.  $\vec{v} = \langle 3, 7 \rangle$
22.  $\vec{v} = \langle 6, 8 \rangle$
23.  $\vec{v} = \langle 1, -2, 2 \rangle$
24.  $\vec{v} = \langle 2, -2, 2 \rangle$
25. Find the unit vector in the first quadrant of  $\mathbb{R}^2$  that makes a  $50^\circ$  angle with the  $x$ -axis.

26. Find the unit vector in the second quadrant of  $\mathbb{R}^2$  that makes a  $30^\circ$  angle with the  $y$ -axis.
27. Verify, from Key Idea 1, that  $\vec{u} = \langle \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta \rangle$  is a unit vector for all angles  $\theta$  and  $\varphi$ .
31.  $\theta = 0^\circ, \varphi = 0^\circ$

A weight of 100lb is suspended from two chains, making angles with the vertical of  $\theta$  and  $\varphi$  as shown in the figure below.



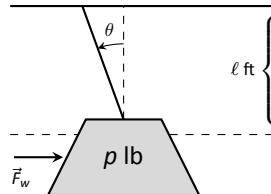
In Exercises 28 – 31, angles  $\theta$  and  $\varphi$  are given. Find the force applied to each chain.

28.  $\theta = 30^\circ, \varphi = 30^\circ$

29.  $\theta = 60^\circ, \varphi = 60^\circ$

30.  $\theta = 20^\circ, \varphi = 15^\circ$

A weight of  $p$ lb is suspended from a chain of length  $\ell$  while a constant force of  $\vec{F}_w$  pushes the weight to the right, making an angle of  $\theta$  with the vertical, as shown in the figure below.



In Exercises 32 – 35, a force  $\vec{F}_w$  and length  $\ell$  are given. Find the angle  $\theta$  and the height the weight is lifted as it moves to the right.

32.  $\vec{F}_w = 1\text{lb}, \ell = 1\text{ft}, p = 1\text{lb}$

33.  $\vec{F}_w = 1\text{lb}, \ell = 1\text{ft}, p = 10\text{lb}$

34.  $\vec{F}_w = 1\text{lb}, \ell = 10\text{ft}, p = 1\text{lb}$

35.  $\vec{F}_w = 10\text{lb}, \ell = 10\text{ft}, p = 1\text{lb}$

## 1.2 The Dot Product

The previous section introduced vectors and described how to add them together and how to multiply them by scalars. This section introduces a multiplication on vectors called the **dot product**.

### Definition 7 Dot Product

- Let  $\vec{u} = \langle u_1, u_2 \rangle$  and  $\vec{v} = \langle v_1, v_2 \rangle$  in  $\mathbb{R}^2$ . The **dot product** of  $\vec{u}$  and  $\vec{v}$ , denoted  $\vec{u} \cdot \vec{v}$ , is

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2.$$

- Let  $\vec{u} = \langle u_1, u_2, u_3 \rangle$  and  $\vec{v} = \langle v_1, v_2, v_3 \rangle$  in  $\mathbb{R}^3$ . The **dot product** of  $\vec{u}$  and  $\vec{v}$ , denoted  $\vec{u} \cdot \vec{v}$ , is

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Note how this product of vectors returns a *scalar*, not another vector. We practice evaluating a dot product in the following example, then we will discuss why this product is useful.

### Example 9 Evaluating dot products

- Let  $\vec{u} = \langle 1, 2 \rangle$ ,  $\vec{v} = \langle 3, -1 \rangle$  in  $\mathbb{R}^2$ . Find  $\vec{u} \cdot \vec{v}$ .

- Let  $\vec{x} = \langle 2, -2, 5 \rangle$  and  $\vec{y} = \langle -1, 0, 3 \rangle$  in  $\mathbb{R}^3$ . Find  $\vec{x} \cdot \vec{y}$ .

### SOLUTION

- Using Definition 7, we have

$$\vec{u} \cdot \vec{v} = 1(3) + 2(-1) = 1.$$

- Using the definition, we have

$$\vec{x} \cdot \vec{y} = 2(-1) - 2(0) + 5(3) = 13.$$

The dot product, as shown by the preceding example, is very simple to evaluate. It is only the sum of products. While the definition gives no hint as to why we would care about this operation, there is an amazing connection between the dot product and angles formed by the vectors. Before stating this connection, we give a theorem stating some of the properties of the dot product.

**Theorem 2 Properties of the Dot Product**

Let  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  be vectors in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  and let  $c$  be a scalar.

1.  $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$  Commutative Property
2.  $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$  Distributive Property
3.  $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
4.  $\vec{0} \cdot \vec{v} = 0$
5.  $\vec{v} \cdot \vec{v} = ||\vec{v}||^2$

The last statement of the theorem makes a handy connection between the magnitude of a vector and the dot product with itself. Our definition and theorem give properties of the dot product, but we are still likely wondering “What does the dot product *mean*?” It is helpful to understand that the dot product of a vector with itself is connected to its magnitude.

The next theorem extends this understanding by connecting the dot product to magnitudes and angles. Given vectors  $\vec{u}$  and  $\vec{v}$  in the plane, an angle  $\theta$  is clearly formed when  $\vec{u}$  and  $\vec{v}$  are drawn with the same initial point as illustrated in Figure 1.13(a). (We always take  $\theta$  to be the angle in  $[0, \pi]$  as two angles are actually created.)

The same is also true of 2 vectors in space: given  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  with the same initial point, there is a plane that contains both  $\vec{u}$  and  $\vec{v}$ . (When  $\vec{u}$  and  $\vec{v}$  are collinear, there are infinite planes that contain both vectors.) In that plane, we can again find an angle  $\theta$  between them (and again,  $0 \leq \theta \leq \pi$ ). This is illustrated in Figure 1.13(b).

The following theorem connects this angle  $\theta$  to the dot product of  $\vec{u}$  and  $\vec{v}$ .

**Theorem 3 The Dot Product and Angles**

Let  $\vec{u}$  and  $\vec{v}$  be vectors in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . Then

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| ||\vec{v}|| \cos \theta,$$

where  $\theta$ ,  $0 \leq \theta \leq \pi$ , is the angle between  $\vec{u}$  and  $\vec{v}$ .

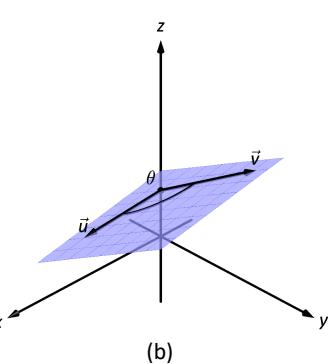
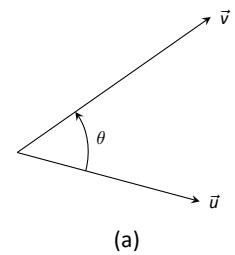


Figure 1.13: Illustrating the angle formed by two vectors with the same initial point.

When  $\theta$  is an acute angle (i.e.,  $0 \leq \theta < \pi/2$ ),  $\cos \theta$  is positive; when  $\theta = \pi/2$ ,  $\cos \theta = 0$ ; when  $\theta$  is an obtuse angle ( $\pi/2 < \theta \leq \pi$ ),  $\cos \theta$  is negative. Thus the sign of the dot product gives a general indication of the angle between the vectors, illustrated in Figure 1.12.

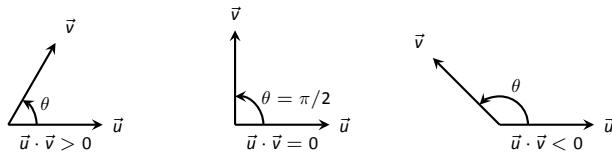


Figure 1.12: Illustrating the relationship between the angle between vectors and the sign of their dot product.

We can use Theorem 3 to compute the dot product, but generally this the-

orem is used to find the angle between known vectors (since the dot product is generally easy to compute). To this end, we rewrite the theorem's equation as

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \Leftrightarrow \theta = \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right).$$

We practice using this theorem in the following example.

**Example 10 Using the dot product to find angles**

Let  $\vec{u} = \langle 3, 1 \rangle$ ,  $\vec{v} = \langle -2, 6 \rangle$  and  $\vec{w} = \langle -4, 3 \rangle$ , as shown in Figure 1.14. Find the angles  $\alpha$ ,  $\beta$  and  $\theta$ .

**SOLUTION** We start by computing the magnitude of each vector.

$$\|\vec{u}\| = \sqrt{10}; \quad \|\vec{v}\| = 2\sqrt{10}; \quad \|\vec{w}\| = 5.$$

We now apply Theorem 3 to find the angles.

$$\begin{aligned} \alpha &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{(\sqrt{10})(2\sqrt{10})} \right) \\ &= \cos^{-1}(0) = \frac{\pi}{2} = 90^\circ. \end{aligned}$$

$$\begin{aligned} \beta &= \cos^{-1} \left( \frac{\vec{v} \cdot \vec{w}}{(2\sqrt{10})(5)} \right) \\ &= \cos^{-1} \left( \frac{26}{10\sqrt{10}} \right) \\ &\approx 0.6055 \approx 34.7^\circ. \end{aligned}$$

$$\begin{aligned} \theta &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{w}}{(\sqrt{10})(5)} \right) \\ &= \cos^{-1} \left( \frac{-9}{5\sqrt{10}} \right) \\ &\approx 2.1763 \approx 124.7^\circ \end{aligned}$$

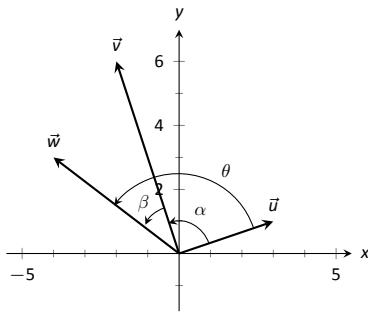


Figure 1.14: Vectors used in Example 10.

We see from our computation that  $\alpha + \beta = \theta$ , as indicated by Figure 1.14. While we knew this should be the case, it is nice to see that this non-intuitive formula indeed returns the results we expected.

We do a similar example next in the context of vectors in space.

**Example 11 Using the dot product to find angles**

Let  $\vec{u} = \langle 1, 1, 1 \rangle$ ,  $\vec{v} = \langle -1, 3, -2 \rangle$  and  $\vec{w} = \langle -5, 1, 4 \rangle$ , as illustrated in Figure 1.15. Find the angle between each pair of vectors.

**SOLUTION**

1. Between  $\vec{u}$  and  $\vec{v}$ :

$$\begin{aligned} \theta &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right) \\ &= \cos^{-1} \left( \frac{0}{\sqrt{3}\sqrt{14}} \right) \\ &= \frac{\pi}{2}. \end{aligned}$$

2. Between  $\vec{u}$  and  $\vec{w}$ :

$$\begin{aligned}\theta &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\| \|\vec{w}\|} \right) \\ &= \cos^{-1} \left( \frac{0}{\sqrt{3}\sqrt{42}} \right) \\ &= \frac{\pi}{2}.\end{aligned}$$

3. Between  $\vec{v}$  and  $\vec{w}$ :

$$\begin{aligned}\theta &= \cos^{-1} \left( \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} \right) \\ &= \cos^{-1} \left( \frac{0}{\sqrt{14}\sqrt{42}} \right) \\ &= \frac{\pi}{2}.\end{aligned}$$

While our work shows that each angle is  $\pi/2$ , i.e.,  $90^\circ$ , none of these angles looks to be a right angle in Figure 1.15. Such is the case when drawing three-dimensional objects on the page.

All three angles between these vectors was  $\pi/2$ , or  $90^\circ$ . We know from geometry and everyday life that  $90^\circ$  angles are “nice” for a variety of reasons, so it should seem significant that these angles are all  $\pi/2$ . Notice the common feature in each calculation (and also the calculation of  $\alpha$  in Example 10): the dot products of each pair of angles was 0. We use this as a basis for a definition of the term **orthogonal**, which is essentially synonymous to *perpendicular*.

### Definition 8 Orthogonal

Vectors  $\vec{u}$  and  $\vec{v}$  are **orthogonal** if their dot product is 0.

### Example 12 Finding orthogonal vectors

Let  $\vec{u} = \langle 3, 5 \rangle$  and  $\vec{v} = \langle 1, 2, 3 \rangle$ .

1. Find two vectors in  $\mathbb{R}^2$  that are orthogonal to  $\vec{u}$ .
2. Find two non-parallel vectors in  $\mathbb{R}^3$  that are orthogonal to  $\vec{v}$ .

### SOLUTION

1. Recall that a line perpendicular to a line with slope  $m$  has slope  $-1/m$ , the “opposite reciprocal slope.” We can think of the slope of  $\vec{u}$  as  $5/3$ , its “rise over run.” A vector orthogonal to  $\vec{u}$  will have slope  $-3/5$ . There are many such choices, though all parallel:

$$\langle -5, 3 \rangle \quad \text{or} \quad \langle 5, -3 \rangle \quad \text{or} \quad \langle -10, 6 \rangle \quad \text{or} \quad \langle 15, -9 \rangle, \text{ etc.}$$

2. There are infinite directions in space orthogonal to any given direction, so there are an infinite number of non-parallel vectors orthogonal to  $\vec{v}$ . Since there are so many, we have great leeway in finding some.

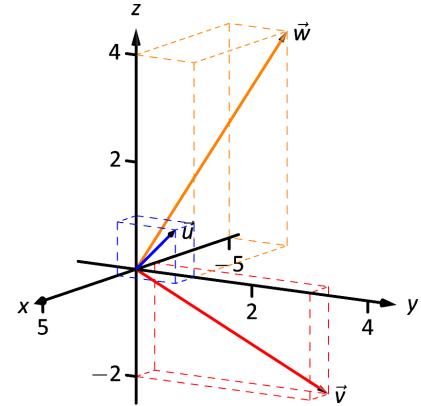


Figure 1.15: Vectors used in Example 11.

**Note:** The term *perpendicular* originally referred to lines. As mathematics progressed, the concept of “being at right angles to” was applied to other objects, such as vectors and planes, and the term *orthogonal* was introduced. It is especially used when discussing objects that are hard, or impossible, to visualize: two vectors in 5-dimensional space are orthogonal if their dot product is 0. It is not wrong to say they are *perpendicular*, but common convention gives preference to the word *orthogonal*.

One way is to arbitrarily pick values for the first two components, leaving the third unknown. For instance, let  $\vec{v}_1 = \langle 2, 7, z \rangle$ . If  $\vec{v}_1$  is to be orthogonal to  $\vec{v}$ , then  $\vec{v}_1 \cdot \vec{v} = 0$ , so

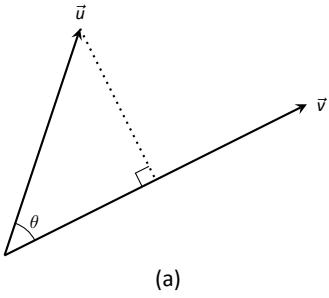
$$2 + 14 + 3z = 0 \Rightarrow z = -\frac{16}{3}.$$

So  $\vec{v}_1 = \langle 2, 7, -16/3 \rangle$  is orthogonal to  $\vec{v}$ . We can apply a similar technique by leaving the first or second component unknown.

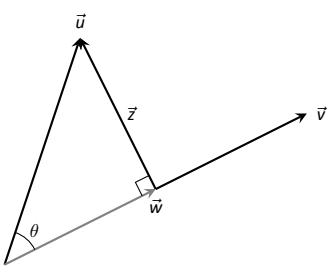
Another method of finding a vector orthogonal to  $\vec{v}$  mirrors what we did in part 1. Let  $\vec{v}_2 = \langle -2, 1, 0 \rangle$ . Here we switched the first two components of  $\vec{v}$ , changing the sign of one of them (similar to the “opposite reciprocal” concept before). Letting the third component be 0 effectively ignores the third component of  $\vec{v}$ , and it is easy to see that

$$\vec{v}_2 \cdot \vec{v} = \langle -2, 1, 0 \rangle \cdot \langle 1, 2, 3 \rangle = 0.$$

Clearly  $\vec{v}_1$  and  $\vec{v}_2$  are not parallel.



(a)



(b)

Figure 1.16: Developing the construction of the *orthogonal projection*.

