Math 2565 Accelerated Calculus II

Fall 2018 Edition, University of Lethbridge

An adaptation of the APEX Calculus textbook, edited by Sean Fitzpatrick

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Preface

This a custom textbook that covers the entire curriculum for the course Math 2565 (Accelerated Calculus II) at the University of Lethbridge at minimal cost to the student. It is also an *Open Education Resource*. As a student, you are free to keep as many copies as you want, for as long as you want. You can print it, in whole or in part, or share it with a friend. As an instructor, I am free to modify the content as I see fit, whether this means editing to fit our curriculum, or simply correcting typos.

Math 2565 covers elements of calculus of both one and several variables. Students who have purchased a commercial textbook on single-variable calculus for Math 1560 may find that this text does not contain everything that they need for Math 2565. There are many commercially available textbooks that cover this material, but none are affordable, especially for a one semester course.

Most of this textbook is adapted from the *APEX Calculus* textbook project, which originated in the Department of Applied Mathematics at the Virginia Military Institute. (See apexcalculus.com.) On the following page you'll find the original preface from their text, which explains their project in more detail. They have produced calculus textbook that is **free** in two regards: it's free to you, the student, in the sense that you can download the PDF from their website at no cost, and do with it as you wish (share it, print it, etc.) It's also free in the sense of being an *open source* textbook: the authors have made all the files needed to produce the textbook freely available, and allow others (such as myself) to edit the text to suit the needs of various courses (such as Math 2565).

What's even better is that the textbook is of remarkably high production quality: unlike many free texts, it is polished and professionally produced, with graphics on almost every page, and a large collection of exercises (with selected answers!). If you're using the electronic version of the textbook, you'll even find that some of the graphics (those of 3D images) are interactive! Clicking on an image allows you to rotate it, and zoom in or out, to get a better understanding of the geometry involved. (This feature is only supported in Acrobat Reader, and not other PDF readers.)

Since Math 2565 includes content on differential equations, but the APEX Calculus textbook does not, I've also relied on the text *Notes on Diffy Qs: Differential Equations for Engineers*, by Jiří Lebl. (See jirka.org/diffyqs.) This is another open source textbook that is free to use and adapt as needed. I have borrowed the first chapter of this text, and included it as a chapter in the APEX Calculus textbook, to provide you with a single textbook for the course. The integration is not entirely seamless: although I made changes to the formatting of the differential equations chapter so that it agrees in appearance with the rest of the textbook, I have not made any changes to Jiří's writing style, which is more conversational and familiar than the rest of the text. I felt it was more appropriate to preserve the voice of the original author.

I hope that you find this textbook useful. If you find any errors, or if you have any suggestions as to how the material could be better arranged or presented, please let me know. (One advantage of an open source textbook is that it can be edited at any time!)

Sean Fitzpatrick
Department of Mathematics and Computer Science
University of Lethbridge
May, 2018

Preface to APEX Calculus

A Note on Using this Text

Thank you for reading this short preface. Allow us to share a few key points about the text so that you may better understand what you will find beyond this page.

This text is Part II of a three–text series on Calculus. The first part covers material taught in many "Calc 1" courses: limits, derivatives, and the basics of integration, found in Chapters 1 through 6.1. The second text covers material often taught in "Calc 2:" integration and its applications, along with an introduction to sequences, series and Taylor Polynomials, found in Chapters 5 through 8. The third text covers topics common in "Calc 3" or "multivariable calc:" parametric equations, polar coordinates, vector–valued functions, and functions of more than one variable, found in Chapters 9 through 13. All three are available separately for free at `www.vmi.edu/APEX`. These three texts are intended to work together and make one cohesive text, *APEX Calculus*, which can also be downloaded from the website.

Printing the entire text as one volume makes for a large, heavy, cumbersome book. One can certainly only print the pages they currently need, but some prefer to have a nice, bound copy of the text. Therefore this text has been split into these three manageable parts, each of which can be purchased for under $15 at Amazon.com.

A result of this splitting is that sometimes a concept is said to be explored in an "earlier/later section," though that section does not actually appear in this particular text. Also, the index makes reference to topics, and page numbers, that do not appear in this text. This is done intentionally to show the reader what topics are available for study. Downloading the .pdf of *APEX Calculus* will ensure that you have all the content.

For Students: How to Read this Text

Mathematics textbooks have a reputation for being hard to read. High–level mathematical writing often seeks to say much with few words, and this style often seeps into texts of lower–level topics. This book was written with the goal of being easier to read than many other calculus textbooks, without becoming too verbose.

Each chapter and section starts with an introduction of the coming material, hopefully setting the stage for "why you should care," and ends with a look ahead to see how the just–learned material helps address future problems.

Please read the text; it is written to explain the concepts of Calculus. There are numerous examples to demonstrate the meaning of definitions, the truth of theorems, and the application of mathematical techniques. When you encounter a sentence you don't understand, read it again. If it still doesn't make sense, read on anyway, as sometimes confusing sentences are explained by later sentences.

You don't have to read every equation. The examples generally show "all" the steps needed to solve a problem. Sometimes reading through each step is helpful; sometimes it is confusing. When the steps are illustrating a new technique, one probably should follow each step closely to learn the new technique. When the steps are showing the mathematics needed to find a number to be used later, one can usually skip ahead and see how that number is being used, instead of getting bogged down in reading how the number was found.

Most proofs have been omitted. In mathematics, *proving* something is always true is extremely important, and entails much more than testing to see if it works twice. However, students often are confused by the details of a proof, or become concerned that they should have been able to construct this proof on their own. To alleviate this potential problem, we do not include the proofs to most theorems in the text. The interested reader is highly encouraged to find proofs online or from their instructor. In most cases, one is very capable of understanding what a theorem *means* and *how to apply it* without knowing fully *why* it is true.

Interactive, 3D Graphics

New to Version 3.0 is the addition of interactive, 3D graphics in the .pdf version. Nearly all graphs of objects in space can be rotated, shifted, and zoomed in/out so the reader can better understand the object illustrated.

As of this writing, the only pdf viewers that support these 3D graphics are Adobe Reader & Acrobat (and only the versions for PC/Mac/Unix/Linux computers, not tablets or smartphones). To activate the interactive mode, click on the image. Once activated, one can click/drag to rotate the object and use the scroll wheel on a mouse to zoom in/out. (A great way to investigate an image is to first zoom in on the page of the pdf viewer so the graphic itself takes up much of the screen, then zoom inside the graphic itself.) A CTRL-click/drag pans the object left/right or up/down. By right-clicking on the graph one can access a menu of other options, such as changing the lighting scheme or perspective. One can also revert the graph back to its default view. If you wish to deactive the interactivity, one can right-click and choose the “Disable Content” option.

Thanks

There are many people who deserve recognition for the important role they have played in the development of this text. First, I thank Michelle for her support and encouragement, even as this “project from work” occupied my time and attention at home. Many thanks to Troy Siemers, whose most important contributions extend far beyond the sections he wrote or the 227 figures he coded in Asymptote for 3D interaction. He provided incredible support, advice and encouragement for which I am very grateful. My thanks to Brian Heinold and Dimplekumar Chalishajar for their contributions and to Jennifer Bowen for reading through so much material and providing great feedback early on. Thanks to Troy, Lee Dewald, Dan Joseph, Meagan Herald, Bill Lowe, John David, Vonda Walsh, Geoff Cox, Jessica Libertini and other faculty of VMI who have given me numerous suggestions and corrections based on their experience with teaching from the text. (Special thanks to Troy, Lee & Dan for their patience in teaching Calc III while I was still writing the Calc III material.) Thanks to Randy Cone for encouraging his tutors of VMI’s Open Math Lab to read through the text and check the solutions, and thanks to the tutors for spending their time doing so. A very special thanks to Kristi Brown and Paul Janiczek who took this opportunity far above & beyond what I expected, meticulously checking every solution and carefully reading every example. Their comments have been extraordinarily helpful. I am also thankful for the support provided by Wane Schneiter, who as my Dean provided me with extra time to work on this project. I am blessed to have so many people give of their time to make this book better.

APEX – Affordable Print and Electronic teXts

APEX is a consortium of authors who collaborate to produce high–quality, low–cost textbooks. The current textbook–writing paradigm is facing a potential revolution as desktop publishing and electronic formats increase in popularity. However, writing a good textbook is no easy task, as the time requirements alone are substantial. It takes countless hours of work to produce text, write examples and exercises, edit and publish. Through collaboration, however, the cost to any individual can be lessened, allowing us to create texts that we freely distribute electronically and sell in printed form for an incredibly low cost. Having said that, nothing is entirely free; someone always bears some cost. This text "cost" the authors of this book their time, and that was not enough. *APEX Calculus* would not exist had not the Virginia Military Institute, through a generous Jackson–Hope grant, given the lead author significant time away from teaching so he could focus on this text.

We encourage others to adapt this work to fit their own needs. One might add sections that are "missing" or remove sections that your students won't need. The source files can be found at `github.com/APEXCalculus`.

You can learn more at `www.vmi.edu/APEX`.

Version 4.0

Key changes from Version 3.0 to 4.0:

- Numerous typographical and "small" mathematical corrections (again, thanks to all my close readers!).
- "Large" mathematical corrections and adjustments. There were a number of places in Version 3.0 where a definition/theorem was not correct as stated. See `www.apexcalculus.com` for more information.
- More useful numbering of Examples, Theorems, etc. "Definition 11.4.2" refers to the second definition of Chapter 11, Section 4.
- The addition of Section 13.7: Triple Integration with Cylindrical and Spherical Coordinates
- The addition of Chapter 14: Vector Analysis.

6: Techniques of Antidifferentiation

In Calculus I you learned techniques that allow you to compute the derivative of practically any function you can conceive of creating using the elementary functions (polynomial, rational, exponential, logarithmic, trigonometric, etc.). You also learned how to define integration using Riemann sums, and saw how the Fundamental Theorem of Calculus relates integration to the antiderivative.

Computing antiderivatives is generally more difficult than computing derivatives. As an example, finding the derivative of $f(x) = x^2 \sin x$ is simple but we do not yet know how to find an antiderivative of f. Worse, we can find the derivative of $y = e^{x^2}$, but its antiderivatives *cannot* be written in terms of elementary functions.

Despite this latter difficulty, there are still broad classes of functions for which we can find antiderivatives. This chapter is dedicated to learning techniques to enable us to compute the antiderivatives of a wide variety of functions.

6.1 Substitution

We motivate this section with an example. Let $f(x) = (x^2 + 3x - 5)^{10}$. We can compute $f'(x)$ using the Chain Rule. It is:

$$f'(x) = 10(x^2 + 3x - 5)^9 \cdot (2x + 3) = (20x + 30)(x^2 + 3x - 5)^9.$$

Now consider this: What is $\int (20x + 30)(x^2 + 3x - 5)^9 \, dx$? We have the answer in front of us;

$$\int (20x + 30)(x^2 + 3x - 5)^9 \, dx = (x^2 + 3x - 5)^{10} + C.$$

How would we have evaluated this indefinite integral without starting with $f(x)$ as we did?

This section explores *integration by substitution.* It allows us to "undo the Chain Rule." Substitution allows us to evaluate the above integral without knowing the original function first.

The underlying principle is to rewrite a "complicated" integral of the form $\int f(x) \, dx$ as a not–so–complicated integral $\int h(u) \, du$. We'll formally establish later how this is done. First, consider again our introductory indefinite integral, $\int (20x + 30)(x^2 + 3x - 5)^9 \, dx$. Arguably the most "complicated" part of the integrand is $(x^2 + 3x - 5)^9$. We wish to make this simpler; we do so through a substitution. Let $u = x^2 + 3x - 5$. Thus

$$(x^2 + 3x - 5)^9 = u^9.$$

We have established u as a function of x, so now consider the differential of u:

$$du = (2x + 3)dx.$$

Keep in mind that $(2x+3)$ and dx are multiplied; the dx is not "just sitting there."

Return to the original integral and do some substitutions through algebra:

$$\begin{aligned}\int (20x+30)(x^2+3x-5)^9\,dx &= \int 10(2x+3)(x^2+3x-5)^9\,dx\\ &= \int 10(\underbrace{x^2+3x-5}_{u})^9\,\underbrace{(2x+3)\,dx}_{du}\\ &= \int 10u^9\,du\\ &= u^{10}+C \quad \text{(replace } u \text{ with } x^2+3x-5)\\ &= (x^2+3x-5)^{10}+C\end{aligned}$$

One might well look at this and think "I (sort of) followed how that worked, but I could never come up with that on my own," but the process can be learned. This section contains numerous examples through which the reader will gain understanding and mathematical maturity enabling them to regard substitution as a natural tool when evaluating integrals.

We stated before that integration by substitution "undoes" the Chain Rule. Specifically, let $F(x)$ and $g(x)$ be differentiable functions and consider the derivative of their composition:

$$\frac{d}{dx}\Big(F\big(g(x)\big)\Big) = F'(g(x))g'(x).$$

Thus

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

Integration by substitution works by recognizing the "inside" function $g(x)$ and replacing it with a variable. By setting $u = g(x)$, we can rewrite the derivative as

$$\frac{d}{dx}\Big(F(u)\Big) = F'(u)u'.$$

Since $du = g'(x)dx$, we can rewrite the above integral as

$$\int F'(g(x))g'(x)\,dx = \int F'(u)du = F(u) + C = F(g(x)) + C.$$

This concept is important so we restate it in the context of a theorem.

Theorem 6.1.1 Integration by Substitution

Let F and g be differentiable functions, where the range of g is an interval I contained in the domain of F. Then

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

If $u = g(x)$, then $du = g'(x)dx$ and

$$\int F'(g(x))g'(x)\,dx = \int F'(u)\,du = F(u) + C = F(g(x)) + C.$$

The point of substitution is to make the integration step easy. Indeed, the step $\int F'(u)\,du = F(u) + C$ looks easy, as the antiderivative of the derivative of F is just F, plus a constant. The "work" involved is making the proper substitution.

There is not a step–by–step process that one can memorize; rather, experience will be one's guide. To gain experience, we now embark on many examples.

Example 6.1.1 Integrating by substitution

Evaluate $\displaystyle\int x\sin(x^2+5)\,dx$.

Solution Knowing that substitution is related to the Chain Rule, we choose to let u be the "inside" function of $\sin(x^2+5)$. (This is not *always* a good choice, but it is often the best place to start.)

Let $u = x^2+5$, hence $du = 2x\,dx$. The integrand has an $x\,dx$ term, but not a $2x\,dx$ term. (Recall that multiplication is commutative, so the x does not physically have to be next to dx for there to be an $x\,dx$ term.) We can divide both sides of the du expression by 2:

$$du = 2x\,dx \quad\Rightarrow\quad \frac{1}{2}du = x\,dx.$$

We can now substitute.

$$\begin{aligned}\int x\sin(x^2+5)\,dx &= \int \sin(\underbrace{x^2+5}_{u})\,\underbrace{x\,dx}_{\frac{1}{2}du}\\ &= \int \frac{1}{2}\sin u\,du\\ &= -\frac{1}{2}\cos u + C \quad \text{(now replace } u \text{ with } x^2+5\text{)}\\ &= -\frac{1}{2}\cos(x^2+5)+C.\end{aligned}$$

Thus $\int x\sin(x^2+5)\,dx = -\frac{1}{2}\cos(x^2+5)+C$. We can check our work by evaluating the derivative of the right hand side.

Example 6.1.2 Integrating by substitution

Evaluate $\displaystyle\int \cos(5x)\,dx$.

Solution Again let u replace the "inside" function. Letting $u=5x$, we have $du = 5dx$. Since our integrand does not have a $5dx$ term, we can divide the previous equation by 5 to obtain $\frac{1}{5}du = dx$. We can now substitute.

$$\begin{aligned}\int \cos(5x)\,dx &= \int \cos(\underbrace{5x}_{u})\,\underbrace{dx}_{\frac{1}{5}du}\\ &= \int \frac{1}{5}\cos u\,du\\ &= \frac{1}{5}\sin u + C\\ &= \frac{1}{5}\sin(5x)+C.\end{aligned}$$

We can again check our work through differentiation.

The previous example exhibited a common, and simple, type of substitution. The "inside" function was a linear function (in this case, $y = 5x$). When the inside function is linear, the resulting integration is very predictable, outlined here.

Key Idea 6.1.1 Substitution With A Linear Function

Consider $\int F'(ax+b)\,dx$, where $a \neq 0$ and b are constants. Letting $u = ax+b$ gives $du = a \cdot dx$, leading to the result

$$\int F'(ax+b)\,dx = \frac{1}{a}F(ax+b) + C.$$

Thus $\int \sin(7x-4)\,dx = -\frac{1}{7}\cos(7x-4) + C$. Our next example can use Key Idea 6.1.1, but we will only employ it after going through all of the steps.

Example 6.1.3 Integrating by substituting a linear function

Evaluate $\displaystyle\int \frac{7}{-3x+1}\,dx$.

Solution View the integrand as the composition of functions $f(g(x))$, where $f(x) = 7/x$ and $g(x) = -3x+1$. Employing our understanding of substitution, we let $u = -3x+1$, the inside function. Thus $du = -3dx$. The integrand lacks a -3; hence divide the previous equation by -3 to obtain $-du/3 = dx$. We can now evaluate the integral through substitution.

$$\begin{aligned}
\int \frac{7}{-3x+1}\,dx &= \int \frac{7}{u}\frac{du}{-3}\\
&= \frac{-7}{3}\int \frac{du}{u}\\
&= \frac{-7}{3}\ln|u| + C\\
&= -\frac{7}{3}\ln|-3x+1| + C.
\end{aligned}$$

Using Key Idea 6.1.1 is faster, recognizing that u is linear and $a = -3$. One may want to continue writing out all the steps until they are comfortable with this particular shortcut.

Not all integrals that benefit from substitution have a clear "inside" function. Several of the following examples will demonstrate ways in which this occurs.

Example 6.1.4 Integrating by substitution

Evaluate $\displaystyle\int \sin x \cos x\,dx$.

Solution There is not a composition of function here to exploit; rather, just a product of functions. Do not be afraid to experiment; when given an integral to evaluate, it is often beneficial to think "If I let u be *this*, then du must be *that* …" and see if this helps simplify the integral at all.

In this example, let's set $u = \sin x$. Then $du = \cos x\,dx$, which we have as part of the integrand! The substitution becomes very straightforward:

$$\begin{aligned}\int \sin x \cos x \, dx &= \int u \, du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}\sin^2 x + C.\end{aligned}$$

One would do well to ask "What would happen if we let $u = \cos x$?" The result is just as easy to find, yet looks very different. The challenge to the reader is to evaluate the integral letting $u = \cos x$ and discover why the answer is the same, yet looks different.

Our examples so far have required "basic substitution." The next example demonstrates how substitutions can be made that often strike the new learner as being "nonstandard."

Example 6.1.5 Integrating by substitution

Evaluate $\displaystyle\int x\sqrt{x+3}\, dx$.

Solution Recognizing the composition of functions, set $u = x + 3$. Then $du = dx$, giving what seems initially to be a simple substitution. But at this stage, we have:

$$\int x\sqrt{x+3}\, dx = \int x\sqrt{u}\, du.$$

We cannot evaluate an integral that has both an x and an u in it. We need to convert the x to an expression involving just u.

Since we set $u = x+3$, we can also state that $u-3 = x$. Thus we can replace x in the integrand with $u - 3$. It will also be helpful to rewrite $\sqrt{u}$ as $u^{\frac{1}{2}}$.

$$\begin{aligned}\int x\sqrt{x+3}\, dx &= \int (u-3)u^{\frac{1}{2}}\, du \\ &= \int \left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right) du \\ &= \frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + C \\ &= \frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.\end{aligned}$$

Checking your work is always a good idea. In this particular case, some algebra will be needed to make one's answer match the integrand in the original problem.

Example 6.1.6 Integrating by substitution

Evaluate $\displaystyle\int \frac{1}{x \ln x}\, dx$.

Solution This is another example where there does not seem to be an obvious composition of functions. The line of thinking used in Example 6.1.5 is useful here: choose something for u and consider what this implies du must be. If u can be chosen such that du also appears in the integrand, then we have chosen well.

Choosing $u = 1/x$ makes $du = -1/x^2\,dx$; that does not seem helpful. However, setting $u = \ln x$ makes $du = 1/x\,dx$, which is part of the integrand. Thus:

$$\begin{aligned}\int \frac{1}{x\ln x}\,dx &= \int \underbrace{\frac{1}{\ln x}}_{u}\,\underbrace{\frac{1}{x}\,dx}_{du}\\ &= \int \frac{1}{u}\,du\\ &= \ln|u| + C\\ &= \ln|\ln x| + C.\end{aligned}$$

The final answer is interesting; the natural log of the natural log. Take the derivative to confirm this answer is indeed correct.

Integrals Involving Trigonometric Functions

Section 6.3 delves deeper into integrals of a variety of trigonometric functions; here we use substitution to establish a foundation that we will build upon.

The next three examples will help fill in some missing pieces of our antiderivative knowledge. We know the antiderivatives of the sine and cosine functions; what about the other standard functions tangent, cotangent, secant and cosecant? We discover these next.

Example 6.1.7 Integration by substitution: antiderivatives of $\tan x$

Evaluate $\displaystyle\int \tan x\,dx$.

Solution The previous paragraph established that we did not know the antiderivatives of tangent, hence we must assume that we have learned something in this section that can help us evaluate this indefinite integral.

Rewrite $\tan x$ as $\sin x/\cos x$. While the presence of a composition of functions may not be immediately obvious, recognize that $\cos x$ is "inside" the $1/x$ function. Therefore, we see if setting $u = \cos x$ returns usable results. We have that $du = -\sin x\,dx$, hence $-du = \sin x\,dx$. We can integrate:

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx\\ &= \int \underbrace{\frac{1}{\cos x}}_{u}\,\underbrace{\sin x\,dx}_{-du}\\ &= \int \frac{-1}{u}\,du\\ &= -\ln|u| + C\\ &= -\ln|\cos x| + C.\end{aligned}$$

Some texts prefer to bring the -1 inside the logarithm as a power of $\cos x$, as in:

$$\begin{aligned}-\ln|\cos x| + C &= \ln|(\cos x)^{-1}| + C\\ &= \ln\left|\frac{1}{\cos x}\right| + C\\ &= \ln|\sec x| + C.\end{aligned}$$

Thus the result they give is $\int \tan x\,dx = \ln|\sec x| + C$. These two answers are equivalent.

Example 6.1.8 Integrating by substitution: antiderivatives of $\sec x$

Evaluate $\displaystyle\int \sec x\, dx$.

Solution This example employs a wonderful trick: multiply the integrand by "1" so that we see how to integrate more clearly. In this case, we write "1" as

$$1 = \frac{\sec x + \tan x}{\sec x + \tan x}.$$

This may seem like it came out of left field, but it works beautifully. Consider:

$$\begin{aligned}\int \sec x\, dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx.\end{aligned}$$

Now let $u = \sec x + \tan x$; this means $du = (\sec x \tan x + \sec^2 x)\, dx$, which is our numerator. Thus:

$$\begin{aligned}&= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|\sec x + \tan x| + C.\end{aligned}$$

We can use similar techniques to those used in Examples 6.1.7 and 6.1.8 to find antiderivatives of $\cot x$ and $\csc x$ (which the reader can explore in the exercises.) We summarize our results here.

Theorem 6.1.2 Antiderivatives of Trigonometric Functions

1. $\displaystyle\int \sin x\, dx = -\cos x + C$
2. $\displaystyle\int \cos x\, dx = \sin x + C$
3. $\displaystyle\int \tan x\, dx = -\ln|\cos x| + C$
4. $\displaystyle\int \csc x\, dx = -\ln|\csc x + \cot x| + C$
5. $\displaystyle\int \sec x\, dx = \ln|\sec x + \tan x| + C$
6. $\displaystyle\int \cot x\, dx = \ln|\sin x| + C$

We explore one more common trigonometric integral.

Example 6.1.9 Integration by substitution: powers of $\cos x$ **and** $\sin x$

Evaluate $\displaystyle\int \cos^2 x\, dx$.

Solution We have a composition of functions as $\cos^2 x = \big(\cos x\big)^2$. However, setting $u = \cos x$ means $du = -\sin x\, dx$, which we do not have in the integral. Another technique is needed.

The process we'll employ is to use a Power Reducing formula for $\cos^2 x$ (perhaps consult the back of this text for this formula), which states

$$\cos^2 x = \frac{1 + \cos(2x)}{2}.$$

The right hand side of this equation is not difficult to integrate. We have:

$$\begin{aligned}\int \cos^2 x\,dx &= \int \frac{1+\cos(2x)}{2}\,dx \\ &= \int \left(\frac{1}{2}+\frac{1}{2}\cos(2x)\right)\,dx.\end{aligned}$$

Now use Key Idea 6.1.1:

$$\begin{aligned}&= \frac{1}{2}x+\frac{1}{2}\frac{\sin(2x)}{2}+C \\ &= \frac{1}{2}x+\frac{\sin(2x)}{4}+C.\end{aligned}$$

We'll make significant use of this power–reducing technique in future sections.

Simplifying the Integrand

It is common to be reluctant to manipulate the integrand of an integral; at first, our grasp of integration is tenuous and one may think that working with the integrand will improperly change the results. Integration by substitution works using a different logic: as long as *equality* is maintained, the integrand can be manipulated so that its *form* is easier to deal with. The next two examples demonstrate common ways in which using algebra first makes the integration easier to perform.

Example 6.1.10 Integration by substitution: simplifying first

Evaluate $\displaystyle\int \frac{x^3+4x^2+8x+5}{x^2+2x+1}\,dx.$

Solution One may try to start by setting u equal to either the numerator or denominator; in each instance, the result is not workable.

When dealing with rational functions (i.e., quotients made up of polynomial functions), it is an almost universal rule that everything works better when the degree of the numerator is less than the degree of the denominator. Hence we use polynomial division.

We skip the specifics of the steps, but note that when x^2+2x+1 is divided into x^3+4x^2+8x+5, it goes in $x+2$ times with a remainder of $3x+3$. Thus

$$\frac{x^3+4x^2+8x+5}{x^2+2x+1} = x+2+\frac{3x+3}{x^2+2x+1}.$$

Integrating $x+2$ is simple. The fraction can be integrated by setting $u=x^2+2x+1$, giving $du=(2x+2)\,dx$. This is very similar to the numerator. Note that

$du/2 = (x+1)\,dx$ and then consider the following:

$$\begin{aligned}
\int \frac{x^3+4x^2+8x+5}{x^2+2x+1}\,dx &= \int \left(x+2+\frac{3x+3}{x^2+2x+1}\right)\,dx \\
&= \int (x+2)\,dx + \int \frac{3(x+1)}{x^2+2x+1}\,dx \\
&= \frac{1}{2}x^2+2x+C_1+\int \frac{3}{u}\frac{du}{2} \\
&= \frac{1}{2}x^2+2x+C_1+\frac{3}{2}\ln|u|+C_2 \\
&= \frac{1}{2}x^2+2x+\frac{3}{2}\ln|x^2+2x+1|+C.
\end{aligned}$$

In some ways, we "lucked out" in that after dividing, substitution was able to be done. In later sections we'll develop techniques for handling rational functions where substitution is not directly feasible.

Example 6.1.11 Integration by alternate methods

Evaluate $\displaystyle\int \frac{x^2+2x+3}{\sqrt{x}}\,dx$ with, and without, substitution.

Solution We already know how to integrate this particular example. Rewrite $\sqrt{x}$ as $x^{\frac{1}{2}}$ and simplify the fraction:

$$\frac{x^2+2x+3}{x^{1/2}} = x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}}.$$

We can now integrate using the Power Rule:

$$\begin{aligned}
\int \frac{x^2+2x+3}{x^{1/2}}\,dx &= \int \left(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}}\right)\,dx \\
&= \frac{2}{5}x^{\frac{5}{2}}+\frac{4}{3}x^{\frac{3}{2}}+6x^{\frac{1}{2}}+C
\end{aligned}$$

This is a perfectly fine approach. We demonstrate how this can also be solved using substitution as its implementation is rather clever.

Let $u = \sqrt{x} = x^{\frac{1}{2}}$; therefore

$$du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2\sqrt{x}}\,dx \quad \Rightarrow \quad 2du = \frac{1}{\sqrt{x}}\,dx.$$

This gives us $\displaystyle\int \frac{x^2+2x+3}{\sqrt{x}}\,dx = \int (x^2+2x+3)\cdot 2\,du$. What are we to do with the other x terms? Since $u = x^{\frac{1}{2}}$, $u^2 = x$, etc. We can then replace x^2 and x with appropriate powers of u. We thus have

$$\begin{aligned}
\int \frac{x^2+2x+3}{\sqrt{x}}\,dx &= \int (x^2+2x+3)\cdot 2\,du \\
&= \int 2(u^4+2u^2+3)\,du \\
&= \frac{2}{5}u^5+\frac{4}{3}u^3+6u+C \\
&= \frac{2}{5}x^{\frac{5}{2}}+\frac{4}{3}x^{\frac{3}{2}}+6x^{\frac{1}{2}}+C,
\end{aligned}$$

which is obviously the same answer we obtained before. In this situation, substitution is arguably more work than our other method. The fantastic thing is that it works. It demonstrates how flexible integration is.

Substitution and Inverse Trigonometric Functions

When studying derivatives of inverse functions, we learned that

$$\frac{d}{dx}\big(\tan^{-1} x\big) = \frac{1}{1+x^2}.$$

Applying the Chain Rule to this is not difficult; for instance,

$$\frac{d}{dx}\big(\tan^{-1} 5x\big) = \frac{5}{1+25x^2}.$$

We now explore how Substitution can be used to "undo" certain derivatives that are the result of the Chain Rule applied to Inverse Trigonometric functions. We begin with an example.

Example 6.1.12 Integrating by substitution: inverse trigonometric functions

Evaluate $\displaystyle\int \frac{1}{25+x^2}\,dx$.

Solution The integrand looks similar to the derivative of the arctangent function. Note:

$$\begin{aligned}\frac{1}{25+x^2} &= \frac{1}{25(1+\frac{x^2}{25})}\\ &= \frac{1}{25(1+\left(\frac{x}{5}\right)^2)}\\ &= \frac{1}{25}\frac{1}{1+\left(\frac{x}{5}\right)^2}.\end{aligned}$$

Thus

$$\int \frac{1}{25+x^2}\,dx = \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx.$$

This can be integrated using Substitution. Set $u = x/5$, hence $du = dx/5$ or $dx = 5du$. Thus

$$\begin{aligned}\int \frac{1}{25+x^2}\,dx &= \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx\\ &= \frac{1}{5}\int \frac{1}{1+u^2}\,du\\ &= \frac{1}{5}\tan^{-1} u + C\\ &= \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C\end{aligned}$$

Example 6.1.12 demonstrates a general technique that can be applied to other integrands that result in inverse trigonometric functions. The results are summarized here.

Theorem 6.1.3 Integrals Involving Inverse Trigonometric Functions

Let $a > 0$.

1. $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$
2. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right)+C$

Let's practice using Theorem 6.1.3.

Example 6.1.13 Integrating by substitution: inverse trigonometric functions

Evaluate the given indefinite integrals.

$$1.\ \int \frac{1}{9+x^2}\,dx \quad 2.\ \int \frac{1}{x\sqrt{x^2-\frac{1}{100}}}\,dx \quad 3.\ \int \frac{1}{\sqrt{5-x^2}}\,dx.$$

Solution Each can be answered using a straightforward application of Theorem 6.1.3.

1. $\displaystyle\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{3}+C$, as $a=3$.
2. $\displaystyle\int \frac{1}{x\sqrt{x^2-\frac{1}{100}}}\,dx = 10\sec^{-1}10x+C$, as $a=\frac{1}{10}$.
3. $\displaystyle\int \frac{1}{\sqrt{5-x^2}} = \sin^{-1}\frac{x}{\sqrt{5}}+C$, as $a=\sqrt{5}$.

Most applications of Theorem 6.1.3 are not as straightforward. The next examples show some common integrals that can still be approached with this theorem.

Example 6.1.14 Integrating by substitution: completing the square

Evaluate $\displaystyle\int \frac{1}{x^2-4x+13}\,dx$.

Solution Initially, this integral seems to have nothing in common with the integrals in Theorem 6.1.3. As it lacks a square root, it almost certainly is not related to arcsine or arcsecant. It is, however, related to the arctangent function.

We see this by *completing the square* in the denominator. We give a brief reminder of the process here.

Start with a quadratic with a leading coefficient of 1. It will have the form of x^2+bx+c. Take 1/2 of b, square it, and add/subtract it back into the expression. I.e.,

$$\begin{aligned} x^2+bx+c &= \underbrace{x^2+bx+\frac{b^2}{4}}_{(x+b/2)^2}-\frac{b^2}{4}+c \\ &= \left(x+\frac{b}{2}\right)^2+c-\frac{b^2}{4} \end{aligned}$$

In our example, we take half of -4 and square it, getting 4. We add/subtract it into the denominator as follows:

$$\begin{aligned}\frac{1}{x^2-4x+13} &= \frac{1}{\underbrace{x^2-4x+4}_{(x-2)^2}-4+13}\\ &= \frac{1}{(x-2)^2+9}\end{aligned}$$

We can now integrate this using the arctangent rule. Technically, we need to substitute first with $u=x-2$, but we can employ Key Idea 6.1.1 instead. Thus we have

$$\int \frac{1}{x^2-4x+13}\,dx = \int \frac{1}{(x-2)^2+9}\,dx = \frac{1}{3}\tan^{-1}\frac{x-2}{3}+C.$$

Example 6.1.15 Integrals requiring multiple methods

Evaluate $\displaystyle\int \frac{4-x}{\sqrt{16-x^2}}\,dx$.

Solution This integral requires two different methods to evaluate it. We get to those methods by splitting up the integral:

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = \int \frac{4}{\sqrt{16-x^2}}\,dx - \int \frac{x}{\sqrt{16-x^2}}\,dx.$$

The first integral is handled using a straightforward application of Theorem 6.1.3; the second integral is handled by substitution, with $u=16-x^2$. We handle each separately.

$\displaystyle\int \frac{4}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4}+C.$

$\displaystyle\int \frac{x}{\sqrt{16-x^2}}\,dx$: Set $u=16-x^2$, so $du=-2xdx$ and $xdx=-du/2$. We have

$$\begin{aligned}\int \frac{x}{\sqrt{16-x^2}}\,dx &= \int \frac{-du/2}{\sqrt{u}}\\ &= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du\\ &= -\sqrt{u}+C\\ &= -\sqrt{16-x^2}+C.\end{aligned}$$

Combining these together, we have

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4}+\sqrt{16-x^2}+C.$$

Substitution and Definite Integration

This section has focused on evaluating indefinite integrals as we are learning a new technique for finding antiderivatives. However, much of the time integration is used in the context of a definite integral. Definite integrals that require substitution can be calculated using the following workflow:

1. Start with a definite integral $\int_a^b f(x)\,dx$ that requires substitution.

2. Ignore the bounds; use substitution to evaluate $\int f(x)\,dx$ and find an antiderivative $F(x)$.

3. Evaluate $F(x)$ at the bounds; that is, evaluate $F(x)\Big|_a^b = F(b) - F(a)$.

This workflow works fine, but substitution offers an alternative that is powerful and amazing (and a little time saving).

At its heart, (using the notation of Theorem 6.1.1) substitution converts integrals of the form $\int F'(g(x))g'(x)\,dx$ into an integral of the form $\int F'(u)\,du$ with the substitution of $u = g(x)$. The following theorem states how the bounds of a definite integral can be changed as the substitution is performed.

Theorem 6.1.4 Substitution with Definite Integrals

Let F and g be differentiable functions, where the range of g is an interval I that is contained in the domain of F. Then

$$\int_a^b F'\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} F'(u)\,du.$$

In effect, Theorem 6.1.4 states that once you convert to integrating with respect to u, you do not need to switch back to evaluating with respect to x. A few examples will help one understand.

Example 6.1.16 Definite integrals and substitution: changing the bounds

Evaluate $\int_0^2 \cos(3x-1)\,dx$ using Theorem 6.1.4.

Solution Observing the composition of functions, let $u = 3x - 1$, hence $du = 3dx$. As $3dx$ does not appear in the integrand, divide the latter equation by 3 to get $du/3 = dx$.

By setting $u = 3x - 1$, we are implicitly stating that $g(x) = 3x - 1$. Theorem 6.1.4 states that the new lower bound is $g(0) = -1$; the new upper bound is $g(2) = 5$. We now evaluate the definite integral:

$$\begin{aligned}\int_0^2 \cos(3x-1)\,dx &= \int_{-1}^5 \cos u\,\frac{du}{3}\\ &= \frac{1}{3}\sin u\Big|_{-1}^5\\ &= \frac{1}{3}\big(\sin 5 - \sin(-1)\big) \approx -0.039.\end{aligned}$$

Notice how once we converted the integral to be in terms of u, we never went back to using x.

The graphs in Figure 6.1.1 tell more of the story. In (a) the area defined by the original integrand is shaded, whereas in (b) the area defined by the new integrand is shaded. In this particular situation, the areas look very similar; the new region is "shorter" but "wider," giving the same area.

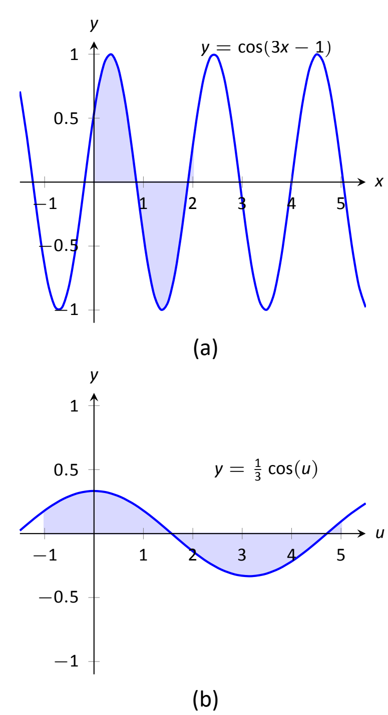


Figure 6.1.1: Graphing the areas defined by the definite integrals of Example 6.1.16.

Example 6.1.17 Definite integrals and substitution: changing the bounds

Evaluate $\displaystyle\int_0^{\pi/2} \sin x\cos x\, dx$ using Theorem 6.1.4.

Solution We saw the corresponding indefinite integral in Example 6.1.4. In that example we set $u = \sin x$ but stated that we could have let $u = \cos x$. For variety, we do the latter here.

Let $u = g(x) = \cos x$, giving $du = -\sin x\, dx$ and hence $\sin x\, dx = -du$. The new upper bound is $g(\pi/2) = 0$; the new lower bound is $g(0) = 1$. Note how the lower bound is actually larger than the upper bound now. We have

$$\begin{aligned}\int_0^{\pi/2} \sin x\cos x\, dx &= \int_1^0 -u\, du \quad \text{(switch bounds \& change sign)}\\ &= \int_0^1 u\, du\\ &= \frac{1}{2}u^2\Big|_0^1 = 1/2.\end{aligned}$$

In Figure 6.1.2 we have again graphed the two regions defined by our definite integrals. Unlike the previous example, they bear no resemblance to each other. However, Theorem 6.1.4 guarantees that they have the same area.

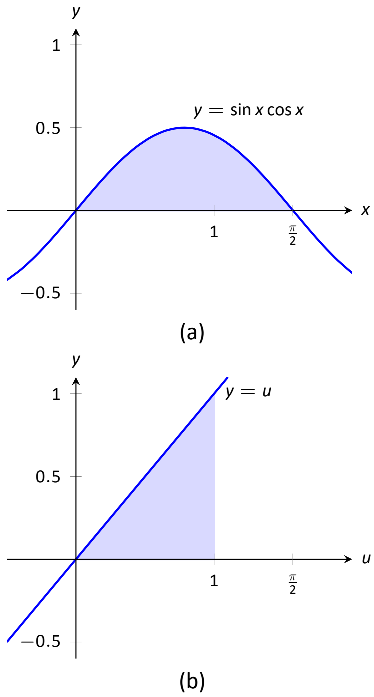


Figure 6.1.2: Graphing the areas defined by the definite integrals of Example 6.1.17.

Integration by substitution is a powerful and useful integration technique. The next section introduces another technique, called Integration by Parts. As substitution "undoes" the Chain Rule, integration by parts "undoes" the Product Rule. Together, these two techniques provide a strong foundation on which most other integration techniques are based.

Exercises 6.1

Terms and Concepts

1. Substitution "undoes" what derivative rule?

2. T/F: One can use algebra to rewrite the integrand of an integral to make it easier to evaluate.

Problems

In Exercises 3 – 14, evaluate the indefinite integral to develop an understanding of Substitution.

3. $\displaystyle\int 3x^2\left(x^3-5\right)^7\,dx$

4. $\displaystyle\int (2x-5)\left(x^2-5x+7\right)^3\,dx$

5. $\displaystyle\int x\left(x^2+1\right)^8\,dx$

6. $\displaystyle\int (12x+14)\left(3x^2+7x-1\right)^5\,dx$

7. $\displaystyle\int \frac{1}{2x+7}dx$

8. $\displaystyle\int \frac{1}{\sqrt{2x+3}}dx$

9. $\displaystyle\int \frac{x}{\sqrt{x+3}}dx$

10. $\displaystyle\int \frac{x^3-x}{\sqrt{x}}dx$

11. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

12. $\displaystyle\int \frac{x^4}{\sqrt{x^5+1}}dx$

13. $\displaystyle\int \frac{\frac{1}{x}+1}{x^2}dx$

14. $\displaystyle\int \frac{\ln(x)}{x}dx$

In Exercises 15 – 24, use Substitution to evaluate the indefinite integral involving trigonometric functions.

15. $\displaystyle\int \sin^2(x)\cos(x)dx$

16. $\displaystyle\int \cos^3(x)\sin(x)dx$

17. $\displaystyle\int \cos(3-6x)dx$

18. $\displaystyle\int \sec^2(4-x)dx$

19. $\displaystyle\int \sec(2x)dx$

20. $\displaystyle\int \tan^2(x)\sec^2(x)dx$

21. $\displaystyle\int x\cos\left(x^2\right)\,dx$

22. $\displaystyle\int \tan^2(x)dx$

23. $\displaystyle\int \cot x\,dx$. Do not just refer to Theorem 6.1.2 for the answer; justify it through Substitution.

24. $\displaystyle\int \csc x\,dx$. Do not just refer to Theorem 6.1.2 for the answer; justify it through Substitution.

In Exercises 25 – 32, use Substitution to evaluate the indefinite integral involving exponential functions.

25. $\displaystyle\int e^{3x-1}dx$

26. $\displaystyle\int e^{x^3}x^2dx$

27. $\displaystyle\int e^{x^2-2x+1}(x-1)dx$

28. $\displaystyle\int \frac{e^x+1}{e^x}dx$

29. $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

30. $\displaystyle\int \frac{e^x-e^{-x}}{e^{2x}}dx$

31. $\displaystyle\int 3^{3x}dx$

32. $\displaystyle\int 4^{2x}dx$

In Exercises 33 – 36, use Substitution to evaluate the indefinite integral involving logarithmic functions.

33. $\displaystyle\int \frac{\ln x}{x}dx$

34. $\displaystyle\int \frac{\left(\ln x\right)^2}{x}dx$

35. $\int \frac{\ln\left(x^3\right)}{x}\,dx$

36. $\int \frac{1}{x\ln\left(x^2\right)}\,dx$

In Exercises 37 – 42, use Substitution to evaluate the indefinite integral involving rational functions.

37. $\int \frac{x^2+3x+1}{x}\,dx$

38. $\int \frac{x^3+x^2+x+1}{x}\,dx$

39. $\int \frac{x^3-1}{x+1}\,dx$

40. $\int \frac{x^2+2x-5}{x-3}\,dx$

41. $\int \frac{3x^2-5x+7}{x+1}\,dx$

42. $\int \frac{x^2+2x+1}{x^3+3x^2+3x}\,dx$

In Exercises 43 – 52, use Substitution to evaluate the indefinite integral involving inverse trigonometric functions.

43. $\int \frac{7}{x^2+7}\,dx$

44. $\int \frac{3}{\sqrt{9-x^2}}\,dx$

45. $\int \frac{14}{\sqrt{5-x^2}}\,dx$

46. $\int \frac{2}{x\sqrt{x^2-9}}\,dx$

47. $\int \frac{5}{\sqrt{x^4-16x^2}}\,dx$

48. $\int \frac{x}{\sqrt{1-x^4}}\,dx$

49. $\int \frac{1}{x^2-2x+8}\,dx$

50. $\int \frac{2}{\sqrt{-x^2+6x+7}}\,dx$

51. $\int \frac{3}{\sqrt{-x^2+8x+9}}\,dx$

52. $\int \frac{5}{x^2+6x+34}\,dx$

In Exercises 53 – 78, evaluate the indefinite integral.

53. $\int \frac{x^2}{\left(x^3+3\right)^2}\,dx$

54. $\int \left(3x^2+2x\right)\left(5x^3+5x^2+2\right)^8\,dx$

55. $\int \frac{x}{\sqrt{1-x^2}}\,dx$

56. $\int x^2\csc^2\left(x^3+1\right)\,dx$

57. $\int \sin(x)\sqrt{\cos(x)}\,dx$

58. $\int \sin\left(5x+1\right)\,dx$

59. $\int \frac{1}{x-5}\,dx$

60. $\int \frac{7}{3x+2}\,dx$

61. $\int \frac{3x^3+4x^2+2x-22}{x^2+3x+5}\,dx$

62. $\int \frac{2x+7}{x^2+7x+3}\,dx$

63. $\int \frac{9(2x+3)}{3x^2+9x+7}\,dx$

64. $\int \frac{-x^3+14x^2-46x-7}{x^2-7x+1}\,dx$

65. $\int \frac{x}{x^4+81}\,dx$

66. $\int \frac{2}{4x^2+1}\,dx$

67. $\int \frac{1}{x\sqrt{4x^2-1}}\,dx$

68. $\int \frac{1}{\sqrt{16-9x^2}}\,dx$

69. $\int \frac{3x-2}{x^2-2x+10}\,dx$

70. $\int \frac{7-2x}{x^2+12x+61}\,dx$

71. $\int \frac{x^2+5x-2}{x^2-10x+32}\,dx$

72. $\int \frac{x^3}{x^2+9}\,dx$

73. $\displaystyle\int \frac{x^3-x}{x^2+4x+9}dx$

74. $\displaystyle\int \frac{\sin(x)}{\cos^2(x)+1}dx$

75. $\displaystyle\int \frac{\cos(x)}{\sin^2(x)+1}dx$

76. $\displaystyle\int \frac{\cos(x)}{1-\sin^2(x)}dx$

77. $\displaystyle\int \frac{3x-3}{\sqrt{x^2-2x-6}}dx$

78. $\displaystyle\int \frac{x-3}{\sqrt{x^2-6x+8}}dx$

In Exercises 79 – 86, evaluate the definite integral.

79. $\displaystyle\int_1^3 \frac{1}{x-5}dx$

80. $\displaystyle\int_2^6 x\sqrt{x-2}dx$

81. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\, dx$

82. $\displaystyle\int_0^1 2x(1-x^2)^4\, dx$

83. $\displaystyle\int_{-2}^{-1} (x+1)e^{x^2+2x+1}\, dx$

84. $\displaystyle\int_{-1}^1 \frac{1}{1+x^2}\, dx$

85. $\displaystyle\int_2^4 \frac{1}{x^2-6x+10}\, dx$

86. $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\, dx$

6.2 Integration by Parts

Here's a simple integral that we can't yet evaluate:

$$\int x \cos x \, dx.$$

It's a simple matter to take the derivative of the integrand using the Product Rule, but there is no Product Rule for integrals. However, this section introduces *Integration by Parts*, a method of integration that is based on the Product Rule for derivatives. It will enable us to evaluate this integral.

The Product Rule says that if u and v are functions of x, then $(uv)' = u'v + uv'$. For simplicity, we've written u for $u(x)$ and v for $v(x)$. Suppose we integrate both sides with respect to x. This gives

$$\int (uv)' \, dx = \int (u'v + uv') \, dx.$$

By the Fundamental Theorem of Calculus, the left side integrates to uv. The right side can be broken up into two integrals, and we have

$$uv = \int u'v \, dx + \int uv' \, dx.$$

Solving for the second integral we have

$$\int uv' \, dx = uv - \int u'v \, dx.$$

Using differential notation, we can write $du = u'(x)dx$ and $dv = v'(x)dx$ and the expression above can be written as follows:

$$\int u \, dv = uv - \int v \, du.$$

This is the Integration by Parts formula. For reference purposes, we state this in a theorem.

Theorem 6.2.1 Integration by Parts

Let u and v be differentiable functions of x on an interval I containing a and b. Then

$$\int u \, dv = uv - \int v \, du,$$

and

$$\int_{x=a}^{x=b} u \, dv = uv\Big|_a^b - \int_{x=a}^{x=b} v \, du.$$

Let's try an example to understand our new technique.

Example 6.2.1 Integrating using Integration by Parts

Evaluate $\displaystyle\int x \cos x \, dx$.

Solution The key to Integration by Parts is to identify part of the integrand as "u" and part as "dv." Regular practice will help one make good identifications, and later we will introduce some principles that help. For now, let $u = x$ and $dv = \cos x \, dx$.

It is generally useful to make a small table of these values as done below. Right now we only know u and dv as shown on the left of Figure 6.2.1; on the right we fill in the rest of what we need. If $u = x$, then $du = dx$. Since $dv = \cos x\, dx$, v is an antiderivative of $\cos x$. We choose $v = \sin x$.

$$\begin{array}{cc} u = x & v = ? \\ du = ? & dv = \cos x\, dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = x & v = \sin x \\ du = dx & dv = \cos x\, dx \end{array}$$

Figure 6.2.1: Setting up Integration by Parts.

Now substitute all of this into the Integration by Parts formula, giving

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx.$$

We can then integrate $\sin x$ to get $-\cos x + C$ and overall our answer is

$$\int x \cos x\, dx = x \sin x + \cos x + C.$$

Note how the antiderivative contains a product, $x \sin x$. This product is what makes Integration by Parts necessary.

The example above demonstrates how Integration by Parts works in general. We try to identify u and dv in the integral we are given, and the key is that we usually want to choose u and dv so that du is simpler than u and v is hopefully not too much more complicated than dv. This will mean that the integral on the right side of the Integration by Parts formula, $\int v\, du$ will be simpler to integrate than the original integral $\int u\, dv$.

In the example above, we chose $u = x$ and $dv = \cos x\, dx$. Then $du = dx$ was simpler than u and $v = \sin x$ is no more complicated than dv. Therefore, instead of integrating $x \cos x\, dx$, we could integrate $\sin x\, dx$, which we knew how to do.

A useful mnemonic for helping to determine u is "LIATE," where

L = **L**ogarithmic, I = **I**nverse Trig., A = **A**lgebraic (polynomials),
T = **T**rigonometric, and E = **E**xponential.

If the integrand contains both a logarithmic and an algebraic term, in general letting u be the logarithmic term works best, as indicated by L coming before A in LIATE.

We now consider another example.

Example 6.2.2 Integrating using Integration by Parts

Evaluate $\displaystyle\int x e^x\, dx$.

Solution The integrand contains an **A**lgebraic term (x) and an **E**xponential term (e^x). Our mnemonic suggests letting u be the algebraic term, so we choose $u = x$ and $dv = e^x\, dx$. Then $du = dx$ and $v = e^x$ as indicated by the tables below.

$$\begin{array}{cc} u = x & v = ? \\ du = ? & dv = e^x\, dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = x & v = e^x \\ du = dx & dv = e^x\, dx \end{array}$$

Figure 6.2.2: Setting up Integration by Parts.

We see du is simpler than u, while there is no change in going from dv to v. This is good. The Integration by Parts formula gives

$$\int xe^x\,dx = xe^x - \int e^x\,dx.$$

The integral on the right is simple; our final answer is

$$\int xe^x\,dx = xe^x - e^x + C.$$

Note again how the antiderivatives contain a product term.

Example 6.2.3 Integrating using Integration by Parts

Evaluate $\displaystyle\int x^2\cos x\,dx$.

Solution The mnemonic suggests letting $u = x^2$ instead of the trigonometric function, hence $dv = \cos x\,dx$. Then $du = 2x\,dx$ and $v = \sin x$ as shown below.

$$\begin{array}{cc} u = x^2 & v = ? \\ du = ? & dv = \cos x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = x^2 & v = \sin x \\ du = 2x\,dx & dv = \cos x\,dx \end{array}$$

Figure 6.2.3: Setting up Integration by Parts.

The Integration by Parts formula gives

$$\int x^2\cos x\,dx = x^2\sin x - \int 2x\sin x\,dx.$$

At this point, the integral on the right is indeed simpler than the one we started with, but to evaluate it, we need to do Integration by Parts again. Here we choose $u = 2x$ and $dv = \sin x$ and fill in the rest below.

$$\begin{array}{cc} u = 2x & v = ? \\ du = ? & dv = \sin x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = 2x & v = -\cos x \\ du = 2\,dx & dv = \sin x\,dx \end{array}$$

Figure 6.2.4: Setting up Integration by Parts (again).

$$\int x^2\cos x\,dx = x^2\sin x - \left(-2x\cos x - \int -2\cos x\,dx\right).$$

The integral all the way on the right is now something we can evaluate. It evaluates to $-2\sin x$. Then going through and simplifying, being careful to keep all the signs straight, our answer is

$$\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x + C.$$

Example 6.2.4 Integrating using Integration by Parts

Evaluate $\displaystyle\int e^x\cos x\,dx$.

Solution This is a classic problem. Our mnemonic suggests letting u be the trigonometric function instead of the exponential. In this particular example, one can let u be either $\cos x$ or e^x; to demonstrate that we do not have to follow LIATE, we choose $u = e^x$ and hence $dv = \cos x\,dx$. Then $du = e^x\,dx$ and $v = \sin x$ as shown below.

$$\begin{array}{rr} u = e^x & v = ? \\ du = ? & dv = \cos x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{rr} u = e^x & v = \sin x \\ du = e^x\,dx & dv = \cos x\,dx \end{array}$$

Figure 6.2.5: Setting up Integration by Parts.

Notice that du is no simpler than u, going against our general rule (but bear with us). The Integration by Parts formula yields

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

The integral on the right is not much different than the one we started with, so it seems like we have gotten nowhere. Let's keep working and apply Integration by Parts to the new integral, using $u = e^x$ and $dv = \sin x\,dx$. This leads us to the following:

$$\begin{array}{rr} u = e^x & v = ? \\ du = ? & dv = \sin x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{rr} u = e^x & v = -\cos x \\ du = e^x\,dx & dv = \sin x\,dx \end{array}$$

Figure 6.2.6: Setting up Integration by Parts (again).

The Integration by Parts formula then gives:

$$\begin{aligned} \int e^x \cos x\,dx &= e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x\,dx\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx. \end{aligned}$$

It seems we are back right where we started, as the right hand side contains $\int e^x \cos x\,dx$. But this is actually a good thing.

Add $\displaystyle\int e^x \cos x\,dx$ to both sides. This gives

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

Now divide both sides by 2:

$$\int e^x \cos x\,dx = \frac{1}{2}\big(e^x \sin x + e^x \cos x\big).$$

Simplifying a little and adding the constant of integration, our answer is thus

$$\int e^x \cos x\,dx = \frac{1}{2}e^x\left(\sin x + \cos x\right) + C.$$

Example 6.2.5 Integrating using Integration by Parts: antiderivative of $\ln x$

Evaluate $\displaystyle\int \ln x\,dx$.

Solution One may have noticed that we have rules for integrating the familiar trigonometric functions and e^x, but we have not yet given a rule for integrating $\ln x$. That is because $\ln x$ can't easily be integrated with any of the rules we have learned up to this point. But we can find its antiderivative by a clever application of Integration by Parts. Set $u = \ln x$ and $dv = dx$. This is a good, sneaky trick to learn as it can help in other situations. This determines $du = (1/x)\,dx$ and $v = x$ as shown below.

$$\begin{array}{ll} u = \ln x & v = ? \\ du = ? & dv = dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = \ln x & v = x \\ du = 1/x\,dx & dv = dx \end{array}$$

Figure 6.2.7: Setting up Integration by Parts.

Putting this all together in the Integration by Parts formula, things work out very nicely:

$$\int \ln x\,dx = x\ln x - \int x\,\frac{1}{x}\,dx.$$

The new integral simplifies to $\int 1\,dx$, which is about as simple as things get. Its integral is $x + C$ and our answer is

$$\int \ln x\,dx = x\ln x - x + C.$$

Example 6.2.6 Integrating using Int. by Parts: antiderivative of $\arctan x$

Evaluate $\displaystyle\int \arctan x\,dx$.

Solution The same sneaky trick we used above works here. Let $u = \arctan x$ and $dv = dx$. Then $du = 1/(1+x^2)\,dx$ and $v = x$. The Integration by Parts formula gives

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx.$$

The integral on the right can be solved by substitution. Taking $u = 1 + x^2$, we get $du = 2x\,dx$. The integral then becomes

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\int \frac{1}{u}\,du.$$

The integral on the right evaluates to $\frac{1}{2}\ln|u| + C$, which becomes $\frac{1}{2}\ln(1+x^2) + C$. Therefore, the answer is

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

Substitution Before Integration

When taking derivatives, it was common to employ multiple rules (such as using both the Quotient and the Chain Rules). It should then come as no surprise that some integrals are best evaluated by combining integration techniques. In particular, here we illustrate making an "unusual" substitution first before using Integration by Parts.

Example 6.2.7 Integration by Parts after substitution

Evaluate $\displaystyle\int \cos(\ln x)\, dx$.

Solution The integrand contains a composition of functions, leading us to think Substitution would be beneficial. Letting $u = \ln x$, we have $du = 1/x\, dx$. This seems problematic, as we do not have a $1/x$ in the integrand. But consider:

$$du = \frac{1}{x}\, dx \Rightarrow x \cdot du = dx.$$

Since $u = \ln x$, we can use inverse functions and conclude that $x = e^u$. Therefore we have that

$$\begin{aligned} dx &= x \cdot du \\ &= e^u\, du. \end{aligned}$$

We can thus replace $\ln x$ with u and dx with $e^u\, du$. Thus we rewrite our integral as

$$\int \cos(\ln x)\, dx = \int e^u \cos u\, du.$$

We evaluated this integral in Example 6.2.4. Using the result there, we have:

$$\begin{aligned} \int \cos(\ln x)\, dx &= \int e^u \cos u\, du \\ &= \frac{1}{2}e^u\big(\sin u + \cos u\big) + C \\ &= \frac{1}{2}e^{\ln x}\big(\sin(\ln x) + \cos(\ln x)\big) + C \\ &= \frac{1}{2}x\big(\sin(\ln x) + \cos(\ln x)\big) + C. \end{aligned}$$

Definite Integrals and Integration By Parts

So far we have focused only on evaluating indefinite integrals. Of course, we can use Integration by Parts to evaluate definite integrals as well, as Theorem 6.2.1 states. We do so in the next example.

Example 6.2.8 Definite integration using Integration by Parts

Evaluate $\displaystyle\int_1^2 x^2 \ln x\, dx$.

Solution Our mnemonic suggests letting $u = \ln x$, hence $dv = x^2\, dx$. We then get $du = (1/x)\, dx$ and $v = x^3/3$ as shown below.

$$\begin{array}{ll} u = \ln x & v = ? \\ du = ? & dv = x^2\,dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = \ln x & v = x^3/3 \\ du = 1/x\,dx & dv = x^2\,dx \end{array}$$

Figure 6.2.8: Setting up Integration by Parts.

The Integration by Parts formula then gives

$$\begin{aligned}
\int_1^2 x^2 \ln x\,dx &= \frac{x^3}{3}\ln x\Big|_1^2 - \int_1^2 \frac{x^3}{3}\frac{1}{x}\,dx \\
&= \frac{x^3}{3}\ln x\Big|_1^2 - \int_1^2 \frac{x^2}{3}\,dx \\
&= \frac{x^3}{3}\ln x\Big|_1^2 - \frac{x^3}{9}\Big|_1^2 \\
&= \left(\frac{x^3}{3}\ln x - \frac{x^3}{9}\right)\Big|_1^2 \\
&= \left(\frac{8}{3}\ln 2 - \frac{8}{9}\right) - \left(\frac{1}{3}\ln 1 - \frac{1}{9}\right) \\
&= \frac{8}{3}\ln 2 - \frac{7}{9} \\
&\approx 1.07.
\end{aligned}$$

In general, Integration by Parts is useful for integrating certain products of functions, like $\int xe^x\,dx$ or $\int x^3 \sin x\,dx$. It is also useful for integrals involving logarithms and inverse trigonometric functions.

As stated before, integration is generally more difficult than derivation. We are developing tools for handling a large array of integrals, and experience will tell us when one tool is preferable/necessary over another. For instance, consider the three similar–looking integrals

$$\int xe^x\,dx, \qquad \int xe^{x^2}\,dx \qquad \text{and} \qquad \int xe^{x^3}\,dx.$$

While the first is calculated easily with Integration by Parts, the second is best approached with Substitution. Taking things one step further, the third integral has no answer in terms of elementary functions, so none of the methods we learn in calculus will get us the exact answer.

Integration by Parts is a very useful method, second only to Substitution. In the following sections of this chapter, we continue to learn other integration techniques. The next section focuses on handling integrals containing trigonometric functions.

Exercises 6.2

Terms and Concepts

1. T/F: Integration by Parts is useful in evaluating integrands that contain products of functions.

2. T/F: Integration by Parts can be thought of as the “opposite of the Chain Rule.”

3. For what is “LIATE” useful?

4. T/F: If the integral that results from Integration by Parts appears to also need Integration by Parts, then a mistake was made in the orginal choice of “u”.

Problems

In Exercises 5 – 34, evaluate the given indefinite integral.

5. $\int x\sin x\,dx$

6. $\int xe^{-x}\,dx$

7. $\int x^2\sin x\,dx$

8. $\int x^3\sin x\,dx$

9. $\int xe^{x^2}\,dx$

10. $\int x^3e^x\,dx$

11. $\int xe^{-2x}\,dx$

12. $\int e^x\sin x\,dx$

13. $\int e^{2x}\cos x\,dx$

14. $\int e^{2x}\sin(3x)\,dx$

15. $\int e^{5x}\cos(5x)\,dx$

16. $\int \sin x\cos x\,dx$

17. $\int \sin^{-1}x\,dx$

18. $\int \tan^{-1}(2x)\,dx$

19. $\int x\tan^{-1}x\,dx$

20. $\int \sin^{-1}x\,dx$

21. $\int x\ln x\,dx$

22. $\int (x-2)\ln x\,dx$

23. $\int x\ln(x-1)\,dx$

24. $\int x\ln(x^2)\,dx$

25. $\int x^2\ln x\,dx$

26. $\int (\ln x)^2\,dx$

27. $\int (\ln(x+1))^2\,dx$

28. $\int x\sec^2x\,dx$

29. $\int x\csc^2x\,dx$

30. $\int x\sqrt{x-2}\,dx$

31. $\int x\sqrt{x^2-2}\,dx$

32. $\int \sec x\tan x\,dx$

33. $\int x\sec x\tan x\,dx$

34. $\int x\csc x\cot x\,dx$

In Exercises 35 – 40, evaluate the indefinite integral after first making a substitution.

35. $\int \sin(\ln x)\,dx$

36. $\int e^{2x}\cos\left(e^x\right)\,dx$

37. $\int \sin(\sqrt{x})\,dx$

38. $\int \ln(\sqrt{x})\,dx$

39. $\int e^{\sqrt{x}}\,dx$

40. $\int e^{\ln x}\,dx$

In Exercises 41 – 49, evaluate the definite integral. Note: the corresponding indefinite integrals appear in Exercises 5 – 13.

41. $\int_0^{\pi} x\sin x\,dx$

42. $\int_{-1}^{1} xe^{-x}\,dx$

43. $\int_{-\pi/4}^{\pi/4} x^2\sin x\,dx$

44. $\int_{-\pi/2}^{\pi/2} x^3\sin x\,dx$

45. $\int_0^{\sqrt{\ln 2}} xe^{x^2}\,dx$

46. $\int_0^1 x^3e^x\,dx$

47. $\int_1^2 xe^{-2x}\,dx$

48. $\int_0^{\pi} e^x\sin x\,dx$

49. $\int_{-\pi/2}^{\pi/2} e^{2x}\cos x\,dx$

6.3 Trigonometric Integrals

Functions involving trigonometric functions are useful as they are good at describing periodic behaviour. This section describes several techniques for finding antiderivatives of certain combinations of trigonometric functions.

Integrals of the form $\int \sin^m x \cos^n x\, dx$

In learning the technique of Substitution, we saw the integral $\int \sin x \cos x\, dx$ in Example 6.1.4. The integration was not difficult, and one could easily evaluate the indefinite integral by letting $u = \sin x$ or by letting $u = \cos x$. This integral is easy since the power of both sine and cosine is 1.

We generalize this integral and consider integrals of the form $\int \sin^m x \cos^n x\, dx$, where m, n are nonnegative integers. Our strategy for evaluating these integrals is to use the identity $\cos^2 x + \sin^2 x = 1$ to convert high powers of one trigonometric function into the other, leaving a single sine or cosine term in the integrand. We summarize the general technique in the following Key Idea.

Key Idea 6.3.1 Integrals Involving Powers of Sine and Cosine

Consider $\int \sin^m x \cos^n x\, dx$, where m, n are nonnegative integers.

1. If m is odd, then $m = 2k + 1$ for some integer k. Rewrite
$$\sin^m x = \sin^{2k+1} x = \sin^{2k} x \sin x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x.$$
Then
$$\int \sin^m x \cos^n x\, dx = \int (1 - \cos^2 x)^k \sin x \cos^n x\, dx = -\int (1 - u^2)^k u^n\, du,$$
where $u = \cos x$ and $du = -\sin x\, dx$.
2. If n is odd, then using substitutions similar to that outlined above we have
$$\int \sin^m x \cos^n x\, dx = \int u^m (1 - u^2)^k\, du,$$
where $u = \sin x$ and $du = \cos x\, dx$.
3. If both m and n are even, use the power–reducing identities
$$\cos^2 x = \frac{1 + \cos(2x)}{2} \quad \text{and} \quad \sin^2 x = \frac{1 - \cos(2x)}{2}$$
to reduce the degree of the integrand. Expand the result and apply the principles of this Key Idea again.

We practice applying Key Idea 6.3.1 in the next examples.

Example 6.3.1 Integrating powers of sine and cosine

Evaluate $\int \sin^5 x \cos^8 x\, dx$.

Solution The power of the sine term is odd, so we rewrite $\sin^5 x$ as
$$\sin^5 x = \sin^4 x \sin x = (\sin^2 x)^2 \sin x = (1 - \cos^2 x)^2 \sin x.$$

Our integral is now $\int (1 - \cos^2 x)^2 \cos^8 x \sin x\, dx$. Let $u = \cos x$, hence $du =$

$-\sin x\,dx$. Making the substitution and expanding the integrand gives

$$\begin{aligned}\int (1-\cos^2)^2\cos^8 x\sin x\,dx &= -\int (1-u^2)^2u^8\,du = -\int \left(1-2u^2+u^4\right)u^8\,du\\ &= -\int \left(u^8-2u^{10}+u^{12}\right)du.\end{aligned}$$

This final integral is not difficult to evaluate, giving

$$\begin{aligned}-\int \left(u^8-2u^{10}+u^{12}\right)du &= -\frac{1}{9}u^9+\frac{2}{11}u^{11}-\frac{1}{13}u^{13}+C\\ &= -\frac{1}{9}\cos^9 x+\frac{2}{11}\cos^{11}x-\frac{1}{13}\cos^{13}x+C.\end{aligned}$$

Example 6.3.2 **Integrating powers of sine and cosine**

Evaluate $\displaystyle\int \sin^5 x\cos^9 x\,dx$.

Solution The powers of both the sine and cosine terms are odd, therefore we can apply the techniques of Key Idea 6.3.1 to either power. We choose to work with the power of the cosine term since the previous example used the sine term's power.

We rewrite $\cos^9 x$ as

$$\begin{aligned}\cos^9 x &= \cos^8 x\cos x\\ &= (\cos^2 x)^4\cos x\\ &= (1-\sin^2 x)^4\cos x\\ &= (1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x)\cos x.\end{aligned}$$

We rewrite the integral as

$$\int \sin^5 x\cos^9 x\,dx = \int \sin^5 x\left(1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x\right)\cos x\,dx.$$

Now substitute and integrate, using $u=\sin x$ and $du=\cos x\,dx$.

$$\begin{aligned}&\int \sin^5 x\left(1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x\right)\cos x\,dx =\\ &\int u^5(1-4u^2+6u^4-4u^6+u^8)\,du = \int \left(u^5-4u^7+6u^9-4u^{11}+u^{13}\right)du\\ &= \frac{1}{6}u^6-\frac{1}{2}u^8+\frac{3}{5}u^{10}-\frac{1}{3}u^{12}+\frac{1}{14}u^{14}+C\\ &= \frac{1}{6}\sin^6 x-\frac{1}{2}\sin^8 x+\frac{3}{5}\sin^{10} x+\ldots\\ &\quad -\frac{1}{3}\sin^{12} x+\frac{1}{14}\sin^{14} x+C.\end{aligned}$$

Technology Note: The work we are doing here can be a bit tedious, but the skills developed (problem solving, algebraic manipulation, etc.) are important. Nowadays problems of this sort are often solved using a computer algebra system. The powerful program *Mathematica*® integrates $\int \sin^5 x\cos^9 x\,dx$ as

$$f(x) = -\frac{45\cos(2x)}{16384}-\frac{5\cos(4x)}{8192}+\frac{19\cos(6x)}{49152}+\frac{\cos(8x)}{4096}-\frac{\cos(10x)}{81920}-\frac{\cos(12x)}{24576}-\frac{\cos(14x)}{114688},$$

which clearly has a different form than our answer in Example 6.3.2, which is

$$g(x) = \frac{1}{6}\sin^6 x-\frac{1}{2}\sin^8 x+\frac{3}{5}\sin^{10} x-\frac{1}{3}\sin^{12} x+\frac{1}{14}\sin^{14} x.$$

Figure 6.3.1 shows a graph of f and g; they are clearly not equal, but they differ *only by a constant*. That is $g(x) = f(x) + C$ for some constant C. So we have two different antiderivatives of the same function, meaning both answers are correct.

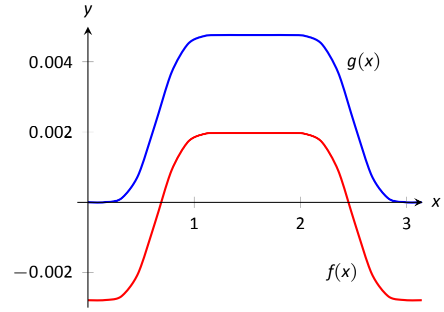


Figure 6.3.1: A plot of $f(x)$ and $g(x)$ from Example 6.3.2 and the Technology Note.

Example 6.3.3 Integrating powers of sine and cosine

Evaluate $\displaystyle\int \cos^4 x \sin^2 x\, dx$.

SOLUTION The powers of sine and cosine are both even, so we employ the power–reducing formulas and algebra as follows.

$$\begin{aligned}\int \cos^4 x \sin^2 x\, dx &= \int \left(\frac{1+\cos(2x)}{2}\right)^2 \left(\frac{1-\cos(2x)}{2}\right) dx\\ &= \int \frac{1+2\cos(2x)+\cos^2(2x)}{4}\cdot\frac{1-\cos(2x)}{2}\, dx\\ &= \int \frac{1}{8}\big(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big)\, dx\end{aligned}$$

The $\cos(2x)$ term is easy to integrate, especially with Key Idea 6.1.1. The $\cos^2(2x)$ term is another trigonometric integral with an even power, requiring the power–reducing formula again. The $\cos^3(2x)$ term is a cosine function with an odd power, requiring a substitution as done before. We integrate each in turn below.

$$\int \cos(2x)\, dx = \frac{1}{2}\sin(2x) + C.$$

$$\int \cos^2(2x)\, dx = \int \frac{1+\cos(4x)}{2}\, dx = \frac{1}{2}\big(x + \frac{1}{4}\sin(4x)\big) + C.$$

Finally, we rewrite $\cos^3(2x)$ as

$$\cos^3(2x) = \cos^2(2x)\cos(2x) = \big(1-\sin^2(2x)\big)\cos(2x).$$

Letting $u = \sin(2x)$, we have $du = 2\cos(2x)\, dx$, hence

$$\begin{aligned}\int \cos^3(2x)\, dx &= \int \big(1-\sin^2(2x)\big)\cos(2x)\, dx\\ &= \int \frac{1}{2}(1-u^2)\, du\\ &= \frac{1}{2}\left(u - \frac{1}{3}u^3\right) + C\\ &= \frac{1}{2}\left(\sin(2x) - \frac{1}{3}\sin^3(2x)\right) + C\end{aligned}$$

Putting all the pieces together, we have

$$\begin{aligned}\int \cos^4 x\sin^2 x\, dx &= \int \frac{1}{8}\big(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big)\, dx\\ &= \frac{1}{8}\left[x + \frac{1}{2}\sin(2x) - \frac{1}{2}\big(x+\frac{1}{4}\sin(4x)\big) - \frac{1}{2}\left(\sin(2x) - \frac{1}{3}\sin^3(2x)\right)\right] + C\\ &= \frac{1}{8}\left[\frac{1}{2}x - \frac{1}{8}\sin(4x) + \frac{1}{6}\sin^3(2x)\right] + C\end{aligned}$$

The process above was a bit long and tedious, but being able to work a problem such as this from start to finish is important.

Integrals of the form $\int \sin(mx)\sin(nx)\ dx$, $\int \cos(mx)\cos(nx)\ dx$, and $\int \sin(mx)\cos(nx)\ dx$.

Functions that contain products of sines and cosines of differing periods are important in many applications including the analysis of sound waves. Integrals of the form

$$\int \sin(mx)\sin(nx)\ dx, \quad \int \cos(mx)\cos(nx)\ dx \quad \text{and} \quad \int \sin(mx)\cos(nx)\ dx$$

are best approached by first applying the Product to Sum Formulas found in the back cover of this text, namely

$$\begin{aligned}
\sin(mx)\sin(nx) &= \frac{1}{2}\Big[\cos\big((m-n)x\big) - \cos\big((m+n)x\big)\Big]\\
\cos(mx)\cos(nx) &= \frac{1}{2}\Big[\cos\big((m-n)x\big) + \cos\big((m+n)x\big)\Big]\\
\sin(mx)\cos(nx) &= \frac{1}{2}\Big[\sin\big((m-n)x\big) + \sin\big((m+n)x\big)\Big]
\end{aligned}$$

Example 6.3.4 **Integrating products of** $\sin(mx)$ **and** $\cos(nx)$

Evaluate $\int \sin(5x)\cos(2x)\ dx$.

Solution The application of the formula and subsequent integration are straightforward:

$$\begin{aligned}
\int \sin(5x)\cos(2x)\ dx &= \int \frac{1}{2}\Big[\sin(3x)+\sin(7x)\Big]\ dx\\
&= -\frac{1}{6}\cos(3x) - \frac{1}{14}\cos(7x) + C
\end{aligned}$$

Integrals of the form $\int \tan^m x \sec^n x\ dx$.

When evaluating integrals of the form $\int \sin^m x\cos^n x\ dx$, the Pythagorean Theorem allowed us to convert even powers of sine into even powers of cosine, and vice–versa. If, for instance, the power of sine was odd, we pulled out one $\sin x$ and converted the remaining even power of $\sin x$ into a function using powers of $\cos x$, leading to an easy substitution.

The same basic strategy applies to integrals of the form $\int \tan^m x\sec^n x\ dx$, albeit a bit more nuanced. The following three facts will prove useful:

- $\frac{d}{dx}(\tan x) = \sec^2 x$,
- $\frac{d}{dx}(\sec x) = \sec x\tan x$, and
- $1+\tan^2 x = \sec^2 x$ (the Pythagorean Theorem).

If the integrand can be manipulated to separate a $\sec^2 x$ term with the remaining secant power even, or if a $\sec x\tan x$ term can be separated with the remaining $\tan x$ power even, the Pythagorean Theorem can be employed, leading to a simple substitution. This strategy is outlined in the following Key Idea.

Key Idea 6.3.2 Integrals Involving Powers of Tangent and Secant

Consider $\int \tan^m x \sec^n x\, dx$, where m, n are nonnegative integers.

1. If n is even, then $n = 2k$ for some integer k. Rewrite $\sec^n x$ as

$$\sec^n x = \sec^{2k} x = \sec^{2k-2} x \sec^2 x = (1 + \tan^2 x)^{k-1} \sec^2 x.$$

Then

$$\int \tan^m x \sec^n x\, dx = \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x\, dx = \int u^m (1 + u^2)^{k-1}\, du,$$

where $u = \tan x$ and $du = \sec^2 x\, dx$.

2. If m is odd, then $m = 2k + 1$ for some integer k. Rewrite $\tan^m x \sec^n x$ as

$$\begin{aligned}\tan^m x \sec^n x &= \tan^{2k+1} x \sec^n x = \tan^{2k} x \sec^{n-1} x \sec x \tan x\\ &= (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x.\end{aligned}$$

Then

$$\int \tan^m x \sec^n x\, dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\, dx = \int (u^2 - 1)^k u^{n-1}\, du,$$

where $u = \sec x$ and $du = \sec x \tan x\, dx$.

3. If n is odd and m is even, then $m = 2k$ for some integer k. Convert $\tan^m x$ to $(\sec^2 x - 1)^k$. Expand the new integrand and use Integration By Parts, with $dv = \sec^2 x\, dx$.

4. If m is even and $n = 0$, rewrite $\tan^m x$ as

$$\tan^m x = \tan^{m-2} x \tan^2 x = \tan^{m-2} x (\sec^2 x - 1) = \tan^{m-2} \sec^2 x - \tan^{m-2} x.$$

So

$$\int \tan^m x\, dx = \underbrace{\int \tan^{m-2} \sec^2 x\, dx}_{\text{apply rule \#1}} - \underbrace{\int \tan^{m-2} x\, dx}_{\text{apply rule \#4 again}}.$$

The techniques described in items 1 and 2 of Key Idea 6.3.2 are relatively straightforward, but the techniques in items 3 and 4 can be rather tedious. A few examples will help with these methods.

Example 6.3.5 Integrating powers of tangent and secant

Evaluate $\int \tan^2 x \sec^6 x\, dx$.

Solution Since the power of secant is even, we use rule #1 from Key Idea 6.3.2 and pull out a $\sec^2 x$ in the integrand. We convert the remaining powers of secant into powers of tangent.

$$\begin{aligned}\int \tan^2 x \sec^6 x\, dx &= \int \tan^2 x \sec^4 x \sec^2 x\, dx\\ &= \int \tan^2 x \left(1 + \tan^2 x\right)^2 \sec^2 x\, dx\end{aligned}$$

Now substitute, with $u = \tan x$, with $du = \sec^2 x\,dx$.

$$= \int u^2\left(1+u^2\right)^2\,du$$

We leave the integration and subsequent substitution to the reader. The final answer is

$$= \frac{1}{3}\tan^3 x + \frac{2}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C.$$

Example 6.3.6 Integrating powers of tangent and secant

Evaluate $\displaystyle\int \sec^3 x\,dx$.

Solution We apply rule #3 from Key Idea 6.3.2 as the power of secant is odd and the power of tangent is even (0 is an even number). We use Integration by Parts; the rule suggests letting $dv = \sec^2 x\,dx$, meaning that $u = \sec x$.

$$\begin{array}{ll} u = \sec x & v = ? \\ du = ? & dv = \sec^2 x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = \sec x & v = \tan x \\ du = \sec x\tan x\,dx & dv = \sec^2 x\,dx \end{array}$$

Figure 6.3.2: Setting up Integration by Parts.

Employing Integration by Parts, we have

$$\begin{aligned} \int \sec^3 x\,dx &= \int \underbrace{\sec x}_{u}\cdot\underbrace{\sec^2 x\,dx}_{dv} \\ &= \sec x\tan x - \int \sec x\tan^2 x\,dx. \end{aligned}$$

This new integral also requires applying rule #3 of Key Idea 6.3.2:

$$\begin{aligned} &= \sec x\tan x - \int \sec x\big(\sec^2 x - 1\big)\,dx \\ &= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\ &= \sec x\tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x| \end{aligned}$$

In previous applications of Integration by Parts, we have seen where the original integral has reappeared in our work. We resolve this by adding $\int \sec^3 x\,dx$ to both sides, giving:

$$\begin{aligned} 2\int \sec^3 x\,dx &= \sec x\tan x + \ln|\sec x + \tan x| \\ \int \sec^3 x\,dx &= \frac{1}{2}\Big(\sec x\tan x + \ln|\sec x + \tan x|\Big) + C \end{aligned}$$

We give one more example.

Example 6.3.7 Integrating powers of tangent and secant

Evaluate $\displaystyle\int \tan^6 x\,dx$.

Solution We employ rule #4 of Key Idea 6.3.2.

$$\begin{aligned}\int \tan^6 x\,dx &= \int \tan^4 x\tan^2 x\,dx\\ &= \int \tan^4 x\big(\sec^2 x - 1\big)\,dx\\ &= \int \tan^4 x\sec^2 x\,dx - \int \tan^4 x\,dx\end{aligned}$$

Integrate the first integral with substitution, $u = \tan x$; integrate the second by employing rule #4 again.

$$\begin{aligned}&= \frac{1}{5}\tan^5 x - \int \tan^2 x\tan^2 x\,dx\\ &= \frac{1}{5}\tan^5 x - \int \tan^2 x\big(\sec^2 x - 1\big)\,dx\\ &= \frac{1}{5}\tan^5 x - \int \tan^2 x\sec^2 x\,dx + \int \tan^2 x\,dx\end{aligned}$$

Again, use substitution for the first integral and rule #4 for the second.

$$\begin{aligned}&= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \int \big(\sec^2 x - 1\big)\,dx\\ &= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C.\end{aligned}$$

These latter examples were admittedly long, with repeated applications of the same rule. Try to not be overwhelmed by the length of the problem, but rather admire how robust this solution method is. A trigonometric function of a high power can be systematically reduced to trigonometric functions of lower powers until all antiderivatives can be computed.

The next section introduces an integration technique known as Trigonometric Substitution, a clever combination of Substitution and the Pythagorean Theorem.

Exercises 6.3

Terms and Concepts

1. T/F: $\int \sin^2 x \cos^2 x\, dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin x$ and $\cos x$ are even.

2. T/F: $\int \sin^3 x \cos^3 x\, dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin x$ and $\cos x$ are odd.

3. T/F: This section addresses how to evaluate indefinite integrals such as $\int \sin^5 x \tan^3 x\, dx$.

4. T/F: Sometimes computer programs evaluate integrals involving trigonometric functions differently than one would using the techniques of this section. When this is the case, the techniques of this section have failed and one should only trust the answer given by the computer.

Problems

In Exercises 5 – 28, evaluate the indefinite integral.

5. $\int \sin x \cos^4 x\, dx$

6. $\int \sin^3 x \cos x\, dx$

7. $\int \sin^3 x \cos^2 x\, dx$

8. $\int \sin^3 x \cos^3 x\, dx$

9. $\int \sin^6 x \cos^5 x\, dx$

10. $\int \sin^2 x \cos^7 x\, dx$

11. $\int \sin^2 x \cos^2 x\, dx$

12. $\int \sin x \cos x\, dx$

13. $\int \sin(5x) \cos(3x)\, dx$

14. $\int \sin(x) \cos(2x)\, dx$

15. $\int \sin(3x) \sin(7x)\, dx$

16. $\int \sin(\pi x) \sin(2\pi x)\, dx$

17. $\int \cos(x) \cos(2x)\, dx$

18. $\int \cos\left(\frac{\pi}{2}x\right) \cos(\pi x)\, dx$

19. $\int \tan^4 x \sec^2 x\, dx$

20. $\int \tan^2 x \sec^4 x\, dx$

21. $\int \tan^3 x \sec^4 x\, dx$

22. $\int \tan^3 x \sec^2 x\, dx$

23. $\int \tan^3 x \sec^3 x\, dx$

24. $\int \tan^5 x \sec^5 x\, dx$

25. $\int \tan^4 x\, dx$

26. $\int \sec^5 x\, dx$

27. $\int \tan^2 x \sec x\, dx$

28. $\int \tan^2 x \sec^3 x\, dx$

In Exercises 29 – 35, evaluate the definite integral. Note: the corresponding indefinite integrals appear in the previous set.

29. $\int_0^{\pi} \sin x \cos^4 x\, dx$

30. $\int_{-\pi}^{\pi} \sin^3 x \cos x\, dx$

31. $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^7 x\, dx$

32. $\int_0^{\pi/2} \sin(5x) \cos(3x)\, dx$

33. $\int_{-\pi/2}^{\pi/2} \cos(x) \cos(2x)\, dx$

34. $\int_0^{\pi/4} \tan^4 x \sec^2 x\, dx$

35. $\int_{-\pi/4}^{\pi/4} \tan^2 x \sec^4 x\, dx$

6.4 Trigonometric and Hyperbolic Substitution

In Section 5.2 we defined the definite integral as the "signed area under the curve." In that section we had not yet learned the Fundamental Theorem of Calculus, so we only evaluated special definite integrals which described nice, geometric shapes. For instance, we were able to evaluate

$$\int_{-3}^{3} \sqrt{9-x^2}\, dx = \frac{9\pi}{2} \tag{6.1}$$

as we recognized that $f(x) = \sqrt{9-x^2}$ described the upper half of a circle with radius 3.

We have since learned a number of integration techniques, including Substitution and Integration by Parts, yet we are still unable to evaluate the above integral without resorting to a geometric interpretation. This section introduces Trigonometric Substitution, a method of integration that fills this gap in our integration skill. This technique works on the same principle as Substitution as found in Section 6.1, though it can feel "backward." In Section 6.1, we set $u = f(x)$, for some function f, and replaced $f(x)$ with u. In this section, we will set $x = f(\theta)$, where f is a trigonometric function, then replace x with $f(\theta)$.

We start by demonstrating this method in evaluating the integral in Equation (6.1). After the example, we will generalize the method and give more examples.

Example 6.4.1 Using Trigonometric Substitution

Evaluate $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\, dx$.

Solution We begin by noting that $9\sin^2\theta + 9\cos^2\theta = 9$, and hence $9\cos^2\theta = 9 - 9\sin^2\theta$. If we let $x = 3\sin\theta$, then $9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$.

Setting $x = 3\sin\theta$ gives $dx = 3\cos\theta\, d\theta$. We are almost ready to substitute. We also wish to change our bounds of integration. The bound $x = -3$ corresponds to $\theta = -\pi/2$ (for when $\theta = -\pi/2$, $x = 3\sin\theta = -3$). Likewise, the bound of $x = 3$ is replaced by the bound $\theta = \pi/2$. Thus

$$\begin{aligned}
\int_{-3}^{3} \sqrt{9-x^2}\, dx &= \int_{-\pi/2}^{\pi/2} \sqrt{9-9\sin^2\theta}(3\cos\theta)\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} 3\sqrt{9\cos^2\theta}\cos\theta\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} 3|3\cos\theta|\cos\theta\, d\theta.
\end{aligned}$$

On $[-\pi/2, \pi/2]$, $\cos\theta$ is always positive, so we can drop the absolute value bars, then employ a power–reducing formula:

$$\begin{aligned}
&= \int_{-\pi/2}^{\pi/2} 9\cos^2\theta\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} \frac{9}{2}\big(1+\cos(2\theta)\big)\, d\theta \\
&= \frac{9}{2}\big(\theta + \frac{1}{2}\sin(2\theta)\big)\Big|_{-\pi/2}^{\pi/2} = \frac{9}{2}\pi.
\end{aligned}$$

This matches our answer from before.

We now describe in detail Trigonometric Substitution. This method excels when dealing with integrands that contain $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ and $\sqrt{x^2+a^2}$. The following Key Idea outlines the procedure for each case, followed by more examples. Each right triangle acts as a reference to help us understand the relationships between x and θ.

Key Idea 6.4.1 Trigonometric Substitution

For the three cases below, we assume that $a > 0$.

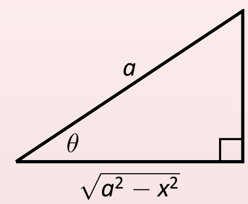
(a) For integrands containing $\sqrt{a^2-x^2}$:

Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$

Thus $\theta = \sin^{-1}(x/a)$, for $-\pi/2 \leq \theta \leq \pi/2$.

On this interval, $\cos\theta \geq 0$, so

$\sqrt{a^2-x^2} = a\cos\theta$



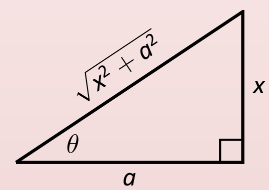
(b) For integrands containing $\sqrt{x^2+a^2}$:

Let $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$

Thus $\theta = \tan^{-1}(x/a)$, for $-\pi/2 < \theta < \pi/2$.

On this interval, $\sec\theta > 0$, so

$\sqrt{x^2+a^2} = a\sec\theta$



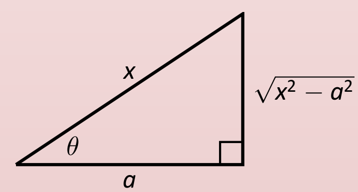
(c) For integrands containing $\sqrt{x^2-a^2}$:

Let $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\, d\theta$

Thus $\theta = \sec^{-1}(x/a)$. Note that $\sqrt{x^2-a^2}$ is defined for $x \geq a$ or $x \leq -a$.
If $x \geq a$, then $x/a \geq 1$ and $0 \leq \theta < \pi/2$; if $x < -a$, then $x/a \leq -1$ and $\pi \leq \theta < \frac{3\pi}{2}$.

On these intervals, $\tan\theta \geq 0$, so

$\sqrt{x^2-a^2} = a\tan\theta$



Example 6.4.2 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{\sqrt{5+x^2}}\, dx$.

Solution Using Key Idea 6.4.1(b), we recognize $a = \sqrt{5}$ and set $x = \sqrt{5}\tan\theta$. This makes $dx = \sqrt{5}\sec^2\theta\, d\theta$. We will use the fact that $\sqrt{5+x^2} = \sqrt{5+5\tan^2\theta} = \sqrt{5\sec^2\theta} = \sqrt{5}\sec\theta$. Substituting, we have:

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\, dx &= \int \frac{1}{\sqrt{5+5\tan^2\theta}}\sqrt{5}\sec^2\theta\, d\theta\\ &= \int \frac{\sqrt{5}\sec^2\theta}{\sqrt{5}\sec\theta}\, d\theta\\ &= \int \sec\theta\, d\theta\\ &= \ln\left|\sec\theta+\tan\theta\right| + C.\end{aligned}$$

While the integration steps are over, we are not yet done. The original problem was stated in terms of x, whereas our answer is given in terms of θ. We must convert back to x.

The reference triangle given in Key Idea 6.4.1(b) helps. With $x = \sqrt{5}\tan\theta$, we have

$$\tan\theta = \frac{x}{\sqrt{5}} \quad \text{and} \quad \sec\theta = \frac{\sqrt{x^2+5}}{\sqrt{5}}.$$

This gives

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \ln\big|\sec\theta+\tan\theta\big| + C\\ &= \ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}} + \frac{x}{\sqrt{5}}\right| + C.\end{aligned}$$

We can leave this answer as is, or we can use a logarithmic identity to simplify it. Note:

$$\begin{aligned}\ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}} + \frac{x}{\sqrt{5}}\right| + C &= \ln\left|\frac{1}{\sqrt{5}}\big(\sqrt{x^2+5}+x\big)\right| + C\\ &= \ln\left|\frac{1}{\sqrt{5}}\right| + \ln\big|\sqrt{x^2+5}+x\big| + C\\ &= \ln\big|\sqrt{x^2+5}+x\big| + C,\end{aligned}$$

where the $\ln\big(1/\sqrt{5}\big)$ term is absorbed into the constant C. (In Example 6.4.8 we will learn another way of approaching this type of problem.)

Example 6.4.3 Using Trigonometric Substitution

Evaluate $\displaystyle\int \sqrt{4x^2-1}\,dx$.

Solution We start by rewriting the integrand so that it looks like $\sqrt{x^2-a^2}$ for some value of a:

$$\begin{aligned}\sqrt{4x^2-1} &= \sqrt{4\left(x^2-\frac{1}{4}\right)}\\ &= 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}.\end{aligned}$$

So we have $a = 1/2$, and following Key Idea 6.4.1(c), we set $x = \frac{1}{2}\sec\theta$, and hence $dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$. We now rewrite the integral with these substitutions:

$$\begin{aligned}\int \sqrt{4x^2-1}\,dx &= \int 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}\,dx\\ &= \int 2\sqrt{\frac{1}{4}\sec^2\theta-\frac{1}{4}}\left(\frac{1}{2}\sec\theta\tan\theta\right)\,d\theta\\ &= \int \sqrt{\frac{1}{4}(\sec^2\theta-1)}\Big(\sec\theta\tan\theta\Big)\,d\theta\\ &= \int \sqrt{\frac{1}{4}\tan^2\theta}\Big(\sec\theta\tan\theta\Big)\,d\theta\\ &= \int \frac{1}{2}\tan^2\theta\sec\theta\,d\theta\\ &= \frac{1}{2}\int\Big(\sec^2\theta-1\Big)\sec\theta\,d\theta\\ &= \frac{1}{2}\int\big(\sec^3\theta-\sec\theta\big)\,d\theta.\end{aligned}$$

We integrated $\sec^3\theta$ in Example 6.3.6, finding its antiderivatives to be

$$\int \sec^3\theta\,d\theta = \frac{1}{2}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)+C.$$

Thus

$$\begin{aligned}\int\sqrt{4x^2-1}\,dx &= \frac{1}{2}\int\big(\sec^3\theta-\sec\theta\big)\,d\theta\\ &= \frac{1}{2}\left(\frac{1}{2}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)-\ln|\sec\theta+\tan\theta|\right)+C\\ &= \frac{1}{4}\left(\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|\right)+C.\end{aligned}$$

We are not yet done. Our original integral is given in terms of x, whereas our final answer, as given, is in terms of θ. We need to rewrite our answer in terms of x. With $a=1/2$, and $x=\frac{1}{2}\sec\theta$, the reference triangle in Key Idea 6.4.1(c) shows that

$$\tan\theta = \sqrt{x^2-1/4}\Big/(1/2) = 2\sqrt{x^2-1/4}\quad\text{and}\quad \sec\theta=2x.$$

Thus

$$\begin{aligned}\frac{1}{4}\Big(\sec\theta\tan\theta-\ln\big|\sec\theta+\tan\theta\big|\Big)+C &= \frac{1}{4}\Big(2x\cdot2\sqrt{x^2-1/4}-\ln\big|2x+2\sqrt{x^2-1/4}\big|\Big)+C\\ &= \frac{1}{4}\Big(4x\sqrt{x^2-1/4}-\ln\big|2x+2\sqrt{x^2-1/4}\big|\Big)+C.\end{aligned}$$

The final answer is given in the last line above, repeated here:

$$\int\sqrt{4x^2-1}\,dx = \frac{1}{4}\Big(4x\sqrt{x^2-1/4}-\ln\big|2x+2\sqrt{x^2-1/4}\big|\Big)+C.$$

Example 6.4.4 Using Trigonometric Substitution

Evaluate $\displaystyle\int\frac{\sqrt{4-x^2}}{x^2}\,dx.$

Solution We use Key Idea 6.4.1(a) with $a=2$, $x=2\sin\theta$, $dx=$

$2\cos\theta$ and hence $\sqrt{4-x^2} = 2\cos\theta$. This gives

$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x^2}\,dx &= \int \frac{2\cos\theta}{4\sin^2\theta}(2\cos\theta)\,d\theta\\ &= \int \cot^2\theta\,d\theta\\ &= \int (\csc^2\theta - 1)\,d\theta\\ &= -\cot\theta - \theta + C.\end{aligned}$$

We need to rewrite our answer in terms of x. Using the reference triangle found in Key Idea 6.4.1(a), we have $\cot\theta = \sqrt{4-x^2}/x$ and $\theta = \sin^{-1}(x/2)$. Thus

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C.$$

Trigonometric Substitution can be applied in many situations, even those not of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ or $\sqrt{x^2+a^2}$. In the following example, we apply it to an integral we already know how to handle.

Example 6.4.5 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{x^2+1}\,dx$.

Solution We know the answer already as $\tan^{-1}x + C$. We apply Trigonometric Substitution here to show that we get the same answer without inherently relying on knowledge of the derivative of the arctangent function.

Using Key Idea 6.4.1(b), let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$ and note that $x^2+1 = \tan^2\theta + 1 = \sec^2\theta$. Thus

$$\begin{aligned}\int \frac{1}{x^2+1}\,dx &= \int \frac{1}{\sec^2\theta}\sec^2\theta\,d\theta\\ &= \int 1\,d\theta\\ &= \theta + C.\end{aligned}$$

Since $x = \tan\theta$, $\theta = \tan^{-1}x$, and we conclude that $\displaystyle\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$.

The next example is similar to the previous one in that it does not involve a square–root. It shows how several techniques and identities can be combined to obtain a solution.

Example 6.4.6 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{(x^2+6x+10)^2}\,dx$.

Solution We start by completing the square, then make the substitution $u = x+3$, followed by the trigonometric substitution of $u = \tan\theta$:

$$\int \frac{1}{(x^2+6x+10)^2}\,dx = \int \frac{1}{\big((x+3)^2+1\big)^2}\,dx = \int \frac{1}{(u^2+1)^2}\,du.$$

Now make the substitution $u = \tan\theta$, $du = \sec^2\theta\, d\theta$:

$$\begin{aligned} &= \int \frac{1}{(\tan^2\theta + 1)^2}\sec^2\theta\, d\theta \\ &= \int \frac{1}{(\sec^2\theta)^2}\sec^2\theta\, d\theta \\ &= \int \cos^2\theta\, d\theta. \end{aligned}$$

Applying a power reducing formula, we have

$$\begin{aligned} &= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\, d\theta \\ &= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C. \quad (6.2) \end{aligned}$$

We need to return to the variable x. As $u = \tan\theta$, $\theta = \tan^{-1} u$. Using the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ and using the reference triangle found in Key Idea 6.4.1(b), we have

$$\frac{1}{4}\sin(2\theta) = \frac{1}{2}\frac{u}{\sqrt{u^2+1}} \cdot \frac{1}{\sqrt{u^2+1}} = \frac{1}{2}\frac{u}{u^2+1}.$$

Finally, we return to x with the substitution $u = x + 3$. We start with the expression in Equation (6.2):

$$\begin{aligned} \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C &= \frac{1}{2}\tan^{-1} u + \frac{1}{2}\frac{u}{u^2+1} + C \\ &= \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C. \end{aligned}$$

Stating our final result in one line,

$$\int \frac{1}{(x^2+6x+10)^2}\, dx = \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C.$$

Our last example using trigonometric substitution returns us to definite integrals, as seen in our first example. Given a definite integral that can be evaluated using Trigonometric Substitution, we could first evaluate the corresponding indefinite integral (by changing from an integral in terms of x to one in terms of θ, then converting back to x) and then evaluate using the original bounds. It is much more straightforward, though, to change the bounds as we substitute.

Example 6.4.7 Definite integration and Trigonometric Substitution

Evaluate $\displaystyle\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\, dx$.

SOLUTION Using Key Idea 6.4.1(b), we set $x = 5\tan\theta$, $dx = 5\sec^2\theta\, d\theta$, and note that $\sqrt{x^2+25} = 5\sec\theta$. As we substitute, we can also change the bounds of integration.

The lower bound of the original integral is $x = 0$. As $x = 5\tan\theta$, we solve for θ and find $\theta = \tan^{-1}(x/5)$. Thus the new lower bound is $\theta = \tan^{-1}(0) = 0$. The original upper bound is $x = 5$, thus the new upper bound is $\theta = \tan^{-1}(5/5) = \pi/4$.

Thus we have

$$\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\,dx = \int_0^{\pi/4} \frac{25\tan^2\theta}{5\sec\theta} 5\sec^2\theta\,d\theta$$
$$= 25\int_0^{\pi/4} \tan^2\theta\sec\theta\,d\theta.$$

We encountered this indefinite integral in Example 6.4.3 where we found

$$\int \tan^2\theta\sec\theta\,d\theta = \frac{1}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big).$$

So

$$25\int_0^{\pi/4} \tan^2\theta\sec\theta\,d\theta = \frac{25}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big)\Big|_0^{\pi/4}$$
$$= \frac{25}{2}\big(\sqrt{2} - \ln(\sqrt{2}+1)\big)$$
$$\approx 6.661.$$

The following equalities are very useful when evaluating integrals using Trigonometric Substitution.

Key Idea 6.4.2 Useful Equalities with Trigonometric Substitution

1. $\sin(2\theta) = 2\sin\theta\cos\theta$
2. $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
3. $\displaystyle\int \sec^3\theta\,d\theta = \frac{1}{2}\Big(\sec\theta\tan\theta + \ln\big|\sec\theta+\tan\theta\big|\Big) + C$
4. $\displaystyle\int \cos^2\theta\,d\theta = \int \frac{1}{2}\big(1+\cos(2\theta)\big)\,d\theta = \frac{1}{2}\big(\theta + \sin\theta\cos\theta\big) + C.$

Hyperbolic substitution

For integrands containing terms of the form $\sqrt{x^2+a^2}$ or $\sqrt{x^2-a^2}$, it is also possible to make use of **hyperbolic substitution** . Recall from Section 5.7 of your Calculus I textbook that the hyperbolic functions are defined by

$$\cosh(x) = \frac{e^x+e^{-x}}{2} \quad \text{and} \quad \sinh(x) = \frac{e^x-e^{-x}}{2},$$

with $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$, and so on. Recall that the hyperbolic functions satisfy the identity

$$\cosh^2(x) - \sinh^2(x) = 1.$$

If we're given $\sqrt{x^2+a^2}$, we can let $x = a\sinh(t)$, then

$$x^2+a^2 = (a\sinh(t))^2 + a^2 = a^2(\sinh^2(t)+1) = a^2\cosh^2(t),$$

and $dx = a\cosh(t)\,dt$. Since $\cosh(t) > 0$ for all real numbers t, we have $\sqrt{a^2+x^2} = \cosh(t)$.

If we're given $\sqrt{x^2-a^2}$, we can let $x = a\cosh(t)$; then

$$x^2-a^2 = (a\cosh(t))^2 - a^2 = a^2(\cosh^2(t)-1) = a^2\sinh^2(t),$$

and $dx = a\sinh(t)$. (One of the convenient aspects of working with hyperbolic functions is that there are no signs to worry about when taking derivatives.) Note that $\cosh(t) > 0$ for all t, so the substitution $x = a\cosh(t)$ works in the case that $x \geq a > 0$. In this case, $\sqrt{x^2 - a^2} = a\sinh(t)$. For $x < -a < 0$, technically we would need to let $x = -a\cosh(t)$. We illustrate this method with a couple of examples.

Example 6.4.8 Using hyperbolic substitution

Use hyperbolic substitution to evaluate the integral $\displaystyle\int \frac{3}{\sqrt{x^2+4}}\,dx$.

SOLUTION Here we encounter the form $\sqrt{x^2+4} = \sqrt{x^2+2^2}$, so we use the substitution $x = 2\sinh(t)$, or $t = \sinh^{-1}(x/2)$. This gives us

$$\sqrt{x^2+4} = \sqrt{4\sinh^2(t)+4} = \sqrt{4\cosh^2(t)} = 2\cosh(t),$$

and $dx = 4\cosh(t)\,dt$. Substituting these into the integral, we have

$$\begin{aligned}\int \frac{3}{\sqrt{x^2+4}}\,dx &= \int \frac{3(2\cosh(t))}{2\cosh(t)}\,dt\\ &= \int 3\,dt = 3t + C\\ &= 3\sinh^{-1}\left(\frac{x}{2}\right) + C.\end{aligned}$$

Of course, we could also evaluate the integral using the substitution $x = 2\tan\theta$. In this case, the hyperbolic substitution turns out to be much simpler. Let's go through the details to confirm this. If $x = 2\tan\theta$, then we have $dx = 2\sec^2\theta\,d\theta$, and

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta+4} = \sqrt{4\sec^2\theta} = 2\sec\theta.$$

In terms of θ, our integral is

$$\begin{aligned}\int \frac{3}{\sqrt{x^2+4}}\,dx &= \int \frac{3(2\sec^2\theta)}{2\sec\theta}\,d\theta\\ &= \int 3\sec\theta\,d\theta\\ &= 3\ln\lvert\sec\theta+\tan\theta\rvert + C\\ &= 3\ln\left(\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right) + C.\end{aligned}$$

(Note that since $\sqrt{x^2+4} > x$ for all values of x, we can drop the absolute value in the argument of the logarithm.)

The answer using the hyperbolic substitution certainly looks simpler, and it was easier to obtain, but are the two results equivalent? That is, is it true that

$$\sinh^{-1}\left(\frac{x}{2}\right) = \ln\left(\frac{\sqrt{x^2+4}+x}{2}\right)?$$

To check, let's take the hyperbolic sine of both sides. On the left, this yields $\frac{x}{2}$.

On the right, we get

$$\begin{aligned}
\sinh\left[\ln\left(\frac{\sqrt{x^2+4}+x}{2}\right)\right] &= \frac{1}{2}\left[e^{\ln\left(\frac{\sqrt{x^2+4}+x}{2}\right)} - e^{\ln\left(\frac{\sqrt{x^2+4}+x}{2}\right)}\right] \\
&= \frac{1}{2}\left[\frac{\sqrt{x^2+4}+x}{2} - \frac{2}{\sqrt{x^2+4}+x}\right] \\
&= \frac{1}{2}\left[\frac{(\sqrt{x^2+4}+x)^2-4}{2(\sqrt{x^2+4}+x)}\right] \\
&= \frac{1}{4}\left[\frac{(x^2+4)+2x\sqrt{x^2+4}+x^2-4}{\sqrt{x^2+4}+x}\right] \\
&= \frac{1}{4}\left[\frac{2x(\sqrt{x^2+4}+x)}{\sqrt{x^2+4}+x}\right] \\
&= \frac{x}{2},
\end{aligned}$$

so the results agree!

Example 6.4.9 Using hyperbolic substitution

Evaluate the integral $\displaystyle\int \frac{x^2}{\sqrt{x^2-16}}\,dx$ using a hyperbolic substitution.

SOLUTION Once again, we *could* use the substitution $x = 4\sec\theta$, but doing so leads us to a $\sec^3\theta$ integral, and those are never fun. (Feel free to try it this way) Instead, we decide to read (and follow) the instructions, and use a hyperbolic substitution.

The form $\sqrt{x^2-16}$ tells us that we should try the substitution $x = 4\cosh(t)$. This gives us $dx = 4\sinh(t)$, and

$$\sqrt{x^2-16} = \sqrt{16(\cosh^2(t)-1)} = \sqrt{16\sinh^2(t)} = 4\sinh(t).$$

Thus,

$$\int \frac{x^2}{\sqrt{x^2-16}}\,dx = \int \frac{16\cosh^2(t)}{4\sinh(t)}(4\sinh(t))\,dt = \int 16\cosh^2(t)\,dt.$$

Now we have to know how to integrate $\cosh^2(t)$. If we recall how $\cosh(t)$ is defined, we have

$$\cosh^2(t) = \left(\frac{e^t+e^{-t}}{2}\right)^2 = \frac{e^{2t}+e^{-2t}+2}{4}.$$

So you could simply write $\cosh^2(t)$ in terms of exponentials as above, and integrate term-by-term. The other option is to notice that there's an identity sitting there: $\dfrac{e^{2t}+e^{-2t}}{4} = \frac{1}{2}\cosh(2t)$, so

$$\int \cosh^2(t)\,dt = \int\left(\frac{1}{2}\cosh(2t)+\frac{1}{2}\right)\,dt = \frac{1}{4}\sinh(2t)+\frac{t}{2}+C.$$

Finally, we have to substitute back in terms of x. Would it surprise you to learn that $\sinh(2t) = 2\sinh(t)\cosh(t)$, just as it is with trig functions? Well, that turns out to be true (verify this!). Since $\sinh(t) = \frac{\sqrt{x^2-16}}{4}$ and $\cosh(t) = \frac{x}{4}$, we get

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2-16}}\,dx &= 16\int \cosh^2(t)\,dt\\ &= \frac{1}{2}x\sqrt{x^2-16} + 8\cosh^{-1}\left(\frac{x}{4}\right) + C.\end{aligned}$$

Again, if you chose the secant substitution route, you would have ended up with a very different-looking answer. This method gives

$$\int \frac{x^2}{\sqrt{x^2-16}}\,dx = \frac{1}{2}x\sqrt{x^2-16} + 8\ln\left(\frac{x+\sqrt{x^2-16}}{4}\right),$$

and as with the last example, you might be wondering is whether the two answers are the same. It's a good exercise to see if you can show that

$$\ln\left(\frac{x+\sqrt{x^2-16}}{4}\right) = \cosh^{-1}(x/4).$$

You can perform algebraic manipulations as before, or check that the derivatives of both sides are the same (it's enough for the answers to agree up to a constant).

The next section introduces Partial Fraction Decomposition, which is an algebraic technique that turns "complicated" fractions into sums of "simpler" fractions, making integration easier.

Exercises 6.4

Terms and Concepts

1. Trigonometric Substitution works on the same principles as Integration by Substitution, though it can feel "________".

2. If one uses Trigonometric Substitution on an integrand containing $\sqrt{25-x^2}$, then one should set $x =$________.

3. Consider the Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1$.

   (a) What identity is obtained when both sides are divided by $\cos^2\theta$?

   (b) Use the new identity to simplify $9\tan^2\theta + 9$.

4. Why does Key Idea 6.4.1(a) state that $\sqrt{a^2-x^2} = a\cos\theta$, and not $|a\cos\theta|$?

Problems

In Exercises 5 – 16, apply Trigonometric Substitution to evaluate the indefinite integrals. Where appropriate, you may also want to try a hyperbolic substitution.

5. $\displaystyle\int \sqrt{x^2+1}\,dx$

6. $\displaystyle\int \sqrt{x^2+4}\,dx$

7. $\displaystyle\int \sqrt{1-x^2}\,dx$

8. $\displaystyle\int \sqrt{9-x^2}\,dx$

9. $\displaystyle\int \sqrt{x^2-1}\,dx$

10. $\displaystyle\int \sqrt{x^2-16}\,dx$

11. $\displaystyle\int \sqrt{4x^2+1}\,dx$

12. $\displaystyle\int \sqrt{1-9x^2}\,dx$

13. $\displaystyle\int \sqrt{16x^2-1}\,dx$

14. $\displaystyle\int \frac{8}{\sqrt{x^2+2}}\,dx$

15. $\displaystyle\int \frac{3}{\sqrt{7-x^2}}\,dx$

16. $\displaystyle\int \frac{5}{\sqrt{x^2-8}}\,dx$

In Exercises 17 – 26, evaluate the indefinite integrals. Some may be evaluated without Trigonometric Substitution.

17. $\displaystyle\int \frac{\sqrt{x^2-11}}{x}\,dx$

18. $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx$

19. $\displaystyle\int \frac{x}{\sqrt{x^2-3}}\,dx$

20. $\displaystyle\int x^2\sqrt{1-x^2}\,dx$

21. $\displaystyle\int \frac{x}{(x^2+9)^{3/2}}\,dx$

22. $\displaystyle\int \frac{5x^2}{\sqrt{x^2-10}}\,dx$

23. $\displaystyle\int \frac{1}{(x^2+4x+13)^2}\,dx$

24. $\displaystyle\int x^2(1-x^2)^{-3/2}\,dx$

25. $\displaystyle\int \frac{\sqrt{5-x^2}}{7x^2}\,dx$

26. $\displaystyle\int \frac{x^2}{\sqrt{x^2+3}}\,dx$

In Exercises 27 – 32, evaluate the definite integrals by making the proper trigonometric substitution *and* changing the bounds of integration. (Note: each of the corresponding indefinite integrals has appeared previously in this Exercise set.)

27. $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$

28. $\displaystyle\int_{4}^{8} \sqrt{x^2-16}\,dx$

29. $\displaystyle\int_{0}^{2} \sqrt{x^2+4}\,dx$

30. $\displaystyle\int_{-1}^{1} \frac{1}{(x^2+1)^2}\,dx$

31. $\displaystyle\int_{-1}^{1} \sqrt{9-x^2}\,dx$

32. $\displaystyle\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx$

6.5 Partial Fraction Decomposition

In this section we investigate the antiderivatives of rational functions. Recall that rational functions are functions of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. Such functions arise in many contexts, one of which is the solving of certain fundamental differential equations.

We begin with an example that demonstrates the motivation behind this section. Consider the integral $\displaystyle\int \frac{1}{x^2-1}\,dx$. We do not have a simple formula for this (if the denominator were x^2+1, we would recognize the antiderivative as being the arctangent function). It can be solved using Trigonometric Substitution, but note how the integral is easy to evaluate once we realize:

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Thus

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \int \frac{1/2}{x-1}\,dx - \int \frac{1/2}{x+1}\,dx\\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

This section teaches how to *decompose*

$$\frac{1}{x^2-1} \quad \text{into} \quad \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

We start with a rational function $f(x) = \dfrac{p(x)}{q(x)}$, where p and q do not have any common factors and the degree of p is less than the degree of q. Note that in the case of a function $f(x) = \dfrac{p(x)}{q(x)}$ where the degree of p is greater than or equal to that of q, we can perform polynomial long division to rewrite $f(x)$ in the form

$$f(x) = Q(x) + \frac{R(x)}{q(x)},$$

where $Q(x)$ is the quotient polynomial, and $R(x)$, the remainder, always has degree less than that of q.

It can be shown that any polynomial, and hence q, can be factored into a product of linear and irreducible quadratic terms. The following Key Idea states how to decompose a rational function into a sum of rational functions whose denominators are all of lower degree than q.

Key Idea 6.5.1 Partial Fraction Decomposition

Let $\dfrac{p(x)}{q(x)}$ be a rational function, where the degree of p is less than the degree of q.

1. **Linear Terms:** Let $(x-a)$ divide $q(x)$, where $(x-a)^n$ is the highest power of $(x-a)$ that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_n}{(x-a)^n}.$$

2. **Quadratic Terms:** Let x^2+bx+c divide $q(x)$, where $(x^2+bx+c)^n$ is the highest power of x^2+bx+c that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{B_1x+C_1}{x^2+bx+c}+\frac{B_2x+C_2}{(x^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(x^2+bx+c)^n}.$$

To find the coefficients A_i, B_i and C_i:

1. Multiply all fractions by $q(x)$, clearing the denominators. Collect like terms.
2. Equate the resulting coefficients of the powers of x and solve the resulting system of linear equations.

The following examples will demonstrate how to put this Key Idea into practice. Example 6.5.1 stresses the decomposition aspect of the Key Idea.

Example 6.5.1 Decomposing into partial fractions

Decompose $f(x)=\dfrac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2}$ without solving for the resulting coefficients.

Solution The denominator is already factored, as both x^2+x+2 and x^2+x+7 cannot be factored further. We need to decompose $f(x)$ properly. Since $(x+5)$ is a linear term that divides the denominator, there will be a

$$\frac{A}{x+5}$$

term in the decomposition.

As $(x-2)^3$ divides the denominator, we will have the following terms in the decomposition:

$$\frac{B}{x-2},\quad \frac{C}{(x-2)^2}\quad \text{and}\quad \frac{D}{(x-2)^3}.$$

The x^2+x+2 term in the denominator results in a $\dfrac{Ex+F}{x^2+x+2}$ term.

Finally, the $(x^2+x+7)^2$ term results in the terms

$$\frac{Gx+H}{x^2+x+7}\quad \text{and}\quad \frac{Ix+J}{(x^2+x+7)^2}.$$

All together, we have

$$\frac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2} = \frac{A}{x+5} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3} + \frac{Ex+F}{x^2+x+2} + \frac{Gx+H}{x^2+x+7} + \frac{Ix+J}{(x^2+x+7)^2}$$

Solving for the coefficients A, $B \ldots J$ would be a bit tedious but not "hard."

Example 6.5.2 Decomposing into partial fractions

Perform the partial fraction decomposition of $\dfrac{1}{x^2-1}$.

Solution The denominator factors into two linear terms: $x^2 - 1 = (x-1)(x+1)$. Thus

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}.$$

To solve for A and B, first multiply through by $x^2 - 1 = (x-1)(x+1)$:

$$\begin{aligned} 1 &= \frac{A(x-1)(x+1)}{x-1} + \frac{B(x-1)(x+1)}{x+1} \\ &= A(x+1) + B(x-1) \\ &= Ax + A + Bx - B \end{aligned}$$

Now collect like terms.

$$= (A+B)x + (A-B).$$

The next step is key. Note the equality we have:

$$1 = (A+B)x + (A-B).$$

For clarity's sake, rewrite the left hand side as

$$0x + 1 = (A+B)x + (A-B).$$

On the left, the coefficient of the x term is 0; on the right, it is $(A+B)$. Since both sides are equal, we must have that $0 = A + B$.

Likewise, on the left, we have a constant term of 1; on the right, the constant term is $(A-B)$. Therefore we have $1 = A - B$.

We have two linear equations with two unknowns. This one is easy to solve by hand, leading to

$$\begin{aligned} A + B &= 0 \\ A - B &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= 1/2 \\ B &= -1/2 \end{aligned}.$$

Thus

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Example 6.5.3 Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{1}{(x-1)(x+2)^2}\, dx$.

Solution We decompose the integrand as follows, as described by Key Idea 6.5.1:

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

To solve for A, B and C, we multiply both sides by $(x-1)(x+2)^2$ and collect like terms:

$$\begin{aligned} 1 &= A(x+2)^2 + B(x-1)(x+2) + C(x-1) \qquad (6.3) \\ &= Ax^2 + 4Ax + 4A + Bx^2 + Bx - 2B + Cx - C \\ &= (A+B)x^2 + (4A+B+C)x + (4A-2B-C) \end{aligned}$$

We have

$$0x^2 + 0x + 1 = (A+B)x^2 + (4A+B+C)x + (4A-2B-C)$$

leading to the equations

$$A+B=0, \quad 4A+B+C=0 \quad \text{and} \quad 4A-2B-C=1.$$

These three equations of three unknowns lead to a unique solution:

$$A = 1/9, \quad B = -1/9 \quad \text{and} \quad C = -1/3.$$

Thus

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \int \frac{1/9}{x-1}\,dx + \int \frac{-1/9}{x+2}\,dx + \int \frac{-1/3}{(x+2)^2}\,dx.$$

Each can be integrated with a simple substitution with $u = x-1$ or $u = x+2$ (or by directly applying Key Idea 6.1.1 as the denominators are linear functions). The end result is

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \frac{1}{9}\ln|x-1| - \frac{1}{9}\ln|x+2| + \frac{1}{3(x+2)} + C.$$

Note: Equation 6.3 offers a direct route to finding the values of A, B and C. Since the equation holds for all values of x, it holds in particular when $x=1$. However, when $x=1$, the right hand side simplifies to $A(1+2)^2 = 9A$. Since the left hand side is still 1, we have $1 = 9A$. Hence $A = 1/9$. Likewise, the equality holds when $x = -2$; this leads to the equation $1 = -3C$. Thus $C = -1/3$.
Knowing A and C, we can find the value of B by choosing yet another value of x, such as $x=0$, and solving for B.

Example 6.5.4 Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{x^3}{(x-5)(x+3)}\,dx$.

Solution Key Idea 6.5.1 presumes that the degree of the numerator is less than the degree of the denominator. Since this is not the case here, we begin by using polynomial division to reduce the degree of the numerator. We omit the steps, but encourage the reader to verify that

$$\frac{x^3}{(x-5)(x+3)} = x+2+\frac{19x+30}{(x-5)(x+3)}.$$

Using Key Idea 6.5.1, we can rewrite the new rational function as:

$$\frac{19x+30}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}$$

for appropriate values of A and B. Clearing denominators, we have

$$\begin{aligned} 19x+30 &= A(x+3) + B(x-5) \\ &= (A+B)x + (3A-5B). \end{aligned}$$

This implies that:

$$\begin{aligned} 19 &= A+B \\ 30 &= 3A-5B. \end{aligned}$$

Note: The values of A and B can be quickly found using the technique described in the margin of Example 6.5.3.

Solving this system of linear equations gives

$$\begin{aligned} 125/8 &= A \\ 27/8 &= B. \end{aligned}$$

We can now integrate.

$$\begin{aligned} \int \frac{x^3}{(x-5)(x+3)}\,dx &= \int \left(x+2+\frac{125/8}{x-5}+\frac{27/8}{x+3}\right)\,dx \\ &= \frac{x^2}{2}+2x+\frac{125}{8}\ln|x-5|+\frac{27}{8}\ln|x+3|+C. \end{aligned}$$

Example 6.5.5 Integrating using partial fractions

Use partial fraction decomposition to evaluate $\displaystyle\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx$.

Solution The degree of the numerator is less than the degree of the denominator so we begin by applying Key Idea 6.5.1. We have:

$$\frac{7x^2+31x+54}{(x+1)(x^2+6x+11)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+6x+11}.$$

Now clear the denominators.

$$\begin{aligned} 7x^2+31x+54 &= A(x^2+6x+11)+(Bx+C)(x+1) \\ &= (A+B)x^2+(6A+B+C)x+(11A+C). \end{aligned}$$

This implies that:

$$\begin{aligned} 7 &= A+B \\ 31 &= 6A+B+C \\ 54 &= 11A+C. \end{aligned}$$

Solving this system of linear equations gives the nice result of $A=5$, $B=2$ and $C=-1$. Thus

$$\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx = \int\left(\frac{5}{x+1}+\frac{2x-1}{x^2+6x+11}\right)\,dx.$$

The first term of this new integrand is easy to evaluate; it leads to a $5\ln|x+1|$ term. The second term is not hard, but takes several steps and uses substitution techniques.

The integrand $\displaystyle\frac{2x-1}{x^2+6x+11}$ has a quadratic in the denominator and a linear term in the numerator. This leads us to try substitution. Let $u=x^2+6x+11$, so $du=(2x+6)\,dx$. The numerator is $2x-1$, not $2x+6$, but we can get a $2x+6$ term in the numerator by adding 0 in the form of "$7-7$."

$$\begin{aligned} \frac{2x-1}{x^2+6x+11} &= \frac{2x-1+7-7}{x^2+6x+11} \\ &= \frac{2x+6}{x^2+6x+11}-\frac{7}{x^2+6x+11}. \end{aligned}$$

We can now integrate the first term with substitution, leading to a $\ln|x^2+6x+11|$ term. The final term can be integrated using arctangent. First, complete the square in the denominator:

$$\frac{7}{x^2+6x+11} = \frac{7}{(x+3)^2+2}.$$

An antiderivative of the latter term can be found using Theorem 6.1.3 and substitution:

$$\int \frac{7}{x^2+6x+11}\,dx = \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right)+C.$$

Let's start at the beginning and put all of the steps together.

$$\begin{aligned}\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx &= \int\left(\frac{5}{x+1}+\frac{2x-1}{x^2+6x+11}\right)\,dx\\ &= \int\frac{5}{x+1}\,dx + \int\frac{2x+6}{x^2+6x+11}\,dx - \int\frac{7}{x^2+6x+11}\,dx\\ &= 5\ln|x+1| + \ln|x^2+6x+11| - \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right)+C.\end{aligned}$$

As with many other problems in calculus, it is important to remember that one is not expected to "see" the final answer immediately after seeing the problem. Rather, given the initial problem, we break it down into smaller problems that are easier to solve. The final answer is a combination of the answers of the smaller problems.

Partial Fraction Decomposition is an important tool when dealing with rational functions. Note that at its heart, it is a technique of algebra, not calculus, as we are rewriting a fraction in a new form. Regardless, it is very useful in the realm of calculus as it lets us evaluate a certain set of "complicated" integrals.

The next section introduces new functions, called the Hyperbolic Functions. They will allow us to make substitutions similar to those found when studying Trigonometric Substitution, allowing us to approach even more integration problems.

Exercises 6.5

Terms and Concepts

1. Fill in the blank: Partial Fraction Decomposition is a method of rewriting ________ functions.

2. T/F: It is sometimes necessary to use polynomial division before using Partial Fraction Decomposition.

3. Decompose $\dfrac{1}{x^2-3x}$ without solving for the coefficients, as done in Example 6.5.1.

4. Decompose $\dfrac{7-x}{x^2-9}$ without solving for the coefficients, as done in Example 6.5.1.

5. Decompose $\dfrac{x-3}{x^2-7}$ without solving for the coefficients, as done in Example 6.5.1.

6. Decompose $\dfrac{2x+5}{x^3+7x}$ without solving for the coefficients, as done in Example 6.5.1.

Problems

In Exercises 7 – 26, evaluate the indefinite integral.

7. $\displaystyle\int \frac{7x+7}{x^2+3x-10}\,dx$

8. $\displaystyle\int \frac{7x-2}{x^2+x}\,dx$

9. $\displaystyle\int \frac{-4}{3x^2-12}\,dx$

10. $\displaystyle\int \frac{6x+4}{3x^2+4x+1}\,dx$

11. $\displaystyle\int \frac{x+7}{(x+5)^2}\,dx$

12. $\displaystyle\int \frac{-3x-20}{(x+8)^2}\,dx$

13. $\displaystyle\int \frac{9x^2+11x+7}{x(x+1)^2}\,dx$

14. $\displaystyle\int \frac{-12x^2-x+33}{(x-1)(x+3)(3-2x)}\,dx$

15. $\displaystyle\int \frac{94x^2-10x}{(7x+3)(5x-1)(3x-1)}\,dx$

16. $\displaystyle\int \frac{x^2+x+1}{x^2+x-2}\,dx$

17. $\displaystyle\int \frac{x^3}{x^2-x-20}\,dx$

18. $\displaystyle\int \frac{2x^2-4x+6}{x^2-2x+3}\,dx$

19. $\displaystyle\int \frac{1}{x^3+2x^2+3x}\,dx$

20. $\displaystyle\int \frac{x^2+x+5}{x^2+4x+10}\,dx$

21. $\displaystyle\int \frac{12x^2+21x+3}{(x+1)(3x^2+5x-1)}\,dx$

22. $\displaystyle\int \frac{6x^2+8x-4}{(x-3)(x^2+6x+10)}\,dx$

23. $\displaystyle\int \frac{2x^2+x+1}{(x+1)(x^2+9)}\,dx$

24. $\displaystyle\int \frac{x^2-20x-69}{(x-7)(x^2+2x+17)}\,dx$

25. $\displaystyle\int \frac{9x^2-60x+33}{(x-9)(x^2-2x+11)}\,dx$

26. $\displaystyle\int \frac{6x^2+45x+121}{(x+2)(x^2+10x+27)}\,dx$

In Exercises 27 – 30, evaluate the definite integral.

27. $\displaystyle\int_1^2 \frac{8x+21}{(x+2)(x+3)}\,dx$

28. $\displaystyle\int_0^5 \frac{14x+6}{(3x+2)(x+4)}\,dx$

29. $\displaystyle\int_{-1}^1 \frac{x^2+5x-5}{(x-10)(x^2+4x+5)}\,dx$

30. $\displaystyle\int_0^1 \frac{x}{(x+1)(x^2+2x+1)}\,dx$

6.6 Improper Integration

We begin this section by considering the following definite integrals:

- $\displaystyle\int_0^{100} \frac{1}{1+x^2}\,dx \approx 1.5608,$

- $\displaystyle\int_0^{1000} \frac{1}{1+x^2}\,dx \approx 1.5698,$

- $\displaystyle\int_0^{10,000} \frac{1}{1+x^2}\,dx \approx 1.5707.$

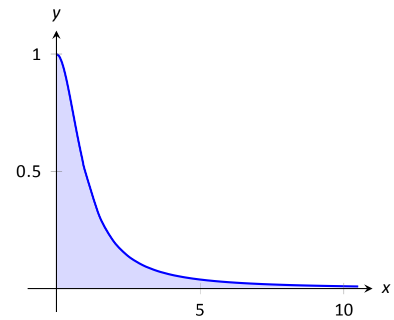


Figure 6.6.1: Graphing $f(x) = \dfrac{1}{1+x^2}$.

Notice how the integrand is $1/(1+x^2)$ in each integral (which is sketched in Figure 6.6.1). As the upper bound gets larger, one would expect the "area under the curve" would also grow. While the definite integrals do increase in value as the upper bound grows, they are not increasing by much. In fact, consider:

$$\int_0^b \frac{1}{1+x^2}\,dx = \tan^{-1} x\Big|_0^b = \tan^{-1} b - \tan^{-1} 0 = \tan^{-1} b.$$

As $b \to \infty$, $\tan^{-1} b \to \pi/2$. Therefore it seems that as the upper bound b grows, the value of the definite integral $\displaystyle\int_0^b \frac{1}{1+x^2}\,dx$ approaches $\pi/2 \approx 1.5708$. This should strike the reader as being a bit amazing: even though the curve extends "to infinity," it has a finite amount of area underneath it.

When we defined the definite integral $\displaystyle\int_a^b f(x)\,dx$, we made two stipulations:

1. The interval over which we integrated, $[a,b]$, was a finite interval, and

2. The function $f(x)$ was continuous on $[a,b]$ (ensuring that the range of f was finite).

In this section we consider integrals where one or both of the above conditions do not hold. Such integrals are called **improper integrals.**

Improper Integrals with Infinite Bounds

Definition 6.6.1 Improper Integrals with Infinite Bounds; Converge, Diverge

1. Let f be a continuous function on $[a, \infty)$. Define
$$\int_a^\infty f(x)\,dx \quad \text{to be} \quad \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. Let f be a continuous function on $(-\infty, b]$. Define
$$\int_{-\infty}^b f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. Let f be a continuous function on $(-\infty, \infty)$. Let c be any real number; define
$$\int_{-\infty}^\infty f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx.$$

An improper integral is said to **converge** if its corresponding limit exists; otherwise, it **diverges**. The improper integral in part 3 converges if and only if both of its limits exist.

Example 6.6.1 Evaluating improper integrals

Evaluate the following improper integrals.

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
3. $\displaystyle\int_{-\infty}^0 e^x\,dx$
4. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

Solution

1. $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left.\frac{-1}{x}\right|_1^b$$
$$= \lim_{b\to\infty}\frac{-1}{b} + 1$$
$$= 1.$$

A graph of the area defined by this integral is given in Figure 6.6.2.

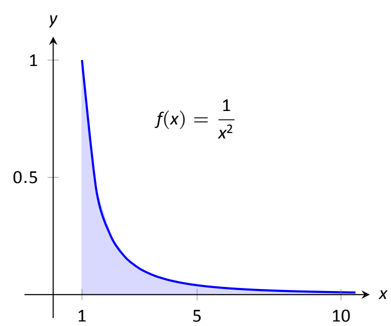


Figure 6.6.2: A graph of $f(x) = \frac{1}{x^2}$ in Example 6.6.1.

2. $$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx$$
$$= \lim_{b\to\infty} \ln|x|\Big|_1^b$$
$$= \lim_{b\to\infty} \ln(b)$$
$$= \infty.$$

The limit does not exist, hence the improper integral $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges.

Compare the graphs in Figures 6.6.2 and 6.6.3; notice how the graph of $f(x) = 1/x$ is noticeably larger. This difference is enough to cause the improper integral to diverge.

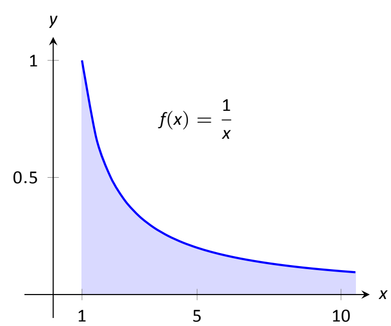


Figure 6.6.3: A graph of $f(x) = \frac{1}{x}$ in Example 6.6.1.

3.
$$\begin{aligned}\int_{-\infty}^{0} e^x\,dx &= \lim_{a\to-\infty}\int_a^0 e^x\,dx\\ &= \lim_{a\to-\infty} e^x\Big|_a^0\\ &= \lim_{a\to-\infty} e^0 - e^a\\ &= 1.\end{aligned}$$

A graph of the area defined by this integral is given in Figure 6.6.4.

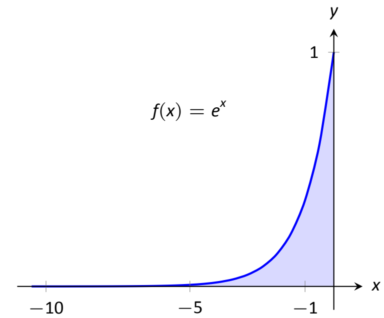


Figure 6.6.4: A graph of $f(x) = e^x$ in Example 6.6.1.

4. We will need to break this into two improper integrals and choose a value of c as in part 3 of Definition 6.6.1. Any value of c is fine; we choose $c = 0$.

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty}\int_a^0\frac{1}{1+x^2}\,dx + \lim_{b\to\infty}\int_0^b\frac{1}{1+x^2}\,dx\\ &= \lim_{a\to-\infty}\tan^{-1}x\Big|_a^0 + \lim_{b\to\infty}\tan^{-1}x\Big|_0^b\\ &= \lim_{a\to-\infty}\left(\tan^{-1}0-\tan^{-1}a\right) + \lim_{b\to\infty}\left(\tan^{-1}b-\tan^{-1}0\right)\\ &= \left(0-\frac{-\pi}{2}\right)+\left(\frac{\pi}{2}-0\right).\end{aligned}$$

Each limit exists, hence the original integral converges and has value:

$$= \pi.$$

A graph of the area defined by this integral is given in Figure 6.6.5.

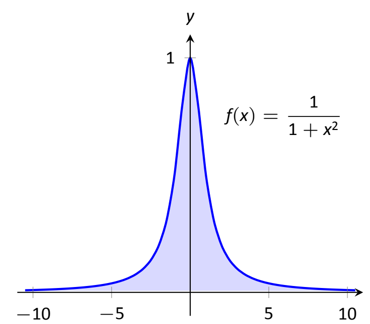


Figure 6.6.5: A graph of $f(x) = \frac{1}{1+x^2}$ in Example 6.6.1.

Recall that many limits that result in indeterminate forms can be handled using **l'Hospital's Rule**. We briefly recall the statement of the theorem: suppose that functions f and g are differentiable on an open interval containing a, and that either

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to a} g(x) = \infty.$$

Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided that the latter limit exists. It is not uncommon for the limits resulting from improper integrals to need this rule as demonstrated next.

Note that l'Hosptial's rule can also be applied in the case of limits where $x \to \pm\infty$.

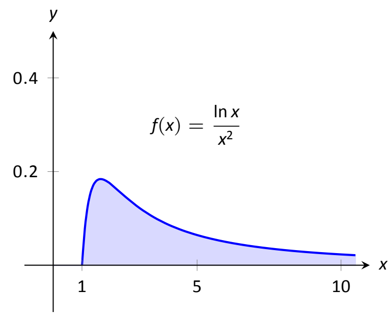


Figure 6.6.6: A graph of $f(x) = \frac{\ln x}{x^2}$ in Example 6.6.2.

Note: In Definition 6.6.2, c can be one of the endpoints (a or b). In that case, there is only one limit to consider as part of the definition.

Example 6.6.2 Improper integration and l'Hospital's Rule

Evaluate the improper integral $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\, dx$.

Solution This integral will require the use of Integration by Parts. Let $u = \ln x$ and $dv = 1/x^2\, dx$. Then

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^2}\, dx &= \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\, dx \\
&= \lim_{b\to\infty} \left(-\frac{\ln x}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\, dx \right) \\
&= \lim_{b\to\infty} \left(-\frac{\ln x}{x} - \frac{1}{x} \right)\Big|_1^b \\
&= \lim_{b\to\infty} \left(-\frac{\ln b}{b} - \frac{1}{b} - (-\ln 1 - 1) \right).
\end{aligned}$$

The $1/b$ and $\ln 1$ terms go to 0, leaving $\displaystyle\lim_{b\to\infty} -\frac{\ln b}{b} + 1$. We need to evaluate $\displaystyle\lim_{b\to\infty} \frac{\ln b}{b}$ with l'Hospital's Rule. We have:

$$\begin{aligned}
\lim_{b\to\infty} \frac{\ln b}{b} &\overset{\text{by LHR}}{=} \lim_{b\to\infty} \frac{1/b}{1} \\
&= 0.
\end{aligned}$$

Thus the improper integral evaluates as:

$$\int_1^\infty \frac{\ln x}{x^2}\, dx = 1.$$

Improper Integrals with Infinite Range

We have just considered definite integrals where the interval of integration was infinite. We now consider another type of improper integration, where the range of the integrand is infinite.

Definition 6.6.2 Improper Integration with Infinite Range

Let $f(x)$ be a continuous function on $[a, b]$ except at c, $a \le c \le b$, where $x = c$ is a vertical asymptote of f. Define

$$\int_a^b f(x)\, dx = \lim_{t\to c^-} \int_a^t f(x)\, dx + \lim_{t\to c^+} \int_t^b f(x)\, dx.$$

Example 6.6.3 Improper integration of functions with infinite range

Evaluate the following improper integrals:

1. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$
2. $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx.$

SOLUTION

1. A graph of $f(x) = 1/\sqrt{x}$ is given in Figure 6.6.7. Notice that f has a vertical asymptote at $x = 0$; in some sense, we are trying to compute the area of a region that has no "top." Could this have a finite value?

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx\\ &= \lim_{a\to 0^+} 2\sqrt{x}\Big|_a^1\\ &= \lim_{a\to 0^+} 2\left(\sqrt{1}-\sqrt{a}\right)\\ &= 2.\end{aligned}$$

It turns out that the region does have a finite area even though it has no upper bound (strange things can occur in mathematics when considering the infinite).

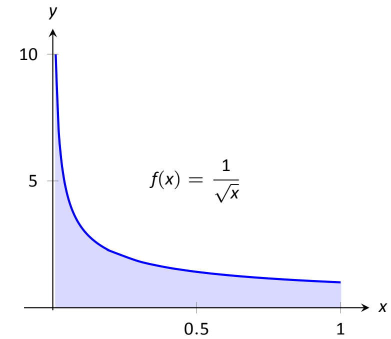


Figure 6.6.7: A graph of $f(x) = \frac{1}{\sqrt{x}}$ in Example 6.6.3.

2. The function $f(x) = 1/x^2$ has a vertical asymptote at $x = 0$, as shown in Figure 6.6.8, so this integral is an improper integral. Let's eschew using limits for a moment and proceed without recognizing the improper nature of the integral. This leads to:

$$\begin{aligned}\int_{-1}^1 \frac{1}{x^2}\,dx &= -\frac{1}{x}\Big|_{-1}^1\\ &= -1-(1)\\ &= -2.\ (!)\end{aligned}$$

Clearly the area in question is above the x-axis, yet the area is supposedly negative! Why does our answer not match our intuition? To answer this, evaluate the integral using Definition 6.6.2.

$$\begin{aligned}\int_{-1}^1 \frac{1}{x^2}\,dx &= \lim_{t\to 0^-}\int_{-1}^t \frac{1}{x^2}\,dx + \lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}\,dx\\ &= \lim_{t\to 0^-} -\frac{1}{x}\Big|_{-1}^t + \lim_{t\to 0^+} -\frac{1}{x}\Big|_t^1\\ &= \lim_{t\to 0^-} -\frac{1}{t} - 1 + \lim_{t\to 0^+} -1 + \frac{1}{t}\\ &\Rightarrow \Big(\infty - 1\Big) + \Big(-1+\infty\Big).\end{aligned}$$

Neither limit converges hence the original improper integral diverges. The nonsensical answer we obtained by ignoring the improper nature of the integral is just that: nonsensical.

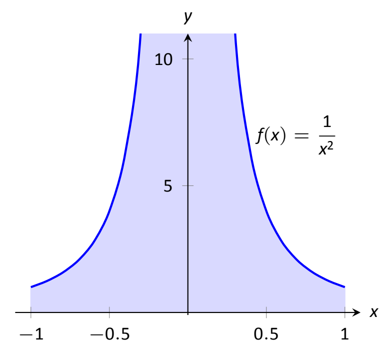


Figure 6.6.8: A graph of $f(x) = \frac{1}{x^2}$ in Example 6.6.3.

Understanding Convergence and Divergence

Oftentimes we are interested in knowing simply whether or not an improper integral converges, and not necessarily the value of a convergent integral. We provide here several tools that help determine the convergence or divergence of improper integrals without integrating.

Our first tool is to understand the behaviour of functions of the form $\dfrac{1}{x^p}$.

Example 6.6.4 Improper integration of $1/x^p$

Determine the values of p for which $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges.

Solution We begin by integrating and then evaluating the limit.

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx\\ &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \qquad (\text{assume } p\neq 1)\\ &= \lim_{b\to\infty}\frac{1}{-p+1}x^{-p+1}\Big|_1^b\\ &= \lim_{b\to\infty}\frac{1}{1-p}\big(b^{1-p}-1^{1-p}\big).\end{aligned}$$

When does this limit converge – i.e., when is this limit *not* ∞? This limit converges precisely when the power of b is less than 0: when $1-p<0 \Rightarrow 1<p$.

Our analysis shows that if $p>1$, then $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges. When $p<1$ the improper integral diverges; we showed in Example 6.6.1 that when $p=1$ the integral also diverges.

Figure 6.6.9 graphs $y=1/x$ with a dashed line, along with graphs of $y=1/x^p$, $p<1$, and $y=1/x^q$, $q>1$. Somehow the dashed line forms a dividing line between convergence and divergence.

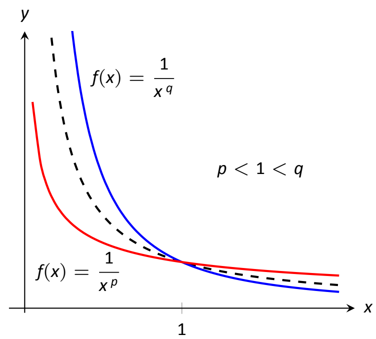


Figure 6.6.9: Plotting functions of the form $1/x^p$ in Example 6.6.4.

The result of Example 6.6.4 provides an important tool in determining the convergence of other integrals. A similar result is proved in the exercises about improper integrals of the form $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$. These results are summarized in the following Key Idea.

Key Idea 6.6.1 Convergence of Improper Integrals $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ and $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$.

1. The improper integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges when $p>1$ and diverges when $p\leq 1$.
2. The improper integral $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges when $p<1$ and diverges when $p\geq 1$.

Note: We used the upper and lower bound of "1" in Key Idea 6.6.1 for convenience. It can be replaced by any a where $a>0$.

A basic technique in determining convergence of improper integrals is to compare an integrand whose convergence is unknown to an integrand whose convergence is known. We often use integrands of the form $1/x^p$ to compare to as their convergence on certain intervals is known. This is described in the following theorem.

Theorem 6.6.1 Direct Comparison Test for Improper Integrals

Let f and g be continuous on $[a,\infty)$ where $0 \leq f(x) \leq g(x)$ for all x in $[a,\infty)$.

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ converges.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ diverges.

Example 6.6.5 Determining convergence of improper integrals

Determine the convergence of the following improper integrals.

1. $\displaystyle\int_1^\infty e^{-x^2}\,dx$ 2. $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2-x}}\,dx$

Solution

1. The function $f(x) = e^{-x^2}$ does not have an antiderivative expressible in terms of elementary functions, so we cannot integrate directly. It is comparable to $g(x) = 1/x^2$, and as demonstrated in Figure 6.6.10, $e^{-x^2} < 1/x^2$ on $[1,\infty)$. We know from Key Idea 6.6.1 that $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges, hence $\displaystyle\int_1^\infty e^{-x^2}\,dx$ also converges.

2. Note that for large values of x, $\displaystyle\frac{1}{\sqrt{x^2-x}} \approx \frac{1}{\sqrt{x^2}} = \frac{1}{x}$. We know from Key Idea 6.6.1 and the subsequent note that $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, so we seek to compare the original integrand to $1/x$.

   It is easy to see that when $x > 0$, we have $x = \sqrt{x^2} > \sqrt{x^2-x}$. Taking reciprocals reverses the inequality, giving

   $$\frac{1}{x} < \frac{1}{\sqrt{x^2-x}}.$$

   Using Theorem 6.6.1, we conclude that since $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2-x}}\,dx$ diverges as well. Figure 6.6.11 illustrates this.

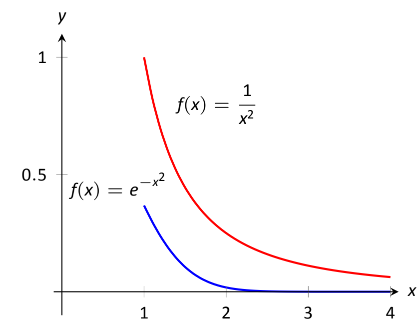


Figure 6.6.10: Graphs of $f(x) = e^{-x^2}$ and $f(x) = 1/x^2$ in Example 6.6.5.

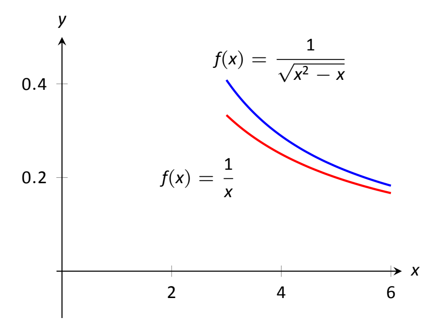


Figure 6.6.11: Graphs of $f(x) = 1/\sqrt{x^2-x}$ and $f(x) = 1/x$ in Example 6.6.5.

Being able to compare "unknown" integrals to "known" integrals is very useful in determining convergence. However, some of our examples were a little "too nice." For instance, it was convenient that $\displaystyle\frac{1}{x} < \frac{1}{\sqrt{x^2-x}}$, but what if the "$-x$" were replaced with a "$+2x+5$"? That is, what can we say about the convergence of $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$? We have $\displaystyle\frac{1}{x} > \frac{1}{\sqrt{x^2+2x+5}}$, so we cannot use Theorem 6.6.1.

In cases like this (and many more) it is useful to employ the following theorem.

Theorem 6.6.2 Limit Comparison Test for Improper Integrals

Let f and g be continuous functions on $[a, \infty)$ where $f(x) > 0$ and $g(x) > 0$ for all x. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either both converge or both diverge.

Example 6.6.6 Determining convergence of improper integrals

Determine the convergence of $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$.

Solution As x gets large, the denominator of the integrand will begin to behave much like $y = x$. So we compare $\dfrac{1}{\sqrt{x^2+2x+5}}$ to $\dfrac{1}{x}$ with the Limit Comparison Test:

$$\lim_{x\to\infty} \frac{1/\sqrt{x^2+2x+5}}{1/x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+2x+5}}.$$

The immediate evaluation of this limit returns ∞/∞, an indeterminate form. Using l'Hospital's Rule seems appropriate, but in this situation, it does not lead to useful results. (We encourage the reader to employ l'Hospital's Rule at least once to verify this.)

The trouble is the square root function. To get rid of it, we employ the following fact: If $\lim\limits_{x\to c} f(x) = L$, then $\lim\limits_{x\to c} f(x)^2 = L^2$. (This is true when either c or L is ∞.) So we consider now the limit

$$\lim_{x\to\infty} \frac{x^2}{x^2+2x+5}.$$

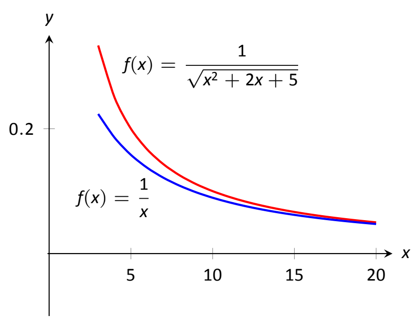


Figure 6.6.12: Graphing $f(x) = \frac{1}{\sqrt{x^2+2x+5}}$ and $f(x) = \frac{1}{x}$ in Example 6.6.6.

This converges to 1, meaning the original limit also converged to 1. As x gets very large, the function $\dfrac{1}{\sqrt{x^2+2x+5}}$ looks very much like $\dfrac{1}{x}$. Since we know that $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, by the Limit Comparison Test we know that $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$ also diverges. Figure 6.6.12 graphs $f(x) = 1/\sqrt{x^2+2x+5}$ and $f(x) = 1/x$, illustrating that as x gets large, the functions become indistinguishable.

Both the Direct and Limit Comparison Tests were given in terms of integrals over an infinite interval. There are versions that apply to improper integrals with an infinite range, but as they are a bit wordy and a little more difficult to employ, they are omitted from this text.

This chapter has explored many integration techniques. We learned Substitution, which "undoes" the Chain Rule of differentiation, as well as Integration by Parts, which "undoes" the Product Rule. We learned specialized techniques for handling trigonometric functions and introduced the hyperbolic functions, which are closely related to the trigonometric functions. All techniques effectively have this goal in common: rewrite the integrand in a new way so that the integration step is easier to see and implement.

As stated before, integration is, in general, hard. It is easy to write a function whose antiderivative is impossible to write in terms of elementary functions, and even when a function does have an antiderivative expressible by elementary functions, it may be really hard to discover what it is. The powerful computer algebra system *Mathematica*® has approximately 1,000 pages of code dedicated to integration.

Do not let this difficulty discourage you. There is great value in learning integration techniques, as they allow one to manipulate an integral in ways that can illuminate a concept for greater understanding. There is also great value in understanding the need for good numerical techniques: the Trapezoidal and Simpson's Rules are just the beginning of powerful techniques for approximating the value of integration.

The next chapter stresses the uses of integration. We generally do not find antiderivatives for antiderivative's sake, but rather because they provide the solution to some type of problem. The following chapter introduces us to a number of different problems whose solution is provided by integration.

Exercises 6.6

Terms and Concepts

1. The definite integral was defined with what two stipulations?

2. If $\lim\limits_{b\to\infty}\int_0^b f(x)\,dx$ exists, then the integral $\int_0^\infty f(x)\,dx$ is said to ________________.

3. If $\int_1^\infty f(x)\,dx = 10$, and $0 \leq g(x) \leq f(x)$ for all x, then we know that $\int_1^\infty g(x)\,dx$ ________________.

4. For what values of p will $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

5. For what values of p will $\int_{10}^\infty \frac{1}{x^p}\,dx$ converge?

6. For what values of p will $\int_0^1 \frac{1}{x^p}\,dx$ converge?

Problems

In Exercises 7 – 34, evaluate the given improper integral.

7. $\int_0^\infty e^{5-2x}\,dx$

8. $\int_1^\infty \frac{1}{x^3}\,dx$

9. $\int_1^\infty x^{-4}\,dx$

10. $\int_{-\infty}^\infty \frac{1}{x^2+9}\,dx$

11. $\int_{-\infty}^0 2^x\,dx$

12. $\int_{-\infty}^0 \left(\frac{1}{2}\right)^x\,dx$

13. $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

14. $\int_3^\infty \frac{1}{x^2-4}\,dx$

15. $\int_2^\infty \frac{1}{(x-1)^2}\,dx$

16. $\int_1^2 \frac{1}{(x-1)^2}\,dx$

17. $\int_2^\infty \frac{1}{x-1}\,dx$

18. $\int_1^2 \frac{1}{x-1}\,dx$

19. $\int_{-1}^1 \frac{1}{x}\,dx$

20. $\int_1^3 \frac{1}{x-2}\,dx$

21. $\int_0^\pi \sec^2 x\,dx$

22. $\int_{-2}^1 \frac{1}{\sqrt{|x|}}\,dx$

23. $\int_0^\infty xe^{-x}\,dx$

24. $\int_0^\infty xe^{-x^2}\,dx$

25. $\int_{-\infty}^\infty xe^{-x^2}\,dx$

26. $\int_{-\infty}^\infty \frac{1}{e^x+e^{-x}}\,dx$

27. $\int_0^1 x\ln x\,dx$

28. $\int_0^1 x^2\ln x\,dx$

29. $\int_1^\infty \frac{\ln x}{x}\,dx$

30. $\int_0^1 \ln x\,dx$

31. $\int_1^\infty \frac{\ln x}{x^2}\,dx$

32. $\int_1^\infty \frac{\ln x}{\sqrt{x}}\,dx$

33. $\int_0^\infty e^{-x}\sin x\,dx$

34. $\int_0^\infty e^{-x}\cos x\,dx$

In Exercises 35 – 44, use the Direct Comparison Test or the Limit Comparison Test to determine whether the given definite integral converges or diverges. Clearly state what test is being used and what function the integrand is being compared to.

35. $\displaystyle\int_{10}^{\infty} \frac{3}{\sqrt{3x^2+2x-5}}\,dx$

36. $\displaystyle\int_{2}^{\infty} \frac{4}{\sqrt{7x^3-x}}\,dx$

37. $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x+3}}{\sqrt{x^3-x^2+x+1}}\,dx$

38. $\displaystyle\int_{1}^{\infty} e^{-x}\ln x\,dx$

39. $\displaystyle\int_{5}^{\infty} e^{-x^2+3x+1}\,dx$

40. $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}}{e^x}\,dx$

41. $\displaystyle\int_{2}^{\infty} \frac{1}{x^2+\sin x}\,dx$

42. $\displaystyle\int_{0}^{\infty} \frac{x}{x^2+\cos x}\,dx$

43. $\displaystyle\int_{0}^{\infty} \frac{1}{x+e^x}\,dx$

44. $\displaystyle\int_{0}^{\infty} \frac{1}{e^x-x}\,dx$

7: Applications of Integration

We begin this chapter with a reminder of a few key concepts from Chapter 5. Let f be a continuous function on $[a, b]$ which is partitioned into n equally spaced subintervals as

$$a < x_1 < x_2 < \cdots < x_n < x_{n+1} = b.$$

Let $\Delta x = (b - a)/n$ denote the length of the subintervals, and let c_i be any x-value in the i^{th} subinterval. Definition 5.3.2 states that the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x$$

is a *Riemann Sum.* Riemann Sums are often used to approximate some quantity (area, volume, work, pressure, etc.). The *approximation* becomes *exact* by taking the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

Theorem 5.3.2 connects limits of Riemann Sums to definite integrals:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\, dx.$$

Finally, the Fundamental Theorem of Calculus states how definite integrals can be evaluated using antiderivatives.

This chapter employs the following technique to a variety of applications. Suppose the value Q of a quantity is to be calculated. We first approximate the value of Q using a Riemann Sum, then find the exact value via a definite integral. We spell out this technique in the following Key Idea.

Key Idea 7.0.1 Application of Definite Integrals Strategy

Let a quantity be given whose value Q is to be computed.

1. Divide the quantity into n smaller "subquantities" of value Q_i.
2. Identify a variable x and function $f(x)$ such that each subquantity can be approximated with the product $f(c_i)\Delta x$, where Δx represents a small change in x. Thus $Q_i \approx f(c_i)\Delta x$. A sample approximation $f(c_i)\Delta x$ of Q_i is called a *differential element*.
3. Recognize that $Q = \displaystyle\sum_{i=1}^{n} Q_i \approx \sum_{i=1}^{n} f(c_i)\Delta x$, which is a Riemann Sum.
4. Taking the appropriate limit gives $Q = \displaystyle\int_a^b f(x)\, dx$

This Key Idea will make more sense after we have had a chance to use it several times. We begin with Area Between Curves, which we addressed briefly in Section 5.5.4.

7.1 Area Between Curves

We are often interested in knowing the area of a region. Forget momentarily that we addressed this already in Section 5.5.4 and approach it instead using the technique described in Key Idea 7.0.1.

Let Q be the area of a region bounded by continuous functions f and g. If we break the region into many subregions, we have an obvious equation:

Total Area = sum of the areas of the subregions.

The issue to address next is how to systematically break a region into subregions. A graph will help. Consider Figure 7.1.1 (a) where a region between two curves is shaded. While there are many ways to break this into subregions, one particularly efficient way is to "slice" it vertically, as shown in Figure 7.1.1 (b), into n equally spaced slices.

We now approximate the area of a slice. Again, we have many options, but using a rectangle seems simplest. Picking any x-value c_i in the i^{th} slice, we set the height of the rectangle to be $f(c_i) - g(c_i)$, the difference of the corresponding y-values. The width of the rectangle is a small difference in x-values, which we represent with Δx. Figure 7.1.1 (c) shows sample points c_i chosen in each subinterval and appropriate rectangles drawn. (Each of these rectangles represents a differential element.) Each slice has an area approximately equal to $\big(f(c_i) - g(c_i)\big)\Delta x$; hence, the total area is approximately the Riemann Sum

$$Q = \sum_{i=1}^{n} \big(f(c_i) - g(c_i)\big)\Delta x.$$

Taking the limit as $n \to \infty$ gives the exact area as $\int_a^b \big(f(x) - g(x)\big)\,dx$.

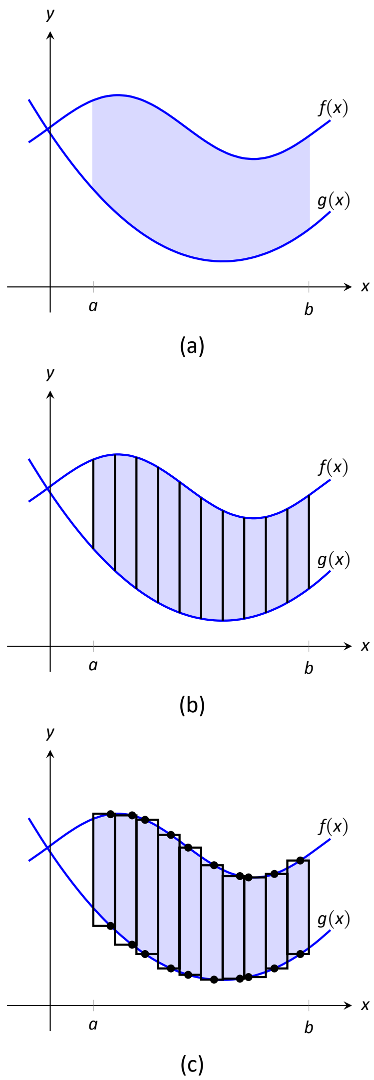


Figure 7.1.1: Subdividing a region into vertical slices and approximating the areas with rectangles.

Theorem 7.1.1 Area Between Curves (restatement of Theorem 5.4.3)

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$ where $f(x) \geq g(x)$ for all x in $[a, b]$. The area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

Example 7.1.1 Finding area enclosed by curves

Find the area of the region bounded by $f(x) = \sin x + 2$, $g(x) = \frac{1}{2}\cos(2x) - 1$, $x = 0$ and $x = 4\pi$, as shown in Figure 7.1.2.

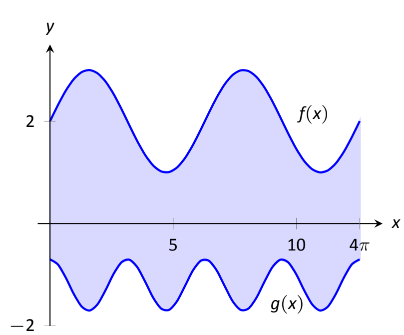


Figure 7.1.2: Graphing an enclosed region in Example 7.1.1.

Solution The graph verifies that the upper boundary of the region is given by f and the lower bound is given by g. Therefore the area of the region is the value of the integral

$$\begin{aligned}\int_0^{4\pi} \big(f(x) - g(x)\big)\,dx &= \int_0^{4\pi} \Big(\sin x + 2 - \big(\frac{1}{2}\cos(2x) - 1\big)\Big)\,dx \\ &= -\cos x - \frac{1}{4}\sin(2x) + 3x\Big|_0^{4\pi} \\ &= 12\pi \approx 37.7 \text{ units}^2.\end{aligned}$$

Example 7.1.2 Finding total area enclosed by curves

Find the total area of the region enclosed by the functions $f(x) = -2x + 5$ and $g(x) = x^3 - 7x^2 + 12x - 3$ as shown in Figure 7.1.3.

Solution A quick calculation shows that $f = g$ at $x = 1, 2$ and 4. One can proceed thoughtlessly by computing $\int_1^4 \big(f(x) - g(x)\big)\,dx$, but this ignores the fact that on $[1, 2]$, $g(x) > f(x)$. (In fact, the thoughtless integration returns $-9/4$, hardly the expected value of an *area*.) Thus we compute the total area by breaking the interval $[1, 4]$ into two subintervals, $[1, 2]$ and $[2, 4]$ and using the proper integrand in each.

$$\begin{aligned}
\text{Total Area} &= \int_1^2 \big(g(x) - f(x)\big)\,dx + \int_2^4 \big(f(x) - g(x)\big)\,dx \\
&= \int_1^2 \big(x^3 - 7x^2 + 14x - 8\big)\,dx + \int_2^4 \big(-x^3 + 7x^2 - 14x + 8\big)\,dx \\
&= 5/12 + 8/3 \\
&= 37/12 = 3.083\ \text{units}^2.
\end{aligned}$$

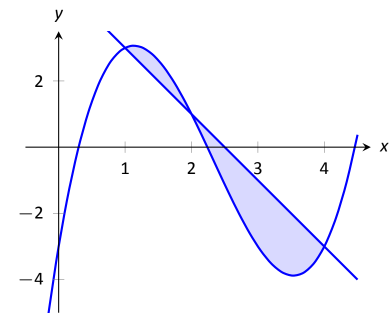


Figure 7.1.3: Graphing a region enclosed by two functions in Example 7.1.2.

The previous example makes note that we are expecting area to be *positive*. When first learning about the definite integral, we interpreted it as "signed area under the curve," allowing for "negative area." That doesn't apply here; area is to be positive.

The previous example also demonstrates that we often have to break a given region into subregions before applying Theorem 7.1.1. The following example shows another situation where this is applicable, along with an alternate view of applying the Theorem.

Example 7.1.3 Finding area: integrating with respect to y

Find the area of the region enclosed by the functions $y = \sqrt{x} + 2$, $y = -(x-1)^2 + 3$ and $y = 2$, as shown in Figure 7.1.4.

Solution We give two approaches to this problem. In the first approach, we notice that the region's "top" is defined by two different curves. On $[0, 1]$, the top function is $y = \sqrt{x} + 2$; on $[1, 2]$, the top function is $y = -(x-1)^2 + 3$. Thus we compute the area as the sum of two integrals:

$$\begin{aligned}
\text{Total Area} &= \int_0^1 \Big(\big(\sqrt{x} + 2\big) - 2\Big)\,dx + \int_1^2 \Big(\big(-(x-1)^2 + 3\big) - 2\Big)\,dx \\
&= 2/3 + 2/3 \\
&= 4/3.
\end{aligned}$$

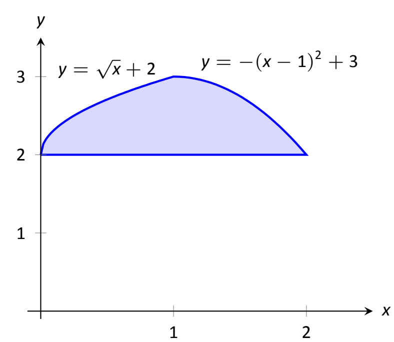


Figure 7.1.4: Graphing a region for Example 7.1.3.

The second approach is clever and very useful in certain situations. We are used to viewing curves as functions of x; we input an x-value and a y-value is returned. Some curves can also be described as functions of y: input a y-value and an x-value is returned. We can rewrite the equations describing the boundary by solving for x:

$$\begin{aligned}
y = \sqrt{x} + 2 \quad &\Rightarrow \quad x = (y-2)^2 \\
y = -(x-1)^2 + 3 \quad &\Rightarrow \quad x = \sqrt{3-y} + 1.
\end{aligned}$$

Figure 7.1.5 shows the region with the boundaries relabelled. A differential element, a horizontal rectangle, is also pictured. The width of the rectangle is

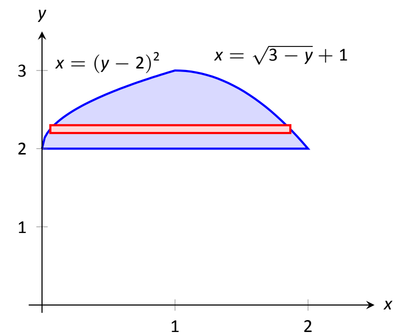


Figure 7.1.5: The region used in Example 7.1.3 with boundaries relabelled as functions of y.

a small change in y: Δy. The height of the rectangle is a difference in x-values. The "top" x-value is the largest value, i.e., the rightmost. The "bottom" x-value is the smaller, i.e., the leftmost. Therefore the height of the rectangle is

$$\big(\sqrt{3-y}+1\big)-(y-2)^2.$$

The area is found by integrating the above function with respect to y with the appropriate bounds. We determine these by considering the y-values the region occupies. It is bounded below by $y=2$, and bounded above by $y=3$. That is, both the "top" and "bottom" functions exist on the y interval $[2,3]$. Thus

$$\begin{aligned}\text{Total Area} &= \int_2^3 \big(\sqrt{3-y}+1-(y-2)^2\big)\,dy\\ &= \Big(-\frac{2}{3}(3-y)^{3/2}+y-\frac{1}{3}(y-2)^3\Big)\Big|_2^3\\ &= 4/3.\end{aligned}$$

This calculus–based technique of finding area can be useful even with shapes that we normally think of as "easy." Example 7.1.4 computes the area of a triangle. While the formula "$\frac{1}{2}\times$ base $\times$ height" is well known, in arbitrary triangles it can be nontrivial to compute the height. Calculus makes the problem simple.

Example 7.1.4 Finding the area of a triangle

Compute the area of the regions bounded by the lines
$y=x+1$, $y=-2x+7$ and $y=-\frac{1}{2}x+\frac{5}{2}$, as shown in Figure 7.1.6.

Solution Recognize that there are two "top" functions to this region, causing us to use two definite integrals.

$$\begin{aligned}\text{Total Area} &= \int_1^2 \big((x+1)-(-\frac{1}{2}x+\frac{5}{2})\big)\,dx+\int_2^3 \big((-2x+7)-(-\frac{1}{2}x+\frac{5}{2})\big)\,dx\\ &= 3/4+3/4\\ &= 3/2.\end{aligned}$$

We can also approach this by converting each function into a function of y. This also requires 2 integrals, so there isn't really any advantage to doing so. We do it here for demonstration purposes.

The "top" function is always $x=\frac{7-y}{2}$ while there are two "bottom" functions. Being mindful of the proper integration bounds, we have

$$\begin{aligned}\text{Total Area} &= \int_1^2 \big(\frac{7-y}{2}-(5-2y)\big)\,dy+\int_2^3 \big(\frac{7-y}{2}-(y-1)\big)\,dy\\ &= 3/4+3/4\\ &= 3/2.\end{aligned}$$

Of course, the final answer is the same. (It is interesting to note that the area of all 4 subregions used is 3/4. This is coincidental.)

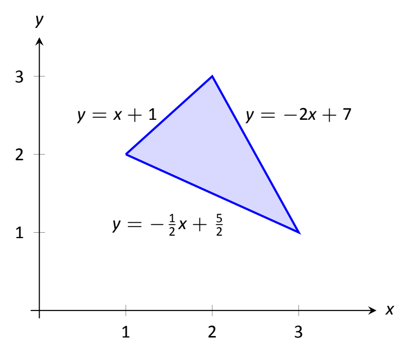


Figure 7.1.6: Graphing a triangular region in Example 7.1.4.

While we have focused on producing exact answers, we are also able to make approximations using the principle of Theorem 7.1.1. The integrand in the theorem is a distance ("top minus bottom"); integrating this distance function gives an area. By taking discrete measurements of distance, we can approximate an area using numerical integration techniques developed in Section 5.5. The following example demonstrates this.

Example 7.1.5 Numerically approximating area

To approximate the area of a lake, shown in Figure 7.1.7 (a), the "length" of the lake is measured at 200-foot increments as shown in Figure 7.1.7 (b), where the lengths are given in hundreds of feet. Approximate the area of the lake.

Solution The measurements of length can be viewed as measuring "top minus bottom" of two functions. The exact answer is found by integrating $\int_0^{12} \big(f(x) - g(x)\big)\,dx$, but of course we don't know the functions f and g. Our discrete measurements instead allow us to approximate.

We have the following data points:

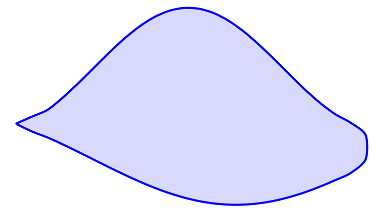
$$(0,0),\ (2,2.25),\ (4,5.08),\ (6,6.35),\ (8,5.21),\ (10,2.76),\ (12,0).$$

We also have that $\Delta x = \frac{b-a}{n} = 2$, so Simpson's Rule gives

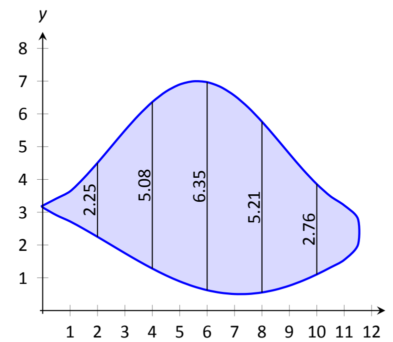
$$\begin{aligned}\text{Area} &\approx \frac{2}{3}\Big(1\cdot 0 + 4\cdot 2.25 + 2\cdot 5.08 + 4\cdot 6.35 + 2\cdot 5.21 + 4\cdot 2.76 + 1\cdot 0\Big)\\ &= 44.01\overline{3}\ \text{units}^2.\end{aligned}$$

Since the measurements are in hundreds of feet, $\text{units}^2 = (100\ \text{ft})^2 = 10,000\ \text{ft}^2$, giving a total area of $440,133\ \text{ft}^2$. (Since we are approximating, we'd likely say the area was about $440,000\ \text{ft}^2$, which is a little more than 10 acres.)

In the next section we apply our applications–of–integration techniques to finding the volumes of certain solids.



(a)



(b)

Figure 7.1.7: (a) A sketch of a lake, and (b) the lake with length measurements.

Exercises 7.1

Terms and Concepts

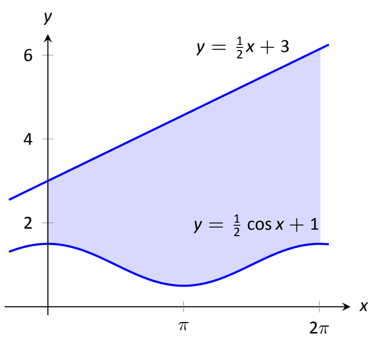
1. T/F: The area between curves is always positive.

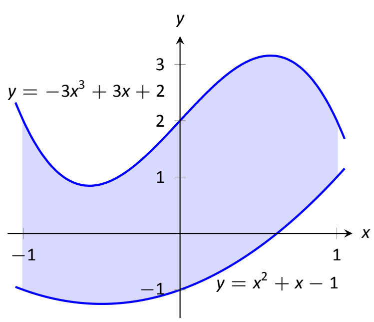
2. T/F: Calculus can be used to find the area of basic geometric shapes.

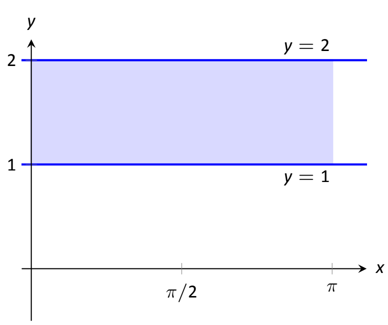
3. In your own words, describe how to find the total area enclosed by $y = f(x)$ and $y = g(x)$.

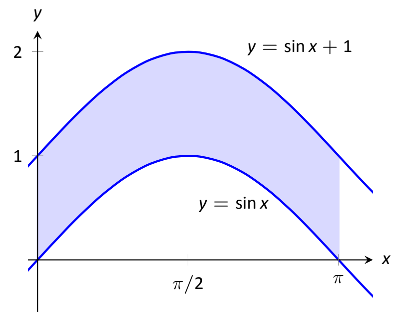
Problems

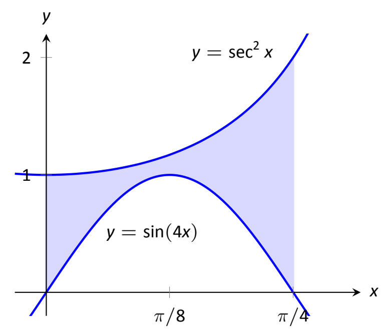
In Exercises 4 – 11, find the area of the shaded region in the given graph.

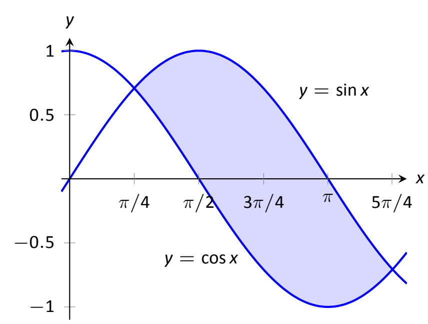
4. 

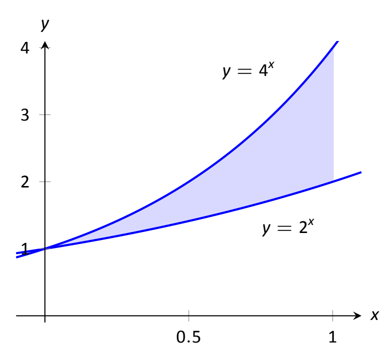
5. 

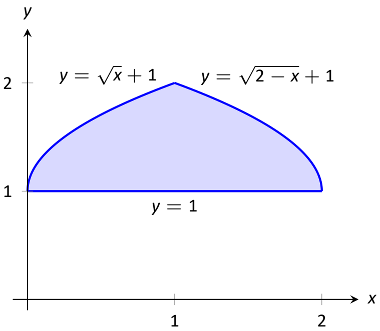
6. 

7. 

8. 

9. 

10. 

11. 

In Exercises 12 – 19, find the total area enclosed by the functions f and g.

12. $f(x) = 2x^2 + 5x - 3$, $g(x) = x^2 + 4x - 1$

13. $f(x) = x^2 - 3x + 2$, $g(x) = -3x + 3$

14. $f(x) = \sin x$, $g(x) = 2x/\pi$

15. $f(x) = x^3 - 4x^2 + x - 1$, $g(x) = -x^2 + 2x - 4$

16. $f(x) = x$, $g(x) = \sqrt{x}$

17. $f(x) = -x^3 + 5x^2 + 2x + 1$, $g(x) = 3x^2 + x + 3$

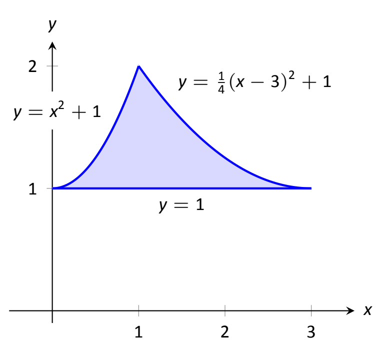
18. The functions $f(x) = \cos(x)$ and $g(x) = \sin x$ intersect infinitely many times, forming an infinite number of repeated, enclosed regions. Find the areas of these regions.

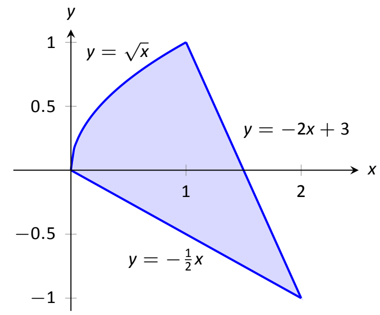
19. The functions $f(x) = \cos(2x)$ and $g(x) = \sin x$ intersect infinitely many times, forming an infinite number of repeated, enclosed regions. Find the areas of these regions.

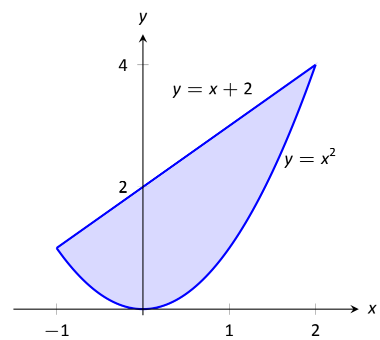
20. The functions $f(x) = \cos(2x)$ and $g(x) = \sin x$ intersect infinitely many times, forming an infinite number of repeated, enclosed regions. Find the areas of these regions.

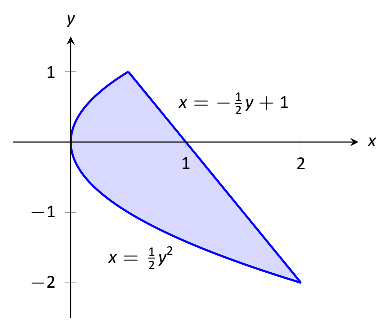
In Exercises 21 – 26, find the area of the enclosed region in two ways:

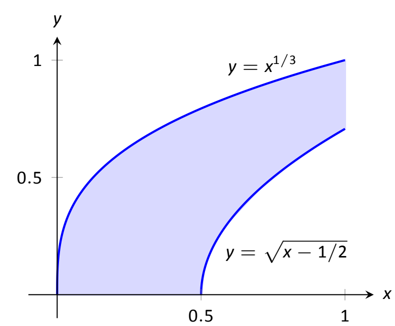
1. **by treating the boundaries as functions of x, and**
2. **by treating the boundaries as functions of y.**

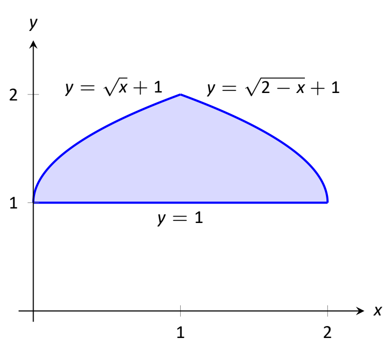
21. 

22. 

23. 

24. 

25. 

26. 

In Exercises 27 – 30, find the area triangle formed by the given three points.

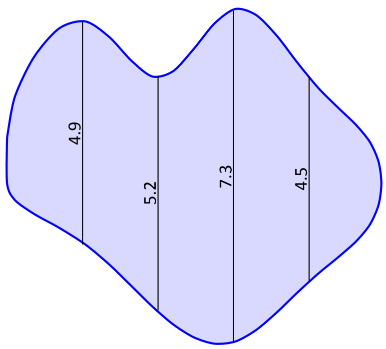
27. $(1,1)$, $(2,3)$, and $(3,3)$

28. $(-1,1)$, $(1,3)$, and $(2,-1)$

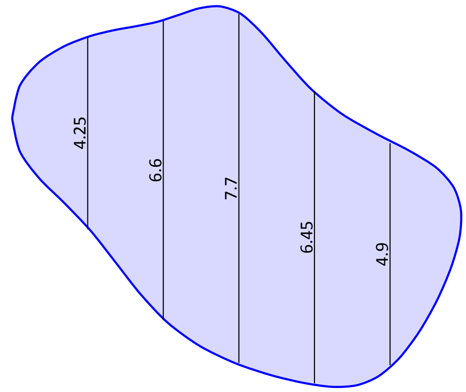
29. $(1,1)$, $(3,3)$, and $(3,3)$

30. $(0,0)$, $(2,5)$, and $(5,2)$

31. Use the Trapezoidal Rule to approximate the area of the pictured lake whose lengths, in hundreds of feet, are measured in 100-foot increments.



32. Use Simpson's Rule to approximate the area of the pictured lake whose lengths, in hundreds of feet, are measured in 200-foot increments.



7.2 Volume by Cross-Sectional Area; Disk and Washer Methods

The volume of a general right cylinder, as shown in Figure 7.2.1, is

Area of the base $\times$ height.

We can use this fact as the building block in finding volumes of a variety of shapes.

Given an arbitrary solid, we can *approximate* its volume by cutting it into n thin slices. When the slices are thin, each slice can be approximated well by a general right cylinder. Thus the volume of each slice is approximately its cross-sectional area $\times$ thickness. (These slices are the differential elements.)

By orienting a solid along the x-axis, we can let $A(x_i)$ represent the cross-sectional area of the i^{th} slice, and let Δx_i represent the thickness of this slice (the thickness is a small change in x). The total volume of the solid is approximately:

$$\begin{aligned}\text{Volume} &\approx \sum_{i=1}^{n}\Big[\text{Area} \;\times\; \text{thickness}\Big]\\ &= \sum_{i=1}^{n} A(x_i)\Delta x_i.\end{aligned}$$

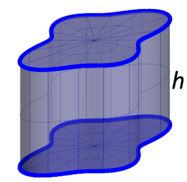


Figure 7.2.1: The volume of a general right cylinder

Recognize that this is a Riemann Sum. By taking a limit (as the thickness of the slices goes to 0) we can find the volume exactly.

Theorem 7.2.1 Volume By Cross-Sectional Area

The volume V of a solid, oriented along the x-axis with cross-sectional area $A(x)$ from $x = a$ to $x = b$, is

$$V = \int_a^b A(x)\, dx.$$

Example 7.2.1 Finding the volume of a solid

Find the volume of a pyramid with a square base of side length 10 in and a height of 5 in.

Solution There are many ways to "orient" the pyramid along the x-axis; Figure 7.2.2 gives one such way, with the pointed top of the pyramid at the origin and the x-axis going through the center of the base.

Each cross section of the pyramid is a square; this is a sample differential element. To determine its area $A(x)$, we need to determine the side lengths of the square.

When $x = 5$, the square has side length 10; when $x = 0$, the square has side length 0. Since the edges of the pyramid are lines, it is easy to figure that each cross-sectional square has side length $2x$, giving $A(x) = (2x)^2 = 4x^2$.