An important construction is illustrated in Figure 1.16, where vectors  $\vec{u}$  and  $\vec{v}$  are sketched. In part (a), a dotted line is drawn from the tip of  $\vec{u}$  to the line containing  $\vec{v}$ , where the dotted line is orthogonal to  $\vec{v}$ . In part (b), the dotted line is replaced with the vector  $\vec{z}$  and  $\vec{w}$  is formed, parallel to  $\vec{v}$ . It is clear by the diagram that  $\vec{u} = \vec{w} + \vec{z}$ . What is important about this construction is this:  $\vec{u}$  is *decomposed* as the sum of two vectors, one of which is parallel to  $\vec{v}$  and one that is perpendicular to  $\vec{v}$ . It is hard to overstate the importance of this construction (as we'll see in upcoming examples).

The vectors  $\vec{w}$ ,  $\vec{z}$  and  $\vec{u}$  as shown in Figure 1.16 (b) form a right triangle, where the angle between  $\vec{v}$  and  $\vec{u}$  is labeled  $\theta$ . We can find  $\vec{w}$  in terms of  $\vec{v}$  and  $\vec{u}$ .

Using trigonometry, we can state that

$$\|\vec{w}\| = \|\vec{u}\| \cos \theta. \quad (1.2)$$

We also know that  $\vec{w}$  is parallel to  $\vec{v}$ ; that is, the direction of  $\vec{w}$  is the direction of  $\vec{v}$ , described by the unit vector  $\frac{1}{\|\vec{v}\|} \vec{v}$ . The vector  $\vec{w}$  is the vector in the direction  $\frac{1}{\|\vec{v}\|} \vec{v}$  with magnitude  $\|\vec{u}\| \cos \theta$ :

$$\vec{w} = \left( \|\vec{u}\| \cos \theta \right) \frac{1}{\|\vec{v}\|} \vec{v}.$$

Replace  $\cos \theta$  using Theorem 3:

$$\begin{aligned} &= \left( \|\vec{u}\| \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right) \frac{1}{\|\vec{v}\|} \vec{v} \\ &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}. \end{aligned}$$

Now apply Theorem 2.

$$= \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}.$$

Since this construction is so important, it is given a special name.

**Definition 9 Orthogonal Projection**

Let  $\vec{u}$  and  $\vec{v}$  be given. The **orthogonal projection of  $\vec{u}$  onto  $\vec{v}$** , denoted  $\text{proj}_{\vec{v}} \vec{u}$ , is

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}.$$

**Example 13 Computing the orthogonal projection**

1. Let  $\vec{u} = \langle -2, 1 \rangle$  and  $\vec{v} = \langle 3, 1 \rangle$ . Find  $\text{proj}_{\vec{v}} \vec{u}$ , and sketch all three vectors with initial points at the origin.
2. Let  $\vec{w} = \langle 2, 1, 3 \rangle$  and  $\vec{x} = \langle 1, 1, 1 \rangle$ . Find  $\text{proj}_{\vec{x}} \vec{w}$ , and sketch all three vectors with initial points at the origin.

**SOLUTION**

1. Applying Definition 9, we have

$$\begin{aligned}\text{proj}_{\vec{v}} \vec{u} &= \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} \\ &= \frac{-5}{10} \langle 3, 1 \rangle \\ &= \left\langle -\frac{3}{2}, -\frac{1}{2} \right\rangle.\end{aligned}$$

Vectors  $\vec{u}$ ,  $\vec{v}$  and  $\text{proj}_{\vec{v}} \vec{u}$  are sketched in Figure 1.17(a). Note how the projection is parallel to  $\vec{v}$ ; that is, it lies on the same line through the origin as  $\vec{v}$ , although it points in the opposite direction. That is because the angle between  $\vec{u}$  and  $\vec{v}$  is obtuse (i.e., greater than  $90^\circ$ ).

2. Apply the definition:

$$\begin{aligned}\text{proj}_{\vec{x}} \vec{w} &= \frac{\vec{w} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \vec{x} \\ &= \frac{6}{3} \langle 1, 1, 1 \rangle \\ &= \langle 2, 2, 2 \rangle.\end{aligned}$$

These vectors are sketched in Figure 1.17(b), and again in part (c) from a different perspective. Because of the nature of graphing these vectors, the sketch in part (b) makes it difficult to recognize that the drawn projection has the geometric properties it should. The graph shown in part (c) illustrates these properties better.

Consider Figure 1.18 where the concept of the orthogonal projection is again illustrated. It is clear that

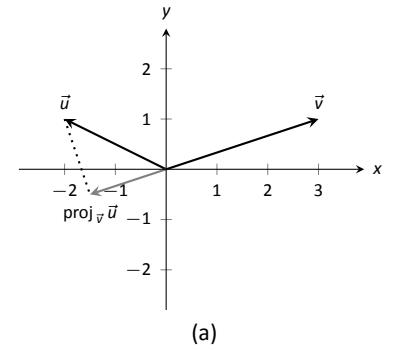
$$\vec{u} = \text{proj}_{\vec{v}} \vec{u} + \vec{z}. \quad (1.3)$$

As we know what  $\vec{u}$  and  $\text{proj}_{\vec{v}} \vec{u}$  are, we can solve for  $\vec{z}$  and state that

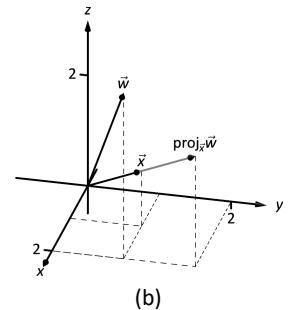
$$\vec{z} = \vec{u} - \text{proj}_{\vec{v}} \vec{u}.$$

This leads us to rewrite Equation (1.3) in a seemingly silly way:

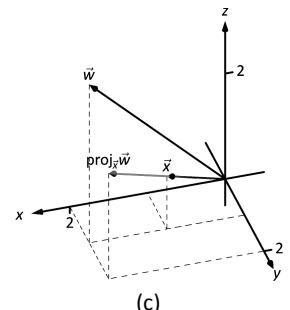
$$\vec{u} = \text{proj}_{\vec{v}} \vec{u} + (\vec{u} - \text{proj}_{\vec{v}} \vec{u}).$$



(a)



(b)



(c)

Figure 1.17: Graphing the vectors used in Example 13.

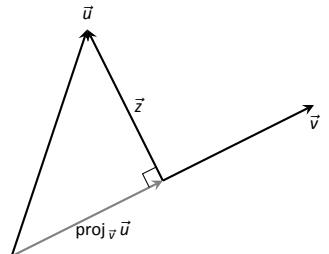


Figure 1.18: Illustrating the orthogonal projection.

This is not nonsense, as pointed out in the following Key Idea. (Notation note: the expression “ $\parallel \vec{y}$ ” means “is parallel to  $\vec{y}$ .” We can use this notation to state “ $\vec{x} \parallel \vec{y}$ ” which means “ $\vec{x}$  is parallel to  $\vec{y}$ .” The expression “ $\perp \vec{y}$ ” means “is orthogonal to  $\vec{y}$ ,” and is used similarly.)

**Key Idea 2 Orthogonal Decomposition of Vectors**

Let  $\vec{u}$  and  $\vec{v}$  be given. Then  $\vec{u}$  can be written as the sum of two vectors, one of which is parallel to  $\vec{v}$ , and one of which is orthogonal to  $\vec{v}$ :

$$\vec{u} = \underbrace{\text{proj}_{\vec{v}} \vec{u}}_{\parallel \vec{v}} + \underbrace{(\vec{u} - \text{proj}_{\vec{v}} \vec{u})}_{\perp \vec{v}}.$$

We illustrate the use of this equality in the following example.

**Example 14 Orthogonal decomposition of vectors**

1. Let  $\vec{u} = \langle -2, 1 \rangle$  and  $\vec{v} = \langle 3, 1 \rangle$  as in Example 13. Decompose  $\vec{u}$  as the sum of a vector parallel to  $\vec{v}$  and a vector orthogonal to  $\vec{v}$ .
2. Let  $\vec{w} = \langle 2, 1, 3 \rangle$  and  $\vec{x} = \langle 1, 1, 1 \rangle$  as in Example 13. Decompose  $\vec{w}$  as the sum of a vector parallel to  $\vec{x}$  and a vector orthogonal to  $\vec{x}$ .

**SOLUTION**

1. In Example 13, we found that  $\text{proj}_{\vec{v}} \vec{u} = \langle -1.5, -0.5 \rangle$ . Let

$$\vec{z} = \vec{u} - \text{proj}_{\vec{v}} \vec{u} = \langle -2, 1 \rangle - \langle -1.5, -0.5 \rangle = \langle -0.5, 1.5 \rangle.$$

Is  $\vec{z}$  orthogonal to  $\vec{v}$ ? (I.e., is  $\vec{z} \perp \vec{v}$ ?) We check for orthogonality with the dot product:

$$\vec{z} \cdot \vec{v} = \langle -0.5, 1.5 \rangle \cdot \langle 3, 1 \rangle = 0.$$

Since the dot product is 0, we know  $\vec{z} \perp \vec{v}$ . Thus:

$$\begin{aligned} \vec{u} &= \text{proj}_{\vec{v}} \vec{u} + (\vec{u} - \text{proj}_{\vec{v}} \vec{u}) \\ \langle -2, 1 \rangle &= \underbrace{\langle -1.5, -0.5 \rangle}_{\parallel \vec{v}} + \underbrace{\langle -0.5, 1.5 \rangle}_{\perp \vec{v}}. \end{aligned}$$

2. We found in Example 13 that  $\text{proj}_{\vec{x}} \vec{w} = \langle 2, 2, 2 \rangle$ . Applying the Key Idea, we have:

$$\vec{z} = \vec{w} - \text{proj}_{\vec{x}} \vec{w} = \langle 2, 1, 3 \rangle - \langle 2, 2, 2 \rangle = \langle 0, -1, 1 \rangle.$$

We check to see if  $\vec{z} \perp \vec{x}$ :

$$\vec{z} \cdot \vec{x} = \langle 0, -1, 1 \rangle \cdot \langle 1, 1, 1 \rangle = 0.$$

Since the dot product is 0, we know the two vectors are orthogonal. We now write  $\vec{w}$  as the sum of two vectors, one parallel and one orthogonal to  $\vec{x}$ :

$$\begin{aligned} \vec{w} &= \text{proj}_{\vec{x}} \vec{w} + (\vec{w} - \text{proj}_{\vec{x}} \vec{w}) \\ \langle 2, 1, 3 \rangle &= \underbrace{\langle 2, 2, 2 \rangle}_{\parallel \vec{x}} + \underbrace{\langle 0, -1, 1 \rangle}_{\perp \vec{x}} \end{aligned}$$

We give an example of where this decomposition is useful.

**Example 15 Orthogonally decomposing a force vector**

Consider Figure 1.19(a), showing a box weighing 50lb on a ramp that rises 5ft over a span of 20ft. Find the components of force, and their magnitudes, acting on the box (as sketched in part (b) of the figure):

1. in the direction of the ramp, and
2. orthogonal to the ramp.

**SOLUTION** As the ramp rises 5ft over a horizontal distance of 20ft, we can represent the direction of the ramp with the vector  $\vec{r} = \langle 20, 5 \rangle$ . Gravity pulls down with a force of 50lb, which we represent with  $\vec{g} = \langle 0, -50 \rangle$ .

1. To find the force of gravity in the direction of the ramp, we compute  $\text{proj}_{\vec{r}} \vec{g}$ :

$$\begin{aligned}\text{proj}_{\vec{r}} \vec{g} &= \frac{\vec{g} \cdot \vec{r}}{\vec{r} \cdot \vec{r}} \vec{r} \\ &= \frac{-250}{425} \langle 20, 5 \rangle \\ &= \left\langle -\frac{200}{17}, -\frac{50}{17} \right\rangle \approx \langle -11.76, -2.94 \rangle.\end{aligned}$$

The magnitude of  $\text{proj}_{\vec{r}} \vec{g}$  is  $\| \text{proj}_{\vec{r}} \vec{g} \| = 50/\sqrt{17} \approx 12.13$ lb. Though the box weighs 50lb, a force of about 12lb is enough to keep the box from sliding down the ramp.

2. To find the component  $\vec{z}$  of gravity orthogonal to the ramp, we use Key Idea 2.

$$\begin{aligned}\vec{z} &= \vec{g} - \text{proj}_{\vec{r}} \vec{g} \\ &= \left\langle \frac{200}{17}, -\frac{800}{17} \right\rangle \approx \langle 11.76, -47.06 \rangle.\end{aligned}$$

The magnitude of this force is  $\| \vec{z} \| \approx 48.51$ lb. In physics and engineering, knowing this force is important when computing things like static frictional force. (For instance, we could easily compute if the static frictional force alone was enough to keep the box from sliding down the ramp.)

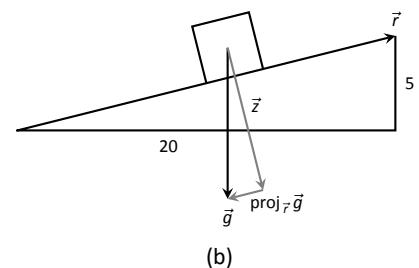
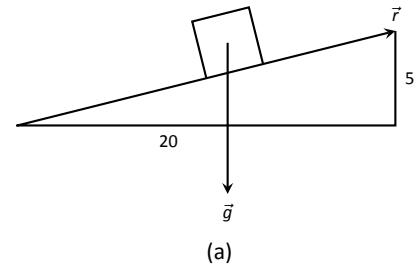


Figure 1.19: Sketching the ramp and box in Example 15. Note: *The vectors are not drawn to scale.*

## Application to Work

In physics, the application of a force  $F$  to move an object in a straight line a distance  $d$  produces *work*; the amount of work  $W$  is  $W = Fd$ , (where  $F$  is in the direction of travel). The orthogonal projection allows us to compute work when the force is not in the direction of travel.

Consider Figure 1.20, where a force  $\vec{F}$  is being applied to an object moving in the direction of  $\vec{d}$ . (The distance the object travels is the magnitude of  $\vec{d}$ .) The work done is the amount of force in the direction of  $\vec{d}$ ,  $\|\text{proj}_{\vec{d}} \vec{F}\|$ , times  $\|\vec{d}\|$ :

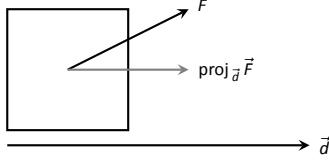


Figure 1.20: Finding work when the force and direction of travel are given as vectors.

$$\begin{aligned} \|\text{proj}_{\vec{d}} \vec{F}\| \cdot \|\vec{d}\| &= \left\| \frac{\vec{F} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} \right\| \cdot \|\vec{d}\| \\ &= \left| \frac{\vec{F} \cdot \vec{d}}{\|\vec{d}\|^2} \right| \cdot \|\vec{d}\| \cdot \|\vec{d}\| \\ &= \frac{|\vec{F} \cdot \vec{d}|}{\|\vec{d}\|^2} \|\vec{d}\|^2 \\ &= |\vec{F} \cdot \vec{d}|. \end{aligned}$$

The expression  $\vec{F} \cdot \vec{d}$  will be positive if the angle between  $\vec{F}$  and  $\vec{d}$  is acute; when the angle is obtuse (hence  $\vec{F} \cdot \vec{d}$  is negative), the force is causing motion in the opposite direction of  $\vec{d}$ , resulting in “negative work.” We want to capture this sign, so we drop the absolute value and find that  $W = \vec{F} \cdot \vec{d}$ .

#### Definition 10 Work

Let  $\vec{F}$  be a constant force that moves an object in a straight line from point  $P$  to point  $Q$ . Let  $\vec{d} = \vec{PQ}$ . The **work**  $W$  done by  $\vec{F}$  along  $\vec{d}$  is  $W = \vec{F} \cdot \vec{d}$ .

#### Example 16 Computing work

A man slides a box along a ramp that rises 3ft over a distance of 15ft by applying 50lb of force as shown in Figure 1.21. Compute the work done.

**SOLUTION** The figure indicates that the force applied makes a  $30^\circ$  angle with the horizontal, so  $\vec{F} = 50 \langle \cos 30^\circ, \sin 30^\circ \rangle \approx \langle 43.3, 25 \rangle$ . The ramp is represented by  $\vec{d} = \langle 15, 3 \rangle$ . The work done is simply

$$\vec{F} \cdot \vec{d} = 50 \langle \cos 30^\circ, \sin 30^\circ \rangle \cdot \langle 15, 3 \rangle \approx 724.5 \text{ ft-lb.}$$

Note how we did not actually compute the distance the object traveled, nor the magnitude of the force in the direction of travel; this is all inherently computed by the dot product!