If one were to cut a slice out of the pyramid at $x = 3$, as shown in Figure 7.2.3, one would have a shape with square bottom and top with sloped sides. If the slice were thin, both the bottom and top squares would have sides lengths of about 6, and thus the cross–sectional area of the bottom and top would be about 36in^2. Letting Δx_i represent the thickness of the slice, the volume of this slice would then be about $36\Delta x_i \text{in}^3$.

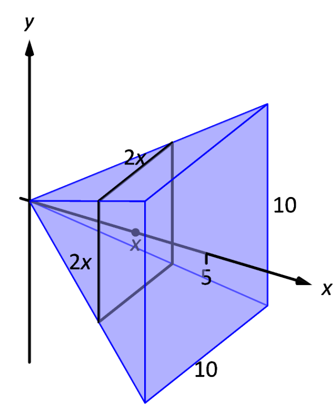


Figure 7.2.2: Orienting a pyramid along the x-axis in Example 7.2.1.

Cutting the pyramid into n slices divides the total volume into n equally–spaced smaller pieces, each with volume $(2x_i)^2\Delta x$, where x_i is the approximate

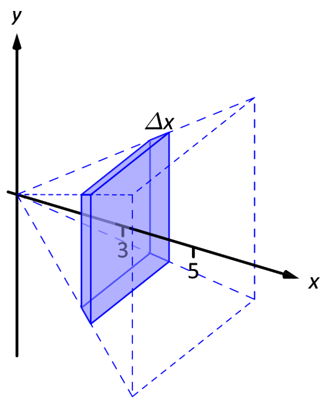


Figure 7.2.3: Cutting a slice in they pyramid in Example 7.2.1 at $x = 3$.

location of the slice along the x-axis and Δx represents the thickness of each slice. One can approximate total volume of the pyramid by summing up the volumes of these slices:

$$\text{Approximate volume } = \sum_{i=1}^{n} (2x_i)^2 \Delta x.$$

Taking the limit as $n \to \infty$ gives the actual volume of the pyramid; recoginizing this sum as a Riemann Sum allows us to find the exact answer using a definite integral, matching the definite integral given by Theorem 7.2.1.

We have

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} (2x_i)^2 \Delta x \\ &= \int_0^5 4x^2\, dx \\ &= \frac{4}{3}x^3\Big|_0^5 \\ &= \frac{500}{3} \text{ in}^3 \approx 166.67 \text{ in}^3. \end{aligned}$$

We can check our work by consulting the general equation for the volume of a pyramid (see the back cover under "Volume of A General Cone"):

$$\tfrac{1}{3} \times \text{area of base} \times \text{height}.$$

Certainly, using this formula from geometry is faster than our new method, but the calculus–based method can be applied to much more than just cones.

An important special case of Theorem 7.2.1 is when the solid is a **solid of revolution**, that is, when the solid is formed by rotating a shape around an axis.

Start with a function $y = f(x)$ from $x = a$ to $x = b$. Revolving this curve about a horizontal axis creates a three-dimensional solid whose cross sections are disks (thin circles). Let $R(x)$ represent the radius of the cross-sectional disk at x; the area of this disk is $\pi R(x)^2$. Applying Theorem 7.2.1 gives the Disk Method.

Key Idea 7.2.1 The Disk Method

Let a solid be formed by revolving the curve $y = f(x)$ from $x = a$ to $x = b$ around a horizontal axis, and let $R(x)$ be the radius of the cross-sectional disk at x. The volume of the solid is

$$V = \pi \int_a^b R(x)^2\, dx.$$

Example 7.2.2 Finding volume using the Disk Method

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, around the x-axis.

Solution A sketch can help us understand this problem. In Figure 7.2.4(a) the curve $y = 1/x$ is sketched along with the differential element – a disk – at x with radius $R(x) = 1/x$. In Figure 7.2.4 (b) the whole solid is pictured, along with the differential element.

The volume of the differential element shown in part (a) of the figure is approximately $\pi R(x_i)^2 \Delta x$, where $R(x_i)$ is the radius of the disk shown and Δx is the thickness of that slice. The radius $R(x_i)$ is the distance from the x-axis to the curve, hence $R(x_i) = 1/x_i$.

Slicing the solid into n equally–spaced slices, we can approximate the total volume by adding up the approximate volume of each slice:

$$\text{Approximate volume} = \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x.$$

Taking the limit of the above sum as $n \to \infty$ gives the actual volume; recognizing this sum as a Riemann sum allows us to evaluate the limit with a definite integral, which matches the formula given in Key Idea 7.2.1:

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x \\ &= \pi \int_1^2 \left(\frac{1}{x}\right)^2 dx \\ &= \pi \int_1^2 \frac{1}{x^2}\, dx \\ &= \pi \left[-\frac{1}{x}\right]\Big|_1^2 \\ &= \pi \left[-\frac{1}{2} - (-1)\right] \\ &= \frac{\pi}{2} \text{ units}^3. \end{aligned}$$

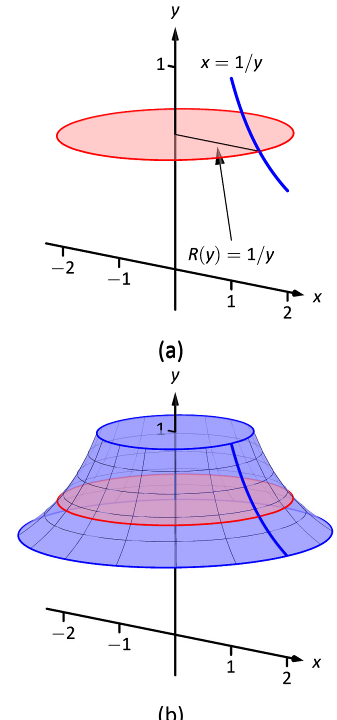


Figure 7.2.4: Sketching a solid in Example 7.2.2.

Figure 7.2.5: Sketching a solid in Example 7.2.3.

While Key Idea 7.2.1 is given in terms of functions of x, the principle involved can be applied to functions of y when the axis of rotation is vertical, not horizontal. We demonstrate this in the next example.

Example 7.2.3 Finding volume using the Disk Method

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, about the y-axis.

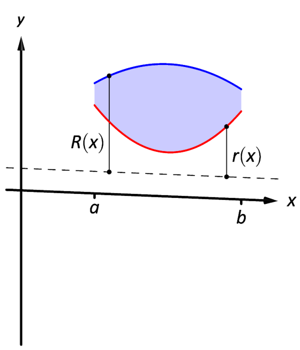
Solution Since the axis of rotation is vertical, we need to convert the function into a function of y and convert the x-bounds to y-bounds. Since $y = 1/x$ defines the curve, we rewrite it as $x = 1/y$. The bound $x = 1$ corresponds to the y-bound $y = 1$, and the bound $x = 2$ corresponds to the y-bound $y = 1/2$.

Thus we are rotating the curve $x = 1/y$, from $y = 1/2$ to $y = 1$ about the y-axis to form a solid. The curve and sample differential element are sketched in Figure 7.2.5 (a), with a full sketch of the solid in Figure 7.2.5 (b). We integrate to find the volume:

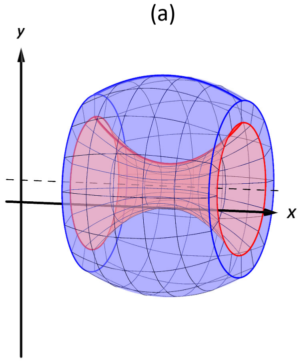
$$V = \pi \int_{1/2}^{1} \frac{1}{y^2}\, dy$$
$$= -\frac{\pi}{y}\Big|_{1/2}^{1}$$
$$= \pi \text{ units}^3.$$

We can also compute the volume of solids of revolution that have a hole in the center. The general principle is simple: compute the volume of the solid irrespective of the hole, then subtract the volume of the hole. If the outside radius of the solid is $R(x)$ and the inside radius (defining the hole) is $r(x)$, then the volume is

$$V = \pi \int_a^b R(x)^2\, dx - \pi \int_a^b r(x)^2\, dx = \pi \int_a^b \left(R(x)^2 - r(x)^2\right)\, dx.$$



(a)



(b)

Figure 7.2.6: Establishing the Washer Method; see also Figure 7.2.7.

One can generate a solid of revolution with a hole in the middle by revolving a region about an axis. Consider Figure 7.2.6(a), where a region is sketched along with a dashed, horizontal axis of rotation. By rotating the region about the axis, a solid is formed as sketched in Figure 7.2.6(b). The outside of the solid has radius $R(x)$, whereas the inside has radius $r(x)$. Each cross section of this solid will be a washer (a disk with a hole in the center) as sketched in Figure 7.2.7. This leads us to the Washer Method.

Key Idea 7.2.2 The Washer Method

Let a region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ be rotated about a horizontal axis that does not intersect the region, forming a solid. Each cross section at x will be a washer with outside radius $R(x)$ and inside radius $r(x)$. The volume of the solid is

$$V = \pi \int_a^b \left(R(x)^2 - r(x)^2\right)\, dx.$$

Even though we introduced it first, the Disk Method is just a special case of the Washer Method with an inside radius of $r(x) = 0$.

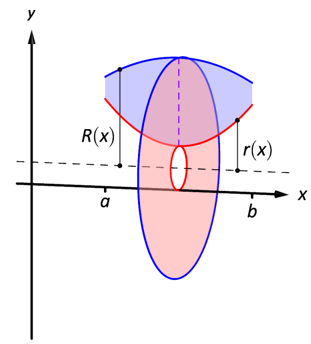


Figure 7.2.7: Establishing the Washer Method; see also Figure 7.2.6.

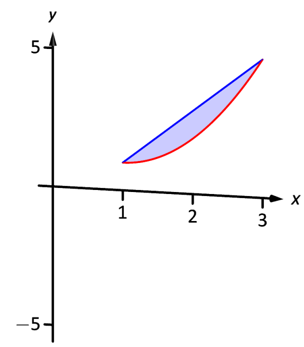
Example 7.2.4 Finding volume with the Washer Method

Find the volume of the solid formed by rotating the region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$ about the x-axis.

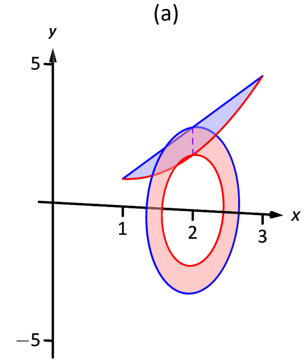
Solution A sketch of the region will help, as given in Figure 7.2.8(a). Rotating about the x-axis will produce cross sections in the shape of washers, as shown in Figure 7.2.8(b); the complete solid is shown in part (c). The outside radius of this washer is $R(x) = 2x + 1$; the inside radius is $r(x) = x^2 - 2x + 2$. As the region is bounded from $x = 1$ to $x = 3$, we integrate as follows to compute the volume.

$$\begin{aligned} V &= \pi \int_1^3 \Big((2x-1)^2 - (x^2 - 2x + 2)^2\Big)\,dx \\ &= \pi \int_1^3 \big(-x^4 + 4x^3 - 4x^2 + 4x - 3\big)\,dx \\ &= \pi \Big[-\frac{1}{5}x^5 + x^4 - \frac{4}{3}x^3 + 2x^2 - 3x\Big]\Big|_1^3 \\ &= \frac{104}{15}\pi \approx 21.78 \text{ units}^3. \end{aligned}$$

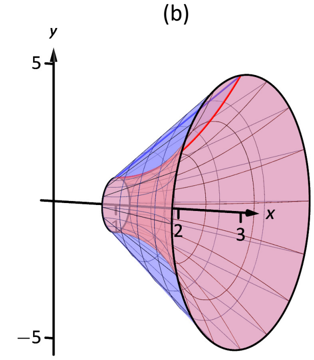
When rotating about a vertical axis, the outside and inside radius functions must be functions of y.



(a)



(b)



(c)

Figure 7.2.8: Sketching the differential element and solid in Example 7.2.4.

Example 7.2.5 Finding volume with the Washer Method

Find the volume of the solid formed by rotating the triangular region with vertices at $(1, 1)$, $(2, 1)$ and $(2, 3)$ about the y-axis.

Solution The triangular region is sketched in Figure 7.2.9(a); the differential element is sketched in (b) and the full solid is drawn in (c). They help us establish the outside and inside radii. Since the axis of rotation is vertical, each radius is a function of y.

The outside radius $R(y)$ is formed by the line connecting $(2, 1)$ and $(2, 3)$; it is a constant function, as regardless of the y-value the distance from the line to the axis of rotation is 2. Thus $R(y) = 2$.

The inside radius is formed by the line connecting $(1, 1)$ and $(2, 3)$. The equation of this line is $y = 2x - 1$, but we need to refer to it as a function of y. Solving for x gives $r(y) = \frac{1}{2}(y + 1)$.

We integrate over the y-bounds of $y = 1$ to $y = 3$. Thus the volume is

$$\begin{aligned} V &= \pi \int_1^3 \Big(2^2 - \big(\frac{1}{2}(y+1)\big)^2\Big)\,dy \\ &= \pi \int_1^3 \Big(-\frac{1}{4}y^2 - \frac{1}{2}y + \frac{15}{4}\Big)\,dy \\ &= \pi \Big[-\frac{1}{12}y^3 - \frac{1}{4}y^2 + \frac{15}{4}y\Big]\Big|_1^3 \\ &= \frac{10}{3}\pi \approx 10.47 \text{ units}^3. \end{aligned}$$

This section introduced a new application of the definite integral. Our default view of the definite integral is that it gives "the area under the curve." However, we can establish definite integrals that represent other quantities; in this section, we computed volume.

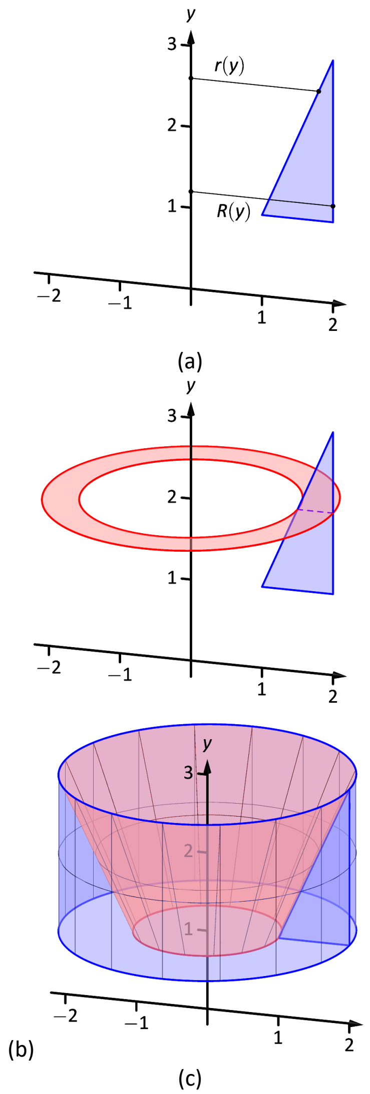


Figure 7.2.9: Sketching the solid in Example 7.2.5.

The ultimate goal of this section is not to compute volumes of solids. That can be useful, but what is more useful is the understanding of this basic principle of integral calculus, outlined in Key Idea 7.0.1: to find the exact value of some quantity,

- we start with an approximation (in this section, slice the solid and approximate the volume of each slice),
- then make the approximation better by refining our original approximation (i.e., use more slices),
- then use limits to establish a definite integral which gives the exact value.

We practice this principle in the next section where we find volumes by slicing solids in a different way.

Exercises 7.2

Terms and Concepts

1. T/F: A solid of revolution is formed by revolving a shape around an axis.

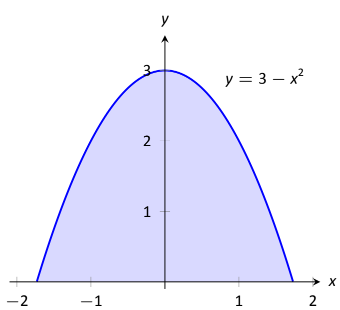
2. In your own words, explain how the Disk and Washer Methods are related.

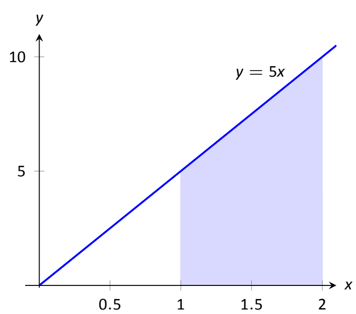
3. Explain the how the units of volume are found in the integral of Theorem 7.2.1: if $A(x)$ has units of in^2, how does $\int A(x)\,dx$ have units of in^3?

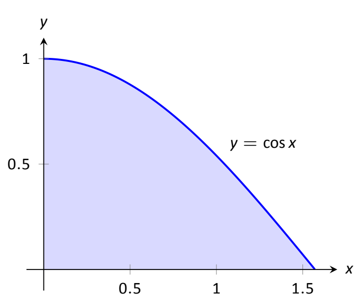
4. A fundamental principle of this section is "________ can be found by integrating an area function."

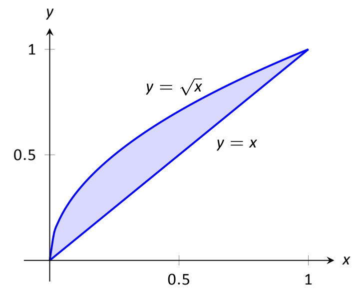
Problems

In Exercises 5 – 8, a region of the Cartesian plane is shaded. Use the Disk/Washer Method to find the volume of the solid of revolution formed by revolving the region about the x-axis.

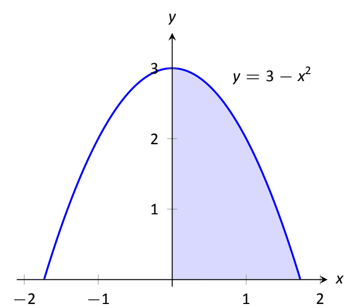
5. 

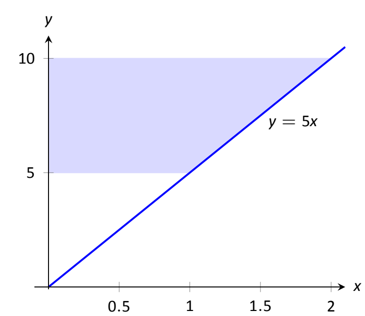
6. 

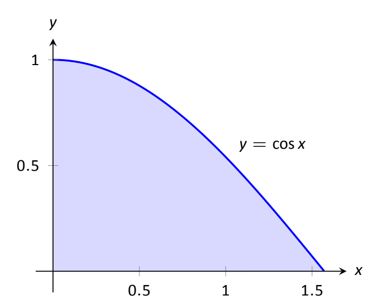
7. 

8. 

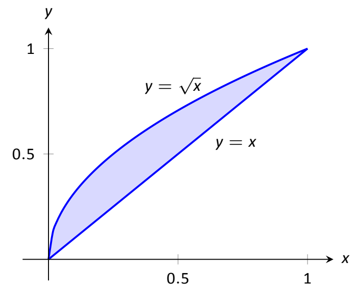
In Exercises 9 – 12, a region of the Cartesian plane is shaded. Use the Disk/Washer Method to find the volume of the solid of revolution formed by revolving the region about the y-axis.

9. 

10. 

11. 

(Hint: Integration By Parts will be necessary, twice. First let $u = \arccos^2 x$, then let $u = \arccos x$.)

12. 

In Exercises 13 – 18, a region of the Cartesian plane is described. Use the Disk/Washer Method to find the volume of the solid of revolution formed by rotating the region about each of the given axes.

13. Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.
Rotate about:
(a) the x-axis
(b) $y = 1$
(c) the y-axis
(d) $x = 1$

14. Region bounded by: $y = 4 - x^2$ and $y = 0$.
Rotate about:
(a) the x-axis
(b) $y = 4$
(c) $y = -1$
(d) $x = 2$

15. The triangle with vertices $(1,1)$, $(1,2)$ and $(2,1)$.
Rotate about:
(a) the x-axis
(b) $y = 2$
(c) the y-axis
(d) $x = 1$

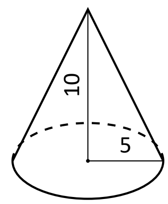
16. Region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$.
Rotate about:
(a) the x-axis
(b) $y = 1$
(c) $y = 5$

17. Region bounded by $y = 1/\sqrt{x^2+1}$, $x = -1$, $x = 1$ and the x-axis.
Rotate about:
(a) the x-axis
(b) $y = 1$
(c) $y = -1$

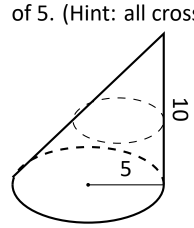
18. Region bounded by $y = 2x$, $y = x$ and $x = 2$.
Rotate about:
(a) the x-axis
(b) $y = 4$
(c) the y-axis
(d) $x = 2$

In Exercises 19 – 22, a solid is described. Orient the solid along the x-axis such that a cross-sectional area function $A(x)$ can be obtained, then apply Theorem 7.2.1 to find the volume of the solid.

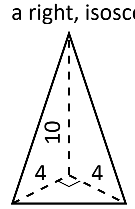
19. A right circular cone with height of 10 and base radius of 5.



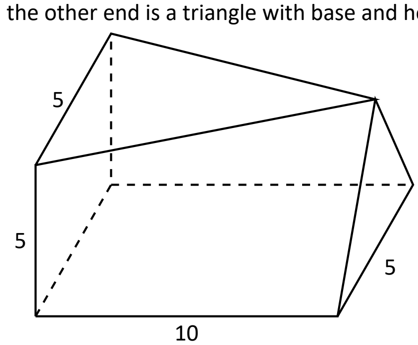
20. A skew right circular cone with height of 10 and base radius of 5. (Hint: all cross-sections are circles.)



21. A right triangular cone with height of 10 and whose base is a right, isosceles triangle with side length 4.



22. A solid with length 10 with a rectangular base and triangular top, wherein one end is a square with side length 5 and the other end is a triangle with base and height of 5.



7.3 The Shell Method

Often a given problem can be solved in more than one way. A particular method may be chosen out of convenience, personal preference, or perhaps necessity. Ultimately, it is good to have options.

The previous section introduced the Disk and Washer Methods, which computed the volume of solids of revolution by integrating the cross–sectional area of the solid. This section develops another method of computing volume, the **Shell Method.** Instead of slicing the solid perpendicular to the axis of rotation creating cross-sections, we now slice it parallel to the axis of rotation, creating "shells."

Consider Figure 7.3.1, where the region shown in (a) is rotated around the y-axis forming the solid shown in (b). A small slice of the region is drawn in (a), parallel to the axis of rotation. When the region is rotated, this thin slice forms a **cylindrical shell**, as pictured in part (c) of the figure. The previous section approximated a solid with lots of thin disks (or washers); we now approximate a solid with many thin cylindrical shells.

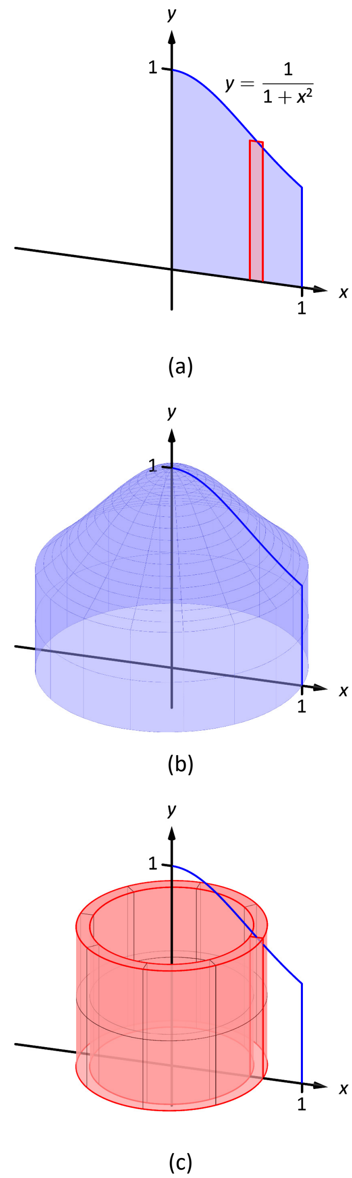


Figure 7.3.1: Introducing the Shell Method.

To compute the volume of one shell, first consider the paper label on a soup can with radius r and height h. What is the area of this label? A simple way of determining this is to cut the label and lay it out flat, forming a rectangle with height h and length $2\pi r$. Thus the area is $A = 2\pi rh$; see Figure 7.3.2(a).

Do a similar process with a cylindrical shell, with height h, thickness Δx, and approximate radius r. Cutting the shell and laying it flat forms a rectangular solid with length $2\pi r$, height h and depth Δx. Thus the volume is $V \approx 2\pi rh\Delta x$; see Figure 7.3.2(b). (We say "approximately" since our radius was an approximation.)

By breaking the solid into n cylindrical shells, we can approximate the volume of the solid as

$$V = \sum_{i=1}^{n} 2\pi r_i h_i \Delta x_i,$$

where r_i, h_i and Δx_i are the radius, height and thickness of the i^{th} shell, respectively.

This is a Riemann Sum. Taking a limit as the thickness of the shells approaches 0 leads to a definite integral.

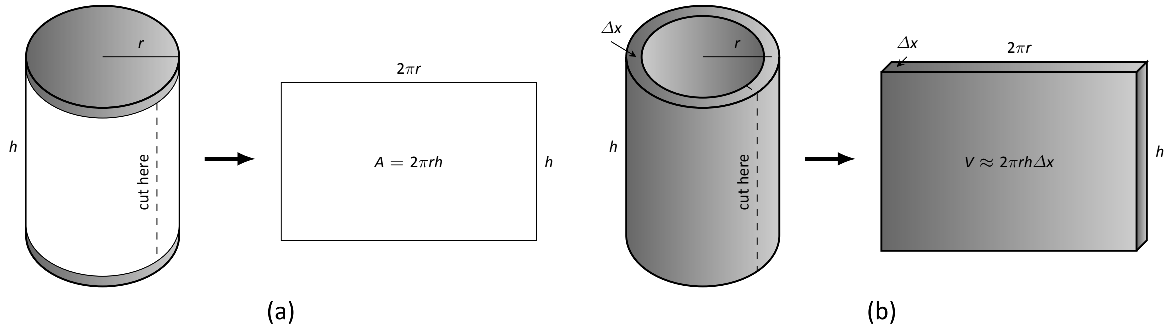


Figure 7.3.2: Determining the volume of a thin cylindrical shell.

Key Idea 7.3.1 The Shell Method

Let a solid be formed by revolving a region R, bounded by $x = a$ and $x = b$, around a vertical axis. Let $r(x)$ represent the distance from the axis of rotation to x (i.e., the radius of a sample shell) and let $h(x)$ represent the height of the solid at x (i.e., the height of the shell). The volume of the solid is

$$V = 2\pi \int_a^b r(x)h(x)\,dx.$$

Special Cases:

1. When the region R is bounded above by $y = f(x)$ and below by $y = g(x)$, then $h(x) = f(x) - g(x)$.
2. When the axis of rotation is the y-axis (i.e., $x = 0$) then $r(x) = x$.

Let's practice using the Shell Method.

Example 7.3.1 Finding volume using the Shell Method

Find the volume of the solid formed by rotating the region bounded by $y = 0$, $y = 1/(1 + x^2)$, $x = 0$ and $x = 1$ about the y-axis.

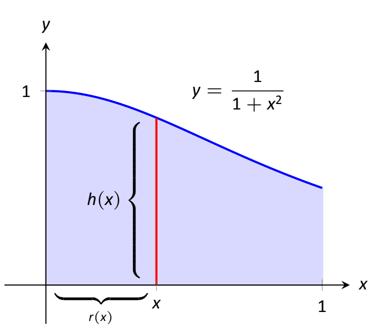


Figure 7.3.3: Graphing a region in Example 7.3.1.

Solution This is the region used to introduce the Shell Method in Figure 7.3.1, but is sketched again in Figure 7.3.3 for closer reference. A line is drawn in the region parallel to the axis of rotation representing a shell that will be carved out as the region is rotated about the y-axis. (This is the differential element.)

The distance this line is from the axis of rotation determines $r(x)$; as the distance from x to the y-axis is x, we have $r(x) = x$. The height of this line determines $h(x)$; the top of the line is at $y = 1/(1 + x^2)$, whereas the bottom of the line is at $y = 0$. Thus $h(x) = 1/(1 + x^2) - 0 = 1/(1 + x^2)$. The region is bounded from $x = 0$ to $x = 1$, so the volume is

$$V = 2\pi \int_0^1 \frac{x}{1 + x^2}\,dx.$$

This requires substitution. Let $u = 1 + x^2$, so $du = 2x\,dx$. We also change the bounds: $u(0) = 1$ and $u(1) = 2$. Thus we have:

$$\begin{aligned} &= \pi \int_1^2 \frac{1}{u}\,du \\ &= \pi \ln u \Big|_1^2 \\ &= \pi \ln 2 \approx 2.178 \text{ units}^3. \end{aligned}$$

Note: in order to find this volume using the Disk Method, two integrals would be needed to account for the regions above and below $y = 1/2$.

With the Shell Method, nothing special needs to be accounted for to compute the volume of a solid that has a hole in the middle, as demonstrated next.

Example 7.3.2 Finding volume using the Shell Method

Find the volume of the solid formed by rotating the triangular region determined by the points $(0, 1)$, $(1, 1)$ and $(1, 3)$ about the line $x = 3$.

Solution The region is sketched in Figure 7.3.4(a) along with the differential element, a line within the region parallel to the axis of rotation. In part (b) of the figure, we see the shell traced out by the differential element, and in part (c) the whole solid is shown.

The height of the differential element is the distance from $y = 1$ to $y = 2x + 1$, the line that connects the points $(0, 1)$ and $(1, 3)$. Thus $h(x) = 2x+1-1 = 2x$. The radius of the shell formed by the differential element is the distance from x to $x = 3$; that is, it is $r(x) = 3 - x$. The x-bounds of the region are $x = 0$ to $x = 1$, giving

$$\begin{aligned} V &= 2\pi \int_0^1 (3 - x)(2x)\,dx \\ &= 2\pi \int_0^1 \left(6x - 2x^2\right)\,dx \\ &= 2\pi \left(3x^2 - \frac{2}{3}x^3\right)\Big|_0^1 \\ &= \frac{14}{3}\pi \approx 14.66 \text{ units}^3. \end{aligned}$$

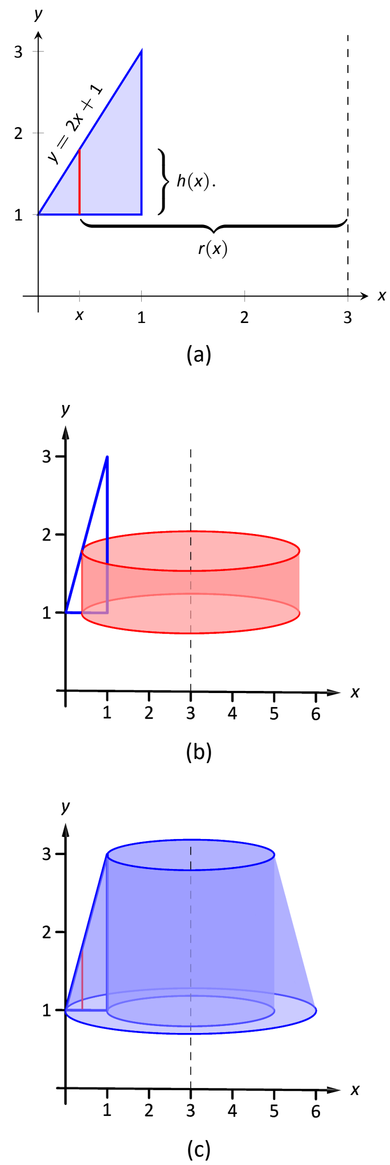


Figure 7.3.4: Graphing a region in Example 7.3.2.

When revolving a region around a horizontal axis, we must consider the radius and height functions in terms of y, not x.

Example 7.3.3 Finding volume using the Shell Method

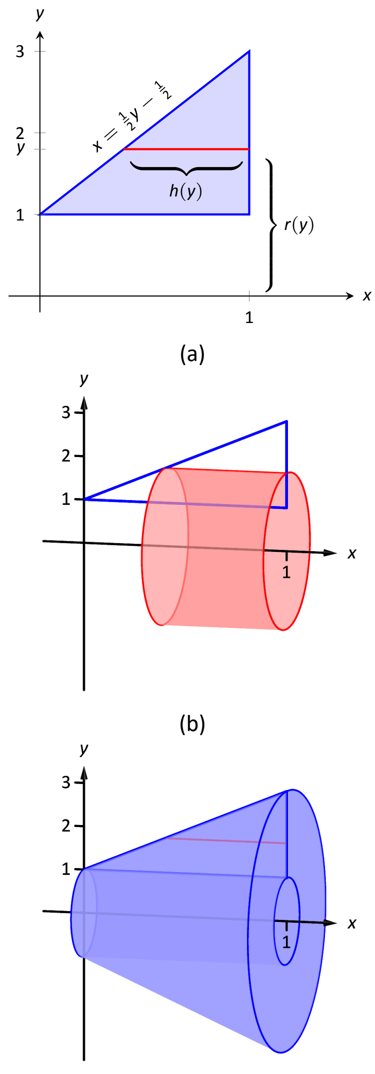
Find the volume of the solid formed by rotating the region given in Example 7.3.2 about the x-axis.

Solution The region is sketched in Figure 7.3.5(a) with a sample differential element. In part (b) of the figure the shell formed by the differential element is drawn, and the solid is sketched in (c). (Note that the triangular region looks "short and wide" here, whereas in the previous example the same region looked "tall and narrow." This is because the bounds on the graphs are different.)

The height of the differential element is an x-distance, between $x = \frac{1}{2}y - \frac{1}{2}$ and $x = 1$. Thus $h(y) = 1-(\frac{1}{2}y-\frac{1}{2}) = -\frac{1}{2}y+\frac{3}{2}$. The radius is the distance from

y to the x-axis, so $r(y) = y$. The y bounds of the region are $y = 1$ and $y = 3$, leading to the integral

$$\begin{aligned}
V &= 2\pi \int_1^3 \left[y\left(-\frac{1}{2}y + \frac{3}{2}\right)\right]\,dy \\
&= 2\pi \int_1^3 \left[-\frac{1}{2}y^2 + \frac{3}{2}y\right]\,dy \\
&= 2\pi \left[-\frac{1}{6}y^3 + \frac{3}{4}y^2\right]\Bigg|_1^3 \\
&= 2\pi \left[\frac{9}{4} - \frac{7}{12}\right] \\
&= \frac{10}{3}\pi \approx 10.472 \text{ units}^3.
\end{aligned}$$



(a) (b) (c)

Figure 7.3.5: Graphing a region in Example 7.3.3.

At the beginning of this section it was stated that "it is good to have options." The next example finds the volume of a solid rather easily with the Shell Method, but using the Washer Method would be quite a chore.

Example 7.3.4 Finding volume using the Shell Method

Find the volume of the solid formed by revolving the region bounded by $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$ about the y-axis.

Solution The region and a differential element, the shell formed by this differential element, and the resulting solid are given in Figure 7.3.6.

The radius of a sample shell is $r(x) = x$; the height of a sample shell is $h(x) = \sin x$, each from $x = 0$ to $x = \pi$. Thus the volume of the solid is

$$V = 2\pi \int_0^\pi x \sin x\,dx.$$

This requires Integration By Parts. Set $u = x$ and $dv = \sin x\,dx$; we leave it to the reader to fill in the rest. We have:

$$\begin{aligned}
&= 2\pi\left[-x\cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx\right] \\
&= 2\pi\left[\pi + \sin x\Big|_0^\pi\right] \\
&= 2\pi\left[\pi + 0\right] \\
&= 2\pi^2 \approx 19.74 \text{ units}^3.
\end{aligned}$$

Note that in order to use the Washer Method, we would need to solve $y = \sin x$ for x, requiring the use of the arcsine function. We leave it to the reader to verify that the outside radius function is $R(y) = \pi - \arcsin y$ and the inside radius function is $r(y) = \arcsin y$. Thus the volume can be computed as

$$\pi \int_0^1 \left[(\pi - \arcsin y)^2 - (\arcsin y)^2\right]\,dy.$$

This integral isn't terrible given that the $\arcsin^2 y$ terms cancel, but it is more onerous than the integral created by the Shell Method.

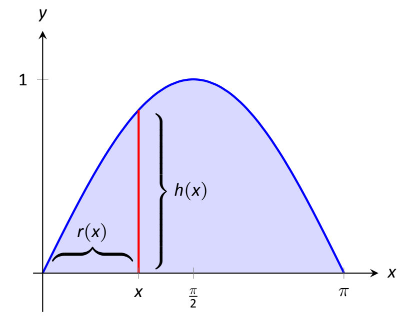
We end this section with a table summarizing the usage of the Washer and Shell Methods.

Key Idea 7.3.2 Summary of the Washer and Shell Methods

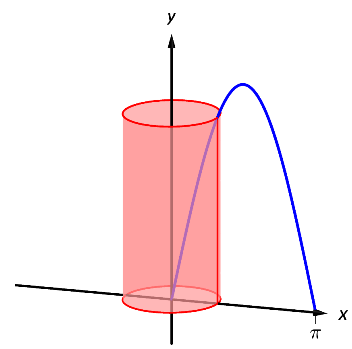
Let a region R be given with x-bounds $x = a$ and $x = b$ and y-bounds $y = c$ and $y = d$.

	Washer Method	Shell Method
Horizontal Axis	$\pi \int_a^b \left(R(x)^2 - r(x)^2\right)\, dx$	$2\pi \int_c^d r(y)h(y)\, dy$
Vertical Axis	$\pi \int_c^d \left(R(y)^2 - r(y)^2\right)\, dy$	$2\pi \int_a^b r(x)h(x)\, dx$

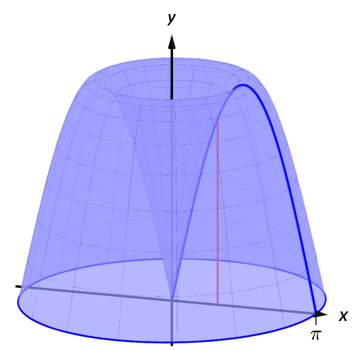
As in the previous section, the real goal of this section is not to be able to compute volumes of certain solids. Rather, it is to be able to solve a problem by first approximating, then using limits to refine the approximation to give the exact value. In this section, we approximate the volume of a solid by cutting it into thin cylindrical shells. By summing up the volumes of each shell, we get an approximation of the volume. By taking a limit as the number of equally spaced shells goes to infinity, our summation can be evaluated as a definite integral, giving the exact value. We use this same principle again in the next section, where we find the length of curves in the plane.



(a)



(b)



(c)

Figure 7.3.6: Graphing a region in Example 7.3.4.

Exercises 7.3

Terms and Concepts

1. T/F: A solid of revolution is formed by revolving a shape around an axis.

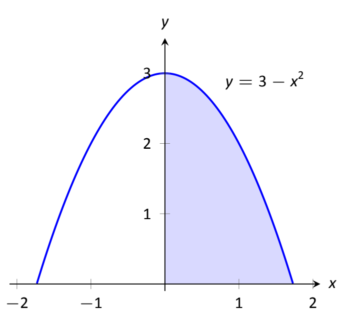
2. T/F: The Shell Method can only be used when the Washer Method fails.

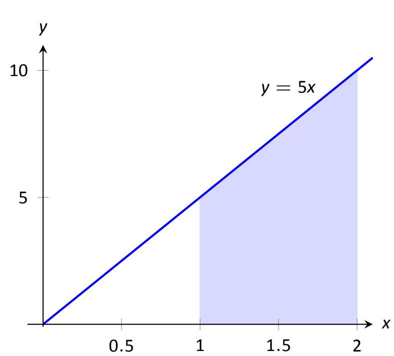
3. T/F: The Shell Method works by integrating cross–sectional areas of a solid.

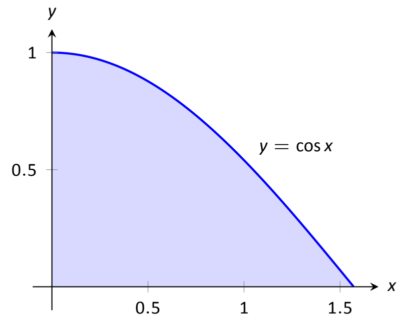
4. T/F: When finding the volume of a solid of revolution that was revolved around a vertical axis, the Shell Method integrates with respect to x.

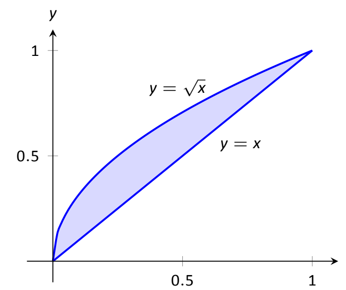
Problems

In Exercises 5 – 8, a region of the Cartesian plane is shaded. Use the Shell Method to find the volume of the solid of revolution formed by revolving the region about the *y*-axis.

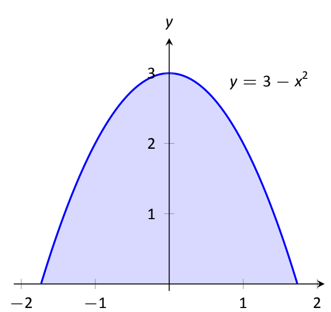
5. 

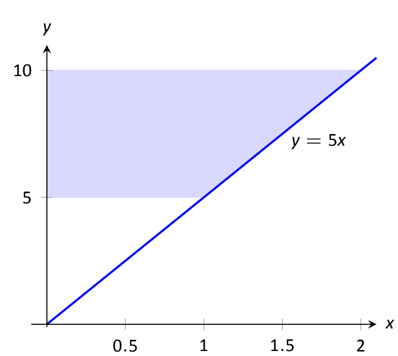
6. 

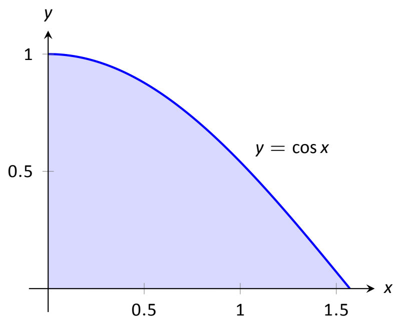
7. 

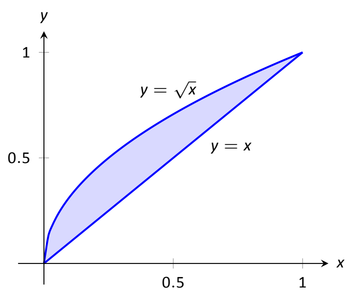
8. 

In Exercises 9 – 12, a region of the Cartesian plane is shaded. Use the Shell Method to find the volume of the solid of revolution formed by revolving the region about the *x*-axis.

9. 

10. 

11. 

12. 

In Exercises 13 – 18, a region of the Cartesian plane is described. Use the Shell Method to find the volume of the solid of revolution formed by rotating the region about each of the given axes.

13. Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.
Rotate about:
   (a) the y-axis
   (b) $x = 1$
   (c) the x-axis
   (d) $y = 1$

14. Region bounded by: $y = 4 - x^2$ and $y = 0$.
Rotate about:
   (a) $x = 2$
   (b) $x = -2$
   (c) the x-axis
   (d) $y = 4$

15. The triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$.
Rotate about:
   (a) the y-axis
   (b) $x = 1$
   (c) the x-axis
   (d) $y = 2$

16. Region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$.
Rotate about:
   (a) the y-axis
   (b) $x = 1$
   (c) $x = -1$

17. Region bounded by $y = 1/\sqrt{x^2 + 1}$, $x = 1$ and the x and y-axes.
Rotate about:
   (a) the y-axis
   (b) $x = 1$

18. Region bounded by $y = 2x$, $y = x$ and $x = 2$.
Rotate about:
   (a) the y-axis
   (b) $x = 2$
   (c) the x-axis
   (d) $y = 4$

7.4 Arc Length and Surface Area

In previous sections we have used integration to answer the following questions:

1. Given a region, what is its area?
2. Given a solid, what is its volume?

In this section, we address a related question: Given a curve, what is its length? This is often referred to as **arc length**.

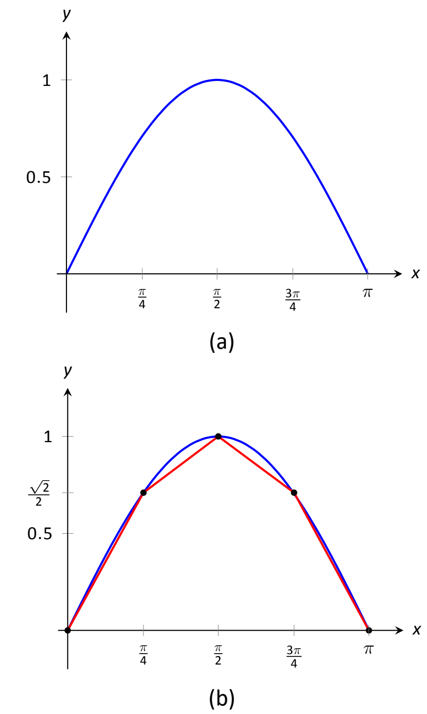


Figure 7.4.1: Graphing $y = \sin x$ on $[0, \pi]$ and approximating the curve with line segments.

Consider the graph of $y = \sin x$ on $[0, \pi]$ given in Figure 7.4.1(a). How long is this curve? That is, if we were to use a piece of string to exactly match the shape of this curve, how long would the string be?

As we have done in the past, we start by approximating; later, we will refine our answer using limits to get an exact solution.

The length of straight–line segments is easy to compute using the Distance Formula. We can approximate the length of the given curve by approximating the curve with straight lines and measuring their lengths.

In Figure 7.4.1(b), the curve $y = \sin x$ has been approximated with 4 line segments (the interval $[0, \pi]$ has been divided into 4 equally–lengthed subintervals). It is clear that these four line segments approximate $y = \sin x$ very well on the first and last subinterval, though not so well in the middle. Regardless, the sum of the lengths of the line segments is 3.79, so we approximate the arc length of $y = \sin x$ on $[0, \pi]$ to be 3.79.

In general, we can approximate the arc length of $y = f(x)$ on $[a, b]$ in the following manner. Let $a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b$ be a partition of $[a, b]$ into n subintervals. Let Δx_i represent the length of the i^{th} subinterval $[x_i, x_{i+1}]$.

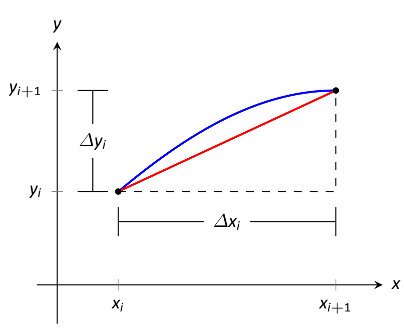


Figure 7.4.2: Zooming in on the i^{th} subinterval $[x_i, x_{i+1}]$ of a partition of $[a, b]$.

Figure 7.4.2 zooms in on the i^{th} subinterval where $y = f(x)$ is approximated by a straight line segment. The dashed lines show that we can view this line segment as the hypotenuse of a right triangle whose sides have length Δx_i and Δy_i. Using the Pythagorean Theorem, the length of this line segment is $\sqrt{\Delta x_i^2 + \Delta y_i^2}$. Summing over all subintervals gives an arc length approximation

$$L \approx \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}.$$

As shown here, this is *not* a Riemann Sum. While we could conclude that taking a limit as the subinterval length goes to zero gives the exact arc length, we would not be able to compute the answer with a definite integral. We need first to do a little algebra.

In the above expression factor out a Δx_i^2 term:

$$\sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_{i=1}^{n} \sqrt{\Delta x_i^2 \left(1 + \frac{\Delta y_i^2}{\Delta x_i^2}\right)}.$$

Now pull the Δx_i^2 term out of the square root:

$$= \sum_{i=1}^{n} \sqrt{1 + \frac{\Delta y_i^2}{\Delta x_i^2}}\, \Delta x_i.$$

This is nearly a Riemann Sum. Consider the $\Delta y_i^2/\Delta x_i^2$ term. The expression $\Delta y_i/\Delta x_i$ measures the "change in y/change in x," that is, the "rise over run" of f on the i^{th} subinterval. The Mean Value Theorem of Differentiation (Theorem 3.2.1) states that there is a c_i in the i^{th} subinterval where $f'(c_i) = \Delta y_i/\Delta x_i$. Thus we can rewrite our above expression as:

$$= \sum_{i=1}^{n} \sqrt{1+f'(c_i)^2}\,\Delta x_i.$$

This *is* a Riemann Sum. As long as f' is continuous, we can invoke Theorem 5.3.2 and conclude

$$= \int_a^b \sqrt{1+f'(x)^2}\,dx.$$

Theorem 7.4.1 Arc Length

Let f be differentiable on $[a,b]$, where f' is also continuous on $[a,b]$. Then the arc length of f from $x=a$ to $x=b$ is

$$L = \int_a^b \sqrt{1+f'(x)^2}\,dx.$$

Note: This is our first use of differentiability on a closed interval since Section 2.1.

The theorem also requires that f' be continuous on $[a,b]$; while examples are arcane, it is possible for f to be differentiable yet f' is not continuous.

As the integrand contains a square root, it is often difficult to use the formula in Theorem 7.4.1 to find the length exactly. When exact answers are difficult to come by, we resort to using numerical methods of approximating definite integrals. The following examples will demonstrate this.

Example 7.4.1 Finding arc length

Find the arc length of $f(x) = x^{3/2}$ from $x=0$ to $x=4$.

Solution We find $f'(x) = \frac{3}{2}x^{1/2}$; note that on $[0,4]$, f is differentiable and f' is also continuous. Using the formula, we find the arc length L as

$$\begin{aligned}
L &= \int_0^4 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx \\
&= \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx \\
&= \int_0^4 \left(1+\frac{9}{4}x\right)^{1/2}\,dx \\
&= \frac{2}{3}\cdot\frac{4}{9}\cdot\left(1+\frac{9}{4}x\right)^{3/2}\Bigg|_0^4 \\
&= \frac{8}{27}\left(10^{3/2}-1\right) \approx 9.07\text{units}.
\end{aligned}$$

A graph of f is given in Figure 7.4.3.

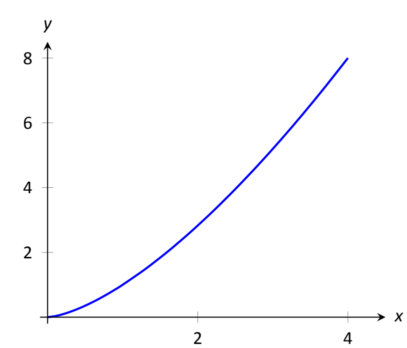


Figure 7.4.3: A graph of $f(x) = x^{3/2}$ from Example 7.4.1.

Example 7.4.2 Finding arc length

Find the arc length of $f(x) = \dfrac{1}{8}x^2 - \ln x$ from $x=1$ to $x=2$.

Solution This function was chosen specifically because the resulting integral can be evaluated exactly. We begin by finding $f'(x) = x/4 - 1/x$. The arc length is

$$\begin{aligned} L &= \int_1^2 \sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}\,dx \\ &= \int_1^2 \sqrt{1+\frac{x^2}{16}-\frac{1}{2}+\frac{1}{x^2}}\,dx \\ &= \int_1^2 \sqrt{\frac{x^2}{16}+\frac{1}{2}+\frac{1}{x^2}}\,dx \\ &= \int_1^2 \sqrt{\left(\frac{x}{4}+\frac{1}{x}\right)^2}\,dx \\ &= \int_1^2 \left(\frac{x}{4}+\frac{1}{x}\right)\,dx \\ &= \left(\frac{x^2}{8}+\ln x\right)\Bigg|_1^2 \\ &= \frac{3}{8}+\ln 2 \approx 1.07 \text{ units.} \end{aligned}$$

A graph of f is given in Figure 7.4.4; the portion of the curve measured in this problem is in bold.

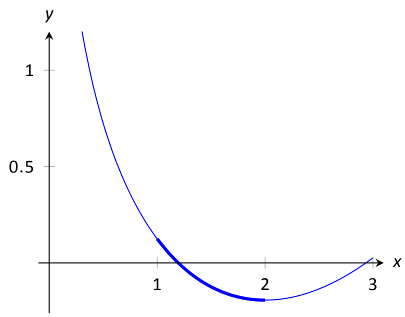


Figure 7.4.4: A graph of $f(x) = \frac{1}{8}x^2 - \ln x$ from Example 7.4.2.

The previous examples found the arc length exactly through careful choice of the functions. In general, exact answers are much more difficult to come by and numerical approximations are necessary.

Example 7.4.3 Approximating arc length numerically

Find the length of the sine curve from $x = 0$ to $x = \pi$.

Solution This is somewhat of a mathematical curiosity; in Example 5.4.3 we found the area under one "hump" of the sine curve is 2 square units; now we are measuring its arc length.

The setup is straightforward: $f(x) = \sin x$ and $f'(x) = \cos x$. Thus

$$L = \int_0^\pi \sqrt{1+\cos^2 x}\,dx.$$

This integral *cannot* be evaluated in terms of elementary functions so we will approximate it with Simpson's Method with $n = 4$. Figure 7.4.5 gives $\sqrt{1+\cos^2 x}$ evaluated at 5 evenly spaced points in $[0, \pi]$. Simpson's Rule then states that

$$\begin{aligned} \int_0^\pi \sqrt{1+\cos^2 x}\,dx &\approx \frac{\pi-0}{4\cdot 3}\left(\sqrt{2}+4\sqrt{3/2}+2(1)+4\sqrt{3/2}+\sqrt{2}\right) \\ &= 3.82918. \end{aligned}$$

Using a computer with $n = 100$ the approximation is $L \approx 3.8202$; our approximation with $n = 4$ is quite good.

x	$\sqrt{1+\cos^2 x}$
0	$\sqrt{2}$
$\pi/4$	$\sqrt{3/2}$
$\pi/2$	1
$3\pi/4$	$\sqrt{3/2}$
π	$\sqrt{2}$

Figure 7.4.5: A table of values of $y = \sqrt{1+\cos^2 x}$ to evaluate a definite integral in Example 7.4.3.

Surface Area of Solids of Revolution

We have already seen how a curve $y = f(x)$ on $[a, b]$ can be revolved around an axis to form a solid. Instead of computing its volume, we now consider its surface area.

We begin as we have in the previous sections: we partition the interval $[a, b]$ with n subintervals, where the i^{th} subinterval is $[x_i, x_{i+1}]$. On each subinterval, we can approximate the curve $y = f(x)$ with a straight line that connects $f(x_i)$ and $f(x_{i+1})$ as shown in Figure 7.4.6(a). Revolving this line segment about the x-axis creates part of a cone (called a *frustum* of a cone) as shown in Figure 7.4.6(b). The surface area of a frustum of a cone is

$$2\pi \cdot \text{ length } \cdot \text{average of the two radii } R \text{ and } r.$$

The length is given by L; we use the material just covered by arc length to state that

$$L \approx \sqrt{1 + f'(c_i)}\Delta x_i$$

for some c_i in the i^{th} subinterval. The radii are just the function evaluated at the endpoints of the interval. That is,

$$R = f(x_{i+1}) \quad \text{and} \quad r = f(x_i).$$

Thus the surface area of this sample frustum of the cone is approximately

$$2\pi\frac{f(x_i) + f(x_{i+1})}{2}\sqrt{1 + f'(c_i)^2}\Delta x_i.$$

Since f is a continuous function, the Intermediate Value Theorem states there is some d_i in $[x_i, x_{i+1}]$ such that $f(d_i) = \dfrac{f(x_i) + f(x_{i+1})}{2}$; we can use this to rewrite the above equation as

$$2\pi f(d_i)\sqrt{1 + f'(c_i)^2}\Delta x_i.$$

Summing over all the subintervals we get the total surface area to be approximately

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(d_i)\sqrt{1 + f'(c_i)^2}\Delta x_i,$$

which is a Riemann Sum. Taking the limit as the subinterval lengths go to zero gives us the exact surface area, given in the following Key Idea.

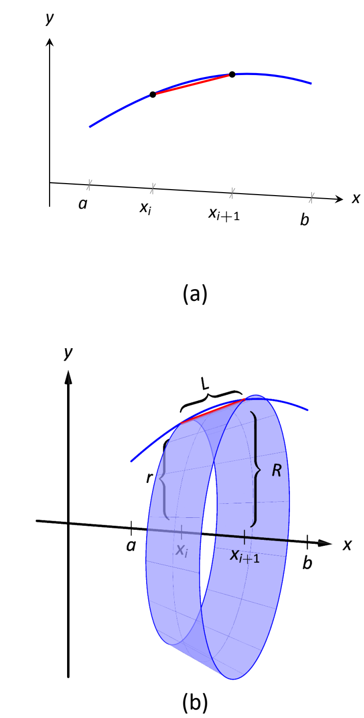


Figure 7.4.6: Establishing the formula for surface area.

Theorem 7.4.2 Surface Area of a Solid of Revolution

Let f be differentiable on $[a, b]$, where f' is also continuous on $[a, b]$.

1. The surface area of the solid formed by revolving the graph of $y = f(x)$, where $f(x) \geq 0$, about the x-axis is

$$\text{Surface Area} = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx.$$

2. The surface area of the solid formed by revolving the graph of $y = f(x)$ about the y-axis, where $a, b \geq 0$, is

$$\text{Surface Area} = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\, dx.$$

(When revolving $y = f(x)$ about the y-axis, the radii of the resulting frustum are x_i and x_{i+1}; their average value is simply the midpoint of the interval. In the limit, this midpoint is just x. This gives the second part of Theorem 7.4.2.)

Example 7.4.4 Finding surface area of a solid of revolution

Find the surface area of the solid formed by revolving $y = \sin x$ on $[0, \pi]$ around the x-axis, as shown in Figure 7.4.7.

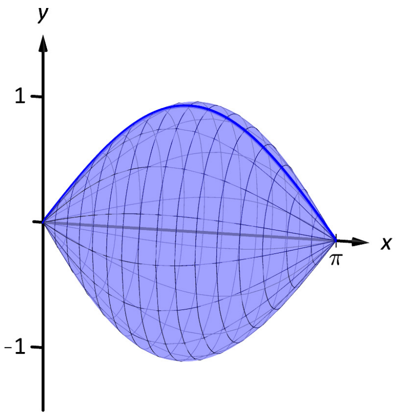


Figure 7.4.7: Revolving $y = \sin x$ on $[0, \pi]$ about the x-axis.

Solution The setup is relatively straightforward. Using Theorem 7.4.2, we have the surface area SA is:

$$\begin{aligned} SA &= 2\pi \int_0^{\pi} \sin x\sqrt{1+\cos^2 x}\,dx \\ &= -2\pi\frac{1}{2}\left(\sinh^{-1}(\cos x) + \cos x\sqrt{1+\cos^2 x}\right)\Big|_0^{\pi} \\ &= 2\pi\left(\sqrt{2} + \sinh^{-1} 1\right) \approx 14.42 \text{ units}^2. \end{aligned}$$

The integration step above is nontrivial, utilizing an integration method called Trigonometric Substitution.

It is interesting to see that the surface area of a solid, whose shape is defined by a trigonometric function, involves both a square root and an inverse hyperbolic trigonometric function.

Example 7.4.5 Finding surface area of a solid of revolution

Find the surface area of the solid formed by revolving the curve $y = x^2$ on $[0, 1]$ about the x-axis and the y-axis.

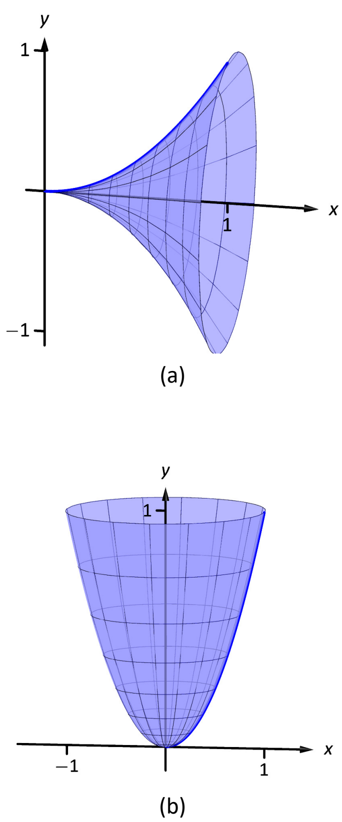


Figure 7.4.8: The solids used in Example 7.4.5.

Solution About the x-axis: the integral is straightforward to setup:

$$SA = 2\pi \int_0^1 x^2\sqrt{1+(2x)^2}\,dx.$$

Like the integral in Example 7.4.4, this requires Trigonometric Substitution.

$$\begin{aligned} &= \frac{\pi}{32}\left(2(8x^3+x)\sqrt{1+4x^2} - \sinh^{-1}(2x)\right)\Big|_0^1 \\ &= \frac{\pi}{32}\left(18\sqrt{5} - \sinh^{-1} 2\right) \\ &\approx 3.81 \text{ units}^2. \end{aligned}$$

The solid formed by revolving $y = x^2$ around the x-axis is graphed in Figure 7.4.8 (a).

About the y-axis: since we are revolving around the y-axis, the "radius" of the solid is not $f(x)$ but rather x. Thus the integral to compute the surface area is:

$$SA = 2\pi \int_0^1 x\sqrt{1+(2x)^2}\,dx.$$

This integral can be solved using substitution. Set $u = 1 + 4x^2$; the new bounds are $u = 1$ to $u = 5$. We then have

$$\begin{aligned} &= \frac{\pi}{4}\int_1^5 \sqrt{u}\,du \\ &= \frac{\pi}{4}\frac{2}{3}u^{3/2}\Big|_1^5 \\ &= \frac{\pi}{6}\left(5\sqrt{5}-1\right) \\ &\approx 5.33 \text{ units}^2. \end{aligned}$$

The solid formed by revolving $y = x^2$ about the y-axis is graphed in Figure 7.4.8 (b).

Our final example is a famous mathematical "paradox."

Example 7.4.6 The surface area and volume of Gabriel's Horn

Consider the solid formed by revolving $y = 1/x$ about the x-axis on $[1, \infty)$. Find the volume and surface area of this solid. (This shape, as graphed in Figure 7.4.9, is known as "Gabriel's Horn" since it looks like a very long horn that only a supernatural person, such as an angel, could play.)

Solution To compute the volume it is natural to use the Disk Method. We have:

$$\begin{aligned} V &= \pi\int_1^\infty \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi\int_1^b \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi\left(\frac{-1}{x}\right)\Big|_1^b \\ &= \lim_{b\to\infty} \pi\left(1-\frac{1}{b}\right) \\ &= \pi \text{ units}^3. \end{aligned}$$

Gabriel's Horn has a finite volume of π cubic units. Since we have already seen that regions with infinite length can have a finite area, this is not too difficult to accept.

We now consider its surface area. The integral is straightforward to setup:

$$SA = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+1/x^4}\,dx.$$

Integrating this expression is not trivial. We can, however, compare it to other improper integrals. Since $1 < \sqrt{1+1/x^4}$ on $[1, \infty)$, we can state that

$$2\pi\int_1^\infty \frac{1}{x}\,dx < 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+1/x^4}\,dx.$$

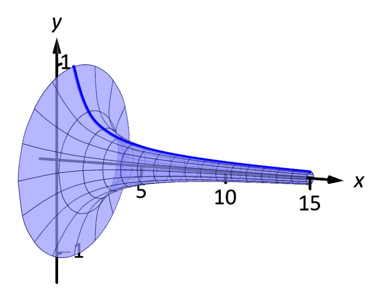


Figure 7.4.9: A graph of Gabriel's Horn.

By Key Idea 6.6.1, the improper integral on the left diverges. Since the integral on the right is larger, we conclude it also diverges, meaning Gabriel's Horn has infinite surface area.

Hence the "paradox": we can fill Gabriel's Horn with a finite amount of paint, but since it has infinite surface area, we can never paint it.

Somehow this paradox is striking when we think about it in terms of volume and area. However, we have seen a similar paradox before, as referenced above. We know that the area under the curve $y = 1/x^2$ on $[1, \infty)$ is finite, yet the shape has an infinite perimeter. Strange things can occur when we deal with the infinite.

A standard equation from physics is "Work = force $\times$ distance", when the force applied is constant. In the next section we learn how to compute work when the force applied is variable.

Exercises 7.4

Terms and Concepts

1. T/F: The integral formula for computing Arc Length was found by first approximating arc length with straight line segments.

2. T/F: The integral formula for computing Arc Length includes a square–root, meaning the integration is probably easy.

Problems

In Exercises 3 – 12, find the arc length of the function on the given interval.

3. $f(x) = x$ on $[0, 1]$.

4. $f(x) = \sqrt{8}x$ on $[-1, 1]$.

5. $f(x) = \frac{1}{3}x^{3/2} - x^{1/2}$ on $[0, 1]$.

6. $f(x) = \frac{1}{12}x^3 + \frac{1}{x}$ on $[1, 4]$.

7. $f(x) = 2x^{3/2} - \frac{1}{6}\sqrt{x}$ on $[0, 9]$.

8. $f(x) = \cosh x$ on $[-\ln 2, \ln 2]$.

9. $f(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$ on $[0, \ln 5]$.

10. $f(x) = \frac{1}{12}x^5 + \frac{1}{5x^3}$ on $[.1, 1]$.

11. $f(x) = \ln\left(\sin x\right)$ on $[\pi/6, \pi/2]$.

12. $f(x) = \ln\left(\cos x\right)$ on $[0, \pi/4]$.

In Exercises 13 – 20, set up the integral to compute the arc length of the function on the given interval. Do not evaluate the integral.

13. $f(x) = x^2$ on $[0, 1]$.

14. $f(x) = x^{10}$ on $[0, 1]$.

15. $f(x) = \sqrt{x}$ on $[0, 1]$.

16. $f(x) = \ln x$ on $[1, e]$.

17. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$. (Note: this describes the top half of a circle with radius 1.)

18. $f(x) = \sqrt{1 - x^2/9}$ on $[-3, 3]$. (Note: this describes the top half of an ellipse with a major axis of length 6 and a minor axis of length 2.)

19. $f(x) = \frac{1}{x}$ on $[1, 2]$.

20. $f(x) = \sec x$ on $[-\pi/4, \pi/4]$.

In Exercises 21 – 28, use Simpson's Rule, with $n = 4$, to approximate the arc length of the function on the given interval. Note: these are the same problems as in Exercises 13–20.

21. $f(x) = x^2$ on $[0, 1]$.

22. $f(x) = x^{10}$ on $[0, 1]$.

23. $f(x) = \sqrt{x}$ on $[0, 1]$. (Note: $f'(x)$ is not defined at $x = 0$.)

24. $f(x) = \ln x$ on $[1, e]$.

25. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$. (Note: $f'(x)$ is not defined at the endpoints.)

26. $f(x) = \sqrt{1 - x^2/9}$ on $[-3, 3]$. (Note: $f'(x)$ is not defined at the endpoints.)

27. $f(x) = \frac{1}{x}$ on $[1, 2]$.

28. $f(x) = \sec x$ on $[-\pi/4, \pi/4]$.

In Exercises 29 – 33, find the surface area of the described solid of revolution.

29. The solid formed by revolving $y = 2x$ on $[0, 1]$ about the x-axis.

30. The solid formed by revolving $y = x^2$ on $[0, 1]$ about the y-axis.

31. The solid formed by revolving $y = x^3$ on $[0, 1]$ about the x-axis.

32. The solid formed by revolving $y = \sqrt{x}$ on $[0, 1]$ about the x-axis.

33. The sphere formed by revolving $y = \sqrt{1 - x^2}$ on $[-1, 1]$ about the x-axis.

7.5 Work

Work is the scientific term used to describe the action of a force which moves an object. When a constant force F is applied to move an object a distance d, the amount of work performed is $W = F \cdot d$.

Note: *Mass* and *weight* are closely related, yet different, concepts. The mass m of an object is a quantitative measure of that object's resistance to acceleration. The weight w of an object is a measurement of the force applied to the object by the acceleration of gravity g.

Since the two measurements are proportional, $w = m \cdot g$, they are often used interchangeably in everyday conversation. When computing work, one must be careful to note which is being referred to. When mass is given, it must be multiplied by the acceleration of gravity to reference the related force.

The SI unit of force is the Newton, (kg·m/s^2), and the SI unit of distance is a meter (m). The fundamental unit of work is one Newton–meter, or a joule (J). That is, applying a force of one Newton for one meter performs one joule of work. In Imperial units (as used in the United States), force is measured in pounds (lb) and distance is measured in feet (ft), hence work is measured in ft–lb.

When force is constant, the measurement of work is straightforward. For instance, lifting a 200 lb object 5 ft performs $200 \cdot 5 = 1000$ ft–lb of work.

What if the force applied is variable? For instance, imagine a climber pulling a 200 ft rope up a vertical face. The rope becomes lighter as more is pulled in, requiring less force and hence the climber performs less work.

In general, let $F(x)$ be a force function on an interval $[a, b]$. We want to measure the amount of work done applying the force F from $x = a$ to $x = b$. We can approximate the amount of work being done by partitioning $[a, b]$ into subintervals $a = x_1 < x_2 < \cdots < x_{n+1} = b$ and assuming that F is constant on each subinterval. Let c_i be a value in the i^{th} subinterval $[x_i, x_{i+1}]$. Then the work done on this interval is approximately $W_i \approx F(c_i) \cdot (x_{i+1} - x_i) = F(c_i)\Delta x_i$, a constant force $\times$ the distance over which it is applied. The total work is

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(c_i)\Delta x_i.$$

This, of course, is a Riemann sum. Taking a limit as the subinterval lengths go to zero give an exact value of work which can be evaluated through a definite integral.

Key Idea 7.5.1 Work

Let $F(x)$ be a continuous function on $[a, b]$ describing the amount of force being applied to an object in the direction of travel from distance $x = a$ to distance $x = b$. The total work W done on $[a, b]$ is

$$W = \int_a^b F(x)\, dx.$$

Example 7.5.1 Computing work performed: applying variable force

A 60m climbing rope is hanging over the side of a tall cliff. How much work is performed in pulling the rope up to the top, where the rope has a mass of 66g/m?

Solution We need to create a force function $F(x)$ on the interval $[0, 60]$. To do so, we must first decide what x is measuring: it is the length of the rope still hanging or is it the amount of rope pulled in? As long as we are consistent, either approach is fine. We adopt for this example the convention that x is the amount of rope pulled in. This seems to match intuition better; pulling up the first 10 meters of rope involves $x = 0$ to $x = 10$ instead of $x = 60$ to $x = 50$.

As x is the amount of rope pulled in, the amount of rope still hanging is $60-x$. This length of rope has a mass of 66 g/m, or 0.066 kg/m. The mass of the rope still hanging is $0.066(60 - x)$ kg; multiplying this mass by the acceleration of gravity, 9.8 m/s^2, gives our variable force function

$$F(x) = (9.8)(0.066)(60 - x) = 0.6468(60 - x).$$

Thus the total work performed in pulling up the rope is

$$W = \int_0^{60} 0.6468(60 - x)\, dx = 1,164.24 \text{ J}.$$

By comparison, consider the work done in lifting the entire rope 60 meters. The rope weighs $60 \times 0.066 \times 9.8 = 38.808$ N, so the work applying this force for 60 meters is $60 \times 38.808 = 2,328.48$ J. This is exactly twice the work calculated before (and we leave it to the reader to understand why.)

Example 7.5.2 Computing work performed: applying variable force

Consider again pulling a 60 m rope up a cliff face, where the rope has a mass of 66 g/m. At what point is exactly half the work performed?

Solution From Example 7.5.1 we know the total work performed is $1,164.24$ J. We want to find a height h such that the work in pulling the rope from a height of $x = 0$ to a height of $x = h$ is 582.12, half the total work. Thus we want to solve the equation

$$\int_0^{h} 0.6468(60 - x)\, dx = 582.12$$

for h.

$$\begin{aligned} \int_0^{h} 0.6468(60 - x)\, dx &= 582.12 \\ \left(38.808x - 0.3234x^2\right)\Big|_0^h &= 582.12 \\ 38.808h - 0.3234h^2 &= 582.12 \\ -0.3234h^2 + 38.808h - 582.12 &= 0. \end{aligned}$$

Note: In Example 7.5.2, we find that half of the work performed in pulling up a 60 m rope is done in the last 42.43 m. Why is it not coincidental that $60/\sqrt{2} = 42.43$?

Apply the Quadratic Formula.

$$h = 17.57 \text{ and } 102.43$$

As the rope is only 60 m long, the only sensible answer is $h = 17.57$. Thus about half the work is done pulling up the first 17.5 m the other half of the work is done pulling up the remaining 42.43 m.

Example 7.5.3 Computing work performed: applying variable force

A box of 100 lb of sand is being pulled up at a uniform rate a distance of 50 ft over 1 minute. The sand is leaking from the box at a rate of 1 lb/s. The box itself weighs 5 lb and is pulled by a rope weighing .2 lb/ft.

1. How much work is done lifting just the rope?
2. How much work is done lifting just the box and sand?
3. What is the total amount of work performed?

SOLUTION

1. We start by forming the force function $F_r(x)$ for the rope (where the subscript denotes we are considering the rope). As in the previous example, let x denote the amount of rope, in feet, pulled in. (This is the same as saying x denotes the height of the box.) The weight of the rope with x feet pulled in is $F_r(x) = 0.2(50-x) = 10-0.2x$. (Note that we do not have to include the acceleration of gravity here, for the *weight* of the rope per foot is given, not its *mass* per meter as before.) The work performed lifting the rope is

$$W_r = \int_0^{50} (10-0.2x)\,dx = 250 \text{ ft–lb}.$$

2. The sand is leaving the box at a rate of 1 lb/s. As the vertical trip is to take one minute, we know that 60 lb will have left when the box reaches its final height of 50 ft. Again letting x represent the height of the box, we have two points on the line that describes the weight of the sand: when $x=0$, the sand weight is 100 lb, producing the point $(0,100)$; when $x=50$, the sand in the box weighs 40 lb, producing the point $(50,40)$. The slope of this line is $\frac{100-40}{0-50} = -1.2$, giving the equation of the weight of the sand at height x as $w(x) = -1.2x+100$. The box itself weighs a constant 5 lb, so the total force function is $F_b(x) = -1.2x+105$. Integrating from $x=0$ to $x=50$ gives the work performed in lifting box and sand:

$$W_b = \int_0^{50} (-1.2x+105)\,dx = 3750 \text{ ft–lb}.$$

3. The total work is the sum of W_r and W_b: $250+3750 = 4000$ ft–lb. We can also arrive at this via integration:

$$\begin{aligned} W &= \int_0^{50} \big(F_r(x)+F_b(x)\big)\,dx \\ &= \int_0^{50} (10-0.2x-1.2x+105)\,dx \\ &= \int_0^{50} (-1.4x+115)\,dx \\ &= 4000 \text{ ft–lb}. \end{aligned}$$

Hooke's Law and Springs

Hooke's Law states that the force required to compress or stretch a spring x units from its natural length is proportional to x; that is, this force is $F(x) = kx$ for some constant k. For example, if a force of 1 N stretches a given spring 2 cm, then a force of 5 N will stretch the spring 10 cm. Converting the distances to meters, we have that stretching this spring 0.02 m requires a force of $F(0.02) = k(0.02) = 1$ N, hence $k = 1/0.02 = 50$ N/m.

Example 7.5.4 Computing work performed: stretching a spring

A force of 20 lb stretches a spring from a natural length of 7 inches to a length of 12 inches. How much work was performed in stretching the spring to this length?

Solution In many ways, we are not at all concerned with the actual length of the spring, only with the amount of its change. Hence, we do not care that 20 lb of force stretches the spring to a length of 12 inches, but rather that a force of 20 lb stretches the spring by 5 in. This is illustrated in Figure 7.5.1; we only measure the change in the spring's length, not the overall length of the spring.

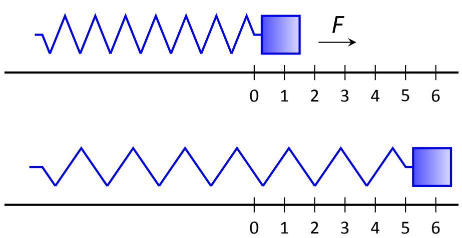


Figure 7.5.1: Illustrating the important aspects of stretching a spring in computing work in Example 7.5.4.

Converting the units of length to feet, we have

$$F(5/12) = 5/12k = 20 \text{ lb}.$$

Thus $k = 48$ lb/ft and $F(x) = 48x$.

We compute the total work performed by integrating $F(x)$ from $x = 0$ to $x = 5/12$:

$$\begin{aligned} W &= \int_0^{5/12} 48x\, dx \\ &= 24x^2\Big|_0^{5/12} \\ &= 25/6 \approx 4.1667 \text{ ft–lb}. \end{aligned}$$

Fluid	lb/ft^3	kg/m^3
Concrete	150	2400
Fuel Oil	55.46	890.13
Gasoline	45.93	737.22
Iodine	307	4927
Methanol	49.3	791.3
Mercury	844	13546
Milk	63.6–65.4	1020 – 1050
Water	62.4	1000

Figure 7.5.2: Weight and Mass densities

Pumping Fluids

Another useful example of the application of integration to compute work comes in the pumping of fluids, often illustrated in the context of emptying a storage tank by pumping the fluid out the top. This situation is different than our previous examples for the forces involved are constant. After all, the force required to move one cubic foot of water (about 62.4 lb) is the same regardless of its location in the tank. What is variable is the distance that cubic foot of water has to travel; water closer to the top travels less distance than water at the bottom, producing less work.

We demonstrate how to compute the total work done in pumping a fluid out of the top of a tank in the next two examples.

Example 7.5.5 Computing work performed: pumping fluids

A cylindrical storage tank with a radius of 10 ft and a height of 30 ft is filled with water, which weighs approximately 62.4 lb/ft^3. Compute the amount of work performed by pumping the water up to a point 5 feet above the top of the tank.

Solution We will refer often to Figure 7.5.3 which illustrates the salient aspects of this problem.

We start as we often do: we partition an interval into subintervals. We orient our tank vertically since this makes intuitive sense with the base of the tank at $y = 0$. Hence the top of the water is at $y = 30$, meaning we are interested in subdividing the y-interval $[0, 30]$ into n subintervals as

$$0 = y_1 < y_2 < \cdots < y_{n+1} = 30.$$

Consider the work W_i of pumping only the water residing in the i^{th} subinterval, illustrated in Figure 7.5.3. The force required to move this water is equal to its weight which we calculate as volume $\times$ density. The volume of water in this subinterval is $V_i = 10^2\pi\Delta y_i$; its density is 62.4 lb/ft^3. Thus the required force is $6240\pi\Delta y_i$ lb.

We approximate the distance the force is applied by using any y-value contained in the i^{th} subinterval; for simplicity, we arbitrarily use y_i for now (it will not matter later on). The water will be pumped to a point 5 feet above the top of the tank, that is, to the height of $y = 35$ ft. Thus the distance the water at height y_i travels is $35 - y_i$ ft.

In all, the approximate work W_i performed in moving the water in the i^{th} subinterval to a point 5 feet above the tank is

$$W_i \approx 6240\pi\Delta y_i(35 - y_i).$$

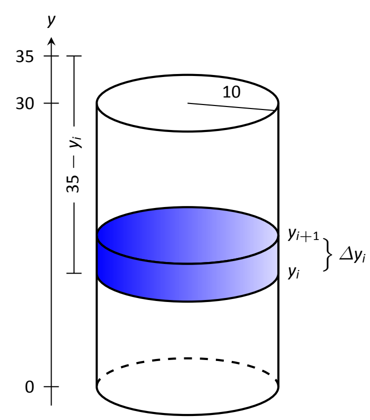


Figure 7.5.3: Illustrating a water tank in order to compute the work required to empty it in Example 7.5.5.

To approximate the total work performed in pumping out all the water from the tank, we sum all the work W_i performed in pumping the water from each of the n subintervals of $[0, 30]$:

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} 6240\pi\Delta y_i(35 - y_i).$$

This is a Riemann sum. Taking the limit as the subinterval length goes to 0 gives

$$\begin{aligned} W &= \int_0^{30} 6240\pi(35 - y)\,dy \\ &= (6240\pi\left(35y - 1/2y^2\right)\Big|_0^{30} \\ &= 11,762,123 \text{ ft–lb} \\ &\approx 1.176 \times 10^7 \text{ ft–lb}. \end{aligned}$$

We can "streamline" the above process a bit as we may now recognize what the important features of the problem are. Figure 7.5.4 shows the tank from Example 7.5.5 without the i^{th} subinterval identified. Instead, we just draw one differential element. This helps establish the height a small amount of water must travel along with the force required to move it (where the force is volume $\times$ density).

We demonstrate the concepts again in the next examples.

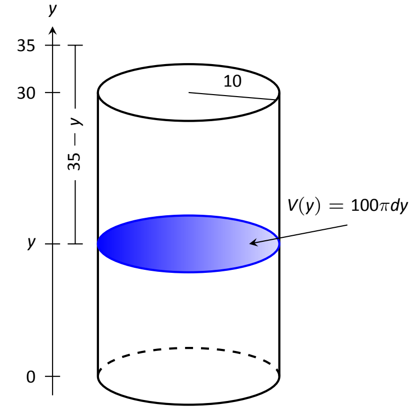


Figure 7.5.4: A simplified illustration for computing work.

Example 7.5.6 Computing work performed: pumping fluids

A conical water tank has its top at ground level and its base 10 feet below ground. The radius of the cone at ground level is 2 ft. It is filled with water weighing 62.4 lb/ft^3 and is to be emptied by pumping the water to a spigot 3 feet above ground level. Find the total amount of work performed in emptying the tank.

Solution The conical tank is sketched in Figure 7.5.5. We can orient the tank in a variety of ways; we could let $y = 0$ represent the base of the tank and $y = 10$ represent the top of the tank, but we choose to keep the convention of the wording given in the problem and let $y = 0$ represent ground level and hence $y = -10$ represents the bottom of the tank. The actual "height" of the water does not matter; rather, we are concerned with the distance the water travels.

The figure also sketches a differential element, a cross–sectional circle. The radius of this circle is variable, depending on y. When $y = -10$, the circle has radius 0; when $y = 0$, the circle has radius 2. These two points, $(-10, 0)$ and $(0, 2)$, allow us to find the equation of the line that gives the radius of the cross–sectional circle, which is $r(y) = 1/5y + 2$. Hence the volume of water at this height is $V(y) = \pi(1/5y + 2)^2 dy$, where dy represents a very small height of the differential element. The force required to move the water at height y is $F(y) = 62.4 \times V(y)$.

The distance the water at height y travels is given by $h(y) = 3 - y$. Thus the total work done in pumping the water from the tank is

$$\begin{aligned}
W &= \int_{-10}^{0} 62.4\pi(1/5y + 2)^2(3 - y)\, dy \\
&= 62.4\pi \int_{-10}^{0} \left(-\frac{1}{25}y^3 - \frac{17}{25}y^2 - \frac{8}{5}y + 12\right)\, dy \\
&= 62.2\pi \cdot \frac{220}{3} \approx 14,376 \text{ ft–lb}.
\end{aligned}$$

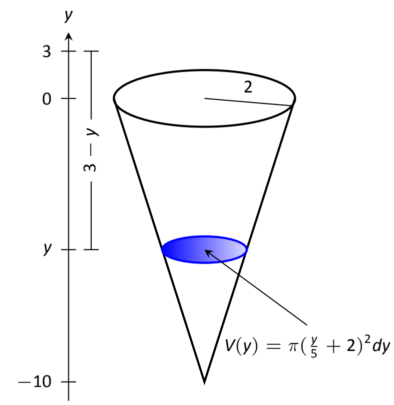


Figure 7.5.5: A graph of the conical water tank in Example 7.5.6.

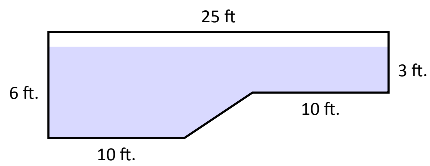


Figure 7.5.6: The cross–section of a swimming pool filled with water in Example 7.5.7.

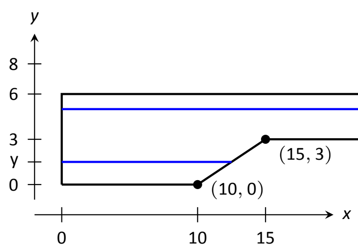


Figure 7.5.7: Orienting the pool and showing differential elements for Example 7.5.7.

Example 7.5.7 Computing work performed: pumping fluids

A rectangular swimming pool is 20 ft wide and has a 3 ft "shallow end" and a 6 ft "deep end." It is to have its water pumped out to a point 2 ft above the current top of the water. The cross–sectional dimensions of the water in the pool are given in Figure 7.5.6; note that the dimensions are for the water, not the pool itself. Compute the amount of work performed in draining the pool.

Solution For the purposes of this problem we choose to set $y = 0$ to represent the bottom of the pool, meaning the top of the water is at $y = 6$. Figure 7.5.7 shows the pool oriented with this y-axis, along with 2 differential elements as the pool must be split into two different regions.

The top region lies in the y-interval of $[3, 6]$, where the length of the differential element is 25 ft as shown. As the pool is 20 ft wide, this differential element represents a thin slice of water with volume $V(y) = 20 \cdot 25 \cdot dy$. The water is to be pumped to a height of $y = 8$, so the height function is $h(y) = 8 - y$. The work done in pumping this top region of water is

$$W_t = 62.4 \int_3^6 500(8 - y)\,dy = 327,600 \text{ ft–lb}.$$

The bottom region lies in the y-interval of $[0, 3]$; we need to compute the length of the differential element in this interval.

One end of the differential element is at $x = 0$ and the other is along the line segment joining the points $(10, 0)$ and $(15, 3)$. The equation of this line is $y = 3/5(x - 10)$; as we will be integrating with respect to y, we rewrite this equation as $x = 5/3y + 10$. So the length of the differential element is a difference of x-values: $x = 0$ and $x = 5/3y + 10$, giving a length of $x = 5/3y + 10$.

Again, as the pool is 20 ft wide, this differential element represents a thin slice of water with volume $V(y) = 20 \cdot (5/3y + 10) \cdot dy$; the height function is the same as before at $h(y) = 8 - y$. The work performed in emptying this part of the pool is

$$W_b = 62.4 \int_0^3 20(5/3y + 10)(8 - y)\,dy = 299,520 \text{ ft–lb}.$$

The total work in emptying the pool is

$$W = W_b + W_t = 327,600 + 299,520 = 627,120 \text{ ft–lb}.$$

Notice how the emptying of the bottom of the pool performs almost as much work as emptying the top. The top portion travels a shorter distance but has more water. In the end, this extra water produces more work.

The next section introduces one final application of the definite integral, the calculation of fluid force on a plate.

Exercises 7.5

Terms and Concepts

1. What are the typical units of work?

2. If a man has a mass of 80 kg on Earth, will his mass on the moon be bigger, smaller, or the same?

3. If a woman weighs 130 lb on Earth, will her weight on the moon be bigger, smaller, or the same?

4. Fill in the blanks:

   Some integrals in this section are set up by multiplying a variable ________ by a constant distance; others are set up by multiplying a constant force by a variable ________.

Problems

5. A 100 ft rope, weighing 0.1 lb/ft, hangs over the edge of a tall building.
   (a) How much work is done pulling the entire rope to the top of the building?
   (b) How much rope is pulled in when half of the total work is done?

6. A 50 m rope, with a mass density of 0.2 kg/m, hangs over the edge of a tall building.
   (a) How much work is done pulling the entire rope to the top of the building?
   (b) How much work is done pulling in the first 20 m?

7. A rope of length ℓ ft hangs over the edge of tall cliff. (Assume the cliff is taller than the length of the rope.) The rope has a weight density of d lb/ft.
   (a) How much work is done pulling the entire rope to the top of the cliff?
   (b) What percentage of the total work is done pulling in the first half of the rope?
   (c) How much rope is pulled in when half of the total work is done?

8. A 20 m rope with mass density of 0.5 kg/m hangs over the edge of a 10 m building. How much work is done pulling the rope to the top?

9. A crane lifts a 2,000 lb load vertically 30 ft with a 1” cable weighing 1.68 lb/ft.
   (a) How much work is done lifting the cable alone?
   (b) How much work is done lifting the load alone?
   (c) Could one conclude that the work done lifting the cable is negligible compared to the work done lifting the load?

10. A 100 lb bag of sand is lifted uniformly 120 ft in one minute. Sand leaks from the bag at a rate of 1/4 lb/s. What is the total work done in lifting the bag?

11. A box weighing 2 lb lifts 10 lb of sand vertically 50 ft. A crack in the box allows the sand to leak out such that 9 lb of sand is in the box at the end of the trip. Assume the sand leaked out at a uniform rate. What is the total work done in lifting the box and sand?

12. A force of 1000 lb compresses a spring 3 in. How much work is performed in compressing the spring?

13. A force of 2 N stretches a spring 5 cm. How much work is performed in stretching the spring?

14. A force of 50 lb compresses a spring from a natural length of 18 in to 12 in. How much work is performed in compressing the spring?

15. A force of 20 lb stretches a spring from a natural length of 6 in to 8 in. How much work is performed in stretching the spring?

16. A force of 7 N stretches a spring from a natural length of 11 cm to 21 cm. How much work is performed in stretching the spring from a length of 16 cm to 21 cm?

17. A force of f N stretches a spring d m from its natural length. How much work is performed in stretching the spring?

18. A 20 lb weight is attached to a spring. The weight rests on the spring, compressing the spring from a natural length of 1 ft to 6 in.

    How much work is done in lifting the box 1.5 ft (i.e, the spring will be stretched 1 ft beyond its natural length)?

19. A 20 lb weight is attached to a spring. The weight rests on the spring, compressing the spring from a natural length of 1 ft to 6 in.

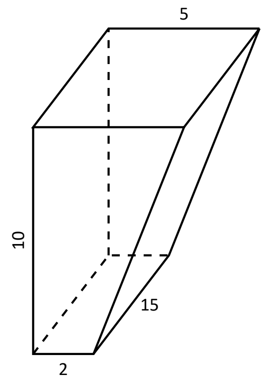
    How much work is done in lifting the box 6 in (i.e, bringing the spring back to its natural length)?

20. A 5 m tall cylindrical tank with radius of 2 m is filled with 3 m of gasoline, with a mass density of 737.22 kg/m^3. Compute the total work performed in pumping all the gasoline to the top of the tank.

21. A 6 ft cylindrical tank with a radius of 3 ft is filled with water, which has a weight density of 62.4 lb/ft^3. The water is to be pumped to a point 2 ft above the top of the tank.
    (a) How much work is performed in pumping all the water from the tank?
    (b) How much work is performed in pumping 3 ft of water from the tank?
    (c) At what point is 1/2 of the total work done?