The dot product is a powerful way of evaluating computations that depend on angles without actually using angles. The next section explores another “product” on vectors, the *cross product*. Once again, angles play an important role, though in a much different way.

Figure 1.21: Computing work when sliding a box up a ramp in Example 16.

# Exercises 1.2

## Terms and Concepts

1. The dot product of two vectors is a \_\_\_\_\_, not a vector.
2. How are the concepts of the dot product and vector magnitude related?
3. How can one quickly tell if the angle between two vectors is acute or obtuse?
4. Give a synonym for “orthogonal.”

## Problems

In Exercises 5 – 10, find the dot product of the given vectors.

5.  $\vec{u} = \langle 2, -4 \rangle, \vec{v} = \langle 3, 7 \rangle$
6.  $\vec{u} = \langle 5, 3 \rangle, \vec{v} = \langle 6, 1 \rangle$
7.  $\vec{u} = \langle 1, -1, 2 \rangle, \vec{v} = \langle 2, 5, 3 \rangle$
8.  $\vec{u} = \langle 3, 5, -1 \rangle, \vec{v} = \langle 4, -1, 7 \rangle$
9.  $\vec{u} = \langle 1, 1 \rangle, \vec{v} = \langle 1, 2, 3 \rangle$
10.  $\vec{u} = \langle 1, 2, 3 \rangle, \vec{v} = \langle 0, 0, 0 \rangle$
11. Create your own vectors  $\vec{u}, \vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^2$  and show that  $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ .
12. Create your own vectors  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  and scalar  $c$  and show that  $c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$ .

In Exercises 13 – 16, find the measure of the angle between the two vectors in both radians and degrees.

13.  $\vec{u} = \langle 1, 1 \rangle, \vec{v} = \langle 1, 2 \rangle$
14.  $\vec{u} = \langle -2, 1 \rangle, \vec{v} = \langle 3, 5 \rangle$
15.  $\vec{u} = \langle 8, 1, -4 \rangle, \vec{v} = \langle 2, 2, 0 \rangle$
16.  $\vec{u} = \langle 1, 7, 2 \rangle, \vec{v} = \langle 4, -2, 5 \rangle$

In Exercises 17 – 20, a vector  $\vec{v}$  is given. Give two vectors that are orthogonal to  $\vec{v}$ .

17.  $\vec{v} = \langle 4, 7 \rangle$
18.  $\vec{v} = \langle -3, 5 \rangle$
19.  $\vec{v} = \langle 1, 1, 1 \rangle$
20.  $\vec{v} = \langle 1, -2, 3 \rangle$

In Exercises 21 – 26, vectors  $\vec{u}$  and  $\vec{v}$  are given. Find  $\text{proj}_{\vec{v}} \vec{u}$ , the orthogonal projection of  $\vec{u}$  onto  $\vec{v}$ , and sketch all three vectors on the same axes.

21.  $\vec{u} = \langle 1, 2 \rangle, \vec{v} = \langle -1, 3 \rangle$
22.  $\vec{u} = \langle 5, 5 \rangle, \vec{v} = \langle 1, 3 \rangle$
23.  $\vec{u} = \langle -3, 2 \rangle, \vec{v} = \langle 1, 1 \rangle$
24.  $\vec{u} = \langle -3, 2 \rangle, \vec{v} = \langle 2, 3 \rangle$
25.  $\vec{u} = \langle 1, 5, 1 \rangle, \vec{v} = \langle 1, 2, 3 \rangle$
26.  $\vec{u} = \langle 3, -1, 2 \rangle, \vec{v} = \langle 2, 2, 1 \rangle$

In Exercises 27 – 32, vectors  $\vec{u}$  and  $\vec{v}$  are given. Write  $\vec{u}$  as the sum of two vectors, one of which is parallel to  $\vec{v}$  and one of which is perpendicular to  $\vec{v}$ . Note: these are the same pairs of vectors as found in Exercises 21 – 26.

27.  $\vec{u} = \langle 1, 2 \rangle, \vec{v} = \langle -1, 3 \rangle$
28.  $\vec{u} = \langle 5, 5 \rangle, \vec{v} = \langle 1, 3 \rangle$
29.  $\vec{u} = \langle -3, 2 \rangle, \vec{v} = \langle 1, 1 \rangle$
30.  $\vec{u} = \langle -3, 2 \rangle, \vec{v} = \langle 2, 3 \rangle$
31.  $\vec{u} = \langle 1, 5, 1 \rangle, \vec{v} = \langle 1, 2, 3 \rangle$
32.  $\vec{u} = \langle 3, -1, 2 \rangle, \vec{v} = \langle 2, 2, 1 \rangle$
33. A 10lb box sits on a ramp that rises 4ft over a distance of 20ft. How much force is required to keep the box from sliding down the ramp?
34. A 10lb box sits on a 15ft ramp that makes a  $30^\circ$  angle with the horizontal. How much force is required to keep the box from sliding down the ramp?
35. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of  $45^\circ$  to the horizontal?
36. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of  $10^\circ$  to the horizontal?
37. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied horizontally?
38. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied at an angle of  $45^\circ$  to the horizontal?
39. How much work is performed in moving a box up the length of a 10ft ramp that makes a  $5^\circ$  angle with the horizontal, with 50lb of force applied in the direction of the ramp?

### 1.3 The Cross Product

“Orthogonality” is immensely important. A quick scan of your current environment will undoubtedly reveal numerous surfaces and edges that are perpendicular to each other (including the edges of this page). The dot product provides a quick test for orthogonality: vectors  $\vec{u}$  and  $\vec{v}$  are perpendicular if, and only if,  $\vec{u} \cdot \vec{v} = 0$ .

Given two non-parallel, nonzero vectors  $\vec{u}$  and  $\vec{v}$  in space, it is very useful to find a vector  $\vec{w}$  that is perpendicular to both  $\vec{u}$  and  $\vec{v}$ . There is a operation, called the **cross product**, that creates such a vector. This section defines the cross product, then explores its properties and applications.

#### Definition 11      Cross Product

Let  $\vec{u} = \langle u_1, u_2, u_3 \rangle$  and  $\vec{v} = \langle v_1, v_2, v_3 \rangle$  be vectors in  $\mathbb{R}^3$ . The **cross product of  $\vec{u}$  and  $\vec{v}$** , denoted  $\vec{u} \times \vec{v}$ , is the vector

$$\vec{u} \times \vec{v} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle.$$

This definition can be a bit cumbersome to remember. After an example we will give a convenient method for computing the cross product. For now, careful examination of the products and differences given in the definition should reveal a pattern that is not too difficult to remember. (For instance, in the first component only 2 and 3 appear as subscripts; in the second component, only 1 and 3 appear as subscripts. Further study reveals the order in which they appear.)

Let’s practice using this definition by computing a cross product.

#### Example 17      Computing a cross product

Let  $\vec{u} = \langle 2, -1, 4 \rangle$  and  $\vec{v} = \langle 3, 2, 5 \rangle$ . Find  $\vec{u} \times \vec{v}$ , and verify that it is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

#### SOLUTION

Using Definition 11, we have

$$\vec{u} \times \vec{v} = \langle (-1)5 - (4)2, -(2)5 - (4)3, (2)2 - (-1)3 \rangle = \langle -13, 2, 7 \rangle.$$

(We encourage the reader to compute this product on their own, then verify their result.)

We test whether or not  $\vec{u} \times \vec{v}$  is orthogonal to  $\vec{u}$  and  $\vec{v}$  using the dot product:

$$(\vec{u} \times \vec{v}) \cdot \vec{u} = \langle -13, 2, 7 \rangle \cdot \langle 2, -1, 4 \rangle = 0,$$

$$(\vec{u} \times \vec{v}) \cdot \vec{v} = \langle -13, 2, 7 \rangle \cdot \langle 3, 2, 5 \rangle = 0.$$

Since both dot products are zero,  $\vec{u} \times \vec{v}$  is indeed orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

A convenient method of computing the cross product starts with forming a particular  $3 \times 3$  *matrix*, or rectangular array. The first row comprises the standard unit vectors  $\vec{i}$ ,  $\vec{j}$ , and  $\vec{k}$ . The second and third rows are the vectors  $\vec{u}$  and  $\vec{v}$ , respectively. Using  $\vec{u}$  and  $\vec{v}$  from Example 17, we begin with:

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{array}$$

Now repeat the first two columns after the original three:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 2 & -1 & 4 & 2 & -1 \\ 3 & 2 & 5 & 3 & 2 \end{array}$$

This gives three full “upper left to lower right” diagonals, and three full “upper right to lower left” diagonals, as shown. Compute the products along each diagonal, then add the products on the right and subtract the products on the left:

$$\vec{u} \times \vec{v} = (-5\vec{i} + 12\vec{j} + 4\vec{k}) - (-3\vec{k} + 8\vec{i} + 10\vec{j}) = -13\vec{i} + 2\vec{j} + 7\vec{k} = \langle -13, 2, 7 \rangle.$$

We practice using this method.

### Example 18 Computing a cross product

Let  $\vec{u} = \langle 1, 3, 6 \rangle$  and  $\vec{v} = \langle -1, 2, 1 \rangle$ . Compute both  $\vec{u} \times \vec{v}$  and  $\vec{v} \times \vec{u}$ .

**SOLUTION** To compute  $\vec{u} \times \vec{v}$ , we form the matrix as prescribed above, complete with repeated first columns:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 3 & 6 & 1 & 3 \\ -1 & 2 & 1 & -1 & 2 \end{array}$$

We let the reader compute the products of the diagonals; we give the result:

$$\vec{u} \times \vec{v} = (3\vec{i} - 6\vec{j} + 2\vec{k}) - (-3\vec{k} + 12\vec{i} + \vec{j}) = \langle -9, -7, 5 \rangle.$$

To compute  $\vec{v} \times \vec{u}$ , we switch the second and third rows of the above matrix, then multiply along diagonals and subtract:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ -1 & 2 & 1 & -1 & 2 \\ 1 & 3 & 6 & 1 & 3 \end{array}$$

Note how with the rows being switched, the products that once appeared on the right now appear on the left, and vice-versa. Thus the result is:

$$\vec{v} \times \vec{u} = (12\vec{i} + \vec{j} - 3\vec{k}) - (2\vec{k} + 3\vec{i} - 6\vec{j}) = \langle 9, 7, -5 \rangle,$$

which is the opposite of  $\vec{u} \times \vec{v}$ . We leave it to the reader to verify that each of these vectors is orthogonal to  $\vec{u}$  and  $\vec{v}$ .

### Properties of the Cross Product

It is not coincidence that  $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$  in the preceding example; one can show using Definition 11 that this will always be the case. The following theorem states several useful properties of the cross product, each of which can be verified by referring to the definition.

**Theorem 4 Properties of the Cross Product**

Let  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  be vectors in  $\mathbb{R}^3$  and let  $c$  be a scalar. The following identities hold:

1.  $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$  Anticommutative Property
2. (a)  $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$  Distributive Properties  
(b)  $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
3.  $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
4. (a)  $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$  Orthogonality Properties  
(b)  $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
5.  $\vec{u} \times \vec{u} = \vec{0}$
6.  $\vec{u} \times \vec{0} = \vec{0}$
7.  $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$  Triple Scalar Product

**Note:** Definition 8 (through Theorem 3) defines  $\vec{u}$  and  $\vec{v}$  to be orthogonal if  $\vec{u} \cdot \vec{v} = 0$ . We could use Theorem 5 to define  $\vec{u}$  and  $\vec{v}$  are parallel if  $\vec{u} \times \vec{v} = 0$ . By such a definition,  $\vec{0}$  would be both orthogonal and parallel to every vector. Apparent paradoxes such as this are not uncommon in mathematics and can be very useful. (See also the marginal note on page 10.)

We introduced the cross product as a way to find a vector orthogonal to two given vectors, but we did not give a proof that the construction given in Definition 11 satisfies this property. Theorem 4 asserts this property holds; we leave it as a problem in the Exercise section to verify this.

Property 5 from the theorem is also left to the reader to prove in the Exercise section, but it reveals something more interesting than “the cross product of a vector with itself is  $\vec{0}$ .” Let  $\vec{u}$  and  $\vec{v}$  be parallel vectors; that is, let there be a scalar  $c$  such that  $\vec{v} = c\vec{u}$ . Consider their cross product:

$$\begin{aligned}\vec{u} \times \vec{v} &= \vec{u} \times (c\vec{u}) \\ &= c(\vec{u} \times \vec{u}) \quad (\text{by Property 3 of Theorem 4}) \\ &= \vec{0}. \quad (\text{by Property 5 of Theorem 4})\end{aligned}$$

We have just shown that the cross product of parallel vectors is  $\vec{0}$ . This hints at something deeper. Theorem 3 related the angle between two vectors and their dot product; there is a similar relationship relating the cross product of two vectors and the angle between them, given by the following theorem.

**Theorem 5 The Cross Product and Angles**

Let  $\vec{u}$  and  $\vec{v}$  be vectors in  $\mathbb{R}^3$ . Then

$$\| \vec{u} \times \vec{v} \| = \| \vec{u} \| \| \vec{v} \| \sin \theta,$$

where  $\theta$ ,  $0 \leq \theta \leq \pi$ , is the angle between  $\vec{u}$  and  $\vec{v}$ .

Note that this theorem makes a statement about the *magnitude* of the cross product. When the angle between  $\vec{u}$  and  $\vec{v}$  is 0 or  $\pi$  (i.e., the vectors are parallel), the magnitude of the cross product is 0. The only vector with a magnitude of 0 is  $\vec{0}$  (see Property 9 of Theorem 1), hence the cross product of parallel vectors is  $\vec{0}$ .

We demonstrate the truth of this theorem in the following example.

### Example 19 The cross product and angles

Let  $\vec{u} = \langle 1, 3, 6 \rangle$  and  $\vec{v} = \langle -1, 2, 1 \rangle$  as in Example 18. Verify Theorem 5 by finding  $\theta$ , the angle between  $\vec{u}$  and  $\vec{v}$ , and the magnitude of  $\vec{u} \times \vec{v}$ .

**SOLUTION** We use Theorem 3 to find the angle between  $\vec{u}$  and  $\vec{v}$ .

$$\begin{aligned}\theta &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right) \\ &= \cos^{-1} \left( \frac{11}{\sqrt{46}\sqrt{6}} \right) \\ &\approx 0.8471 = 48.54^\circ.\end{aligned}$$

Our work in Example 18 showed that  $\vec{u} \times \vec{v} = \langle -9, -7, 5 \rangle$ , hence  $\|\vec{u} \times \vec{v}\| = \sqrt{155}$ . Is  $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$ ? Using numerical approximations, we find:

$$\begin{aligned}\|\vec{u} \times \vec{v}\| &= \sqrt{155} \\ &\approx 12.45.\end{aligned}\quad \begin{aligned}\|\vec{u}\| \|\vec{v}\| \sin \theta &= \sqrt{46}\sqrt{6} \sin 0.8471 \\ &\approx 12.45.\end{aligned}$$

Numerically, they seem equal. Using a right triangle, one can show that

$$\sin \left( \cos^{-1} \left( \frac{11}{\sqrt{46}\sqrt{6}} \right) \right) = \frac{\sqrt{155}}{\sqrt{46}\sqrt{6}},$$

which allows us to verify the theorem exactly.

### Right Hand Rule

The anticommutative property of the cross product demonstrates that  $\vec{u} \times \vec{v}$  and  $\vec{v} \times \vec{u}$  differ only by a sign – these vectors have the same magnitude but point in the opposite direction. When seeking a vector perpendicular to  $\vec{u}$  and  $\vec{v}$ , we essentially have two directions to choose from, one in the direction of  $\vec{u} \times \vec{v}$  and one in the direction of  $\vec{v} \times \vec{u}$ . Does it matter which we choose? How can we tell which one we will get without graphing, etc.?

Another wonderful property of the cross product, as defined, is that it follows the **right hand rule**. Given  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  with the same initial point, point the index finger of your right hand in the direction of  $\vec{u}$  and let your middle finger point in the direction of  $\vec{v}$  (much as we did when establishing the right hand rule for the 3-dimensional coordinate system). Your thumb will naturally extend in the direction of  $\vec{u} \times \vec{v}$ . One can “practice” this using Figure 1.22. If you switch, and point the index finger in the direction of  $\vec{v}$  and the middle finger in the direction of  $\vec{u}$ , your thumb will now point in the opposite direction, allowing you to “visualize” the anticommutative property of the cross product.

### Applications of the Cross Product

There are a number of ways in which the cross product is useful in mathematics, physics and other areas of science beyond “just” finding a vector perpendicular to two others. We highlight a few here.

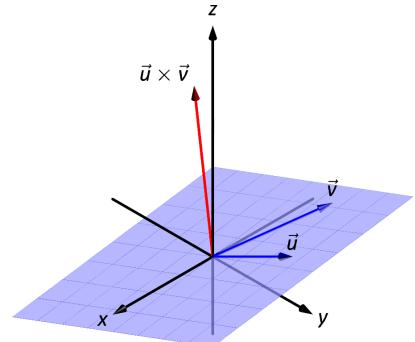


Figure 1.22: Illustrating the Right Hand Rule of the cross product.

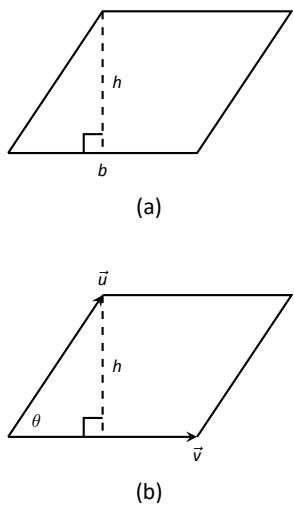


Figure 1.23: Using the cross product to find the area of a parallelogram.

### Area of a Parallelogram

It is a standard geometry fact that the area of a parallelogram is  $A = bh$ , where  $b$  is the length of the base and  $h$  is the height of the parallelogram, as illustrated in Figure 1.23(a). As shown when defining the Parallelogram Law of vector addition, two vectors  $\vec{u}$  and  $\vec{v}$  define a parallelogram when drawn from the same initial point, as illustrated in Figure 1.23(b). Trigonometry tells us that  $h = \|\vec{u}\| \sin \theta$ , hence the area of the parallelogram is

$$A = \|\vec{u}\| \|\vec{v}\| \sin \theta = \|\vec{u} \times \vec{v}\|, \quad (1.4)$$

where the second equality comes from Theorem 5. We illustrate using Equation (1.4) in the following example.

#### Example 20 Finding the area of a parallelogram

- Find the area of the parallelogram defined by the vectors  $\vec{u} = \langle 2, 1 \rangle$  and  $\vec{v} = \langle 1, 3 \rangle$ .
- Verify that the points  $A = (1, 1, 1)$ ,  $B = (2, 3, 2)$ ,  $C = (4, 5, 3)$  and  $D = (3, 3, 2)$  are the vertices of a parallelogram. Find the area of the parallelogram.

#### SOLUTION

- Figure 1.24(a) sketches the parallelogram defined by the vectors  $\vec{u}$  and  $\vec{v}$ . We have a slight problem in that our vectors exist in  $\mathbb{R}^2$ , not  $\mathbb{R}^3$ , and the cross product is only defined on vectors in  $\mathbb{R}^3$ . We skirt this issue by viewing  $\vec{u}$  and  $\vec{v}$  as vectors in the  $x-y$  plane of  $\mathbb{R}^3$ , and rewrite them as  $\vec{u} = \langle 2, 1, 0 \rangle$  and  $\vec{v} = \langle 1, 3, 0 \rangle$ . We can now compute the cross product. It is easy to show that  $\vec{u} \times \vec{v} = \langle 0, 0, 5 \rangle$ ; therefore the area of the parallelogram is  $A = \|\vec{u} \times \vec{v}\| = 5$ .
- To show that the quadrilateral  $ABCD$  is a parallelogram (shown in Figure 1.24(b)), we need to show that the opposite sides are parallel. We can quickly show that  $\vec{AB} = \vec{DC} = \langle 1, 2, 1 \rangle$  and  $\vec{BC} = \vec{AD} = \langle 2, 2, 1 \rangle$ . We find the area by computing the magnitude of the cross product of  $\vec{AB}$  and  $\vec{BC}$ :

$$\vec{AB} \times \vec{BC} = \langle 0, 1, -2 \rangle \Rightarrow \|\vec{AB} \times \vec{BC}\| = \sqrt{5} \approx 2.236.$$

This application is perhaps more useful in finding the area of a triangle (in short, triangles are used more often than parallelograms). We illustrate this in the following example.

#### Example 21 Area of a triangle

Find the area of the triangle with vertices  $A = (1, 2)$ ,  $B = (2, 3)$  and  $C = (3, 1)$ , as pictured in Figure 1.25.

**SOLUTION** We found the area of this triangle in Example 202 to be 1.5 using integration. There we discussed the fact that finding the area of a triangle can be inconvenient using the  $\frac{1}{2}bh$  formula as one has to compute the height, which generally involves finding angles, etc. Using a cross product is much more direct.

We can choose any two sides of the triangle to use to form vectors; we choose  $\vec{AB} = \langle 1, 1 \rangle$  and  $\vec{AC} = \langle 2, -1 \rangle$ . As in the previous example, we will

Figure 1.24: Sketching the parallelograms in Example 20.

rewrite these vectors with a third component of 0 so that we can apply the cross product. The area of the triangle is

$$\frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2} \|\langle 1, 1, 0 \rangle \times \langle 2, -1, 0 \rangle\| = \frac{1}{2} \|\langle 0, 0, -3 \rangle\| = \frac{3}{2}.$$

We arrive at the same answer as before with less work.

### Volume of a Parallelepiped

The three dimensional analogue to the parallelogram is the **parallelepiped**. Each face is parallel to the face opposite face, as illustrated in Figure 1.26. By crossing  $\vec{v}$  and  $\vec{w}$ , one gets a vector whose magnitude is the area of the base. Dotting this vector with  $\vec{u}$  computes the volume of parallelepiped! (Up to a sign; take the absolute value.)

Thus the volume of a parallelepiped defined by vectors  $\vec{u}, \vec{v}$  and  $\vec{w}$  is

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|. \quad (1.5)$$

Note how this is the Triple Scalar Product, first seen in Theorem 4. Applying the identities given in the theorem shows that we can apply the Triple Scalar Product in any “order” we choose to find the volume. That is,

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| = |\vec{u} \cdot (\vec{w} \times \vec{v})| = |(\vec{u} \times \vec{v}) \cdot \vec{w}|, \quad \text{etc.}$$

### Example 22 Finding the volume of parallelepiped

Find the volume of the parallelepiped defined by the vectors  $\vec{u} = \langle 1, 1, 0 \rangle$ ,  $\vec{v} = \langle -1, 1, 0 \rangle$  and  $\vec{w} = \langle 0, 1, 1 \rangle$ .

**SOLUTION**  
Then

$$|\vec{u} \cdot (\vec{v} \times \vec{w})| = |\langle 1, 1, 0 \rangle \cdot \langle 1, 1, -1 \rangle| = 2.$$

So the volume of the parallelepiped is 2 cubic units.

While this application of the Triple Scalar Product is interesting, it is not used all that often: parallelepipeds are not a common shape in physics and engineering. The last application of the cross product is very applicable in engineering.

### Torque

**Torque** is a measure of the turning force applied to an object. A classic scenario involving torque is the application of a wrench to a bolt. When a force is applied to the wrench, the bolt turns. When we represent the force and wrench with vectors  $\vec{F}$  and  $\vec{l}$ , we see that the bolt moves (because of the threads) in a direction orthogonal to  $\vec{F}$  and  $\vec{l}$ . Torque is usually represented by the Greek letter  $\tau$ , or tau, and has units of N·m, a Newton–meter, or ft·lb, a foot–pound.

While a full understanding of torque is beyond the purposes of this book, when a force  $\vec{F}$  is applied to a lever arm  $\vec{l}$ , the resulting torque is

$$\vec{\tau} = \vec{l} \times \vec{F}. \quad (1.6)$$

### Example 23 Computing torque

A lever of length 2ft makes an angle with the horizontal of  $45^\circ$ . Find the resulting torque when a force of 10lb is applied to the end of the level where:

1. the force is perpendicular to the lever, and
2. the force makes an angle of  $60^\circ$  with the lever, as shown in Figure 1.28.

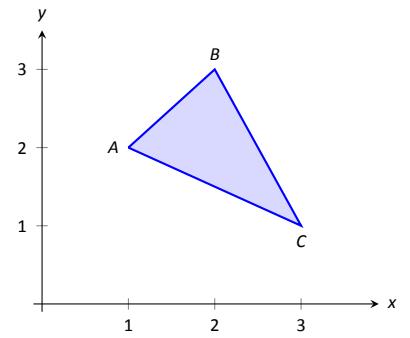


Figure 1.25: Finding the area of a triangle in Example 21.

**Note:** The word “parallelepiped” is pronounced “parallel–eh–pipe–ed.”

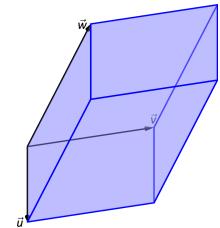


Figure 1.26: A parallelepiped is the three dimensional analogue to the parallelogram.

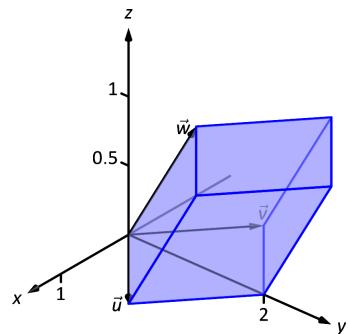


Figure 1.27: A parallelepiped in Example 22.

**SOLUTION**

1. We start by determining vectors for the force and lever arm. Since the lever arm makes a  $45^\circ$  angle with the horizontal and is 2ft long, we can state that  $\vec{\ell} = 2 \langle \cos 45^\circ, \sin 45^\circ \rangle = \langle \sqrt{2}, \sqrt{2} \rangle$ .

Since the force vector is perpendicular to the lever arm (as seen in the left hand side of Figure 1.28), we can conclude it is making an angle of  $-45^\circ$  with the horizontal. As it has a magnitude of 10lb, we can state  $\vec{F} = 10 \langle \cos(-45^\circ), \sin(-45^\circ) \rangle = \langle 5\sqrt{2}, -5\sqrt{2} \rangle$ .

Using Equation (1.6) to find the torque requires a cross product. We again let the third component of each vector be 0 and compute the cross product:

$$\begin{aligned}\vec{\tau} &= \vec{\ell} \times \vec{F} \\ &= \langle \sqrt{2}, \sqrt{2}, 0 \rangle \times \langle 5\sqrt{2}, -5\sqrt{2}, 0 \rangle \\ &= \langle 0, 0, -20 \rangle\end{aligned}$$

This clearly has a magnitude of 20 ft-lb.

We can view the force and lever arm vectors as lying “on the page”; our computation of  $\vec{\tau}$  shows that the torque goes “into the page.” This follows the Right Hand Rule of the cross product, and it also matches well with the example of the wrench turning the bolt. Turning a bolt clockwise moves it in.

2. Our lever arm can still be represented by  $\vec{\ell} = \langle \sqrt{2}, \sqrt{2} \rangle$ . As our force vector makes a  $60^\circ$  angle with  $\vec{\ell}$ , we can see (referencing the right hand side of the figure) that  $\vec{F}$  makes a  $-15^\circ$  angle with the horizontal. Thus

$$\begin{aligned}\vec{F} &= 10 \langle \cos -15^\circ, \sin -15^\circ \rangle = \left\langle \frac{5(1 + \sqrt{3})}{\sqrt{2}}, -\frac{5(1 + \sqrt{3})}{\sqrt{2}} \right\rangle \\ &\approx \langle 9.659, -2.588 \rangle.\end{aligned}$$

We again make the third component 0 and take the cross product to find the torque:

$$\begin{aligned}\vec{\tau} &= \vec{\ell} \times \vec{F} \\ &= \langle \sqrt{2}, \sqrt{2}, 0 \rangle \times \left\langle \frac{5(1 + \sqrt{3})}{\sqrt{2}}, -\frac{5(1 + \sqrt{3})}{\sqrt{2}}, 0 \right\rangle \\ &= \langle 0, 0, -10\sqrt{3} \rangle \\ &\approx \langle 0, 0, -17.321 \rangle.\end{aligned}$$

As one might expect, when the force and lever arm vectors *are* orthogonal, the magnitude of force is greater than when the vectors *are not* orthogonal.

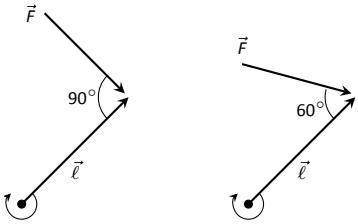


Figure 1.28: Showing a force being applied to a lever in Example 23.

While the cross product has a variety of applications (as noted in this chapter), its fundamental use is finding a vector perpendicular to two others. Knowing a vector is orthogonal to two others is of incredible importance, as it allows us to find the equations of lines and planes in a variety of contexts. The importance of the cross product, in some sense, relies on the importance of lines and planes, which see widespread use throughout engineering, physics and mathematics. We study lines and planes in the next two sections.

# Exercises 1.3

## Terms and Concepts

1. The cross product of two vectors is a \_\_\_\_\_, not a scalar.
2. One can visualize the direction of  $\vec{u} \times \vec{v}$  using the \_\_\_\_\_.
3. Give a synonym for “orthogonal.”
4. T/F: A fundamental principle of the cross product is that  $\vec{u} \times \vec{v}$  is orthogonal to  $\vec{u}$  and  $\vec{v}$ .
5. \_\_\_\_\_ is a measure of the turning force applied to an object.

## Problems

In Exercises 6 – 14, vectors  $\vec{u}$  and  $\vec{v}$  are given. Compute  $\vec{u} \times \vec{v}$  and show this is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

6.  $\vec{u} = \langle 3, 2, -2 \rangle, \quad \vec{v} = \langle 0, 1, 5 \rangle$
7.  $\vec{u} = \langle 5, -4, 3 \rangle, \quad \vec{v} = \langle 2, -5, 1 \rangle$
8.  $\vec{u} = \langle 4, -5, -5 \rangle, \quad \vec{v} = \langle 3, 3, 4 \rangle$
9.  $\vec{u} = \langle -4, 7, -10 \rangle, \quad \vec{v} = \langle 4, 4, 1 \rangle$
10.  $\vec{u} = \langle 1, 0, 1 \rangle, \quad \vec{v} = \langle 5, 0, 7 \rangle$
11.  $\vec{u} = \langle 1, 5, -4 \rangle, \quad \vec{v} = \langle -2, -10, 8 \rangle$
12.  $\vec{u} = \vec{i}, \quad \vec{v} = \vec{j}$
13.  $\vec{u} = \vec{i}, \quad \vec{v} = \vec{k}$
14.  $\vec{u} = \vec{j}, \quad \vec{v} = \vec{k}$

15. Pick any vectors  $\vec{u}, \vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^3$  and show that  $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ .
16. Pick any vectors  $\vec{u}, \vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^3$  and show that  $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ .

In Exercises 17 – 20, the magnitudes of vectors  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  are given, along with the angle  $\theta$  between them. Use this information to find the magnitude of  $\vec{u} \times \vec{v}$ .

17.  $\|\vec{u}\| = 2, \quad \|\vec{v}\| = 5, \quad \theta = 30^\circ$
18.  $\|\vec{u}\| = 3, \quad \|\vec{v}\| = 7, \quad \theta = \pi/2$
19.  $\|\vec{u}\| = 3, \quad \|\vec{v}\| = 4, \quad \theta = \pi$
20.  $\|\vec{u}\| = 2, \quad \|\vec{v}\| = 5, \quad \theta = 5\pi/6$

In Exercises 21 – 24, find the area of the parallelogram defined by the given vectors.

21.  $\vec{u} = \langle 1, 1, 2 \rangle, \quad \vec{v} = \langle 2, 0, 3 \rangle$
22.  $\vec{u} = \langle -2, 1, 5 \rangle, \quad \vec{v} = \langle -1, 3, 1 \rangle$
23.  $\vec{u} = \langle 1, 2 \rangle, \quad \vec{v} = \langle 2, 1 \rangle$
24.  $\vec{u} = \langle 2, 0 \rangle, \quad \vec{v} = \langle 0, 3 \rangle$

In Exercises 25 – 28, find the area of the triangle with the given vertices.