22. A gasoline tanker is filled with gasoline with a weight density of 45.93 lb/ft^3. The dispensing valve at the base is jammed shut, forcing the operator to empty the tank via pumping the gas to a point 1 ft above the top of the tank. Assume the tank is a perfect cylinder, 20 ft long with a diameter of 7.5 ft. How much work is performed in pumping all the gasoline from the tank?

23. A fuel oil storage tank is 10 ft deep with trapezoidal sides, 5 ft at the top and 2 ft at the bottom, and is 15 ft wide (see diagram below). Given that fuel oil weighs 55.46 lb/ft^3, find the work performed in pumping all the oil from the tank to a point 3 ft above the top of the tank.



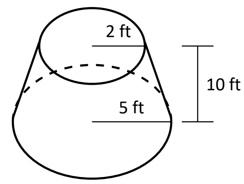
24. A conical water tank is 5 m deep with a top radius of 3 m. (This is similar to Example 7.5.6.) The tank is filled with pure water, with a mass density of 1000 kg/m^3.

    (a) Find the work performed in pumping all the water to the top of the tank.

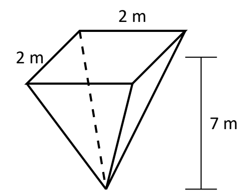
    (b) Find the work performed in pumping the top 2.5 m of water to the top of the tank.

    (c) Find the work performed in pumping the top half of the water, by volume, to the top of the tank.

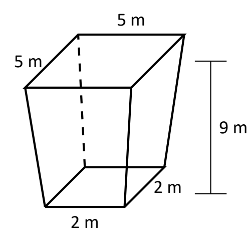
25. A water tank has the shape of a truncated cone, with dimensions given below, and is filled with water with a weight density of 62.4 lb/ft^3. Find the work performed in pumping all water to a point 1 ft above the top of the tank.



26. A water tank has the shape of an inverted pyramid, with dimensions given below, and is filled with water with a mass density of 1000 kg/m^3. Find the work performed in pumping all water to a point 5 m above the top of the tank.



27. A water tank has the shape of an truncated, inverted pyramid, with dimensions given below, and is filled with water with a mass density of 1000 kg/m^3. Find the work performed in pumping all water to a point 1 m above the top of the tank.



7.6 Fluid Forces

In the unfortunate situation of a car driving into a body of water, the conventional wisdom is that the water pressure on the doors will quickly be so great that they will be effectively unopenable. (Survival techniques suggest immediately opening the door, rolling down or breaking the window, or waiting until the water fills up the interior at which point the pressure is equalized and the door will open. See Mythbusters episode #72 to watch Adam Savage test these options.)

How can this be true? How much force does it take to open the door of a submerged car? In this section we will find the answer to this question by examining the forces exerted by fluids.

We start with **pressure**, which is related to **force** by the following equations:

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} \quad \Leftrightarrow \quad \text{Force} = \text{Pressure} \times \text{Area}.$$

In the context of fluids, we have the following definition.

Definition 7.6.1 Fluid Pressure

Let w be the weight–density of a fluid. The **pressure** p exerted on an object at depth d in the fluid is $p = w \cdot d$.

We use this definition to find the **force** exerted on a horizontal sheet by considering the sheet's area.

Example 7.6.1 Computing fluid force

1. A cylindrical storage tank has a radius of 2 ft and holds 10 ft of a fluid with a weight–density of 50 lb/ft^3. (See Figure 7.6.1(a).) What is the force exerted on the base of the cylinder by the fluid?
2. A rectangular tank whose base is a 5 ft square has a circular hatch at the bottom with a radius of 2 ft. The tank holds 10 ft of a fluid with a weight–density of 50 lb/ft^3. (See Figure 7.6.1(b).) What is the force exerted on the hatch by the fluid?

SOLUTION

1. Using Definition 7.6.1, we calculate that the pressure exerted on the cylinder's base is $w \cdot d = 50$ lb/ft$^3 \times 10$ ft $= 500$ lb/ft^2. The area of the base is $\pi \cdot 2^2 = 4\pi$ ft^2. So the force exerted by the fluid is
$$F = 500 \times 4\pi = 6283 \text{ lb}.$$
Note that we effectively just computed the *weight* of the fluid in the tank.

2. The dimensions of the tank in this problem are irrelevant. All we are concerned with are the dimensions of the hatch and the depth of the fluid. Since the dimensions of the hatch are the same as the base of the tank in the previous part of this example, as is the depth, we see that the fluid force is the same. That is, $F = 6283$ lb.

   A key concept to understand here is that we are effectively measuring the weight of a 10 ft column of water above the hatch. The size of the tank holding the fluid does not matter.

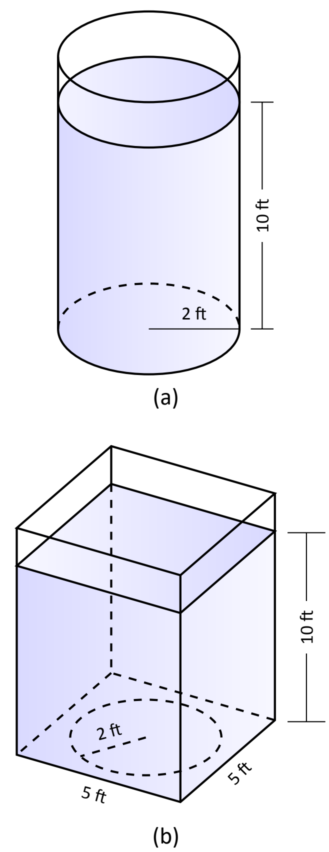


Figure 7.6.1: The cylindrical and rectangular tank in Example 7.6.1.

The previous example demonstrates that computing the force exerted on a horizontally oriented plate is relatively easy to compute. What about a vertically oriented plate? For instance, suppose we have a circular porthole located on the side of a submarine. How do we compute the fluid force exerted on it?

Pascal's Principle states that the pressure exerted by a fluid at a depth is equal in all directions. Thus the pressure on any portion of a plate that is 1 ft below the surface of water is the same no matter how the plate is oriented. (Thus a hollow cube submerged at a great depth will not simply be "crushed" from above, but the sides will also crumple in. The fluid will exert force on *all* sides of the cube.)

So consider a vertically oriented plate as shown in Figure 7.6.2 submerged in a fluid with weight–density w. What is the total fluid force exerted on this plate? We find this force by first approximating the force on small horizontal strips.

Let the top of the plate be at depth b and let the bottom be at depth a. (For now we assume that surface of the fluid is at depth 0, so if the bottom of the plate is 3 ft under the surface, we have $a = -3$. We will come back to this later.) We partition the interval $[a, b]$ into n subintervals

$$a = y_1 < y_2 < \cdots < y_{n+1} = b,$$

with the i^{th} subinterval having length Δy_i. The force F_i exerted on the plate in the i^{th} subinterval is $F_i = \text{Pressure} \times \text{Area}$.

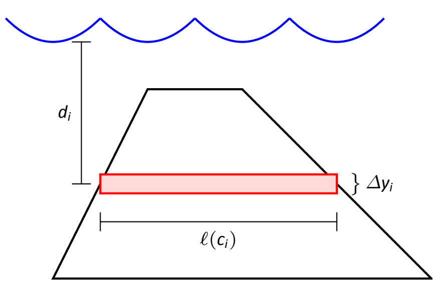


Figure 7.6.2: A thin, vertically oriented plate submerged in a fluid with weight–density w.

The pressure is depth $\times w$. We approximate the depth of this thin strip by choosing any value d_i in $[y_i, y_{i+1}]$; the depth is approximately $-d_i$. (Our convention has d_i being a negative number, so $-d_i$ is positive.) For convenience, we let d_i be an endpoint of the subinterval; we let $d_i = y_i$.

The area of the thin strip is approximately length $\times$ width. The width is Δy_i. The length is a function of some y-value c_i in the i^{th} subinterval. We state the length is $\ell(c_i)$. Thus

$$\begin{aligned} F_i &= \text{Pressure} \times \text{Area} \\ &= -y_i \cdot w \times \ell(c_i) \cdot \Delta y_i. \end{aligned}$$

To approximate the total force, we add up the approximate forces on each of the n thin strips:

$$F = \sum_{i=1}^{n} F_i \approx \sum_{i=1}^{n} -w \cdot y_i \cdot \ell(c_i) \cdot \Delta y_i.$$

This is, of course, another Riemann Sum. We can find the exact force by taking a limit as the subinterval lengths go to 0; we evaluate this limit with a definite integral.

Key Idea 7.6.1 Fluid Force on a Vertically Oriented Plate

Let a vertically oriented plate be submerged in a fluid with weight–density w where the top of the plate is at $y = b$ and the bottom is at $y = a$. Let $\ell(y)$ be the length of the plate at y.

1. If $y = 0$ corresponds to the surface of the fluid, then the force exerted on the plate by the fluid is
$$F = \int_a^b w \cdot (-y) \cdot \ell(y)\,dy.$$
2. In general, let $d(y)$ represent the distance between the surface of the fluid and the plate at y. Then the force exerted on the plate by the fluid is
$$F = \int_a^b w \cdot d(y) \cdot \ell(y)\,dy.$$

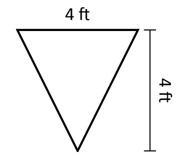


Figure 7.6.3: A thin plate in the shape of an isosceles triangle in Example 7.6.2.

Example 7.6.2 Finding fluid force

Consider a thin plate in the shape of an isosceles triangle as shown in Figure 7.6.3 submerged in water with a weight–density of 62.4 lb/ft^3. If the bottom of the plate is 10 ft below the surface of the water, what is the total fluid force exerted on this plate?

Solution We approach this problem in two different ways to illustrate the different ways Key Idea 7.6.1 can be implemented. First we will let $y = 0$ represent the surface of the water, then we will consider an alternate convention.

1. We let $y = 0$ represent the surface of the water; therefore the bottom of the plate is at $y = -10$. We center the triangle on the y-axis as shown in Figure 7.6.4. The depth of the plate at y is $-y$ as indicated by the Key Idea. We now consider the length of the plate at y.

   We need to find equations of the left and right edges of the plate. The right hand side is a line that connects the points $(0, -10)$ and $(2, -6)$: that line has equation $x = 1/2(y + 10)$. (Find the equation in the familiar $y = mx+b$ format and solve for x.) Likewise, the left hand side is described by the line $x = -1/2(y + 10)$. The total length is the distance between these two lines: $\ell(y) = 1/2(y + 10) - (-1/2(y + 10)) = y + 10$.

   The total fluid force is then:
$$\begin{aligned} F &= \int_{-10}^{-6} 62.4(-y)(y + 10)\,dy \\ &= 62.4 \cdot \frac{176}{3} \approx 3660.8 \text{ lb.} \end{aligned}$$

2. Sometimes it seems easier to orient the thin plate nearer the origin. For instance, consider the convention that the bottom of the triangular plate is at $(0, 0)$, as shown in Figure 7.6.5. The equations of the left and right hand sides are easy to find. They are $y = 2x$ and $y = -2x$, respectively, which we rewrite as $x = 1/2y$ and $x = -1/2y$. Thus the length function is $\ell(y) = 1/2y - (-1/2y) = y$.

   As the surface of the water is 10 ft above the base of the plate, we have that the surface of the water is at $y = 10$. Thus the depth function is the

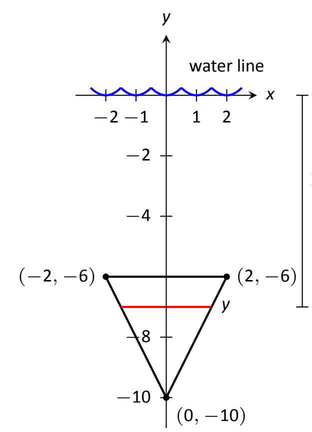


Figure 7.6.4: Sketching the triangular plate in Example 7.6.2 with the convention that the water level is at $y = 0$.

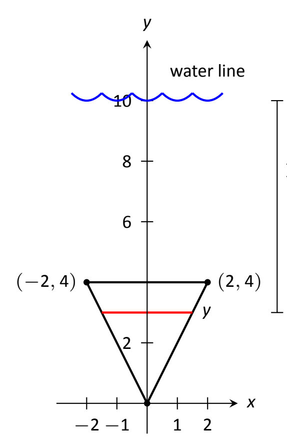


Figure 7.6.5: Sketching the triangular plate in Example 7.6.2 with the convention that the base of the triangle is at $(0, 0)$.

distance between $y = 10$ and y; $d(y) = 10 - y$. We compute the total fluid force as:

$$\begin{aligned} F &= \int_0^4 62.4(10-y)(y)\,dy \\ &\approx 3660.8 \text{ lb.} \end{aligned}$$

The correct answer is, of course, independent of the placement of the plate in the coordinate plane as long as we are consistent.

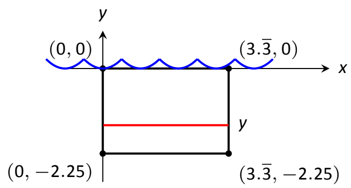


Figure 7.6.6: Sketching a submerged car door in Example 7.6.3.

Example 7.6.3 Finding fluid force

Find the total fluid force on a car door submerged up to the bottom of its window in water, where the car door is a rectangle 40" long and 27" high (based on the dimensions of a 2005 Fiat Grande Punto.)

Solution The car door, as a rectangle, is drawn in Figure 7.6.6. Its length is $10/3$ ft and its height is 2.25 ft. We adopt the convention that the top of the door is at the surface of the water, both of which are at $y = 0$. Using the weight–density of water of 62.4 lb/ft^3, we have the total force as

$$\begin{aligned} F &= \int_{-2.25}^{0} 62.4(-y)10/3\,dy \\ &= \int_{-2.25}^{0} -208y\,dy \\ &= -104y^2\Big|_{-2.25}^{0} \\ &= 526.5 \text{ lb.} \end{aligned}$$

Most adults would find it very difficult to apply over 500 lb of force to a car door while seated inside, making the door effectively impossible to open. This is counter–intuitive as most assume that the door would be relatively easy to open. The truth is that it is not, hence the survival tips mentioned at the beginning of this section.

Example 7.6.4 Finding fluid force

An underwater observation tower is being built with circular viewing portholes enabling visitors to see underwater life. Each vertically oriented porthole is to have a 3 ft diameter whose center is to be located 50 ft underwater. Find the total fluid force exerted on each porthole. Also, compute the fluid force on a horizontally oriented porthole that is under 50 ft of water.

SOLUTION We place the center of the porthole at the origin, meaning the surface of the water is at $y = 50$ and the depth function will be $d(y) = 50 - y$; see Figure 7.6.7

The equation of a circle with a radius of 1.5 is $x^2 + y^2 = 2.25$; solving for x we have $x = \pm\sqrt{2.25 - y^2}$, where the positive square root corresponds to the right side of the circle and the negative square root corresponds to the left side of the circle. Thus the length function at depth y is $\ell(y) = 2\sqrt{2.25 - y^2}$. Integrating on $[-1.5, 1.5]$ we have:

$$\begin{aligned} F &= 62.4 \int_{-1.5}^{1.5} 2(50 - y)\sqrt{2.25 - y^2}\, dy \\ &= 62.4 \int_{-1.5}^{1.5} \left(100\sqrt{2.25 - y^2} - 2y\sqrt{2.25 - y^2}\right) dy \\ &= 6240 \int_{-1.5}^{1.5} \left(\sqrt{2.25 - y^2}\right) dy - 62.4 \int_{-1.5}^{1.5} \left(2y\sqrt{2.25 - y^2}\right) dy. \end{aligned}$$

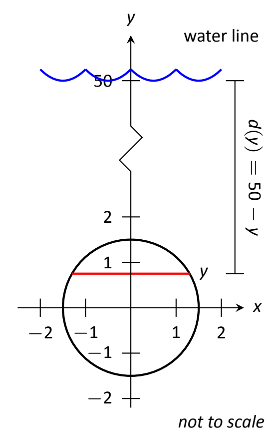


Figure 7.6.7: Measuring the fluid force on an underwater porthole in Example 7.6.4.

The second integral above can be evaluated using substitution. Let $u = 2.25 - y^2$ with $du = -2y\, dy$. The new bounds are: $u(-1.5) = 0$ and $u(1.5) = 0$; the new integral will integrate from $u = 0$ to $u = 0$, hence the integral is 0.

The first integral above finds the area of half a circle of radius 1.5, thus the first integral evaluates to $6240 \cdot \pi \cdot 1.5^2/2 = 22,054$. Thus the total fluid force on a vertically oriented porthole is 22,054 lb.

Finding the force on a horizontally oriented porthole is more straightforward:

$$F = \text{Pressure} \times \text{Area} = 62.4 \cdot 50 \times \pi \cdot 1.5^2 = 22,054 \text{ lb.}$$

That these two forces are equal is not coincidental; it turns out that the fluid force applied to a vertically oriented circle whose center is at depth d is the same as force applied to a horizontally oriented circle at depth d.

We end this chapter with a reminder of the true skills meant to be developed here. We are not truly concerned with an ability to find fluid forces or the volumes of solids of revolution. Work done by a variable force is important, though measuring the work done in pulling a rope up a cliff is probably not.

What we are actually concerned with is the ability to solve certain problems by first approximating the solution, then refining the approximation, then recognizing if/when this refining process results in a definite integral through a limit. Knowing the formulas found inside the special boxes within this chapter is beneficial as it helps solve problems found in the exercises, and other mathematical skills are strengthened by properly applying these formulas. However, more importantly, understand how each of these formulas was constructed. Each is the result of a summation of approximations; each summation was a Riemann sum, allowing us to take a limit and find the exact answer through a definite integral.

The next chapter addresses an entirely different topic: sequences and series. In short, a sequence is a list of numbers, where a series is the summation of a list of numbers. These seemingly–simple ideas lead to very powerful mathematics.

Exercises 7.6

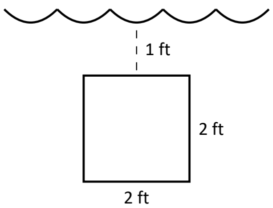
Terms and Concepts

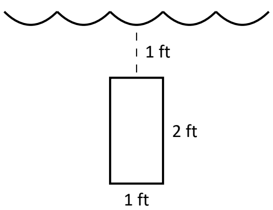
1. State in your own words Pascal's Principle.

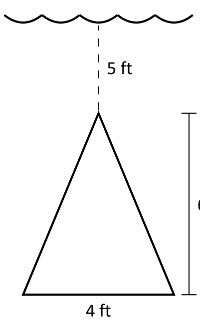
2. State in your own words how pressure is different from force.

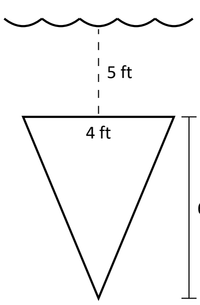
Problems

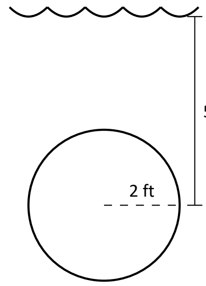
In Exercises 3 – 12, find the fluid force exerted on the given plate, submerged in water with a weight density of 62.4 lb/ft^3.

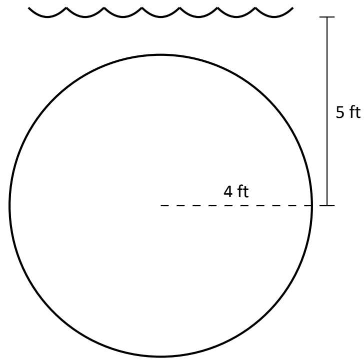
3. 

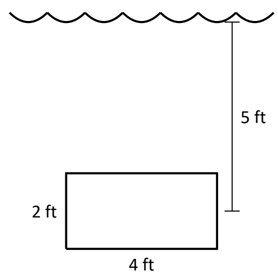
4. 

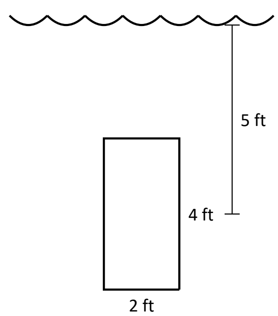
5. 

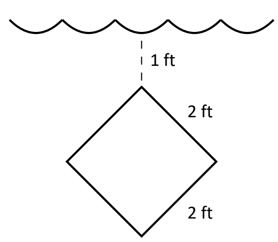
6. 

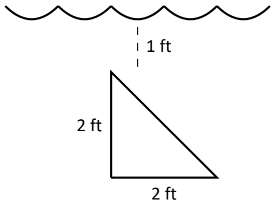
7. 

8. 

9. 

10. 

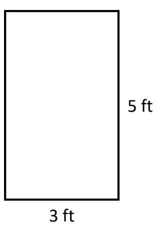
11. 

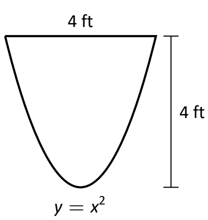
12. 

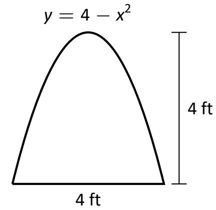
In Exercises 13 – 18, the side of a container is pictured. Find the fluid force exerted on this plate when the container is full of:

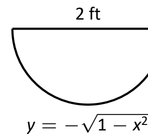
1. **water, with a weight density of 62.4 lb/ft^3, and**

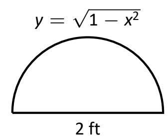
2. **concrete, with a weight density of 150 lb/ft^3.**

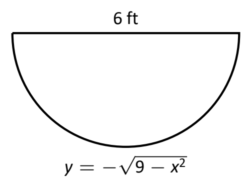
13. 

14. 

15. 

16. 

17. 

18. 

19. How deep must the center of a vertically oriented circular plate with a radius of 1 ft be submerged in water, with a weight density of 62.4 lb/ft^3, for the fluid force on the plate to reach 1,000 lb?

20. How deep must the center of a vertically oriented square plate with a side length of 2 ft be submerged in water, with a weight density of 62.4 lb/ft^3, for the fluid force on the plate to reach 1,000 lb?

8: Differential Equations

8.1 Introduction to differential equations

Differential equations

The laws of physics are generally written down as differential equations. Therefore, all of science and engineering use differential equations to some degree. Understanding differential equations is essential to understanding almost anything you will study in your science and engineering classes. You can think of mathematics as the language of science, and differential equations are one of the most important parts of this language as far as science and engineering are concerned. As an analogy, suppose all your classes from now on were given in Swahili. It would be important to first learn Swahili, or you would have a very tough time getting a good grade in your classes.

You have already seen many differential equations without perhaps knowing about it. And you have even solved simple differential equations when you were taking calculus. Let us see an example you may not have seen:

$$\frac{dx}{dt} + x = 2\cos t. \tag{8.1}$$

Here x is the **dependent variable** and t is the **independent variable**. Equation (8.1) is a basic example of a **differential equation**. In fact it is an example of a **first order differential equation**, since it involves only the first derivative of the dependent variable. This equation arises from Newton's law of cooling where the ambient temperature oscillates with time.

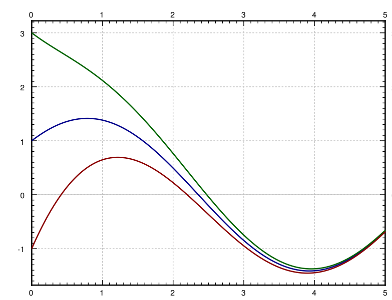


Figure 8.1.1: A few solutions of $\frac{dx}{dt} + x = 2\cos t$.

Solutions of differential equations

Solving the differential equation means finding x in terms of t. That is, we want to find a function of t, which we will call x, such that when we plug x, t, and $\frac{dx}{dt}$ into (8.1), the equation holds. It is the same idea as it would be for a normal (algebraic) equation of just x and t. We claim that

$$x = x(t) = \cos t + \sin t$$

is a **solution**. How do we check? We simply plug x into equation (8.1)! First we need to compute $\frac{dx}{dt}$. We find that $\frac{dx}{dt} = -\sin t + \cos t$. Now let us compute the left hand side of (8.1).

$$\frac{dx}{dt} + x = (-\sin t + \cos t) + (\cos t + \sin t) = 2\cos t.$$

Hooray! We got precisely the right hand side. But there is more! We claim $x = \cos t + \sin t + e^{-t}$ is also a solution. Let us try,

$$\frac{dx}{dt} = -\sin t + \cos t - e^{-t}.$$

Again plugging into the left hand side of (8.1)

$$\frac{dx}{dt} + x = (-\sin t + \cos t - e^{-t}) + (\cos t + \sin t + e^{-t}) = 2\cos t.$$

And it works yet again!

So there can be many different solutions. In fact, for this equation all solutions can be written in the form

$$x = \cos t + \sin t + Ce^{-t}$$

for some constant C. See 8.1.1 for the graph of a few of these solutions. We will see how we find these solutions a few lectures from now.

It turns out that solving differential equations can be quite hard. There is no general method that solves every differential equation. We will generally focus on how to get exact formulas for solutions of certain differential equations, but we will also spend a little bit of time on getting approximate solutions.

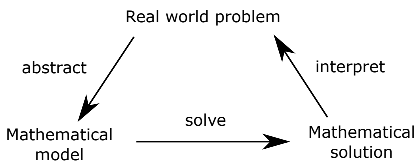


Figure 8.1.2: Mathematical modelling process

For most of the course we will look at **ordinary differential equation** (often abbreviated **ODEs**, by which we mean that there is only one independent variable and derivatives are only with respect to this one variable. If there are several independent variables, we will get **partial differential equations** or **PDEs**.

Even for ODEs, which are very well understood, it is not a simple question of turning a crank to get answers. It is important to know when it is easy to find solutions and how to do so. Although in real applications you will leave much of the actual calculations to computers, you need to understand what they are doing. It is often necessary to simplify or transform your equations into something that a computer can understand and solve. You may need to make certain assumptions and changes in your model to achieve this.

To be a successful engineer or scientist, you will be required to solve problems in your job that you have never seen before. It is important to learn problem solving techniques, so that you may apply those techniques to new problems. A common mistake is to expect to learn some prescription for solving all the problems you will encounter in your later career. This course is no exception.

Differential equations in practice

So how do we use differential equations in science and engineering? First, we have some **real world problem** we wish to understand. We make some simplifying assumptions and create a **mathematical model** That is, we translate the real world situation into a set of differential equations. Then we apply mathematics to get some sort of a **mathematical solution** to the model. There is still something left to do. We have to interpret the results. We have to figure out what the mathematical solution says about the real world problem we started with.

Learning how to formulate the mathematical model and how to interpret the results is what your physics and engineering classes do. In this course we will focus mostly on the mathematical analysis. Sometimes we will work with simple real world examples, so that we have some intuition and motivation about what we are doing.

Let us look at an example of this process. One of the most basic differential equations is the standard **exponential growth model** Let P denote the population of some bacteria on a Petri dish. We assume that there is enough food and enough space. Then the rate of growth of bacteria is proportional to the population—a large population grows quicker. Let t denote time (say in seconds) and P the population. Our model is

$$\frac{dP}{dt} = kP,$$

for some positive constant $k > 0$.

Example 8.1.1 Model for bacterial growth

Suppose there are 100 bacteria at time 0 and 200 bacteria 10 seconds later. How many bacteria will there be 1 minute from time 0 (in 60 seconds)?

Solution First we have to solve the equation. We claim that a solution is given by

$$P(t) = Ce^{kt},$$

where C is a constant. Let us try:

$$\frac{dP}{dt} = Cke^{kt} = kP.$$

And it really is a solution.

OK, so what now? We do not know C and we do not know k. But we know something. We know $P(0) = 100$, and we also know $P(10) = 200$. Let us plug these conditions in and see what happens.

$$100 = P(0) = Ce^{k0} = C,$$
$$200 = P(10) = 100\,e^{k10}.$$

Therefore, $2 = e^{10k}$ or $\frac{\ln 2}{10} = k \approx 0.069$. So we know that

$$P(t) = 100\,e^{(\ln 2)t/10} \approx 100\,e^{0.069t}.$$

At one minute, $t = 60$, the population is $P(60) = 6400$. See Figure 8.1.3.

Let us talk about the interpretation of the results. Does our solution mean that there must be exactly 6400 bacteria on the plate at 60s? No! We made assumptions that might not be true exactly, just approximately. If our assumptions are reasonable, then there will be approximately 6400 bacteria. Also, in real life P is a discrete quantity, not a real number. However, our model has no problem saying that for example at 61 seconds, $P(61) \approx 6859.35$.

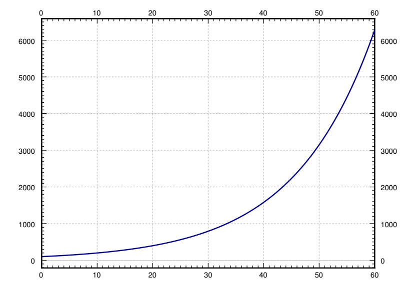


Figure 8.1.3: Bacteria growth in the first 60 seconds.

Normally, the k in $P' = kP$ is known, and we want to solve the equation for different **initial conditions**. What does that mean? Take $k = 1$ for simplicity. Now suppose we want to solve the equation $\frac{dP}{dt} = P$ subject to $P(0) = 1000$ (the initial condition). Then the solution turns out to be (exercise)

$$P(t) = 1000\,e^{t}.$$

We call $P(t) = Ce^{t}$ **general solution**, as every solution of the equation can be written in this form for some constant C. You will need an initial condition to find out what C is, in order to find the **particular solution** we are looking for. Generally, when we say "particular solution," we just mean some solution.

Let us get to what we will call the four fundamental equations. These equations appear very often and it is useful to just memorize what their solutions are. These solutions are reasonably easy to guess by recalling properties of exponentials, sines, and cosines. They are also simple to check, which is something that you should always do. There is no need to wonder if you have remembered the solution correctly.

First such equation is,

$$\frac{dy}{dx} = ky,$$

for some constant $k > 0$. Here y is the dependent and x the independent variable. The general solution for this equation is

$$y(x) = Ce^{kx}.$$

We have already seen that this function is a solution above with different variable names.

Next,

$$\frac{dy}{dx} = -ky,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = Ce^{-kx}.$$

Exercise: Check that the y given is really a solution to the equation.

Next, take the **second order differential equation**

$$\frac{d^2y}{dx^2} = -k^2y,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = C_1 \cos(kx) + C_2 \sin(kx).$$

Note that because we have a second order differential equation, we have two constants in our general solution.

Exercise: Check that the y given is really a solution to the equation.

And finally, take the second order differential equation

$$\frac{d^2y}{dx^2} = k^2y,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = C_1e^{kx} + C_2e^{-kx},$$

or

$$y(x) = D_1 \cosh(kx) + D_2 \sinh(kx).$$

For those that do not know, cosh and sinh are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2},$$
$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

These functions are sometimes easier to work with than exponentials. They have some nice familiar properties such as $\cosh 0 = 1$, $\sinh 0 = 0$, and $\frac{d}{dx} \cosh x = \sinh x$ (no that is not a typo) and $\frac{d}{dx} \sinh x = \cosh x$.

Exercise: Check that both forms of the y given are really solutions to the equation.

An interesting note about cosh: The graph of cosh is the exact shape a hanging chain will make. This shape is called a **catenary** Contrary to popular belief this is not a parabola. If you invert the graph of cosh it is also the ideal arch for supporting its own weight. For example, the gateway arch in Saint Louis is an inverted graph of cosh—if it were just a parabola it might fall down. The formula used in the design is inscribed inside the arch:

$$y = -127.7 \text{ ft} \cdot \cosh(x/127.7 \text{ ft}) + 757.7 \text{ ft}.$$

Exercises 8.1

Problems

1. Show that $x = e^{4t}$ is a solution to $x''' - 12x'' + 48x' - 64x = 0$.

2. Show that $x = e^{t}$ is not a solution to $x''' - 12x'' + 48x' - 64x = 0$.

3. Is $y = \sin t$ a solution to $\left(\frac{dy}{dt}\right)^2 = 1 - y^2$? Justify.

4. Let $y'' + 2y' - 8y = 0$. Now try a solution of the form $y = e^{rx}$ for some (unknown) constant r. Is this a solution for some r? If so, find all such r.

5. Verify that $x = Ce^{-2t}$ is a solution to $x' = -2x$. Find C to solve for the initial condition $x(0) = 100$.

6. Verify that $x = C_1e^{-t} + C_2e^{2t}$ is a solution to $x'' - x' - 2x = 0$. Find C_1 and C_2 to solve for the initial conditions $x(0) = 10$ and $x'(0) = 0$.

7. Find a solution to $(x')^2 + x^2 = 4$ using your knowledge of derivatives of functions that you know from basic calculus.

8. Solve:

   (a) $\frac{dA}{dt} = -10A,\ A(0) = 5$

   (b) $\frac{dH}{dx} = 3H,\ H(0) = 1$

   (c) $\frac{d^2y}{dx^2} = 4y,\ y(0) = 0,\ y'(0) = 1$

   (d) $\frac{d^2x}{dy^2} = -9x,\ x(0) = 1,\ x'(0) = 0$

9. Is there a solution to $y' = y$, such that $y(0) = y(1)$?

10. Show that $x = e^{-2t}$ is a solution to $x'' + 4x' + 4x = 0$.

11. Is $y = x^2$ a solution to $x^2y'' - 2y = 0$? Justify.

12. Let $xy'' - y' = 0$. Try a solution of the form $y = x^r$. Is this a solution for some r? If so, find all such r.

13. Verify that $x = C_1e^{t} + C_2$ is a solution to $x'' - x' = 0$. Find C_1 and C_2 so that x satisfies $x(0) = 10$ and $x'(0) = 100$.

14. Solve $\frac{d\varphi}{ds} = 8\varphi$ and $\varphi(0) = -9$.

8.2 Integrals as solutions

A first order ODE is an equation of the form

$$\frac{dy}{dx} = f(x, y),$$

or just

$$y' = f(x, y).$$

In general, there is no simple formula or procedure one can follow to find solutions. In the next few lectures we will look at special cases where solutions are not difficult to obtain. In this section, let us assume that f is a function of x alone, that is, the equation is

$$y' = f(x). \tag{8.2}$$

We could just integrate (antidifferentiate) both sides with respect to x.

$$\int y'(x)\, dx = \int f(x)\, dx + C,$$

that is

$$y(x) = \int f(x)\, dx + C.$$

This $y(x)$ is actually the general solution. So to solve (8.2), we find some antiderivative of $f(x)$ and then we add an arbitrary constant to get the general solution.

Now is a good time to discuss a point about calculus notation and terminology. Calculus textbooks muddy the waters by talking about the integral as primarily the so-called indefinite integral. The is really the **antiderivative** (in fact the whole one-parameter family of antiderivatives). There really exists only one integral and that is the definite integral. The only reason for the indefinite integral notation is that we can always write an antiderivative as a (definite) integral. That is, by the fundamental theorem of calculus we can always write $\int f(x)\, dx + C$ as

$$\int_{x_0}^{x} f(t)\, dt + C.$$

Hence the terminology *to integrate* when we may really mean *to antidifferentiate*. Integration is just one way to compute the antiderivative (and it is a way that always works, see the following examples). Integration is defined as the area under the graph, it only happens to also compute antiderivatives. For sake of consistency, we will keep using the indefinite integral notation when we want an antiderivative, and you should *always* think of the definite integral.

Example 8.2.1 Finding a general solution

Find the general solution of $y' = 3x^2$.

Solution Elementary calculus tells us that the general solution must be $y = x^3 + C$. Let us check by differentiating: $y' = 3x^2$. We have gotten *precisely* our equation back.

Normally, we also have an initial condition such as $y(x_0) = y_0$ for some two numbers x_0 and y_0 (x_0 is usually 0, but not always). We can then write the solution as a definite integral in a nice way. Suppose our problem is $y' = f(x)$, $y(x_0) = y_0$. Then the solution is

$$y(x) = \int_{x_0}^{x} f(s)\, ds + y_0. \tag{8.3}$$

Let us check! We compute $y' = f(x)$, via the fundamental theorem of calculus, and by Jupiter, y is a solution. Is it the one satisfying the initial condition? Well, $y(x_0) = \int_{x_0}^{x_0} f(x)\,dx + y_0 = y_0$. It is!

Do note that the definite integral and the indefinite integral (antidifferentiation) are completely different beasts. The definite integral always evaluates to a number. Therefore, (8.3) is a formula we can plug into the calculator or a computer, and it will be happy to calculate specific values for us. We will easily be able to plot the solution and work with it just like with any other function. It is not so crucial to always find a closed form for the antiderivative.

Example 8.2.2 **An ODE with no closed-form solution**

Solve

$$y' = e^{-x^2}, \qquad y(0) = 1.$$

Solution By the preceding discussion, the solution must be

$$y(x) = \int_0^x e^{-s^2}\,ds + 1.$$

Here is a good way to make fun of your friends taking second semester calculus. Tell them to find the closed form solution. Ha ha ha (bad math joke). It is not possible (in closed form). There is absolutely nothing wrong with writing the solution as a definite integral. This particular integral is in fact very important in statistics.

Using this method, we can also solve equations of the form

$$y' = f(y).$$

Let us write the equation in .

$$\frac{dy}{dx} = f(y).$$

Now we use the inverse function theorem from calculus to switch the roles of x and y to obtain

$$\frac{dx}{dy} = \frac{1}{f(y)}.$$

What we are doing seems like algebra with dx and dy. It is tempting to just do algebra with dx and dy as if they were numbers. And in this case it does work. Be careful, however, as this sort of hand-waving calculation can lead to trouble, especially when more than one independent variable is involved. At this point we can simply integrate,

$$x(y) = \int \frac{1}{f(y)}\,dy + C.$$

Finally, we try to solve for y.

Example 8.2.3 **Solving the exponential growth equation**

Previously, we guessed $y' = ky$ (for some $k > 0$) has the solution $y = Ce^{kx}$. We can now find the solution without guessing.

Solution First we note that $y = 0$ is a solution. Henceforth, we assume $y \neq 0$. We write

$$\frac{dx}{dy} = \frac{1}{ky}.$$

We integrate to obtain

$$x(y) = x = \frac{1}{k} \ln |y| + D,$$

where D is an arbitrary constant. Now we solve for y (actually for $|y|$).

$$|y| = e^{kx-kD} = e^{-kD}e^{kx}.$$

If we replace e^{-kD} with an arbitrary constant C we can get rid of the absolute value bars (which we can do as D was arbitrary). In this way, we also incorporate the solution $y = 0$. We get the same general solution as we guessed before, $y = Ce^{kx}$.

Example 8.2.4 Solving an ODE by integration

Find the general solution of $y' = y^2$.

Solution First we note that $y = 0$ is a solution. We can now assume that $y \neq 0$. Write

$$\frac{dx}{dy} = \frac{1}{y^2}.$$

We integrate to get

$$x = \frac{-1}{y} + C.$$

We solve for $y = \frac{1}{C-x}$. So the general solution is

$$y = \frac{1}{C-x} \qquad \text{or} \qquad y = 0.$$

Note the singularities of the solution. If for example $C = 1$, then the solution "blows up" as we approach $x = 1$. Generally, it is hard to tell from just looking at the equation itself how the solution is going to behave. The equation $y' = y^2$ is very nice and defined everywhere, but the solution is only defined on some interval $(-\infty, C)$ or (C, ∞).

Classical problems leading to differential equations solvable by integration are problems dealing with , and . You have surely seen these problems before in your calculus class.

Example 8.2.5 Finding the distance travelled

Suppose a car drives at a speed $e^{t/2}$ meters per second, where t is time in seconds. How far did the car get in 2 seconds (starting at $t = 0$)? How far in 10 seconds?

Solution Let x denote the distance the car travelled. The equation is

$$x' = e^{t/2}.$$

We can just integrate this equation to get that

$$x(t) = 2e^{t/2} + C.$$

We still need to figure out C. We know that when $t = 0$, then $x = 0$. That is, $x(0) = 0$. So

$$0 = x(0) = 2e^{0/2} + C = 2 + C.$$

Thus $C = -2$ and

$$x(t) = 2e^{t/2} - 2.$$

Now we just plug in to get where the car is at 2 and at 10 seconds. We obtain

$$x(2) = 2e^{2/2} - 2 \approx 3.44 \text{ meters}, \qquad x(10) = 2e^{10/2} - 2 \approx 294 \text{ meters}.$$

Example 8.2.6 Another car problem

Suppose that the car accelerates at a rate of t^2 m/s². At time $t = 0$ the car is at the 1 meter mark and is travelling at 10 m/s. Where is the car at time $t = 10$?

Solution Well this is actually a second order problem. If x is the distance travelled, then x' is the velocity, and x'' is the acceleration. The equation with initial conditions is

$$x'' = t^2, \qquad x(0) = 1, \qquad x'(0) = 10.$$

What if we say $x' = v$. Then we have the problem

$$v' = t^2, \qquad v(0) = 10.$$

Once we solve for v, we can integrate and find x.

Exercise: Solve for v, and then solve for x. Find $x(10)$ to answer the question.

Exercises 8.2

Problems

1. Solve $\frac{dy}{dx} = x^2 + x$ for $y(1) = 3$.

2. Solve $\frac{dy}{dx} = \sin(5x)$ for $y(0) = 2$.

3. Solve $\frac{dy}{dx} = \frac{1}{x^2 - 1}$ for $y(0) = 0$.

4. Solve $y' = y^3$ for $y(0) = 1$.

5. (A little harder) Solve $y' = (y - 1)(y + 1)$ for $y(0) = 3$.

6. Solve $\frac{dy}{dx} = \frac{1}{y + 1}$ for $y(0) = 0$.

7. (Harder) Solve $y'' = \sin x$ for $y(0) = 0$, $y'(0) = 2$.

8. A spaceship is travelling at the speed $2t^2 + 1$ km/s (t is time in seconds). It is pointing directly away from Earth and at time $t = 0$ it is 1000 kilometres from earth. How far from earth is it at one minute from time $t = 0$?

9. Solve $\frac{dx}{dt} = \sin(t^2) + t$, $x(0) = 20$. It is OK to leave your answer as a definite integral.

10. Solve $\frac{dy}{dx} = e^x + x$ and $y(0) = 10$.

11. Solve $x' = \frac{1}{x^2}$, $x(1) = 1$.

12. Solve $x' = \frac{1}{\cos(x)}$, $x(0) = \frac{\pi}{2}$.

13. Sid is in a car travelling at speed $10t + 70$ miles per hour away from Las Vegas, where t is in hours. At $t = 0$, Sid is 10 miles away from Vegas. How far from Vegas is Sid 2 hours later?

14. Solve $y' = y^n$, $y(0) = 1$, where n is a positive integer. Hint: You have to consider different cases.

8.3 Slope fields

Note: you might find the software *DFIELD* and *PPLANE* useful. You can download the programs at http://math.rice.edu/~dfield/dfpp.html. These used to be available as in-browser Java applets, but due to changes in Java security settings, you need to download the programs and run them locally. Both Java and MATLAB versions are available.

Another option is the IODE software which accompanies the lecture notes by Jiří Lebl from which we've borrowed the text for this chapter.

As we said, the general first order equation we are studying looks like

$$y' = f(x, y).$$

In general, we cannot simply solve these kinds of equations explicitly. It would be nice if we could at least figure out the shape and behaviour of the solutions, or if we could find approximate solutions.

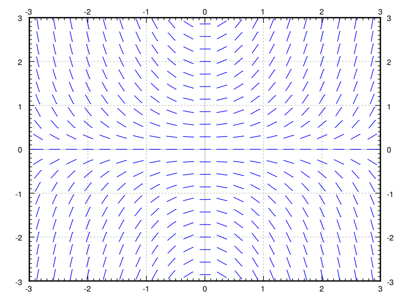


Figure 8.3.1: Slope field for the equation $y' = xy$

Slope fields

Suppose we are able to solve a first order equation of the form $y' = f(x, y)$, obtaining a solution $y = g(x)$. Differential calculus tells us that $y' = g'(x)$ gives us the slope of the tangent line to the curve $y = g(x)$ at the point $(x, g(x))$. Thus, the equation $y' = f(x, y)$ gives you a slope at each point in the (x, y)-plane. We can plot the slope at lots of points as a short line through the point (x, y) with the slope $f(x, y)$. See Figure 8.3.1.

We call this picture the **slope field** of the equation. If we are given a specific initial condition $y(x_0) = y_0$, we can look at the location (x_0, y_0) and follow the slopes. See Figure 8.3.2.

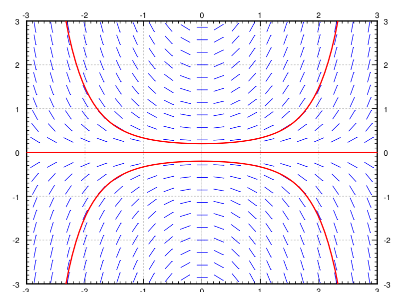


Figure 8.3.2: Slope field of $y' = xy$ with a graph of solutions satisfying $y(0) = 0.2$, $y(0) = 0$, and $y(0) = -0.2$.

By looking at the slope field we can get a lot of information about the behaviour of solutions. For example, in Figure 8.3.2 we can see what the solutions do when the initial conditions are $y(0) > 0$, $y(0) = 0$ and $y(0) < 0$. Note that a small change in the initial condition causes quite different behaviour. On the other hand, plotting a few solutions of the equation $y' = -y$, we see that no matter what $y(0)$ is, all solutions tend to zero as x tends to infinity. See Figure 8.3.3.

Existence and uniqueness

We wish to ask two fundamental questions about the problem

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

(i) Does a solution *exist*?

(ii) Is the solution *unique* (if it exists)?

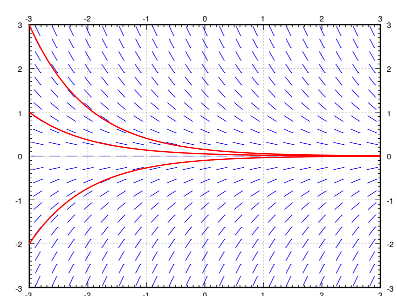


Figure 8.3.3: Slope field of $y' = -y$ with a graph of a few solutions.

What do you think is the answer? The answer seems to be yes to both does it not? Well, pretty much. But there are cases when the answer to either question can be no.

Since generally the equations we encounter in applications come from real life situations, it seems logical that a solution always exists. It also has to be unique if we believe our universe is deterministic. If the solution does not exist, or if it is not unique, we have probably not devised the correct model. Hence, it is good to know when things go wrong and why.

Example 8.3.1 An initial value problem with no solution

Attempt to solve:

$$y' = \frac{1}{x}, \qquad y(0) = 0.$$

Solution Integrate to find the general solution $y = \ln|x| + C$. Note that the solution does not exist at $x = 0$. See Figure 8.3.4.

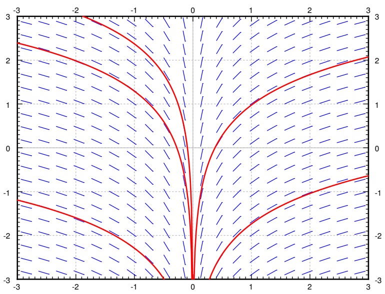


Figure 8.3.4: Slope field of $y' = \dfrac{1}{x}$.

Example 8.3.2 An initial value problem without a unique solution

Solve:

$$y' = 2\sqrt{|y|}, \qquad y(0) = 0.$$

Solution See Figure 8.3.5. Note that $y = 0$ is a solution. But another solution is the function

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

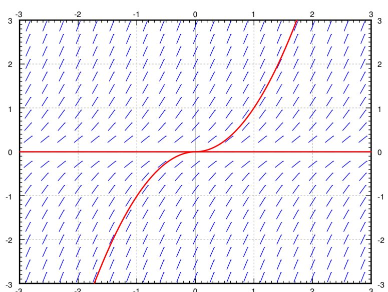


Figure 8.3.5: Slope field of $y' = 2\sqrt{|y|}$ with two solutions satisfying $y(0) = 0$.

It is hard to tell by staring at the slope field that the solution is not unique. Is there any hope? Of course there is. We have the following theorem, known as Picard's theorem

Theorem 8.3.1 Picard's theorem on existence and uniqueness

If $f(x, y)$ is continuous (as a function of two variables) and $\frac{\partial f}{\partial y}$ exists and is continuous near some (x_0, y_0), then a solution to

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

exists (at least for some small interval of x's) and is unique.

Note that the problems $y' = \frac{1}{x}$, $y(0) = 0$ and $y' = 2\sqrt{|y|}$, $y(0) = 0$ do not satisfy the hypothesis of the theorem. Even if we can use the theorem, we ought to be careful about this existence business. It is quite possible that the solution only exists for a short while.

Picard's Theorem is named after the French mathematician Charles Émile Picard (1856 – 1941)

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Example 8.3.3 An initial value problem with a "finite time" solution

For some constant A, solve:

$$y' = y^2, \qquad y(0) = A.$$

Solution We know how to solve this equation. First assume that $A \neq 0$, so y is not equal to zero at least for some x near 0. So $x' = \frac{1}{y^2}$, so $x = \frac{-1}{y} + C$, so $y = \frac{1}{C-x}$. If $y(0) = A$, then $C = \frac{1}{A}$ so

$$y = \frac{1}{\frac{1}{A} - x}.$$

If $A = 0$, then $y = 0$ is a solution.

For example, when $A = 1$ the solution "blows up" at $x = 1$. Hence, the solution does not exist for all x even if the equation is nice everywhere. The equation $y' = y^2$ certainly looks nice.

For most of this course we will be interested in equations where existence and uniqueness holds, and in fact holds "globally" unlike for the equation $y' = y^2$.

Exercises 8.3

Problems

1. Sketch slope field for $y' = e^{x-y}$. How do the solutions behave as x grows? Can you guess a particular solution by looking at the slope field?

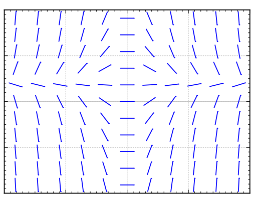
2. Sketch slope field for $y' = x^2$.

3. Sketch slope field for $y' = y^2$.

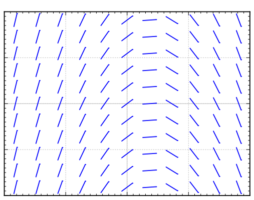
4. Is it possible to solve the equation $y' = \dfrac{xy}{\cos x}$ for $y(0) = 1$? Justify.

5. Is it possible to solve the equation $y' = y\sqrt{|x|}$ for $y(0) = 0$? Is the solution unique? Justify.

6. Match the following equations to their slope fields.
(i) $y' = 1 - x$
(ii) $y' = x - 2y$
(iii) $y' = x(1 - y)$
Justify.



(a)



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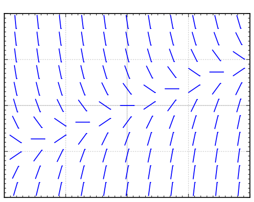


(c)

7. (Challenging) Take $y' = f(x, y)$, $y(0) = 0$, where $f(x, y) > 1$ for all x and y. If the solution exists for all x, can you say what happens to $y(x)$ as x goes to positive infinity? Explain.

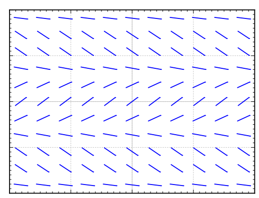
8. (Challenging) Take $(y - x)y' = 0$, $y(0) = 0$. a) Find two distinct solutions. b) Explain why this does not violate Picard's theorem.

9. Sketch the slope field of $y' = y^3$. Can you visually find the solution that satisfies $y(0) = 0$?

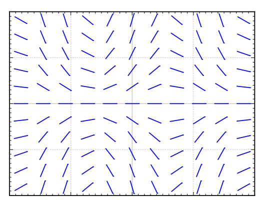
10. Is it possible to solve $y' = xy$ for $y(0) = 0$? Is the solution unique?

11. Is it possible to solve $y' = \dfrac{x}{x^2 - 1}$ for $y(1) = 0$?

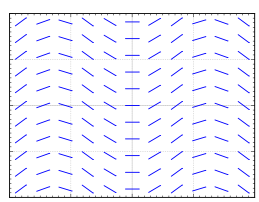
12. Match the following equations to their slope fields:
(i) $y' = \sin x$
(ii) $y' = \cos y$
(iii) $y' = y\cos(x)$
Justify.



(a)



(b)



(c)

8.4 Separable equations

When a differential equation is of the form $y' = f(x)$, we can just integrate: $y = \int f(x)\,dx + C$. Unfortunately this method no longer works for the general form of the equation $y' = f(x, y)$. Integrating both sides yields

$$y = \int f(x, y)\,dx + C.$$

Notice the dependence on y in the integral.

Separable equations

Let us suppose that the equation is **separable**. That is, let us consider

$$y' = f(x)g(y),$$

for some functions $f(x)$ and $g(y)$. Let us write the equation in the **Leibniz notation**

$$\frac{dy}{dx} = f(x)g(y).$$

Then we rewrite the equation as

$$\frac{dy}{g(y)} = f(x)\,dx.$$

Now both sides look like something we can integrate. We obtain

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C.$$

If we can find closed form expressions for these two integrals, we can, perhaps, solve for y.

Example 8.4.1 A separable ODE

Solve the equation

$$y' = xy.$$

Solution First note that $y = 0$ is a solution, so assume $y \neq 0$ from now on. Write the equation as $\frac{dy}{dx} = xy$, then

$$\int \frac{dy}{y} = \int x\,dx + C.$$

We compute the antiderivatives to get

$$\ln|y| = \frac{x^2}{2} + C.$$

Or

$$|y| = e^{\frac{x^2}{2}+C} = e^{\frac{x^2}{2}}e^{C} = De^{\frac{x^2}{2}},$$

where $D > 0$ is some constant. Because $y = 0$ is a solution and because of the absolute value we actually can write:

$$y = De^{\frac{x^2}{2}},$$

for any number D (including zero or negative).

We check:
$$y' = Dxe^{\frac{x^2}{2}} = x\left(De^{\frac{x^2}{2}}\right) = xy.$$
It works!

We should be a little bit more careful with this method. You may be worried that we were integrating in two different variables. We seemed to be doing a different operation to each side. Let us work this method out more rigorously. Take
$$\frac{dy}{dx} = f(x)g(y).$$
We rewrite the equation as follows. Note that $y = y(x)$ is a function of x and so is $\frac{dy}{dx}$!
$$\frac{1}{g(y)}\,\frac{dy}{dx} = f(x).$$
We integrate both sides with respect to x.
$$\int \frac{1}{g(y)}\,\frac{dy}{dx}\,dx = \int f(x)\,dx + C.$$
We can use the change of variables formula.
$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + C.$$
And we are done.

Implicit solutions

It is clear that we might sometimes get stuck even if we can do the integration. For example, take the separable equation
$$y' = \frac{xy}{y^2+1}.$$
We separate variables,
$$\frac{y^2+1}{y}\,dy = \left(y + \frac{1}{y}\right)\,dy = x\,dx.$$
We integrate to get
$$\frac{y^2}{2} + \ln|y| = \frac{x^2}{2} + C,$$
or perhaps the easier looking expression (where $D = 2C$)
$$y^2 + 2\ln|y| = x^2 + D.$$
It is not easy to find the solution explicitly as it is hard to solve for y. We, therefore, leave the solution in this form and call it an **implicit solution**. It is still easy to check that an implicit solution satisfies the differential equation. In this case, we differentiate with respect to x to get
$$y'\left(2y + \frac{2}{y}\right) = 2x.$$
It is simple to see that the differential equation holds. If you want to compute values for y, you might have to be tricky. For example, you can graph x as a

function of y, and then flip your paper. Computers are also good at some of these tricks.

We note that the above equation also has the solution $y = 0$. The general solution is $y^2 + 2\ln|y| = x^2 + C$ together with $y = 0$. These outlying solutions such as $y = 0$ are sometimes called *singular solutions*.

Example 8.4.2 An example with initial conditions

Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2$, $y(1) = 0$.

Solution First factor the right hand side to obtain

$$x^2y' = (1 - x^2)(1 + y^2).$$

Separate variables, integrate, and solve for y.

$$\begin{aligned}
\frac{y'}{1+y^2} &= \frac{1-x^2}{x^2}, \\
\frac{y'}{1+y^2} &= \frac{1}{x^2} - 1, \\
\arctan(y) &= \frac{-1}{x} - x + C, \\
y &= \tan\left(\frac{-1}{x} - x + C\right).
\end{aligned}$$

Now solve for the initial condition, $0 = \tan(-2 + C)$ to get $C = 2$ (or $2 + \pi$, etc…). The solution we are seeking is, therefore,

$$y = \tan\left(\frac{-1}{x} - x + 2\right).$$

Example 8.4.3 Cooling a cup of coffee

Bob made a cup of coffee, and Bob likes to drink coffee only once it will not burn him at 60 degrees. Initially at time $t = 0$ minutes, Bob measured the temperature and the coffee was 89 degrees Celsius. One minute later, Bob measured the coffee again and it had 85 degrees. The temperature of the room (the ambient temperature) is 22 degrees. When should Bob start drinking?

Solution Let T be the temperature of the coffee, and let A be the ambient (room) temperature. **Newton's law of cooling** states that the rate at which the temperature of the coffee is changing is proportional to the difference between the ambient temperature and the temperature of the coffee. That is,

$$\frac{dT}{dt} = k(A - T),$$

for some constant k. For our setup $A = 22$, $T(0) = 89$, $T(1) = 85$. We separate variables and integrate (let C and D denote arbitrary constants)

$$\begin{aligned}
\frac{1}{T-A}\,\frac{dT}{dt} &= -k, \\
\ln(T - A) &= -kt + C, \qquad \text{(note that } T - A > 0\text{)} \\
T - A &= D\,e^{-kt}, \\
T &= A + D\,e^{-kt}.
\end{aligned}$$

That is, $T = 22 + D\,e^{-kt}$. We plug in the first condition: $89 = T(0) = 22 + D$, and hence $D = 67$. So $T = 22 + 67\,e^{-kt}$. The second condition says $85 = T(1) = 22 + 67\,e^{-k}$. Solving for k we get $k = -\ln\frac{85-22}{67} \approx 0.0616$. Now we solve for the time t that gives us a temperature of 60 degrees. That is, we solve $60 = 22 + 67e^{-0.0616t}$ to get $t = -\frac{\ln\frac{60-22}{67}}{0.0616} \approx 9.21$ minutes. So Bob can begin to drink the coffee at just over 9 minutes from the time Bob made it. That is probably about the amount of time it took us to calculate how long it would take.

Example 8.4.4 Finding all possible solutions

Find the general solution to $y' = \frac{-xy^2}{3}$ (including singular solutions).

Solution First note that $y = 0$ is a solution (a singular solution). So assume that $y \neq 0$ and write

$$\begin{aligned} \frac{-3}{y^2}y' &= x, \\ \frac{3}{y} &= \frac{x^2}{2} + C, \\ y &= \frac{3}{x^2/2 + C} = \frac{6}{x^2 + 2C}. \end{aligned}$$

Exercises 8.4

Problems

1. Solve $y' = \dfrac{x}{y}$.

2. Solve $y' = x^2 y$.

3. Solve $\dfrac{dx}{dt} = (x^2 - 1)\,t$, for $x(0) = 0$.

4. Solve $\dfrac{dx}{dt} = x\ \sin(t)$, for $x(0) = 1$.

5. Solve $\dfrac{dy}{dx} = xy + x + y + 1$. Hint: Factor the right hand side.

6. Solve $xy' = y + 2x^2 y$, where $y(1) = 1$.

7. Solve $\dfrac{dy}{dx} = \dfrac{y^2 + 1}{x^2 + 1}$, for $y(0) = 1$.

8. Find an implicit solution for $\dfrac{dy}{dx} = \dfrac{x^2 + 1}{y^2 + 1}$, for $y(0) = 1$.

9. Find an explicit solution for $y' = xe^{-y}$, $y(0) = 1$.

10. Find an explicit solution for $xy' = e^{-y}$, for $y(1) = 1$.

11. Find an explicit solution for $y' = ye^{-x^2}$, $y(0) = 1$. It is all right to leave a definite integral in your answer.

12. Suppose a cup of coffee is at 100 degrees Celsius at time $t = 0$, it is at 70 degrees at $t = 10$ minutes, and it is at 50 degrees at $t = 20$ minutes. Compute the ambient temperature.

13. Solve $y' = 2xy$.

14. Solve $x' = 3xt^2 - 3t^2$, $x(0) = 2$.

15. Find an implicit solution for $x' = \frac{1}{3x^2+1}$, $x(0) = 1$.

16. Find an explicit solution to $xy' = y^2$, $y(1) = 1$.

17. Find an implicit solution to $y' = \frac{\sin(x)}{\cos(y)}$.

18. Take Example 8.4.3 with the same numbers: 89 degrees at $t = 0$, 85 degrees at $t = 1$, and ambient temperature of 22 degrees. Suppose these temperatures were measured with precision of ± 0.5 degrees. Given this imprecision, the time it takes the coffee to cool to (exactly) 60 degrees is also only known in a certain range. Find this range. Hint: Think about what kind of error makes the cooling time longer and what shorter.

8.5 Linear equations and the integrating factor

One of the most important types of equations we will learn how to solve are the so-called **linear equations** In this lecture we focus on the **first order linear equation**. A first order equation is linear if we can put it into the form:

$$y' + p(x)y = f(x). \tag{8.4}$$

Here the word "linear" means linear in y and y'; no higher powers nor functions of y or y' appear. The dependence on x can be more complicated.

Solutions of linear equations have nice properties. For example, the solution exists wherever $p(x)$ and $f(x)$ are defined, and has the same regularity (read: it is just as nice). But most importantly for us right now, there is a method for solving linear first order equations.

The trick is to rewrite the left hand side of (8.4) as a derivative of a product of y with another function. To this end we find a function $r(x)$ such that

$$r(x)y' + r(x)p(x)y = \frac{d}{dx}\Big[r(x)y\Big].$$

This is the left hand side of (8.4) multiplied by $r(x)$. So if we multiply (8.4) by $r(x)$, we obtain

$$\frac{d}{dx}\Big[r(x)y\Big] = r(x)f(x).$$

Now we integrate both sides. The right hand side does not depend on y and the left hand side is written as a derivative of a function. Afterwards, we solve for y. The function $r(x)$ is called the **integrating factor** and the method is called the **integrating factor method**.

We are looking for a function $r(x)$, such that if we differentiate it, we get the same function back multiplied by $p(x)$. That seems like a job for the exponential function! Let

$$r(x) = e^{\int p(x)\,dx}.$$

We compute:

$$\begin{aligned}
y' + p(x)y &= f(x), \\
e^{\int p(x)\,dx}y' + e^{\int p(x)\,dx}p(x)y &= e^{\int p(x)\,dx}f(x), \\
\frac{d}{dx}\Big[e^{\int p(x)\,dx}y\Big] &= e^{\int p(x)\,dx}f(x), \\
e^{\int p(x)\,dx}y &= \int e^{\int p(x)\,dx}f(x)\,dx + C, \\
y &= e^{-\int p(x)\,dx}\left(\int e^{\int p(x)\,dx}f(x)\,dx + C\right).
\end{aligned}$$

Of course, to get a closed form formula for y, we need to be able to find a closed form formula for the integrals appearing above.

Example 8.5.1 A linear equation with a closed form solution

Solve

$$y' + 2xy = e^{x-x^2}, \qquad y(0) = -1.$$

Solution First note that $p(x) = 2x$ and $f(x) = e^{x-x^2}$. The integrating factor is $r(x) = e^{\int p(x)\,dx} = e^{x^2}$. We multiply both sides of the equation by $r(x)$

to get

$$e^{x^2}y' + 2xe^{x^2}y = e^{x-x^2}e^{x^2},$$
$$\frac{d}{dx}\left[e^{x^2}y\right] = e^x.$$

We integrate

$$e^{x^2}y = e^x + C,$$
$$y = e^{x-x^2} + Ce^{-x^2}.$$

Next, we solve for the initial condition $-1 = y(0) = 1 + C$, so $C = -2$. The solution is

$$y = e^{x-x^2} - 2e^{-x^2}.$$

Note that we do not care which antiderivative we take when computing $e^{\int p(x)dx}$. You can always add a constant of integration, but those constants will not matter in the end.

Exercise: Try it! Add a constant of integration to the integral in the integrating factor and show that the solution you get in the end is the same as what we got above.

A piece of advice: Do not try to remember the formula itself, that is way too hard. It is easier to remember the process and repeat it.

Since we cannot always evaluate the integrals in closed form, it is useful to know how to write the solution in definite integral form. A definite integral is something that you can plug into a computer or a calculator. Suppose we are given

$$y' + p(x)y = f(x), \qquad y(x_0) = y_0.$$

Look at the solution and write the integrals as definite integrals.

$$\boxed{y(x) = e^{-\int_{x_0}^{x} p(s)\,ds}\left(\int_{x_0}^{x} e^{\int_{x_0}^{t} p(s)\,ds} f(t)\,dt + y_0\right).} \tag{8.5}$$

You should be careful to properly use dummy variables here. If you now plug such a formula into a computer or a calculator, it will be happy to give you numerical answers.

Exercise: Check that $y(x_0) = y_0$ in formula (8.5).

Exercise: Write the solution of the following problem as a definite integral, but try to simplify as far as you can. You will not be able to find the solution in closed form.

$$y' + y = e^{x^2-x}, \qquad y(0) = 10.$$

Remark: Before we move on, we should note some interesting properties of linear equations. First, for the linear initial value problem $y' + p(x)y = f(x)$, $y(x_0) = y_0$, there is always an explicit formula (8.5) for the solution. Second, it follows from the formula (8.5) that if $p(x)$ and $f(x)$ are continuous on some interval (a, b), then the solution $y(x)$ exists and is differentiable on (a, b). Compare with the simple nonlinear example we have seen previously, $y' = y^2$, and compare to Theorem 8.3.1.

Let us discuss a common simple application of linear equations. This type of problem is used often in real life. For example, linear equations are used in figuring out the concentration of chemicals in bodies of water (rivers and lakes).

Example 8.5.2 An application of linear ODEs

A 100 litre tank contains 10 kilograms of salt dissolved in 60 litres of water. Solution of water and salt (brine) with concentration of 0.1 kilograms per litre is flowing in at the rate of 5 litres a minute. The solution in the tank is well stirred and flows out at a rate of 3 litres a minute. How much salt is in the tank when the tank is full?

Solution Let us come up with the equation. Let x denote the kilograms of salt in the tank, let t denote the time in minutes. For a small change Δt in time, the change in x (denoted Δx) is approximately

$$\Delta x \approx (\text{rate in} \times \text{concentration in})\Delta t - (\text{rate out} \times \text{concentration out})\Delta t.$$

Dividing through by Δt and taking the limit $\Delta t \to 0$ we see that

$$\frac{dx}{dt} = (\text{rate in} \times \text{concentration in}) - (\text{rate out} \times \text{concentration out}).$$

In our example, we have

$$\begin{aligned} \text{rate in} &= 5, \\ \text{concentration in} &= 0.1, \\ \text{rate out} &= 3, \\ \text{concentration out} &= \frac{x}{\text{volume}} = \frac{x}{60 + (5-3)t}. \end{aligned}$$

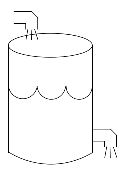


Figure 8.5.1: The tank in Example 8.5.2

Our equation is, therefore,

$$\frac{dx}{dt} = (5 \times 0.1) - \left(3\frac{x}{60+2t}\right).$$

Or in the form (8.4)

$$\frac{dx}{dt} + \frac{3}{60+2t}x = 0.5.$$

Let us solve. The integrating factor is

$$r(t) = \exp\left(\int \frac{3}{60+2t}dt\right) = \exp\left(\frac{3}{2}\ln(60+2t)\right) = (60+2t)^{3/2}.$$

We multiply both sides of the equation to get

$$(60+2t)^{3/2}\frac{dx}{dt} + (60+2t)^{3/2}\frac{3}{60+2t}x = 0.5(60+2t)^{3/2},$$

and reversing the product rule gives us

$$\frac{d}{dt}\left[(60+2t)^{3/2}x\right] = 0.5(60+2t)^{3/2},$$

so

$$(60+2t)^{3/2}x = \int 0.5(60+2t)^{3/2}dt + C.$$

Thus,

$$\begin{aligned} x &= (60+2t)^{-3/2}\int \frac{(60+2t)^{3/2}}{2}dt + C(60+2t)^{-3/2}, \\ &= (60+2t)^{-3/2}\frac{1}{10}(60+2t)^{5/2} + C(60+2t)^{-3/2}, \\ &= \frac{60+2t}{10} + C(60+2t)^{-3/2}. \end{aligned}$$

We need to find C. We know that at $t = 0$, $x = 10$. So

$$10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2},$$

or

$$C = 4(60^{3/2}) \approx 1859.03.$$

We are interested in x when the tank is full. So we note that the tank is full when $60 + 2t = 100$, or when $t = 20$. So

$$x(20) = \frac{60+40}{10} + C(60+40)^{-3/2} \approx 10 + 1859.03(100)^{-3/2} \approx 11.86.$$

The concentration at the end is approximately 0.1186 $^{\text{kg}}/_{\text{litre}}$ and we started with $\frac{1}{6}$ or 0.167 $^{\text{kg}}/_{\text{litre}}$.

Exercises 8.5

Problems

In the exercises, feel free to leave answer as a definite integral if a closed form solution cannot be found. If you can find a closed form solution, you should give that.

1. Solve $y' + xy = x$.

2. Solve $y' + 6y = e^x$.

3. Solve $y' + 3x^2y = \sin(x)\, e^{-x^3}$, with $y(0) = 1$.

4. Solve $y' + \cos(x)y = \cos(x)$.

5. Solve $\frac{1}{x^2+1}\, y' + xy = 3$, with $y(0) = 0$.

6. Suppose there are two lakes located on a stream. Clean water flows into the first lake, then the water from the first lake flows into the second lake, and then water from the second lake flows further downstream. The in and out flow from each lake is 500 litres per hour. The first lake contains 100 thousand litres of water and the second lake contains 200 thousand litres of water. A truck with 500 kg of toxic substance crashes into the first lake. Assume that the water is being continually mixed perfectly by the stream. a) Find the concentration of toxic substance as a function of time in both lakes. b) When will the concentration in the first lake be below 0.001 kg per litre? c) When will the concentration in the second lake be maximal?

7. **Newton's law of cooling** states that $\frac{dx}{dt} = -k(x - A)$ where x is the temperature, t is time, A is the ambient temperature, and $k > 0$ is a constant. Suppose that $A = A_0 \cos(\omega t)$ for some constants A_0 and ω. That is, the ambient temperature oscillates (for example night and day temperatures). a) Find the general solution. b) In the long term, will the initial conditions make much of a difference? Why or why not?

8. Initially 5 grams of salt are dissolved in 20 litres of water. Brine with concentration of salt 2 grams of salt per litre is added at a rate of 3 litres a minute. The tank is mixed well and is drained at 3 litres a minute. How long does the process have to continue until there are 20 grams of salt in the tank?

9. Initially a tank contains 10 litres of pure water. Brine of unknown (but constant) concentration of salt is flowing in at 1 litre per minute. The water is mixed well and drained at 1 litre per minute. In 20 minutes there are 15 grams of salt in the tank. What is the concentration of salt in the incoming brine?

10. Solve $y' + 3x^2y = x^2$.

11. Solve $y' + 2\sin(2x)y = 2\sin(2x)$, $y(\pi/2) = 3$.

12. Suppose a water tank is being pumped out at 3 $^{\mathrm{L}}\!/_{\mathrm{min}}$. The water tank starts at 10 L of clean water. Water with toxic substance is flowing into the tank at 2 $^{\mathrm{L}}\!/_{\mathrm{min}}$, with concentration $20t$ $^{\mathrm{g}}\!/_{\mathrm{L}}$ at time t. When the tank is half empty, how many grams of toxic substance are in the tank (assuming perfect mixing)?

13. Suppose we have bacteria on a plate and suppose that we are slowly adding a toxic substance such that the rate of growth is slowing down. That is, suppose that $\frac{dP}{dt} = (2 - 0.1t)P$. If $P(0) = 1000$, find the population at $t = 5$.

8.6 Numerical methods: Euler's method

At this point it may be good to first try the Lab II and/or Project II from the IODE website: http://www.math.uiuc.edu/iode/materials.html. (This is completely optional, and you're free to look for your own software solutions online, or try using Maple or similar software. But it is generally a good idea to have the computer's help when exploring Euler's method.)

As we said before, unless $f(x,y)$ is of a special form, it is generally very hard if not impossible to get a nice formula for the solution of the problem

$$y' = f(x,y), \qquad y(x_0) = y_0.$$

What if we want to find the value of the solution at some particular x? Or perhaps we want to produce a graph of the solution to inspect the behaviour. In this section we will learn about the basics of numerical approximation of solutions.

The simplest method for approximating a solution is **Euler's method**

.

It works as follows: We take x_0 and compute the slope $k = f(x_0, y_0)$. The slope is the change in y per unit change in x. We follow the line for an interval of length h on the x axis. Hence if $y = y_0$ at x_0, then we will say that y_1 (the approximate value of y at $x_1 = x_0 + h$) will be $y_1 = y_0 + hk$. Rinse, repeat! That is, compute x_2 and y_2 using x_1 and y_1. For an example of the first two steps of the method see Figure 8.6.1.

More abstractly, for any $i = 1, 2, 3, \ldots$, we compute

$$x_{i+1} = x_i + h, \qquad y_{i+1} = y_i + h f(x_i, y_i).$$

The line segments we get are an approximate graph of the solution. Generally it is not exactly the solution. See Figure 8.6.2 for the plot of the real solution and the approximation.

Let us see what happens with the equation $y' = \frac{y^2}{3}$, $y(0) = 1$. Let us try to approximate $y(2)$ using Euler's method. In Figures 8.6.1 and 8.6.2 we have graphically approximated $y(2)$ with step size 1. With step size 1 we have $y(2) \approx 1.926$. The real answer is 3. So we are approximately 1.074 off. Let us halve the step size. Computing y_4 with $h = 0.5$, we find that $y(2) \approx 2.209$, so an error of about 0.791. Table 8.1 gives the values computed for various parameters.

Exercise: Solve this equation exactly and show that $y(2) = 3$.

The difference between the actual solution and the approximate solution we will call the error. We will usually talk about just the size of the error and we do not care much about its sign. The main point is, that we usually do not know the real solution, so we only have a vague understanding of the error. If we knew the error exactly …what is the point of doing the approximation?

We notice that except for the first few times, every time we halved the interval the error approximately halved. This halving of the error is a general feature of Euler's method as it is a **first order method**. In the IODE Project II you are asked to implement a **second order method**. A second order method reduces the error to approximately one quarter every time we halve the interval (second order as $\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$).

To get the error to be within 0.1 of the answer we had to already do 64 steps. To get it to within 0.01 we would have to halve another three or four times,

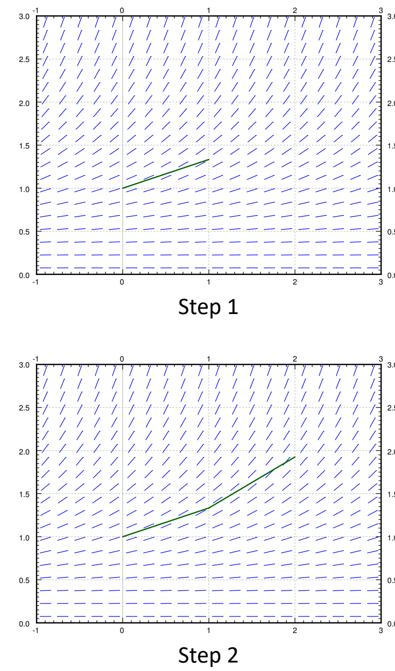


Figure 8.6.1: First two steps of Euler's method with $h = 1$ for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

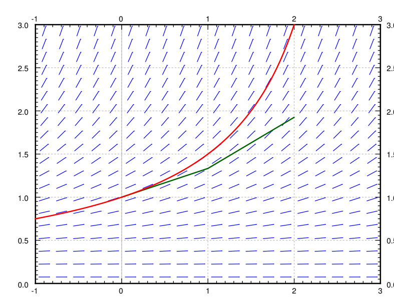


Figure 8.6.2: Two steps of Euler's method (step size 1) and the exact solution for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

Euler's Method is named after the Swiss mathematician Leonhard Paul Euler (1707 – 1783). Do note the correct pronunciation of the name sounds more like "oiler."

h	Approximate $y(2)$	Error	$\frac{\text{Error}}{\text{Previous error}}$
1	1.92593	1.07407	
0.5	2.20861	0.79139	0.73681
0.25	2.47250	0.52751	0.66656
0.125	2.68034	0.31966	0.60599
0.0625	2.82040	0.17960	0.56184
0.03125	2.90412	0.09588	0.53385
0.015625	2.95035	0.04965	0.51779
0.0078125	2.97472	0.02528	0.50913

Table 8.1: Euler's method approximation of $y(2)$ where of $y' = \frac{y^2}{3}$, $y(0) = 1$.

h	Approximate $y(3)$
1	3.16232
0.5	4.54329
0.25	6.86079
0.125	10.80321
0.0625	17.59893
0.03125	29.46004
0.015625	50.40121
0.0078125	87.75769

Figure 8.6.3: Attempts to use Euler's to approximate $y(3)$ where of $y' = \frac{y^2}{3}$, $y(0) = 1$.

meaning doing 512 to 1024 steps. That is quite a bit to do by hand. The improved Euler method from IODE Project II should quarter the error every time we halve the interval, so we would have to approximately do half as many "halvings" to get the same error. This reduction can be a big deal. With 10 halvings (starting at $h = 1$) we have 1024 steps, whereas with 5 halvings we only have to do 32 steps, assuming that the error was comparable to start with. A computer may not care about this difference for a problem this simple, but suppose each step would take a second to compute (the function may be substantially more difficult to compute than $\frac{y^2}{3}$). Then the difference is 32 seconds versus about 17 minutes. Note: We are not being altogether fair, a second order method would probably double the time to do each step. Even so, it is 1 minute versus 17 minutes. Next, suppose that we have to repeat such a calculation for different parameters a thousand times. You get the idea.

Note that in practice we do not know how large the error is! How do we know what is the right step size? Well, essentially we keep halving the interval, and if we are lucky, we can estimate the error from a few of these calculations and the assumption that the error goes down by a factor of one half each time (if we are using standard Euler).

Exercise: In the table above, suppose you do not know the error. Take the approximate values of the function in the last two lines, assume that the error goes down by a factor of 2. Can you estimate the error in the last time from this? Does it (approximately) agree with the table? Now do it for the first two rows. Does this agree with the table?

Let us talk a little bit more about the example $y' = \frac{y^2}{3}$, $y(0) = 1$. Suppose that instead of the value $y(2)$ we wish to find $y(3)$. The results of this effort are listed in Table 8.6.3 for successive halvings of h. What is going on here? Well, you should solve the equation exactly and you will notice that the solution does not exist at $x = 3$. In fact, the solution goes to infinity when you approach $x = 3$.

Another case where things go bad is if the solution oscillates wildly near some point. Such an example is given in IODE Project II. The solution may exist at all points, but even a much better numerical method than Euler would need an insanely small step size to approximate the solution with reasonable precision. And computers might not be able to easily handle such a small step size.

In real applications we would not use a simple method such as Euler's. The simplest method that would probably be used in a real application is the standard Runge-Kutta method. That is a fourth order method, meaning that if we halve the interval, the error generally goes down by a factor of 16 (it is fourth order as $\frac{1}{16} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$).

Choosing the right method to use and the right step size can be very tricky. There are several competing factors to consider.

- Computational time: Each step takes computer time. Even if the function f is simple to compute, we do it many times over. Large step size means faster computation, but perhaps not the right precision.

- Roundoff errors: Computers only compute with a certain number of significant digits. Errors introduced by rounding numbers off during our computations become noticeable when the step size becomes too small relative to the quantities we are working with. So reducing step size may in fact make errors worse. There is a certain optimum step size such that the precision increases as we approach it, but then starts getting worse as we make our step size smaller still. Trouble is: this optimum may be hard to find.

- Stability: Certain equations may be numerically unstable. What may happen is that the numbers never seem to stabilize no matter how many times we halve the interval. We may need a ridiculously small interval size, which may not be practical due to roundoff errors or computational time considerations. Such problems are sometimes called *stiff*. In the worst case, the numerical computations might be giving us bogus numbers that look like a correct answer. Just because the numbers seem to have stabilized after successive halving, does not mean that we must have the right answer.

We have seen just the beginnings of the challenges that appear in real applications. Numerical approximation of solutions to differential equations is an active research area for engineers and mathematicians. For example, the general purpose method used for the ODE solver in Matlab and Octave (as of this writing) is a method that appeared in the literature only in the 1980s.

Exercises 8.6

Problems

1. Consider $\frac{dx}{dt} = (2t-x)^2$, $x(0) = 2$. Use Euler's method with step size $h = 0.5$ to approximate $x(1)$.

2. Consider $\frac{dx}{dt} = t - x$, $x(0) = 1$. a) Use Euler's method with step sizes $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ to approximate $x(1)$. b) Solve the equation exactly. c) Describe what happens to the errors for each h you used. That is, find the factor by which the error changed each time you halved the interval.

3. Approximate the value of e by looking at the initial value problem $y' = y$ with $y(0) = 1$ and approximating $y(1)$ using Euler's method with a step size of 0.2.

4. Example of numerical instability: Take $y' = -5y$, $y(0) = 1$. We know that the solution should decay to zero as x grows. Using Euler's method, start with $h = 1$ and compute y_1, y_2, y_3, y_4 to try to approximate $y(4)$. What happened? Now halve the interval. Keep halving the interval and approximating $y(4)$ until the numbers you are getting start to stabilize (that is, until they start going towards zero). Note: You might want to use a calculator.

5. Let $x' = \sin(xt)$, and $x(0) = 1$. Approximate $x(1)$ using Euler's method with step sizes 1, 0.5, 0.25. Use a calculator and compute up to 4 decimal digits.

6. Let $x' = 2t$, and $x(0) = 0$. a) Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1. b) Solve exactly, and compute the errors. c) Compute the factor by which the errors changed.

7. Let $x' = xe^{xt+1}$, and $x(0) = 0$. (a) Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1. (b) Guess an exact solution based on part (a) and compute the errors.

9: Sequences and Series

This chapter introduces **sequences** and **series**, important mathematical constructions that are useful when solving a large variety of mathematical problems. The content of this chapter is considerably different from the content of the chapters before it. While the material we learn here definitely falls under the scope of "calculus," we will make very little use of derivatives or integrals. Limits are extremely important, though, especially limits that involve infinity.

Notation: We use $\mathbb{N}$ to describe the set of natural numbers, that is, the integers 1, 2, 3, ...

One of the problems addressed by this chapter is this: suppose we know information about a function and its derivatives at a point, such as $f(1) = 3$, $f'(1) = 1$, $f''(1) = -2$, $f'''(1) = 7$, and so on. What can I say about $f(x)$ itself? Is there any reasonable approximation of the value of $f(2)$? The topic of Taylor Series addresses this problem, and allows us to make excellent approximations of functions when limited knowledge of the function is available.

Factorial: The expression 4! refers to the number $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

In general, $n! = n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1$, where n is a natural number.

We define $0! = 1$. While this does not immediately make sense, it makes many mathematical formulas work properly.

9.1 Sequences

We commonly refer to a set of events that occur one after the other as a *sequence* of events. In mathematics, we use the word *sequence* to refer to an ordered set of numbers, i.e., a set of numbers that "occur one after the other."

For instance, the numbers 2, 4, 6, 8, ..., form a sequence. The order is important; the first number is 2, the second is 4, etc. It seems natural to seek a formula that describes a given sequence, and often this can be done. For instance, the sequence above could be described by the function $a(n) = 2n$, for the values of $n = 1, 2, \ldots$ To find the 10^{th} term in the sequence, we would compute $a(10)$. This leads us to the following, formal definition of a sequence.

Definition 9.1.1 Sequence

A **sequence** is a function $a(n)$ whose domain is $\mathbb{N}$. The **range** of a sequence is the set of all distinct values of $a(n)$.

The **terms** of a sequence are the values $a(1)$, $a(2)$, ..., which are usually denoted with subscripts as a_1, a_2, ...

A sequence $a(n)$ is often denoted as $\{a_n\}$.

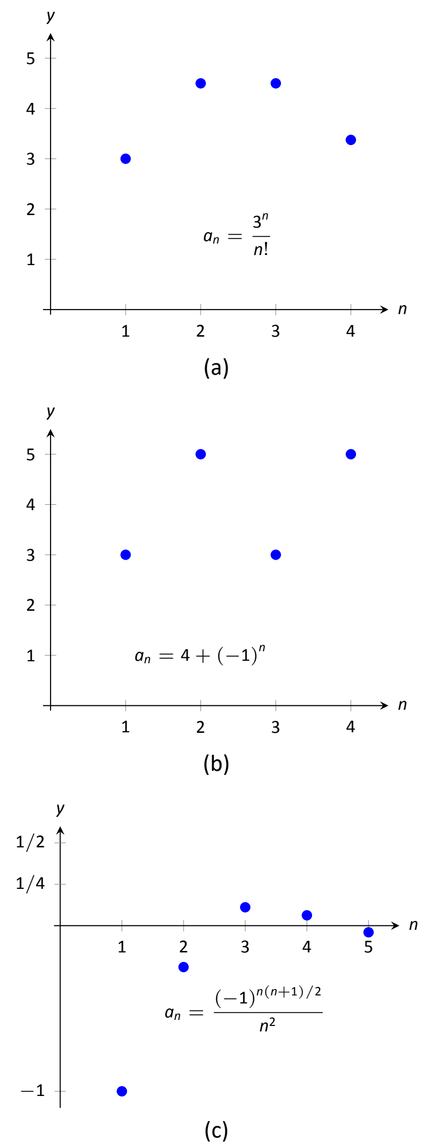


Figure 9.1.1: Plotting sequences in Example 9.1.1.

Example 9.1.1 Listing terms of a sequence

List the first four terms of the following sequences.

1. $\{a_n\} = \left\{\frac{3^n}{n!}\right\}$ 2. $\{a_n\} = \{4+(-1)^n\}$ 3. $\{a_n\} = \left\{\frac{(-1)^{n(n+1)/2}}{n^2}\right\}$

Solution

1. $a_1 = \frac{3^1}{1!} = 3;\quad a_2 = \frac{3^2}{2!} = \frac{9}{2};\quad a_3 = \frac{3^3}{3!} = \frac{9}{2};\quad a_4 = \frac{3^4}{4!} = \frac{27}{8}$

   We can plot the terms of a sequence with a scatter plot. The "x"-axis is used for the values of n, and the values of the terms are plotted on the y-axis. To visualize this sequence, see Figure 9.1.1(a).

2. $a_1 = 4+(-1)^1 = 3;\quad a_2 = 4+(-1)^2 = 5;$

   $a_3 = 4+(-1)^3 = 3;\quad a_4 = 4+(-1)^4 = 5$. Note that the range of this sequence is finite, consisting of only the values 3 and 5. This sequence is plotted in Figure 9.1.1(b).

3. $a_1 = \frac{(-1)^{1(2)/2}}{1^2} = -1;\quad a_2 = \frac{(-1)^{2(3)/2}}{2^2} = -\frac{1}{4}$

   $a_3 = \frac{(-1)^{3(4)/2}}{3^2} = \frac{1}{9}\quad a_4 = \frac{(-1)^{4(5)/2}}{4^2} = \frac{1}{16};$

   $a_5 = \frac{(-1)^{5(6)/2}}{5^2} = -\frac{1}{25}.$

   We gave one extra term to begin to show the pattern of signs is "$-, -, +, +, -, -, \ldots$" due to the fact that the exponent of -1 is a special quadratic. This sequence is plotted in Figure 9.1.1(c).

Example 9.1.2 Determining a formula for a sequence

Find the n^{th} term of the following sequences, i.e., find a function that describes each of the given sequences.

1. $2, 5, 8, 11, 14, \ldots$
2. $2, -5, 10, -17, 26, -37, \ldots$
3. $1, 1, 2, 6, 24, 120, 720, \ldots$
4. $\frac{5}{2}, \frac{5}{2}, \frac{15}{8}, \frac{5}{4}, \frac{25}{32}, \ldots$

Solution We should first note that there is never exactly one function that describes a finite set of numbers as a sequence. There are many sequences that start with 2, then 5, as our first example does. We are looking for a simple formula that describes the terms given, knowing there is possibly more than one answer.

1. Note how each term is 3 more than the previous one. This implies a linear function would be appropriate: $a(n) = a_n = 3n + b$ for some appropriate value of b. As we want $a_1 = 2$, we set $b = -1$. Thus $a_n = 3n - 1$.

2. First notice how the sign changes from term to term. This is most commonly accomplished by multiplying the terms by either $(-1)^n$ or $(-1)^{n+1}$. Using $(-1)^n$ multiplies the odd terms by (-1); using $(-1)^{n+1}$ multiplies

the even terms by (-1). As this sequence has negative even terms, we will multiply by $(-1)^{n+1}$.

After this, we might feel a bit stuck as to how to proceed. At this point, we are just looking for a pattern of some sort: what do the numbers 2, 5, 10, 17, etc., have in common? There are many correct answers, but the one that we'll use here is that each is one more than a perfect square. That is, $2 = 1^2 + 1$, $5 = 2^2 + 1$, $10 = 3^2 + 1$, etc. Thus our formula is $a_n = (-1)^{n+1}(n^2 + 1)$.

3. One who is familiar with the factorial function will readily recognize these numbers. They are 0!, 1!, 2!, 3!, etc. Since our sequences start with $n = 1$, we cannot write $a_n = n!$, for this misses the 0! term. Instead, we shift by 1, and write $a_n = (n-1)!$.

4. This one may appear difficult, especially as the first two terms are the same, but a little "sleuthing" will help. Notice how the terms in the numerator are always multiples of 5, and the terms in the denominator are always powers of 2. Does something as simple as $a_n = \frac{5n}{2^n}$ work?

   When $n = 1$, we see that we indeed get $5/2$ as desired. When $n = 2$, we get $10/4 = 5/2$. Further checking shows that this formula indeed matches the other terms of the sequence.

A common mathematical endeavour is to create a new mathematical object (for instance, a sequence) and then apply previously known mathematics to the new object. We do so here. The fundamental concept of calculus is the limit, so we will investigate what it means to find the limit of a sequence.

Definition 9.1.2 Limit of a Sequence, Convergent, Divergent

Let $\{a_n\}$ be a sequence and let L be a real number. Given any $\varepsilon > 0$, if an m can be found such that $|a_n - L| < \varepsilon$ for all $n > m$, then we say the **limit of $\{a_n\}$, as n approaches infinity, is L**, denoted

$$\lim_{n\to\infty} a_n = L.$$

If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges**; otherwise, the sequence **diverges**.

This definition states, informally, that if the limit of a sequence is L, then if you go far enough out along the sequence, all subsequent terms will be *really close* to L. Of course, the terms "far enough" and "really close" are subjective terms, but hopefully the intent is clear.