25. Vertices:  $(0, 0, 0), (1, 3, -1)$  and  $(2, 1, 1)$ .
26. Vertices:  $(5, 2, -1), (3, 6, 2)$  and  $(1, 0, 4)$ .
27. Vertices:  $(1, 1), (1, 3)$  and  $(2, 2)$ .
28. Vertices:  $(3, 1), (1, 2)$  and  $(4, 3)$ .

In Exercises 29 – 30, find the area of the quadrilateral with the given vertices. (Hint: break the quadrilateral into 2 triangles.)

29. Vertices:  $(0, 0), (1, 2), (3, 0)$  and  $(4, 3)$ .
30. Vertices:  $(0, 0, 0), (2, 1, 1), (-1, 2, -8)$  and  $(1, -1, 5)$ .

In Exercises 31 – 32, find the volume of the parallelepiped defined by the given vectors.

31.  $\vec{u} = \langle 1, 1, 1 \rangle, \quad \vec{v} = \langle 1, 2, 3 \rangle, \quad \vec{w} = \langle 1, 0, 1 \rangle$
32.  $\vec{u} = \langle -1, 2, 1 \rangle, \quad \vec{v} = \langle 2, 2, 1 \rangle, \quad \vec{w} = \langle 3, 1, 3 \rangle$

In Exercises 33 – 36, find a unit vector orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

33.  $\vec{u} = \langle 1, 1, 1 \rangle, \quad \vec{v} = \langle 2, 0, 1 \rangle$
34.  $\vec{u} = \langle 1, -2, 1 \rangle, \quad \vec{v} = \langle 3, 2, 1 \rangle$
35.  $\vec{u} = \langle 5, 0, 2 \rangle, \quad \vec{v} = \langle -3, 0, 7 \rangle$

36.  $\vec{u} = \langle 1, -2, 1 \rangle, \quad \vec{v} = \langle -2, 4, -2 \rangle$
37. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in horizontally from the crankshaft. Find the magnitude of the torque applied to the crankshaft.
38. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in from the crankshaft, making a  $30^\circ$  angle with the horizontal. Find the magnitude of the torque applied to the crankshaft.

39. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench. What is the maximum amount of torque that can be applied to the bolt?
40. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench in a confined space, where the direction of applied force makes a  $10^\circ$  angle with the wrench. How much torque is subsequently applied to the wrench?
41. Show, using the definition of the Cross Product, that  $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ ; that is, that  $\vec{u}$  is orthogonal to the cross product of  $\vec{u}$  and  $\vec{v}$ .
42. Show, using the definition of the Cross Product, that  $\vec{u} \times \vec{u} = \vec{0}$ .

## 1.4 Lines

To find the equation of a line in the  $x$ - $y$  plane, we need two pieces of information: a point and the slope. The slope conveys *direction* information. As vertical lines have an undefined slope, the following statement is more accurate:

To define a line, one needs a point on the line and the direction of the line.

This holds true for lines in space.

Let  $P$  be a point in space, let  $\vec{p}$  be the vector with initial point at the origin and terminal point at  $P$  (i.e.,  $\vec{p}$  “points” to  $P$ ), and let  $\vec{d}$  be a vector. Consider the points on the line through  $P$  in the direction of  $\vec{d}$ .

Clearly one point on the line is  $P$ ; we can say that the vector  $\vec{p}$  lies at this point on the line. To find another point on the line, we can start at  $\vec{p}$  and move in a direction parallel to  $\vec{d}$ . For instance, starting at  $\vec{p}$  and traveling one length of  $\vec{d}$  places one at another point on the line. Consider Figure 1.30 where certain points along the line are indicated.

The figure illustrates how every point on the line can be obtained by starting with  $\vec{p}$  and moving a certain distance in the direction of  $\vec{d}$ . That is, we can define the line as a function of  $t$ :

$$\ell(t) = \vec{p} + t\vec{d}. \quad (1.7)$$

In many ways, this is *not* a new concept. Compare Equation (1.7) to the familiar “ $y = mx + b$ ” equation of a line:

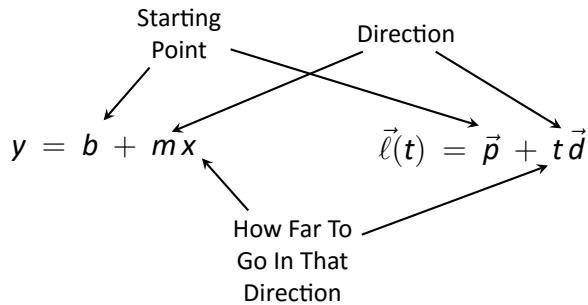


Figure 1.29: Understanding the vector equation of a line.

The equations exhibit the same structure: they give a starting point, define a direction, and state how far in that direction to travel.

Equation (1.7) is an example of a **vector-valued function**; the input of the function is a real number and the output is a vector. We will cover vector-valued functions extensively in the next chapter.

There are other ways to represent a line. Let  $\vec{p} = \langle x_0, y_0, z_0 \rangle$  and let  $\vec{d} = \langle a, b, c \rangle$ . Then the equation of the line through  $\vec{p}$  in the direction of  $\vec{d}$  is:

$$\begin{aligned}\ell(t) &= \vec{p} + t\vec{d} \\ &= \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle \\ &= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle.\end{aligned}$$

The last line states the the  $x$  values of the line are given by  $x = x_0 + at$ , the  $y$  values are given by  $y = y_0 + bt$ , and the  $z$  values are given by  $z = z_0 + ct$ . These three equations, taken together, are the **parametric equations of the line** through  $\vec{p}$  in the direction of  $\vec{d}$ .

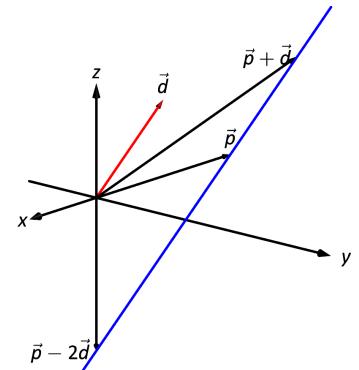


Figure 1.30: Defining a line in space.

Finally, each of the equations for  $x$ ,  $y$  and  $z$  above contain the variable  $t$ . We can solve for  $t$  in each equation:

$$\begin{aligned}x = x_0 + at &\Rightarrow t = \frac{x - x_0}{a}, \\y = y_0 + bt &\Rightarrow t = \frac{y - y_0}{b}, \\z = z_0 + ct &\Rightarrow t = \frac{z - z_0}{c},\end{aligned}$$

assuming  $a, b, c \neq 0$ . Since  $t$  is equal to each expression on the right, we can set these equal to each other, forming the **symmetric equations of the line** through  $\vec{p}$  in the direction of  $\vec{d}$ :

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Each representation has its own advantages, depending on the context. We summarize these three forms in the following definition, then give examples of their use.

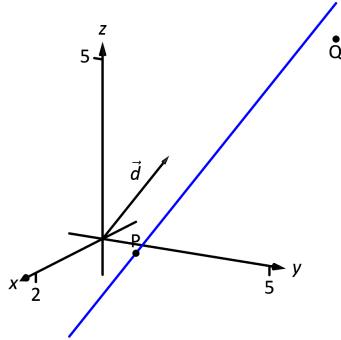


Figure 1.31: Graphing a line in Example 24.

### Definition 12 Equations of Lines in Space

Consider the line in space that passes through  $\vec{p} = \langle x_0, y_0, z_0 \rangle$  in the direction of  $\vec{d} = \langle a, b, c \rangle$ .

1. The **vector equation** of the line is

$$\vec{\ell}(t) = \vec{p} + t\vec{d}.$$

2. The **parametric equations** of the line are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

3. The **symmetric equations** of the line are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

### Example 24 Finding the equation of a line

Give all three equations, as given in Definition 12, of the line through  $P = (2, 3, 1)$  in the direction of  $\vec{d} = \langle -1, 1, 2 \rangle$ . Does the point  $Q = (-1, 6, 6)$  lie on this line?

**SOLUTION** We identify the point  $P = (2, 3, 1)$  with the vector  $\vec{p} = \langle 2, 3, 1 \rangle$ . Following the definition, we have

- the vector equation of the line is  $\vec{\ell}(t) = \langle 2, 3, 1 \rangle + t \langle -1, 1, 2 \rangle$ ;
- the parametric equations of the line are

$$x = 2 - t, \quad y = 3 + t, \quad z = 1 + 2t; \text{ and}$$

- the symmetric equations of the line are

$$\frac{x - 2}{-1} = \frac{y - 3}{1} = \frac{z - 1}{2}.$$

The first two equations of the line are useful when a  $t$  value is given: one can immediately find the corresponding point on the line. These forms are good when calculating with a computer; most software programs easily handle equations in these formats. (For instance, to make Figure 1.31, a certain graphics program was given the input  $(2-x, 3+x, 1+2*x)$ . This particular program requires the variable always be “ $x$ ” instead of “ $t$ ”).

Does the point  $Q = (-1, 6, 6)$  lie on the line? The graph in Figure 1.31 makes it clear that it does not. We can answer this question without the graph using any of the three equation forms. Of the three, the symmetric equations are probably best suited for this task. Simply plug in the values of  $x$ ,  $y$  and  $z$  and see if equality is maintained:

$$\frac{-1 - 2}{-1} = \frac{6 - 3}{1} = \frac{6 - 1}{2} \Rightarrow 3 = 3 \neq 2.5.$$

We see that  $Q$  does not lie on the line as it did not satisfy the symmetric equations.

### Example 25 Finding the equation of a line through two points

Find the parametric equations of the line through the points  $P = (2, -1, 2)$  and  $Q = (1, 3, -1)$ .

**SOLUTION** Recall the statement made at the beginning of this section: to find the equation of a line, we need a point and a direction. We have two points; either one will suffice. The direction of the line can be found by the vector with initial point  $P$  and terminal point  $Q$ :  $\vec{PQ} = \langle -1, 4, -3 \rangle$ .

The parametric equations of the line  $\ell$  through  $P$  in the direction of  $\vec{PQ}$  are:

$$\ell : x = 2 - t \quad y = -1 + 4t \quad z = 2 - 3t.$$

A graph of the points and line are given in Figure 1.32. Note how in the given parametrization of the line,  $t = 0$  corresponds to the point  $P$ , and  $t = 1$  corresponds to the point  $Q$ . This relates to the understanding of the vector equation of a line described in Figure 1.29. The parametric equations “start” at the point  $P$ , and  $t$  determines how far in the direction of  $\vec{PQ}$  to travel. When  $t = 0$ , we travel 0 lengths of  $\vec{PQ}$ ; when  $t = 1$ , we travel one length of  $\vec{PQ}$ , resulting in the point  $Q$ .

## Parallel, Intersecting and Skew Lines

In the plane, two *distinct* lines can either be parallel or they will intersect at exactly one point. In space, given equations of two lines, it can sometimes be difficult to tell whether the lines are distinct or not (i.e., the same line can be represented in different ways). Given lines  $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$  and  $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$ , we have four possibilities:  $\vec{\ell}_1$  and  $\vec{\ell}_2$  are

the same line	they share all points;
intersecting lines	share only 1 point;
parallel lines	$\vec{d}_1 \parallel \vec{d}_2$ , no points in common; or
skew lines	$\vec{d}_1 \not\parallel \vec{d}_2$ , no points in common.

The next two examples investigate these possibilities.

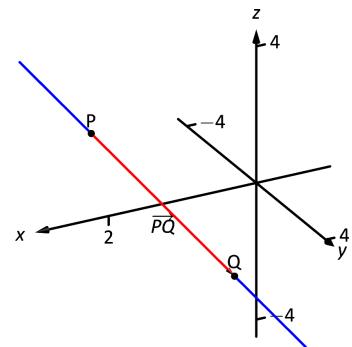


Figure 1.32: A graph of the line in Example 25.

**Example 26 Comparing lines**

Consider lines  $\ell_1$  and  $\ell_2$ , given in parametric equation form:

$$\begin{array}{ll} \ell_1: \begin{aligned} x &= 1 + 3t \\ y &= 2 - t \\ z &= t \end{aligned} & \ell_2: \begin{aligned} x &= -2 + 4s \\ y &= 3 + s \\ z &= 5 + 2s. \end{aligned} \end{array}$$

Determine whether  $\ell_1$  and  $\ell_2$  are the same line, intersect, are parallel, or skew.

**SOLUTION** We start by looking at the directions of each line. Line  $\ell_1$  has the direction given by  $\vec{d}_1 = \langle 3, -1, 1 \rangle$  and line  $\ell_2$  has the direction given by  $\vec{d}_2 = \langle 4, 1, 2 \rangle$ . It should be clear that  $\vec{d}_1$  and  $\vec{d}_2$  are not parallel, hence  $\ell_1$  and  $\ell_2$  are not the same line, nor are they parallel. Figure 1.33 verifies this fact (where the points and directions indicated by the equations of each line are identified).

We next check to see if they intersect (if they do not, they are skew lines). To find if they intersect, we look for  $t$  and  $s$  values such that the respective  $x$ ,  $y$  and  $z$  values are the same. That is, we want  $s$  and  $t$  such that:

$$\begin{aligned} 1 + 3t &= -2 + 4s \\ 2 - t &= 3 + s \\ t &= 5 + 2s. \end{aligned}$$

This is a relatively simple system of linear equations. Since the last equation is already solved for  $t$ , substitute that value of  $t$  into the equation above it:

$$2 - (5 + 2s) = 3 + s \Rightarrow s = -2, t = 1.$$

A key to remember is that we have *three* equations; we need to check if  $s = -2$ ,  $t = 1$  satisfies the first equation as well:

$$1 + 3(1) \neq -2 + 4(-2).$$

It does not. Therefore, we conclude that the lines  $\ell_1$  and  $\ell_2$  are skew.

**Example 27 Comparing lines**

Consider lines  $\ell_1$  and  $\ell_2$ , given in parametric equation form:

$$\begin{array}{ll} \ell_1: \begin{aligned} x &= -0.7 + 1.6t \\ y &= 4.2 + 2.72t \\ z &= 2.3 - 3.36t \end{aligned} & \ell_2: \begin{aligned} x &= 2.8 - 2.9s \\ y &= 10.15 - 4.93s \\ z &= -5.05 + 6.09s. \end{aligned} \end{array}$$

Determine whether  $\ell_1$  and  $\ell_2$  are the same line, intersect, are parallel, or skew.

**SOLUTION** It is obviously very difficult to simply look at these equations and discern anything. This is done intentionally. In the “real world,” most equations that are used do not have nice, integer coefficients. Rather, there are lots of digits after the decimal and the equations can look “messy.”

We again start by deciding whether or not each line has the same direction. The direction of  $\ell_1$  is given by  $\vec{d}_1 = \langle 1.6, 2.72, -3.36 \rangle$  and the direction of  $\ell_2$  is given by  $\vec{d}_2 = \langle -2.9, -4.93, 6.09 \rangle$ . When it is not clear through observation whether two vectors are parallel or not, the standard way of determining this is by comparing their respective unit vectors. Using a calculator, we find:

$$\begin{aligned} \vec{u}_1 &= \frac{\vec{d}_1}{\|\vec{d}_1\|} = \langle 0.3471, 0.5901, -0.7289 \rangle \\ \vec{u}_2 &= \frac{\vec{d}_2}{\|\vec{d}_2\|} = \langle -0.3471, -0.5901, 0.7289 \rangle. \end{aligned}$$

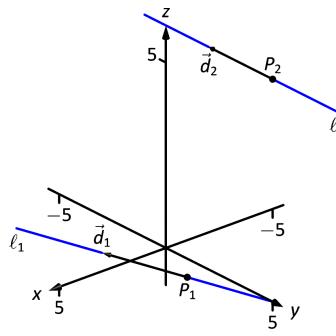


Figure 1.33: Sketching the lines from Example 26.

The two vectors seem to be parallel (at least, their components are equal to 4 decimal places). In most situations, it would suffice to conclude that the lines are at least parallel, if not the same. One way to be sure is to rewrite  $\vec{d}_1$  and  $\vec{d}_2$  in terms of fractions, not decimals. We have

$$\vec{d}_1 = \left\langle \frac{16}{10}, \frac{272}{100}, -\frac{336}{100} \right\rangle \quad \vec{d}_2 = \left\langle -\frac{29}{10}, -\frac{493}{100}, \frac{609}{100} \right\rangle.$$

One can then find the magnitudes of each vector in terms of fractions, then compute the unit vectors likewise. After a lot of manual arithmetic (or after briefly using a computer algebra system), one finds that

$$\vec{u}_1 = \left\langle \sqrt{\frac{10}{83}}, \frac{17}{\sqrt{830}}, -\frac{21}{\sqrt{830}} \right\rangle \quad \vec{u}_2 = \left\langle -\sqrt{\frac{10}{83}}, -\frac{17}{\sqrt{830}}, \frac{21}{\sqrt{830}} \right\rangle.$$

We can now say without equivocation that these lines are parallel.

Are they the same line? The parametric equations for a line describe one point that lies on the line, so we know that the point  $P_1 = (-0.7, 4.2, 2.3)$  lies on  $\ell_1$ . To determine if this point also lies on  $\ell_2$ , plug in the  $x$ ,  $y$  and  $z$  values of  $P_1$  into the symmetric equations for  $\ell_2$ :

$$\frac{(-0.7) - 2.8}{-2.9} \stackrel{?}{=} \frac{(4.2) - 10.15}{-4.93} \stackrel{?}{=} \frac{(2.3) - (-5.05)}{6.09} \Rightarrow 1.2069 = 1.2069 = 1.2069.$$

The point  $P_1$  lies on both lines, so we conclude they are the same line, just parametrized differently. Figure 1.34 graphs this line along with the points and vectors described by the parametric equations. Note how  $\vec{d}_1$  and  $\vec{d}_2$  are parallel, though point in opposite directions (as indicated by their unit vectors above).

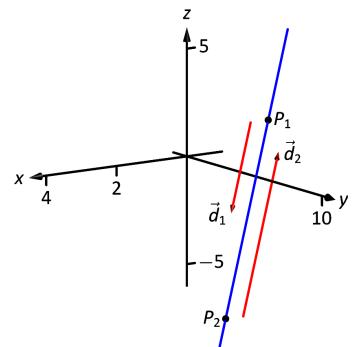


Figure 1.34: Graphing the lines in Example 27.

## Distances

Given a point  $Q$  and a line  $\vec{\ell}(t) = \vec{p} + t\vec{d}$  in space, it is often useful to know the distance from the point to the line. (Here we use the standard definition of “distance,” i.e., the length of the shortest line segment from the point to the line.) Identifying  $\vec{p}$  with the point  $P$ , Figure 1.35 will help establish a general method of computing this distance  $h$ .

From trigonometry, we know  $h = \|\overrightarrow{PQ}\| \sin \theta$ . We have a similar identity involving the cross product:  $\|\overrightarrow{PQ} \times \vec{d}\| = \|\overrightarrow{PQ}\| \|\vec{d}\| \sin \theta$ . Divide both sides of this latter equation by  $\|\vec{d}\|$  to obtain  $h$ :

$$h = \frac{\|\overrightarrow{PQ} \times \vec{d}\|}{\|\vec{d}\|}. \quad (1.8)$$

It is also useful to determine the distance between lines, which we define as the length of the shortest line segment that connects the two lines (an argument from geometry shows that this line segments is perpendicular to both lines). Let lines  $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$  and  $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$  be given, as shown in Figure 1.36. To find the direction orthogonal to both  $\vec{d}_1$  and  $\vec{d}_2$ , we take the cross product:  $\vec{c} = \vec{d}_1 \times \vec{d}_2$ . The magnitude of the orthogonal projection of  $\overrightarrow{P_1P_2}$  onto  $\vec{c}$  is the

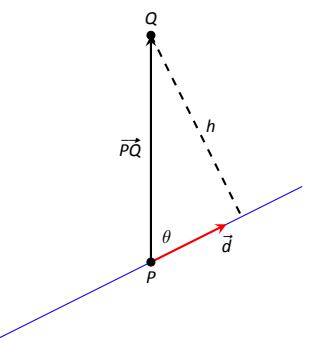


Figure 1.35: Establishing the distance from a point to a line.

distance  $h$  we seek:

$$\begin{aligned} h &= \left\| \text{proj}_{\vec{c}} \overrightarrow{P_1 P_2} \right\| \\ &= \left\| \frac{\overrightarrow{P_1 P_2} \cdot \vec{c}}{\vec{c} \cdot \vec{c}} \vec{c} \right\| \\ &= \frac{|\overrightarrow{P_1 P_2} \cdot \vec{c}|}{\|\vec{c}\|^2} \|\vec{c}\| \\ &= \frac{|\overrightarrow{P_1 P_2} \cdot \vec{c}|}{\|\vec{c}\|}. \end{aligned}$$

A problem in the Exercise section is to show that this distance is 0 when the lines intersect. Note the use of the Triple Scalar Product:  $\overrightarrow{P_1 P_2} \cdot \vec{c} = \overrightarrow{P_1 P_2} \cdot (\vec{d}_1 \times \vec{d}_2)$ .

The following Key Idea restates these two distance formulas.

### Key Idea 3 Distances to Lines

- Let  $P$  be a point on a line  $\ell$  that is parallel to  $\vec{d}$ . The distance  $h$  from a point  $Q$  to the line  $\ell$  is:

$$h = \frac{\|\overrightarrow{PQ} \times \vec{d}\|}{\|\vec{d}\|}.$$

- Let  $P_1$  be a point on line  $\ell_1$  that is parallel to  $\vec{d}_1$ , and let  $P_2$  be a point on line  $\ell_2$  parallel to  $\vec{d}_2$ , and let  $\vec{c} = \vec{d}_1 \times \vec{d}_2$ , where lines  $\ell_1$  and  $\ell_2$  are not parallel. The distance  $h$  between the two lines is:

$$h = \frac{|\overrightarrow{P_1 P_2} \cdot \vec{c}|}{\|\vec{c}\|}.$$

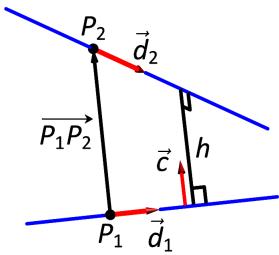


Figure 1.36: Establishing the distance between lines.

### Example 28 Finding the distance from a point to a line

Find the distance from the point  $Q = (1, 1, 3)$  to the line  $\ell(t) = \langle 1, -1, 1 \rangle + t \langle 2, 3, 1 \rangle$ .

**SOLUTION** The equation of the line gives us the point  $P = (1, -1, 1)$  that lies on the line, hence  $\overrightarrow{PQ} = \langle 0, 2, 2 \rangle$ . The equation also gives  $\vec{d} = \langle 2, 3, 1 \rangle$ . Following Key Idea 3, we have the distance as

$$\begin{aligned} h &= \frac{\|\overrightarrow{PQ} \times \vec{d}\|}{\|\vec{d}\|} \\ &= \frac{\|\langle -4, 4, -4 \rangle\|}{\sqrt{14}} \\ &= \frac{4\sqrt{3}}{\sqrt{14}} \approx 1.852. \end{aligned}$$

The point  $Q$  is approximately 1.852 units from the line  $\ell(t)$ .

### Example 29 Finding the distance between lines

Find the distance between the lines

$$\begin{array}{ll} \ell_1 : \begin{array}{l} x = 1 + 3t \\ y = 2 - t \\ z = t \end{array} & \ell_2 : \begin{array}{l} x = -2 + 4s \\ y = 3 + s \\ z = 5 + 2s \end{array} \end{array}$$

**SOLUTION** These are the same lines as given in Example 26, where we showed them to be skew. The equations allow us to identify the following points and vectors:

$$P_1 = (1, 2, 0) \quad P_2 = (-2, 3, 5) \quad \Rightarrow \quad \overrightarrow{P_1 P_2} = \langle -3, 1, 5 \rangle.$$

$$\vec{d}_1 = \langle 3, -1, 1 \rangle \quad \vec{d}_2 = \langle 4, 1, 2 \rangle \quad \Rightarrow \quad \vec{c} = \vec{d}_1 \times \vec{d}_2 = \langle -3, -2, 7 \rangle.$$

From Key Idea 3 we have the distance  $h$  between the two lines is

$$\begin{aligned} h &= \frac{|\overrightarrow{P_1 P_2} \cdot \vec{c}|}{\|\vec{c}\|} \\ &= \frac{42}{\sqrt{62}} \approx 5.334. \end{aligned}$$

The lines are approximately 5.334 units apart.

One of the key points to understand from this section is this: to describe a line, we need a point and a direction. Whenever a problem is posed concerning a line, one needs to take whatever information is offered and glean point and direction information. Many questions can be asked (and are asked in the Exercise section) whose answer immediately follows from this understanding.

Lines are one of two fundamental objects of study in space. The other fundamental object is the *plane*, which we study in detail in the next section. Many complex three dimensional objects are studied by approximating their surfaces with lines and planes.

# Exercises 1.4

## Terms and Concepts

1. To find an equation of a line, what two pieces of information are needed?
2. Two distinct lines in the plane can intersect or be \_\_\_\_\_.
3. Two distinct lines in space can intersect, be \_\_\_\_\_ or be \_\_\_\_\_.
4. Use your own words to describe what it means for two lines in space to be skew.

## Problems

**In Exercises 5 – 14, write the vector, parametric and symmetric equations of the lines described.**

5. Passes through  $P = (2, -4, 1)$ , parallel to  $\vec{d} = \langle 9, 2, 5 \rangle$ .
6. Passes through  $P = (6, 1, 7)$ , parallel to  $\vec{d} = \langle -3, 2, 5 \rangle$ .
7. Passes through  $P = (2, 1, 5)$  and  $Q = (7, -2, 4)$ .
8. Passes through  $P = (1, -2, 3)$  and  $Q = (5, 5, 5)$ .
9. Passes through  $P = (0, 1, 2)$  and orthogonal to both  $\vec{d}_1 = \langle 2, -1, 7 \rangle$  and  $\vec{d}_2 = \langle 7, 1, 3 \rangle$ .
10. Passes through  $P = (5, 1, 9)$  and orthogonal to both  $\vec{d}_1 = \langle 1, 0, 1 \rangle$  and  $\vec{d}_2 = \langle 2, 0, 3 \rangle$ .
11. Passes through the point of intersection of  $\vec{\ell}_1(t)$  and  $\vec{\ell}_2(t)$  and orthogonal to both lines, where  
 $\vec{\ell}_1(t) = \langle 2, 1, 1 \rangle + t \langle 5, 1, -2 \rangle$  and  
 $\vec{\ell}_2(t) = \langle -2, -1, 2 \rangle + t \langle 3, 1, -1 \rangle$ .
12. Passes through the point of intersection of  $\ell_1(t)$  and  $\ell_2(t)$  and orthogonal to both lines, where  
 $\ell_1 = \begin{cases} x = t \\ y = -2 + 2t \\ z = 1 + t \end{cases}$  and  $\ell_2 = \begin{cases} x = 2 + t \\ y = 2 - t \\ z = 3 + 2t \end{cases}$ .
13. Passes through  $P = (1, 1)$ , parallel to  $\vec{d} = \langle 2, 3 \rangle$ .
14. Passes through  $P = (-2, 5)$ , parallel to  $\vec{d} = \langle 0, 1 \rangle$ .

**In Exercises 15 – 22, determine if the described lines are the same line, parallel lines, intersecting or skew lines. If intersecting, give the point of intersection.**

15.  $\vec{\ell}_1(t) = \langle 1, 2, 1 \rangle + t \langle 2, -1, 1 \rangle$ ,  
 $\vec{\ell}_2(t) = \langle 3, 3, 3 \rangle + t \langle -4, 2, -2 \rangle$ .

16.  $\vec{\ell}_1(t) = \langle 2, 1, 1 \rangle + t \langle 5, 1, 3 \rangle$ ,  
 $\vec{\ell}_2(t) = \langle 14, 5, 9 \rangle + t \langle 1, 1, 1 \rangle$ .

17.  $\vec{\ell}_1(t) = \langle 3, 4, 1 \rangle + t \langle 2, -3, 4 \rangle$ ,  
 $\vec{\ell}_2(t) = \langle -3, 3, -3 \rangle + t \langle 3, -2, 4 \rangle$ .

18.  $\vec{\ell}_1(t) = \langle 1, 1, 1 \rangle + t \langle 3, 1, 3 \rangle$ ,  
 $\vec{\ell}_2(t) = \langle 7, 3, 7 \rangle + t \langle 6, 2, 6 \rangle$ .

19.  $\ell_1 = \begin{cases} x = 1 + 2t \\ y = 3 - 2t \\ z = t \end{cases}$  and  $\ell_2 = \begin{cases} x = 3 - t \\ y = 3 + 5t \\ z = 2 + 7t \end{cases}$

20.  $\ell_1 = \begin{cases} x = 1.1 + 0.6t \\ y = 3.77 + 0.9t \\ z = -2.3 + 1.5t \end{cases}$  and  $\ell_2 = \begin{cases} x = 3.11 + 3.4t \\ y = 2 + 5.1t \\ z = 2.5 + 8.5t \end{cases}$

21.  $\ell_1 = \begin{cases} x = 0.2 + 0.6t \\ y = 1.33 - 0.45t \\ z = -4.2 + 1.05t \end{cases}$  and  $\ell_2 = \begin{cases} x = 0.86 + 9.2t \\ y = 0.835 - 6.9t \\ z = -3.045 + 16.1t \end{cases}$

22.  $\ell_1 = \begin{cases} x = 0.1 + 1.1t \\ y = 2.9 - 1.5t \\ z = 3.2 + 1.6t \end{cases}$  and  $\ell_2 = \begin{cases} x = 4 - 2.1t \\ y = 1.8 + 7.2t \\ z = 3.1 + 1.1t \end{cases}$

**In Exercises 23 – 26, find the distance from the point to the line.**

23.  $P = (1, 1, 1)$ ,  $\vec{\ell}(t) = \langle 2, 1, 3 \rangle + t \langle 2, 1, -2 \rangle$

24.  $P = (2, 5, 6)$ ,  $\vec{\ell}(t) = \langle -1, 1, 1 \rangle + t \langle 1, 0, 1 \rangle$

25.  $P = (0, 3)$ ,  $\vec{\ell}(t) = \langle 2, 0 \rangle + t \langle 1, 1 \rangle$

26.  $P = (1, 1)$ ,  $\vec{\ell}(t) = \langle 4, 5 \rangle + t \langle -4, 3 \rangle$

**In Exercises 27 – 28, find the distance between the two lines.**

27.  $\vec{\ell}_1(t) = \langle 1, 2, 1 \rangle + t \langle 2, -1, 1 \rangle$ ,  
 $\vec{\ell}_2(t) = \langle 3, 3, 3 \rangle + t \langle 4, 2, -2 \rangle$ .

28.  $\vec{\ell}_1(t) = \langle 0, 0, 1 \rangle + t \langle 1, 0, 0 \rangle$ ,  
 $\vec{\ell}_2(t) = \langle 0, 0, 3 \rangle + t \langle 0, 1, 0 \rangle$ .

**Exercises 29 – 31 explore special cases of the distance formulas found in Key Idea 3.**

29. Let  $Q$  be a point on the line  $\ell(t)$ . Show why the distance formula correctly gives the distance from the point to the line as 0.

30. Let lines  $\ell_1(t)$  and  $\ell_2(t)$  be intersecting lines. Show why the distance formula correctly gives the distance between these lines as 0.

31. Let lines  $\ell_1(t)$  and  $\ell_2(t)$  be parallel.
- (a) Show why the distance formula for distance between lines cannot be used as stated to find the distance between the lines.
  - (b) Show why letting  $c = (\overrightarrow{P_1P_2} \times \vec{d}_2) \times \vec{d}_2$  allows one to use the formula.
  - (c) Show how one can use the formula for the distance between a point and a line to find the distance between parallel lines.

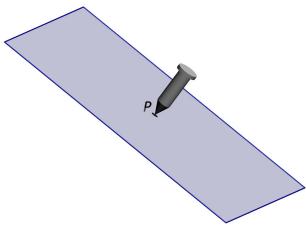


Figure 1.37: Illustrating defining a plane with a sheet of cardboard and a nail.