This definition is reminiscent of the ε–δ proofs of Chapter 1. In that chapter we developed other tools to evaluate limits apart from the formal definition; we do so here as well.

Theorem 9.1.1 Limit of a Sequence

Let $\{a_n\}$ be a sequence and let $f(x)$ be a function whose domain contains the positive real numbers where $f(n) = a_n$ for all n in $\mathbb{N}$.

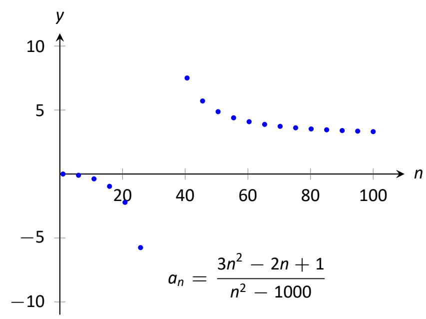
If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

Theorem 9.1.1 allows us, in certain cases, to apply the tools developed in Chapter 1 to limits of sequences. Note two things *not* stated by the theorem:

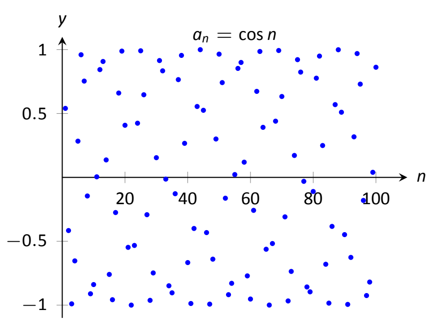
1. If $\lim_{x\to\infty} f(x)$ does not exist, we cannot conclude that $\lim_{n\to\infty} a_n$ does not exist. It may, or may not, exist. For instance, we can define a sequence $\{a_n\} = \{\cos(2\pi n)\}$. Let $f(x) = \cos(2\pi x)$. Since the cosine function oscillates over the real numbers, the limit $\lim_{x\to\infty} f(x)$ does not exist.

   However, for every positive integer n, $\cos(2\pi n) = 1$, so $\lim_{n\to\infty} a_n = 1$.

2. If we cannot find a function $f(x)$ whose domain contains the positive real numbers where $f(n) = a_n$ for all n in $\mathbb{N}$, we cannot conclude $\lim_{n\to\infty} a_n$ does not exist. It may, or may not, exist.



(a)



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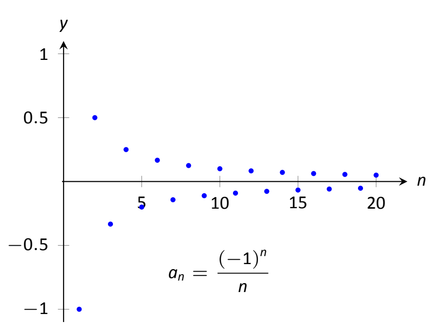


(c)

Figure 9.1.2: Scatter plots of the sequences in Example 9.1.3.

Example 9.1.3 Determining convergence/divergence of a sequence

Determine the convergence or divergence of the following sequences.

1. $\{a_n\} = \left\{\dfrac{3n^2 - 2n + 1}{n^2 - 1000}\right\}$ 2. $\{a_n\} = \{\cos n\}$ 3. $\{a_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$

SOLUTION

1. Using Theorem 1.5.1, we can state that $\lim_{x\to\infty} \dfrac{3x^2 - 2x + 1}{x^2 - 1000} = 3$. (We could have also directly applied l'Hospital's Rule.) Thus the sequence $\{a_n\}$ converges, and its limit is 3. A scatter plot of every 5 values of a_n is given in Figure 9.1.2 (a). The values of a_n vary widely near $n = 30$, ranging from about -73 to 125, but as n grows, the values approach 3.

2. The limit $\lim_{x\to\infty} \cos x$ does not exist, as $\cos x$ oscillates (and takes on every value in $[-1, 1]$ infinitely many times). Thus we cannot apply Theorem 9.1.1.

   The fact that the cosine function oscillates strongly hints that $\cos n$, when n is restricted to $\mathbb{N}$, will also oscillate. Figure 9.1.2 (b), where the sequence is plotted, implies that this is true. Because only discrete values of cosine are plotted, it does not bear strong resemblance to the familiar cosine wave. The proof of the following statement is beyond the scope of this text, but it is true: there are infinitely many integers n that are arbitrarily (i.e., *very*) close to an even multiple of π, so that $\cos n \approx 1$. Similarly, there are infinitely many integers m that are arbitrarily close to an odd multiple of π, so that $\cos m \approx -1$. As the sequence takes on values near 1 and -1 infinitely many times, we conclude that $\lim_{n\to\infty} a_n$ does not exist.

3. We cannot actually apply Theorem 9.1.1 here, as the function $f(x) = (-1)^x/x$ is not well defined. (What does $(-1)^{\sqrt{2}}$ mean? In actuality, there is an answer, but it involves *complex analysis*, beyond the scope of this text.)

   Instead, we invoke the definition of the limit of a sequence. By looking at the plot in Figure 9.1.2 (c), we would like to conclude that the sequence converges to $L = 0$. Let $\varepsilon > 0$ be given. We can find a natural number m

such that $1/m < \varepsilon$. Let $n > m$, and consider $|a_n - L|$:

$$\begin{aligned} |a_n - L| &= \left|\frac{(-1)^n}{n} - 0\right| \\ &= \frac{1}{n} \\ &< \frac{1}{m} \quad \text{(since } n > m\text{)} \\ &< \varepsilon. \end{aligned}$$

We have shown that by picking m large enough, we can ensure that a_n is arbitrarily close to our limit, $L = 0$, hence by the definition of the limit of a sequence, we can say $\lim\limits_{n\to\infty} a_n = 0$.

In the previous example we used the definition of the limit of a sequence to determine the convergence of a sequence as we could not apply Theorem 9.1.1. In general, we like to avoid invoking the definition of a limit, and the following theorem gives us tool that we could use in that example instead.

Theorem 9.1.2 Absolute Value Theorem

Let $\{a_n\}$ be a sequence. If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$

Example 9.1.4 Determining the convergence/divergence of a sequence

Determine the convergence or divergence of the following sequences.

1. $\{a_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$ 2. $\{a_n\} = \left\{\dfrac{(-1)^n(n+1)}{n}\right\}$

Solution

1. This appeared in Example 9.1.3. We want to apply Theorem 9.1.2, so consider the limit of $\{|a_n|\}$:

$$\begin{aligned} \lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| \\ &= \lim_{n\to\infty} \frac{1}{n} \\ &= 0. \end{aligned}$$

Since this limit is 0, we can apply Theorem 9.1.2 and state that $\lim\limits_{n\to\infty} a_n = 0$.

2. Because of the alternating nature of this sequence (i.e., every other term is multiplied by -1), we cannot simply look at the limit $\lim\limits_{x\to\infty} \dfrac{(-1)^x(x+1)}{x}$. We can try to apply the techniques of Theorem 9.1.2:

$$\begin{aligned} \lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \left|\frac{(-1)^n(n+1)}{n}\right| \\ &= \lim_{n\to\infty} \frac{n+1}{n} \\ &= 1. \end{aligned}$$

We have concluded that when we ignore the alternating sign, the sequence approaches 1. This means we cannot apply Theorem 9.1.2; it states the the limit must be 0 in order to conclude anything.

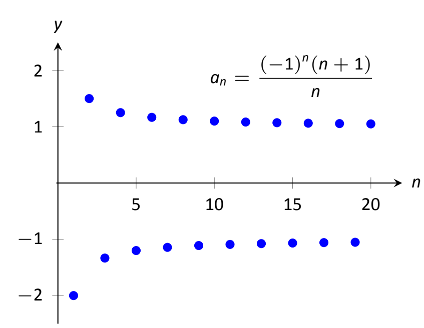


Figure 9.1.3: A plot of a sequence in Example 9.1.4, part 2.

Since we know that the signs of the terms alternate *and* we know that the limit of $|a_n|$ is 1, we know that as n approaches infinity, the terms will alternate between values close to 1 and -1, meaning the sequence diverges. A plot of this sequence is given in Figure 9.1.3.

We continue our study of the limits of sequences by considering some of the properties of these limits.

Theorem 9.1.3 Properties of the Limits of Sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} b_n = K$, and let c be a real number.

1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = L \pm K$
2. $\lim\limits_{n\to\infty} (a_n \cdot b_n) = L \cdot K$
3. $\lim\limits_{n\to\infty} (a_n/b_n) = L/K$, $K \neq 0$
4. $\lim\limits_{n\to\infty} c \cdot a_n = c \cdot L$

Example 9.1.5 Applying properties of limits of sequences

Let the following sequences, and their limits, be given:

- $\{a_n\} = \left\{\dfrac{n+1}{n^2}\right\}$, and $\lim\limits_{n\to\infty} a_n = 0$;
- $\{b_n\} = \left\{\left(1+\dfrac{1}{n}\right)^n\right\}$, and $\lim\limits_{n\to\infty} b_n = e$; and
- $\{c_n\} = \{n \cdot \sin(5/n)\}$, and $\lim\limits_{n\to\infty} c_n = 5$.

Evaluate the following limits.

1. $\lim\limits_{n\to\infty} (a_n + b_n)$
2. $\lim\limits_{n\to\infty} (b_n \cdot c_n)$
3. $\lim\limits_{n\to\infty} (1000 \cdot a_n)$

SOLUTION We will use Theorem 9.1.3 to answer each of these.

1. Since $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = e$, we conclude that $\lim\limits_{n\to\infty} (a_n + b_n) = 0 + e = e$. So even though we are adding something to each term of the sequence b_n, we are adding something so small that the final limit is the same as before.

2. Since $\lim\limits_{n\to\infty} b_n = e$ and $\lim\limits_{n\to\infty} c_n = 5$, we conclude that $\lim\limits_{n\to\infty} (b_n \cdot c_n) = e \cdot 5 = 5e$.

3. Since $\lim\limits_{n\to\infty} a_n = 0$, we have $\lim\limits_{n\to\infty} 1000a_n = 1000 \cdot 0 = 0$. It does not matter that we multiply each term by 1000; the sequence still approaches 0. (It just takes longer to get close to 0.)

There is more to learn about sequences than just their limits. We will also study their range and the relationships terms have with the terms that follow. We start with some definitions describing properties of the range.

Definition 9.1.3 Bounded and Unbounded Sequences

A sequence $\{a_n\}$ is said to be **bounded** if there exist real numbers m and M such that $m < a_n < M$ for all n in $\mathbb{N}$.

A sequence $\{a_n\}$ is said to be **unbounded** if it is not bounded.

A sequence $\{a_n\}$ is said to be **bounded above** if there exists an M such that $a_n < M$ for all n in $\mathbb{N}$; it is **bounded below** if there exists an m such that $m < a_n$ for all n in $\mathbb{N}$.

It follows from this definition that an unbounded sequence may be bounded above or bounded below; a sequence that is both bounded above and below is simply a bounded sequence.

Example 9.1.6 Determining boundedness of sequences

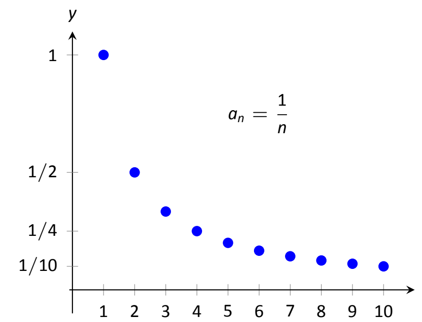
Determine the boundedness of the following sequences.

1. $\{a_n\} = \left\{\dfrac{1}{n}\right\}$ 2. $\{a_n\} = \{2^n\}$

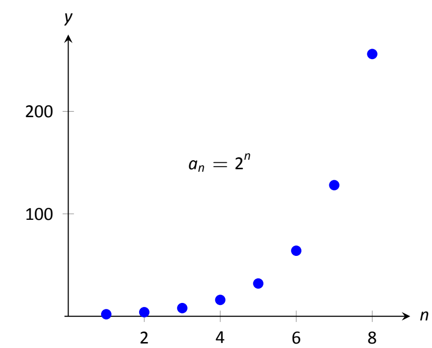
Solution

1. The terms of this sequence are always positive but are decreasing, so we have $0 < a_n < 2$ for all n. Thus this sequence is bounded. Figure 9.1.4(a) illustrates this.

2. The terms of this sequence obviously grow without bound. However, it is also true that these terms are all positive, meaning $0 < a_n$. Thus we can say the sequence is unbounded, but also bounded below. Figure 9.1.4(b) illustrates this.



(a)



(b)

Figure 9.1.4: A plot of $\{a_n\} = \{1/n\}$ and $\{a_n\} = \{2^n\}$ from Example 9.1.6.

The previous example produces some interesting concepts. First, we can recognize that the sequence $\{1/n\}$ converges to 0. This says, informally, that "most" of the terms of the sequence are "really close" to 0. This implies that the sequence is bounded, using the following logic. First, "most" terms are near 0, so we could find some sort of bound on these terms (using Definition 9.1.2, the bound is ε). That leaves a "few" terms that are not near 0 (i.e., a *finite* number of terms). A finite list of numbers is always bounded.

This logic implies that if a sequence converges, it must be bounded. This is indeed true, as stated by the following theorem.

Theorem 9.1.4 Convergent Sequences are Bounded

Let $\{a_n\}$ be a convergent sequence. Then $\{a_n\}$ is bounded.

Note: Keep in mind what Theorem 9.1.4 does *not* say. It does not say that bounded sequences must converge, nor does it say that if a sequence does not converge, it is not bounded.

In Example 9.1.5 we saw the sequence $\{b_n\} = \left\{(1+1/n)^n\right\}$, where it was stated that $\lim\limits_{n\to\infty} b_n = e$. (Note that this is simply restating part of Theorem 1.3.5.) Even though it may be difficult to intuitively grasp the behaviour of this sequence, we know immediately that it is bounded.

Another interesting concept to come out of Example 9.1.6 again involves the sequence $\{1/n\}$. We stated, without proof, that the terms of the sequence were decreasing. That is, that $a_{n+1} < a_n$ for all n. (This is easy to show. Clearly $n < n+1$. Taking reciprocals flips the inequality: $1/n > 1/(n+1)$. This is the

Note: It is sometimes useful to call a monotonically increasing sequence *strictly increasing* if $a_n < a_{n+1}$ for all n; i.e, we remove the possibility that subsequent terms are equal.
A similar statement holds for *strictly decreasing.*

same as $a_n > a_{n+1}$.) Sequences that either steadily increase or decrease are important, so we give this property a name.

Definition 9.1.4 Monotonic Sequences

1. A sequence $\{a_n\}$ is **monotonically increasing** if $a_n \leq a_{n+1}$ for all n, i.e.,
$$a_1 \leq a_2 \leq a_3 \leq \cdots a_n \leq a_{n+1} \cdots$$
2. A sequence $\{a_n\}$ is **monotonically decreasing** if $a_n \geq a_{n+1}$ for all n, i.e.,
$$a_1 \geq a_2 \geq a_3 \geq \cdots a_n \geq a_{n+1} \cdots$$
3. A sequence is **monotonic** if it is monotonically increasing or monotonically decreasing.

Example 9.1.7 Determining monotonicity

Determine the monotonicity of the following sequences.

1. $\{a_n\} = \left\{\dfrac{n+1}{n}\right\}$
2. $\{a_n\} = \left\{\dfrac{n^2+1}{n+1}\right\}$
3. $\{a_n\} = \left\{\dfrac{n^2-9}{n^2-10n+26}\right\}$
4. $\{a_n\} = \left\{\dfrac{n^2}{n!}\right\}$

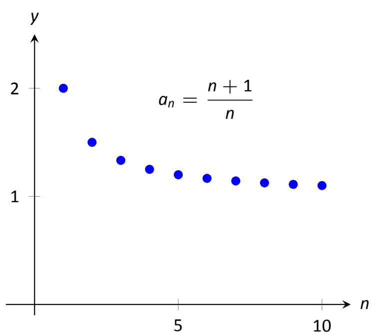


Figure 9.1.5: A plot of $\{a_n\} = \{(n+1)/n\}$ in Example 9.1.7.

SOLUTION In each of the following, we will examine $a_{n+1} - a_n$. If $a_{n+1} - a_n > 0$, we conclude that $a_n < a_{n+1}$ and hence the sequence is increasing. If $a_{n+1} - a_n < 0$, we conclude that $a_n > a_{n+1}$ and the sequence is decreasing. Of course, a sequence need not be monotonic and perhaps neither of the above will apply.

We also give a scatter plot of each sequence. These are useful as they suggest a pattern of monotonicity, but analytic work should be done to confirm a graphical trend.

1.
$$\begin{aligned} a_{n+1} - a_n &= \frac{n+2}{n+1} - \frac{n+1}{n} \\ &= \frac{(n+2)(n) - (n+1)^2}{(n+1)n} \\ &= \frac{-1}{n(n+1)} \\ &< 0 \quad \text{for all } n. \end{aligned}$$

Since $a_{n+1} - a_n < 0$ for all n, we conclude that the sequence is decreasing.

2.
$$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)^2+1}{n+2} - \frac{n^2+1}{n+1} \\ &= \frac{\big((n+1)^2+1\big)(n+1) - (n^2+1)(n+2)}{(n+1)(n+2)} \\ &= \frac{n^2+4n+1}{(n+1)(n+2)} \\ &> 0 \quad \text{for all } n. \end{aligned}$$

Since $a_{n+1} - a_n > 0$ for all n, we conclude the sequence is increasing; see Figure 9.1.6(a).

3. We can clearly see in Figure 9.1.6(b), where the sequence is plotted, that it is not monotonic. However, it does seem that after the first 4 terms it is decreasing. To understand why, perform the same analysis as done before:

$$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)^2 - 9}{(n+1)^2 - 10(n+1) + 26} - \frac{n^2 - 9}{n^2 - 10n + 26} \\ &= \frac{n^2 + 2n - 8}{n^2 - 8n + 17} - \frac{n^2 - 9}{n^2 - 10n + 26} \\ &= \frac{(n^2 + 2n - 8)(n^2 - 10n + 26) - (n^2 - 9)(n^2 - 8n + 17)}{(n^2 - 8n + 17)(n^2 - 10n + 26)} \\ &= \frac{-10n^2 + 60n - 55}{(n^2 - 8n + 17)(n^2 - 10n + 26)}. \end{aligned}$$

We want to know when this is greater than, or less than, 0. The denominator is always positive, therefore we are only concerned with the numerator. For small values of n, the numerator is positive. As n grows large, the numerator is dominated by $-10n^2$, meaning the entire fraction will be negative; i.e., for large enough n, $a_{n+1} - a_n < 0$. Using the quadratic formula we can determine that the numerator is negative for $n \geq 5$.

In short, the sequence is simply not monotonic, though it is useful to note that for $n \geq 5$, the sequence is monotonically decreasing.

4. Again, the plot in Figure 9.1.6(c) shows that the sequence is not monotonic, but it suggests that it is monotonically decreasing after the first term. We perform the usual analysis to confirm this.

$$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)^2}{(n+1)!} - \frac{n^2}{n!} \\ &= \frac{(n+1)^2 - n^2(n+1)}{(n+1)!} \\ &= \frac{-n^3 + 2n + 1}{(n+1)!} \end{aligned}$$

When $n = 1$, the above expression is > 0; for $n \geq 2$, the above expression is < 0. Thus this sequence is not monotonic, but it is monotonically decreasing after the first term.

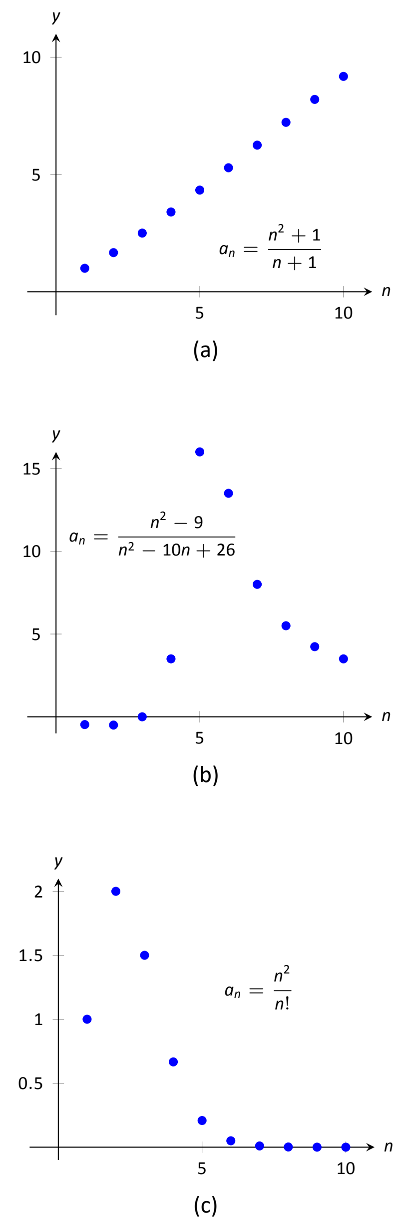


Figure 9.1.6: Plots of sequences in Example 9.1.7.

Knowing that a sequence is monotonic can be useful. Consider, for example, a sequence that is monotonically decreasing and is bounded below. We know the sequence is always getting smaller, but that there is a bound to how small it can become. This is enough to prove that the sequence will converge, as stated in the following theorem.

Theorem 9.1.5 Bounded Monotonic Sequences are Convergent

1. Let $\{a_n\}$ be a monotonically increasing sequence that is bounded above. Then $\{a_n\}$ converges.
2. Let $\{a_n\}$ be a monotonically decreasing sequence that is bounded below. Then $\{a_n\}$ converges.

Consider once again the sequence $\{a_n\} = \{1/n\}$. It is easy to show it is monotonically decreasing and that it is always positive (i.e., bounded below by 0). Therefore we can conclude by Theorem 9.1.5 that the sequence converges. We already knew this by other means, but in the following section this theorem will become very useful.

We can replace Theorem 9.1.5 with the statement "Let $\{a_n\}$ be a bounded, monotonic sequence. Then $\{a_n\}$ converges; i.e., $\lim\limits_{n\to\infty} a_n$ exists." We leave it to the reader in the exercises to show the theorem and the above statement are equivalent.

Sequences are a great source of mathematical inquiry. The On-Line Encyclopedia of Integer Sequences (`http://oeis.org`) contains thousands of sequences and their formulae. (As of this writing, there are 297,573 sequences in the database.) Perusing this database quickly demonstrates that a single sequence can represent several different "real life" phenomena.

Interesting as this is, our interest actually lies elsewhere. We are more interested in the *sum* of a sequence. That is, given a sequence $\{a_n\}$, we are very interested in $a_1 + a_2 + a_3 + \cdots$. Of course, one might immediately counter with "Doesn't this just add up to 'infinity'?" Many times, yes, but there are many important cases where the answer is no. This is the topic of *series*, which we begin to investigate in the next section.

Exercises 9.1

Terms and Concepts

1. Use your own words to define a *sequence.*

2. The domain of a sequence is the ________ numbers.

3. Use your own words to describe the *range* of a sequence.

4. Describe what it means for a sequence to be *bounded*.

Problems

In Exercises 5 – 8, give the first five terms of the given sequence.

5. $\{a_n\} = \left\{\dfrac{4^n}{(n+1)!}\right\}$

6. $\{b_n\} = \left\{\left(-\dfrac{3}{2}\right)^n\right\}$

7. $\{c_n\} = \left\{-\dfrac{n^{n+1}}{n+2}\right\}$

8. $\{d_n\} = \left\{\dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right)\right\}$

In Exercises 9 – 12, determine the n^{th} term of the given sequence.

9. $4, 7, 10, 13, 16, \ldots$

10. $3, -\dfrac{3}{2}, \dfrac{3}{4}, -\dfrac{3}{8}, \ldots$

11. $10, 20, 40, 80, 160, \ldots$

12. $1, 1, \dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{24}, \dfrac{1}{120}, \ldots$

In Exercises 13 – 16, use the following information to determine the limit of the given sequences.

- $\{a_n\} = \left\{\dfrac{2^n - 20}{2^n}\right\}$; $\quad \lim\limits_{n\to\infty} a_n = 1$
- $\{b_n\} = \left\{\left(1+\dfrac{2}{n}\right)^n\right\}$; $\quad \lim\limits_{n\to\infty} b_n = e^2$
- $\{c_n\} = \{\sin(3/n)\}$; $\quad \lim\limits_{n\to\infty} c_n = 0$

13. $\{a_n\} = \left\{\dfrac{2^n - 20}{7\cdot 2^n}\right\}$

14. $\{a_n\} = \{3b_n - a_n\}$

15. $\{a_n\} = \left\{\sin(3/n)\left(1+\dfrac{2}{n}\right)^n\right\}$

16. $\{a_n\} = \left\{\left(1+\dfrac{2}{n}\right)^{2n}\right\}$

In Exercises 17 – 28, determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

17. $\{a_n\} = \left\{(-1)^n\dfrac{n}{n+1}\right\}$

18. $\{a_n\} = \left\{\dfrac{4n^2 - n + 5}{3n^2 + 1}\right\}$

19. $\{a_n\} = \left\{\dfrac{4^n}{5^n}\right\}$

20. $\{a_n\} = \left\{\dfrac{n-1}{n} - \dfrac{n}{n-1}\right\}, n \geq 2$

21. $\{a_n\} = \{\ln(n)\}$

22. $\{a_n\} = \left\{\dfrac{3n}{\sqrt{n^2+1}}\right\}$

23. $\{a_n\} = \left\{\left(1+\dfrac{1}{n}\right)^n\right\}$

24. $\{a_n\} = \left\{5 - \dfrac{1}{n}\right\}$

25. $\{a_n\} = \left\{\dfrac{(-1)^{n+1}}{n}\right\}$

26. $\{a_n\} = \left\{\dfrac{1.1^n}{n}\right\}$

27. $\{a_n\} = \left\{\dfrac{2n}{n+1}\right\}$

28. $\{a_n\} = \left\{(-1)^n\dfrac{n^2}{2^n - 1}\right\}$

In Exercises 29 – 34, determine whether the sequence is bounded, bounded above, bounded below, or none of the above.

29. $\{a_n\} = \{\sin n\}$

30. $\{a_n\} = \{\tan n\}$

31. $\{a_n\} = \left\{(-1)^n\dfrac{3n-1}{n}\right\}$

32. $\{a_n\} = \left\{\dfrac{3n^2-1}{n}\right\}$

33. $\{a_n\} = \{n\cos n\}$

34. $\{a_n\} = \{2^n - n!\}$

In Exercises 35 – 38, determine whether the sequence is monotonically increasing or decreasing. If it is not, determine if there is an m such that it is monotonic for all $n \geq m$.

35. $\{a_n\} = \left\{\dfrac{n}{n+2}\right\}$

36. $\{a_n\} = \left\{\dfrac{n^2 - 6n + 9}{n}\right\}$

37. $\{a_n\} = \left\{(-1)^n \dfrac{1}{n^3}\right\}$

38. $\{a_n\} = \left\{\dfrac{n^2}{2^n}\right\}$

Exercises 39 – 42 explore further the theory of sequences.

39. Prove Theorem 9.1.2; that is, use the definition of the limit of a sequence to show that if $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

40. Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = K$.

    (a) Show that if $a_n < b_n$ for all n, then $L \leq K$.

    (b) Give an example where $L = K$.

41. Prove the Squeeze Theorem for sequences: Let $\{a_n\}$ and $\{b_n\}$ be such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = L$, and let $\{c_n\}$ be such that $a_n \leq c_n \leq b_n$ for all n. Then $\lim\limits_{n\to\infty} c_n = L$

42. Prove the statement "Let $\{a_n\}$ be a bounded, monotonic sequence. Then $\{a_n\}$ converges; i.e., $\lim\limits_{n\to\infty} a_n$ exists." is equivalent to Theorem 9.1.5. That is,

    (a) Show that if Theorem 9.1.5 is true, then above statement is true, and

    (b) Show that if the above statement is true, then Theorem 9.1.5 is true.

9.2 Infinite Series

Given the sequence $\{a_n\} = \{1/2^n\} = 1/2,\ 1/4,\ 1/8,\ \ldots$, consider the following sums:

$$\begin{array}{ccccc} a_1 & = & 1/2 & = & 1/2 \\ a_1 + a_2 & = & 1/2 + 1/4 & = & 3/4 \\ a_1 + a_2 + a_3 & = & 1/2 + 1/4 + 1/8 & = & 7/8 \\ a_1 + a_2 + a_3 + a_4 & = & 1/2 + 1/4 + 1/8 + 1/16 & = & 15/16 \end{array}$$

In general, we can show that

$$a_1 + a_2 + a_3 + \cdots + a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

Let S_n be the sum of the first n terms of the sequence $\{1/2^n\}$. From the above, we see that $S_1 = 1/2$, $S_2 = 3/4$, etc. Our formula at the end shows that $S_n = 1 - 1/2^n$.

Now consider the following limit: $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \big(1 - 1/2^n\big) = 1$. This limit can be interpreted as saying something amazing: *the sum of* all *the terms of the sequence* $\{1/2^n\}$ *is 1.*

This example illustrates some interesting concepts that we explore in this section. We begin this exploration with some definitions.

Definition 9.2.1 Infinite Series, n^{th} Partial Sums, Convergence, Divergence

Let $\{a_n\}$ be a sequence.

1. The sum $\displaystyle\sum_{n=1}^{\infty} a_n$ is an **infinite series** (or, simply **series**).
2. Let $S_n = \displaystyle\sum_{i=1}^{n} a_i$; the sequence $\{S_n\}$ is the sequence of n^{th} **partial sums** of $\{a_n\}$.
3. If the sequence $\{S_n\}$ converges to L, we say the series $\displaystyle\sum_{n=1}^{\infty} a_n$ **converges** to L, and we write $\displaystyle\sum_{n=1}^{\infty} a_n = L$.
4. If the sequence $\{S_n\}$ diverges, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ **diverges**.

Using our new terminology, we can state that the series $\displaystyle\sum_{n=1}^{\infty} 1/2^n$ converges, and $\displaystyle\sum_{n=1}^{\infty} 1/2^n = 1$.

We will explore a variety of series in this section. We start with two series that diverge, showing how we might discern divergence.

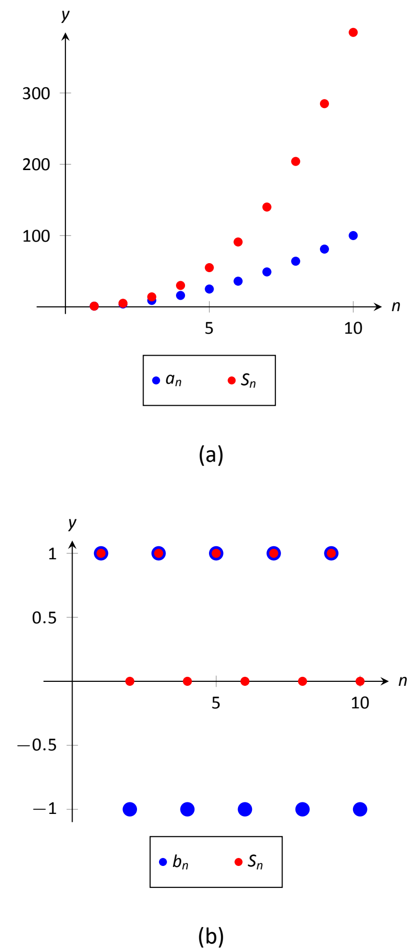


Figure 9.2.1: Scatter plots relating to Example 9.2.1.

Example 9.2.1 Showing series diverge

1. Let $\{a_n\} = \{n^2\}$. Show $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

2. Let $\{b_n\} = \{(-1)^{n+1}\}$. Show $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

Solution

1. Consider S_n, the n^{th} partial sum.

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= 1^2 + 2^2 + 3^2 \cdots + n^2. \end{aligned}$$

By Theorem 5.3.1, this is

$$= \frac{n(n+1)(2n+1)}{6}.$$

Since $\displaystyle\lim_{n\to\infty} S_n = \infty$, we conclude that the series $\displaystyle\sum_{n=1}^{\infty} n^2$ diverges. It is instructive to write $\displaystyle\sum_{n=1}^{\infty} n^2 = \infty$ for this tells us *how* the series diverges: it grows without bound.

A scatter plot of the sequences $\{a_n\}$ and $\{S_n\}$ is given in Figure 9.2.1(a). The terms of $\{a_n\}$ are growing, so the terms of the partial sums $\{S_n\}$ are growing even faster, illustrating that the series diverges.

2. The sequence $\{b_n\}$ starts with 1, −1, 1, −1, ... Consider some of the partial sums S_n of $\{b_n\}$:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 0 \\ S_3 &= 1 \\ S_4 &= 0 \end{aligned}$$

This pattern repeats; we find that $S_n = \begin{cases} 1 & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$. As $\{S_n\}$ oscillates, repeating 1, 0, 1, 0, ..., we conclude that $\displaystyle\lim_{n\to\infty} S_n$ does not exist, hence $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges.

A scatter plot of the sequence $\{b_n\}$ and the partial sums $\{S_n\}$ is given in Figure 9.2.1(b). When n is odd, $b_n = S_n$ so the marks for b_n are drawn oversized to show they coincide.

While it is important to recognize when a series diverges, we are generally more interested in the series that converge. In this section we will demonstrate a few general techniques for determining convergence; later sections will delve deeper into this topic.

Geometric Series

One important type of series is a *geometric series*.

Definition 9.2.2 Geometric Series

A **geometric series** is a series of the form

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots$$

Note that the index starts at $n = 0$, not $n = 1$.

We started this section with a geometric series, although we dropped the first term of 1. One reason geometric series are important is that they have nice convergence properties.

Theorem 9.2.1 Geometric Series Test

Consider the geometric series $\sum_{n=0}^{\infty} r^n$.

1. The n^{th} partial sum is: $S_n = \dfrac{1 - r^{n+1}}{1 - r}$.
2. The series converges if, and only if, $|r| < 1$. When $|r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

According to Theorem 9.2.1, the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^2 = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

converges as $r = 1/2$, and $\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - 1/2} = 2$. This concurs with our introductory example; while there we got a sum of 1, we skipped the first term of 1.

Example 9.2.2 Exploring geometric series

Check the convergence of the following series. If the series converges, find its sum.

1. $\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n$ 2. $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n$ 3. $\sum_{n=0}^{\infty} 3^n$

Solution

1. Since $r = 3/4 < 1$, this series converges. By Theorem 9.2.1, we have that

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1 - 3/4} = 4.$$

However, note the subscript of the summation in the given series: we are

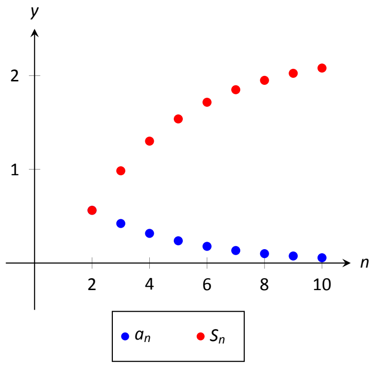


Figure 9.2.2: Scatter plots relating to the series in Example 9.2.2.

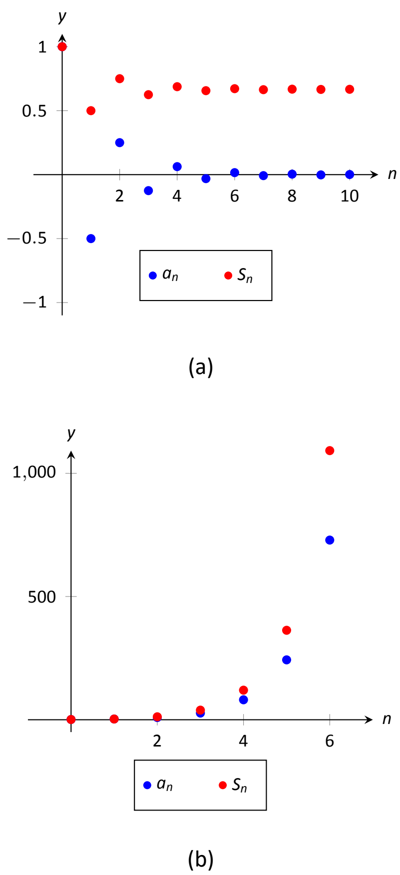


Figure 9.2.3: Scatter plots relating to the series in Example 9.2.2.

Note: Theorem 9.2.2 assumes that $an+b\neq 0$ for all n. If $an+b=0$ for some n, then of course the series does not converge regardless of p as not all of the terms of the sequence are defined.

to start with $n=2$. Therefore we subtract off the first two terms, giving:

$$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n = 4-1-\frac{3}{4}=\frac{9}{4}.$$

This is illustrated in Figure 9.2.2.

2. Since $|r|=1/2<1$, this series converges, and by Theorem 9.2.1,

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n=\frac{1}{1-(-1/2)}=\frac{2}{3}.$$

The partial sums of this series are plotted in Figure 9.2.3(a). Note how the partial sums are not purely increasing as some of the terms of the sequence $\{(-1/2)^n\}$ are negative.

3. Since $r>1$, the series diverges. (This makes "common sense"; we expect the sum

$$1+3+9+27+81+243+\cdots$$

to diverge.) This is illustrated in Figure 9.2.3(b).

p–Series

Another important type of series is the *p-series*.

Definition 9.2.3 *p*–Series, General *p*–Series

1. A ***p*–series** is a series of the form

$$\sum_{n=1}^{\infty}\frac{1}{n^p},\qquad \text{where } p>0.$$

2. A **general *p*–series** is a series of the form

$$\sum_{n=1}^{\infty}\frac{1}{(an+b)^p},\qquad \text{where } p>0 \text{ and } a, b \text{ are real numbers.}$$

Like geometric series, one of the nice things about p–series is that they have easy to determine convergence properties.

Theorem 9.2.2 *p*–Series Test

A general *p*–series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(an+b)^p}$ will converge if, and only if, $p>1$.

Example 9.2.3 Determining convergence of series

Determine the convergence of the following series.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$
2. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$
4. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$
5. $\displaystyle\sum_{n=11}^{\infty}\frac{1}{(\frac{1}{2}n-5)^3}$
6. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$

Solution

1. This is a p–series with $p = 1$. By Theorem 9.2.2, this series diverges.

   This series is a famous series, called the *Harmonic Series*, so named because of its relationship to *harmonics* in the study of music and sound.

2. This is a p–series with $p = 2$. By Theorem 9.2.2, it converges. Note that the theorem does not give a formula by which we can determine *what* the series converges to; we just know it converges. A famous, unexpected result is that this series converges to $\pi^2/6$.

3. This is a p–series with $p = 1/2$; the theorem states that it diverges.

4. This is not a p–series; the definition does not allow for alternating signs. Therefore we cannot apply Theorem 9.2.2. (Another famous result states that this series, the *Alternating Harmonic Series*, converges to $\ln 2$.)

5. This is a general p–series with $p = 3$, therefore it converges.

6. This is not a p–series, but a geometric series with $r = 1/2$. It converges.

Later sections will provide tests by which we can determine whether or not a given series converges. This, in general, is much easier than determining *what* a given series converges to. There are many cases, though, where the sum can be determined.

Example 9.2.4 Telescoping series

Evaluate the sum $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

Solution It will help to write down some of the first few partial sums of this series.

$$\begin{aligned}
S_1 &= \frac{1}{1}-\frac{1}{2} && = 1-\frac{1}{2}\\
S_2 &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right) && = 1-\frac{1}{3}\\
S_3 &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right) && = 1-\frac{1}{4}\\
S_4 &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right) && = 1-\frac{1}{5}
\end{aligned}$$

Note how most of the terms in each partial sum are cancelled out! In general, we see that $S_n = 1-\dfrac{1}{n+1}$. The sequence $\{S_n\}$ converges, as $\displaystyle\lim_{n\to\infty} S_n =$

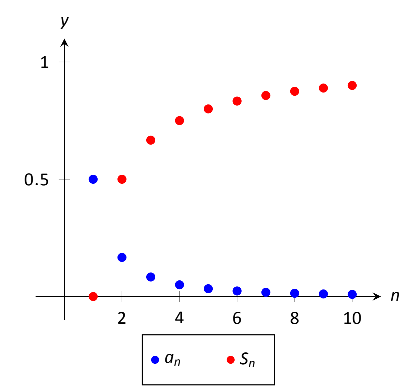


Figure 9.2.4: Scatter plots relating to the series of Example 9.2.4.

$\lim\limits_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$, and so we conclude that $\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1$. Partial sums of the series are plotted in Figure 9.2.4.

The series in Example 9.2.4 is an example of a **telescoping series**. Informally, a telescoping series is one in which most terms cancel with preceding or following terms, reducing the number of terms in each partial sum. The partial sum S_n did not contain n terms, but rather just two: 1 and $1/(n+1)$.

When possible, seek a way to write an explicit formula for the n^{th} partial sum S_n. This makes evaluating the limit $\lim\limits_{n\to\infty} S_n$ much more approachable. We do so in the next example.

Example 9.2.5 Evaluating series

Evaluate each of the following infinite series.

1. $\sum\limits_{n=1}^{\infty}\frac{2}{n^2+2n}$ 2. $\sum\limits_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)$

Solution

1. We can decompose the fraction $2/(n^2+2n)$ as

$$\frac{2}{n^2+2n}=\frac{1}{n}-\frac{1}{n+2}.$$

(See Section 6.5, Partial Fraction Decomposition, to recall how this is done, if necessary.)

Expressing the terms of $\{S_n\}$ is now more instructive:

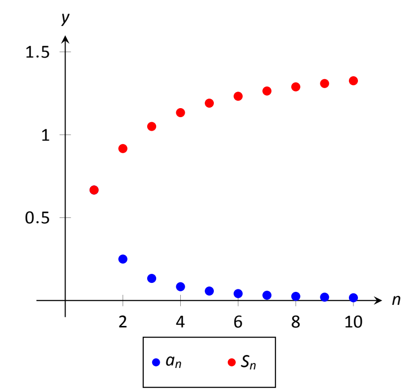
$$\begin{aligned}
S_1 &= 1-\frac{1}{3} && =1-\frac{1}{3}\\
S_2 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right) && =1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}\\
S_3 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right) && =1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}\\
S_4 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right) && =1+\frac{1}{2}-\frac{1}{5}-\frac{1}{6}\\
S_5 &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right) && =1+\frac{1}{2}-\frac{1}{6}-\frac{1}{7}
\end{aligned}$$

We again have a telescoping series. In each partial sum, most of the terms cancel and we obtain the formula $S_n=1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$. Taking limits allows us to determine the convergence of the series:

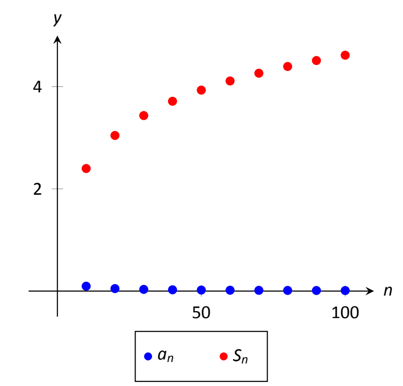
$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{3}{2},\quad\text{so}\quad\sum_{n=1}^{\infty}\frac{1}{n^2+2n}=\frac{3}{2}.$$

This is illustrated in Figure 9.2.5(a).

2. We begin by writing the first few partial sums of the series:



(a)



(b)

Figure 9.2.5: Scatter plots relating to the series in Example 9.2.5.

$$S_1 = \ln(2)$$
$$S_2 = \ln(2) + \ln\left(\frac{3}{2}\right)$$
$$S_3 = \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right)$$
$$S_4 = \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right)$$

At first, this does not seem helpful, but recall the logarithmic identity: $\ln x + \ln y = \ln(xy)$. Applying this to S_4 gives:

$$S_4 = \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right) = \ln\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\right) = \ln(5).$$

We can conclude that $\{S_n\} = \{\ln(n+1)\}$. This sequence does not converge, as $\lim\limits_{n\to\infty} S_n = \infty$. Therefore $\sum\limits_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \infty$; the series diverges. Note in Figure 9.2.5(b) how the sequence of partial sums grows slowly; after 100 terms, it is not yet over 5. Graphically we may be fooled into thinking the series converges, but our analysis above shows that it does not.

We are learning about a new mathematical object, the series. As done before, we apply "old" mathematics to this new topic.

Theorem 9.2.3 Properties of Infinite Series

Let $\sum\limits_{n=1}^{\infty} a_n = L$, $\sum\limits_{n=1}^{\infty} b_n = K$, and let c be a constant.

1. Constant Multiple Rule: $\sum\limits_{n=1}^{\infty} c\cdot a_n = c\cdot\sum\limits_{n=1}^{\infty} a_n = c\cdot L.$
2. Sum/Difference Rule: $\sum\limits_{n=1}^{\infty} (a_n \pm b_n) = \sum\limits_{n=1}^{\infty} a_n \pm \sum\limits_{n=1}^{\infty} b_n = L \pm K.$

Before using this theorem, we provide a few "famous" series.

Key Idea 9.2.1 Important Series

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e.$ (Note that the index starts with $n = 0$.)

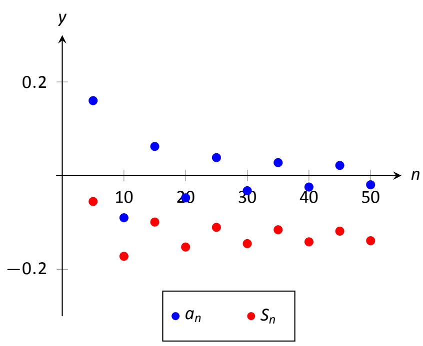
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$

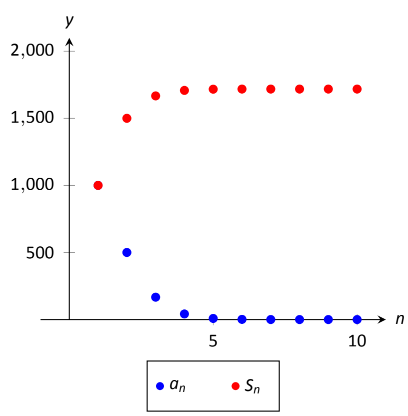
4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2n+1} = \frac{\pi}{4}.$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (This is called the *Harmonic Series*.)

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$ (This is called the *Alternating Harmonic Series*.)



(a)



(b)

Figure 9.2.6: Scatter plots relating to the series in Example 9.2.6.

Example 9.2.6 Evaluating series

Evaluate the given series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(n^2-n\right)}{n^3}$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{1000}{n!}$ 3. $\displaystyle\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\frac{1}{49}+\cdots$

SOLUTION

1. We start by using algebra to break the series apart:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(n^2-n\right)}{n^3} &= \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}n^2}{n^3}-\frac{(-1)^{n+1}n}{n^3}\right)\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\\ &= \ln(2)-\frac{\pi^2}{12} \approx -0.1293.\end{aligned}$$

This is illustrated in Figure 9.2.6(a).

2. This looks very similar to the series that involves e in Key Idea 9.2.1. Note, however, that the series given in this example starts with $n = 1$ and not $n = 0$. The first term of the series in the Key Idea is $1/0! = 1$, so we will subtract this from our result below:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1000}{n!} &= 1000 \cdot \sum_{n=1}^{\infty}\frac{1}{n!}\\ &= 1000 \cdot (e-1) \approx 1718.28.\end{aligned}$$

This is illustrated in Figure 9.2.6(b). The graph shows how this particular series converges very rapidly.

3. The denominators in each term are perfect squares; we are adding $\displaystyle\sum_{n=4}^{\infty}\frac{1}{n^2}$ (note we start with $n=4$, not $n=1$). This series will converge. Using the formula from Key Idea 9.2.1, we have the following:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^2} &= \sum_{n=1}^{3}\frac{1}{n^2}+\sum_{n=4}^{\infty}\frac{1}{n^2}\\
\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{3}\frac{1}{n^2} &= \sum_{n=4}^{\infty}\frac{1}{n^2}\\
\frac{\pi^2}{6}-\left(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}\right) &= \sum_{n=4}^{\infty}\frac{1}{n^2}\\
\frac{\pi^2}{6}-\frac{49}{36} &= \sum_{n=4}^{\infty}\frac{1}{n^2}\\
0.2838 &\approx \sum_{n=4}^{\infty}\frac{1}{n^2}
\end{aligned}$$

It may take a while before one is comfortable with this statement, whose truth lies at the heart of the study of infinite series: *it is possible that the sum of an infinite list of nonzero numbers is finite.* We have seen this repeatedly in this section, yet it still may "take some getting used to."

As one contemplates the behaviour of series, a few facts become clear.

1. In order to add an infinite list of nonzero numbers and get a finite result, "most" of those numbers must be "very near" 0.

2. If a series diverges, it means that the sum of an infinite list of numbers is not finite (it may approach $\pm\infty$ or it may oscillate), and:

   (a) The series will still diverge if the first term is removed.

   (b) The series will still diverge if the first 10 terms are removed.

   (c) The series will still diverge if the first $1,000,000$ terms are removed.

   (d) The series will still diverge if any finite number of terms from anywhere in the series are removed.

These concepts are very important and lie at the heart of the next two theorems.

Theorem 9.2.4 **n^{th}–Term Test for Divergence**

Consider the series $\displaystyle\sum_{n=1}^{\infty}a_n$. If $\displaystyle\lim_{n\to\infty}a_n\neq 0$, then $\displaystyle\sum_{n=1}^{\infty}a_n$ diverges.

Important! This theorem *does not state* that if $\displaystyle\lim_{n\to\infty}a_n=0$ then $\displaystyle\sum_{n=1}^{\infty}a_n$ converges. The standard example of this is the Harmonic Series, as given in Key Idea 9.2.1. The Harmonic Sequence, $\{1/n\}$, converges to 0; the Harmonic Series, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$, diverges.

Looking back, we can apply this theorem to the series in Example 9.2.1. In that example, the n^{th} terms of both sequences do not converge to 0, therefore we can quickly conclude that each series diverges.

One can rewrite Theorem 9.2.4 to state "If a series converges, then the underlying sequence converges to 0." While it is important to understand the truth of this statement, in practice it is rarely used. It is generally far easier to prove the convergence of a sequence than the convergence of a series.

Theorem 9.2.5 Infinite Nature of Series

The convergence or divergence of an infinite series remains unchanged by the addition or subtraction of any finite number of terms. That is:

1. A divergent series will remain divergent with the addition or subtraction of any finite number of terms.
2. A convergent series will remain convergent with the addition or subtraction of any finite number of terms. (Of course, the *sum* will likely change.)

Consider once more the Harmonic Series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges; that is, the sequence of partial sums $\{S_n\}$ grows (very, very slowly) without bound. One might think that by removing the "large" terms of the sequence that perhaps the series will converge. This is simply not the case. For instance, the sum of the first 10 million terms of the Harmonic Series is about 16.7. Removing the first 10 million terms from the Harmonic Series changes the n^{th} partial sums, effectively subtracting 16.7 from the sum. However, a sequence that is growing without bound will still grow without bound when 16.7 is subtracted from it.

The equations below illustrate this. The first line shows the infinite sum of the Harmonic Series split into the sum of the first 10 million terms plus the sum of "everything else." The next equation shows us subtracting these first 10 million terms from both sides. The final equation employs a bit of "psuedo–math": subtracting 16.7 from "infinity" still leaves one with "infinity."

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{1}{n} &= \sum_{n=1}^{10,000,000}\frac{1}{n} + \sum_{n=10,000,001}^{\infty}\frac{1}{n} \\ \sum_{n=1}^{\infty}\frac{1}{n} - \sum_{n=1}^{10,000,000}\frac{1}{n} &= \sum_{n=10,000,001}^{\infty}\frac{1}{n} \\ \infty - 16.7 &= \infty. \end{aligned}$$

This section introduced us to series and defined a few special types of series whose convergence properties are well known: we know when a p-series or a geometric series converges or diverges. Most series that we encounter are not one of these types, but we are still interested in knowing whether or not they converge. The next three sections introduce tests that help us determine whether or not a given series converges.

Exercises 9.2

Terms and Concepts

1. Use your own words to describe how sequences and series are related.

2. Use your own words to define a *partial sum*.

3. Given a series $\sum_{n=1}^{\infty} a_n$, describe the two sequences related to the series that are important.

4. Use your own words to explain what a geometric series is.

5. T/F: If $\{a_n\}$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

6. T/F: If $\{a_n\}$ converges to 0, then $\sum_{n=0}^{\infty} a_n$ converges.

Problems

In Exercises 7 – 14, a series $\sum_{n=1}^{\infty} a_n$ is given.

(a) Give the first 5 partial sums of the series.

(b) Give a graph of the first 5 terms of a_n and S_n on the same axes.

7. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

8. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

9. $\sum_{n=1}^{\infty} \cos(\pi n)$

10. $\sum_{n=1}^{\infty} n$

11. $\sum_{n=1}^{\infty} \frac{1}{n!}$

12. $\sum_{n=1}^{\infty} \frac{1}{3^n}$

13. $\sum_{n=1}^{\infty} \left(-\frac{9}{10}\right)^n$

14. $\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$

In Exercises 15 – 20, use Theorem 9.2.4 to show the given series diverges.

15. $\sum_{n=1}^{\infty} \frac{3n^2}{n(n+2)}$

16. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

17. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

18. $\sum_{n=1}^{\infty} \frac{5^n - n^5}{5^n + n^5}$

19. $\sum_{n=1}^{\infty} \frac{2^n + 1}{2^{n+1}}$

20. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

In Exercises 21 – 30, state whether the given series converges or diverges.

21. $\sum_{n=1}^{\infty} \frac{1}{n^5}$

22. $\sum_{n=0}^{\infty} \frac{1}{5^n}$

23. $\sum_{n=0}^{\infty} \frac{6^n}{5^n}$

24. $\sum_{n=1}^{\infty} n^{-4}$

25. $\sum_{n=1}^{\infty} \sqrt{n}$

26. $\sum_{n=1}^{\infty} \frac{10}{n!}$

27. T/F: If $\{a_n\}$ converges to 0, then $\sum_{n=0}^{\infty} a_n$ converges.

28. $\sum_{n=1}^{\infty} \frac{2}{(2n+8)^2}$

29. $\sum_{n=1}^{\infty} \frac{1}{2n}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

In Exercises 31 – 46, a series is given.

(a) Find a formula for S_n, the n^{th} partial sum of the series.

(b) Determine whether the series converges or diverges. If it converges, state what it converges to.

31. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n}$

32. $\displaystyle\sum_{n=1}^{\infty} 2$

33. $1^3 + 2^3 + 3^3 + 4^3 + \cdots$

34. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n$

35. $\displaystyle\sum_{n=0}^{\infty} \frac{5}{2^n}$

36. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

37. $1 - \dfrac{1}{3} + \dfrac{1}{9} - \dfrac{1}{27} + \dfrac{1}{81} + \cdots$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+2)}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$

41. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

43. $\dfrac{1}{1\cdot 4} + \dfrac{1}{2\cdot 5} + \dfrac{1}{3\cdot 6} + \dfrac{1}{4\cdot 7} + \cdots$

44. $2 + \left(\dfrac{1}{2} + \dfrac{1}{3}\right) + \left(\dfrac{1}{4} + \dfrac{1}{9}\right) + \left(\dfrac{1}{8} + \dfrac{1}{27}\right) + \cdots$

45. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

46. $\displaystyle\sum_{n=0}^{\infty} \big(\sin 1\big)^n$

47. Break the Harmonic Series into the sum of the odd and even terms:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{2n-1} + \sum_{n=1}^{\infty} \frac{1}{2n}.$$

The goal is to show that each of the series on the right diverge.

(a) Show why $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1} > \sum_{n=1}^{\infty} \frac{1}{2n}$.

(Compare each n^{th} partial sum.)

(b) Show why $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1} < 1 + \sum_{n=1}^{\infty} \frac{1}{2n}$

(c) Explain why (a) and (b) demonstrate that the series of odd terms is convergent, if, and only if, the series of even terms is also convergent. (That is, show both converge or both diverge.)

(d) Explain why knowing the Harmonic Series is divergent determines that the even and odd series are also divergent.

48. Show the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2n-1)(2n+1)}$ diverges.

9.3 Integral and Comparison Tests

Knowing whether or not a series converges is very important, especially when we discuss Power Series in Section 9.6. Theorems 9.2.1 and 9.2.2 give criteria for when Geometric and p-series converge, and Theorem 9.2.4 gives a quick test to determine if a series diverges. There are many important series whose convergence cannot be determined by these theorems, though, so we introduce a set of tests that allow us to handle a broad range of series. We start with the Integral Test.

Integral Test

We stated in Section 9.1 that a sequence $\{a_n\}$ is a function $a(n)$ whose domain is $\mathbb{N}$, the set of natural numbers. If we can extend $a(n)$ to $\mathbb{R}$, the real numbers, and it is both positive and decreasing on $[1,\infty)$, then the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$ is the same as $\displaystyle\int_1^{\infty} a(x)\,dx$.

Note: Theorem 9.3.1 does not state that the integral and the summation have the same value.

Theorem 9.3.1 Integral Test

Let a sequence $\{a_n\}$ be defined by $a_n = a(n)$, where $a(n)$ is continuous, positive and decreasing on $[1,\infty)$. Then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, if, and only if, $\displaystyle\int_1^{\infty} a(x)\,dx$ converges.

We can demonstrate the truth of the Integral Test with two simple graphs. In Figure 9.3.1(a), the height of each rectangle is $a(n) = a_n$ for $n = 1, 2, \ldots$, and clearly the rectangles enclose more area than the area under $y = a(x)$. Therefore we can conclude that

$$\int_1^{\infty} a(x)\,dx < \sum_{n=1}^{\infty} a_n. \tag{9.1}$$

In Figure 9.3.1(b), we draw rectangles under $y = a(x)$ with the Right-Hand rule, starting with $n = 2$. This time, the area of the rectangles is less than the area under $y = a(x)$, so $\displaystyle\sum_{n=2}^{\infty} a_n < \int_1^{\infty} a(x)\,dx$. Note how this summation starts with $n = 2$; adding a_1 to both sides lets us rewrite the summation starting with $n = 1$:

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx. \tag{9.2}$$

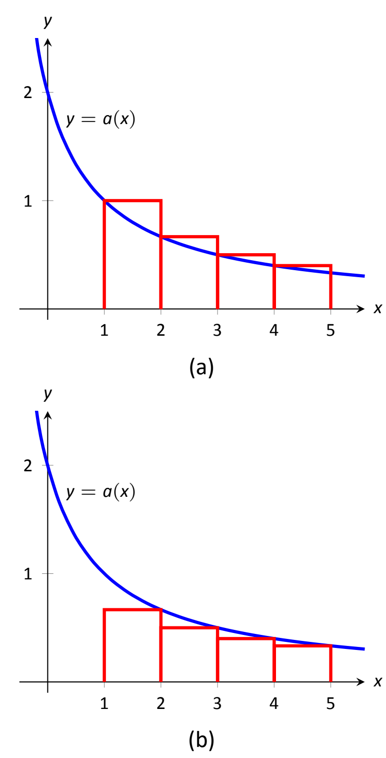


Figure 9.3.1: Illustrating the truth of the Integral Test.

Combining Equations (9.1) and (9.2), we have

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx < a_1 + \sum_{n=1}^{\infty} a_n. \tag{9.3}$$

From Equation (9.3) we can make the following two statements:

1. If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, so does $\displaystyle\int_1^{\infty} a(x)\,dx$ (because $\displaystyle\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx$)

2. If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, so does $\displaystyle\int_1^{\infty} a(x)\,dx$ (because $\displaystyle\int_1^{\infty} a(x)\,dx < \sum_{n=1}^{\infty} a_n$.)

Therefore the series and integral either both converge or both diverge. Theorem 9.2.5 allows us to extend this theorem to series where $a(n)$ is positive and decreasing on $[b,\infty)$ for some $b > 1$.

Example 9.3.1 Using the Integral Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$. (The terms of the sequence $\{a_n\} = \{\ln n/n^2\}$ and the n[th] partial sums are given in Figure 9.3.2.)

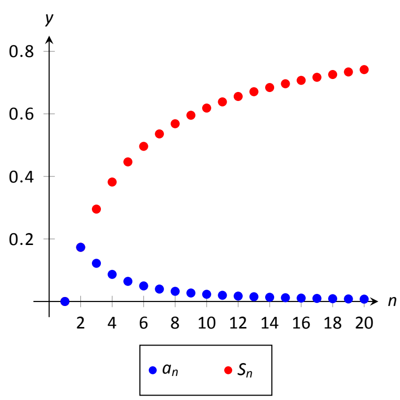


Figure 9.3.2: Plotting the sequence and series in Example 9.3.1.

Solution Figure 9.3.2 implies that $a(n) = (\ln n)/n^2$ is positive and decreasing on $[2,\infty)$. We can determine this analytically, too. We know $a(n)$ is positive as both $\ln n$ and n^2 are positive on $[2,\infty)$. To determine that $a(n)$ is decreasing, consider $a'(n) = (1 - 2\ln n)/n^3$, which is negative for $n \geq 2$. Since $a'(n)$ is negative, $a(n)$ is decreasing.

Applying the Integral Test, we test the convergence of $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$. Integrating this improper integral requires the use of Integration by Parts, with $u = \ln x$ and $dv = 1/x^2\,dx$.

$$\begin{aligned}
\int_1^{\infty} \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx \\
&= \lim_{b\to\infty} -\frac{1}{x}\ln x\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx
\end{aligned}$$

$$\begin{aligned}
&= \lim_{b\to\infty} -\frac{1}{x}\ln x - \frac{1}{x}\Big|_1^b \\
&= \lim_{b\to\infty} 1 - \frac{1}{b} - \frac{\ln b}{b}. \qquad \text{Apply L'Hospital's Rule:} \\
&= 1.
\end{aligned}$$

Since $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$ converges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$.

Theorem 9.2.2 was given without justification, stating that the general p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$. In the following example, we prove this to be true by applying the Integral Test.

Example 9.3.2 Using the Integral Test to establish Theorem 9.2.2.

Use the Integral Test to prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$.

Solution Consider the integral $\displaystyle\int_1^\infty \frac{1}{(ax+b)^p}\,dx$; assuming $p \neq 1$,

$$\begin{aligned}\int_1^\infty \frac{1}{(ax+b)^p}\,dx &= \lim_{c\to\infty}\int_1^c \frac{1}{(ax+b)^p}\,dx\\ &= \lim_{c\to\infty}\frac{1}{a(1-p)}(ax+b)^{1-p}\Big|_1^c\\ &= \lim_{c\to\infty}\frac{1}{a(1-p)}\big((ac+b)^{1-p}-(a+b)^{1-p}\big).\end{aligned}$$

This limit converges if, and only if, $p > 1$. It is easy to show that the integral also diverges in the case of $p = 1$. (This result is similar to the work preceding Key Idea 6.6.1.)

Therefore $\displaystyle\sum_{n=1}^\infty \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$.

Note: A sequence $\{a_n\}$ is a **positive sequence** if $a_n > 0$ for all n.

Because of Theorem 9.2.5, any theorem that relies on a positive sequence still holds true when $a_n > 0$ for all but a finite number of values of n.

We consider two more convergence tests in this section, both *comparison* tests. That is, we determine the convergence of one series by comparing it to another series with known convergence.

Direct Comparison Test

Theorem 9.3.2 Direct Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences where $a_n \leq b_n$ for all $n \geq N$, for some $N \geq 1$.

1. If $\displaystyle\sum_{n=1}^\infty b_n$ converges, then $\displaystyle\sum_{n=1}^\infty a_n$ converges.
2. If $\displaystyle\sum_{n=1}^\infty a_n$ diverges, then $\displaystyle\sum_{n=1}^\infty b_n$ diverges.

Example 9.3.3 Applying the Direct Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^\infty \frac{1}{3^n+n^2}$.

Solution This series is neither a geometric or p-series, but seems related. We predict it will converge, so we look for a series with larger terms that converges. (Note too that the Integral Test seems difficult to apply here.)

Since $3^n < 3^n + n^2$, $\displaystyle\frac{1}{3^n} > \frac{1}{3^n+n^2}$ for all $n \geq 1$. The series $\displaystyle\sum_{n=1}^\infty \frac{1}{3^n}$ is a convergent geometric series; by Theorem 9.3.2, $\displaystyle\sum_{n=1}^\infty \frac{1}{3^n+n^2}$ converges.

Example 9.3.4 Applying the Direct Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^\infty \frac{1}{n-\ln n}$.

Solution We know the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and it seems that the given series is closely related to it, hence we predict it will diverge.

Since $n \geq n - \ln n$ for all $n \geq 1$, $\frac{1}{n} \leq \frac{1}{n - \ln n}$ for all $n \geq 1$.

The Harmonic Series diverges, so we conclude that $\sum_{n=1}^{\infty} \frac{1}{n - \ln n}$ diverges as well.

The concept of direct comparison is powerful and often relatively easy to apply. Practice helps one develop the necessary intuition to quickly pick a proper series with which to compare. However, it is easy to construct a series for which it is difficult to apply the Direct Comparison Test.

Consider $\sum_{n=1}^{\infty} \frac{1}{n + \ln n}$. It is very similar to the divergent series given in Example 9.3.4. We suspect that it also diverges, as $\frac{1}{n} \approx \frac{1}{n + \ln n}$ for large n. However, the inequality that we naturally want to use "goes the wrong way": since $n \leq n + \ln n$ for all $n \geq 1$, $\frac{1}{n} \geq \frac{1}{n + \ln n}$ for all $n \geq 1$. The given series has terms *less than* the terms of a divergent series, and we cannot conclude anything from this.

Fortunately, we can apply another test to the given series to determine its convergence.

Limit Comparison Test

Theorem 9.3.3 Limit Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences.

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is a positive real number, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then if $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.

Theorem 9.3.3 is most useful when the convergence of the series from $\{b_n\}$ is known and we are trying to determine the convergence of the series from $\{a_n\}$.

We use the Limit Comparison Test in the next example to examine the series $\sum_{n=1}^{\infty} \frac{1}{n + \ln n}$ which motivated this new test.

Example 9.3.5 Applying the Limit Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$ using the Limit Comparison Test.

Solution We compare the terms of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$ to the terms of the Harmonic Sequence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$:

$$\begin{aligned} \lim_{n\to\infty} \frac{1/(n+\ln n)}{1/n} &= \lim_{n\to\infty} \frac{n}{n+\ln n} \\ &= 1 \quad \text{(after applying L'Hôpital's Rule).} \end{aligned}$$

Since the Harmonic Series diverges, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$ diverges as well.

Example 9.3.6 Applying the Limit Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n-n^2}$

Solution This series is similar to the one in Example 9.3.3, but now we are considering "$3^n - n^2$" instead of "$3^n + n^2$." This difference makes applying the Direct Comparison Test difficult.

Instead, we use the Limit Comparison Test and compare with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$:

$$\begin{aligned} \lim_{n\to\infty} \frac{1/(3^n-n^2)}{1/3^n} &= \lim_{n\to\infty} \frac{3^n}{3^n-n^2} \\ &= 1 \quad \text{(after applying L'Hospital's Rule twice).} \end{aligned}$$

We know $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series, hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n-n^2}$ converges as well.

As mentioned before, practice helps one develop the intuition to quickly choose a series with which to compare. A general rule of thumb is to pick a series based on the dominant term in the expression of $\{a_n\}$. It is also helpful to note that factorials dominate exponentials, which dominate algebraic functions (e.g., polynomials), which dominate logarithms. In the previous example, the dominant term of $\dfrac{1}{3^n-n^2}$ was 3^n, so we compared the series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$. It is hard to apply the Limit Comparison Test to series containing factorials, though, as we have not learned how to apply L'Hospital's Rule to $n!$.

Example 9.3.7 Applying the Limit Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2-n+1}$.

Solution We naïvely attempt to apply the rule of thumb given above and note that the dominant term in the expression of the series is $1/n^2$. Knowing that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, we attempt to apply the Limit Comparison Test:

$$\lim_{n\to\infty}\frac{(\sqrt{n}+3)/(n^2-n+1)}{1/n^2} = \lim_{n\to\infty}\frac{n^2(\sqrt{n}+3)}{n^2-n+1}$$
$$= \infty \quad \text{(Apply L'Hôpital's Rule).}$$

Theorem 9.3.3 part (3) only applies when $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges; in our case, it converges. Ultimately, our test has not revealed anything about the convergence of our series.

The problem is that we chose a poor series with which to compare. Since the numerator and denominator of the terms of the series are both algebraic functions, we should have compared our series to the dominant term of the numerator divided by the dominant term of the denominator.

The dominant term of the numerator is $n^{1/2}$ and the dominant term of the denominator is n^2. Thus we should compare the terms of the given series to $n^{1/2}/n^2 = 1/n^{3/2}$:

$$\lim_{n\to\infty}\frac{(\sqrt{n}+3)/(n^2-n+1)}{1/n^{3/2}} = \lim_{n\to\infty}\frac{n^{3/2}(\sqrt{n}+3)}{n^2-n+1}$$
$$= 1 \quad \text{(Apply L'Hôpital's Rule).}$$

Since the p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, we conclude that $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}+3}{n^2-n+1}$ converges as well.

We mentioned earlier that the Integral Test did not work well with series containing factorial terms. The next section introduces the Ratio Test, which does handle such series well. We also introduce the Root Test, which is good for series where each term is raised to a power.

Exercises 9.3

Terms and Concepts

1. In order to apply the Integral Test to a sequence $\{a_n\}$, the function $a(n) = a_n$ must be ______, ______ and ______.

2. T/F: The Integral Test can be used to determine the sum of a convergent series.

3. What test(s) in this section do not work well with factorials?

4. Suppose $\displaystyle\sum_{n=0}^{\infty} a_n$ is convergent, and there are sequences $\{b_n\}$ and $\{c_n\}$ such that $0 \le b_n \le a_n \le c_n$ for all n. What can be said about the series $\displaystyle\sum_{n=0}^{\infty} b_n$ and $\displaystyle\sum_{n=0}^{\infty} c_n$?

Problems

In Exercises 5 – 12, use the Integral Test to determine the convergence of the given series.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

10. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

In Exercises 13 – 22, use the Direct Comparison Test to determine the convergence of the given series; state what series is used for comparison.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n-5}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n+n^2-n}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!+n}$

17. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$

18. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt{n}-2}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3-5}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^n+10}$

21. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2-1}$

22. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$

In Exercises 23 – 32, use the Limit Comparison Test to determine the convergence of the given series; state what series is used for comparison.

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-3n+5}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n-n^2}$

25. $\displaystyle\sum_{n=4}^{\infty} \frac{\ln n}{n-3}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{n-10}{n^2+10n+10}$

29. $\displaystyle\sum_{n=1}^{\infty} \sin\left(1/n\right)$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3-5}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2+17}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+100}$

In Exercises 33 – 40, determine the convergence of the given series. State the test used; more than one test may be appropriate.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+5)^3}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n!}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+n}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{10n+5}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(1/n)}{\sqrt{n}}$

41. Given that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, state which of the following series converges, may converge, or does not converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} a_n a_{n+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} (a_n)^2$

(d) $\displaystyle\sum_{n=1}^{\infty} n a_n$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$

9.4 Ratio and Root Tests

The n^{th}–Term Test of Theorem 9.2.4 states that in order for a series $\sum_{n=1}^{\infty} a_n$ to converge, $\lim_{n\to\infty} a_n = 0$. That is, the terms of $\{a_n\}$ must get very small. Not only must the terms approach 0, they must approach 0 "fast enough": while $\lim_{n\to\infty} 1/n = 0$, the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as the terms of $\{1/n\}$ do not approach 0 "fast enough."

The comparison tests of the previous section determine convergence by comparing terms of a series to terms of another series whose convergence is known. This section introduces the Ratio and Root Tests, which determine convergence by analyzing the terms of a series to see if they approach 0 "fast enough."

Ratio Test

Theorem 9.4.1 Ratio Test

Let $\{a_n\}$ be a positive sequence where $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the Ratio Test is inconclusive.

Note: Theorem 9.2.5 allows us to apply the Ratio Test to series where $\{a_n\}$ is positive for all but a finite number of terms.

The principle of the Ratio Test is this: if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, then for large n, each term of $\{a_n\}$ is significantly smaller than its previous term which is enough to ensure convergence.

Example 9.4.1 Applying the Ratio Test

Use the Ratio Test to determine the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$
2. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$
3. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Solution

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} &= \lim_{n\to\infty} \frac{2^{n+1}n!}{2^n(n+1)!} \\
&= \lim_{n\to\infty} \frac{2}{n+1} \\
&= 0.
\end{aligned}$$

Since the limit is $0 < 1$, by the Ratio Test $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{3^{n+1}/(n+1)^3}{3^n/n^3} &= \lim_{n\to\infty} \frac{3^{n+1}n^3}{3^n(n+1)^3}\\ &= \lim_{n\to\infty} \frac{3n^3}{(n+1)^3}\\ &= 3.\end{aligned}$$

Since the limit is $3 > 1$, by the Ratio Test $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ diverges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{1/\big((n+1)^2+1\big)}{1/(n^2+1)} &= \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1}\\ &= 1.\end{aligned}$$

Since the limit is 1, the Ratio Test is inconclusive. We can easily show this series converges using the Direct or Limit Comparison Tests, with each comparing to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

The Ratio Test is not effective when the terms of a series *only* contain algebraic functions (e.g., polynomials). It is most effective when the terms contain some factorials or exponentials. The previous example also reinforces our developing intuition: factorials dominate exponentials, which dominate algebraic functions, which dominate logarithmic functions. In Part 1 of the example, the factorial in the denominator dominated the exponential in the numerator, causing the series to converge. In Part 2, the exponential in the numerator dominated the algebraic function in the denominator, causing the series to diverge.

While we have used factorials in previous sections, we have not explored them closely and one is likely to not yet have a strong intuitive sense for how they behave. The following example gives more practice with factorials.

Example 9.4.2 Applying the Ratio Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$.

Solution Before we begin, be sure to note the difference between $(2n)!$ and $2n!$. When $n = 4$, the former is $8! = 8\cdot 7\cdot \ldots \cdot 2\cdot 1 = 40,320$, whereas the latter is $2(4\cdot 3\cdot 2\cdot 1) = 48$.

Applying the Ratio Test:

$$\lim_{n\to\infty} \frac{(n+1)!(n+1)!/\big(2(n+1)\big)!}{n!n!/(2n)!} = \lim_{n\to\infty} \frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)!}$$

Noting that $(2n+2)! = (2n+2)\cdot(2n+1)\cdot(2n)!$, we have

$$\begin{aligned}&= \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)}\\ &= 1/4.\end{aligned}$$

Since the limit is $1/4 < 1$, by the Ratio Test we conclude $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$ converges.

Root Test

The final test we introduce is the Root Test, which works particularly well on series where each term is raised to a power, and does not work well with terms containing factorials.

Note: Theorem 9.2.5 allows us to apply the Root Test to series where $\{a_n\}$ is positive for all but a finite number of terms.

Theorem 9.4.2 Root Test

Let $\{a_n\}$ be a positive sequence, and let $\lim\limits_{n\to\infty} (a_n)^{1/n} = L$.

1. If $L < 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$ or $L = \infty$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the Root Test is inconclusive.

Example 9.4.3 Applying the Root Test

Determine the convergence of the following series using the Root Test:

1. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n+1}{5n-2}\right)^n$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{(\ln n)^n}$ 3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$.

Solution

1. $\displaystyle\lim_{n\to\infty} \left(\left(\frac{3n+1}{5n-2}\right)^n\right)^{1/n} = \lim_{n\to\infty} \frac{3n+1}{5n-2} = \frac{3}{5}.$

   Since the limit is less than 1, we conclude the series converges. Note: it is difficult to apply the Ratio Test to this series.

2. $\displaystyle\lim_{n\to\infty} \left(\frac{n^4}{(\ln n)^n}\right)^{1/n} = \lim_{n\to\infty} \frac{\left(n^{1/n}\right)^4}{\ln n}.$

   As n grows, the numerator approaches 1 (apply L'Hospital's Rule) and the denominator grows to infinity. Thus

   $$\lim_{n\to\infty} \frac{\left(n^{1/n}\right)^4}{\ln n} = 0.$$

   Since the limit is less than 1, we conclude the series converges.

3. $\displaystyle\lim_{n\to\infty} \left(\frac{2^n}{n^2}\right)^{1/n} = \lim_{n\to\infty} \frac{2}{\left(n^{1/n}\right)^2} = 2.$

   Since this is greater than 1, we conclude the series diverges.

Each of the tests we have encountered so far has required that we analyze series from *positive* sequences. The next section relaxes this restriction by considering *alternating series*, where the underlying sequence has terms that alternate between being positive and negative.

Exercises 9.4

Terms and Concepts

1. The Ratio Test is not effective when the terms of a sequence only contain _____ functions.

2. The Ratio Test is most effective when the terms of a sequence contains _____ and/or _____ functions.

3. What three convergence tests do not work well with terms containing factorials?

4. The Root Test works particularly well on series where each term is _____ to a _____.

Problems

In Exercises 5 – 14, determine the convergence of the given series using the Ratio Test. If the Ratio Test is inconclusive, state so and determine convergence with another test.

5. $\displaystyle\sum_{n=0}^{\infty} \frac{2n}{n!}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n - 3n}{4^n}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{n!10^n}{(2n)!}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n + n^4}{7^n + n^2}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^3 + 7}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{10 \cdot 5^n}{7^n - 3}$

12. $\displaystyle\sum_{n=1}^{\infty} n \cdot \left(\frac{3}{5}\right)^n$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n}{3 \cdot 6 \cdot 9 \cdot 12 \cdots 3n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5 \cdot 10 \cdot 15 \cdots (5n)}$

In Exercises 15 – 24, determine the convergence of the given series using the Root Test. If the Root Test is inconclusive, state so and determine convergence with another test.

15. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+5}{3n+11}\right)^n$

16. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{.9n^2 - n - 3}{n^2 + n + 3}\right)^n$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^2}{3^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2 2^{n+1}}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+7}}{7^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2 - n}{n^2 + n}\right)^n$

22. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^n}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(\ln n)^n}$

In Exercises 25 – 34, determine the convergence of the given series. State the test used; more than one test may be appropriate.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 4n - 2}{n^3 + 4n^2 - 3n + 7}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^4 4^n}{n!}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n + n}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^n}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 4n + 1}}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!n!}{(3n)!}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln n}$

32. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+2}{n+1}\right)^{n}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{\left(\ln n\right)^n}$

34. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

9.5 Alternating Series and Absolute Convergence

All of the series convergence tests we have used require that the underlying sequence $\{a_n\}$ be a positive sequence. (We can relax this with Theorem 9.2.5 and state that there must be an $N > 0$ such that $a_n > 0$ for all $n > N$; that is, $\{a_n\}$ is positive for all but a finite number of values of n.)

In this section we explore series whose summation includes negative terms. We start with a very specific form of series, where the terms of the summation alternate between being positive and negative.

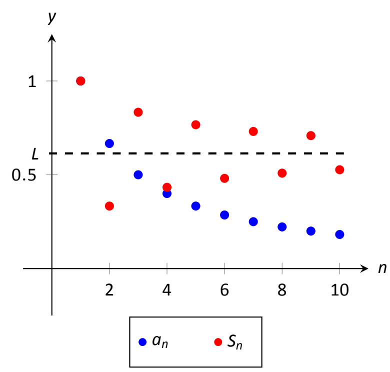


Figure 9.5.1: Illustrating convergence with the Alternating Series Test.

Definition 9.5.1 Alternating Series

Let $\{a_n\}$ be a positive sequence. An **alternating series** is a series of either the form

$$\sum_{n=1}^{\infty}(-1)^n a_n \qquad \text{or} \qquad \sum_{n=1}^{\infty}(-1)^{n+1}a_n.$$

Recall the terms of Harmonic Series come from the Harmonic Sequence $\{a_n\} = \{1/n\}$. An important alternating series is the **Alternating Harmonic Series**:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Geometric Series can also be alternating series when $r < 0$. For instance, if $r = -1/2$, the geometric series is

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$$

Theorem 9.2.1 states that geometric series converge when $|r| < 1$ and gives the sum: $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. When $r = -1/2$ as above, we find

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}.$$

A powerful convergence theorem exists for other alternating series that meet a few conditions.

Theorem 9.5.1 Alternating Series Test

Let $\{a_n\}$ be a positive, decreasing sequence where $\displaystyle\lim_{n\to\infty} a_n = 0$. Then

$$\sum_{n=1}^{\infty}(-1)^n a_n \qquad \text{and} \qquad \sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

converge.

The basic idea behind Theorem 9.5.1 is illustrated in Figure 9.5.1. A positive, decreasing sequence $\{a_n\}$ is shown along with the partial sums

$$S_n = \sum_{i=1}^{n}(-1)^{i+1}a_i = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1}a_n.$$

Because $\{a_n\}$ is decreasing, the amount by which S_n bounces up/down decreases. Moreover, the odd terms of S_n form a decreasing, bounded sequence, while the even terms of S_n form an increasing, bounded sequence. Since bounded, monotonic sequences converge (see Theorem 9.1.5) and the terms of $\{a_n\}$ approach 0, one can show the odd and even terms of S_n converge to the same common limit L, the sum of the series.

Example 9.5.1 Applying the Alternating Series Test

Determine if the Alternating Series Test applies to each of the following series.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ 2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{\ln n}{n}$ 3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{|\sin n|}{n^2}$

Solution

1. This is the Alternating Harmonic Series as seen previously. The underlying sequence is $\{a_n\} = \{1/n\}$, which is positive, decreasing, and approaches 0 as $n \to \infty$. Therefore we can apply the Alternating Series Test and conclude this series converges.

   While the test does not state what the series converges to, we will see later that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2$.

2. The underlying sequence is $\{a_n\} = \{\ln n/n\}$. This is positive and approaches 0 as $n \to \infty$ (use L'Hospital's Rule). However, the sequence is not decreasing for all n. It is straightforward to compute $a_1 = 0$, $a_2 \approx 0.347$, $a_3 \approx 0.366$, and $a_4 \approx 0.347$: the sequence is increasing for at least the first 3 terms.

   We do not immediately conclude that we cannot apply the Alternating Series Test. Rather, consider the long–term behaviour of $\{a_n\}$. Treating $a_n = a(n)$ as a continuous function of n defined on $[1, \infty)$, we can take its derivative:
   $$a'(n) = \frac{1 - \ln n}{n^2}.$$
   The derivative is negative for all $n \geq 3$ (actually, for all $n > e$), meaning $a(n) = a_n$ is decreasing on $[3, \infty)$. We can apply the Alternating Series Test to the series when we start with $n = 3$ and conclude that $\displaystyle\sum_{n=3}^{\infty}(-1)^{n}\frac{\ln n}{n}$ converges; adding the terms with $n = 1$ and $n = 2$ do not change the convergence (i.e., we apply Theorem 9.2.5).

   The important lesson here is that as before, if a series fails to meet the criteria of the Alternating Series Test on only a finite number of terms, we can still apply the test.

3. The underlying sequence is $\{a_n\} = |\sin n|/n$. This sequence is positive and approaches 0 as $n \to \infty$. However, it is not a decreasing sequence; the value of $|\sin n|$ oscillates between 0 and 1 as $n \to \infty$. We cannot remove a finite number of terms to make $\{a_n\}$ decreasing, therefore we cannot apply the Alternating Series Test.

   Keep in mind that this does not mean we conclude the series diverges; in fact, it does converge. We are just unable to conclude this based on Theorem 9.5.1.

Key Idea 9.2.1 gives the sum of some important series. Two of these are

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}\approx 1.64493 \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}\approx 0.82247.$$

These two series converge to their sums at different rates. To be accurate to two places after the decimal, we need 202 terms of the first series though only 13 of the second. To get 3 places of accuracy, we need 1069 terms of the first series though only 33 of the second. Why is it that the second series converges so much faster than the first?

While there are many factors involved when studying rates of convergence, the alternating structure of an alternating series gives us a powerful tool when approximating the sum of a convergent series.

Theorem 9.5.2 The Alternating Series Approximation Theorem

Let $\{a_n\}$ be a sequence that satisfies the hypotheses of the Alternating Series Test, and let S_n and L be the n^{th} partial sums and sum, respectively, of either $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ or $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}a_n$. Then

1. $|S_n - L| < a_{n+1}$, and
2. L is between S_n and S_{n+1}.

Part 1 of Theorem 9.5.2 states that the n^{th} partial sum of a convergent alternating series will be within a_{n+1} of its total sum. Consider the alternating series we looked at before the statement of the theorem, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. Since $a_{14}=1/14^2\approx 0.0051$, we know that S_{13} is within 0.0051 of the total sum.

Moreover, Part 2 of the theorem states that since $S_{13}\approx 0.8252$ and $S_{14}\approx 0.8201$, we know the sum L lies between 0.8201 and 0.8252. One use of this is the knowledge that S_{14} is accurate to two places after the decimal.

Some alternating series converge slowly. In Example 9.5.1 we determined the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$ converged. With $n=1001$, we find $\ln n/n\approx 0.0069$, meaning that $S_{1000}\approx 0.1633$ is accurate to one, maybe two, places after the decimal. Since $S_{1001}\approx 0.1564$, we know the sum L is $0.1564\leq L\leq 0.1633$.

Example 9.5.2 Approximating the sum of convergent alternating series

Approximate the sum of the following series, accurate to within 0.001.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}$ 2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$.

Solution

1. Using Theorem 9.5.2, we want to find n where $1/n^3 < 0.001$:

$$\begin{aligned}\frac{1}{n^3} &\leq 0.001 = \frac{1}{1000}\\ n^3 &\geq 1000\\ n &\geq \sqrt[3]{1000}\\ n &\geq 10.\end{aligned}$$

   Let L be the sum of this series. By Part 1 of the theorem, $|S_9 - L| < a_{10} = 1/1000$. We can compute $S_9 = 0.902116$, which our theorem states is within 0.001 of the total sum.

   We can use Part 2 of the theorem to obtain an even more accurate result. As we know the 10$^{\text{th}}$ term of the series is $-1/1000$, we can easily compute $S_{10} = 0.901116$. Part 2 of the theorem states that L is between S_9 and S_{10}, so $0.901116 < L < 0.902116$.

2. We want to find n where $\ln(n)/n < 0.001$. We start by solving $\ln(n)/n = 0.001$ for n. This cannot be solved algebraically, so we will use Newton's Method to approximate a solution.

   Let $f(x) = \ln(x)/x - 0.001$; we want to know where $f(x) = 0$. We make a guess that x must be "large," so our initial guess will be $x_1 = 1000$. Recall how Newton's Method works: given an approximate solution x_n, our next approximation x_{n+1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

   We find $f'(x) = \big(1 - \ln(x)\big)/x^2$. This gives

$$\begin{aligned}x_2 &= 1000 - \frac{\ln(1000)/1000 - 0.001}{\big(1 - \ln(1000)\big)/1000^2}\\ &= 2000.\end{aligned}$$

   Using a computer, we find that Newton's Method seems to converge to a solution $x = 9118.01$ after 8 iterations. Taking the next integer higher, we have $n = 9119$, where $\ln(9119)/9119 = 0.000999903 < 0.001$.

   Again using a computer, we find $S_{9118} = -0.160369$. Part 1 of the theorem states that this is within 0.001 of the actual sum L. Already knowing the 9,119$^{\text{th}}$ term, we can compute $S_{9119} = -0.159369$, meaning $-0.159369 < L < -0.160369$.

Notice how the first series converged quite quickly, where we needed only 10 terms to reach the desired accuracy, whereas the second series took over 9,000 terms.

One of the famous results of mathematics is that the Harmonic Series, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, yet the Alternating Harmonic Series, $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$, converges. The notion that alternating the signs of the terms in a series can make a series converge leads us to the following definitions.

Definition 9.5.2 Absolute and Conditional Convergence

1. A series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\sum_{n=1}^{\infty} |a_n|$ converges.
2. A series $\sum_{n=1}^{\infty} a_n$ **converges conditionally** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Note: In Definition 9.5.2, $\sum_{n=1}^{\infty} a_n$ is not necessarily an alternating series; it just may have some negative terms.

Thus we say the Alternating Harmonic Series converges conditionally.

Example 9.5.3 Determining absolute and conditional convergence.
Determine if the following series converge absolutely, conditionally, or diverge.

1. $\sum_{n=1}^{\infty} (-1)^n \dfrac{n+3}{n^2+2n+5}$
2. $\sum_{n=1}^{\infty} (-1)^n \dfrac{n^2+2n+5}{2^n}$
3. $\sum_{n=3}^{\infty} (-1)^n \dfrac{3n-3}{5n-10}$

Solution

1. We can show the series
$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{n+3}{n^2+2n+5} \right| = \sum_{n=1}^{\infty} \frac{n+3}{n^2+2n+5}$$
diverges using the Limit Comparison Test, comparing with $1/n$.
The series $\sum_{n=1}^{\infty} (-1)^n \dfrac{n+3}{n^2+2n+5}$ converges using the Alternating Series Test; we conclude it converges conditionally.
2. We can show the series
$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{n^2+2n+5}{2^n} \right| = \sum_{n=1}^{\infty} \frac{n^2+2n+5}{2^n}$$
converges using the Ratio Test.
Therefore we conclude $\sum_{n=1}^{\infty} (-1)^n \dfrac{n^2+2n+5}{2^n}$ converges absolutely.
3. The series
$$\sum_{n=3}^{\infty} \left| (-1)^n \frac{3n-3}{5n-10} \right| = \sum_{n=3}^{\infty} \frac{3n-3}{5n-10}$$
diverges using the n^{th} Term Test, so it does not converge absolutely.
The series $\sum_{n=3}^{\infty} (-1)^n \dfrac{3n-3}{5n-10}$ fails the conditions of the Alternating Series Test as $(3n-3)/(5n-10)$ does not approach 0 as $n \to \infty$. We can state further that this series diverges; as $n \to \infty$, the series effectively adds and subtracts $3/5$ over and over. This causes the sequence of partial sums to oscillate and not converge.
Therefore the series $\sum_{n=1}^{\infty} (-1)^n \dfrac{3n-3}{5n-10}$ diverges.

Knowing that a series converges absolutely allows us to make two important statements, given in the following theorem. The first is that absolute convergence is "stronger" than regular convergence. That is, just because $\sum_{n=1}^{\infty} a_n$ converges, we cannot conclude that $\sum_{n=1}^{\infty} |a_n|$ will converge, but knowing a series converges absolutely tells us that $\sum_{n=1}^{\infty} a_n$ will converge.

One reason this is important is that our convergence tests all require that the underlying sequence of terms be positive. By taking the absolute value of the terms of a series where not all terms are positive, we are often able to apply an appropriate test and determine absolute convergence. This, in turn, determines that the series we are given also converges.

The second statement relates to **rearrangements** of series. When dealing with a finite set of numbers, the sum of the numbers does not depend on the order which they are added. (So $1+2+3 = 3+1+2$.) One may be surprised to find out that when dealing with an infinite set of numbers, the same statement does not always hold true: some infinite lists of numbers may be rearranged in different orders to achieve different sums. The theorem states that the terms of an absolutely convergent series can be rearranged in any way without affecting the sum.