## 1.5 Planes

Any flat surface, such as a wall, table top or stiff piece of cardboard can be thought of as representing part of a plane. Consider a piece of cardboard with a point  $P$  marked on it. One can take a nail and stick it into the cardboard at  $P$  such that the nail is perpendicular to the cardboard; see Figure 1.37

This nail provides a “handle” for the cardboard. Moving the cardboard around moves  $P$  to different locations in space. Tilting the nail (but keeping  $P$  fixed) tilts the cardboard. Both moving and tilting the cardboard defines a different plane in space. In fact, we can define a plane by: 1) the location of  $P$  in space, and 2) the direction of the nail.

The previous section showed that one can define a line given a point on the line and the direction of the line (usually given by a vector). One can make a similar statement about planes: we can define a plane in space given a point on the plane and the direction the plane “faces” (using the description above, the direction of the nail). Once again, the direction information will be supplied by a vector, called a **normal vector**, that is orthogonal to the plane.

What exactly does “orthogonal to the plane” mean? Choose any two points  $P$  and  $Q$  in the plane, and consider the vector  $\vec{PQ}$ . We say a vector  $\vec{n}$  is orthogonal to the plane if  $\vec{n}$  is perpendicular to  $\vec{PQ}$  for all choices of  $P$  and  $Q$ ; that is, if  $\vec{n} \cdot \vec{PQ} = 0$  for all  $P$  and  $Q$ .

This gives us way of writing an equation describing the plane. Let  $P = (x_0, y_0, z_0)$  be a point in the plane and let  $\vec{n} = \langle a, b, c \rangle$  be a normal vector to the plane. A point  $Q = (x, y, z)$  lies in the plane defined by  $P$  and  $\vec{n}$  if, and only if,  $\vec{PQ}$  is orthogonal to  $\vec{n}$ . Knowing  $\vec{PQ} = \langle x - x_0, y - y_0, z - z_0 \rangle$ , consider:

$$\begin{aligned}\vec{PQ} \cdot \vec{n} &= 0 \\ \langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0\end{aligned}\tag{1.9}$$

Equation (1.9) defines an *implicit* function describing the plane. More algebra produces:

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

The right hand side is just a number, so we replace it with  $d$ :

$$ax + by + cz = d.\tag{1.10}$$

As long as  $c \neq 0$ , we can solve for  $z$ :

$$z = \frac{1}{c}(d - ax - by).\tag{1.11}$$

Equation (1.11) is especially useful as many computer programs can graph functions in this form. Equations (1.9) and (1.10) have specific names, given next.

**Definition 13 Equations of a Plane in Standard and General Forms**

The plane passing through the point  $P = (x_0, y_0, z_0)$  with normal vector  $\vec{n} = \langle a, b, c \rangle$  can be described by an equation with **standard form**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0;$$

the equation's **general form** is

$$ax + by + cz = d.$$

A key to remember throughout this section is this: to find the equation of a plane, we need a point and a normal vector. We will give several examples of finding the equation of a plane, and in each one different types of information are given. In each case, we need to use the given information to find a point on the plane and a normal vector.

**Example 30 Finding the equation of a plane.**

Write the equation of the plane that passes through the points  $P = (1, 1, 0)$ ,  $Q = (1, 2, -1)$  and  $R = (0, 1, 2)$  in standard form.

**SOLUTION** We need a vector  $\vec{n}$  that is orthogonal to the plane. Since  $P$ ,  $Q$  and  $R$  are in the plane, so are the vectors  $\vec{PQ}$  and  $\vec{PR}$ ;  $\vec{PQ} \times \vec{PR}$  is orthogonal to  $\vec{PQ}$  and  $\vec{PR}$  and hence the plane itself.

It is straightforward to compute  $\vec{n} = \vec{PQ} \times \vec{PR} = \langle 2, 1, 1 \rangle$ . We can use any point we wish in the plane (any of  $P$ ,  $Q$  or  $R$  will do) and we arbitrarily choose  $P$ . Following Definition 13, the equation of the plane in standard form is

$$2(x - 1) + (y - 1) + z = 0.$$

The plane is sketched in Figure 1.38.

We have just demonstrated the fact that any three non-collinear points define a plane. (This is why a three-legged stool does not "rock;" it's three feet always lie in a plane. A four-legged stool will rock unless all four feet lie in the same plane.)

**Example 31 Finding the equation of a plane.**

Verify that lines  $\ell_1$  and  $\ell_2$ , whose parametric equations are given below, intersect, then give the equation of the plane that contains these two lines in general form.

$$\begin{array}{ll} \ell_1: \begin{aligned} x &= -5 + 2s \\ y &= 1 + s \\ z &= -4 + 2s \end{aligned} & \ell_2: \begin{aligned} x &= 2 + 3t \\ y &= 1 - 2t \\ z &= 1 + t \end{aligned} \end{array}$$

**SOLUTION** The lines clearly are not parallel. If they do not intersect, they are skew, meaning there is not a plane that contains them both. If they do intersect, there is such a plane.

To find their point of intersection, we set the  $x$ ,  $y$  and  $z$  equations equal to each other and solve for  $s$  and  $t$ :

$$\begin{aligned} -5 + 2s &= 2 + 3t \\ 1 + s &= 1 - 2t \quad \Rightarrow \quad s = 2, \quad t = -1. \\ -4 + 2s &= 1 + t \end{aligned}$$

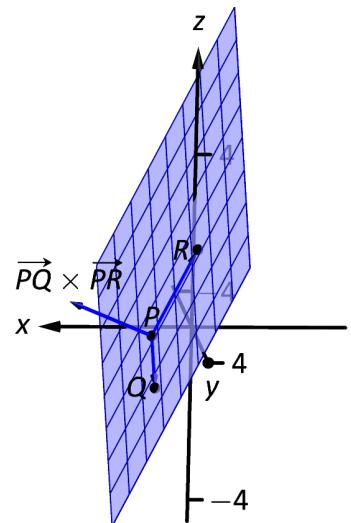


Figure 1.38: Sketching the plane in Example 30.

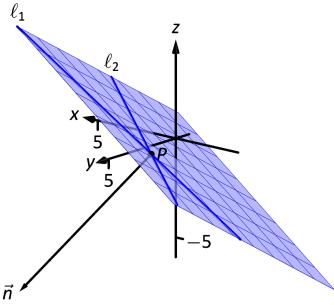


Figure 1.39: Sketching the plane in Example 31.

When  $s = 2$  and  $t = -1$ , the lines intersect at the point  $P = (-1, 3, 0)$ .

Let  $\vec{d}_1 = \langle 2, 1, 2 \rangle$  and  $\vec{d}_2 = \langle 3, -2, 1 \rangle$  be the directions of lines  $\ell_1$  and  $\ell_2$ , respectively. A normal vector to the plane containing these two lines will also be orthogonal to  $\vec{d}_1$  and  $\vec{d}_2$ . Thus we find a normal vector  $\vec{n}$  by computing  $\vec{n} = \vec{d}_1 \times \vec{d}_2 = \langle 5, 4, -7 \rangle$ .

We can pick any point in the plane with which to write our equation; each line gives us infinite choices of points. We choose  $P$ , the point of intersection. We follow Definition 13 to write the plane's equation in general form:

$$\begin{aligned} 5(x + 1) + 4(y - 3) - 7z &= 0 \\ 5x + 5 + 4y - 12 - 7z &= 0 \\ 5x + 4y - 7z &= 7. \end{aligned}$$

The plane's equation in general form is  $5x + 4y - 7z = 7$ ; it is sketched in Figure 1.39.

### Example 32 Finding the equation of a plane

Give the equation, in standard form, of the plane that passes through the point  $P = (-1, 0, 1)$  and is orthogonal to the line with vector equation  $\vec{\ell}(t) = \langle -1, 0, 1 \rangle + t \langle 1, 2, 2 \rangle$ .

**SOLUTION** As the plane is to be orthogonal to the line, the plane must be orthogonal to the direction of the line given by  $\vec{d} = \langle 1, 2, 2 \rangle$ . We use this as our normal vector. Thus the plane's equation, in standard form, is

$$(x + 1) + 2y + 2(z - 1) = 0.$$

The line and plane are sketched in Figure 1.40.

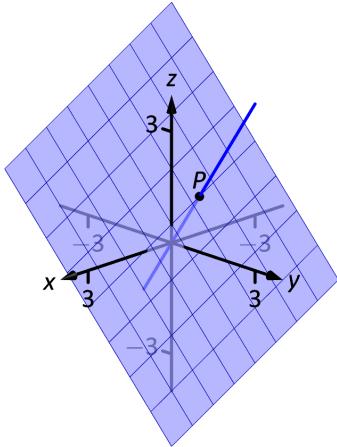


Figure 1.40: The line and plane in Example 32.

### Example 33 Finding the intersection of two planes

Give the parametric equations of the line that is the intersection of the planes  $p_1$  and  $p_2$ , where:

$$\begin{aligned} p_1 : x - (y - 2) + (z - 1) &= 0 \\ p_2 : -2(x - 2) + (y + 1) + (z - 3) &= 0 \end{aligned}$$

**SOLUTION** To find an equation of a line, we need a point on the line and the direction of the line.

We can find a point on the line by solving each equation of the planes for  $z$ :

$$\begin{aligned} p_1 : z &= -x + y - 1 \\ p_2 : z &= 2x - y - 2 \end{aligned}$$

We can now set these two equations equal to each other (i.e., we are finding values of  $x$  and  $y$  where the planes have the same  $z$  value):

$$\begin{aligned} -x + y - 1 &= 2x - y - 2 \\ 2y &= 3x - 1 \\ y &= \frac{1}{2}(3x - 1) \end{aligned}$$

We can choose any value for  $x$ ; we choose  $x = 1$ . This determines that  $y = 1$ . We can now use the equations of either plane to find  $z$ : when  $x = 1$  and  $y = 1$ ,  $z = -1$  on both planes. We have found a point  $P$  on the line:  $P = (1, 1, -1)$ .

We now need the direction of the line. Since the line lies in each plane, its direction is orthogonal to a normal vector for each plane. Considering the

equations for  $p_1$  and  $p_2$ , we can quickly determine their normal vectors. For  $p_1$ ,  $\vec{n}_1 = \langle 1, -1, 1 \rangle$  and for  $p_2$ ,  $\vec{n}_2 = \langle -2, 1, 1 \rangle$ . A direction orthogonal to both of these directions is their cross product:  $\vec{d} = \vec{n}_1 \times \vec{n}_2 = \langle -2, -3, -1 \rangle$ .

The parametric equations of the line through  $P = (1, 1, -1)$  in the direction of  $\vec{d} = \langle -2, -3, -1 \rangle$  is:

$$\ell : \quad x = -2t + 1 \quad y = -3t + 1 \quad z = -t - 1.$$

The planes and line are graphed in Figure 1.41.

#### Example 34 Finding the intersection of a plane and a line

Find the point of intersection, if any, of the line  $\ell(t) = \langle 3, -3, -1 \rangle + t \langle -1, 2, 1 \rangle$  and the plane with equation in general form  $2x + y + z = 4$ .

**SOLUTION** The equation of the plane shows that the vector  $\vec{n} = \langle 2, 1, 1 \rangle$  is a normal vector to the plane, and the equation of the line shows that the line moves parallel to  $\vec{d} = \langle -1, 2, 1 \rangle$ . Since these are not orthogonal, we know there is a point of intersection. (If there were orthogonal, it would mean that the plane and line were parallel to each other, either never intersecting or the line was in the plane itself.)

To find the point of intersection, we need to find a  $t$  value such that  $\ell(t)$  satisfies the equation of the plane. Rewriting the equation of the line with parametric equations will help:

$$\ell(t) = \begin{cases} x = 3 - t \\ y = -3 + 2t \\ z = -1 + t \end{cases}$$

Replacing  $x$ ,  $y$  and  $z$  in the equation of the plane with the expressions containing  $t$  found in the equation of the line allows us to determine a  $t$  value that indicates the point of intersection:

$$\begin{aligned} 2x + y + z &= 4 \\ 2(3 - t) + (-3 + 2t) + (-1 + t) &= 4 \\ t &= 2. \end{aligned}$$

When  $t = 2$ , the point on the line satisfies the equation of the plane; that point is  $\ell(2) = \langle 1, 1, 1 \rangle$ . Thus the point  $(1, 1, 1)$  is the point of intersection between the plane and the line, illustrated in Figure 1.42.

### Distances

Just as it was useful to find distances between points and lines in the previous section, it is also often necessary to find the distance from a point to a plane.

Consider Figure 1.43, where a plane with normal vector  $\vec{n}$  is sketched containing a point  $P$  and a point  $Q$ , not on the plane, is given. We measure the distance from  $Q$  to the plane by measuring the length of the projection of  $\overrightarrow{PQ}$  onto  $\vec{n}$ . That is, we want:

$$\| \text{proj}_{\vec{n}} \overrightarrow{PQ} \| = \left\| \frac{\vec{n} \cdot \overrightarrow{PQ}}{\| \vec{n} \|^2} \vec{n} \right\| = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{\| \vec{n} \|} \quad (1.12)$$

Equation (1.12) is important as it does more than just give the distance between a point and a plane. We will see how it allows us to find several other distances

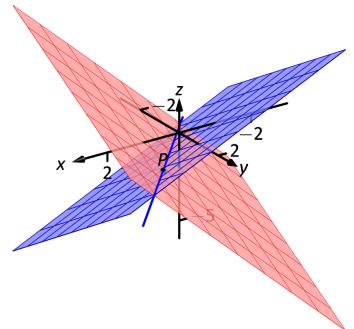


Figure 1.41: Graphing the planes and their line of intersection in Example 33.

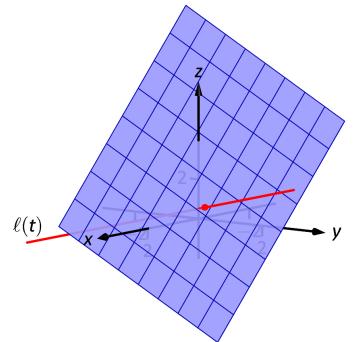


Figure 1.42: Illustrating the intersection of a line and a plane in Example 34.

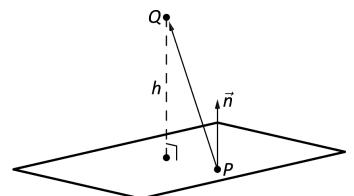


Figure 1.43: Illustrating finding the distance from a point to a plane.

as well: the distance between parallel planes and the distance from a line and a plane. Because Equation (1.12) is important, we restate it as a Key Idea.

**Key Idea 4 Distance from a Point to a Plane**

Let a plane with normal vector  $\vec{n}$  be given, and let  $Q$  be a point. The distance  $h$  from  $Q$  to the plane is

$$h = \frac{|\vec{n} \cdot \vec{PQ}|}{\|\vec{n}\|},$$

where  $P$  is any point in the plane.

**Example 35 Distance between a point and a plane**

Find the distance between the point  $Q = (2, 1, 4)$  and the plane with equation  $2x - 5y + 6z = 9$ .

**SOLUTION** Using the equation of the plane, we find the normal vector  $\vec{n} = \langle 2, -5, 6 \rangle$ . To find a point on the plane, we can let  $x$  and  $y$  be anything we choose, then let  $z$  be whatever satisfies the equation. Letting  $x$  and  $y$  be 0 seems simple; this makes  $z = 1.5$ . Thus we let  $P = \langle 0, 0, 1.5 \rangle$ , and  $\vec{PQ} = \langle 2, 1, 2.5 \rangle$ .

The distance  $h$  from  $Q$  to the plane is given by Key Idea 4:

$$\begin{aligned} h &= \frac{|\vec{n} \cdot \vec{PQ}|}{\|\vec{n}\|} \\ &= \frac{|\langle 2, -5, 6 \rangle \cdot \langle 2, 1, 2.5 \rangle|}{\|\langle 2, -5, 6 \rangle\|} \\ &= \frac{|-16|}{\sqrt{65}} \\ &\approx 1.98. \end{aligned}$$

We can use Key Idea 4 to find other distances. Given two parallel planes, we can find the distance between these planes by letting  $P$  be a point on one plane and  $Q$  a point on the other. If  $\ell$  is a line parallel to a plane, we can use the Key Idea to find the distance between them as well: again, let  $P$  be a point in the plane and let  $Q$  be any point on the line. (One can also use Key Idea 3.) The Exercise section contains problems of these types.