Theorem 9.5.3 Absolute Convergence Theorem

Let $\sum_{n=1}^{\infty} a_n$ be a series that converges absolutely.

1. $\sum_{n=1}^{\infty} a_n$ converges.
2. Let $\{b_n\}$ be any rearrangement of the sequence $\{a_n\}$. Then

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

In Example 9.5.3, we determined the series in part 2 converges absolutely. Theorem 9.5.3 tells us the series converges (which we could also determine using the Alternating Series Test).

The theorem states that rearranging the terms of an absolutely convergent series does not affect its sum. This implies that perhaps the sum of a conditionally convergent series can change based on the arrangement of terms. Indeed, it can. The Riemann Rearrangement Theorem (named after Bernhard Riemann) states that any conditionally convergent series can have its terms rearranged so that the sum is any desired value, including ∞!

As an example, consider the Alternating Harmonic Series once more. We have stated that

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \cdots = \ln 2,$$

(see Key Idea 9.2.1 or Example 9.5.1).

Consider the rearrangement where every positive term is followed by two negative terms:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} \cdots$$

(Convince yourself that these are exactly the same numbers as appear in the Alternating Harmonic Series, just in a different order.) Now group some terms and simplify:

$$\begin{aligned} \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots &= \\ \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots &= \\ \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right) &= \frac{1}{2}\ln 2. \end{aligned}$$

By rearranging the terms of the series, we have arrived at a different sum! (One could *try* to argue that the Alternating Harmonic Series does not actually converge to $\ln 2$, because rearranging the terms of the series *shouldn't* change the sum. However, the Alternating Series Test proves this series converges to L, for some number L, and if the rearrangement does not change the sum, then $L = L/2$, implying $L = 0$. But the Alternating Series Approximation Theorem quickly shows that $L > 0$. The only conclusion is that the rearrangement *did* change the sum.) This is an incredible result.

We end here our study of tests to determine convergence. The end of this text contains a table summarizing the tests that one may find useful.

While series are worthy of study in and of themselves, our ultimate goal within calculus is the study of Power Series, which we will consider in the next section. We will use power series to create functions where the output is the result of an infinite summation.

Exercises 9.5

Terms and Concepts

1. Why is $\sum_{n=1}^{\infty} \sin n$ not an alternating series?

2. A series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges when $\{a_n\}$ is ______, ______ and $\lim_{n\to\infty} a_n =$______.

3. Give an example of a series where $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} |a_n|$ does not.

4. The sum of a ______ convergent series can be changed by rearranging the order of its terms.

Problems

In Exercises 5 – 20, an alternating series $\sum_{n=i}^{\infty} a_n$ is given.

(a) Determine if the series converges or diverges.

(b) Determine if $\sum_{n=0}^{\infty} |a_n|$ converges or diverges.

(c) If $\sum_{n=0}^{\infty} a_n$ converges, determine if the convergence is conditional or absolute.

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n!}}$

7. $\sum_{n=0}^{\infty} (-1)^n \frac{n+5}{3n-5}$

8. $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2}$

9. $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{3n+5}{n^2-3n+1}$

10. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n+1}$

11. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$

12. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+3+5+\cdots+(2n-1)}$

13. $\sum_{n=1}^{\infty} \cos(\pi n)$

14. $\sum_{n=2}^{\infty} \frac{\sin\left((n+1/2)\pi\right)}{n \ln n}$

15. $\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n$

16. $\sum_{n=0}^{\infty} (-e)^{-n}$

17. $\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{n!}$

18. $\sum_{n=0}^{\infty} (-1)^n 2^{-n^2}$

19. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

20. $\sum_{n=1}^{\infty} \frac{(-1000)^n}{n!}$

Let S_n be the n^{th} partial sum of a series. In Exercises 21 – 24, a convergent alternating series is given and a value of n. Compute S_n and S_{n+1} and use these values to find bounds on the sum of the series.

21. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}, \quad n=5$

22. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}, \quad n=4$

23. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}, \quad n=6$

24. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n, \quad n=9$

In Exercises 25 – 28, a convergent alternating series is given along with its sum and a value of ε. Use Theorem 9.5.2 to find n such that the n^{th} partial sum of the series is within ε of the sum of the series.

25. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}, \quad \varepsilon = 0.001$

26. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}, \quad \varepsilon = 0.0001$

27. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}, \quad \varepsilon = 0.001$

28. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = \cos 1, \quad \varepsilon = 10^{-8}$

9.6 Power Series

So far, our study of series has examined the question of "Is the sum of these infinite terms finite?," i.e., "Does the series converge?" We now approach series from a different perspective: as a function. Given a value of x, we evaluate $f(x)$ by finding the sum of a particular series that depends on x (assuming the series converges). We start this new approach to series with a definition.

Definition 9.6.1 Power Series

Let $\{a_n\}$ be a sequence, let x be a variable, and let c be a real number.

1. The **power series in** x is the series
$$\sum_{n=0}^{\infty} a_nx^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$$
2. The **power series in** x **centred at** c is the series
$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$$

Example 9.6.1 Examples of power series

Write out the first five terms of the following power series:

1. $\displaystyle\sum_{n=0}^{\infty} x^n$ 2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x+1)^n}{n}$ 3. $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}$.

Solution

1. One of the conventions we adopt is that $x^0 = 1$ regardless of the value of x. Therefore
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$$
This is a geometric series in x.
2. This series is centred at $c = -1$. Note how this series starts with $n = 1$. We could rewrite this series starting at $n = 0$ with the understanding that $a_0 = 0$, and hence the first term is 0.
$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x+1)^n}{n} = (x+1) - \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} - \frac{(x+1)^4}{4} + \frac{(x+1)^5}{5}\ldots$$
3. This series is centred at $c = \pi$. Recall that $0! = 1$.
$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!} = -1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24} + \frac{(x-\pi)^6}{6!} - \frac{(x-\pi)^8}{8!}\ldots$$

We introduced power series as a type of function, where a value of x is given and the sum of a series is returned. Of course, not every series converges. For instance, in part 1 of Example 9.6.1, we recognized the series $\displaystyle\sum_{n=0}^{\infty} x^n$ as a geometric series in x. Theorem 9.2.1 states that this series converges only when $|x| < 1$.

This raises the question: "For what values of x will a given power series converge?," which leads us to a theorem and definition.

Theorem 9.6.1 Convergence of Power Series

Let a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ be given. Then one of the following is true:

1. The series converges only at $x = c$.
2. There is an $R > 0$ such that the series converges for all x in $(c - R, c + R)$ and diverges for all $x < c - R$ and $x > c + R$.
3. The series converges for all x.

The value of R is important when understanding a power series, hence it is given a name in the following definition. Also, note that part 2 of Theorem 9.6.1 makes a statement about the interval $(c - R, c + R)$, but the not the endpoints of that interval. A series may/may not converge at these endpoints.

Definition 9.6.2 Radius and Interval of Convergence

1. The number R given in Theorem 9.6.1 is the **radius of convergence** of a given series. When a series converges for only $x = c$, we say the radius of convergence is 0, i.e., $R = 0$. When a series converges for all x, we say the series has an infinite radius of convergence, i.e., $R = \infty$.
2. The **interval of convergence** is the set of all values of x for which the series converges.

To find the values of x for which a given series converges, we will use the convergence tests we studied previously (especially the Ratio Test). However, the tests all required that the terms of a series be positive. The following theorem gives us a work–around to this problem.

Theorem 9.6.2 The Radius of Convergence of a Series and Absolute Convergence

The series $\sum_{n=0}^{\infty} a_n(x-c)^n$ and $\sum_{n=0}^{\infty} \left|a_n(x-c)^n\right|$ have the same radius of convergence R.

Theorem 9.6.2 allows us to find the radius of convergence R of a series by applying the Ratio Test (or any applicable test) to the absolute value of the terms of the series. We practice this in the following example.

Example 9.6.2 Determining the radius and interval of convergence.
Find the radius and interval of convergence for each of the following series:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ 2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$ 3. $\displaystyle\sum_{n=0}^{\infty} 2^n(x-3)^n$ 4. $\displaystyle\sum_{n=0}^{\infty} n!x^n$

SOLUTION

1. We apply the Ratio Test to the series $\displaystyle\sum_{n=0}^{\infty} \left|\frac{x^n}{n!}\right|$:

$$\begin{aligned}\lim_{n\to\infty} \frac{\left|x^{n+1}/(n+1)!\right|}{\left|x^n/n!\right|} &= \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\cdot\frac{n!}{(n+1)!}\right| \\ &= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| \\ &= 0 \text{ for all } x.\end{aligned}$$

The Ratio Test shows us that regardless of the choice of x, the series converges. Therefore the radius of convergence is $R=\infty$, and the interval of convergence is $(-\infty,\infty)$.

2. We apply the Ratio Test to the series $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{x^n}{n}\right| = \sum_{n=1}^{\infty}\left|\frac{x^n}{n}\right|$:

$$\begin{aligned}\lim_{n\to\infty} \frac{\left|x^{n+1}/(n+1)\right|}{\left|x^n/n\right|} &= \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\cdot\frac{n}{n+1}\right| \\ &= \lim_{n\to\infty}|x|\frac{n}{n+1} \\ &= |x|.\end{aligned}$$

The Ratio Test states a series converges if the limit of $|a_{n+1}/a_n| = L < 1$. We found the limit above to be $|x|$; therefore, the power series converges when $|x|<1$, or when x is in $(-1,1)$. Thus the radius of convergence is $R=1$.

To determine the interval of convergence, we need to check the endpoints of $(-1,1)$. When $x=-1$, we have the opposite of the Harmonic Series:

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n} &= \sum_{n=1}^{\infty}\frac{-1}{n} \\ &= -\infty.\end{aligned}$$

The series diverges when $x=-1$.

When $x=1$, we have the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1)^n}{n}$, which is the Alternating Harmonic Series, which converges. Therefore the interval of convergence is $(-1,1]$.

3. We apply the Ratio Test to the series $\displaystyle\sum_{n=0}^{\infty}\left|2^n(x-3)^n\right|$:

$$\begin{aligned}\lim_{n\to\infty}\frac{\left|2^{n+1}(x-3)^{n+1}\right|}{\left|2^n(x-3)^n\right|} &= \lim_{n\to\infty}\left|\frac{2^{n+1}}{2^n}\cdot\frac{(x-3)^{n+1}}{(x-3)^n}\right| \\ &= \lim_{n\to\infty}\left|2(x-3)\right|.\end{aligned}$$

According to the Ratio Test, the series converges when $\big|2(x-3)\big| < 1 \implies \big|x-3\big| < 1/2$. The series is centred at 3, and x must be within $1/2$ of 3 in order for the series to converge. Therefore the radius of convergence is $R = 1/2$, and we know that the series converges absolutely for all x in $(3-1/2, 3+1/2) = (2.5, 3.5)$.

We check for convergence at the endpoints to find the interval of convergence. When $x = 2.5$, we have:

$$\begin{aligned}\sum_{n=0}^{\infty} 2^n(2.5-3)^n &= \sum_{n=0}^{\infty} 2^n(-1/2)^n \\ &= \sum_{n=0}^{\infty}(-1)^n,\end{aligned}$$

which diverges. A similar process shows that the series also diverges at $x = 3.5$. Therefore the interval of convergence is $(2.5, 3.5)$.

4. We apply the Ratio Test to $\displaystyle\sum_{n=0}^{\infty}\big|n!x^n\big|$:

$$\begin{aligned}\lim_{n\to\infty}\frac{\big|(n+1)!x^{n+1}\big|}{\big|n!x^n\big|} &= \lim_{n\to\infty}\big|(n+1)x\big| \\ &= \infty \text{ for all } x, \text{ except } x = 0.\end{aligned}$$

The Ratio Test shows that the series diverges for all x except $x = 0$. Therefore the radius of convergence is $R = 0$.

We can use a power series to define a function:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where the domain of f is a subset of the interval of convergence of the power series. One can apply calculus techniques to such functions; in particular, we can find derivatives and antiderivatives.

Theorem 9.6.3 Derivatives and Indefinite Integrals of Power Series Functions

Let $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n$ be a function defined by a power series, with radius of convergence R.

1. $f(x)$ is continuous and differentiable on $(c-R, c+R)$.
2. $f'(x) = \displaystyle\sum_{n=1}^{\infty} a_n \cdot n \cdot (x-c)^{n-1}$, with radius of convergence R.
3. $\displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n\frac{(x-c)^{n+1}}{n+1}$, with radius of convergence R.

A few notes about Theorem 9.6.3:

1. The theorem states that differentiation and integration do not change the radius of convergence. It does not state anything about the *interval* of convergence. They are not always the same.

2. Notice how the summation for $f'(x)$ starts with $n = 1$. This is because the constant term a_0 of $f(x)$ goes to 0.

3. Differentiation and integration are simply calculated term–by–term using the Power Rules.

Example 9.6.3 Derivatives and indefinite integrals of power series

Let $f(x) = \displaystyle\sum_{n=0}^{\infty} x^n$. Find $f'(x)$ and $F(x) = \displaystyle\int f(x)\,dx$, along with their respective intervals of convergence.

SOLUTION We find the derivative and indefinite integral of $f(x)$, following Theorem 9.6.3.

1. $f'(x) = \displaystyle\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$.

   In Example 9.6.1, we recognized that $\displaystyle\sum_{n=0}^{\infty} x^n$ is a geometric series in x. We know that such a geometric series converges when $|x| < 1$; that is, the interval of convergence is $(-1, 1)$.

   To determine the interval of convergence of $f'(x)$, we consider the endpoints of $(-1, 1)$:

   $$f'(-1) = 1 - 2 + 3 - 4 + \cdots, \quad \text{which diverges.}$$

   $$f'(1) = 1 + 2 + 3 + 4 + \cdots, \quad \text{which diverges.}$$

   Therefore, the interval of convergence of $f'(x)$ is $(-1, 1)$.

2. $F(x) = \displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

   To find the interval of convergence of $F(x)$, we again consider the endpoints of $(-1, 1)$:

   $$F(-1) = C - 1 + 1/2 - 1/3 + 1/4 + \cdots$$

   The value of C is irrelevant; notice that the rest of the series is an Alternating Series that whose terms converge to 0. By the Alternating Series Test, this series converges. (In fact, we can recognize that the terms of the series after C are the opposite of the Alternating Harmonic Series. We can thus say that $F(-1) = C - \ln 2$.)

   $$F(1) = C + 1 + 1/2 + 1/3 + 1/4 + \cdots$$

   Notice that this summation is $C +$ the Harmonic Series, which diverges. Since F converges for $x = -1$ and diverges for $x = 1$, the interval of convergence of $F(x)$ is $[-1, 1)$.

The previous example showed how to take the derivative and indefinite integral of a power series without motivation for why we care about such operations. We may care for the sheer mathematical enjoyment "that we can", which is motivation enough for many. However, we would be remiss to not recognize that we can learn a great deal from taking derivatives and indefinite integrals.

Recall that $f(x) = \displaystyle\sum_{n=0}^{\infty} x^n$ in Example 9.6.3 is a geometric series. According to Theorem 9.2.1, this series converges to $1/(1-x)$ when $|x| < 1$. Thus we can say

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{on} \quad (-1,1).$$

Integrating the power series, (as done in Example 9.6.3,) we find

$$F(x) = C_1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \tag{9.4}$$

while integrating the function $f(x) = 1/(1-x)$ gives

$$F(x) = -\ln|1-x| + C_2. \tag{9.5}$$

Equating Equations (9.4) and (9.5), we have

$$F(x) = C_1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x| + C_2.$$

Letting $x = 0$, we have $F(0) = C_1 = C_2$. This implies that we can drop the constants and conclude

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x|.$$

We established in Example 9.6.3 that the series on the left converges at $x = -1$; substituting $x = -1$ on both sides of the above equality gives

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = -\ln 2.$$

On the left we have the opposite of the Alternating Harmonic Series; on the right, we have $-\ln 2$. We conclude that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

Important: We stated in Key Idea 9.2.1 (in Section 9.2) that the Alternating Harmonic Series converges to $\ln 2$, and referred to this fact again in Example 9.5.1 of Section 9.5. However, we never gave an argument for why this was the case. The work above finally shows how we conclude that the Alternating Harmonic Series converges to $\ln 2$.

We use this type of analysis in the next example.

Example 9.6.4 Analyzing power series functions

Let $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Find $f'(x)$ and $\displaystyle\int f(x)\,dx$, and use these to analyze the behaviour of $f(x)$.

Solution We start by making two notes: first, in Example 9.6.2, we found the interval of convergence of this power series is $(-\infty, \infty)$. Second, we will find it useful later to have a few terms of the series written out:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \tag{9.6}$$

We now find the derivative:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n\frac{x^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + x + \frac{x^2}{2!} + \cdots . \end{aligned}$$

Since the series starts at $n = 1$ and each term refers to $(n-1)$, we can re-index the series starting with $n = 0$:

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= f(x). \end{aligned}$$

We found the derivative of $f(x)$ is $f(x)$. The only functions for which this is true are of the form $y = ce^x$ for some constant c. As $f(0) = 1$ (see Equation (9.6)), c must be 1. Therefore we conclude that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

for all x.

We can also find $\int f(x)\, dx$:

$$\begin{aligned} \int f(x)\, dx &= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)} \\ &= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \end{aligned}$$

We write out a few terms of this last series:

$$C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = C + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

The integral of $f(x)$ differs from $f(x)$ only by a constant, again indicating that $f(x) = e^x$.

Example 9.6.4 and the work following Example 9.6.3 established relationships between a power series function and "regular" functions that we have dealt with in the past. In general, given a power series function, it is difficult (if not impossible) to express the function in terms of elementary functions. We chose examples where things worked out nicely.

In this section's last example, we show how to solve a simple differential equation with a power series.

Example 9.6.5 Solving a differential equation with a power series.
Give the first 4 terms of the power series solution to $y' = 2y$, where $y(0) = 1$.

SOLUTION The differential equation $y' = 2y$ describes a function $y = f(x)$ where the derivative of y is twice y and $y(0) = 1$. This is a rather simple differential equation; with a bit of thought one should realize that if $y = Ce^{2x}$, then $y' = 2Ce^{2x}$, and hence $y' = 2y$. By letting $C = 1$ we satisfy the initial condition of $y(0) = 1$.

Let's ignore the fact that we already know the solution and find a power series function that satisfies the equation. The solution we seek will have the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

for unknown coefficients a_n. We can find $f'(x)$ using Theorem 9.6.3:

$$f'(x) = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots .$$

Since $f'(x) = 2f(x)$, we have

$$\begin{aligned} a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots &= 2\big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\big) \\ &= 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + \cdots \end{aligned}$$

The coefficients of like powers of x must be equal, so we find that

$$a_1 = 2a_0, \quad 2a_2 = 2a_1, \quad 3a_3 = 2a_2, \quad 4a_4 = 2a_3, \quad \text{etc.}$$

The initial condition $y(0) = f(0) = 1$ indicates that $a_0 = 1$; with this, we can find the values of the other coefficients:

$$\begin{aligned} a_0 = 1 \text{ and } a_1 = 2a_0 &\Rightarrow a_1 = 2; \\ a_1 = 2 \text{ and } 2a_2 = 2a_1 &\Rightarrow a_2 = 4/2 = 2; \\ a_2 = 2 \text{ and } 3a_3 = 2a_2 &\Rightarrow a_3 = 8/(2 \cdot 3) = 4/3; \\ a_3 = 4/3 \text{ and } 4a_4 = 2a_3 &\Rightarrow a_4 = 16/(2 \cdot 3 \cdot 4) = 2/3. \end{aligned}$$

Thus the first 5 terms of the power series solution to the differential equation $y' = 2y$ is

$$f(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

In Section 9.8, as we study Taylor Series, we will learn how to recognize this series as describing $y = e^{2x}$.

Our last example illustrates that it can be difficult to recognize an elementary function by its power series expansion. It is far easier to start with a known function, expressed in terms of elementary functions, and represent it as a power series function. One may wonder why we would bother doing so, as the latter function probably seems more complicated. In the next two sections, we show both *how* to do this and *why* such a process can be beneficial.

Exercises 9.6

Terms and Concepts

1. We adopt the convention that $x^0 =$______, regardless of the value of x.

2. What is the difference between the radius of convergence and the interval of convergence?

3. If the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ is 5, what is the radius of convergence of $\displaystyle\sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$?

4. If the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ is 5, what is the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} (-1)^n a_n x^n$?

Problems

In Exercises 5 – 8, write out the sum of the first 5 terms of the given power series.

5. $\displaystyle\sum_{n=0}^{\infty} 2^n x^n$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

In Exercises 9 – 24, a power series is given.

(a) Find the radius of convergence.

(b) Find the interval of convergence.

9. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} x^n$

10. $\displaystyle\sum_{n=0}^{\infty} n x^n$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n}$

12. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+4)^n}{n!}$

13. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

14. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-5)^n}{10^n}$

15. $\displaystyle\sum_{n=0}^{\infty} 5^n (x-1)^n$

16. $\displaystyle\sum_{n=0}^{\infty} (-2)^n x^n$

17. $\displaystyle\sum_{n=0}^{\infty} \sqrt{n} x^n$

18. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{3^n} x^n$

19. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!} (x-5)^n$

20. $\displaystyle\sum_{n=0}^{\infty} (-1)^n n! (x-10)^n$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^3}$

23. $\displaystyle\sum_{n=0}^{\infty} n! \left(\frac{x}{10}\right)^n$

24. $\displaystyle\sum_{n=0}^{\infty} n^2 \left(\frac{x+4}{4}\right)^n$

In Exercises 25 – 30, a function $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ is given.

(a) Give a power series for $f'(x)$ and its interval of convergence.

(b) Give a power series for $\int f(x)\,dx$ and its interval of convergence.

25. $\displaystyle\sum_{n=0}^{\infty} n x^n$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

27. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

28. $\sum_{n=0}^{\infty}(-3x)^n$

29. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

30. $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$

In Exercises 31 – 36, give the first 5 terms of the series that is a solution to the given differential equation.

31. $y' = 3y, \quad y(0) = 1$

32. $y' = 5y, \quad y(0) = 5$

33. $y' = y^2, \quad y(0) = 1$

34. $y' = y + 1, \quad y(0) = 1$

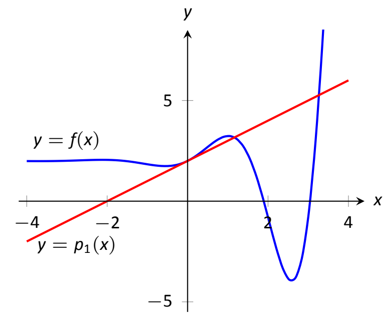
35. $y'' = -y, \quad y(0) = 0, y'(0) = 1$

36. $y'' = 2y, \quad y(0) = 1, y'(0) = 1$

9.7 Taylor Polynomials

Consider a function $y = f(x)$ and a point $\big(c, f(c)\big)$. The derivative, $f'(c)$, gives the instantaneous rate of change of f at $x = c$. Of all lines that pass through the point $\big(c, f(c)\big)$, the line that best approximates f at this point is the tangent line; that is, the line whose slope (rate of change) is $f'(c)$.

In Figure 9.7.1, we see a function $y = f(x)$ graphed. The table below the graph shows that $f(0) = 2$ and $f'(0) = 1$; therefore, the tangent line to f at $x = 0$ is $p_1(x) = 1(x-0)+2 = x+2$. The tangent line is also given in the figure. Note that "near" $x = 0$, $p_1(x) \approx f(x)$; that is, the tangent line approximates f well.



$$f(0) = 2 \qquad f'''(0) = -1$$
$$f'(0) = 1 \qquad f^{(4)}(0) = -12$$
$$f''(0) = 2 \qquad f^{(5)}(0) = -19$$

Figure 9.7.1: Plotting $y = f(x)$ and a table of derivatives of f evaluated at 0.

One shortcoming of this approximation is that the tangent line only matches the slope of f; it does not, for instance, match the concavity of f. We can find a polynomial, $p_2(x)$, that does match the concavity without much difficulty, though. The table in Figure 9.7.1 gives the following information:

$$f(0) = 2 \qquad f'(0) = 1 \qquad f''(0) = 2.$$

Therefore, we want our polynomial $p_2(x)$ to have these same properties. That is, we need

$$p_2(0) = 2 \qquad p_2'(0) = 1 \qquad p_2''(0) = 2.$$

Let's start with a general quadratic function

$$p(x) = a_0 + a_1x + a_2x^2$$

We find the following:

$$\begin{aligned} p_2(x) &= a_0 + a_1x + a_2x^2 & \qquad p_2(0) &= a_0 \\ p_2{}'(x) &= a_1 + 2a_2x & p_2{}'(0) &= a_1 \\ p_2{}''(x) &= 2a_2 & p_2{}''(0) &= 2a_2. \end{aligned}$$

To get the desired properties above, we must have

$$a_0 = f(0) = 2,\ a_1 = f'(0) = 1,\ 2a_2 = f''(0) = 2,$$

so $a_0 = 2$, $a_1 = 1$, and $a_2 = 2/2 = 1$, giving us the polynomial

$$p_2(x) = 2 + x + x^2.$$

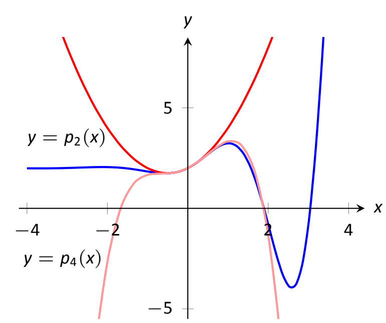


Figure 9.7.2: Plotting f, p_2 and p_4.

We can repeat this approximation process by creating polynomials of higher degree that match more of the derivatives of f at $x = 0$. In general, a polynomial of degree n can be created to match the first n derivatives of f. Figure 9.7.2 also shows $p_4(x) = -x^4/2 - x^3/6 + x^2 + x + 2$, whose first four derivatives at 0 match those of f.

How do we ensure that the derivatives of our polynomial match those of f? We simply begin with a polynomial of the desired degree, compute its derivatives, and compare them to those of f! Recall that each term in a polynomial consists of a power of x, and a coefficient, like so: a_nx^n. Our goal is to determine the value for each coefficient a_n so that the derivatives of our polynomial match those of our function f. If we take k derivatives of the term a_nx^n, with $k \leq n$, we obtain

$$\frac{d^k}{dx^k}(a_nx^n) = n(n-1)\cdots(n-k+1)a_nx^{n-k}.$$

For $k < n$, the expression above vanishes when we set $x = 0$. However, for $n = k$, we obtain the constant value

$$\frac{d^k}{dx^k}(a_kx^k) = k\cdot(k-1)\cdots 2\cdot 1a_k. \tag{9.7}$$

Consider a polynomial

$$p_n(x) = a_0 + a_1x + \cdots + a_kx^k + \cdots + a_nx^n$$

The notation $k!$ is read as "k factorial". By convention, we also define $0! = 1$, mostly because it makes our formulas look a lot nicer.

of degree n. If we take k derivatives, all of the terms involving powers of x less than k disappear, and when we set $x = 0$, all of the terms involving powers of x larger than k disappear, leaving us with the single constant given in (9.7).

Recalling the notation $k! = 1 \cdot 2 \cdot 3 \cdots k$ for the product of the first k integers, we have shown that

$$p_n^{(k)}(0) = k!a_k.$$

If we want the derivatives of p_n to agree with some unknown function f when $x = 0$, then we must have

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

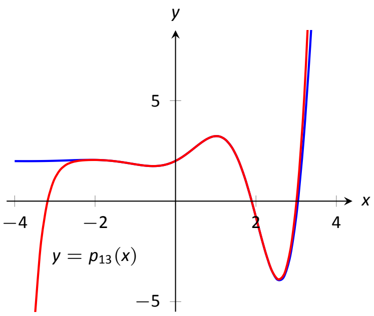


Figure 9.7.3: Plotting f and p_{13}.

As we use more and more derivatives, our polynomial approximation to f gets better and better. In this example, the interval on which the approximation is "good" gets bigger and bigger. Figure 9.7.3 shows $p_{13}(x)$; we can visually affirm that this polynomial approximates f very well on $[-2, 3]$. (The polynomial $p_{13}(x)$ is not particularly "nice". It is

$$\frac{16901x^{13}}{6227020800}+\frac{13x^{12}}{1209600}-\frac{1321x^{11}}{39916800}-\frac{779x^{10}}{1814400}-\frac{359x^9}{362880}+\frac{x^8}{240}+\frac{139x^7}{5040}+\frac{11x^6}{360}-\frac{19x^5}{120}-\frac{x^4}{2}-\frac{x^3}{6}+x^2+x+2.)$$

Historical note: Colin Maclaurin was a Scottish mathematician, born in 1698. He lived until 1746, and made a number of contributions to the development of mathematics and physics. His election as professor of mathematics at the University of Aberdeen at the age of 19 made him the world's youngest professor, a record he held until 2008! He was also a staunch foe of the Jacobite Rebellion, and was instrumental in the defence of Edinburgh against the army of Bonnie Prince Charlie. (For more details, see Wikipedia.)

The polynomials we have created are examples of *Taylor polynomials*, named after the British mathematician Brook Taylor who made important discoveries about such functions. In the discussion above, we concentrated on evaluating the derivatives of f at 0; however, there is nothing special about this point. Just as we can consider the linear approximation of a function near any point, so too can we determine a polynomial approximation about any value c in the domain of f. The only catch is that our polynomial will then be given in terms of powers of $x - c$, rather than powers of x, as we see in the following definition.

Definition 9.7.1 Taylor Polynomial, Maclaurin Polynomial

Let f be a function whose first n derivatives exist at $x = c$.

1. The **Taylor polynomial of degree n of f at $x = c$** is

$$p_n(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n.$$

2. A special case of the Taylor polynomial is the Maclaurin polynomial, where $c = 0$. That is, the **Maclaurin polynomial of degree n of f** is

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

We will practice creating Taylor and Maclaurin polynomials in the following examples.

Example 9.7.1 Finding and using Maclaurin polynomials

1. Find the n^{th} Maclaurin polynomial for $f(x) = e^x$.
2. Use $p_5(x)$ to approximate the value of e.

Solution

1. We start with creating a table of the derivatives of e^x evaluated at $x = 0$. In this particular case, this is relatively simple, as shown in Figure 9.7.4. By the definition of the Maclaurin series, we have

$$\begin{aligned} p_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n. \end{aligned}$$

$$\begin{array}{lcl} f(x) = e^x & \Rightarrow & f(0) = 1 \\ f'(x) = e^x & \Rightarrow & f'(0) = 1 \\ f''(x) = e^x & \Rightarrow & f''(0) = 1 \\ \vdots & & \vdots \\ f^{(n)}(x) = e^x & \Rightarrow & f^{(n)}(0) = 1 \end{array}$$

Figure 9.7.4: The derivatives of $f(x) = e^x$ evaluated at $x = 0$.

2. Using our answer from part 1, we have

$$p_5 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5.$$

To approximate the value of e, note that $e = e^1 = f(1) \approx p_5(1)$. It is very straightforward to evaluate $p_5(1)$:

$$p_5(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60} \approx 2.71667.$$

A plot of $f(x) = e^x$ and $p_5(x)$ is given in Figure 9.7.5.

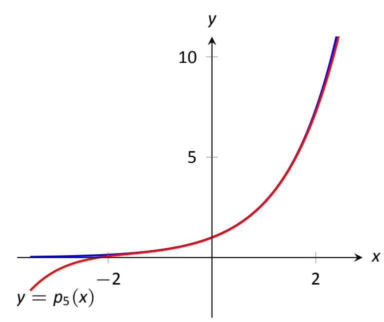


Figure 9.7.5: A plot of $f(x) = e^x$ and its 5^{th} degree Maclaurin polynomial $p_5(x)$.

Example 9.7.2 Finding and using Taylor polynomials

1. Find the n^{th} Taylor polynomial of $y = \ln x$ at $x = 1$.
2. Use $p_6(x)$ to approximate the value of $\ln 1.5$.
3. Use $p_6(x)$ to approximate the value of $\ln 2$.

Solution

1. We begin by creating a table of derivatives of $\ln x$ evaluated at $x = 1$. While this is not as straightforward as it was in the previous example, a pattern does emerge, as shown in Figure 9.7.6.

Using Definition 9.7.1, we have

$$\begin{aligned} p_n(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots + \frac{(-1)^{n+1}}{n}(x-1)^n. \end{aligned}$$

Note how the coefficients of the $(x - 1)$ terms turn out to be "nice."

$$\begin{array}{lcl} f(x) = \ln x & \Rightarrow & f(1) = 0 \\ f'(x) = 1/x & \Rightarrow & f'(1) = 1 \\ f''(x) = -1/x^2 & \Rightarrow & f''(1) = -1 \\ f'''(x) = 2/x^3 & \Rightarrow & f'''(1) = 2 \\ f^{(4)}(x) = -6/x^4 & \Rightarrow & f^{(4)}(1) = -6 \\ \vdots & & \vdots \\ f^{(n)}(x) = & \Rightarrow & f^{(n)}(1) = \\ \dfrac{(-1)^{n+1}(n-1)!}{x^n} & & (-1)^{n+1}(n-1)! \end{array}$$

Figure 9.7.6: Derivatives of $\ln x$ evaluated at $x = 1$.

2. We can compute $p_6(x)$ using our work above:

$$p_6(x) = (x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\frac{1}{5}(x-1)^5-\frac{1}{6}(x-1)^6.$$

Since $p_6(x)$ approximates $\ln x$ well near $x = 1$, we approximate $\ln 1.5 \approx p_6(1.5)$:

$$\begin{aligned} p_6(1.5) &= (1.5-1)-\frac{1}{2}(1.5-1)^2+\frac{1}{3}(1.5-1)^3-\frac{1}{4}(1.5-1)^4+\cdots \\ &\quad \cdots+\frac{1}{5}(1.5-1)^5-\frac{1}{6}(1.5-1)^6 \\ &= \frac{259}{640} \\ &\approx 0.404688. \end{aligned}$$

This is a good approximation as a calculator shows that $\ln 1.5 \approx 0.4055$. Figure 9.7.7 plots $y = \ln x$ with $y = p_6(x)$. We can see that $\ln 1.5 \approx p_6(1.5)$.

3. We approximate $\ln 2$ with $p_6(2)$:

$$\begin{aligned} p_6(2) &= (2-1)-\frac{1}{2}(2-1)^2+\frac{1}{3}(2-1)^3-\frac{1}{4}(2-1)^4+\cdots \\ &\quad \cdots+\frac{1}{5}(2-1)^5-\frac{1}{6}(2-1)^6 \\ &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6} \\ &= \frac{37}{60} \\ &\approx 0.616667. \end{aligned}$$

This approximation is not terribly impressive: a hand held calculator shows that $\ln 2 \approx 0.693147$. The graph in Figure 9.7.7 shows that $p_6(x)$ provides less accurate approximations of $\ln x$ as x gets close to 0 or 2.

Surprisingly enough, even the 20th degree Taylor polynomial fails to approximate $\ln x$ for $x > 2$, as shown in Figure 9.7.8. We'll soon discuss why this is.

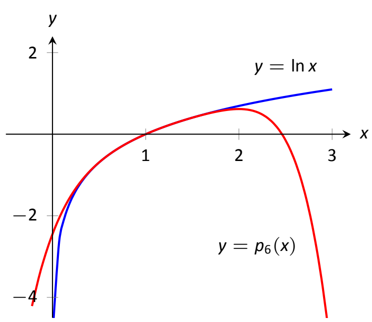


Figure 9.7.7: A plot of $y = \ln x$ and its 6th degree Taylor polynomial at $x = 1$.

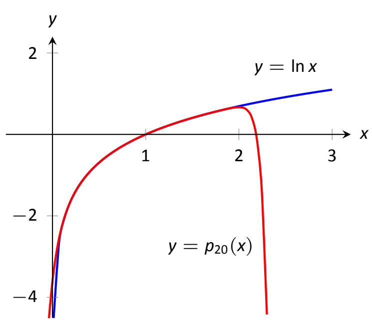


Figure 9.7.8: A plot of $y = \ln x$ and its 20th degree Taylor polynomial at $x = 1$.

Taylor polynomials are used to approximate functions $f(x)$ in mainly two situations:

1. When $f(x)$ is known, but perhaps "hard" to compute directly. For instance, we can define $y = \cos x$ as either the ratio of sides of a right triangle ("adjacent over hypotenuse") or with the unit circle. However, neither of these provides a convenient way of computing $\cos 2$. A Taylor polynomial of sufficiently high degree can provide a reasonable method of computing such values using only operations usually hard–wired into a computer ($+$, $-$, $\times$ and $\div$).

2. When $f(x)$ is not known, but information about its derivatives is known. This occurs more often than one might think, especially in the study of differential equations.

Note: Even though Taylor polynomials *could* be used in calculators and computers to calculate values of trigonometric functions, in practice they generally aren't. Other more efficient and accurate methods have been developed, such as the CORDIC algorithm.

In both situations, a critical piece of information to have is "How good is my approximation?" If we use a Taylor polynomial to compute $\cos 2$, how do we know how accurate the approximation is?

Although much of the content presented in Calculus concerns the search for exact answers to problems such as integration and differentiation, many practical applications of calculus involve attempts to find *approximations*; for example, using Newton's Method to approximate the zeros of a function or numerical integration to approximate the value of an integral that cannot be solved exactly. Whenever an approximation is used, one naturally wishes to know how good the approximation is. In other words, we look for a bound on the error introduced by working with an approximation. The following theorem gives bounds on the error introduced in using a Taylor (and hence Maclaurin) polynomial to approximate a function.

Theorem 9.7.1 Taylor's Theorem

1. Let f be a function whose $n+1^{\text{th}}$ derivative exists on an interval I and let c be in I. Then, for each x in I, there exists z_x between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x),$$

where $R_n(x) = \dfrac{f^{(n+1)}(z_x)}{(n+1)!}(x-c)^{(n+1)}$.

2. $|R_n(x)| \leq \dfrac{\max \left| f^{(n+1)}(z) \right|}{(n+1)!} \left| (x-c)^{(n+1)} \right|$

The first part of Taylor's Theorem states that $f(x) = p_n(x) + R_n(x)$, where $p_n(x)$ is the n^{th} order Taylor polynomial and $R_n(x)$ is the remainder, or error, in the Taylor approximation. The second part gives bounds on how big that error can be. If the $(n+1)^{\text{th}}$ derivative is large on I, the error may be large; if x is far from c, the error may also be large. However, the $(n+1)!$ term in the denominator tends to ensure that the error gets smaller as n increases.

The following example computes error estimates for the approximations of $\ln 1.5$ and $\ln 2$ made in Example 9.7.2.

Example 9.7.3 Finding error bounds of a Taylor polynomial

Use Theorem 9.7.1 to find error bounds when approximating $\ln 1.5$ and $\ln 2$ with $p_6(x)$, the Taylor polynomial of degree 6 of $f(x) = \ln x$ at $x = 1$, as calculated in Example 9.7.2.

Solution

1. We start with the approximation of $\ln 1.5$ with $p_6(1.5)$. The theorem references an open interval I that contains both x and c. The smaller the interval we use the better; it will give us a more accurate (and smaller!) approximation of the error. We let $I = (0.9, 1.6)$, as this interval contains both $c = 1$ and $x = 1.5$.

   The theorem references $\max \left| f^{(n+1)}(z) \right|$. In our situation, this is asking "How big can the 7^{th} derivative of $y = \ln x$ be on the interval $(0.9, 1.6)$?" The seventh derivative is $y = -6!/x^7$. The largest value it attains on I is about 1506. Thus we can bound the error as:

$$\begin{aligned} \left|R_6(1.5)\right| &\le \frac{\max\left|f^{(7)}(z)\right|}{7!}\left|(1.5-1)^7\right| \\ &\le \frac{1506}{5040}\cdot\frac{1}{2^7} \\ &\approx 0.0023. \end{aligned}$$

We computed $p_6(1.5) = 0.404688$; using a calculator, we find $\ln 1.5 \approx 0.405465$, so the actual error is about 0.000778, which is less than our bound of 0.0023. This affirms Taylor's Theorem; the theorem states that our approximation would be within about 2 thousandths of the actual value, whereas the approximation was actually closer.

2. We again find an interval I that contains both $c = 1$ and $x = 2$; we choose $I = (0.9, 2.1)$. The maximum value of the seventh derivative of f on this interval is again about 1506 (as the largest values come near $x = 0.9$). Thus

$$\begin{aligned} \left|R_6(2)\right| &\le \frac{\max\left|f^{(7)}(z)\right|}{7!}\left|(2-1)^7\right| \\ &\le \frac{1506}{5040}\cdot 1^7 \\ &\approx 0.30. \end{aligned}$$

This bound is not as nearly as good as before. Using the degree 6 Taylor polynomial at $x = 1$ will bring us within 0.3 of the correct answer. As $p_6(2) \approx 0.61667$, our error estimate guarantees that the actual value of $\ln 2$ is somewhere between 0.31667 and 0.91667. These bounds are not particularly useful.

In reality, our approximation was only off by about 0.07. However, we are approximating ostensibly because we do not know the real answer. In order to be assured that we have a good approximation, we would have to resort to using a polynomial of higher degree.

We practice again. This time, we use Taylor's theorem to find n that guarantees our approximation is within a certain amount.

Example 9.7.4 Finding sufficiently accurate Taylor polynomials
Find n such that the n^{th} Taylor polynomial of $f(x) = \cos x$ at $x = 0$ approximates $\cos 2$ to within 0.001 of the actual answer. What is $p_n(2)$?

Solution Following Taylor's theorem, we need bounds on the size of the derivatives of $f(x) = \cos x$. In the case of this trigonometric function, this is easy. All derivatives of cosine are $\pm\sin x$ or $\pm\cos x$. In all cases, these functions are never greater than 1 in absolute value. We want the error to be less than 0.001. To find the appropriate n, consider the following inequalities:

$$\begin{aligned} \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(2-0)^{(n+1)}\right| &\le 0.001 \\ \frac{1}{(n+1)!}\cdot 2^{(n+1)} &\le 0.001 \end{aligned}$$

We find an n that satisfies this last inequality with trial–and–error. When $n = 8$, we have $\dfrac{2^{8+1}}{(8+1)!} \approx 0.0014$; when $n = 9$, we have $\dfrac{2^{9+1}}{(9+1)!} \approx 0.000282 <$

$$\begin{aligned}
f(x) = \cos x &\Rightarrow f(0) = 1 \\
f'(x) = -\sin x &\Rightarrow f'(0) = 0 \\
f''(x) = -\cos x &\Rightarrow f''(0) = -1 \\
f'''(x) = \sin x &\Rightarrow f'''(0) = 0 \\
f^{(4)}(x) = \cos x &\Rightarrow f^{(4)}(0) = 1 \\
f^{(5)}(x) = -\sin x &\Rightarrow f^{(5)}(0) = 0 \\
f^{(6)}(x) = -\cos x &\Rightarrow f^{(6)}(0) = -1 \\
f^{(7)}(x) = \sin x &\Rightarrow f^{(7)}(0) = 0 \\
f^{(8)}(x) = \cos x &\Rightarrow f^{(8)}(0) = 1 \\
f^{(9)}(x) = -\sin x &\Rightarrow f^{(9)}(0) = 0
\end{aligned}$$

Figure 9.7.9: A table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$.

0.001. Thus we want to approximate $\cos 2$ with $p_9(2)$.

We now set out to compute $p_9(x)$. We again need a table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$. A table of these values is given in Figure 9.7.9. Notice how the derivatives, evaluated at $x = 0$, follow a certain pattern. All the odd powers of x in the Taylor polynomial will disappear as their coefficient is 0. While our error bounds state that we need $p_9(x)$, our work shows that this will be the same as $p_8(x)$.

Since we are forming our polynomial at $x = 0$, we are creating a Maclaurin polynomial, and:

$$\begin{aligned} p_8(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(8)}(0)}{8!}x^8 \\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \end{aligned}$$

We finally approximate $\cos 2$:

$$\cos 2 \approx p_8(2) = -\frac{131}{315} \approx -0.41587.$$

Our error bound guarantee that this approximation is within 0.001 of the correct answer. Technology shows us that our approximation is actually within about 0.0003 of the correct answer.

Figure 9.7.10 shows a graph of $y = p_8(x)$ and $y = \cos x$. Note how well the two functions agree on about $(-\pi, \pi)$.

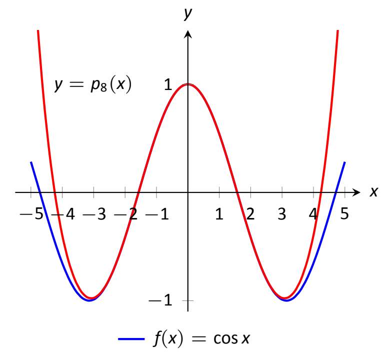


Figure 9.7.10: A graph of $f(x) = \cos x$ and its degree 8 Maclaurin polynomial.

Example 9.7.5 Finding and using Taylor polynomials

1. Find the degree 4 Taylor polynomial, $p_4(x)$, for $f(x) = \sqrt{x}$ at $x = 4$.
2. Use $p_4(x)$ to approximate $\sqrt{3}$.
3. Find bounds on the error when approximating $\sqrt{3}$ with $p_4(3)$.

Solution

1. We begin by evaluating the derivatives of f at $x = 4$. This is done in Figure 9.7.11. These values allow us to form the Taylor polynomial $p_4(x)$:

$$p_4(x) = 2 + \frac{1}{4}(x-4) + \frac{-1/32}{2!}(x-4)^2 + \frac{3/256}{3!}(x-4)^3 + \frac{-15/2048}{4!}(x-4)^4.$$

2. As $p_4(x) \approx \sqrt{x}$ near $x = 4$, we approximate $\sqrt{3}$ with $p_4(3) = 1.73212$.

3. To find a bound on the error, we need an open interval that contains $x = 3$ and $x = 4$. We set $I = (2.9, 4.1)$. The largest value the fifth derivative of $f(x) = \sqrt{x}$ takes on this interval is near $x = 2.9$, at about 0.0273. Thus

$$\left|R_4(3)\right| \le \frac{0.0273}{5!}\left|(3-4)^5\right| \approx 0.00023.$$

This shows our approximation is accurate to at least the first 2 places after the decimal. (It turns out that our approximation is actually accurate to 4 places after the decimal.) A graph of $f(x) = \sqrt{x}$ and $p_4(x)$ is given in Figure 9.7.12. Note how the two functions are nearly indistinguishable on $(2, 7)$.

$$\begin{aligned} f(x) &= \sqrt{x} & &\Rightarrow & f(4) &= 2 \\ f'(x) &= \frac{1}{2\sqrt{x}} & &\Rightarrow & f'(4) &= \frac{1}{4} \\ f''(x) &= \frac{-1}{4x^{3/2}} & &\Rightarrow & f''(4) &= \frac{-1}{32} \\ f'''(x) &= \frac{3}{8x^{5/2}} & &\Rightarrow & f'''(4) &= \frac{3}{256} \\ f^{(4)}(x) &= \frac{-15}{16x^{7/2}} & &\Rightarrow & f^{(4)}(4) &= \frac{-15}{2048} \end{aligned}$$

Figure 9.7.11: A table of the derivatives of $f(x) = \sqrt{x}$ evaluated at $x = 4$.

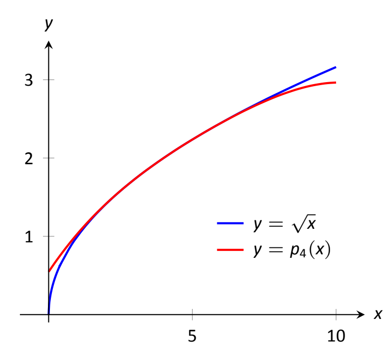


Figure 9.7.12: A graph of $f(x) = \sqrt{x}$ and its degree 4 Taylor polynomial at $x = 4$.

Our final example gives a brief introduction to using Taylor polynomials to solve differential equations.

Example 9.7.6 Approximating an unknown function

A function $y = f(x)$ is unknown save for the following two facts.

1. $y(0) = f(0) = 1$, and
2. $y' = y^2$

(This second fact says that amazingly, the derivative of the function is actually the function squared!)

Find the degree 3 Maclaurin polynomial $p_3(x)$ of $y = f(x)$.

Solution One might initially think that not enough information is given to find $p_3(x)$. However, note how the second fact above actually lets us know what $y'(0)$ is:

$$y' = y^2 \Rightarrow y'(0) = y^2(0).$$

Since $y(0) = 1$, we conclude that $y'(0) = 1$.

Now we find information about y''. Starting with $y' = y^2$, take derivatives of both sides, *with respect to* x. That means we must use implicit differentiation.

$$\begin{aligned} y' &= y^2 \\ \frac{d}{dx}\left(y'\right) &= \frac{d}{dx}\left(y^2\right) \\ y'' &= 2y \cdot y'. \end{aligned}$$

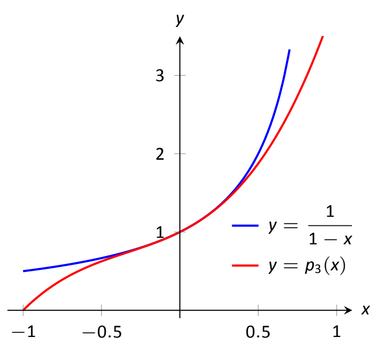


Figure 9.7.13: A graph of $y = -1/(x-1)$ and $y = p_3(x)$ from Example 9.7.6.

Now evaluate both sides at $x = 0$:

$$\begin{aligned} y''(0) &= 2y(0) \cdot y'(0) \\ y''(0) &= 2 \end{aligned}$$

We repeat this once more to find $y'''(0)$. We again use implicit differentiation; this time the Product Rule is also required.

$$\begin{aligned} \frac{d}{dx}\left(y''\right) &= \frac{d}{dx}\left(2yy'\right) \\ y''' &= 2y' \cdot y' + 2y \cdot y''. \end{aligned}$$

Now evaluate both sides at $x = 0$:

$$\begin{aligned} y'''(0) &= 2y'(0)^2 + 2y(0)y''(0) \\ y'''(0) &= 2 + 4 = 6 \end{aligned}$$

In summary, we have:

$$y(0) = 1 \qquad y'(0) = 1 \qquad y''(0) = 2 \qquad y'''(0) = 6.$$

We can now form $p_3(x)$:

$$\begin{aligned} p_3(x) &= 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 \\ &= 1 + x + x^2 + x^3. \end{aligned}$$

It turns out that the differential equation we started with, $y' = y^2$, where $y(0) = 1$, can be solved without too much difficulty: $y = \frac{1}{1-x}$. Figure 9.7.13 shows this function plotted with $p_3(x)$. Note how similar they are near $x = 0$.

It is beyond the scope of this text to pursue error analysis when using Taylor polynomials to approximate solutions to differential equations. This topic is often broached in introductory Differential Equations courses and usually covered in depth in Numerical Analysis courses. Such an analysis is very important; one needs to know how good their approximation is. We explored this example simply to demonstrate the usefulness of Taylor polynomials.

Most of this chapter has been devoted to the study of infinite series. This section has taken a step back from this study, focusing instead on finite summation of terms. In the next section, we explore **Taylor Series**, where we represent a function with an infinite series.

Exercises 9.7

Terms and Concepts

1. What is the difference between a Taylor polynomial and a Maclaurin polynomial?

2. T/F: In general, $p_n(x)$ approximates $f(x)$ better and better as n gets larger.

3. For some function $f(x)$, the Maclaurin polynomial of degree 4 is $p_4(x) = 6 + 3x - 4x^2 + 5x^3 - 7x^4$. What is $p_2(x)$?

4. For some function $f(x)$, the Maclaurin polynomial of degree 4 is $p_4(x) = 6 + 3x - 4x^2 + 5x^3 - 7x^4$. What is $f'''(0)$?

Problems

In Exercises 5 – 12, find the Maclaurin polynomial of degree *n* for the given function.

5. $f(x) = e^{-x}, \quad n = 3$

6. $f(x) = \sin x, \quad n = 8$

7. $f(x) = x \cdot e^x, \quad n = 5$

8. $f(x) = \tan x, \quad n = 6$

9. $f(x) = e^{2x}, \quad n = 4$

10. $f(x) = \dfrac{1}{1-x}, \quad n = 4$

11. $f(x) = \dfrac{1}{1+x}, \quad n = 4$

12. $f(x) = \dfrac{1}{1+x}, \quad n = 7$

In Exercises 13 – 20, find the Taylor polynomial of degree *n*, at $x = c$, for the given function.

13. $f(x) = \sqrt{x}, \quad n = 4, \quad c = 1$

14. $f(x) = \ln(x+1), \quad n = 4, \quad c = 1$

15. $f(x) = \cos x, \quad n = 6, \quad c = \pi/4$

16. $f(x) = \sin x, \quad n = 5, \quad c = \pi/6$

17. $f(x) = \dfrac{1}{x}, \quad n = 5, \quad c = 2$

18. $f(x) = \dfrac{1}{x^2}, \quad n = 8, \quad c = 1$

19. $f(x) = \dfrac{1}{x^2+1}, \quad n = 3, \quad c = -1$

20. $f(x) = x^2 \cos x, \quad n = 2, \quad c = \pi$

In Exercises 21 – 24, approximate the function value with the indicated Taylor polynomial and give approximate bounds on the error.

21. Approximate $\sin 0.1$ with the Maclaurin polynomial of degree 3.

22. Approximate $\cos 1$ with the Maclaurin polynomial of degree 4.

23. Approximate $\sqrt{10}$ with the Taylor polynomial of degree 2 centered at $x = 9$.

24. Approximate $\ln 1.5$ with the Taylor polynomial of degree 3 centered at $x = 1$.

Exercises 25 – 28 ask for an *n* to be found such that $p_n(x)$ approximates $f(x)$ within a certain bound of accuracy.

25. Find n such that the Maclaurin polynomial of degree n of $f(x) = e^x$ approximates e within 0.0001 of the actual value.

26. Find n such that the Taylor polynomial of degree n of $f(x) = \sqrt{x}$, centered at $x = 4$, approximates $\sqrt{3}$ within 0.0001 of the actual value.

27. Find n such that the Maclaurin polynomial of degree n of $f(x) = \cos x$ approximates $\cos \pi/3$ within 0.0001 of the actual value.

28. Find n such that the Maclaurin polynomial of degree n of $f(x) = \sin x$ approximates $\cos \pi$ within 0.0001 of the actual value.

In Exercises 29 – 34, find the n^{th} term of the indicated Taylor polynomial.

29. Find a formula for the n^{th} term of the Maclaurin polynomial for $f(x) = e^x$.

30. Find a formula for the n^{th} term of the Maclaurin polynomial for $f(x) = \cos x$.

31. Find a formula for the n^{th} term of the Maclaurin polynomial for $f(x) = \sin x$.

32. Find a formula for the n^{th} term of the Maclaurin polynomial for $f(x) = \dfrac{1}{1-x}$.

33. Find a formula for the n^{th} term of the Maclaurin polynomial for $f(x) = \dfrac{1}{1+x}$.

34. Find a formula for the n^{th} term of the Taylor polynomial for $f(x) = \ln x$ centred at $x = 1$.

In Exercises 35 – 37, approximate the solution to the given differential equation with a degree 4 Maclaurin polynomial.

35. $y' = y, \qquad y(0) = 1$

36. $y' = 5y, \qquad y(0) = 3$

37. $y' = \dfrac{2}{y}, \qquad y(0) = 1$

9.8 Taylor Series

In Section 9.6, we showed how certain functions can be represented by a power series function. In Section 9.7, we showed how we can approximate functions with polynomials, given that enough derivative information is available. In this section we combine these concepts: if a function $f(x)$ is infinitely differentiable, we show how to represent it with a power series function.

Definition 9.8.1 Taylor and Maclaurin Series

Let $f(x)$ have derivatives of all orders at $x = c$.

1. The **Taylor Series of $f(x)$, centred at** c is
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$
2. Setting $c = 0$ gives the **Maclaurin Series of** $f(x)$:
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

If $p_n(x)$ is the n^{th} degree Taylor polynomial for $f(x)$ centred at $x = c$, we saw how $f(x)$ is *approximately equal* to $p_n(x)$ near $x = c$. We also saw how increasing the degree of the polynomial generally reduced the error.

We are now considering *series*, where we sum an infinite set of terms. Our ultimate hope is to see the error vanish and claim a function is *equal* to its Taylor series.

When creating the Taylor polynomial of degree n for a function $f(x)$ at $x = c$, we needed to evaluate f, and the first n derivatives of f, at $x = c$. When creating the Taylor series of f, it helps to find a pattern that describes the n^{th} derivative of f at $x = c$. We demonstrate this in the next two examples.

Example 9.8.1 The Maclaurin series of $f(x) = \cos x$

Find the Maclaurin series of $f(x) = \cos x$.

$f(x) = \cos x$	$\Rightarrow$	$f(0) = 1$
$f'(x) = -\sin x$	$\Rightarrow$	$f'(0) = 0$
$f''(x) = -\cos x$	$\Rightarrow$	$f''(0) = -1$
$f'''(x) = \sin x$	$\Rightarrow$	$f'''(0) = 0$
$f^{(4)}(x) = \cos x$	$\Rightarrow$	$f^{(4)}(0) = 1$
$f^{(5)}(x) = -\sin x$	$\Rightarrow$	$f^{(5)}(0) = 0$
$f^{(6)}(x) = -\cos x$	$\Rightarrow$	$f^{(6)}(0) = -1$
$f^{(7)}(x) = \sin x$	$\Rightarrow$	$f^{(7)}(0) = 0$
$f^{(8)}(x) = \cos x$	$\Rightarrow$	$f^{(8)}(0) = 1$
$f^{(9)}(x) = -\sin x$	$\Rightarrow$	$f^{(9)}(0) = 0$

Figure 9.8.1: A table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$.

SOLUTION In Example 9.7.4 we found the 8^{th} degree Maclaurin polynomial of $\cos x$. In doing so, we created the table shown in Figure 9.8.1. Notice how $f^{(n)}(0) = 0$ when n is odd, $f^{(n)}(0) = 1$ when n is divisible by 4, and $f^{(n)}(0) = -1$ when n is even but not divisible by 4. Thus the Maclaurin series of $\cos x$ is
$$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$
We can go further and write this as a summation. Since we only need the terms where the power of x is even, we write the power series in terms of x^{2n}:
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Example 9.8.2 The Taylor series of $f(x) = \ln x$ at $x = 1$

Find the Taylor series of $f(x) = \ln x$ centred at $x = 1$.

Solution Figure 9.8.2 shows the n^{th} derivative of $\ln x$ evaluated at $x = 1$ for $n = 0, \ldots, 5$, along with an expression for the n^{th} term:

$$f^{(n)}(1) = (-1)^{n+1}(n-1)! \quad \text{for } n \geq 1.$$

Remember that this is what distinguishes Taylor series from Taylor polynomials; we are very interested in finding a pattern for the n^{th} term, not just finding a finite set of coefficients for a polynomial. Since $f(1) = \ln 1 = 0$, we skip the first term and start the summation with $n = 1$, giving the Taylor series for $\ln x$, centred at $x = 1$, as

$$\sum_{n=1}^{\infty}(-1)^{n+1}(n-1)!\frac{1}{n!}(x-1)^n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}.$$

$f(x) = \ln x$	$\Rightarrow$	$f(1) = 0$
$f'(x) = 1/x$	$\Rightarrow$	$f'(1) = 1$
$f''(x) = -1/x^2$	$\Rightarrow$	$f''(1) = -1$
$f'''(x) = 2/x^3$	$\Rightarrow$	$f'''(1) = 2$
$f^{(4)}(x) = -6/x^4$	$\Rightarrow$	$f^{(4)}(1) = -6$
$f^{(5)}(x) = 24/x^5$	$\Rightarrow$	$f^{(5)}(1) = 24$
$\vdots$		$\vdots$
$f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{x^n}$	$\Rightarrow$	$f^{(n)}(1) = (-1)^{n+1}(n-1)!$

Figure 9.8.2: Derivatives of $\ln x$ evaluated at $x = 1$.

It is important to note that Definition 9.8.1 defines a Taylor series given a function $f(x)$; however, we *cannot* yet state that $f(x)$ *is equal* to its Taylor series. We will find that "most of the time" they are equal, but we need to consider the conditions that allow us to conclude this.

Theorem 9.7.1 states that the error between a function $f(x)$ and its n^{th}–degree Taylor polynomial $p_n(x)$ is $R_n(x)$, where

$$|R_n(x)| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(x-c)^{(n+1)}\right|.$$

If $R_n(x)$ goes to 0 for each x in an interval I as n approaches infinity, we conclude that the function is equal to its Taylor series expansion.

Theorem 9.8.1 Function and Taylor Series Equality

Let $f(x)$ have derivatives of all orders at $x = c$, let $R_n(x)$ be as stated in Theorem 9.7.1, and let I be an interval on which the Taylor series of $f(x)$ converges. If $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x in I, then

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n \text{ on } I.$$

We demonstrate the use of this theorem in an example.

Example 9.8.3 Establishing equality of a function and its Taylor series

Show that $f(x) = \cos x$ is equal to its Maclaurin series, as found in Example 9.8.1, for all x.

Solution Given a value x, the magnitude of the error term $R_n(x)$ is bounded by

$$|R_n(x)| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x^{n+1}\right|.$$

Since all derivatives of $\cos x$ are $\pm\sin x$ or $\pm\cos x$, whose magnitudes are bounded by 1, we can state

$$|R_n(x)| \leq \frac{1}{(n+1)!}\left|x^{n+1}\right|$$

which implies

$$-\frac{|x^{n+1}|}{(n+1)!} \leq R_n(x) \leq \frac{|x^{n+1}|}{(n+1)!}. \tag{9.8}$$

For any x, $\lim\limits_{n\to\infty} \dfrac{x^{n+1}}{(n+1)!} = 0$. Applying the Squeeze Theorem to Equation (9.8), we conclude that $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x, and hence

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x.$$

It is natural to assume that a function is equal to its Taylor series on the series' interval of convergence, but this is not always the case. In order to properly establish equality, one must use Theorem 9.8.1. This is a bit disappointing, as we developed beautiful techniques for determining the interval of convergence of a power series, and proving that $R_n(x) \to 0$ can be difficult. For instance, it is not a simple task to show that $\ln x$ equals its Taylor series on $(0, 2]$ as found in Example 9.8.2; in the Exercises, the reader is only asked to show equality on $(1, 2)$, which is simpler.

There is good news. A function $f(x)$ that is equal to its Taylor series, centred at any point the domain of $f(x)$, is said to be an **analytic function**, and most, if not all, functions that we encounter within this course are analytic functions. Generally speaking, any function that one creates with elementary functions (polynomials, exponentials, trigonometric functions, etc.) that is not piecewise defined is probably analytic. For most functions, we assume the function is equal to its Taylor series on the series' interval of convergence and only use Theorem 9.8.1 when we suspect something may not work as expected.

We develop the Taylor series for one more important function, then give a table of the Taylor series for a number of common functions.

Example 9.8.4 The Binomial Series

Find the Maclaurin series of $f(x) = (1+x)^k$, $k \neq 0$.

Solution When k is a positive integer, the Maclaurin series is finite. For instance, when $k = 4$, we have

$$f(x) = (1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

The coefficients of x when k is a positive integer are known as the *binomial coefficients*, giving the series we are developing its name.

When $k = 1/2$, we have $f(x) = \sqrt{1+x}$. Knowing a series representation of this function would give a useful way of approximating $\sqrt{1.3}$, for instance.

To develop the Maclaurin series for $f(x) = (1+x)^k$ for any value of $k \neq 0$, we consider the derivatives of f evaluated at $x = 0$:

$$\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1\\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k\\
f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1)\\
f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2)\\
&\vdots & &\vdots\\
f^{(n)}(x) &= k(k-1)\cdots\big(k-(n-1)\big)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots\big(k-(n-1)\big)
\end{aligned}$$

Thus the Maclaurin series for $f(x) = (1+x)^k$ is

$$1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\ldots+\frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n+\ldots$$

It is important to determine the interval of convergence of this series. With

$$a_n = \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n,$$

we apply the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty}\left|\frac{k(k-1)\cdots(k-n)}{(n+1)!}x^{n+1}\right| \Bigg/ \left|\frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n\right| \\
&= \lim_{n\to\infty}\left|\frac{k-n}{n}x\right| \\
&= |x|.
\end{aligned}$$

The series converges absolutely when the limit of the Ratio Test is less than 1; therefore, we have absolute convergence when $|x| < 1$.

While outside the scope of this text, the interval of convergence depends on the value of k. When $k > 0$, the interval of convergence is $[-1, 1]$. When $-1 < k < 0$, the interval of convergence is $[-1, 1)$. If $k \le -1$, the interval of convergence is $(-1, 1)$.

We learned that Taylor polynomials offer a way of approximating a "difficult to compute" function with a polynomial. Taylor series offer a way of exactly representing a function with a series. One probably can see the use of a good approximation; is there any use of representing a function exactly as a series?

While we should not overlook the mathematical beauty of Taylor series (which is reason enough to study them), there are practical uses as well. They provide a valuable tool for solving a variety of problems, including problems relating to integration and differential equations.

In Key Idea 9.8.1 (on the following page) we give a table of the Taylor series of a number of common functions. We then give a theorem about the "algebra of power series," that is, how we can combine power series to create power series of new functions. This allows us to find the Taylor series of functions like $f(x) = e^x \cos x$ by knowing the Taylor series of e^x and $\cos x$.

Before we investigate combining functions, consider the Taylor series for the arctangent function (see Key Idea 9.8.1). Knowing that $\tan^{-1}(1) = \pi/4$, we can use this series to approximate the value of π:

$$\begin{aligned}
\frac{\pi}{4} &= \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \\
\pi &= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)
\end{aligned}$$

Unfortunately, this particular expansion of π converges very slowly. The first 100 terms approximate π as 3.13159, which is not particularly good.

Key Idea 9.8.1 Important Taylor Series Expansions

Function and Series	First Few Terms	Interval of Convergence
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$(-\infty, \infty)$
$\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$	$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$	$(0, 2]$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!} x^n$	$1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots$	$(-1, 1)$[a]
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$[-1, 1]$

[a] Convergence at $x = \pm 1$ depends on the value of k.

Theorem 9.8.2 Algebra of Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and let $h(x)$ be continuous.

1. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n \quad$ for $|x| < R$.
2. $f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \ldots a_n b_0) x^n$ for $|x| < R$.
3. $f\big(h(x)\big) = \sum_{n=0}^{\infty} a_n \big(h(x)\big)^n \quad$ for $|h(x)| < R$.

Example 9.8.5 Combining Taylor series
Write out the first 3 terms of the Taylor Series for $f(x) = e^x \cos x$ using Key Idea 9.8.1 and Theorem 9.8.2.

Solution Key Idea 9.8.1 informs us that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots .$$

Applying Theorem 9.8.2, we find that

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right).$$

Distribute the right hand expression across the left:

$$= 1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$
$$+ \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^4}{4!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \cdots$$

Distribute again and collect like terms.

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \frac{x^7}{630} + \cdots$$

While this process is a bit tedious, it is much faster than evaluating all the necessary derivatives of $e^x \cos x$ and computing the Taylor series directly.

Because the series for e^x and $\cos x$ both converge on $(-\infty, \infty)$, so does the series expansion for $e^x \cos x$.

Note: In Example 9.8.6, one could create a series for $\ln(\sqrt{x})$ by simply recognizing that $\ln(\sqrt{x}) = \ln(x^{1/2}) = 1/2 \ln x$, and hence multiplying the Taylor series for $\ln x$ by $1/2$. This example was chosen to demonstrate other aspects of series, such as the fact that the interval of convergence changes.

Example 9.8.6 Creating new Taylor series
Use Theorem 9.8.2 to create series for $y = \sin(x^2)$ and $y = \ln(\sqrt{x})$.

Solution Given that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ,$$

we simply substitute x^2 for x in the series, giving

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} \cdots .$$

Since the Taylor series for $\sin x$ has an infinite radius of convergence, so does the Taylor series for $\sin(x^2)$.

The Taylor expansion for $\ln x$ given in Key Idea 9.8.1 is centred at $x = 1$, so we will center the series for $\ln(\sqrt{x})$ at $x = 1$ as well. With

$$\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots ,$$

we substitute $\sqrt{x}$ for x to obtain

$$\ln(\sqrt{x}) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\sqrt{x}-1)^n}{n} = (\sqrt{x}-1) - \frac{(\sqrt{x}-1)^2}{2} + \frac{(\sqrt{x}-1)^3}{3} - \cdots .$$

While this is not strictly a power series, it is a series that allows us to study the function $\ln(\sqrt{x})$. Since the interval of convergence of $\ln x$ is $(0,2]$, and the range of $\sqrt{x}$ on $(0,4]$ is $(0,2]$, the interval of convergence of this series expansion of $\ln(\sqrt{x})$ is $(0,4]$.

Example 9.8.7 Using Taylor series to evaluate definite integrals

Use the Taylor series of e^{-x^2} to evaluate $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution We learned, when studying Numerical Integration, that e^{-x^2} does not have an antiderivative expressible in terms of elementary functions. This means any definite integral of this function must have its value approximated, and not computed exactly.

We can quickly write out the Taylor series for e^{-x^2} using the Taylor series of e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and so

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots . \end{aligned}$$

We use Theorem 9.6.3 to integrate:

$$\int e^{-x^2}\,dx = C + x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

This *is* the antiderivative of e^{-x^2}; while we can write it out as a series, we cannot write it out in terms of elementary functions. We can evaluate the definite integral $\displaystyle\int_0^1 e^{-x^2}\,dx$ using this antiderivative; substituting 1 and 0 for x and subtracting gives

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} \cdots .$$

Summing the 5 terms shown above give the approximation of 0.74749. Since this is an alternating series, we can use the Alternating Series Approximation Theorem, (Theorem 9.5.2), to determine how accurate this approximation is. The next term of the series is $1/(11\cdot 5!) \approx 0.00075758$. Thus we know our approximation is within 0.00075758 of the actual value of the integral. This is arguably much less work than using Simpson's Rule to approximate the value of the integral.

Example 9.8.8 Using Taylor series to solve differential equations

Solve the differential equation $y' = 2y$ in terms of a power series, and use the theory of Taylor series to recognize the solution in terms of an elementary function.

Solution We found the first 5 terms of the power series solution to this differential equation in Example 9.6.5 in Section 9.6. These are:

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = \frac{4}{2} = 2, \quad a_3 = \frac{8}{2\cdot 3} = \frac{4}{3}, \quad a_4 = \frac{16}{2\cdot 3\cdot 4} = \frac{2}{3}.$$

We include the "unsimplified" expressions for the coefficients found in Example 9.6.5 as we are looking for a pattern. It can be shown that $a_n = 2^n/n!$. Thus the solution, written as a power series, is

$$y = \sum_{n=0}^{\infty} \frac{2^n}{n!}x^n = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

Using Key Idea 9.8.1 and Theorem 9.8.2, we recognize $f(x) = e^{2x}$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \Rightarrow \qquad e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

Finding a pattern in the coefficients that match the series expansion of a known function, such as those shown in Key Idea 9.8.1, can be difficult. What if the coefficients in the previous example were given in their reduced form; how could we still recover the function $y = e^{2x}$?

Suppose that all we know is that

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 2, \quad a_3 = \frac{4}{3}, \quad a_4 = \frac{2}{3}.$$

Definition 9.8.1 states that each term of the Taylor expansion of a function includes an $n!$. This allows us to say that

$$a_2 = 2 = \frac{b_2}{2!}, \quad a_3 = \frac{4}{3} = \frac{b_3}{3!}, \quad \text{and} \quad a_4 = \frac{2}{3} = \frac{b_4}{4!}$$

for some values b_2, b_3 and b_4. Solving for these values, we see that $b_2 = 4$, $b_3 = 8$ and $b_4 = 16$. That is, we are recovering the pattern we had previously seen, allowing us to write

$$\begin{aligned} f(x) = \sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} \frac{b_n}{n!}x^n \\ &= 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 + \cdots \end{aligned}$$

From here it is easier to recognize that the series is describing an exponential function.

There are simpler, more direct ways of solving the differential equation $y' = 2y$. We applied power series techniques to this equation to demonstrate its utility, and went on to show how *sometimes* we are able to recover the solution in terms of elementary functions using the theory of Taylor series. Most differential equations faced in real scientific and engineering situations are much more complicated than this one, but power series can offer a valuable tool in finding, or at least approximating, the solution.

This chapter introduced sequences, which are ordered lists of numbers, followed by series, wherein we add up the terms of a sequence. We quickly saw that such sums do not always add up to "infinity," but rather converge. We studied tests for convergence, then ended the chapter with a formal way of defining functions based on series. Such "series–defined functions" are a valuable tool in solving a number of different problems throughout science and engineering.

Coming in the next chapters are new ways of defining curves in the plane apart from using functions of the form $y = f(x)$. Curves created by these new methods can be beautiful, useful, and important.

Exercises 9.8

Terms and Concepts

1. What is the difference between a Taylor polynomial and a Taylor series?

2. What theorem must we use to show that a function is equal to its Taylor series?

Problems

Key Idea 9.8.1 gives the n^{th} term of the Taylor series of common functions. In Exercises 3 – 6, verify the formula given in the Key Idea by finding the first few terms of the Taylor series of the given function and identifying a pattern.

3. $f(x) = e^x; \quad c = 0$

4. $f(x) = \sin x; \quad c = 0$

5. $f(x) = 1/(1-x); \quad c = 0$

6. $f(x) = \tan^{-1} x; \quad c = 0$

In Exercises 7 – 12, find a formula for the n^{th} term of the Taylor series of $f(x)$, centered at c, by finding the coefficients of the first few powers of x and looking for a pattern. (The formulas for several of these are found in Key Idea 9.8.1; show work verifying these formula.)

7. $f(x) = \cos x; \quad c = \pi/2$

8. $f(x) = 1/x; \quad c = 1$

9. $f(x) = e^{-x}; \quad c = 0$

10. $f(x) = \ln(1+x); \quad c = 0$

11. $f(x) = x/(x+1); \quad c = 1$

12. $f(x) = \sin x; \quad c = \pi/4$

In Exercises 13 – 16, show that the Taylor series for $f(x)$, as given in Key Idea 9.8.1, is equal to $f(x)$ by applying Theorem 9.8.1; that is, show $\lim\limits_{n\to\infty} R_n(x) = 0$.

13. $f(x) = e^x$

14. $f(x) = \sin x$

15. $f(x) = \ln x$ (show equality only on $(1,2)$)

16. $f(x) = 1/(1-x)$ (show equality only on $(-1,0)$)

In Exercises 17 – 20, use the Taylor series given in Key Idea 9.8.1 to verify the given identity.

17. $\cos(-x) = \cos x$

18. $\sin(-x) = -\sin x$

19. $\frac{d}{dx}\big(\sin x\big) = \cos x$

20. $\frac{d}{dx}\big(\cos x\big) = -\sin x$

In Exercises 21 – 24, write out the first 5 terms of the Binomial series with the given k-value.

21. $k = 1/2$

22. $k = -1/2$

23. $k = 1/3$

24. $k = 4$

In Exercises 25 – 30, use the Taylor series given in Key Idea 9.8.1 to create the Taylor series of the given functions.

25. $f(x) = \cos\left(x^2\right)$

26. $f(x) = e^{-x}$

27. $f(x) = \sin\left(2x+3\right)$

28. $f(x) = \tan^{-1}\left(x/2\right)$

29. $f(x) = e^x \sin x$ (only find the first 4 terms)

30. $f(x) = (1+x)^{1/2}\cos x$ (only find the first 4 terms)

In Exercises 31 – 32, approximate the value of the given definite integral by using the first 4 nonzero terms of the integrand's Taylor series.

31. $\displaystyle\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\,dx$

32. $\displaystyle\int_0^{\pi^2/4} \cos\left(\sqrt{x}\right)\,dx$

10: CURVES IN THE PLANE

We have explored functions of the form $y = f(x)$ closely throughout this text. We have explored their limits, their derivatives and their antiderivatives; we have learned to identify key features of their graphs, such as relative maxima and minima, inflection points and asymptotes; we have found equations of their tangent lines, the areas between portions of their graphs and the x-axis, and the volumes of solids generated by revolving portions of their graphs about a horizontal or vertical axis.

Despite all this, the graphs created by functions of the form $y = f(x)$ are limited. Since each x-value can correspond to only 1 y-value, common shapes like circles cannot be fully described by a function in this form. Fittingly, the "vertical line test" excludes vertical lines from being functions of x, even though these lines are important in mathematics.

In this chapter we'll explore new ways of drawing curves in the plane. We'll still work within the framework of functions, as an input will still only correspond to one output. However, our new techniques of drawing curves will render the vertical line test pointless, and allow us to create important – and beautiful – new curves. Once these curves are defined, we'll apply the concepts of calculus to them, continuing to find equations of tangent lines and the areas of enclosed regions.

10.1 Conic Sections

The ancient Greeks recognized that interesting shapes can be formed by intersecting a plane with a *double napped* cone (i.e., two identical cones placed tip–to–tip as shown in the following figures). As these shapes are formed as sections of conics, they have earned the official name "conic sections."

The three "most interesting" conic sections are given in the top row of Figure 10.1.1. They are the parabola, the ellipse (which includes circles) and the hyperbola. In each of these cases, the plane does not intersect the tips of the cones (usually taken to be the origin).

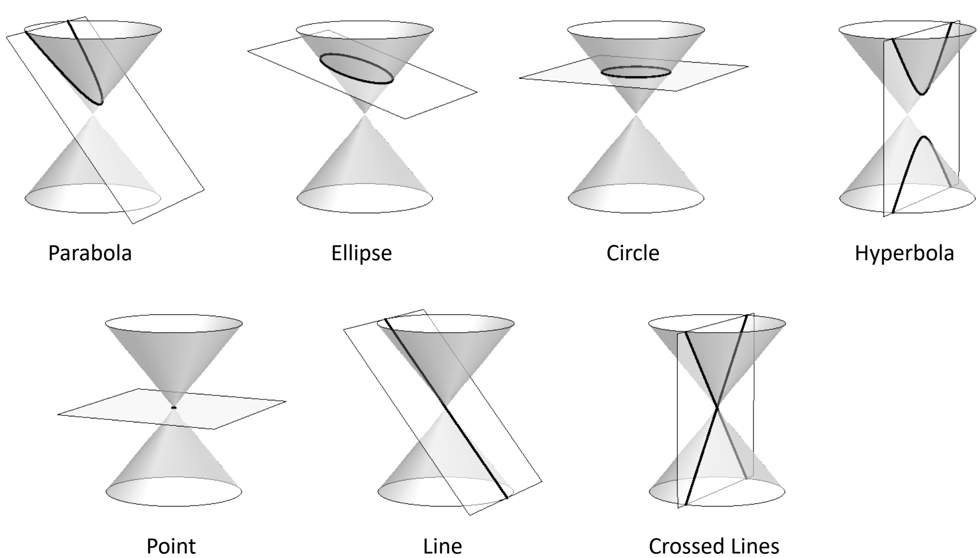


Figure 10.1.1: Conic Sections

When the plane does contain the origin, three **degenerate** cones can be formed as shown the bottom row of Figure 10.1.1: a point, a line, and crossed lines. We focus here on the nondegenerate cases.

While the above geometric constructs define the conics in an intuitive, visual way, these constructs are not very helpful when trying to analyze the shapes algebraically or consider them as the graph of a function. It can be shown that all conics can be defined by the general second–degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

While this algebraic definition has its uses, most find another geometric perspective of the conics more beneficial.

Each nondegenerate conic can be defined as the **locus**, or set, of points that satisfy a certain distance property. These distance properties can be used to generate an algebraic formula, allowing us to study each conic as the graph of a function.

Parabolas

Definition 10.1.1 Parabola

A **parabola** is the locus of all points equidistant from a point (called a **focus**) and a line (called the **directrix**) that does not contain the focus.

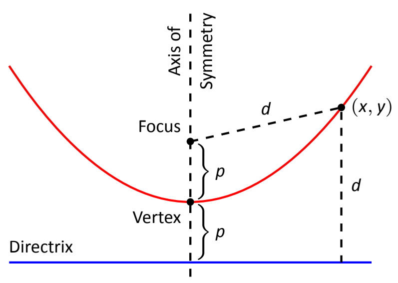


Figure 10.1.2: Illustrating the definition of the parabola and establishing an algebraic formula.

Figure 10.1.2 illustrates this definition. The point halfway between the focus and the directrix is the **vertex**. The line through the focus, perpendicular to the directrix, is the **axis of symmetry**, as the portion of the parabola on one side of this line is the mirror–image of the portion on the opposite side.

The definition leads us to an algebraic formula for the parabola. Let $P = (x, y)$ be a point on a parabola whose focus is at $F = (0, p)$ and whose directrix is at $y = -p$. (We'll assume for now that the focus lies on the y-axis; by placing the focus p units above the x-axis and the directrix p units below this axis, the vertex will be at $(0, 0)$.)

We use the Distance Formula to find the distance d_1 between F and P:

$$d_1 = \sqrt{(x-0)^2 + (y-p)^2}.$$

The distance d_2 from P to the directrix is more straightforward:

$$d_2 = y - (-p) = y + p.$$

These two distances are equal. Setting $d_1 = d_2$, we can solve for y in terms of x:

$$\begin{aligned} d_1 &= d_2 \\ \sqrt{x^2 + (y-p)^2} &= y + p \end{aligned}$$

Now square both sides.

$$\begin{aligned} x^2 + (y-p)^2 &= (y+p)^2 \\ x^2 + y^2 - 2yp + p^2 &= y^2 + 2yp + p^2 \\ x^2 &= 4yp \\ y &= \frac{1}{4p}x^2. \end{aligned}$$

The geometric definition of the parabola has led us to the familiar quadratic function whose graph is a parabola with vertex at the origin. When we allow the vertex to not be at $(0, 0)$, we get the following standard form of the parabola.

Key Idea 10.1.1 General Equation of a Parabola

1. **Vertical Axis of Symmetry:** The equation of the parabola with vertex at (h, k) and directrix $y = k - p$ in standard form is
$$y = \frac{1}{4p}(x - h)^2 + k.$$
The focus is at $(h, k + p)$.
2. **Horizontal Axis of Symmetry:** The equation of the parabola with vertex at (h, k) and directrix $x = h - p$ in standard form is
$$x = \frac{1}{4p}(y - k)^2 + h.$$
The focus is at $(h + p, k)$.

Note: p is not necessarily a positive number.

Example 10.1.1 Finding the equation of a parabola
Give the equation of the parabola with focus at $(1, 2)$ and directrix at $y = 3$.

Solution The vertex is located halfway between the focus and directrix, so $(h, k) = (1, 2.5)$. This gives $p = -0.5$. Using Key Idea 10.1.1 we have the equation of the parabola as
$$y = \frac{1}{4(-0.5)}(x - 1)^2 + 2.5 = -\frac{1}{2}(x - 1)^2 + 2.5.$$
The parabola is sketched in Figure 10.1.3.

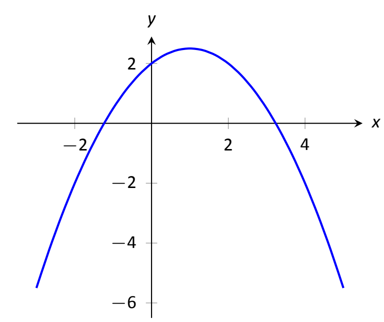


Figure 10.1.3: The parabola described in Example 10.1.1.

Example 10.1.2 Finding the focus and directrix of a parabola
Find the focus and directrix of the parabola $x = \frac{1}{8}y^2 - y + 1$. The point $(7, 12)$ lies on the graph of this parabola; verify that it is equidistant from the focus and directrix.

Solution We need to put the equation of the parabola in its general form. This requires us to complete the square:
$$\begin{aligned} x &= \frac{1}{8}y^2 - y + 1 \\ &= \frac{1}{8}\left(y^2 - 8y + 8\right) \\ &= \frac{1}{8}\left(y^2 - 8y + 16 - 16 + 8\right) \\ &= \frac{1}{8}\left((y - 4)^2 - 8\right) \\ &= \frac{1}{8}(y - 4)^2 - 1. \end{aligned}$$

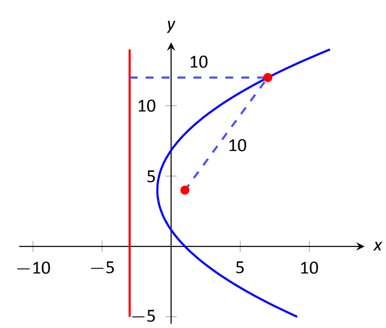


Figure 10.1.4: The parabola described in Example 10.1.2. The distances from a point on the parabola to the focus and directrix is given.

Hence the vertex is located at $(-1, 4)$. We have $\frac{1}{8} = \frac{1}{4p}$, so $p = 2$. We conclude that the focus is located at $(1, 4)$ and the directrix is $x = -3$. The parabola is graphed in Figure 10.1.4, along with its focus and directrix.

The point $(7, 12)$ lies on the graph and is $7 - (-3) = 10$ units from the directrix. The distance from $(7, 12)$ to the focus is:

$$\sqrt{(7-1)^2 + (12-4)^2} = \sqrt{100} = 10.$$

Indeed, the point on the parabola is equidistant from the focus and directrix.

Reflective Property

One of the fascinating things about the nondegenerate conic sections is their reflective properties. Parabolas have the following reflective property:

> Any ray emanating from the focus that intersects the parabola reflects off along a line perpendicular to the directrix.

This is illustrated in Figure 10.1.5. The following theorem states this more rigorously.

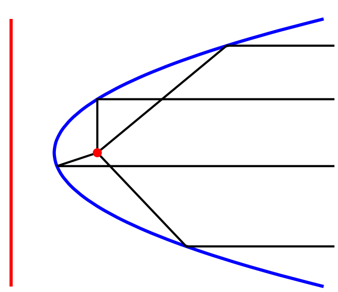


Figure 10.1.5: Illustrating the parabola's reflective property.

Theorem 10.1.1 Reflective Property of the Parabola

Let P be a point on a parabola. The tangent line to the parabola at P makes equal angles with the following two lines:

1. The line containing P and the focus F, and
2. The line perpendicular to the directrix through P.

Because of this reflective property, paraboloids (the 3D analogue of parabolas) make for useful flashlight reflectors as the light from the bulb, ideally located at the focus, is reflected along parallel rays. Satellite dishes also have paraboloid shapes. Signals coming from satellites effectively approach the dish along parallel rays. The dish then *focuses* these rays at the focus, where the sensor is located.

Ellipses

Definition 10.1.2 Ellipse

An **ellipse** is the locus of all points whose sum of distances from two fixed points, each a **focus** of the ellipse, is constant.

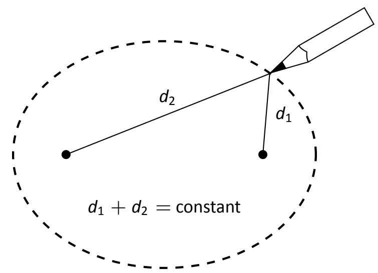


Figure 10.1.6: Illustrating the construction of an ellipse with pins, pencil and string.

An easy way to visualize this construction of an ellipse is to pin both ends of a string to a board. The pins become the foci. Holding a pencil tight against the string places the pencil on the ellipse; the sum of distances from the pencil to the pins is constant: the length of the string. See Figure 10.1.6.

We can again find an algebraic equation for an ellipse using this geometric definition. Let the foci be located along the x-axis, c units from the origin. Let these foci be labeled as $F_1 = (-c, 0)$ and $F_2 = (c, 0)$. Let $P = (x, y)$ be a point on the ellipse. The sum of distances from F_1 to P (d_1) and from F_2 to P (d_2) is a constant d. That is, $d_1 + d_2 = d$. Using the Distance Formula, we have

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = d.$$

Using a fair amount of algebra can produce the following equation of an ellipse (note that the equation is an implicitly defined function; it has to be, as an ellipse fails the Vertical Line Test):

$$\frac{x^2}{\left(\frac{d}{2}\right)^2} + \frac{y^2}{\left(\frac{d}{2}\right)^2 - c^2} = 1.$$

This is not particularly illuminating, but by making the substitution $a = d/2$ and $b = \sqrt{a^2 - c^2}$, we can rewrite the above equation as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This choice of a and b is not without reason; as shown in Figure 10.1.7, the values of a and b have geometric meaning in the graph of the ellipse.

In general, the two foci of an ellipse lie on the **major axis** of the ellipse, and the midpoint of the segment joining the two foci is the **center**. The major axis intersects the ellipse at two points, each of which is a **vertex**. The line segment through the center and perpendicular to the major axis is the **minor axis**. The "constant sum of distances" that defines the ellipse is the length of the major axis, i.e., $2a$.

Allowing for the shifting of the ellipse gives the following standard equations.

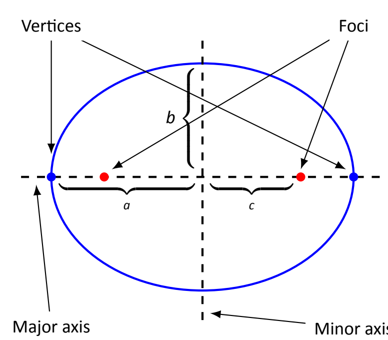


Figure 10.1.7: Labeling the significant features of an ellipse.

Key Idea 10.1.2 Standard Equation of the Ellipse

The equation of an ellipse centered at (h, k) with major axis of length $2a$ and minor axis of length $2b$ in standard form is:

1. **Horizontal major axis:** $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1.$
2. **Vertical major axis:** $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1.$

The foci lie along the major axis, c units from the center, where $c^2 = a^2 - b^2$.

Example 10.1.3 Finding the equation of an ellipse
Find the general equation of the ellipse graphed in Figure 10.1.8.

Solution The center is located at $(-3, 1)$. The distance from the center to a vertex is 5 units, hence $a = 5$. The minor axis seems to have length 4, so $b = 2$. Thus the equation of the ellipse is

$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{25} = 1.$$

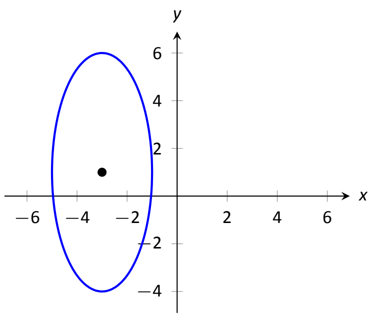


Figure 10.1.8: The ellipse used in Example 10.1.3.

Example 10.1.4 Graphing an ellipse
Graph the ellipse defined by $4x^2 + 9y^2 - 8x - 36y = -4$.