These past two sections have not explored lines and planes in space as an exercise of mathematical curiosity. However, there are many, many applications of these fundamental concepts. Complex shapes can be modeled (or, *approximated*) using planes. For instance, part of the exterior of an aircraft may have a complex, yet smooth, shape, and engineers will want to know how air flows across this piece as well as how heat might build up due to air friction. Many equations that help determine air flow and heat dissipation are difficult to apply to arbitrary surfaces, but simple to apply to planes. By approximating a surface with millions of small planes one can more readily model the needed behavior.

# Exercises 1.5

## Terms and Concepts

- In order to find the equation of a plane, what two pieces of information must one have?
- What is the relationship between a plane and one of its normal vectors?

## Problems

**In Exercises 3 – 6, give any two points in the given plane.**

- $2x - 4y + 7z = 2$
- $3(x + 2) + 5(y - 9) - 4z = 0$
- $x = 2$
- $4(y + 2) - (z - 6) = 0$

**In Exercises 7 – 20, give the equation of the described plane in standard and general forms.**

- Passes through  $(2, 3, 4)$  and has normal vector  $\vec{n} = \langle 3, -1, 7 \rangle$ .
- Passes through  $(1, 3, 5)$  and has normal vector  $\vec{n} = \langle 0, 2, 4 \rangle$ .
- Passes through the points  $(1, 2, 3)$ ,  $(3, -1, 4)$  and  $(1, 0, 1)$ .
- Passes through the points  $(5, 3, 8)$ ,  $(6, 4, 9)$  and  $(3, 3, 3)$ .

- Contains the intersecting lines  
 $\ell_1(t) = \langle 2, 1, 2 \rangle + t \langle 1, 2, 3 \rangle$  and  
 $\ell_2(t) = \langle 2, 1, 2 \rangle + t \langle 2, 5, 4 \rangle$ .

- Contains the intersecting lines  
 $\ell_1(t) = \langle 5, 0, 3 \rangle + t \langle -1, 1, 1 \rangle$  and  
 $\ell_2(t) = \langle 1, 4, 7 \rangle + t \langle 3, 0, -3 \rangle$ .

- Contains the parallel lines  
 $\ell_1(t) = \langle 1, 1, 1 \rangle + t \langle 1, 2, 3 \rangle$  and  
 $\ell_2(t) = \langle 1, 1, 2 \rangle + t \langle 1, 2, 3 \rangle$ .

- Contains the parallel lines  
 $\ell_1(t) = \langle 1, 1, 1 \rangle + t \langle 4, 1, 3 \rangle$  and  
 $\ell_2(t) = \langle 2, 2, 2 \rangle + t \langle 4, 1, 3 \rangle$ .

- Contains the point  $(2, -6, 1)$  and the line  
$$\ell(t) = \begin{cases} x = 2 + 5t \\ y = 2 + 2t \\ z = -1 + 2t \end{cases}$$

- Contains the point  $(5, 7, 3)$  and the line

$$\ell(t) = \begin{cases} x = t \\ y = t \\ z = t \end{cases}$$

- Contains the point  $(5, 7, 3)$  and is orthogonal to the line  
 $\ell(t) = \langle 4, 5, 6 \rangle + t \langle 1, 1, 1 \rangle$ .

- Contains the point  $(4, 1, 1)$  and is orthogonal to the line

$$\ell(t) = \begin{cases} x = 4 + 4t \\ y = 1 + 1t \\ z = 1 + 1t \end{cases}$$

- Contains the point  $(-4, 7, 2)$  and is parallel to the plane  
 $3(x - 2) + 8(y + 1) - 10z = 0$ .

- Contains the point  $(1, 2, 3)$  and is parallel to the plane  
 $x = 5$ .

**In Exercises 21 – 22, give the equation of the line that is the intersection of the given planes.**

- $p1 : 3(x - 2) + (y - 1) + 4z = 0$ , and  
 $p2 : 2(x - 1) - 2(y + 3) + 6(z - 1) = 0$ .

- $p1 : 5(x - 5) + 2(y + 2) + 4(z - 1) = 0$ , and  
 $p2 : 3x - 4(y - 1) + 2(z - 1) = 0$ .

**In Exercises 23 – 26, find the point of intersection between the line and the plane.**

- line:  $\langle 5, 1, -1 \rangle + t \langle 2, 2, 1 \rangle$ ,  
plane:  $5x - y - z = -3$

- line:  $\langle 4, 1, 0 \rangle + t \langle 1, 0, -1 \rangle$ ,  
plane:  $3x + y - 2z = 8$

- line:  $\langle 1, 2, 3 \rangle + t \langle 3, 5, -1 \rangle$ ,  
plane:  $3x - 2y - z = 4$

- line:  $\langle 1, 2, 3 \rangle + t \langle 3, 5, -1 \rangle$ ,  
plane:  $3x - 2y - z = -4$

**In Exercises 27 – 30, find the given distances.**

- The distance from the point  $(1, 2, 3)$  to the plane  
 $3(x - 1) + (y - 2) + 5(z - 2) = 0$ .

- The distance from the point  $(2, 6, 2)$  to the plane  
 $2(x - 1) - y + 4(z + 1) = 0$ .

- The distance between the parallel planes  
 $x + y + z = 0$  and  
 $(x - 2) + (y - 3) + (z + 4) = 0$

30. The distance between the parallel planes  
 $2(x - 1) + 2(y + 1) + (z - 2) = 0$  and  
 $2(x - 3) + 2(y - 1) + (z - 3) = 0$
31. Show why if the point  $Q$  lies in a plane, then the distance formula correctly gives the distance from the point to the plane as 0.
32. How is Exercise 30 in Section 1.4 easier to answer once we have an understanding of planes?

# A: SOLUTIONS TO SELECTED PROBLEMS

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## Chapter 1

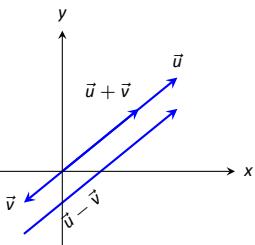
### Section 1.1

1. Answers will vary.  
3. A vector with magnitude 1.  
5. It stretches the vector by a factor of 2, and points it in the opposite direction.

7.  $\vec{PQ} = \langle -4, 4 \rangle = -4\vec{i} + 4\vec{j}$   
9.  $\vec{PQ} = \langle 2, 2, 0 \rangle = 2\vec{i} + 2\vec{j}$

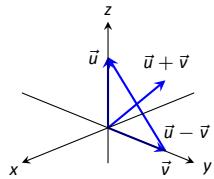
11. (a)  $\vec{u} + \vec{v} = \langle 3, 2, 1 \rangle$ ;  $\vec{u} - \vec{v} = \langle -1, 0, -3 \rangle$ ;  
 $\pi\vec{u} - \sqrt{2}\vec{v} = \langle \pi - 2\sqrt{2}, \pi - \sqrt{2}, -\pi - 2\sqrt{2} \rangle$ .  
(c)  $\vec{x} = \langle -1, 0, -3 \rangle$ .

13.



Sketch of  $\vec{u} - \vec{v}$  shifted for clarity.

15.



17.  $\|\vec{u}\| = \sqrt{17}$ ,  $\|\vec{v}\| = \sqrt{3}$ ,  $\|\vec{u} + \vec{v}\| = \sqrt{14}$ ,  $\|\vec{u} - \vec{v}\| = \sqrt{26}$

19.  $\|\vec{u}\| = 7$ ,  $\|\vec{v}\| = 35$ ,  $\|\vec{u} + \vec{v}\| = 42$ ,  $\|\vec{u} - \vec{v}\| = 28$

21.  $\vec{u} = \langle 3/\sqrt{30}, 7/\sqrt{30} \rangle$

23.  $\vec{u} = \langle 1/3, -2/3, 2/3 \rangle$

25.  $\vec{u} = \langle \cos 50^\circ, \sin 50^\circ \rangle \approx \langle 0.643, 0.766 \rangle$ .

27.

$$\begin{aligned} \|\vec{u}\| &= \sqrt{\sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi + \cos^2 \theta} \\ &= \sqrt{\sin^2 \theta (\cos^2 \varphi + \sin^2 \varphi) + \cos^2 \theta} \\ &= \sqrt{\sin^2 \theta + \cos^2 \theta} \\ &= 1. \end{aligned}$$

29. The force on each chain is 100lb.

31. The force on each chain is 50lb.

33.  $\theta = 5.71^\circ$ ; the weight is lifted 0.005 ft (about 1/16th of an inch).

35.  $\theta = 84.29^\circ$ ; the weight is lifted 9 ft.

### Section 1.2

1. Scalar

3. By considering the sign of the dot product of the two vectors. If the dot product is positive, the angle is acute; if the dot product is negative, the angle is obtuse.

5.  $-22$

7.  $3$

9. not defined

11. Answers will vary.

13.  $\theta = 0.3218 \approx 18.43^\circ$

15.  $\theta = \pi/4 = 45^\circ$

17. Answers will vary; two possible answers are  $\langle -7, 4 \rangle$  and  $\langle 14, -8 \rangle$ .

19. Answers will vary; two possible answers are  $\langle 1, 0, -1 \rangle$  and  $\langle 4, 5, -9 \rangle$ .

21.  $\text{proj}_y \vec{u} = \langle -1/2, 3/2 \rangle$ .

23.  $\text{proj}_y \vec{u} = \langle -1/2, -1/2 \rangle$ .

25.  $\text{proj}_y \vec{u} = \langle 1, 2, 3 \rangle$ .

27.  $\vec{u} = \langle -1/2, 3/2 \rangle + \langle 3/2, 1/2 \rangle$ .

29.  $\vec{u} = \langle -1/2, -1/2 \rangle + \langle -5/2, 5/2 \rangle$ .

31.  $\vec{u} = \langle 1, 2, 3 \rangle + \langle 0, 3, -2 \rangle$ .

33. 1.96lb

35. 141.42ft-lb

37. 500ft-lb

39. 500ft-lb

### Section 1.3

1. vector

3. "Perpendicular" is one answer.

5. Torque

7.  $\vec{u} \times \vec{v} = \langle 11, 1, -17 \rangle$

9.  $\vec{u} \times \vec{v} = \langle 47, -36, -44 \rangle$

11.  $\vec{u} \times \vec{v} = \langle 0, 0, 0 \rangle$

13.  $\vec{i} \times \vec{k} = -\vec{j}$

15. Answers will vary.

17. 5

19. 0

21.  $\sqrt{14}$

23. 3

25.  $5\sqrt{2}/2$

27. 1

29. 7

31. 2

33.  $\pm \frac{1}{\sqrt{6}} \langle 1, 1, -2 \rangle$

35.  $\langle 0, \pm 1, 0 \rangle$

37. 87.5ft-lb

39.  $200/3 \approx 66.67$ ft-lb

41. With  $\vec{u} = \langle u_1, u_2, u_3 \rangle$  and  $\vec{v} = \langle v_1, v_2, v_3 \rangle$ , we have

$$\begin{aligned} \vec{u} \cdot (\vec{u} \times \vec{v}) &= \langle u_1, u_2, u_3 \rangle \cdot ((u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1)) \\ &= u_1(u_2 v_3 - u_3 v_2) - u_2(u_1 v_3 - u_3 v_1) + u_3(u_1 v_2 - u_2 v_1) \\ &= 0. \end{aligned}$$

## Section 1.4

1. A point on the line and the direction of the line.
3. parallel, skew
5. vector:  $\ell(t) = \langle 2, -4, 1 \rangle + t \langle 9, 2, 5 \rangle$   
parametric:  $x = 2 + 9t, y = -4 + 2t, z = 1 + 5t$   
symmetric:  $(x - 2)/9 = (y + 4)/2 = (z - 1)/5$
7. Answers can vary: vector:  $\ell(t) = \langle 2, 1, 5 \rangle + t \langle 5, -3, -1 \rangle$   
parametric:  $x = 2 + 5t, y = 1 - 3t, z = 5 - t$   
symmetric:  $(x - 2)/5 = -(y - 1)/3 = -(z - 5)$
9. Answers can vary; here the direction is given by  $\vec{d}_1 \times \vec{d}_2$ : vector:  
 $\ell(t) = \langle 0, 1, 2 \rangle + t \langle -10, 43, 9 \rangle$   
parametric:  $x = -10t, y = 1 + 43t, z = 2 + 9t$   
symmetric:  $-x/10 = (y - 1)/43 = (z - 2)/9$
11. Answers can vary; here the direction is given by  $\vec{d}_1 \times \vec{d}_2$ : vector:  
 $\ell(t) = \langle 7, 2, -1 \rangle + t \langle 1, -1, 2 \rangle$   
parametric:  $x = 7 + t, y = 2 - t, z = -1 + 2t$   
symmetric:  $x - 7 = 2 - y = (z + 1)/2$
13. vector:  $\ell(t) = \langle 1, 1 \rangle + t \langle 2, 3 \rangle$   
parametric:  $x = 1 + 2t, y = 1 + 3t$   
symmetric:  $(x - 1)/2 = (y - 1)/3$
15. parallel
17. intersecting;  $\vec{\ell}_1(3) = \vec{\ell}_2(4) = \langle 9, -5, 13 \rangle$
19. skew
21. same
23.  $\sqrt{41}/3$
25.  $5\sqrt{2}/2$
27.  $3/\sqrt{2}$
29. Since both  $P$  and  $Q$  are on the line,  $\overrightarrow{PQ}$  is parallel to  $\vec{d}$ . Thus  $\overrightarrow{PQ} \times \vec{d} = \vec{0}$ , giving a distance of 0.
31.
  - The distance formula cannot be used because since  $\vec{d}_1$  and  $\vec{d}_2$  are parallel,  $\vec{c}$  is  $\vec{0}$  and we cannot divide by  $\|\vec{0}\|$ .
  - Since  $\vec{d}_1$  and  $\vec{d}_2$  are parallel,  $\overrightarrow{P_1P_2}$  lies in the plane formed by the two lines. Thus  $\overrightarrow{P_1P_2} \times \vec{d}_2$  is orthogonal to this plane, and  $\vec{c} = (\overrightarrow{P_1P_2} \times \vec{d}_2) \times \vec{d}_2$  is parallel to the plane, but still orthogonal to both  $\vec{d}_1$  and  $\vec{d}_2$ . We desire the length of the projection of  $\overrightarrow{P_1P_2}$  onto  $\vec{c}$ , which is what the formula provides.
  - Since the lines are parallel, one can measure the distance between the lines at any location on either line (just as to find the distance between straight railroad tracks, one can

use a measuring tape anywhere along the track, not just at one specific place.) Let  $P = P_1$  and  $Q = P_2$  as given by the equations of the lines, and apply the formula for distance between a point and a line.

## Section 1.5

1. A point in the plane and a normal vector (i.e., a direction orthogonal to the plane).
3. Answers will vary.
5. Answers will vary.
7. Standard form:  $3(x - 2) - (y - 3) + 7(z - 4) = 0$   
general form:  $3x - y + 7z = 31$
9. Answers may vary;  
Standard form:  $8(x - 1) + 4(y - 2) - 4(z - 3) = 0$   
general form:  $8x + 4y - 4z = 4$
11. Answers may vary;  
Standard form:  $-7(x - 2) + 2(y - 1) + (z - 2) = 0$   
general form:  $-7x + 2y + z = -10$
13. Answers may vary;  
Standard form:  $2(x - 1) - (y - 1) = 0$   
general form:  $2x - y = 1$
15. Answers may vary;  
Standard form:  $2(x - 2) - (y + 6) - 4(z - 1) = 0$   
general form:  $2x - y - 4z = 6$
17. Answers may vary;  
Standard form:  $(x - 5) + (y - 7) + (z - 3) = 0$   
general form:  $x + y + z = 15$
19. Answers may vary;  
Standard form:  $3(x + 4) + 8(y - 7) - 10(z - 2) = 0$   
general form:  $3x + 8y - 10z = 24$
21. Answers may vary:  

$$\ell = \begin{cases} x = 14t \\ y = -1 - 10t \\ z = 2 - 8t \end{cases}$$
23.  $(-3, -7, -5)$
25. No point of intersection; the plane and line are parallel.
27.  $\sqrt{5/7}$
29.  $1/\sqrt{3}$
31. If  $P$  is any point in the plane, and  $Q$  is also in the plane, then  $\overrightarrow{PQ}$  lies parallel to the plane and is orthogonal to  $\vec{n}$ , the normal vector. Thus  $\vec{n} \cdot \overrightarrow{PQ} = 0$ , giving the distance as 0.

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