Solution It is simple to graph an ellipse once it is in standard form. In order to put the given equation in standard form, we must complete the square with both the x and y terms. We first rewrite the equation by regrouping:

$$4x^2 + 9y^2 - 8x - 36y = -4 \quad \Rightarrow \quad (4x^2 - 8x) + (9y^2 - 36y) = -4.$$

Now we complete the squares.

$$\begin{aligned}
(4x^2 - 8x) + (9y^2 - 36y) &= -4 \\
4(x^2 - 2x) + 9(y^2 - 4y) &= -4 \\
4(x^2 - 2x + 1 - 1) + 9(y^2 - 4y + 4 - 4) &= -4 \\
4\big((x-1)^2 - 1\big) + 9\big((y-2)^2 - 4\big) &= -4 \\
4(x-1)^2 - 4 + 9(y-2)^2 - 36 &= -4 \\
4(x-1)^2 + 9(y-2)^2 &= 36 \\
\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} &= 1.
\end{aligned}$$

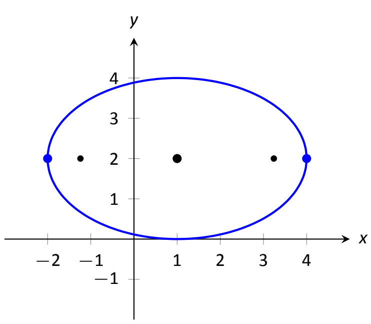


Figure 10.1.9: Graphing the ellipse in Example 10.1.4.

We see the center of the ellipse is at $(1, 2)$. We have $a = 3$ and $b = 2$; the major axis is horizontal, so the vertices are located at $(-2, 2)$ and $(4, 2)$. We find $c = \sqrt{9-4} = \sqrt{5} \approx 2.24$. The foci are located along the major axis, approximately 2.24 units from the center, at $(1 \pm 2.24, 2)$. This is all graphed in Figure 10.1.9 .

Eccentricity

When $a = b$, we have a circle. The general equation becomes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1 \quad \Rightarrow (x-h)^2 + (y-k)^2 = a^2,$$

the familiar equation of the circle centred at (h, k) with radius a. Since $a = b$, $c = \sqrt{a^2 - b^2} = 0$. The circle has "two" foci, but they lie on the same point, the center of the circle.

Consider Figure 10.1.10, where several ellipses are graphed with $a = 1$. In (a), we have $c = 0$ and the ellipse is a circle. As c grows, the resulting ellipses look less and less circular. A measure of this "noncircularness" is *eccentricity*.

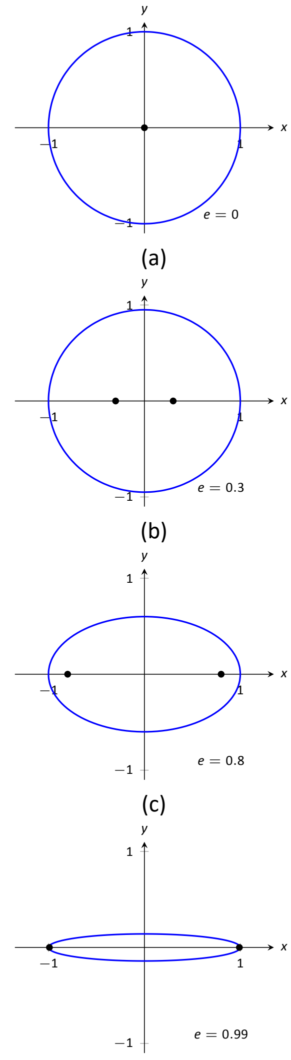


Figure 10.1.10: Understanding the eccentricity of an ellipse.

Definition 10.1.3 Eccentricity of an Ellipse

The eccentricity e of an ellipse is $e = \frac{c}{a}$.

The eccentricity of a circle is 0; that is, a circle has no "noncircularness." As c approaches a, e approaches 1, giving rise to a very noncircular ellipse, as seen in Figure 10.1.10 (d).

It was long assumed that planets had circular orbits. This is known to be incorrect; the orbits are elliptical. Earth has an eccentricity of 0.0167 – it has a nearly circular orbit. Mercury's orbit is the most eccentric, with $e = 0.2056$. (Pluto's eccentricity is greater, at $e = 0.248$, the greatest of all the currently known dwarf planets.) The planet with the most circular orbit is Venus, with $e = 0.0068$. The Earth's moon has an eccentricity of $e = 0.0549$, also very circular.

Reflective Property

The ellipse also possesses an interesting reflective property. Any ray emanating from one focus of an ellipse reflects off the ellipse along a line through the other focus, as illustrated in Figure 10.1.11. This property is given formally in the following theorem.

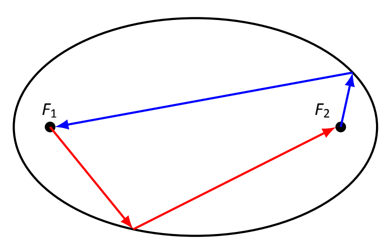


Figure 10.1.11: Illustrating the reflective property of an ellipse.

Theorem 10.1.2 Reflective Property of an Ellipse

Let P be a point on a ellipse with foci F_1 and F_2. The tangent line to the ellipse at P makes equal angles with the following two lines:

1. The line through F_1 and P, and
2. The line through F_2 and P.

This reflective property is useful in optics and is the basis of the phenomena experienced in whispering halls.

Hyperbolas

The definition of a hyperbola is very similar to the definition of an ellipse; we essentially just change the word "sum" to "difference."

Definition 10.1.4 Hyperbola

A **hyperbola** is the locus of all points where the absolute value of difference of distances from two fixed points, each a focus of the hyperbola, is constant.

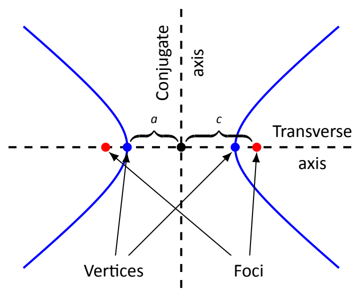


Figure 10.1.12: Labeling the significant features of a hyperbola.

We do not have a convenient way of visualizing the construction of a hyperbola as we did for the ellipse. The geometric definition does allow us to find an algebraic expression that describes it. It will be useful to define some terms first.

The two foci lie on the **transverse axis** of the hyperbola; the midpoint of the line segment joining the foci is the **center** of the hyperbola. The transverse axis intersects the hyperbola at two points, each a **vertex** of the hyperbola. The line through the center and perpendicular to the transverse axis is the **conjugate axis.** This is illustrated in Figure 10.1.12. It is easy to show that the constant difference of distances used in the definition of the hyperbola is the distance between the vertices, i.e., $2a$.

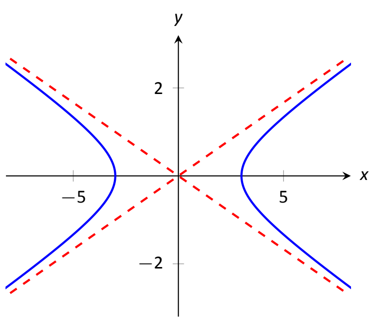


Figure 10.1.13: Graphing the hyperbola $\frac{x^2}{9} - \frac{y^2}{1} = 1$ along with its asymptotes, $y = \pm x/3$.

Key Idea 10.1.3 Standard Equation of a Hyperbola

The equation of a hyperbola centered at (h, k) in standard form is:

1. **Horizontal Transverse Axis:** $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1.$

2. **Vertical Transverse Axis:** $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1.$

The vertices are located a units from the center and the foci are located c units from the center, where $c^2 = a^2 + b^2$.

Graphing Hyperbolas

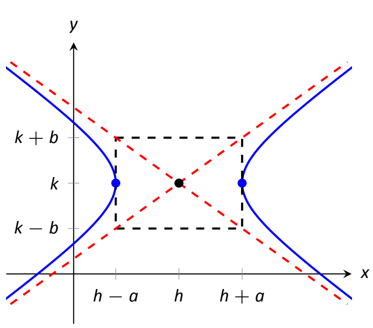


Figure 10.1.14: Using the asymptotes of a hyperbola as a graphing aid.

Consider the hyperbola $\frac{x^2}{9} - \frac{y^2}{1} = 1$. Solving for y, we find $y = \pm\sqrt{x^2/9 - 1}$. As x grows large, the "-1" part of the equation for y becomes less significant and $y \approx \pm\sqrt{x^2/9} = \pm x/3$. That is, as x gets large, the graph of the hyperbola looks very much like the lines $y = \pm x/3$. These lines are asymptotes of the hyperbola, as shown in Figure 10.1.13.

This is a valuable tool in sketching. Given the equation of a hyperbola in general form, draw a rectangle centered at (h, k) with sides of length $2a$ parallel to the transverse axis and sides of length $2b$ parallel to the conjugate axis. (See Figure 10.1.14 for an example with a horizontal transverse axis.) The diagonals of the rectangle lie on the asymptotes.

These lines pass through (h, k). When the transverse axis is horizontal, the slopes are $\pm b/a$; when the transverse axis is vertical, their slopes are $\pm a/b$. This gives equations:

Horizontal Transverse Axis	Vertical Transverse Axis
$y = \pm\frac{b}{a}(x-h)+k$	$y = \pm\frac{a}{b}(x-h)+k.$

Example 10.1.5 Graphing a hyperbola

Sketch the hyperbola given by $\frac{(y-2)^2}{25} - \frac{(x-1)^2}{4} = 1.$

Solution The hyperbola is centred at $(1,2)$; $a = 5$ and $b = 2$. In Figure 10.1.15 we draw the prescribed rectangle centred at $(1,2)$ along with the asymptotes defined by its diagonals. The hyperbola has a vertical transverse axis, so the vertices are located at $(1,7)$ and $(1,-3)$. This is enough to make a good sketch.

We also find the location of the foci: as $c^2 = a^2 + b^2$, we have $c = \sqrt{29} \approx 5.4$. Thus the foci are located at $(1, 2 \pm 5.4)$ as shown in the figure.

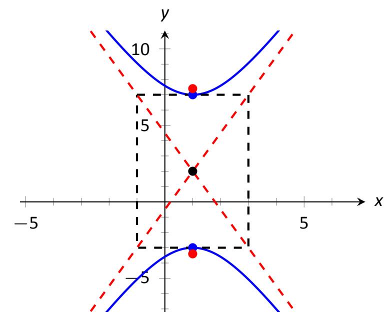


Figure 10.1.15: Graphing the hyperbola in Example 10.1.5.

Example 10.1.6 Graphing a hyperbola

Sketch the hyperbola given by $9x^2 - y^2 + 2y = 10$.

Solution We must complete the square to put the equation in general form. (We recognize this as a hyperbola since it is a general quadratic equation and the x^2 and y^2 terms have opposite signs.)

$$\begin{aligned}
9x^2 - y^2 + 2y &= 10\\
9x^2 - (y^2 - 2y) &= 10\\
9x^2 - (y^2 - 2y + 1 - 1) &= 10\\
9x^2 - \big((y-1)^2 - 1\big) &= 10\\
9x^2 - (y-1)^2 &= 9\\
x^2 - \frac{(y-1)^2}{9} &= 1
\end{aligned}$$

We see the hyperbola is centred at $(0,1)$, with a horizontal transverse axis, where $a = 1$ and $b = 3$. The appropriate rectangle is sketched in Figure 10.1.16 along with the asymptotes of the hyperbola. The vertices are located at $(\pm1, 1)$. We have $c = \sqrt{10} \approx 3.2$, so the foci are located at $(\pm3.2, 1)$ as shown in the figure.

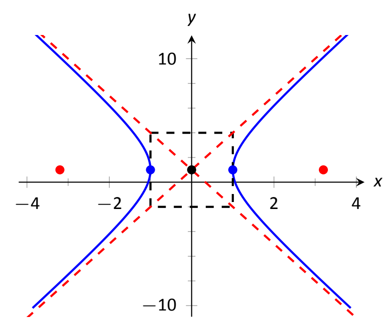


Figure 10.1.16: Graphing the hyperbola in Example 10.1.6.

Eccentricity

Definition 10.1.5 Eccentricity of a Hyperbola

The eccentricity of a hyperbola is $e = \dfrac{c}{a}$.

Note that this is the definition of eccentricity as used for the ellipse. When c is close in value to a (i.e., $e \approx 1$), the hyperbola is very narrow (looking almost like crossed lines). Figure 10.1.17 shows hyperbolas centered at the origin with $a = 1$. The graph in (a) has $c = 1.05$, giving an eccentricity of $e = 1.05$, which is close to 1. As c grows larger, the hyperbola widens and begins to look like parallel lines, as shown in part (d) of the figure.

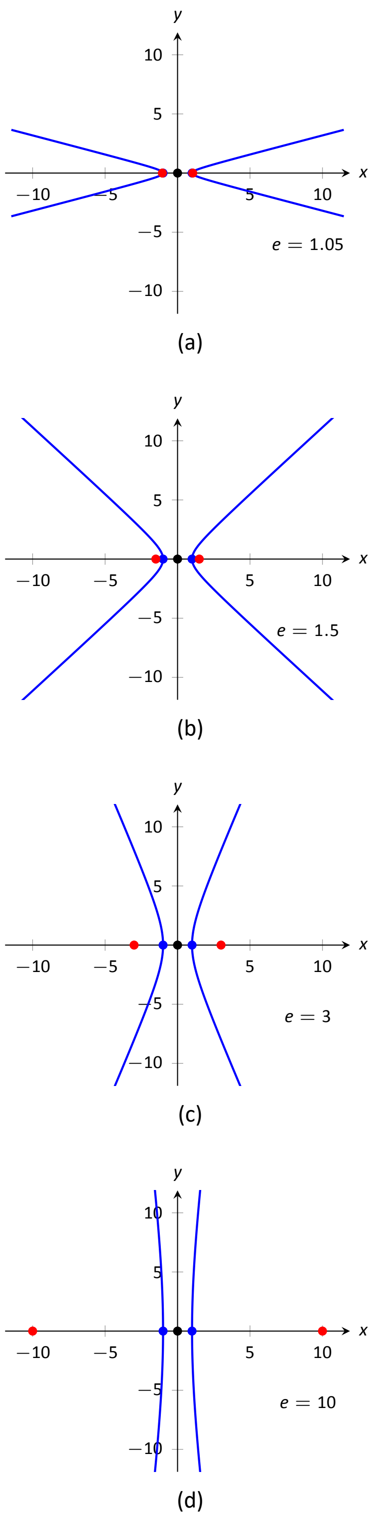


Figure 10.1.17: Understanding the eccentricity of a hyperbola.

Reflective Property

Hyperbolas share a similar reflective property with ellipses. However, in the case of a hyperbola, a ray emanating from a focus that intersects the hyperbola reflects along a line containing the other focus, but moving *away* from that focus. This is illustrated in Figure 10.1.19 (on the next page). Hyperbolic mirrors are commonly used in telescopes because of this reflective property. It is stated formally in the following theorem.

Theorem 10.1.3 Reflective Property of Hyperbolas

Let P be a point on a hyperbola with foci F_1 and F_2. The tangent line to the hyperbola at P makes equal angles with the following two lines:

1. The line through F_1 and P, and
2. The line through F_2 and P.

Location Determination

Determining the location of a known event has many practical uses (locating the epicenter of an earthquake, an airplane crash site, the position of the person speaking in a large room, etc.).

To determine the location of an earthquake's epicenter, seismologists use *trilateration* (not to be confused with *triangulation*). A seismograph allows one to determine how far away the epicenter was; using three separate readings, the location of the epicenter can be approximated.

A key to this method is knowing distances. What if this information is not available? Consider three microphones at positions A, B and C which all record a noise (a person's voice, an explosion, etc.) created at unknown location D. The microphone does not "know" when the sound was *created*, only when the sound was *detected*. How can the location be determined in such a situation?

If each location has a clock set to the same time, hyperbolas can be used to determine the location. Suppose the microphone at position A records the sound at exactly 12:00, location B records the time exactly 1 second later, and location C records the noise exactly 2 seconds after that. We are interested in the *difference* of times. Since the speed of sound is approximately 340 m/s, we can conclude quickly that the sound was created 340 meters closer to position A than position B. If A and B are a known distance apart (as shown in Figure 10.1.18 (a)), then we can determine a hyperbola on which D must lie.

The "difference of distances" is 340; this is also the distance between vertices of the hyperbola. So we know $2a = 340$. Positions A and B lie on the foci, so $2c = 1000$. From this we can find $b \approx 470$ and can sketch the hyperbola, given in part (b) of the figure. We only care about the side closest to A. (Why?)

We can also find the hyperbola defined by positions B and C. In this case, $2a = 680$ as the sound travelled an extra 2 seconds to get to C. We still have $2c = 1000$, centring this hyperbola at $(-500, 500)$. We find $b \approx 367$. This hyperbola is sketched in part (c) of the figure. The intersection point of the two graphs is the location of the sound, at approximately $(188, -222.5)$.

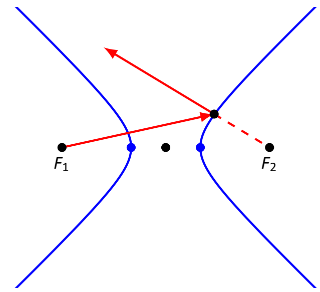


Figure 10.1.19: Illustrating the reflective property of a hyperbola.

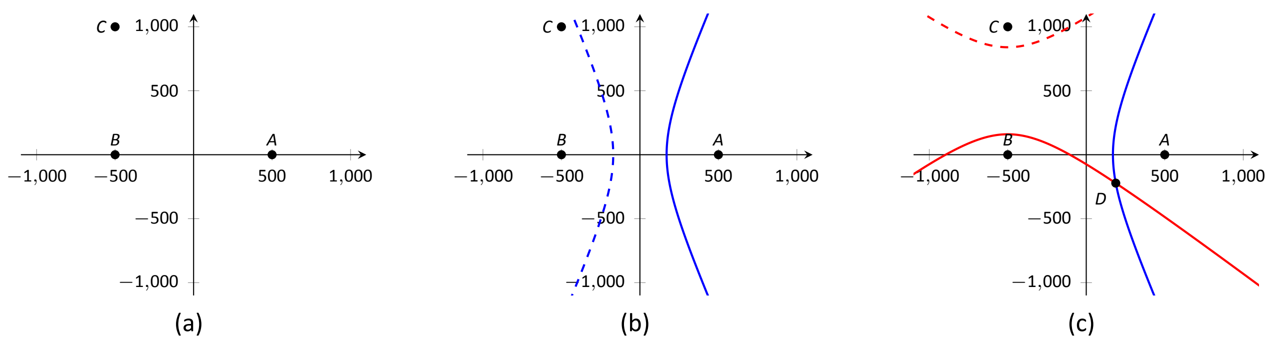


Figure 10.1.18: Using hyperbolas in location detection.

This chapter explores curves in the plane, in particular curves that cannot be described by functions of the form $y = f(x)$. In this section, we learned of ellipses and hyperbolas that are defined implicitly, not explicitly. In the following sections, we will learn completely new ways of describing curves in the plane, using *parametric equations* and *polar coordinates*, then study these curves using calculus techniques.

Exercises 10.1

Terms and Concepts

1. What is the difference between degenerate and nondegenerate conics?

2. Use your own words to explain what the eccentricity of an ellipse measures.

3. What has the largest eccentricity: an ellipse or a hyperbola?

4. Explain why the following is true: "If the coefficient of the x^2 term in the equation of an ellipse in standard form is smaller than the coefficient of the y^2 term, then the ellipse has a horizontal major axis."

5. Explain how one can quickly look at the equation of a hyperbola in standard form and determine whether the transverse axis is horizontal or vertical.

6. Fill in the blank: It can be said that ellipses and hyperbolas share the *same* reflective property: "A ray emanating from one focus will reflect off the conic along a ________ that contains the other focus."

Problems

In Exercises 7 – 14, find the equation of the parabola defined by the given information. Sketch the parabola.

7. Focus: $(3, 2)$; directrix: $y = 1$

8. Focus: $(-1, -4)$; directrix: $y = 2$

9. Focus: $(1, 5)$; directrix: $x = 3$

10. Focus: $(1/4, 0)$; directrix: $x = -1/4$

11. Focus: $(1, 1)$; vertex: $(1, 2)$

12. Focus: $(-3, 0)$; vertex: $(0, 0)$

13. Vertex: $(0, 0)$; directrix: $y = -1/16$

14. Vertex: $(2, 3)$; directrix: $x = 4$

In Exercises 15 – 16, the equation of a parabola and a point on its graph are given. Find the focus and directrix of the parabola, and verify that the given point is equidistant from the focus and directrix.

15. $y = \frac{1}{4}x^2$, $P = (2, 1)$

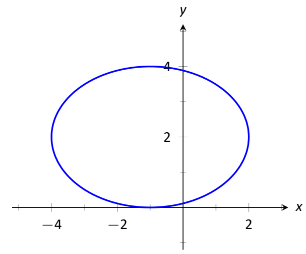
16. $x = \frac{1}{8}(y-2)^2 + 3$, $P = (11, 10)$

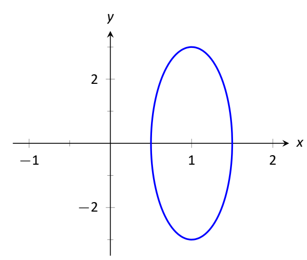
In Exercises 17 – 18, sketch the ellipse defined by the given equation. Label the center, foci and vertices.

17. $\dfrac{(x-1)^2}{3} + \dfrac{(y-2)^2}{5} = 1$

18. $\dfrac{1}{25}x^2 + \dfrac{1}{9}(y+3)^2 = 1$

In Exercises 19 – 20, find the equation of the ellipse shown in the graph. Give the location of the foci and the eccentricity of the ellipse.

19. 

20. 

In Exercises 21 – 24, find the equation of the ellipse defined by the given information. Sketch the elllipse.

21. Foci: $(\pm 2, 0)$; vertices: $(\pm 3, 0)$

22. Foci: $(-1, 3)$ and $(5, 3)$; vertices: $(-3, 3)$ and $(7, 3)$

23. Foci: $(2, \pm 2)$; vertices: $(2, \pm 7)$

24. Focus: $(-1, 5)$; vertex: $(-1, -4)$; center: $(-1, 1)$

In Exercises 25 – 28, write the equation of the given ellipse in standard form.

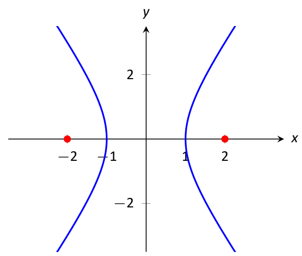
25. $x^2 - 2x + 2y^2 - 8y = -7$

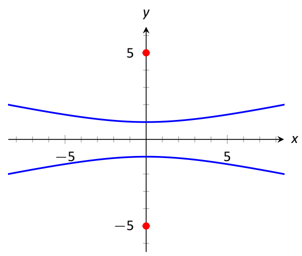
26. $5x^2 + 3y^2 = 15$

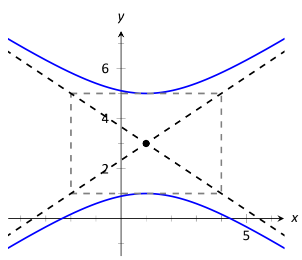
27. $3x^2 + 2y^2 - 12y + 6 = 0$

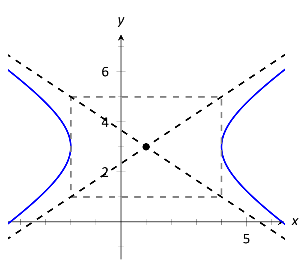
28. $x^2 + y^2 - 4x - 4y + 4 = 0$

In Exercises 29 – 32, find the equation of the hyperbola shown in the graph.

29. 

30. 

31. 

32. 

In Exercises 33 – 34, sketch the hyperbola defined by the given equation. Label the center and foci.

33. $\dfrac{(x-1)^2}{16} - \dfrac{(y+2)^2}{9} = 1$

34. $(y-4)^2 - \dfrac{(x+1)^2}{25} = 1$

In Exercises 35 – 38, find the equation of the hyperbola defined by the given information. Sketch the hyperbola.

35. Foci: $(\pm 3, 0)$; vertices: $(\pm 2, 0)$

36. Foci: $(0, \pm 3)$; vertices: $(0, \pm 2)$

37. Foci: $(-2, 3)$ and $(8, 3)$; vertices: $(-1, 3)$ and $(7, 3)$

38. Foci: $(3, -2)$ and $(3, 8)$; vertices: $(3, 0)$ and $(3, 6)$

In Exercises 39 – 42, write the equation of the hyperbola in standard form.

39. $3x^2 - 4y^2 = 12$

40. $3x^2 - y^2 + 2y = 10$

41. $x^2 - 10y^2 + 40y = 30$

42. $(4y - x)(4y + x) = 4$

43. Consider the ellipse given by $\dfrac{(x-1)^2}{4} + \dfrac{(y-3)^2}{12} = 1$.

    (a) Verify that the foci are located at $(1, 3 \pm 2\sqrt{2})$.

    (b) The points $P_1 = (2, 6)$ and $P_2 = (1+\sqrt{2}, 3+\sqrt{6}) \approx (2.414, 5.449)$ lie on the ellipse. Verify that the sum of distances from each point to the foci is the same.

44. Johannes Kepler discovered that the planets of our solar system have elliptical orbits with the Sun at one focus. The Earth's elliptical orbit is used as a standard unit of distance; the distance from the center of Earth's elliptical orbit to one vertex is 1 Astronomical Unit, or A.U.

    The following table gives information about the orbits of three planets.

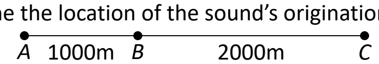
	Distance from center to vertex	eccentricity
Mercury	0.387 A.U.	0.2056
Earth	1 A.U.	0.0167
Mars	1.524 A.U.	0.0934

    (a) In an ellipse, knowing $c^2 = a^2 - b^2$ and $e = c/a$ allows us to find b in terms of a and e. Show $b = a\sqrt{1-e^2}$.

    (b) For each planet, find equations of their elliptical orbit of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. (This places the center at $(0, 0)$, but the Sun is in a different location for each planet.)

    (c) Shift the equations so that the Sun lies at the origin. Plot the three elliptical orbits.

45. A loud sound is recorded at three stations that lie on a line as shown in the figure below. Station A recorded the sound 1 second after Station B, and Station C recorded the sound 3 seconds after B. Using the speed of sound as 340m/s, determine the location of the sound's origination.



10.2 Parametric Equations

We are familiar with sketching shapes, such as parabolas, by following this basic procedure:

Choose x → Use a function f to find y $\big(y = f(x)\big)$ → Plot point (x, y)

The **rectangular equation** $y = f(x)$ works well for some shapes like a parabola with a vertical axis of symmetry, but in the previous section we encountered several shapes that could not be sketched in this manner. (To plot an ellipse using the above procedure, we need to plot the "top" and "bottom" separately.)

In this section we introduce a new sketching procedure:

Choose t → Use a function f to find x $\big(x = f(t)\big)$ → Plot point (x, y)

Choose t → Use a function g to find y $\big(y = g(t)\big)$ → Plot point (x, y)

Here, x and y are found separately but then plotted together. This leads us to a definition.

Definition 10.2.1 Parametric Equations and Curves

Let f and g be continuous functions on an interval I. The set of all points $(x, y) = \big(f(t), g(t)\big)$ in the Cartesian plane, as t varies over I, is the **graph** of the **parametric equations** $x = f(t)$ and $y = g(t)$, where t is the **parameter**. A **curve** is a graph along with the parametric equations that define it.

This is a formal definition of the word *curve*. When a curve lies in a plane (such as the Cartesian plane), it is often referred to as a **plane curve**. Examples will help us understand the concepts introduced in the definition.

Example 10.2.1 Plotting parametric functions

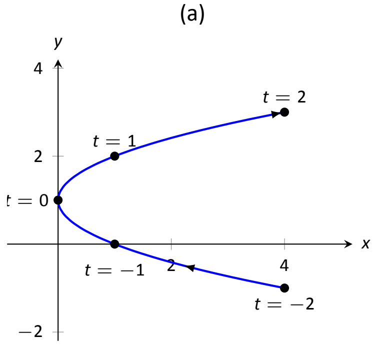
Plot the graph of the parametric equations $x = t^2$, $y = t + 1$ for t in $[-2, 2]$.

Solution We plot the graphs of parametric equations in much the same manner as we plotted graphs of functions like $y = f(x)$: we make a table of values, plot points, then connect these points with a "reasonable" looking curve. Figure 10.2.1(a) shows such a table of values; note how we have 3 columns.

The points (x, y) from the table are plotted in Figure 10.2.1(b). The points have been connected with a smooth curve. Each point has been labeled with its corresponding t-value. These values, along with the two arrows along the curve, are used to indicate the **orientation** of the graph. This information helps us determine the direction in which the graph is "moving."

t	x	y
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3

(a)



(b)

Figure 10.2.1: A table of values of the parametric equations in Example 10.2.1 along with a sketch of their graph.

We often use the letter t as the parameter as we often regard t as representing *time*. Certainly there are many contexts in which the parameter is not time, but it can be helpful to think in terms of time as one makes sense of parametric plots and their orientation (for instance, "At time $t = 0$ the position is $(1, 2)$ and at time $t = 3$ the position is $(5, 1)$.").

Example 10.2.2 Plotting parametric functions

Sketch the graph of the parametric equations $x = \cos^2 t$, $y = \cos t + 1$ for t in $[0, \pi]$.

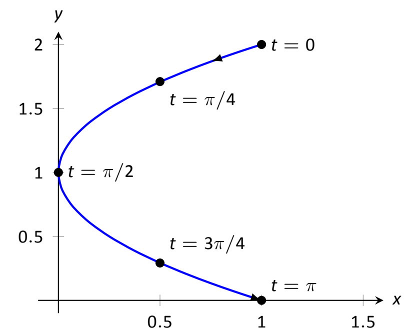
Solution We again start by making a table of values in Figure 10.2.2(a), then plot the points (x, y) on the Cartesian plane in Figure 10.2.2(b).

It is not difficult to show that the curves in Examples 10.2.1 and 10.2.2 are portions of the same parabola. While the *parabola* is the same, the *curves* are different. In Example 10.2.1, if we let t vary over all real numbers, we'd obtain the entire parabola. In this example, letting t vary over all real numbers would still produce the same graph; this portion of the parabola would be traced, and re–traced, infinitely many times. The orientation shown in Figure 10.2.2 shows the orientation on $[0, \pi]$, but this orientation is reversed on $[\pi, 2\pi]$.

These examples begin to illustrate the powerful nature of parametric equations. Their graphs are far more diverse than the graphs of functions produced by "$y = f(x)$" functions.

t	x	y
0	1	2
$\pi/4$	$1/2$	$1 + \sqrt{2}/2$
$\pi/2$	0	1
$3\pi/4$	$1/2$	$1 - \sqrt{2}/2$
π	1	0

(a)



(b)

Figure 10.2.2: A table of values of the parametric equations in Example 10.2.2 along with a sketch of their graph.

Technology Note: Most graphing utilities can graph functions given in parametric form. Often the word "parametric" is abbreviated as "PAR" or "PARAM" in the options. The user usually needs to determine the graphing window (i.e, the minimum and maximum x- and y-values), along with the values of t that are to be plotted. The user is often prompted to give a t minimum, a t maximum, and a "t-step" or "Δt." Graphing utilities effectively plot parametric functions just as we've shown here: they plots lots of points. A smaller t-step plots more points, making for a smoother graph (but may take longer). In Figure 10.2.1, the t-step is 1; in Figure 10.2.2, the t-step is $\pi/4$.

One nice feature of parametric equations is that their graphs are easy to shift. While this is not too difficult in the "$y = f(x)$" context, the resulting function can look rather messy. (Plus, to shift to the right by two, we replace x with $x - 2$, which is counter–intuitive.) The following example demonstrates this.

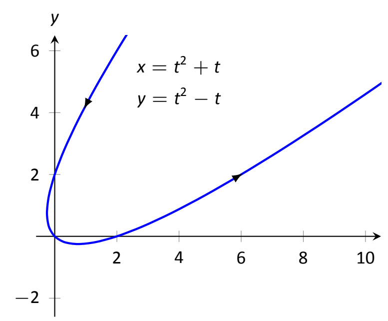
Example 10.2.3 Shifting the graph of parametric functions

Sketch the graph of the parametric equations $x = t^2 + t$, $y = t^2 - t$. Find new parametric equations that shift this graph to the right 3 places and down 2.

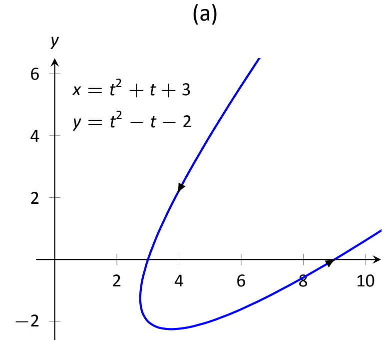
Solution The graph of the parametric equations is given in Figure 10.2.3 (a). It is a parabola with a axis of symmetry along the line $y = x$; the vertex is at $(0, 0)$.

In order to shift the graph to the right 3 units, we need to increase the x-value by 3 for every point. The straightforward way to accomplish this is simply to add 3 to the function defining x: $x = t^2 + t + 3$. To shift the graph down by 2 units, we wish to decrease each y-value by 2, so we subtract 2 from the function defining y: $y = t^2 - t - 2$. Thus our parametric equations for the shifted graph are $x = t^2 + t + 3$, $y = t^2 - t - 2$. This is graphed in Figure 10.2.3 (b). Notice how the vertex is now at $(3, -2)$.

Because the x- and y-values of a graph are determined independently, the



(a)



(b)

Figure 10.2.3: Illustrating how to shift graphs in Example 10.2.3.

graphs of parametric functions often possess features not seen on "$y = f(x)$" type graphs. The next example demonstrates how such graphs can arrive at the same point more than once.

Example 10.2.4 Graphs that cross themselves

Plot the parametric functions $x = t^3 - 5t^2 + 3t + 11$ and $y = t^2 - 2t + 3$ and determine the t-values where the graph crosses itself.

SOLUTION Using the methods developed in this section, we again plot points and graph the parametric equations as shown in Figure 10.2.4. It appears that the graph crosses itself at the point $(2, 6)$, but we'll need to analytically determine this.

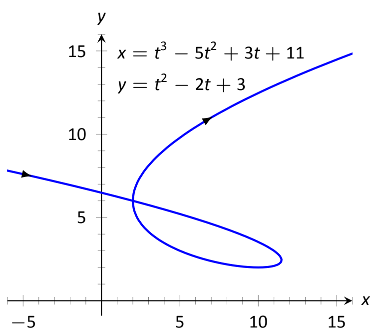


Figure 10.2.4: A graph of the parametric equations from Example 10.2.4.

We are looking for two different values, say, s and t, where $x(s) = x(t)$ and $y(s) = y(t)$. That is, the x-values are the same precisely when the y-values are the same. This gives us a system of 2 equations with 2 unknowns:

$$s^3 - 5s^2 + 3s + 11 = t^3 - 5t^2 + 3t + 11$$
$$s^2 - 2s + 3 = t^2 - 2t + 3$$

Solving this system is not trivial but involves only algebra. Using the quadratic formula, one can solve for t in the second equation and find that $t = 1 \pm \sqrt{s^2 - 2s + 1}$. This can be substituted into the first equation, revealing that the graph crosses itself at $t = -1$ and $t = 3$. We confirm our result by computing $x(-1) = x(3) = 2$ and $y(-1) = y(3) = 6$.

Converting between rectangular and parametric equations

It is sometimes useful to rewrite equations in rectangular form (i.e., $y = f(x)$) into parametric form, and vice–versa. Converting from rectangular to parametric can be very simple: given $y = f(x)$, the parametric equations $x = t$, $y = f(t)$ produce the same graph. As an example, given $y = x^2$, the parametric equations $x = t$, $y = t^2$ produce the familiar parabola. However, other parametrizations can be used. The following example demonstrates one possible alternative.

Example 10.2.5 Converting from rectangular to parametric

Consider $y = x^2$. Find parametric equations $x = f(t)$, $y = g(t)$ for the parabola where $t = \frac{dy}{dx}$. That is, $t = a$ corresponds to the point on the graph whose tangent line has slope a.

SOLUTION We start by computing $\frac{dy}{dx}$: $y' = 2x$. Thus we set $t = 2x$. We can solve for x and find $x = t/2$. Knowing that $y = x^2$, we have $y = t^2/4$. Thus parametric equations for the parabola $y = x^2$ are

$$x = t/2 \quad y = t^2/4.$$

To find the point where the tangent line has a slope of -2, we set $t = -2$. This gives the point $(-1, 1)$. We can verify that the slope of the line tangent to the curve at this point indeed has a slope of -2.

We sometimes choose the parameter to accurately model physical behaviour.

Example 10.2.6 Converting from rectangular to parametric

An object is fired from a height of 0 feet and lands 6 seconds later, 192 feet away. Assuming ideal projectile motion, the height, in feet, of the object can be described by $h(x) = -x^2/64 + 3x$, where x is the distance in feet from the initial location. (Thus $h(0) = h(192) = 0$ ft.) Find parametric equations $x = f(t)$,

$y = g(t)$ for the path of the projectile where x is the horizontal distance the object has travelled at time t (in seconds) and y is the height at time t.

Solution Physics tells us that the horizontal motion of the projectile is linear; that is, the horizontal speed of the projectile is constant. Since the object travels 192 ft in 6 s, we deduce that the object is moving horizontally at a rate of 32 ft/s, giving the equation $x = 32t$. As $y = -x^2/64 + 3x$, we find $y = -16t^2 + 96t$. We can quickly verify that $y'' = -32$ ft/s^2, the acceleration due to gravity, and that the projectile reaches its maximum at $t = 3$, halfway along its path.

These parametric equations make certain determinations about the object's location easy: 2 seconds into the flight the object is at the point $\big(x(2), y(2)\big) = \big(64, 128\big)$. That is, it has travelled horizontally 64 ft and is at a height of 128 ft, as shown in Figure 10.2.5.

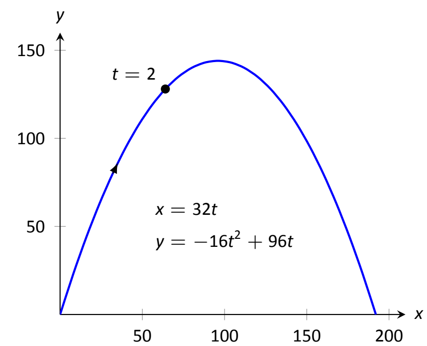


Figure 10.2.5: Graphing projectile motion in Example 10.2.6.

It is sometimes necessary to convert given parametric equations into rectangular form. This can be decidedly more difficult, as some "simple" looking parametric equations can have very "complicated" rectangular equations. This conversion is often referred to as "eliminating the parameter," as we are looking for a relationship between x and y that does not involve the parameter t.

Example 10.2.7 Eliminating the parameter

Find a rectangular equation for the curve described by

$$x = \frac{1}{t^2+1} \quad \text{and} \quad y = \frac{t^2}{t^2+1}.$$

Solution There is not a set way to eliminate a parameter. One method is to solve for t in one equation and then substitute that value in the second. We use that technique here, then show a second, simpler method.

Starting with $x = 1/(t^2+1)$, solve for t: $t = \pm\sqrt{1/x - 1}$. Substitute this value for t in the equation for y:

$$\begin{aligned} y &= \frac{t^2}{t^2+1} \\ &= \frac{1/x - 1}{1/x - 1 + 1} \\ &= \frac{1/x - 1}{1/x} \\ &= \left(\frac{1}{x} - 1\right)\cdot x \\ &= 1 - x. \end{aligned}$$

Thus $y = 1 - x$. One may have recognized this earlier by manipulating the equation for y:

$$y = \frac{t^2}{t^2+1} = 1 - \frac{1}{t^2+1} = 1 - x.$$

This is a shortcut that is very specific to this problem; sometimes shortcuts exist and are worth looking for.

We should be careful to limit the domain of the function $y = 1 - x$. The parametric equations limit x to values in $(0, 1]$, thus to produce the same graph we should limit the domain of $y = 1 - x$ to the same.

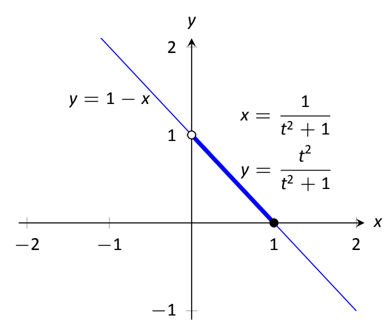


Figure 10.2.6: Graphing parametric and rectangular equations for a graph in Example 10.2.7.

The graphs of these functions is given in Figure 10.2.6. The portion of the graph defined by the parametric equations is given in a thick line; the graph defined by $y = 1 - x$ with unrestricted domain is given in a thin line.

Example 10.2.8 Eliminating the parameter

Eliminate the parameter in $x = 4\cos t + 3$, $y = 2\sin t + 1$

SOLUTION We should not try to solve for t in this situation as the resulting algebra/trig would be messy. Rather, we solve for $\cos t$ and $\sin t$ in each equation, respectively. This gives

$$\cos t = \frac{x-3}{4} \quad \text{and} \quad \sin t = \frac{y-1}{2}.$$

The Pythagorean Theorem gives $\cos^2 t + \sin^2 t = 1$, so:

$$\begin{aligned} \cos^2 t + \sin^2 t &= 1 \\ \left(\frac{x-3}{4}\right)^2 + \left(\frac{y-1}{2}\right)^2 &= 1 \\ \frac{(x-3)^2}{16} + \frac{(y-1)^2}{4} &= 1 \end{aligned}$$

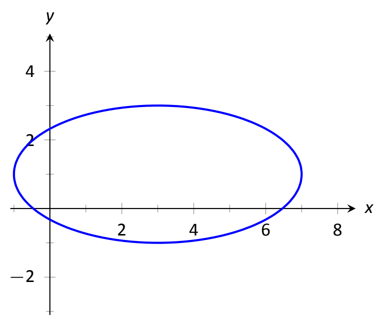


Figure 10.2.7: Graphing the parametric equations $x = 4\cos t + 3$, $y = 2\sin t + 1$ in Example 10.2.8.

This final equation should look familiar – it is the equation of an ellipse! Figure 10.2.7 plots the parametric equations, demonstrating that the graph is indeed of an ellipse with a horizontal major axis and center at $(3, 1)$.

The Pythagorean Theorem can also be used to identify parametric equations for hyperbolas. We give the parametric equations for ellipses and hyperbolas in the following Key Idea.

Key Idea 10.2.1 Parametric Equations of Ellipses and Hyperbolas

- The parametric equations

$$x = a\cos t + h, \quad y = b\sin t + k$$

define an ellipse with horizontal axis of length $2a$ and vertical axis of length $2b$, centred at (h, k).

- The parametric equations

$$x = a\tan t + h, \quad y = \pm b\sec t + k$$

define a hyperbola with vertical transverse axis centred at (h, k), and

$$x = \pm a\sec t + h, \quad y = b\tan t + k$$

defines a hyperbola with horizontal transverse axis. Each has asymptotes at $y = \pm b/a(x - h) + k$.

Special Curves

Figure 10.2.8 gives a small gallery of "interesting" and "famous" curves along with parametric equations that produce them. Interested readers can begin learning more about these curves through internet searches.

One might note a feature shared by two of these graphs: "sharp corners," or **cusps**. We have seen graphs with cusps before and determined that such functions are not differentiable at these points. This leads us to a definition.

> **Definition 10.2.2 Smooth**
>
> A curve C defined by $x = f(t)$, $y = g(t)$ is **smooth** on an interval I if f' and g' are continuous on I and not simultaneously 0 (except possibly at the endpoints of I). A curve is **piecewise smooth** on I if I can be partitioned into subintervals where C is smooth on each subinterval.

Consider the astroid, given by $x = \cos^3 t$, $y = \sin^3 t$. Taking derivatives, we have:

$$x' = -3\cos^2 t \sin t \quad \text{and} \quad y' = 3\sin^2 t \cos t.$$

It is clear that each is 0 when $t = 0,\ \pi/2,\ \pi, \ldots$. Thus the astroid is not smooth at these points, corresponding to the cusps seen in the figure.

We demonstrate this once more.

Example 10.2.9 Determine where a curve is not smooth

Let a curve C be defined by the parametric equations $x = t^3 - 12t + 17$ and $y = t^2 - 4t + 8$. Determine the points, if any, where it is not smooth.

Solution We begin by taking derivatives.

$$x' = 3t^2 - 12, \quad y' = 2t - 4.$$

We set each equal to 0:

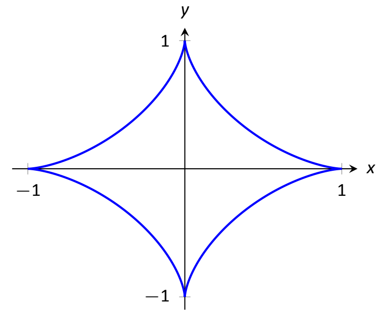
$$\begin{aligned} x' &= 0 \Rightarrow 3t^2 - 12 = 0 \Rightarrow t = \pm 2 \\ y' &= 0 \Rightarrow 2t - 4 = 0 \Rightarrow t = 2 \end{aligned}$$

We see at $t = 2$ both x' and y' are 0; thus C is not smooth at $t = 2$, corresponding to the point $(1, 4)$. The curve is graphed in Figure 10.2.9, illustrating the cusp at $(1, 4)$.

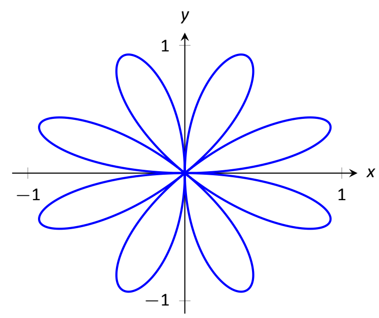
If a curve is not smooth at $t = t_0$, it means that $x'(t_0) = y'(t_0) = 0$ as defined. This, in turn, means that rate of change of x (and y) is 0; that is, at that instant, neither x nor y is changing. If the parametric equations describe the path of some object, this means the object is at rest at t_0. An object at rest can make a "sharp" change in direction, whereas moving objects tend to change direction in a "smooth" fashion.

One should be careful to note that a "sharp corner" does not have to occur when a curve is not smooth. For instance, one can verify that $x = t^3$, $y = t^6$ produce the familiar $y = x^2$ parabola. However, in this parametrization, the curve is not smooth. A particle travelling along the parabola according to the given parametric equations comes to rest at $t = 0$, though no sharp point is created.

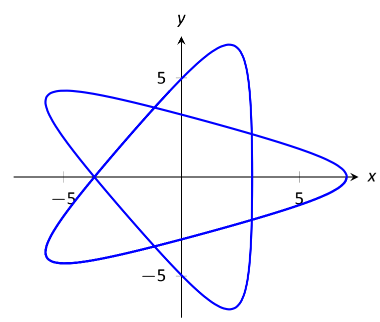
Our previous experience with cusps taught us that a function was not differentiable at a cusp. This can lead us to wonder about derivatives in the context



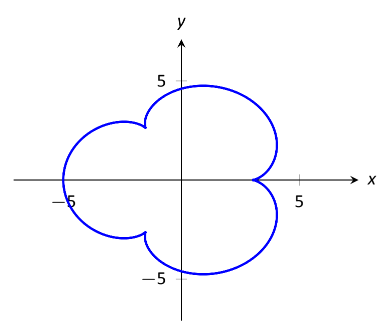
Astroid
$x = \cos^3 t$
$y = \sin^3 t$



Rose Curve
$x = \cos(t)\sin(4t)$
$y = \sin(t)\sin(4t)$



Hypotrochoid
$x = 2\cos(t) + 5\cos(2t/3)$
$y = 2\sin(t) - 5\sin(2t/3)$



Epicycloid
$x = 4\cos(t) - \cos(4t)$
$y = 4\sin(t) - \sin(4t)$

Figure 10.2.8: A gallery of interesting planar curves.

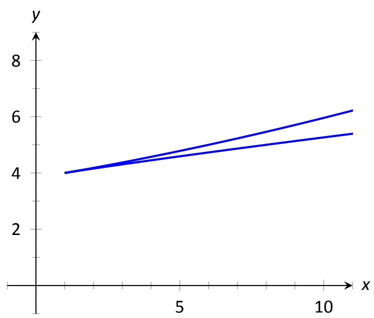


Figure 10.2.9: Graphing the curve in Example 10.2.9; note it is not smooth at $(1,4)$.

of parametric equations and the application of other calculus concepts. Given a curve defined parametrically, how do we find the slopes of tangent lines? Can we determine concavity? We explore these concepts and more in the next section.

Exercises 10.2

Terms and Concepts

1. T/F: When sketching the graph of parametric equations, the x and y values are found separately, then plotted together.

2. The direction in which a graph is "moving" is called the ____ of the graph.

3. An equation written as $y = f(x)$ is written in ____ form.

4. Create parametric equations $x = f(t)$, $y = g(t)$ and sketch their graph. Explain any interesting features of your graph based on the functions f and g.

Problems

In Exercises 5 – 8, sketch the graph of the given parametric equations by hand, making a table of points to plot. Be sure to indicate the orientation of the graph.

5. $x = t^2 + t, \quad y = 1 - t^2, \quad -3 \leq t \leq 3$

6. $x = 1, \quad y = 5\sin t, \quad -\pi/2 \leq t \leq \pi/2$

7. $x = t^2, \quad y = 2, \quad -2 \leq t \leq 2$

8. $x = t^3 - t + 3, \quad y = t^2 + 1, \quad -2 \leq t \leq 2$

In Exercises 9 – 18, sketch the graph of the given parametric equations; using a graphing utility is advisable. Be sure to indicate the orientation of the graph.

9. $x = t^3 - 2t^2, \quad y = t^2, \quad -2 \leq t \leq 3$

10. $x = 1/t, \quad y = \sin t, \quad 0 < t \leq 10$

11. $x = 3\cos t, \quad y = 5\sin t, \quad 0 \leq t \leq 2\pi$

12. $x = 3\cos t + 2, \quad y = 5\sin t + 3, \quad 0 \leq t \leq 2\pi$

13. $x = \cos t, \quad y = \cos(2t), \quad 0 \leq t \leq \pi$

14. $x = \cos t, \quad y = \sin(2t), \quad 0 \leq t \leq 2\pi$

15. $x = 2\sec t, \quad y = 3\tan t, \quad -\pi/2 < t < \pi/2$

16. $x = \cosh t, \quad y = \sinh t, \quad -2 \leq t \leq 2$

17. $x = \cos t + \frac{1}{4}\cos(8t), \quad y = \sin t + \frac{1}{4}\sin(8t), \quad 0 \leq t \leq 2\pi$

18. $x = \cos t + \frac{1}{4}\sin(8t), \quad y = \sin t + \frac{1}{4}\cos(8t), \quad 0 \leq t \leq 2\pi$

In Exercises 19 – 20, four sets of parametric equations are given. Describe how their graphs are similar and different. Be sure to discuss orientation and ranges.

19. (a) $x = t \quad y = t^2, \quad -\infty < t < \infty$
    (b) $x = \sin t \quad y = \sin^2 t, \quad -\infty < t < \infty$
    (c) $x = e^t \quad y = e^{2t}, \quad -\infty < t < \infty$
    (d) $x = -t \quad y = t^2, \quad -\infty < t < \infty$

20. (a) $x = \cos t \quad y = \sin t, \quad 0 \leq t \leq 2\pi$
    (b) $x = \cos(t^2) \quad y = \sin(t^2), \quad 0 \leq t \leq 2\pi$
    (c) $x = \cos(1/t) \quad y = \sin(1/t), \quad 0 < t < 1$
    (d) $x = \cos(\cos t) \quad y = \sin(\cos t), \quad 0 \leq t \leq 2\pi$

In Exercises 21 – 30, eliminate the parameter in the given parametric equations.

21. $x = 2t + 5, \quad y = -3t + 1$

22. $x = \sec t, \quad y = \tan t$

23. $x = 4\sin t + 1, \quad y = 3\cos t - 2$

24. $x = t^2, \quad y = t^3$

25. $x = \dfrac{1}{t+1}, \quad y = \dfrac{3t+5}{t+1}$

26. $x = e^t, \quad y = e^{3t} - 3$

27. $x = \ln t, \quad y = t^2 - 1$

28. $x = \cot t, \quad y = \csc t$

29. $x = \cosh t, \quad y = \sinh t$

30. $x = \cos(2t), \quad y = \sin t$

In Exercises 31 – 34, eliminate the parameter in the given parametric equations. Describe the curve defined by the parametric equations based on its rectangular form.

31. $x = at + x_0, \quad y = bt + y_0$

32. $x = r\cos t, \quad y = r\sin t$

33. $x = a\cos t + h, \quad y = b\sin t + k$

34. $x = a\sec t + h, \quad y = b\tan t + k$

In Exercises 35 – 38, find parametric equations for the given rectangular equation using the parameter $t = \frac{dy}{dx}$. Verify that at $t = 1$, the point on the graph has a tangent line with slope of 1.

35. $y = 3x^2 - 11x + 2$

36. $y = e^x$

37. $y = \sin x$ on $[0, \pi]$

38. $y = \sqrt{x}$ on $[0, \infty)$

In Exercises 39 – 42, find the values of t where the graph of the parametric equations crosses itself.

39. $x = t^3 - t + 3, \quad y = t^2 - 3$

40. $x = t^3 - 4t^2 + t + 7, \quad y = t^2 - t$

41. $x = \cos t, \quad y = \sin(2t)$ on $[0, 2\pi]$

42. $x = \cos t \cos(3t), \quad y = \sin t \cos(3t)$ on $[0, \pi]$

In Exercises 43 – 46, find the value(s) of t where the curve defined by the parametric equations is not smooth.

43. $x = t^3 + t^2 - t, \quad y = t^2 + 2t + 3$

44. $x = t^2 - 4t, \quad y = t^3 - 2t^2 - 4t$

45. $x = \cos t, \quad y = 2\cos t$

46. $x = 2\cos t - \cos(2t), \quad y = 2\sin t - \sin(2t)$

In Exercises 47 – 55, find parametric equations that describe the given situation.

47. A projectile is fired from a height of 0ft, landing 16ft away in 4s.

48. A projectile is fired from a height of 0ft, landing 200ft away in 4s.

49. A projectile is fired from a height of 0ft, landing 200ft away in 20s.

50. A circle of radius 2, centered at the origin, that is traced clockwise once on $[0, 2\pi]$.

51. A circle of radius 3, centered at $(1, 1)$, that is traced once counter–clockwise on $[0, 1]$.

52. An ellipse centered at $(1, 3)$ with vertical major axis of length 6 and minor axis of length 2.

53. An ellipse with foci at $(\pm 1, 0)$ and vertices at $(\pm 5, 0)$.

54. A hyperbola with foci at $(5, -3)$ and $(-1, -3)$, and with vertices at $(1, -3)$ and $(3, -3)$.

55. A hyperbola with vertices at $(0, \pm 6)$ and asymptotes $y = \pm 3x$.

10.3 Calculus and Parametric Equations

The previous section defined curves based on parametric equations. In this section we'll employ the techniques of calculus to study these curves.

We are still interested in lines tangent to points on a curve. They describe how the y-values are changing with respect to the x-values, they are useful in making approximations, and they indicate instantaneous direction of travel.

The slope of the tangent line is still $\frac{dy}{dx}$, and the Chain Rule allows us to calculate this in the context of parametric equations. If $x = f(t)$ and $y = g(t)$, the Chain Rule states that

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} = \frac{g'(t)}{f'(t)},$$

provided that $f'(t) \neq 0$. This is important so we label it a Key Idea.

Key Idea 10.3.1 Finding $\frac{dy}{dx}$ with Parametric Equations.

Let $x = f(t)$ and $y = g(t)$, where f and g are differentiable on some open interval I and $f'(t) \neq 0$ on I. Then

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}.$$

We use this to define the tangent line.

Definition 10.3.1 Tangent and Normal Lines

Let a curve C be parametrized by $x = f(t)$ and $y = g(t)$, where f and g are differentiable functions on some interval I containing $t = t_0$. The **tangent line** to C at $t = t_0$ is the line through $\big(f(t_0), g(t_0)\big)$ with slope $m = g'(t_0)/f'(t_0)$, provided $f'(t_0) \neq 0$.

The **normal line** to C at $t = t_0$ is the line through $\big(f(t_0), g(t_0)\big)$ with slope $m = -f'(t_0)/g'(t_0)$, provided $g'(t_0) \neq 0$.

The definition leaves two special cases to consider. When the tangent line is horizontal, the normal line is undefined by the above definition as $g'(t_0) = 0$. Likewise, when the normal line is horizontal, the tangent line is undefined. It seems reasonable that these lines be defined (one can draw a line tangent to the "right side" of a circle, for instance), so we add the following to the above definition.

1. If the tangent line at $t = t_0$ has a slope of 0, the normal line to C at $t = t_0$ is the line $x = f(t_0)$.

2. If the normal line at $t = t_0$ has a slope of 0, the tangent line to C at $t = t_0$ is the line $x = f(t_0)$.

Example 10.3.1 Tangent and Normal Lines to Curves

Let $x = 5t^2 - 6t + 4$ and $y = t^2 + 6t - 1$, and let C be the curve defined by these equations.

1. Find the equations of the tangent and normal lines to C at $t = 3$.

2. Find where C has vertical and horizontal tangent lines.

SOLUTION

1. We start by computing $f'(t) = 10t - 6$ and $g'(t) = 2t + 6$. Thus

$$\frac{dy}{dx} = \frac{2t + 6}{10t - 6}.$$

Make note of something that might seem unusual: $\frac{dy}{dx}$ is a function of t, not x. Just as points on the curve are found in terms of t, so are the slopes of the tangent lines.

The point on C at $t = 3$ is $(31, 26)$. The slope of the tangent line is $m = 1/2$ and the slope of the normal line is $m = -2$. Thus,

- the equation of the tangent line is $y = \frac{1}{2}(x - 31) + 26$, and
- the equation of the normal line is $y = -2(x - 31) + 26$.

This is illustrated in Figure 10.3.1.

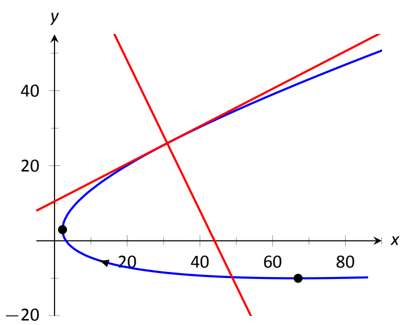


Figure 10.3.1: Graphing tangent and normal lines in Example 10.3.1.

2. To find where C has a horizontal tangent line, we set $\frac{dy}{dx} = 0$ and solve for t. In this case, this amounts to setting $g'(t) = 0$ and solving for t (and making sure that $f'(t) \neq 0$).

$$g'(t) = 0 \quad \Rightarrow \quad 2t + 6 = 0 \quad \Rightarrow \quad t = -3.$$

The point on C corresponding to $t = -3$ is $(67, -10)$; the tangent line at that point is horizontal (hence with equation $y = -10$).

To find where C has a vertical tangent line, we find where it has a horizontal normal line, and set $-\frac{f'(t)}{g'(t)} = 0$. This amounts to setting $f'(t) = 0$ and solving for t (and making sure that $g'(t) \neq 0$).

$$f'(t) = 0 \quad \Rightarrow \quad 10t - 6 = 0 \quad \Rightarrow \quad t = 0.6.$$

The point on C corresponding to $t = 0.6$ is $(2.2, 2.96)$. The tangent line at that point is $x = 2.2$.

The points where the tangent lines are vertical and horizontal are indicated on the graph in Figure 10.3.1.

Example 10.3.2 Tangent and Normal Lines to a Circle

1. Find where the unit circle, defined by $x = \cos t$ and $y = \sin t$ on $[0, 2\pi]$, has vertical and horizontal tangent lines.

2. Find the equation of the normal line at $t = t_0$.

Solution

1. We compute the derivative following Key Idea 10.3.1:

$$\frac{dy}{dx}=\frac{g'(t)}{f'(t)}=-\frac{\cos t}{\sin t}.$$

The derivative is 0 when $\cos t=0$; that is, when $t=\pi/2,\ 3\pi/2$. These are the points $(0,1)$ and $(0,-1)$ on the circle.

The normal line is horizontal (and hence, the tangent line is vertical) when $\sin t=0$; that is, when $t=0,\ \pi,\ 2\pi$, corresponding to the points $(-1,0)$ and $(0,1)$ on the circle. These results should make intuitive sense.

2. The slope of the normal line at $t=t_0$ is $m=\dfrac{\sin t_0}{\cos t_0}=\tan t_0$. This normal line goes through the point $(\cos t_0,\sin t_0)$, giving the line

$$\begin{aligned} y&=\frac{\sin t_0}{\cos t_0}(x-\cos t_0)+\sin t_0\\ &=(\tan t_0)x, \end{aligned}$$

as long as $\cos t_0\neq 0$. It is an important fact to recognize that the normal lines to a circle pass through its center, as illustrated in Figure 10.3.2. Stated in another way, any line that passes through the center of a circle intersects the circle at right angles.

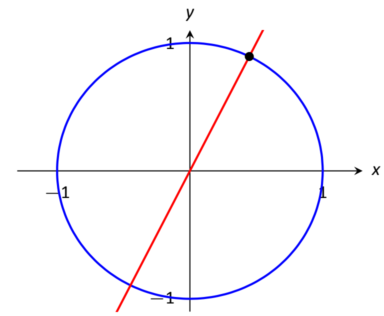


Figure 10.3.2: Illustrating how a circle's normal lines pass through its center.

Example 10.3.3 Tangent lines when $\frac{dy}{dx}$ is not defined

Find the equation of the tangent line to the astroid $x=\cos^3 t$, $y=\sin^3 t$ at $t=0$, shown in Figure 10.3.3.

Solution We start by finding $x'(t)$ and $y'(t)$:

$$x'(t)=-3\sin t\cos^2 t,\qquad y'(t)=3\cos t\sin^2 t.$$

Note that both of these are 0 at $t=0$; the curve is not smooth at $t=0$ forming a cusp on the graph. Evaluating $\frac{dy}{dx}$ at this point returns the indeterminate form of "0/0".

We can, however, examine the slopes of tangent lines near $t=0$, and take the limit as $t\to 0$.

$$\begin{aligned} \lim_{t\to 0}\frac{y'(t)}{x'(t)}&=\lim_{t\to 0}\frac{3\cos t\sin^2 t}{-3\sin t\cos^2 t}\quad\text{(We can cancel as } t\neq 0.)\\ &=\lim_{t\to 0}-\frac{\sin t}{\cos t}\\ &=0. \end{aligned}$$

We have accomplished something significant. When the derivative $\frac{dy}{dx}$ returns an indeterminate form at $t=t_0$, we can define its value by setting it to be $\displaystyle\lim_{t\to t_0}\frac{dy}{dx}$, if that limit exists. This allows us to find slopes of tangent lines at cusps, which can be very beneficial.

We found the slope of the tangent line at $t=0$ to be 0; therefore the tangent line is $y=0$, the x-axis.

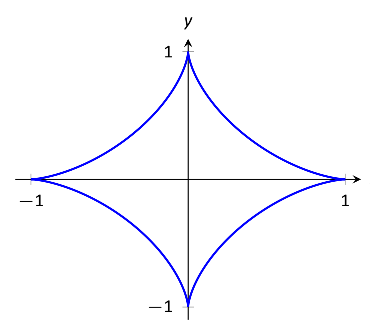


Figure 10.3.3: A graph of an astroid.

Concavity

We continue to analyze curves in the plane by considering their concavity; that is, we are interested in $\frac{d^2y}{dx^2}$, "the second derivative of y with respect to x." To find this, we need to find the derivative of $\frac{dy}{dx}$ with respect to x; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right],$$

but recall that $\frac{dy}{dx}$ is a function of t, not x, making this computation not straightforward.

To make the upcoming notation a bit simpler, let $h(t) = \frac{dy}{dx}$. We want $\frac{d}{dx}[h(t)]$; that is, we want $\frac{dh}{dx}$. We again appeal to the Chain Rule. Note:

$$\frac{dh}{dt} = \frac{dh}{dx}\cdot\frac{dx}{dt} \quad\Rightarrow\quad \frac{dh}{dx} = \frac{dh}{dt}\Big/\frac{dx}{dt}.$$

In words, to find $\frac{d^2y}{dx^2}$, we first take the derivative of $\frac{dy}{dx}$ *with respect to t*, then divide by $x'(t)$. We restate this as a Key Idea.

Key Idea 10.3.2 Finding $\frac{d^2y}{dx^2}$ with Parametric Equations

Let $x = f(t)$ and $y = g(t)$ be twice differentiable functions on an open interval I, where $f'(t) \neq 0$ on I. Then

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dx}\right]\Big/\frac{dx}{dt} = \frac{d}{dt}\left[\frac{dy}{dx}\right]\Big/f'(t).$$

Examples will help us understand this Key Idea.

Example 10.3.4 Concavity of Plane Curves

Let $x = 5t^2 - 6t + 4$ and $y = t^2 + 6t - 1$ as in Example 10.3.1. Determine the t-intervals on which the graph is concave up/down.

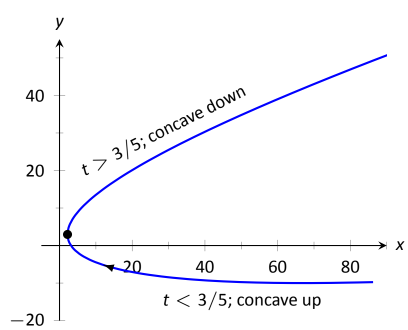


Figure 10.3.4: Graphing the parametric equations in Example 10.3.4 to demonstrate concavity.

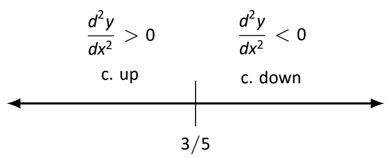
Solution Concavity is determined by the second derivative of y with respect to x, $\frac{d^2y}{dx^2}$, so we compute that here following Key Idea 10.3.2.

In Example 10.3.1, we found $\frac{dy}{dx} = \frac{2t+6}{10t-6}$ and $f'(t) = 10t - 6$. So:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dt}\left[\frac{2t+6}{10t-6}\right]\Big/(10t-6)\\ &= -\frac{72}{(10t-6)^2}\Big/(10t-6)\\ &= -\frac{72}{(10t-6)^3}\\ &= -\frac{9}{(5t-3)^3}\end{aligned}$$

The graph of the parametric functions is concave up when $\frac{d^2y}{dx^2} > 0$ and concave down when $\frac{d^2y}{dx^2} < 0$. We determine the intervals when the second derivative is greater/less than 0 by first finding when it is 0 or undefined.

As the numerator of $-\dfrac{9}{(5t-3)^3}$ is never 0, $\frac{d^2y}{dx^2} \neq 0$ for all t. It is undefined when $5t - 3 = 0$; that is, when $t = 3/5$. Following the work established in Section 3.4, we look at values of t greater/less than $3/5$ on a number line:



Reviewing Example 10.3.1, we see that when $t = 3/5 = 0.6$, the graph of the parametric equations has a vertical tangent line. This point is also a point of inflection for the graph, illustrated in Figure 10.3.4.

Example 10.3.5 Concavity of Plane Curves

Find the points of inflection of the graph of the parametric equations $x = \sqrt{t}$, $y = \sin t$, for $0 \leq t \leq 16$.

Solution We need to compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\cos t}{1/(2\sqrt{t})} = 2\sqrt{t}\cos t.$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{x'(t)} = \frac{\cos t/\sqrt{t} - 2\sqrt{t}\sin t}{1/(2\sqrt{t})} = 2\cos t - 4t\sin t.$$

The points of inflection are found by setting $\frac{d^2y}{dx^2} = 0$. This is not trivial, as equations that mix polynomials and trigonometric functions generally do not have "nice" solutions.

In Figure 10.3.5(a) we see a plot of the second derivative. It shows that it has zeros at approximately $t = 0.5,\ 3.5,\ 6.5,\ 9.5,\ 12.5$ and 16. These approximations are not very good, made only by looking at the graph. Newton's Method provides more accurate approximations. Accurate to 2 decimal places, we have:

$$t = 0.65,\ 3.29,\ 6.36,\ 9.48,\ 12.61 \text{ and } 15.74.$$

The corresponding points have been plotted on the graph of the parametric equations in Figure 10.3.5(b). Note how most occur near the x-axis, but not exactly on the axis.

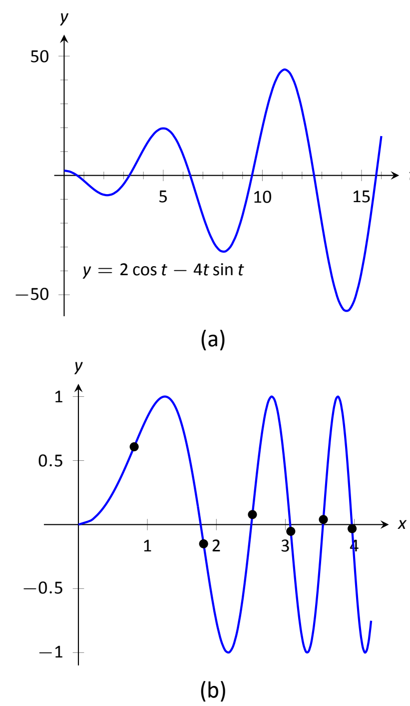


Figure 10.3.5: In (a), a graph of $\frac{d^2y}{dx^2}$, showing where it is approximately 0. In (b), graph of the parametric equations in Example 10.3.5 along with the points of inflection.

Arc Length

We continue our study of the features of the graphs of parametric equations by computing their arc length.

Recall in Section 7.4 we found the arc length of the graph of a function, from $x = a$ to $x = b$, to be

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

We can use this equation and convert it to the parametric equation context. Letting $x = f(t)$ and $y = g(t)$, we know that $\frac{dy}{dx} = g'(t)/f'(t)$. It will also be useful to calculate the differential of x:

$$dx = f'(t)dt \qquad \Rightarrow \qquad dt = \frac{1}{f'(t)} \cdot dx.$$

Starting with the arc length formula above, consider:

$$\begin{aligned} L &= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_a^b \sqrt{1+\frac{g'(t)^2}{f'(t)^2}}\,dx. \end{aligned}$$

Factor out the $f'(t)^2$:

$$\begin{aligned} &= \int_a^b \sqrt{f'(t)^2+g'(t)^2}\cdot \underbrace{\frac{1}{f'(t)}\,dx}_{=dt} \\ &= \int_{t_1}^{t_2} \sqrt{f'(t)^2+g'(t)^2}\,dt. \end{aligned}$$

Note the new bounds (no longer "x" bounds, but "t" bounds). They are found by finding t_1 and t_2 such that $a=f(t_1)$ and $b=f(t_2)$. This formula is important, so we restate it as a theorem.

Theorem 10.3.1 Arc Length of Parametric Curves

Let $x=f(t)$ and $y=g(t)$ be parametric equations with f' and g' continuous on $[t_1,t_2]$, on which the graph traces itself only once. The arc length of the graph, from $t=t_1$ to $t=t_2$, is

$$L=\int_{t_1}^{t_2}\sqrt{f'(t)^2+g'(t)^2}\,dt.$$

Note: Theorem 10.3.1 makes use of differentiability on closed intervals, just as was done in Section 7.4.

As before, these integrals are often not easy to compute. We start with a simple example, then give another where we approximate the solution.

Example 10.3.6 Arc Length of a Circle

Find the arc length of the circle parametrized by $x=3\cos t$, $y=3\sin t$ on $[0,3\pi/2]$.

SOLUTION By direct application of Theorem 10.3.1, we have

$$L=\int_0^{3\pi/2}\sqrt{(-3\sin t)^2+(3\cos t)^2}\,dt.$$

Apply the Pythagorean Theorem.

$$\begin{aligned} &= \int_0^{3\pi/2} 3\,dt \\ &= 3t\Big|_0^{3\pi/2} = 9\pi/2. \end{aligned}$$

This should make sense; we know from geometry that the circumference of a circle with radius 3 is 6π; since we are finding the arc length of $3/4$ of a circle, the arc length is $3/4\cdot 6\pi = 9\pi/2$.

Example 10.3.7 Arc Length of a Parametric Curve

The graph of the parametric equations $x = t(t^2 - 1)$, $y = t^2 - 1$ crosses itself as shown in Figure 10.3.6, forming a "teardrop." Find the arc length of the teardrop.

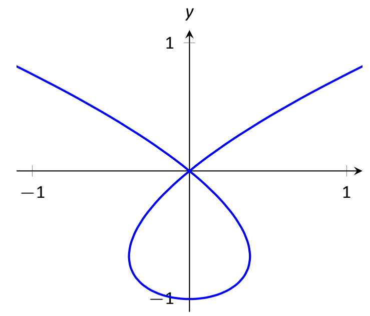


Figure 10.3.6: A graph of the parametric equations in Example 10.3.7, where the arc length of the teardrop is calculated.

Solution We can see by the parametrizations of x and y that when $t = \pm 1$, $x = 0$ and $y = 0$. This means we'll integrate from $t = -1$ to $t = 1$. Applying Theorem 10.3.1, we have

$$\begin{aligned} L &= \int_{-1}^{1} \sqrt{(3t^2 - 1)^2 + (2t)^2}\, dt \\ &= \int_{-1}^{1} \sqrt{9t^4 - 2t^2 + 1}\, dt. \end{aligned}$$

Unfortunately, the integrand does not have an antiderivative expressible by elementary functions. We turn to numerical integration to approximate its value. Using 4 subintervals, Simpson's Rule approximates the value of the integral as 2.65051. Using a computer, more subintervals are easy to employ, and $n = 20$ gives a value of 2.71559. Increasing n shows that this value is stable and a good approximation of the actual value.

Surface Area of a Solid of Revolution

Related to the formula for finding arc length is the formula for finding surface area. We can adapt the formula found in Theorem 7.4.2 from Section 7.4 in a similar way as done to produce the formula for arc length done before.

Theorem 10.3.2 Surface Area of a Solid of Revolution

Consider the graph of the parametric equations $x = f(t)$ and $y = g(t)$, where f' and g' are continuous on an open interval I containing t_1 and t_2 on which the graph does not cross itself.

1. The surface area of the solid formed by revolving the graph about the x-axis is (where $g(t) \geq 0$ on $[t_1, t_2]$):

$$\text{Surface Area} = 2\pi \int_{t_1}^{t_2} g(t)\sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

2. The surface area of the solid formed by revolving the graph about the y-axis is (where $f(t) \geq 0$ on $[t_1, t_2]$):

$$\text{Surface Area} = 2\pi \int_{t_1}^{t_2} f(t)\sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

Example 10.3.8 Surface Area of a Solid of Revolution

Consider the teardrop shape formed by the parametric equations $x = t(t^2 - 1)$, $y = t^2 - 1$ as seen in Example 10.3.7. Find the surface area if this shape is rotated about the x-axis, as shown in Figure 10.3.7.

Solution The teardrop shape is formed between $t = -1$ and $t = 1$. Using Theorem 10.3.2, we see we need for $g(t) \geq 0$ on $[-1, 1]$, and this is not the case. To fix this, we simplify replace $g(t)$ with $-g(t)$, which flips the whole graph about the x-axis (and does not change the surface area of the resulting solid). The surface area is:

$$\begin{aligned} \text{Area } S &= 2\pi \int_{-1}^{1} (1 - t^2)\sqrt{(3t^2 - 1)^2 + (2t)^2}\, dt \\ &= 2\pi \int_{-1}^{1} (1 - t^2)\sqrt{9t^4 - 2t^2 + 1}\, dt. \end{aligned}$$

Once again we arrive at an integral that we cannot compute in terms of elementary functions. Using Simpson's Rule with $n = 20$, we find the area to be $S = 9.44$. Using larger values of n shows this is accurate to 2 places after the decimal.

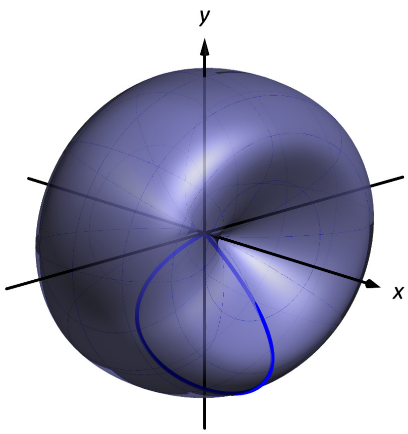


Figure 10.3.7: Rotating a teardrop shape about the x-axis in Example 10.3.8.

After defining a new way of creating curves in the plane, in this section we have applied calculus techniques to the parametric equation defining these curves to study their properties. In the next section, we define another way of forming curves in the plane. To do so, we create a new coordinate system, called *polar coordinates*, that identifies points in the plane in a manner different than from measuring distances from the y- and x- axes.

Exercises 10.3

Terms and Concepts

1. T/F: Given parametric equations $x = f(t)$ and $y = g(t)$, $\frac{dy}{dx} = f'(t)/g'(t)$, as long as $g'(t) \neq 0$.

2. Given parametric equations $x = f(t)$ and $y = g(t)$, the derivative $\frac{dy}{dx}$ as given in Key Idea 10.3.1 is a function of ________?

3. T/F: Given parametric equations $x = f(t)$ and $y = g(t)$, to find $\frac{d^2y}{dx^2}$, one simply computes $\frac{d}{dt}\left(\frac{dy}{dx}\right)$.

4. T/F: If $\frac{dy}{dx} = 0$ at $t = t_0$, then the normal line to the curve at $t = t_0$ is a vertical line.

Problems

In Exercises 5 – 12, parametric equations for a curve are given.

(a) Find $\dfrac{dy}{dx}$.

(b) Find the equations of the tangent and normal line(s) at the point(s) given.

(c) Sketch the graph of the parametric functions along with the found tangent and normal lines.

5. $x = t, y = t^2; \quad t = 1$

6. $x = \sqrt{t}, y = 5t + 2; \quad t = 4$

7. $x = t^2 - t, y = t^2 + t; \quad t = 1$

8. $x = t^2 - 1, y = t^3 - t; \quad t = 0$ and $t = 1$

9. $x = \sec t, y = \tan t$ on $(-\pi/2, \pi/2); \quad t = \pi/4$

10. $x = \cos t, y = \sin(2t)$ on $[0, 2\pi]; \quad t = \pi/4$

11. $x = \cos t \sin(2t), y = \sin t \sin(2t)$ on $[0, 2\pi]; \quad t = 3\pi/4$

12. $x = e^{t/10} \cos t, y = e^{t/10} \sin t; \quad t = \pi/2$

In Exercises 13 – 20, find t-values where the curve defined by the given parametric equations has a horizontal tangent line. Note: these are the same equations as in Exercises 5 – 12.

13. $x = t, y = t^2$

14. $x = \sqrt{t}, y = 5t + 2$

15. $x = t^2 - t, y = t^2 + t$

16. $x = t^2 - 1, y = t^3 - t$

17. $x = \sec t, y = \tan t$ on $(-\pi/2, \pi/2)$

18. $x = \cos t, y = \sin(2t)$ on $[0, 2\pi]$

19. $x = \cos t \sin(2t), y = \sin t \sin(2t)$ on $[0, 2\pi]$

20. $x = e^{t/10} \cos t, y = e^{t/10} \sin t$

In Exercises 21 – 24, find $t = t_0$ where the graph of the given parametric equations is not smooth, then find $\lim\limits_{t\to t_0} \dfrac{dy}{dx}$.

21. $x = \dfrac{1}{t^2 + 1}, \quad y = t^3$

22. $x = -t^3 + 7t^2 - 16t + 13, \quad y = t^3 - 5t^2 + 8t - 2$

23. $x = t^3 - 3t^2 + 3t - 1, \quad y = t^2 - 2t + 1$

24. $x = \cos^2 t, \quad y = 1 - \sin^2 t$

In Exercises 25 – 32, parametric equations for a curve are given. Find $\frac{d^2y}{dx^2}$, then determine the intervals on which the graph of the curve is concave up/down. Note: these are the same equations as in Exercises 5 – 12.

25. $x = t, \quad y = t^2$

26. $x = \sqrt{t}, \quad y = 5t + 2$

27. $x = t^2 - t, \quad y = t^2 + t$

28. $x = t^2 - 1, \quad y = t^3 - t$

29. $x = \sec t, \quad y = \tan t$ on $(-\pi/2, \pi/2)$

30. $x = \cos t, \quad y = \sin(2t)$ on $[0, 2\pi]$

31. $x = \cos t \sin(2t), \quad y = \sin t \sin(2t)$ on $[-\pi/2, \pi/2]$

32. $x = e^{t/10} \cos t, \quad y = e^{t/10} \sin t$

In Exercises 33 – 36, find the arc length of the graph of the parametric equations on the given interval(s).

33. $x = -3\sin(2t), \quad y = 3\cos(2t)$ on $[0, \pi]$

34. $x = e^{t/10} \cos t, \quad y = e^{t/10} \sin t$ on $[0, 2\pi]$ and $[2\pi, 4\pi]$

35. $x = 5t + 2, \quad y = 1 - 3t$ on $[-1, 1]$

36. $x = 2t^{3/2}, \quad y = 3t$ on $[0, 1]$

In Exercises 37 – 40, numerically approximate the given arc length.

37. Approximate the arc length of one petal of the rose curve $x = \cos t \cos(2t), \quad y = \sin t \cos(2t)$ using Simpson's Rule and $n = 4$.

38. Approximate the arc length of the "bow tie curve" $x = \cos t, \quad y = \sin(2t)$ using Simpson's Rule and $n = 6$.

39. Approximate the arc length of the parabola $x = t^2 - t$, $y = t^2 + t$ on $[-1, 1]$ using Simpson's Rule and $n = 4$.

40. A common approximate of the circumference of an ellipse given by $x = a\cos t, \quad y = b\sin t$ is $C \approx 2\pi\sqrt{\dfrac{a^2+b^2}{2}}$. Use this formula to approximate the circumference of $x = 5\cos t, \quad y = 3\sin t$ and compare this to the approximation given by Simpson's Rule and $n = 6$.

In Exercises 41 – 44, a solid of revolution is described. Find or approximate its surface area as specified.

41. Find the surface area of the sphere formed by rotating the circle $x = 2\cos t, \quad y = 2\sin t$ about:

    (a) the x-axis and

    (b) the y-axis.

42. Find the surface area of the torus (or "donut") formed by rotating the circle $x = \cos t + 2, \quad y = \sin t$ about the y-axis.

43. Approximate the surface area of the solid formed by rotating the "upper right half" of the bow tie curve $x = \cos t$, $y = \sin(2t)$ on $[0, \pi/2]$ about the x-axis, using Simpson's Rule and $n = 4$.

44. Approximate the surface area of the solid formed by rotating the one petal of the rose curve $x = \cos t\cos(2t)$, $y = \sin t\cos(2t)$ on $[0, \pi/4]$ about the x-axis, using Simpson's Rule and $n = 4$.

10.4 Introduction to Polar Coordinates

We are generally introduced to the idea of graphing curves by relating x-values to y-values through a function f. That is, we set $y = f(x)$, and plot lots of point pairs (x, y) to get a good notion of how the curve looks. This method is useful but has limitations, not least of which is that curves that "fail the vertical line test" cannot be graphed without using multiple functions.

The previous two sections introduced and studied a new way of plotting points in the x, y-plane. Using parametric equations, x and y values are computed independently and then plotted together. This method allows us to graph an extraordinary range of curves. This section introduces yet another way to plot points in the plane: using **polar coordinates**.

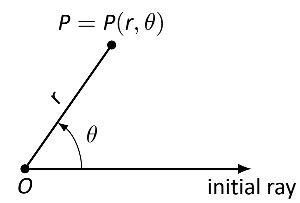


Figure 10.4.1: Illustrating polar coordinates.

Polar Coordinates

Start with a point O in the plane called the **pole** (we will always identify this point with the origin). From the pole, draw a ray, called the **initial ray** (we will always draw this ray horizontally, identifying it with the positive x-axis). A point P in the plane is determined by the distance r that P is from O, and the angle θ formed between the initial ray and the segment $\overline{OP}$ (measured counter-clockwise). We record the distance and angle as an ordered pair (r, θ). To avoid confusion with rectangular coordinates, we will denote polar coordinates with the letter P, as in $P(r, \theta)$. This is illustrated in Figure 10.4.1

Practice will make this process more clear.

Example 10.4.1 Plotting Polar Coordinates
Plot the following polar coordinates:

$$A = P(1, \pi/4) \quad B = P(1.5, \pi) \quad C = P(2, -\pi/3) \quad D = P(-1, \pi/4)$$

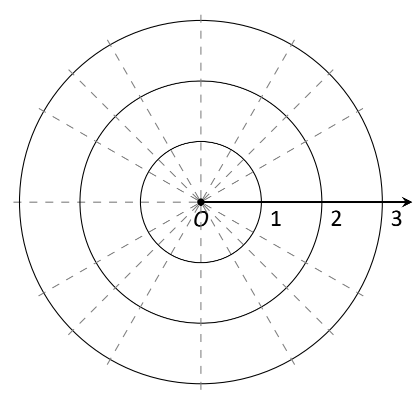


Figure 10.4.2: A polar grid for Example 10.4.1

Solution To aid in the drawing, a polar grid is provided to the right. To place the point A, go out 1 unit along the initial ray (putting you on the inner circle shown on the grid), then rotate counter-clockwise $\pi/4$ radians (or 45°). Alternately, one can consider the rotation first: think about the ray from O that forms an angle of $\pi/4$ with the initial ray, then move out 1 unit along this ray (again placing you on the inner circle of the grid).

To plot B, go out 1.5 units along the initial ray and rotate π radians (180°).

To plot C, go out 2 units along the initial ray then rotate *clockwise* $\pi/3$ radians, as the angle given is negative.

To plot D, move along the initial ray "-1" units – in other words, "back up" 1 unit, then rotate counter-clockwise by $\pi/4$. The results are given in Figure 10.4.3.

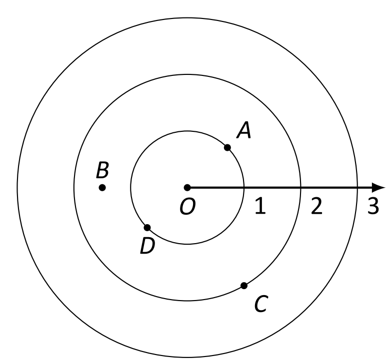


Figure 10.4.3: Plotting polar points in Example 10.4.1.

Consider the following two points: $A = P(1, \pi)$ and $B = P(-1, 0)$. To locate A, go out 1 unit on the initial ray then rotate π radians; to locate B, go out -1 units on the initial ray and don't rotate. One should see that A and B are located at the same point in the plane. We can also consider $C = P(1, 3\pi)$, or $D = P(1, -\pi)$; all four of these points share the same location.

This ability to identify a point in the plane with multiple polar coordinates is both a "blessing" and a "curse." We will see that it is beneficial as we can plot beautiful functions that intersect themselves (much like we saw with parametric functions). The unfortunate part of this is that it can be difficult to determine when this happens. We'll explore this more later in this section.

Polar to Rectangular Conversion

It is useful to recognize both the rectangular (or, Cartesian) coordinates of a point in the plane and its polar coordinates. Figure 10.4.4 shows a point P in the plane with rectangular coordinates (x, y) and polar coordinates $P(r, \theta)$. Using trigonometry, we can make the identities given in the following Key Idea.

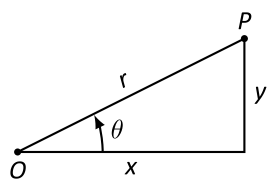


Figure 10.4.4: Converting between rectangular and polar coordinates.

Key Idea 10.4.1 Converting Between Rectangular and Polar Coordinates

Given the polar point $P(r, \theta)$, the rectangular coordinates are determined by

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Given the rectangular coordinates (x, y), the polar coordinates are determined by

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}.$$

Example 10.4.2 Converting Between Polar and Rectangular Coordinates

1. Convert the polar coordinates $P(2, 2\pi/3)$ and $P(-1, 5\pi/4)$ to rectangular coordinates.
2. Convert the rectangular coordinates $(1, 2)$ and $(-1, 1)$ to polar coordinates.

SOLUTION

1. (a) We start with $P(2, 2\pi/3)$. Using Key Idea 10.4.1, we have

$$x = 2\cos(2\pi/3) = -1 \qquad y = 2\sin(2\pi/3) = \sqrt{3}.$$

So the rectangular coordinates are $(-1, \sqrt{3}) \approx (-1, 1.732)$.

(b) The polar point $P(-1, 5\pi/4)$ is converted to rectangular with:

$$x = -1\cos(5\pi/4) = \sqrt{2}/2 \qquad y = -1\sin(5\pi/4) = \sqrt{2}/2.$$

So the rectangular coordinates are $(\sqrt{2}/2, \sqrt{2}/2) \approx (0.707, 0.707)$.

These points are plotted in Figure 10.4.5 (a). The rectangular coordinate system is drawn lightly under the polar coordinate system so that the relationship between the two can be seen.

2. (a) To convert the rectangular point $(1, 2)$ to polar coordinates, we use the Key Idea to form the following two equations:

$$1^2 + 2^2 = r^2 \qquad \tan\theta = \frac{2}{1}.$$

The first equation tells us that $r = \sqrt{5}$. Using the inverse tangent function, we find

$$\tan\theta = 2 \quad \Rightarrow \quad \theta = \tan^{-1} 2 \approx 1.11 \approx 63.43^\circ.$$

Thus polar coordinates of $(1, 2)$ are $P(\sqrt{5}, 1.11)$.

(b) To convert $(-1, 1)$ to polar coordinates, we form the equations

$$(-1)^2 + 1^2 = r^2 \qquad \tan\theta = \frac{1}{-1}.$$

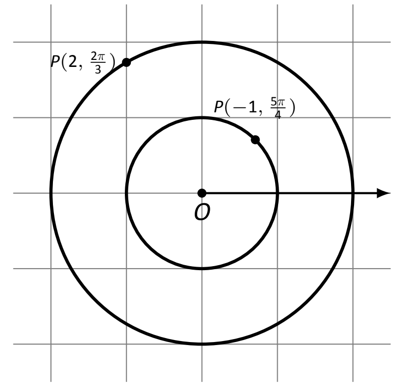
Thus $r = \sqrt{2}$. We need to be careful in computing θ: using the inverse tangent function, we have

$$\tan\theta = -1 \quad \Rightarrow \quad \theta = \tan^{-1}(-1) = -\pi/4 = -45^\circ.$$

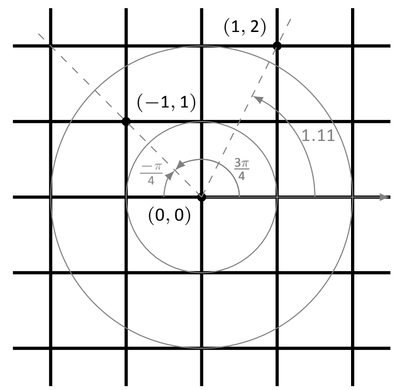
This is not the angle we desire. The range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$; that is, it returns angles that lie in the 1st and 4th quadrants. To find locations in the 2nd and 3rd quadrants, add π to the result of $\tan^{-1} x$. So $\pi + (-\pi/4)$ puts the angle at $3\pi/4$. Thus the polar point is $P(\sqrt{2}, 3\pi/4)$.

An alternate method is to use the angle θ given by arctangent, but change the sign of r. Thus we could also refer to $(-1, 1)$ as $P(-\sqrt{2}, -\pi/4)$.

These points are plotted in Figure 10.4.5 (b). The polar system is drawn lightly under the rectangular grid with rays to demonstrate the angles used.



(a)



(b)

Figure 10.4.5: Plotting rectangular and polar points in Example 10.4.2.

Polar Functions and Polar Graphs

Defining a new coordinate system allows us to create a new kind of function, a **polar function.** Rectangular coordinates lent themselves well to creating functions that related x and y, such as $y = x^2$. Polar coordinates allow us to create functions that relate r and θ. Normally these functions look like $r = f(\theta)$, although we can create functions of the form $\theta = f(r)$. The following examples introduce us to this concept.

Example 10.4.3 Introduction to Graphing Polar Functions

Describe the graphs of the following polar functions.

1. $r = 1.5$
2. $\theta = \pi/4$

SOLUTION

1. The equation $r = 1.5$ describes all points that are 1.5 units from the pole; as the angle is not specified, any θ is allowable. All points 1.5 units from the pole describes a circle of radius 1.5.

   We can consider the rectangular equivalent of this equation; using $r^2 = x^2 + y^2$, we see that $1.5^2 = x^2 + y^2$, which we recognize as the equation of a circle centred at $(0, 0)$ with radius 1.5. This is sketched in Figure 10.4.6.

2. The equation $\theta = \pi/4$ describes all points such that the line through them and the pole make an angle of $\pi/4$ with the initial ray. As the radius r is not specified, it can be any value (even negative). Thus $\theta = \pi/4$ describes the line through the pole that makes an angle of $\pi/4 = 45^\circ$ with the initial ray.

   We can again consider the rectangular equivalent of this equation. Combine $\tan\theta = y/x$ and $\theta = \pi/4$:

$$\tan\pi/4 = y/x \quad \Rightarrow x\tan\pi/4 = y \quad \Rightarrow y = x.$$

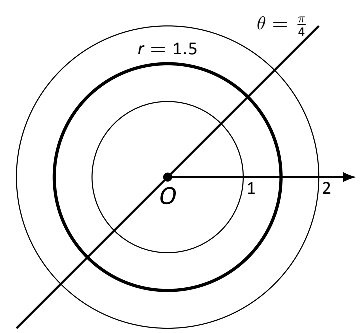


Figure 10.4.6: Plotting standard polar plots.

The basic rectangular equations of the form $x = h$ and $y = k$ create vertical and horizontal lines, respectively; the basic polar equations $r = h$ and $\theta = \alpha$ create circles and lines through the pole, respectively. With this as a foundation, we can create more complicated polar functions of the form $r = f(\theta)$. The input is an angle; the output is a length, how far in the direction of the angle to go out.

We sketch these functions much like we sketch rectangular and parametric functions: we plot lots of points and "connect the dots" with curves. We demonstrate this in the following example.

Example 10.4.4 Sketching Polar Functions
Sketch the polar function $r = 1 + \cos\theta$ on $[0, 2\pi]$ by plotting points.

θ	$r = 1 + \cos\theta$
0	2
$\pi/6$	1.86603
$\pi/2$	1
$4\pi/3$	0.5
$7\pi/4$	1.70711

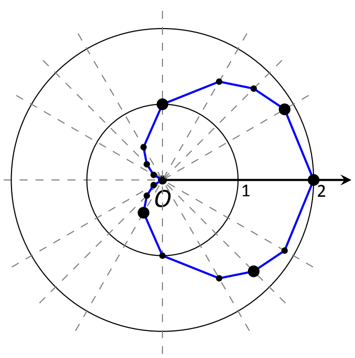


Figure 10.4.8: Graphing a polar function in Example 10.4.4 by plotting points.

Solution A common question when sketching curves by plotting points is "Which points should I plot?" With rectangular equations, we often choose "easy" values – integers, then add more if needed. When plotting polar equations, start with the "common" angles – multiples of $\pi/6$ and $\pi/4$. Figure 10.4.8 gives a table of just a few values of θ in $[0, \pi]$.

Consider the point $P(0, 2)$ determined by the first line of the table. The angle is 0 radians – we do not rotate from the initial ray – then we go out 2 units from the pole. When $\theta = \pi/6$, $r = 1.866$ (actually, it is $1 + \sqrt{3}/2$); so rotate by $\pi/6$ radians and go out 1.866 units.

The graph shown uses more points, connected with straight lines. (The points on the graph that correspond to points in the table are signified with larger dots.) Such a sketch is likely good enough to give one an idea of what the graph looks like.

Technology Note: Plotting functions in this way can be tedious, just as it was with rectangular functions. To obtain very accurate graphs, technology is a great aid. Most graphing calculators can plot polar functions; in the menu, set the plotting mode to something like `polar` or `POL`, depending on one's calculator. As with plotting parametric functions, the viewing "window" no longer determines the x-values that are plotted, so additional information needs to be provided. Often with the "window" settings are the settings for the beginning and ending θ values (often called $\theta_{\texttt{min}}$ and $\theta_{\texttt{max}}$) as well as the $\theta_{\texttt{step}}$ – that is, how far apart the θ values are spaced. The smaller the $\theta_{\texttt{step}}$ value, the more accurate the graph (which also increases plotting time). Using technology, we graphed the polar function $r = 1 + \cos\theta$ from Example 10.4.4 in Figure 10.4.9.

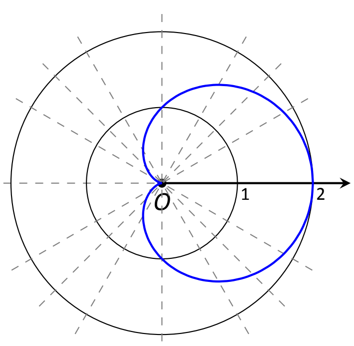


Figure 10.4.9: Using technology to graph a polar function.

Example 10.4.5 Sketching Polar Functions
Sketch the polar function $r = \cos(2\theta)$ on $[0, 2\pi]$ by plotting points.

Solution We start by making a table of $\cos(2\theta)$ evaluated at common angles θ, as shown in Figure 10.4.7. These points are then plotted in Figure 10.4.10 (a). This particular graph "moves" around quite a bit and one can easily forget which points should be connected to each other. To help us with this, we numbered each point in the table and on the graph.

Pt.	θ	$\cos(2\theta)$
1	0	1.
2	$\pi/6$	0.5
3	$\pi/4$	0.
4	$\pi/3$	-0.5
5	$\pi/2$	$-1.$
6	$2\pi/3$	-0.5
7	$3\pi/4$	0.
8	$5\pi/6$	0.5
9	π	1.

Pt.	θ	$\cos(2\theta)$
10	$7\pi/6$	0.5
11	$5\pi/4$	0.
12	$4\pi/3$	-0.5
13	$3\pi/2$	$-1.$
14	$5\pi/3$	-0.5
15	$7\pi/4$	0.
16	$11\pi/6$	0.5
17	2π	1.

Figure 10.4.9: Tables of points for plotting a polar curve.

Using more points (and the aid of technology) a smoother plot can be made as shown in Figure 10.4.10 (b). This plot is an example of a *rose curve*.

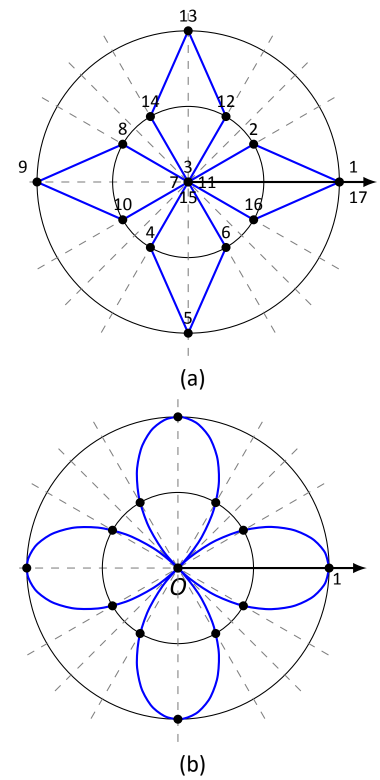


Figure 10.4.10: Polar plots from Example 10.4.5.

It is sometimes desirable to refer to a graph via a polar equation, and other times by a rectangular equation. Therefore it is necessary to be able to convert between polar and rectangular functions, which we practice in the following example. We will make frequent use of the identities found in Key Idea 10.4.1.

Example 10.4.6 Converting between rectangular and polar equations.

Convert from rectangular to polar.

1. $y = x^2$
2. $xy = 1$

Convert from polar to rectangular.

3. $r = \dfrac{2}{\sin\theta - \cos\theta}$
4. $r = 2\cos\theta$

Solution

1. Replace y with $r\sin\theta$ and replace x with $r\cos\theta$, giving:

$$\begin{aligned} y &= x^2 \\ r\sin\theta &= r^2\cos^2\theta \\ \frac{\sin\theta}{\cos^2\theta} &= r \end{aligned}$$

We have found that $r = \sin\theta/\cos^2\theta = \tan\theta\sec\theta$. The domain of this polar function is $(-\pi/2, \pi/2)$; plot a few points to see how the familiar parabola is traced out by the polar equation.

2. We again replace x and y using the standard identities and work to solve for r:

$$\begin{aligned} xy &= 1 \\ r\cos\theta\cdot r\sin\theta &= 1 \\ r^2 &= \frac{1}{\cos\theta\sin\theta} \\ r &= \frac{1}{\sqrt{\cos\theta\sin\theta}} \end{aligned}$$

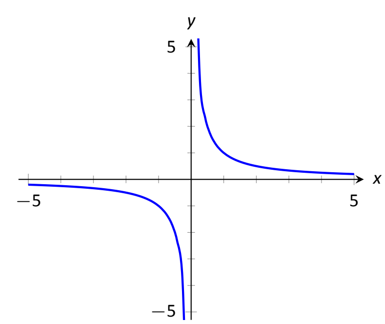


Figure 10.4.11: Graphing $xy = 1$ from Example 10.4.6.

This function is valid only when the product of $\cos\theta\sin\theta$ is positive. This occurs in the first and third quadrants, meaning the domain of this polar function is $(0, \pi/2) \cup (\pi, 3\pi/2)$.

We can rewrite the original rectangular equation $xy = 1$ as $y = 1/x$. This is graphed in Figure 10.4.11; note how it only exists in the first and third quadrants.

3. There is no set way to convert from polar to rectangular; in general, we look to form the products $r\cos\theta$ and $r\sin\theta$, and then replace these with x and y, respectively. We start in this problem by multiplying both sides by $\sin\theta - \cos\theta$:

$$\begin{aligned} r &= \frac{2}{\sin\theta - \cos\theta} \\ r(\sin\theta - \cos\theta) &= 2 \\ r\sin\theta - r\cos\theta &= 2. \qquad \text{Now replace with } y \text{ and } x\text{:} \\ y - x &= 2 \\ y &= x + 2. \end{aligned}$$

The original polar equation, $r = 2/(\sin\theta - \cos\theta)$ does not easily reveal that its graph is simply a line. However, our conversion shows that it is. The upcoming gallery of polar curves gives the general equations of lines in polar form.

4. By multiplying both sides by r, we obtain both an r^2 term and an $r\cos\theta$ term, which we replace with $x^2 + y^2$ and x, respectively.

$$\begin{aligned} r &= 2\cos\theta \\ r^2 &= 2r\cos\theta \\ x^2 + y^2 &= 2x. \end{aligned}$$

We recognize this as a circle; by completing the square we can find its radius and center.

$$\begin{aligned} x^2 - 2x + y^2 &= 0 \\ (x-1)^2 + y^2 &= 1. \end{aligned}$$

The circle is centered at $(1, 0)$ and has radius 1. The upcoming gallery of polar curves gives the equations of *some* circles in polar form; circles with arbitrary centers have a complicated polar equation that we do not consider here.

Some curves have very simple polar equations but rather complicated rectangular ones. For instance, the equation $r = 1 + \cos\theta$ describes a *cardioid* (a shape important to the sensitivity of microphones, among other things; one is graphed in the gallery in the Limaçon section). It's rectangular form is not nearly as simple; it is the implicit equation $x^4 + y^4 + 2x^2y^2 - 2xy^2 - 2x^3 - y^2 = 0$. The conversion is not "hard," but takes several steps, and is left as a problem in the Exercise section.

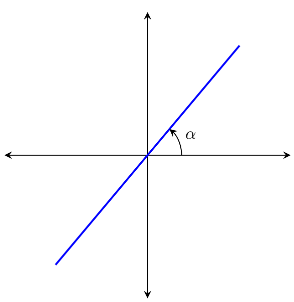
Gallery of Polar Curves

There are a number of basic and “classic” polar curves, famous for their beauty and/or applicability to the sciences. This section ends with a small gallery of some of these graphs. We encourage the reader to understand how these graphs are formed, and to investigate with technology other types of polar functions.

Lines

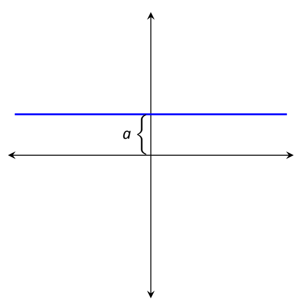
Through the origin:

$\theta = \alpha$



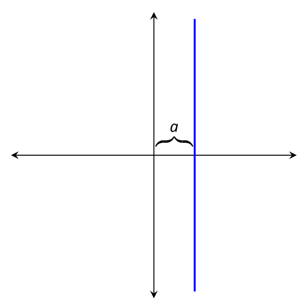
Horizontal line:

$r = a\csc\theta$



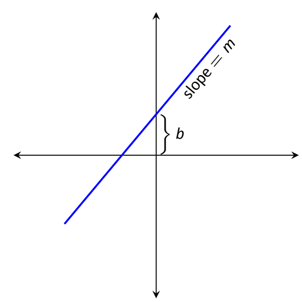
Vertical line:

$r = a\sec\theta$



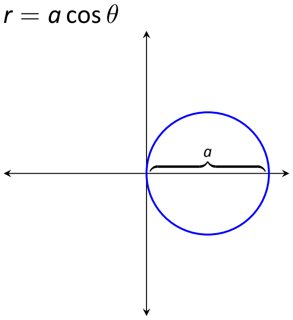
Not through origin:

$r = \dfrac{b}{\sin\theta - m\cos\theta}$

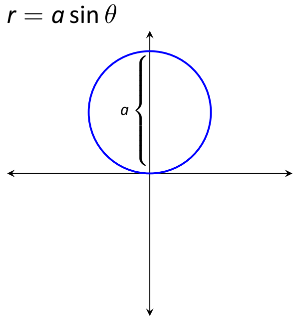


Circles

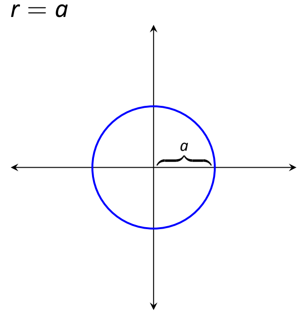
Centered on *x*-axis:
$r = a\cos\theta$



Centered on *y*-axis:
$r = a\sin\theta$

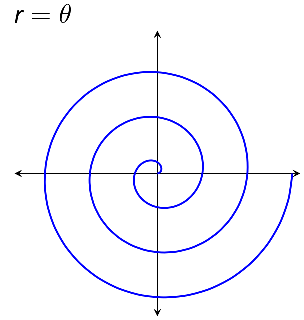


Centered on origin:
$r = a$



Spiral

Archimedean spiral
$r = \theta$

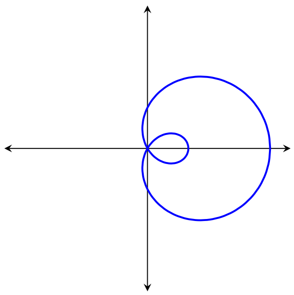


Limaçons

Symmetric about x-axis: $r = a \pm b\cos\theta$; Symmetric about y-axis: $r = a \pm b\sin\theta$; $a, b > 0$

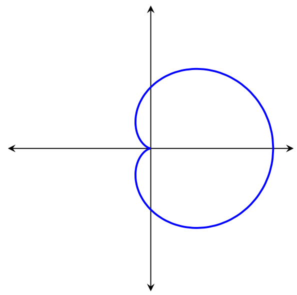
With inner loop:

$\dfrac{a}{b} < 1$



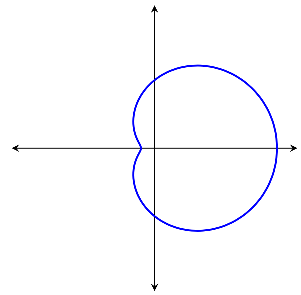
Cardioid:

$\dfrac{a}{b} = 1$



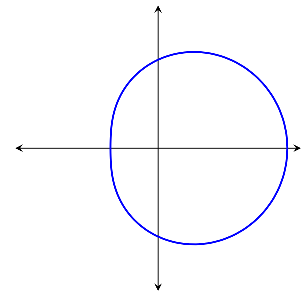
Dimpled:

$1 < \dfrac{a}{b} < 2$



Convex:

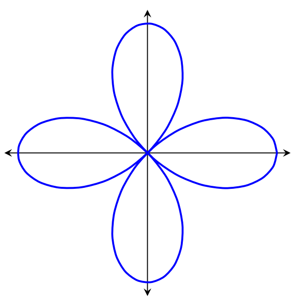
$\dfrac{a}{b} > 2$



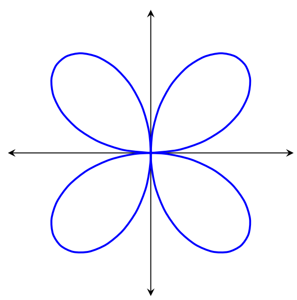
Rose Curves

Symmetric about x-axis: $r = a\cos(n\theta)$; Symmetric about y-axis: $r = a\sin(n\theta)$
Curve contains $2n$ petals when n is even and n petals when n is odd.

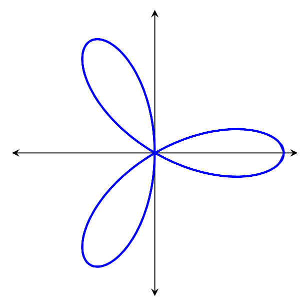
$r = a\cos(2\theta)$



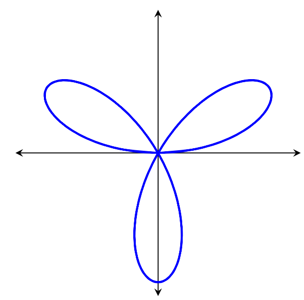
$r = a\sin(2\theta)$



$r = a\cos(3\theta)$



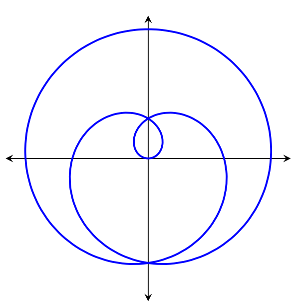
$r = a\sin(3\theta)$



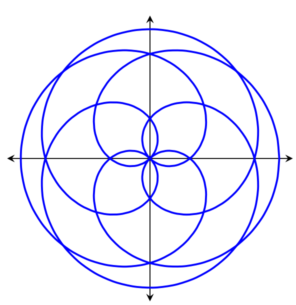
Special Curves

Rose curves

$r = a\sin(\theta/5)$

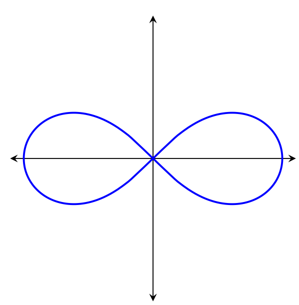


$r = a\sin(2\theta/5)$



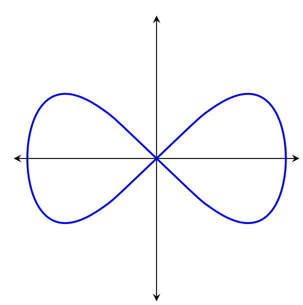
Lemniscate:

$r^2 = a^2\cos(2\theta)$



Eight Curve:

$r^2 = a^2\sec^4\theta\cos(2\theta)$



Earlier we discussed how each point in the plane does not have a unique representation in polar form. This can be a "good" thing, as it allows for the beautiful and interesting curves seen in the preceding gallery. However, it can also be a "bad" thing, as it can be difficult to determine where two curves intersect.

Example 10.4.7 Finding points of intersection with polar curves

Determine where the graphs of the polar equations $r = 1+3\cos\theta$ and $r = \cos\theta$ intersect.

Solution As technology is generally readily available, it is usually a good idea to start with a graph. We have graphed the two functions in Figure 10.4.12(a); to better discern the intersection points, part (b) of the figure zooms in around the origin. We start by setting the two functions equal to each other and solving for θ:

$$\begin{aligned} 1+3\cos\theta &= \cos\theta \\ 2\cos\theta &= -1 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3}. \end{aligned}$$

(There are, of course, infinite solutions to the equation $\cos\theta = -1/2$; as the limaçon is traced out once on $[0, 2\pi]$, we restrict our solutions to this interval.)

We need to analyze this solution. When $\theta = 2\pi/3$ we obtain the point of intersection that lies in the 4$^{\text{th}}$ quadrant. When $\theta = 4\pi/3$, we get the point of intersection that lies in the 2$^{\text{nd}}$ quadrant. There is more to say about this second intersection point, however. The circle defined by $r = \cos\theta$ is traced out once on $[0, \pi]$, meaning that this point of intersection occurs while tracing out the circle a second time. It seems strange to pass by the point once and then recognize it as a point of intersection only when arriving there a "second time." The first time the circle arrives at this point is when $\theta = \pi/3$. It is key to understand that these two points are the same: $(\cos\pi/3, \pi/3)$ and $(\cos 4\pi/3, 4\pi/3)$.

To summarize what we have done so far, we have found two points of intersection: when $\theta = 2\pi/3$ and when $\theta = 4\pi/3$. When referencing the circle $r = \cos\theta$, the latter point is better referenced as when $\theta = \pi/3$.

There is yet another point of intersection: the pole (or, the origin). We did not recognize this intersection point using our work above as each graph arrives at the pole at a different θ value.

A graph intersects the pole when $r = 0$. Considering the circle $r = \cos\theta$, $r = 0$ when $\theta = \pi/2$ (and odd multiples thereof, as the circle is repeatedly traced). The limaçon intersects the pole when $1+3\cos\theta = 0$; this occurs when $\cos\theta = -1/3$, or for $\theta = \cos^{-1}(-1/3)$. This is a nonstandard angle, approximately $\theta = 1.9106 = 109.47^\circ$. The limaçon intersects the pole twice in $[0, 2\pi]$; the other angle at which the limaçon is at the pole is the reflection of the first angle across the x-axis. That is, $\theta = 4.3726 = 250.53^\circ$.

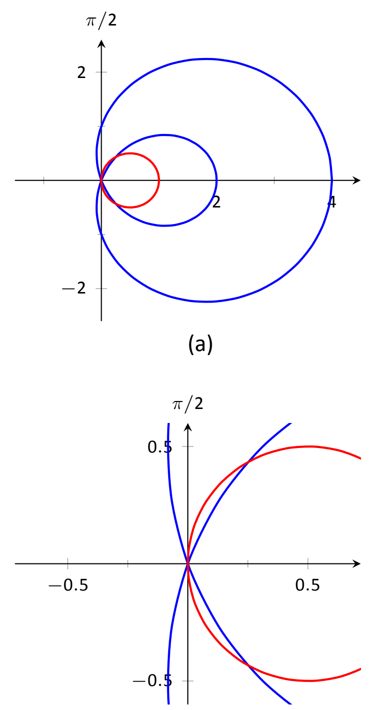


Figure 10.4.12: Graphs to help determine the points of intersection of the polar functions given in Example 10.4.7.

If all one is concerned with is the (x, y) coordinates at which the graphs intersect, much of the above work is extraneous. We know they intersect at $(0, 0)$; we might not care at what θ value. Likewise, using $\theta = 2\pi/3$ and $\theta = 4\pi/3$ can give us the needed rectangular coordinates. However, in the next section we apply calculus concepts to polar functions. When computing the area of a region bounded by polar curves, understanding the nuances of the points of intersection becomes important.

Exercises 10.4

Terms and Concepts

1. In your own words, describe how to plot the polar point $P(r, \theta)$.

2. T/F: When plotting a point with polar coordinate $P(r, \theta)$, r must be positive.

3. T/F: Every point in the Cartesian plane can be represented by a polar coordinate.

4. T/F: Every point in the Cartesian plane can be represented uniquely by a polar coordinate.

Problems

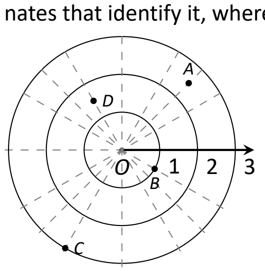
5. Plot the points with the given polar coordinates.

   (a) $A = P(2, 0)$ (c) $C = P(-2, \pi/2)$
   (b) $B = P(1, \pi)$ (d) $D = P(1, \pi/4)$

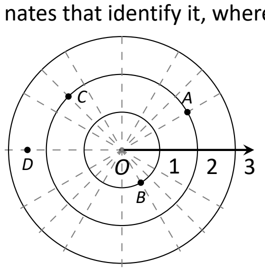
6. Plot the points with the given polar coordinates.

   (a) $A = P(2, 3\pi)$ (c) $C = P(1, 2)$
   (b) $B = P(1, -\pi)$ (d) $D = P(1/2, 5\pi/6)$

7. For each of the given points give two sets of polar coordinates that identify it, where $0 \leq \theta \leq 2\pi$.



8. For each of the given points give two sets of polar coordinates that identify it, where $-\pi \leq \theta \leq \pi$.



9. Convert each of the following polar coordinates to rectangular, and each of the following rectangular coordinates to polar.

   (a) $A = P(2, \pi/4)$ (c) $C = (2, -1)$
   (b) $B = P(2, -\pi/4)$ (d) $D = (-2, 1)$

10. Convert each of the following polar coordinates to rectangular, and each of the following rectangular coordinates to polar.

   (a) $A = P(3, \pi)$ (c) $C = (0, 4)$
   (b) $B = P(1, 2\pi/3)$ (d) $D = (1, -\sqrt{3})$

In Exercises 11 – 30, graph the polar function on the given interval.

11. $r = 2, \quad 0 \leq \theta \leq \pi/2$

12. $\theta = \pi/6, \quad -1 \leq r \leq 2$

13. $r = 1 - \cos\theta, \quad [0, 2\pi]$

14. $r = 2 + \sin\theta, \quad [0, 2\pi]$

15. $r = 2 - \sin\theta, \quad [0, 2\pi]$

16. $r = 1 - 2\sin\theta, \quad [0, 2\pi]$

17. $r = 1 + 2\sin\theta, \quad [0, 2\pi]$

18. $r = \cos(2\theta), \quad [0, 2\pi]$

19. $r = \sin(3\theta), \quad [0, \pi]$

20. $r = \cos(\theta/3), \quad [0, 3\pi]$

21. $r = \cos(2\theta/3), \quad [0, 6\pi]$

22. $r = \theta/2, \quad [0, 4\pi]$

23. $r = 3\sin(\theta), \quad [0, \pi]$

24. $r = 2\cos(\theta), \quad [0, \pi/2]$

25. $r = \cos\theta\sin\theta, \quad [0, 2\pi]$

26. $r = \theta^2 - (\pi/2)^2, \quad [-\pi, \pi]$

27. $r = \dfrac{3}{5\sin\theta - \cos\theta}, \quad [0, 2\pi]$

28. $r = \dfrac{-2}{3\cos\theta - 2\sin\theta}, \quad [0, 2\pi]$

29. $r = 3\sec\theta, \quad (-\pi/2, \pi/2)$

30. $r = 3\csc\theta, \quad (0, \pi)$

In Exercises 31 – 40, convert the polar equation to a rectangular equation.

31. $r = 6\cos\theta$

32. $r = -4\sin\theta$

33. $r = \cos\theta + \sin\theta$

34. $r = \dfrac{7}{5\sin\theta - 2\cos\theta}$

35. $r = \dfrac{3}{\cos\theta}$

36. $r = \dfrac{4}{\sin\theta}$

37. $r = \tan\theta$

38. $r = \cot\theta$

39. $r = 2$

40. $\theta = \pi/6$

In Exercises 41 – 48, convert the rectangular equation to a polar equation.

41. $y = x$

42. $y = 4x + 7$

43. $x = 5$

44. $y = 5$

45. $x = y^2$

46. $x^2 y = 1$

47. $x^2 + y^2 = 7$

48. $(x + 1)^2 + y^2 = 1$

In Exercises 49 – 56, find the points of intersection of the polar graphs.

49. $r = \sin(2\theta)$ and $r = \cos\theta$ on $[0, \pi]$

50. $r = \cos(2\theta)$ and $r = \cos\theta$ on $[0, \pi]$

51. $r = 2\cos\theta$ and $r = 2\sin\theta$ on $[0, \pi]$

52. $r = \sin\theta$ and $r = \sqrt{3} + 3\sin\theta$ on $[0, 2\pi]$

53. $r = \sin(3\theta)$ and $r = \cos(3\theta)$ on $[0, \pi]$

54. $r = 3\cos\theta$ and $r = 1 + \cos\theta$ on $[-\pi, \pi]$

55. $r = 1$ and $r = 2\sin(2\theta)$ on $[0, 2\pi]$

56. $r = 1 - \cos\theta$ and $r = 1 + \sin\theta$ on $[0, 2\pi]$

57. Pick a integer value for n, where $n \neq 2, 3$, and use technology to plot $r = \sin\left(\dfrac{m}{n}\theta\right)$ for three different integer values of m. Sketch these and determine a minimal interval on which the entire graph is shown.

58. Create your own polar function, $r = f(\theta)$ and sketch it. Describe why the graph looks as it does.

10.5 Calculus and Polar Functions

The previous section defined polar coordinates, leading to polar functions. We investigated plotting these functions and solving a fundamental question about their graphs, namely, where do two polar graphs intersect?

We now turn our attention to answering other questions, whose solutions require the use of calculus. A basis for much of what is done in this section is the ability to turn a polar function $r = f(\theta)$ into a set of parametric equations. Using the identities $x = r\cos\theta$ and $y = r\sin\theta$, we can create the parametric equations $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$ and apply the concepts of Section 10.3.

Polar Functions and $\dfrac{dy}{dx}$

We are interested in the lines tangent to a given graph, regardless of whether that graph is produced by rectangular, parametric, or polar equations. In each of these contexts, the slope of the tangent line is $\frac{dy}{dx}$. Given $r = f(\theta)$, we are generally *not* concerned with $r' = f'(\theta)$; that describes how fast r changes with respect to θ. Instead, we will use $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$ to compute $\frac{dy}{dx}$.

Using Key Idea 10.3.1 we have

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta}.$$

Each of the two derivatives on the right hand side of the equality requires the use of the Product Rule. We state the important result as a Key Idea.

Key Idea 10.5.1 Finding $\frac{dy}{dx}$ with Polar Functions

Let $r = f(\theta)$ be a polar function. With $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$,

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Example 10.5.1 Finding $\frac{dy}{dx}$ with polar functions.
Consider the limaçon $r = 1 + 2\sin\theta$ on $[0, 2\pi]$.

1. Find the equations of the tangent and normal lines to the graph at $\theta = \pi/4$.

2. Find where the graph has vertical and horizontal tangent lines.

Solution

1. We start by computing $\frac{dy}{dx}$. With $f'(\theta) = 2\cos\theta$, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{2\cos\theta\sin\theta + \cos\theta(1+2\sin\theta)}{2\cos^2\theta - \sin\theta(1+2\sin\theta)}\\ &= \frac{\cos\theta(4\sin\theta+1)}{2(\cos^2\theta - \sin^2\theta) - \sin\theta}.\end{aligned}$$

When $\theta = \pi/4$, $\frac{dy}{dx} = -2\sqrt{2} - 1$ (this requires a bit of simplification). In rectangular coordinates, the point on the graph at $\theta = \pi/4$ is $(1 +$

$\sqrt{2}/2, 1+\sqrt{2}/2)$. Thus the rectangular equation of the line tangent to the limaçon at $\theta = \pi/4$ is

$$y = (-2\sqrt{2}-1)\big(x-(1+\sqrt{2}/2)\big)+1+\sqrt{2}/2 \approx -3.83x+8.24.$$

The limaçon and the tangent line are graphed in Figure 10.5.1.

The normal line has the opposite–reciprocal slope as the tangent line, so its equation is

$$y \approx \frac{1}{3.83}x + 1.26.$$

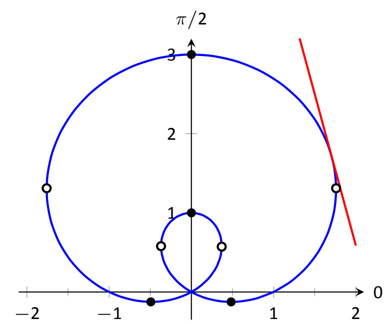


Figure 10.5.1: The limaçon in Example 10.5.1 with its tangent line at $\theta = \pi/4$ and points of vertical and horizontal tangency.

2. To find the horizontal lines of tangency, we find where $\frac{dy}{dx} = 0$; thus we find where the numerator of our equation for $\frac{dy}{dx}$ is 0.

$$\cos\theta(4\sin\theta+1) = 0 \quad \Rightarrow \quad \cos\theta = 0 \quad \text{or} \quad 4\sin\theta+1 = 0.$$

On $[0, 2\pi]$, $\cos\theta = 0$ when $\theta = \pi/2,\ 3\pi/2$.

Setting $4\sin\theta+1 = 0$ gives $\theta = \sin^{-1}(-1/4) \approx -0.2527 = -14.48^\circ$. We want the results in $[0, 2\pi]$; we also recognize there are two solutions, one in the 3rd quadrant and one in the 4th. Using reference angles, we have our two solutions as $\theta = 3.39$ and 6.03 radians. The four points we obtained where the limaçon has a horizontal tangent line are given in Figure 10.5.1 with black–filled dots.

To find the vertical lines of tangency, we set the denominator of $\frac{dy}{dx} = 0$.

$$2(\cos^2\theta - \sin^2\theta) - \sin\theta = 0.$$

Convert the $\cos^2\theta$ term to $1-\sin^2\theta$:

$$\begin{aligned} 2(1-\sin^2\theta-\sin^2\theta)-\sin\theta &= 0\\ 4\sin^2\theta+\sin\theta-2 &= 0. \end{aligned}$$

Recognize this as a quadratic in the variable $\sin\theta$. Using the quadratic formula, we have

$$\sin\theta = \frac{-1\pm\sqrt{33}}{8}.$$

We solve $\sin\theta = \frac{-1+\sqrt{33}}{8}$ and $\sin\theta = \frac{-1-\sqrt{33}}{8}$:

$$\begin{aligned} \sin\theta &= \frac{-1+\sqrt{33}}{8} & \qquad \sin\theta &= \frac{-1-\sqrt{33}}{8}\\ \theta &= \sin^{-1}\left(\frac{-1+\sqrt{33}}{8}\right) & \theta &= \sin^{-1}\left(\frac{-1-\sqrt{33}}{8}\right)\\ \theta &= 0.6349 & \theta &= -1.0030 \end{aligned}$$

In each of the solutions above, we only get one of the possible two solutions as $\sin^{-1}x$ only returns solutions in $[-\pi/2, \pi/2]$, the 4th and 1st quadrants. Again using reference angles, we have:

$$\sin\theta = \frac{-1+\sqrt{33}}{8} \quad \Rightarrow \quad \theta = 0.6349,\ 2.5067 \text{ radians}$$

and

$$\sin\theta = \frac{-1-\sqrt{33}}{8} \quad \Rightarrow \quad \theta = 4.1446,\ 5.2802 \text{ radians.}$$

These points are also shown in Figure 10.5.1 with white–filled dots.

When the graph of the polar function $r = f(\theta)$ intersects the pole, it means that $f(\alpha) = 0$ for some angle α. Thus the formula for $\frac{dy}{dx}$ in such instances is very simple, reducing simply to

$$\frac{dy}{dx} = \tan\alpha.$$

This equation makes an interesting point. It tells us the slope of the tangent line at the pole is $\tan\alpha$; some of our previous work (see, for instance, Example 10.4.3) shows us that the line through the pole with slope $\tan\alpha$ has polar equation $\theta = \alpha$. Thus when a polar graph touches the pole at $\theta = \alpha$, the equation of the tangent line at the pole is $\theta = \alpha$.

Example 10.5.2 Finding tangent lines at the pole.

Let $r = 1 + 2\sin\theta$, a limaçon. Find the equations of the lines tangent to the graph at the pole.

SOLUTION We need to know when $r = 0$.

$$\begin{aligned} 1 + 2\sin\theta &= 0 \\ \sin\theta &= -1/2 \\ \theta &= \frac{7\pi}{6},\ \frac{11\pi}{6}. \end{aligned}$$

Thus the equations of the tangent lines, in polar, are $\theta = 7\pi/6$ and $\theta = 11\pi/6$. In rectangular form, the tangent lines are $y = \tan(7\pi/6)x$ and $y = \tan(11\pi/6)x$. The full limaçon can be seen in Figure 10.5.1; we zoom in on the tangent lines in Figure 10.5.2.

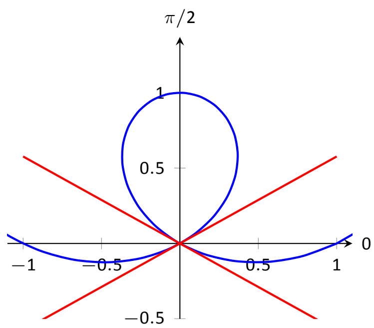
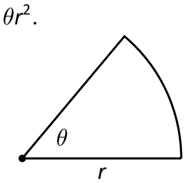


Figure 10.5.2: Graphing the tangent lines at the pole in Example 10.5.2.

Area

When using rectangular coordinates, the equations $x = h$ and $y = k$ defined vertical and horizontal lines, respectively, and combinations of these lines create rectangles (hence the name "rectangular coordinates"). It is then somewhat natural to use rectangles to approximate area as we did when learning about the definite integral.

Note: Recall that the area of a sector of a circle with radius r subtended by an angle θ is $A = \frac{1}{2}\theta r^2$.



When using polar coordinates, the equations $\theta = \alpha$ and $r = c$ form lines through the origin and circles centred at the origin, respectively, and combinations of these curves form sectors of circles. It is then somewhat natural to calculate the area of regions defined by polar functions by first approximating with sectors of circles.

Consider Figure 10.5.3 (a) where a region defined by $r = f(\theta)$ on $[\alpha, \beta]$ is given. (Note how the "sides" of the region are the lines $\theta = \alpha$ and $\theta = \beta$, whereas in rectangular coordinates the "sides" of regions were often the vertical lines $x = a$ and $x = b$.)

Partition the interval $[\alpha, \beta]$ into n equally spaced subintervals as $\alpha = \theta_1 < \theta_2 < \cdots < \theta_{n+1} = \beta$. The length of each subinterval is $\Delta\theta = (\beta - \alpha)/n$, representing a small change in angle. The area of the region defined by the i^{th} subinterval $[\theta_i, \theta_{i+1}]$ can be approximated with a sector of a circle with radius

$f(c_i)$, for some c_i in $[\theta_i, \theta_{i+1}]$. The area of this sector is $\frac{1}{2}f(c_i)^2\Delta\theta$. This is shown in part (b) of the figure, where $[\alpha, \beta]$ has been divided into 4 subintervals. We approximate the area of the whole region by summing the areas of all sectors:

$$\text{Area} \approx \sum_{i=1}^{n} \frac{1}{2} f(c_i)^2 \Delta\theta.$$

This is a Riemann sum. By taking the limit of the sum as $n \to \infty$, we find the exact area of the region in the form of a definite integral.

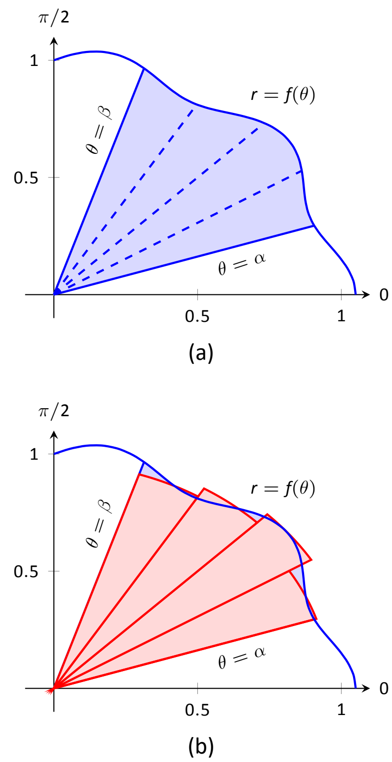


Figure 10.5.3: Computing the area of a polar region.

Theorem 10.5.1 Area of a Polar Region

Let f be continuous and non-negative on $[\alpha, \beta]$, where $0 \le \beta - \alpha \le 2\pi$. The area A of the region bounded by the curve $r = f(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2\, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$$

The theorem states that $0 \le \beta - \alpha \le 2\pi$. This ensures that region does not overlap itself, which would give a result that does not correspond directly to the area.

Example 10.5.3 Area of a polar region

Find the area of the circle defined by $r = \cos\theta$. (Recall this circle has radius $1/2$.)

Solution This is a direct application of Theorem 10.5.1. The circle is traced out on $[0, \pi]$, leading to the integral

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_0^{\pi} \cos^2\theta\, d\theta \\ &= \frac{1}{2}\int_0^{\pi} \frac{1+\cos(2\theta)}{2}\, d\theta \\ &= \frac{1}{4}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi} \\ &= \frac{1}{4}\pi.\end{aligned}$$

Of course, we already knew the area of a circle with radius $1/2$. We did this example to demonstrate that the area formula is correct.

Note: Example 10.5.3 requires the use of the integral $\int \cos^2\theta\, d\theta$. This is handled well by using the power reducing formula as found at the end of this text. Due to the nature of the area formula, integrating $\cos^2\theta$ and $\sin^2\theta$ is required often. We offer here these indefinite integrals as a time–saving measure.

$$\int \cos^2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$

$$\int \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + C$$

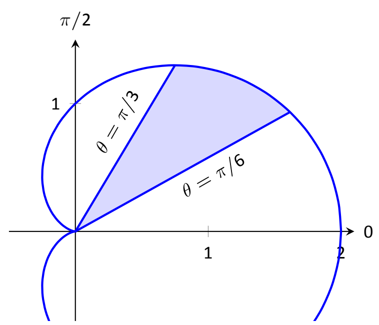


Figure 10.5.4: Finding the area of the shaded region of a cardiod in Example 10.5.4.

Example 10.5.4 Area of a polar region

Find the area of the cardioid $r = 1+\cos\theta$ bound between $\theta = \pi/6$ and $\theta = \pi/3$, as shown in Figure 10.5.4.

SOLUTION This is again a direct application of Theorem 10.5.1.

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_{\pi/6}^{\pi/3} (1+\cos\theta)^2\,d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{\pi/3} (1+2\cos\theta+\cos^2\theta)\,d\theta \\ &= \frac{1}{2}\left(\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right)\Bigg|_{\pi/6}^{\pi/3} \\ &= \frac{1}{8}\left(\pi + 4\sqrt{3} - 4\right) \approx 0.7587.\end{aligned}$$

Area Between Curves

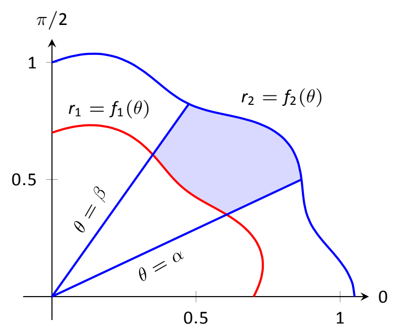


Figure 10.5.5: Illustrating area bound between two polar curves.

Our study of area in the context of rectangular functions led naturally to finding area bounded between curves. We consider the same in the context of polar functions.

Consider the shaded region shown in Figure 10.5.5. We can find the area of this region by computing the area bounded by $r_2 = f_2(\theta)$ and subtracting the area bounded by $r_1 = f_1(\theta)$ on $[\alpha, \beta]$. Thus

$$\text{Area} = \frac{1}{2}\int_\alpha^\beta r_2^2\,d\theta - \frac{1}{2}\int_\alpha^\beta r_1^2\,d\theta = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right)\,d\theta.$$

Key Idea 10.5.2 Area Between Polar Curves

The area A of the region bounded by $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$, $\theta = \alpha$ and $\theta = \beta$, where $f_1(\theta) \leq f_2(\theta)$ on $[\alpha, \beta]$, is

$$A = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right)\,d\theta.$$

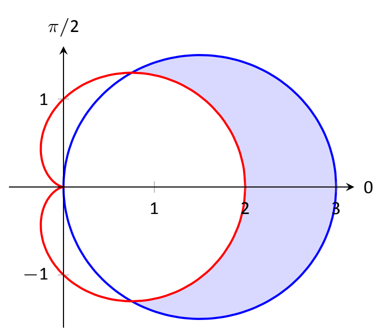


Figure 10.5.6: Finding the area between polar curves in Example 10.5.5.

Example 10.5.5 Area between polar curves

Find the area bounded between the curves $r = 1+\cos\theta$ and $r = 3\cos\theta$, as shown in Figure 10.5.6.

SOLUTION We need to find the points of intersection between these two functions. Setting them equal to each other, we find:

$$\begin{aligned}1+\cos\theta &= 3\cos\theta \\ \cos\theta &= 1/2 \\ \theta &= \pm\pi/3\end{aligned}$$

Thus we integrate $\frac{1}{2}\big((3\cos\theta)^2-(1+\cos\theta)^2\big)$ on $[-\pi/3,\pi/3]$.

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\big((3\cos\theta)^2-(1+\cos\theta)^2\big)\,d\theta\\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\big(8\cos^2\theta-2\cos\theta-1\big)\,d\theta\\ &= \frac{1}{2}\big(2\sin(2\theta)-2\sin\theta+3\theta\big)\Big|_{-\pi/3}^{\pi/3}\\ &= \pi.\end{aligned}$$

Amazingly enough, the area between these curves has a "nice" value.

Example 10.5.6 Area defined by polar curves

Find the area bounded between the polar curves $r=1$ and $r=2\cos(2\theta)$, as shown in Figure 10.5.7 (a).

Solution We need to find the point of intersection between the two curves. Setting the two functions equal to each other, we have

$$2\cos(2\theta)=1 \quad\Rightarrow\quad \cos(2\theta)=\frac{1}{2} \quad\Rightarrow\quad 2\theta=\pi/3 \quad\Rightarrow\quad \theta=\pi/6.$$

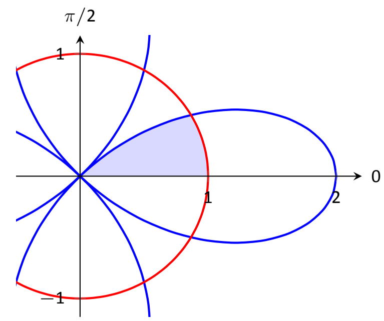
In part (b) of the figure, we zoom in on the region and note that it is not really bounded *between* two polar curves, but rather *by* two polar curves, along with $\theta=0$. The dashed line breaks the region into its component parts. Below the dashed line, the region is defined by $r=1$, $\theta=0$ and $\theta=\pi/6$. (Note: the dashed line lies on the line $\theta=\pi/6$.) Above the dashed line the region is bounded by $r=2\cos(2\theta)$ and $\theta=\pi/6$. Since we have two separate regions, we find the area using two separate integrals.

Call the area below the dashed line A_1 and the area above the dashed line A_2. They are determined by the following integrals:

$$A_1=\frac{1}{2}\int_0^{\pi/6}(1)^2\,d\theta \qquad A_2=\frac{1}{2}\int_{\pi/6}^{\pi/4}\big(2\cos(2\theta)\big)^2\,d\theta.$$

(The upper bound of the integral computing A_2 is $\pi/4$ as $r=2\cos(2\theta)$ is at the pole when $\theta=\pi/4$.)

We omit the integration details and let the reader verify that $A_1=\pi/12$ and $A_2=\pi/12-\sqrt{3}/8$; the total area is $A=\pi/6-\sqrt{3}/8$.



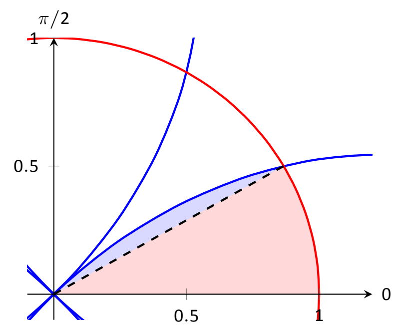


Figure 10.5.7: Graphing the region bounded by the functions in Example 10.5.6.

Arc Length

As we have already considered the arc length of curves defined by rectangular and parametric equations, we now consider it in the context of polar equations. Recall that the arc length L of the graph defined by the parametric equations $x=f(t)$, $y=g(t)$ on $[a,b]$ is

$$L=\int_a^b\sqrt{f'(t)^2+g'(t)^2}\,dt=\int_a^b\sqrt{x'(t)^2+y'(t)^2}\,dt. \tag{10.1}$$

Now consider the polar function $r=f(\theta)$. We again use the identities $x=f(\theta)\cos\theta$ and $y=f(\theta)\sin\theta$ to create parametric equations based on the polar function. We compute $x'(\theta)$ and $y'(\theta)$ as done before when computing $\frac{dy}{dx}$, then apply Equation (10.1).

The expression $x'(\theta)^2 + y'(\theta)^2$ can be simplified a great deal; we leave this as an exercise and state that

$$x'(\theta)^2 + y'(\theta)^2 = f'(\theta)^2 + f(\theta)^2.$$

This leads us to the arc length formula.

Theorem 10.5.2 Arc Length of Polar Curves

Let $r = f(\theta)$ be a polar function with f' continuous on $[\alpha, \beta]$, on which the graph traces itself only once. The arc length L of the graph on $[\alpha, \beta]$ is

$$L = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta = \int_\alpha^\beta \sqrt{(r')^2 + r^2}\, d\theta.$$

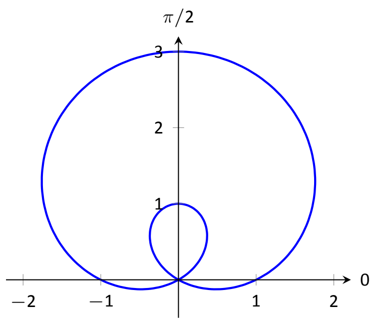


Figure 10.5.8: The limaçon in Example 10.5.7 whose arc length is measured.

Example 10.5.7 Arc length of a limaçon

Find the arc length of the limaçon $r = 1 + 2\sin t$.

Solution With $r = 1 + 2\sin t$, we have $r' = 2\cos t$. The limaçon is traced out once on $[0, 2\pi]$, giving us our bounds of integration. Applying Theorem 10.5.2, we have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(2\cos\theta)^2 + (1 + 2\sin\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\cos^2\theta + 4\sin^2\theta + 4\sin\theta + 1}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin\theta + 5}\, d\theta \\ &\approx 13.3649. \end{aligned}$$

The final integral cannot be solved in terms of elementary functions, so we resorted to a numerical approximation. (Simpson's Rule, with $n = 4$, approximates the value with 13.0608. Using $n = 22$ gives the value above, which is accurate to 4 places after the decimal.)

Surface Area

The formula for arc length leads us to a formula for surface area. The following Theorem is based on Theorem 10.3.2.

Theorem 10.5.3 Surface Area of a Solid of Revolution

Consider the graph of the polar equation $r = f(\theta)$, where f' is continuous on $[\alpha, \beta]$, on which the graph does not cross itself.

1. The surface area of the solid formed by revolving the graph about the initial ray ($\theta = 0$) is:

$$\text{Surface Area} = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin\theta \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta.$$

2. The surface area of the solid formed by revolving the graph about the line $\theta = \pi/2$ is:

$$\text{Surface Area} = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos\theta \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta.$$

Example 10.5.8 Surface area determined by a polar curve

Find the surface area formed by revolving one petal of the rose curve $r = \cos(2\theta)$ about its central axis (see Figure 10.5.9).

SOLUTION We choose, as implied by the figure, to revolve the portion of the curve that lies on $[0, \pi/4]$ about the initial ray. Using Theorem 10.5.3 and the fact that $f'(\theta) = -2\sin(2\theta)$, we have

$$\begin{aligned}\text{Surface Area} &= 2\pi \int_0^{\pi/4} \cos(2\theta)\sin(\theta)\sqrt{\big(-2\sin(2\theta)\big)^2 + \big(\cos(2\theta)\big)^2}\, d\theta \\ &\approx 1.36707.\end{aligned}$$

The integral is another that cannot be evaluated in terms of elementary functions. Simpson's Rule, with $n = 4$, approximates the value at 1.36751.

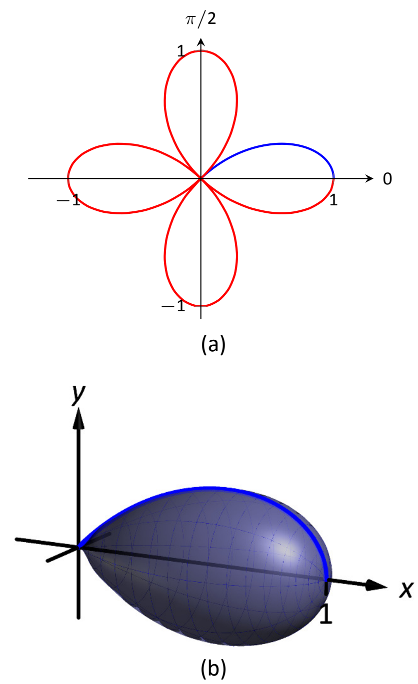


Figure 10.5.9: Finding the surface area of a rose–curve petal that is revolved around its central axis.

This chapter has been about curves in the plane. While there is great mathematics to be discovered in the two dimensions of a plane, we live in a three dimensional world and hence we should also look to do mathematics in 3D – that is, in *space*. The next chapter begins our exploration into space by introducing the topic of *vectors*, which are incredibly useful and powerful mathematical objects.

Exercises 10.5

Terms and Concepts

1. Given polar equation $r = f(\theta)$, how can one create parametric equations of the same curve?

2. With rectangular coordinates, it is natural to approximate area with ________; with polar coordinates, it is natural to approximate area with ________.

Problems

In Exercises 3 – 10, find:

(a) $\dfrac{dy}{dx}$

(b) the equation of the tangent and normal lines to the curve at the indicated θ–value.

3. $r = 1; \quad \theta = \pi/4$

4. $r = \cos\theta; \quad \theta = \pi/4$

5. $r = 1 + \sin\theta; \quad \theta = \pi/6$

6. $r = 1 - 3\cos\theta; \quad \theta = 3\pi/4$

7. $r = \theta; \quad \theta = \pi/2$

8. $r = \cos(3\theta); \quad \theta = \pi/6$

9. $r = \sin(4\theta); \quad \theta = \pi/3$

10. $r = \dfrac{1}{\sin\theta - \cos\theta}; \quad \theta = \pi$

In Exercises 11 – 14, find the values of θ in the given interval where the graph of the polar function has horizontal and vertical tangent lines.

11. $r = 3; \quad [0, 2\pi]$

12. $r = 2\sin\theta; \quad [0, \pi]$

13. $r = \cos(2\theta); \quad [0, 2\pi]$

14. $r = 1 + \cos\theta; \quad [0, 2\pi]$

In Exercises 15 – 16, find the equation of the lines tangent to the graph at the pole.

15. $r = \sin\theta; \quad [0, \pi]$

16. $r = \sin(3\theta); \quad [0, \pi]$

In Exercises 17 – 28, find the area of the described region.

17. Enclosed by the circle: $r = 4\sin\theta$

18. Enclosed by the circle $r = 5$

19. Enclosed by one petal of $r = \sin(3\theta)$

20. Enclosed by one petal of the rose curve $r = \cos(n\,\theta)$, where n is a positive integer.

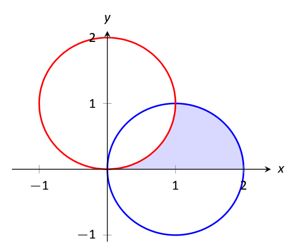
21. Enclosed by the cardioid $r = 1 - \sin\theta$

22. Enclosed by the inner loop of the limaçon $r = 1 + 2\cos\theta$

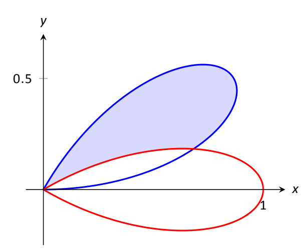
23. Enclosed by the outer loop of the limaçon $r = 1 + 2\cos\theta$ (including area enclosed by the inner loop)

24. Enclosed between the inner and outer loop of the limaçon $r = 1 + 2\cos\theta$

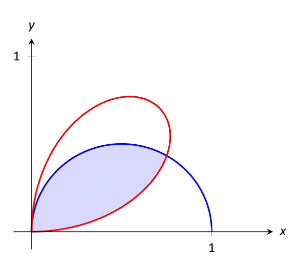
25. Enclosed by $r = 2\cos\theta$ and $r = 2\sin\theta$, as shown:



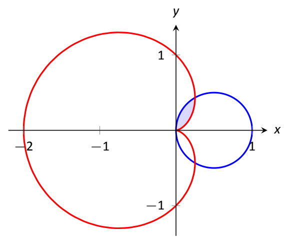
26. Enclosed by $r = \cos(3\theta)$ and $r = \sin(3\theta)$, as shown:



27. Enclosed by $r = \cos\theta$ and $r = \sin(2\theta)$, as shown:



28. Enclosed by $r = \cos\theta$ and $r = 1 - \cos\theta$, as shown:



In Exercises 29 – 34, answer the questions involving arc length.

29. Use the arc length formula to compute the arc length of the circle $r = 2$.

30. Use the arc length formula to compute the arc length of the circle $r = 4\sin\theta$.

31. Use the arc length formula to compute the arc length of $r = \cos\theta + \sin\theta$.

32. Use the arc length formula to compute the arc length of the cardioid $r = 1 + \cos\theta$. (Hint: apply the formula, simplify, then use a Power–Reducing Formula to convert $1 + \cos\theta$ into a square.)

33. Approximate the arc length of one petal of the rose curve $r = \sin(3\theta)$ with Simpson's Rule and $n = 4$.

34. Let $x(\theta) = f(\theta)\cos\theta$ and $y(\theta) = f(\theta)\sin\theta$. Show, as suggested by the text, that

$$x'(\theta)^2 + y'(\theta)^2 = f'(\theta)^2 + f(\theta)^2.$$

In Exercises 35 – 40, answer the questions involving surface area.

35. Find the surface area of the sphere formed by revolving the circle $r = 2$ about the initial ray.

36. Find the surface area of the sphere formed by revolving the circle $r = 2\cos\theta$ about the initial ray.

37. Find the surface area of the solid formed by revolving the cardioid $r = 1 + \cos\theta$ about the initial ray.

38. Find the surface area of the solid formed by revolving the circle $r = 2\cos\theta$ about the line $\theta = \pi/2$.

39. Find the surface area of the solid formed by revolving the line $r = 3\sec\theta$, $-\pi/4 \le \theta \le \pi/4$, about the line $\theta = \pi/2$.

40. Find the surface area of the solid formed by revolving the line $r = 3\sec\theta$, $0 \le \theta \le \pi/4$, about the initial ray.

11: Functions of Several Variables

A function of the form $y = f(x)$ is a function of a single variable; given a value of x, we can find a value y. There are many situations where a desired quantity is a function of two or more variables. For instance, wind chill is measured by knowing the temperature and wind speed; the volume of a gas can be computed knowing the pressure and temperature of the gas; to compute a baseball player's batting average, one needs to know the number of hits and the number of at–bats.

This chapter studies **multivariable** functions, that is, functions with more than one input. Examples from economics, air traffic control, ecology, etc. can involve hundreds, if not thousands of variables, but to keep our lives simple, we'll focus on functions of two or three variables.

You'll recall from Calculus I that a lot can be learned about a function from its graph. The graph of a function of one variable is a subset of the two-dimensional Cartesian plane: we need one coordinate for the input (independent) variable, and one for the output (dependent) variable. To graph a function of two variables we need three dimensions: two for the inputs, and one for the output. This chapter begins with moving our mathematics out of the plane and into "space." That is, we begin to think mathematically not only in two dimensions, but in three.

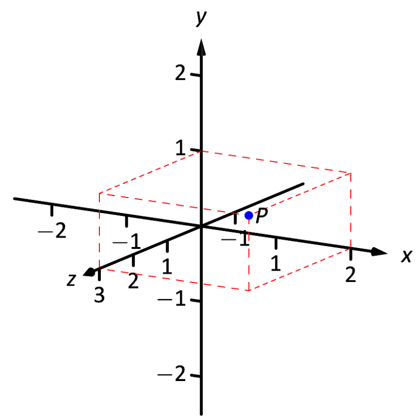


Figure 11.1.1: Plotting the point $P = (2, 1, 3)$ in space.

Graphs of functions of three or more variables are easy to define mathematically, but difficult (if not impossible) to picture. This is because their graphs live in spaces of four dimensions or more, and the human brain is just not set up to visualize objects in more than three dimensions.

11.1 Introduction to Cartesian Coordinates in Space

Up to this point in this text we have considered mathematics in a 2–dimensional world. We have plotted graphs on the x-y plane using rectangular and polar coordinates and found the area of regions in the plane. We have considered properties of *solid* objects, such as volume and surface area, but only by first defining a curve in the plane and then rotating it out of the plane.

While there is wonderful mathematics to explore in "2D," we live in a "3D" world and eventually we will want to apply mathematics involving this third dimension. In this section we introduce Cartesian coordinates in space and explore basic surfaces. This will lay a foundation for much of what we do in the remainder of the text.

Each point P in space can be represented with an ordered triple, $P = (a, b, c)$, where a, b and c represent the relative position of P along the x-, y- and z-axes, respectively. Each axis is perpendicular to the other two.

Visualizing points in space on paper can be problematic, as we are trying to represent a 3-dimensional concept on a 2–dimensional medium. We cannot draw three lines representing the three axes in which each line is perpendicular to the other two. Despite this issue, standard conventions exist for plotting shapes in space that we will discuss that are more than adequate.

One convention is that the axes must conform to the **right hand rule**. This rule states that when the index finger of the right hand is extended in the direction of the positive x-axis, and the middle finger (bent "inward" so it is perpendicular to the palm) points along the positive y-axis, then the extended thumb will point in the direction of the positive z-axis. (It may take some thought to verify this, but this system is inherently different from the one created by using the "left hand rule.")

As long as the coordinate axes are positioned so that they follow this rule, it does not matter how the axes are drawn on paper. There are two popular methods that we briefly discuss.

In Figure 11.1.1 we see the point $P = (2, 1, 3)$ plotted on a set of axes. The basic convention here is that the x-y plane is drawn in its standard way, with the z-axis down to the left. The perspective is that the paper represents the x-y plane and the positive z axis is coming up, off the page. This method is preferred by many engineers. Because it can be hard to tell where a single point lies in relation to all the axes, dashed lines have been added to let one see how far along each axis the point lies.

One can also consider the x-y plane as being a horizontal plane in, say, a room, where the positive z-axis is pointing up. When one steps back and looks at this room, one might draw the axes as shown in Figure 11.1.2. The same point P is drawn, again with dashed lines. This point of view is preferred by most mathematicians, and is the convention adopted by this text.

Just as the x- and y-axes divide the plane into four *quadrants*, the x-, y-, and z-coordinate planes divide space into eight *octants*. The octant in which x, y, and z are positive is called the **first octant**. We do not name the other seven octants in this text.

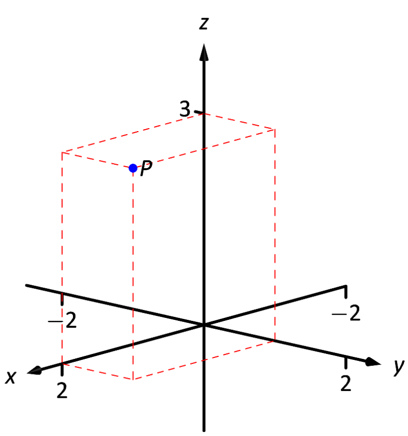


Figure 11.1.2: Plotting the point $P = (2, 1, 3)$ in space with a perspective used in this text.

Measuring Distances

It is of critical importance to know how to measure distances between points in space. The formula for doing so is based on measuring distance in the plane, and is known (in both contexts) as the Euclidean measure of distance.

Definition 11.1.1 Distance In Space

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be points in space. The distance D between P and Q is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

We refer to the line segment that connects points P and Q in space as $\overline{PQ}$, and refer to the length of this segment as $\|\overline{PQ}\|$. The above distance formula allows us to compute the length of this segment.

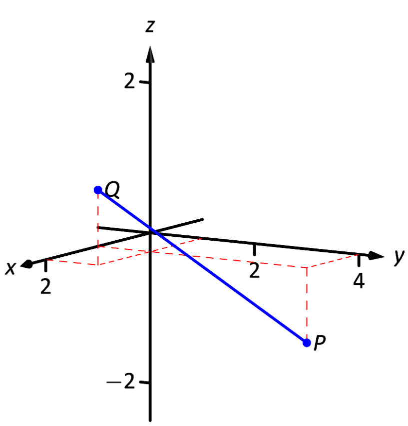


Figure 11.1.3: Plotting points P and Q in Example 11.1.1.

Example 11.1.1 Length of a line segment

Let $P = (1, 4, -1)$ and let $Q = (2, 1, 1)$. Draw the line segment $\overline{PQ}$ and find its length.

Solution The points P and Q are plotted in Figure 11.1.3; no special consideration need be made to draw the line segment connecting these two points; simply connect them with a straight line. One *cannot* actually measure this line on the page and deduce anything meaningful; its true length must be measured analytically. Applying Definition 11.1.1, we have

$$\|\overline{PQ}\| = \sqrt{(2-1)^2 + (1-4)^2 + (1-(-1))^2} = \sqrt{14} \approx 3.74.$$

Spheres

Just as a circle is the set of all points in the *plane* equidistant from a given point (its center), a sphere is the set of all points in *space* that are equidistant from a given point. Definition 11.1.1 allows us to write an equation of the sphere.

We start with a point $C = (a, b, c)$ which is to be the center of a sphere with radius r. If a point $P = (x, y, z)$ lies on the sphere, then P is r units from C; that is,

$$\|\overline{PC}\| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r.$$

Squaring both sides, we get the standard equation of a sphere in space with center at $C = (a, b, c)$ with radius r, as given in the following Key Idea.

Key Idea 11.1.1 Standard Equation of a Sphere in Space

The standard equation of the sphere with radius r, centred at $C = (a, b, c)$, is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

Example 11.1.2 Equation of a sphere

Find the center and radius of the sphere defined by $x^2+2x+y^2-4y+z^2-6z = 2$.

Solution To determine the center and radius, we must put the equation in standard form. This requires us to complete the square (three times).

$$\begin{aligned} x^2 + 2x + y^2 - 4y + z^2 - 6z &= 2 \\ (x^2 + 2x + 1) + (y^2 - 4y + 4) + (z^2 - 6z + 9) - 14 &= 2 \\ (x+1)^2 + (y-2)^2 + (z-3)^2 &= 16 \end{aligned}$$

The sphere is centred at $(-1, 2, 3)$ and has a radius of 4.

The equation of a sphere is an example of an implicit function defining a surface in space. In the case of a sphere, the variables x, y and z are all used. We now consider situations where surfaces are defined where one or two of these variables are absent.

Introduction to Planes in Space

The coordinate axes naturally define three planes (shown in Figure 11.1.4), the **coordinate planes**: the x-y plane, the y-z plane and the x-z plane. The x-y plane is characterized as the set of all points in space where the z-value is 0. This, in fact, gives us an equation that describes this plane: $z = 0$. Likewise, the x-z plane is all points where the y-value is 0, characterized by $y = 0$.

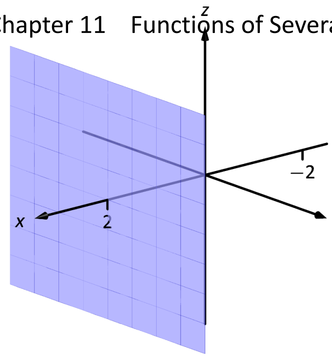
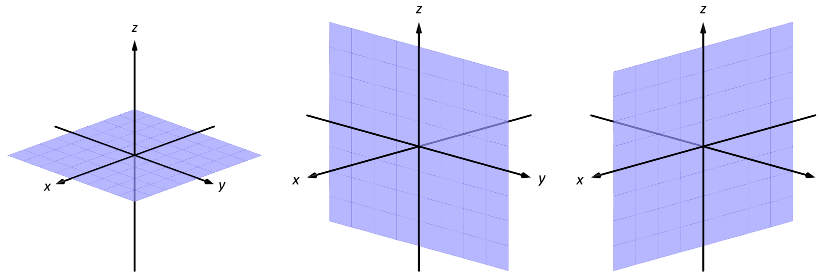


Figure 11.1.5: The plane $x = 2$.



the x-y plane the y-z plane the x-z plane

Figure 11.1.4: The coordinate planes.

The equation $x = 2$ describes all points in space where the x-value is 2. This is a plane, parallel to the y-z coordinate plane, shown in Figure 11.1.5.

Example 11.1.3 Regions defined by planes
Sketch the region defined by the inequalities $-1 \leq y \leq 2$.

Solution The region is all points between the planes $y = -1$ and $y = 2$. These planes are sketched in Figure 11.1.6, which are parallel to the x-z plane. Thus the region extends infinitely in the x and z directions, and is bounded by planes in the y direction.

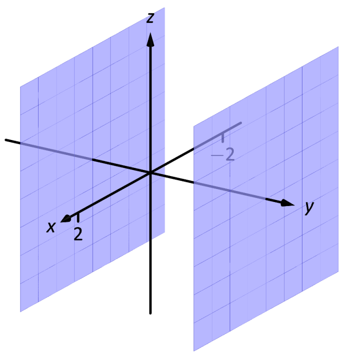


Figure 11.1.6: Sketching the boundaries of a region in Example 11.1.3.

Cylinders

The equation $x = 1$ obviously lacks the y and z variables, meaning it defines points where the y and z coordinates can take on any value. Now consider the equation $x^2 + y^2 = 1$ *in space*. In *the plane*, this equation describes a circle of radius 1, centred at the origin. In space, the z coordinate is not specified, meaning it can take on any value. In Figure 11.1.7 (a), we show part of the graph of the equation $x^2 + y^2 = 1$ by sketching 3 circles: the bottom one has a constant z-value of -1.5, the middle one has a z-value of 0 and the top circle has a z-value of 1. By plotting *all* possible z-values, we get the surface shown in Figure 11.1.7(b). This surface looks like a "tube," or a "cylinder"; mathematicians call this surface a **cylinder** for an entirely different reason.

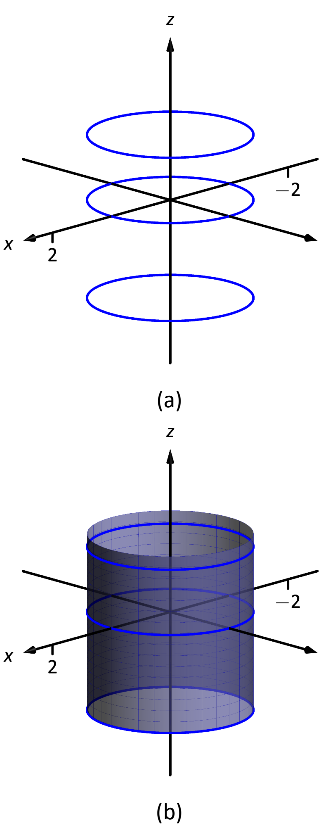


Figure 11.1.7: Sketching $x^2 + y^2 = 1$.

> **Definition 11.1.2 Cylinder**
>
> Let C be a curve in a plane and let L be a line not parallel to C. A **cylinder** is the set of all lines parallel to L that pass through C. The curve C is the **directrix** of the cylinder, and the lines are the **rulings**.

In this text, we consider curves C that lie in planes parallel to one of the coordinate planes, and lines L that are perpendicular to these planes, forming **right cylinders**. Thus the directrix can be defined using equations involving 2 variables, and the rulings will be parallel to the axis of the 3$^{\text{rd}}$ variable.

In the example preceding the definition, the curve $x^2 + y^2 = 1$ in the x-y plane is the directrix and the rulings are lines parallel to the z-axis. (Any circle shown in Figure 11.1.7 can be considered a directrix; we simply choose the one where $z = 0$.) Sample rulings can also be viewed in part (b) of the figure. More examples will help us understand this definition.

Example 11.1.4 Graphing cylinders

Graph the following cylinders.

1. $z = y^2$ 2. $x = \sin z$

SOLUTION

1. We can view the equation $z = y^2$ as a parabola in the y-z plane, as illustrated in Figure 11.1.8(a). As x does not appear in the equation, the rulings are lines through this parabola parallel to the x-axis, shown in (b). These rulings give an idea as to what the surface looks like, drawn in (c).

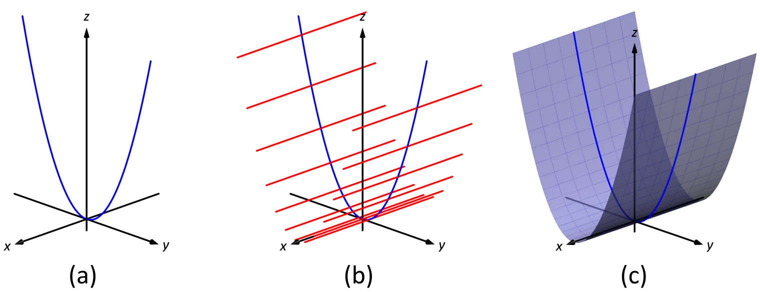


Figure 11.1.8: Sketching the cylinder defined by $z = y^2$.

2. We can view the equation $x = \sin z$ as a sine curve that exists in the x-z plane, as shown in Figure 11.1.9 (a). The rules are parallel to the y axis as the variable y does not appear in the equation $x = \sin z$; some of these are shown in part (b). The surface is shown in part (c) of the figure.

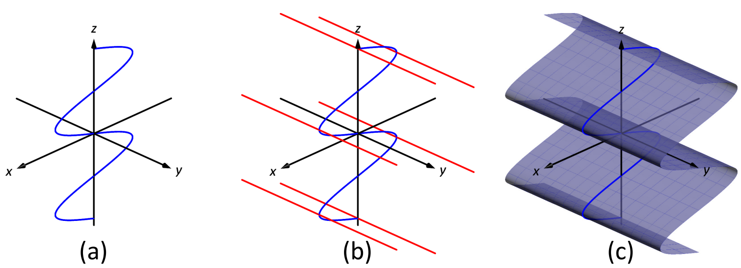


Figure 11.1.9: Sketching the cylinder defined by $x = \sin z$.

Surfaces of Revolution

One of the applications of integration we learned previously was to find the volume of solids of revolution – solids formed by revolving a curve about a horizontal or vertical axis. We now consider how to find the equation of the surface of such a solid.

Consider the surface formed by revolving $y = \sqrt{x}$ about the x-axis. Cross–sections of this surface parallel to the y-z plane are circles, as shown in Figure 11.1.10(a). Each circle has equation of the form $y^2 + z^2 = r^2$ for some radius r. The radius is a function of x; in fact, it is $r(x) = \sqrt{x}$. Thus the equation of the surface shown in Figure 11.1.10b is $y^2 + z^2 = (\sqrt{x})^2$.

We generalize the above principles to give the equations of surfaces formed by revolving curves about the coordinate axes.

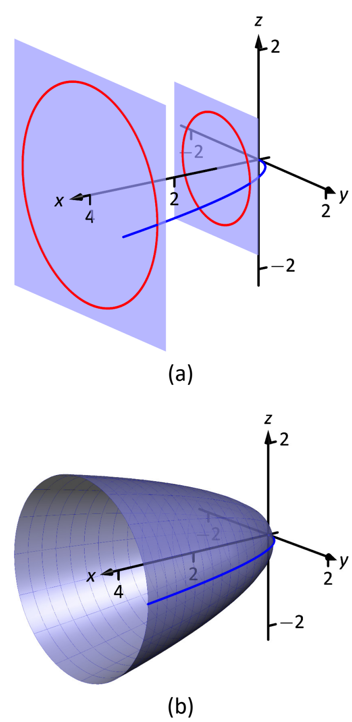


Figure 11.1.10: Introducing surfaces of revolution.

Key Idea 11.1.2 Surfaces of Revolution, Part 1

Let r be a radius function.

1. The equation of the surface formed by revolving $y = r(x)$ or $z = r(x)$ about the x-axis is $y^2 + z^2 = r(x)^2$.
2. The equation of the surface formed by revolving $x = r(y)$ or $z = r(y)$ about the y-axis is $x^2 + z^2 = r(y)^2$.
3. The equation of the surface formed by revolving $x = r(z)$ or $y = r(z)$ about the z-axis is $x^2 + y^2 = r(z)^2$.

Example 11.1.5 Finding equation of a surface of revolution

Let $y = \sin z$ on $[0, \pi]$. Find the equation of the surface of revolution formed by revolving $y = \sin z$ about the z-axis.

Solution Using Key Idea 11.1.2, we find the surface has equation $x^2 + y^2 = \sin^2 z$. The curve is sketched in Figure 11.1.11(a) and the surface is drawn in Figure 11.1.11(b).

Note how the surface (and hence the resulting equation) is the same if we began with the curve $x = \sin z$, which is also drawn in Figure 11.1.11(a).

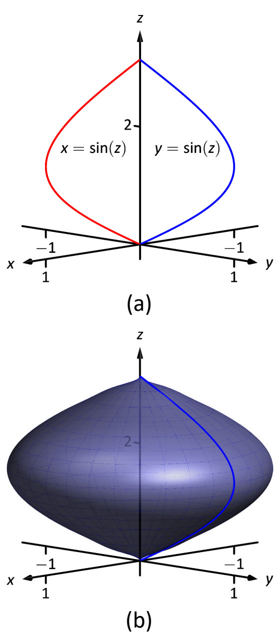


Figure 11.1.11: Revolving $y = \sin z$ about the z-axis in Example 11.1.5.

This particular method of creating surfaces of revolution is limited. For instance, in Example 7.3.4 of Section 7.3 we found the volume of the solid formed by revolving $y = \sin x$ about the y-axis. Our current method of forming surfaces can only rotate $y = \sin x$ about the x-axis. Trying to rewrite $y = \sin x$ as a function of y is not trivial, as simply writing $x = \sin^{-1} y$ only gives part of the region we desire.

What we desire is a way of writing the surface of revolution formed by rotating $y = f(x)$ about the y-axis. We start by first recognizing this surface is the same as revolving $z = f(x)$ about the z-axis. This will give us a more natural way of viewing the surface.

A value of x is a measurement of distance from the z-axis. At the distance r, we plot a z-height of $f(r)$. When rotating $f(x)$ about the z-axis, we want all points a distance of r from the z-axis in the x-y plane to have a z-height of $f(r)$. All such points satisfy the equation $r^2 = x^2 + y^2$; hence $r = \sqrt{x^2 + y^2}$. Replacing r with $\sqrt{x^2 + y^2}$ in $f(r)$ gives $z = f(\sqrt{x^2 + y^2})$. This is the equation of the surface.

Key Idea 11.1.3 Surfaces of Revolution, Part 2

Let $z = f(x)$, $x \geq 0$, be a curve in the x-z plane. The surface formed by revolving this curve about the z-axis has equation $z = f\big(\sqrt{x^2 + y^2}\big)$.

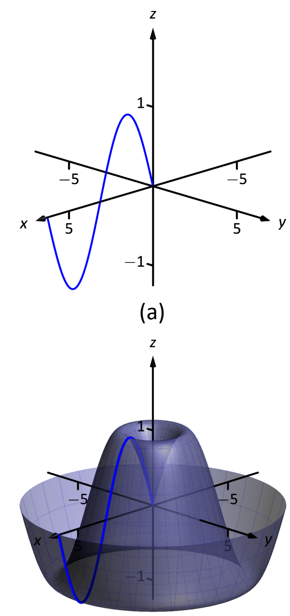
Example 11.1.6 Finding equation of surface of revolution

Find the equation of the surface found by revolving $z = \sin x$ about the z-axis.

SOLUTION Using Key Idea 11.1.3, the surface has equation $z = \sin\left(\sqrt{x^2 + y^2}\right)$. The curve and surface are graphed in Figure 11.1.12.

Quadric Surfaces

Spheres, planes and cylinders are important surfaces to understand. We now consider one last type of surface, a **quadric surface**. The definition may look intimidating, but we will show how to analyze these surfaces in an illuminating way.



(a)

(b)

> **Definition 11.1.3 Quadric Surface**
>
> A **quadric surface** is the graph of the general second–degree equation in three variables:
>
> $$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

When the coefficients D, E or F are not zero, the basic shapes of the quadric surfaces are rotated in space. We will focus on quadric surfaces where these coefficients are 0; we will not consider rotations. There are six basic quadric surfaces: the elliptic paraboloid, elliptic cone, ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, and the hyperbolic paraboloid.

We study each shape by considering **traces**, that is, intersections of each surface with a plane parallel to a coordinate plane. For instance, consider the elliptic paraboloid $z = x^2/4 + y^2$, shown in Figure 11.1.13. If we intersect this shape with the plane $z = d$ (i.e., replace z with d), we have the equation:

$$d = \frac{x^2}{4} + y^2.$$

Divide both sides by d:

$$1 = \frac{x^2}{4d} + \frac{y^2}{d}.$$

This describes an ellipse – so cross sections parallel to the x-y coordinate plane are ellipses. This ellipse is drawn in the figure.

Now consider cross sections parallel to the x-z plane. For instance, letting $y = 0$ gives the equation $z = x^2/4$, clearly a parabola. Intersecting with the plane $x = 0$ gives a cross section defined by $z = y^2$, another parabola. These parabolas are also sketched in the figure.

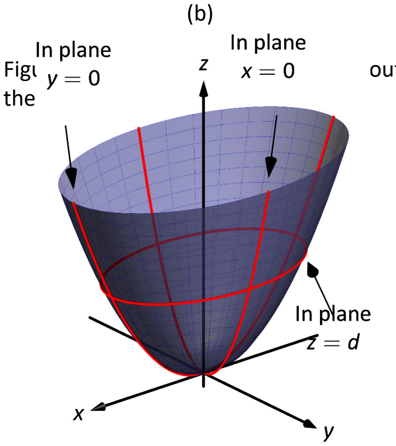
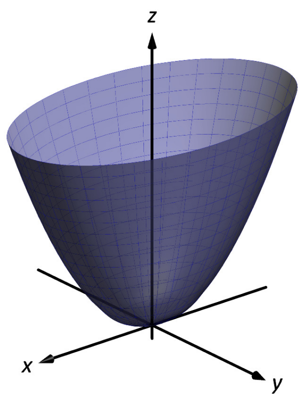


Figure 11.1.13: The elliptic paraboloid $z = x^2/4 + y^2$.

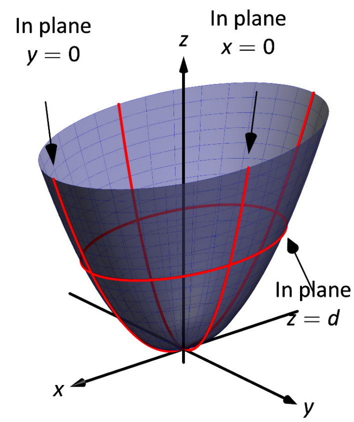
Thus we see where the elliptic paraboloid gets its name: some cross sections are ellipses, and others are parabolas.

Such an analysis can be made with each of the quadric surfaces. We give a sample equation of each, provide a sketch with representative traces, and describe these traces.

Elliptic Paraboloid, $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$



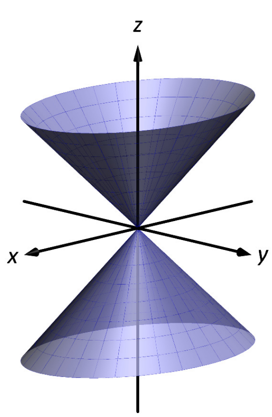
Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Ellipse



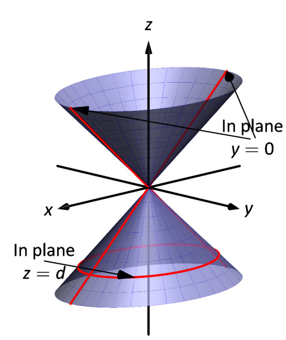
One variable in the equation of the elliptic paraboloid will be raised to the first power; above, this is the z variable. The paraboloid will "open" in the direction of this variable's axis. Thus $x = y^2/a^2 + z^2/b^2$ is an elliptic paraboloid that opens along the x-axis.

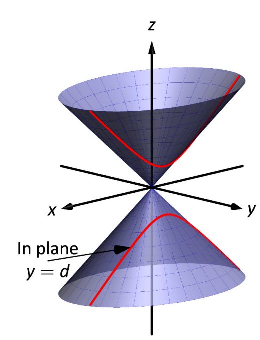
Multiplying the right hand side by (-1) defines an elliptic paraboloid that "opens" in the opposite direction.

Elliptic Cone, $z^2 = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$



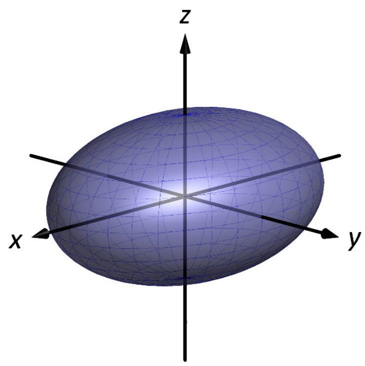
Plane	Trace
$x = 0$	Crossed Lines
$y = 0$	Crossed Lines
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse



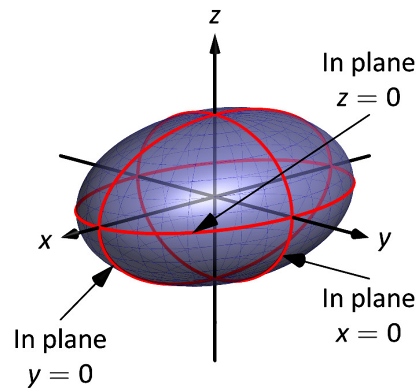


One can rewrite the equation as $z^2 - x^2/a^2 - y^2/b^2 = 0$. The one variable with a positive coefficient corresponds to the axis that the cones "open" along.

Ellipsoid, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

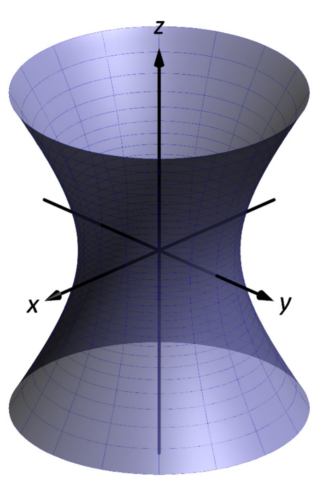


Plane	Trace
$x=d$	Ellipse
$y=d$	Ellipse
$z=d$	Ellipse

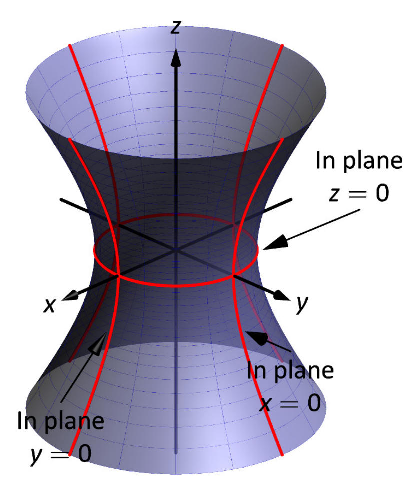


If $a=b=c\neq 0$, the ellipsoid is a sphere with radius a; compare to Key Idea 11.1.1.

Hyperboloid of One Sheet, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$

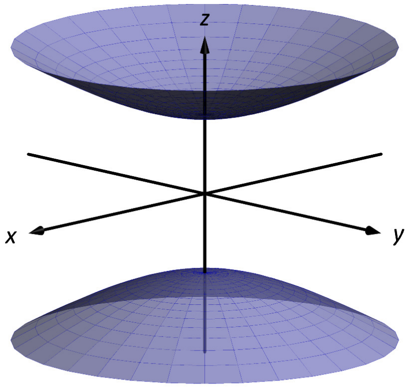


Plane	Trace
$x=d$	Hyperbola
$y=d$	Hyperbola
$z=d$	Ellipse

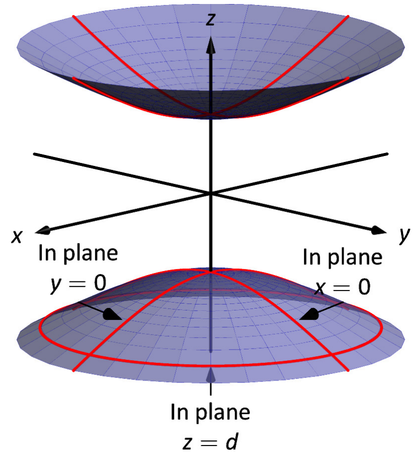


The one variable with a negative coefficient corresponds to the axis that the hyperboloid "opens" along.

Hyperboloid of Two Sheets, $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

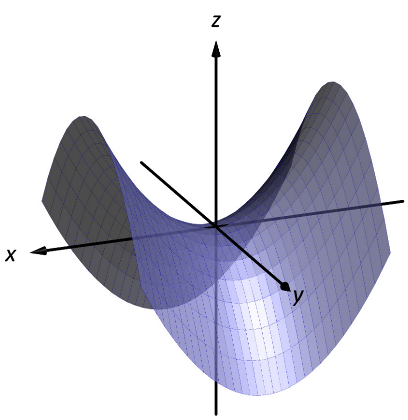


Plane	Trace
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

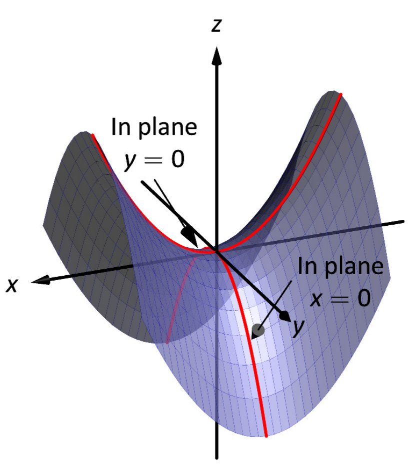


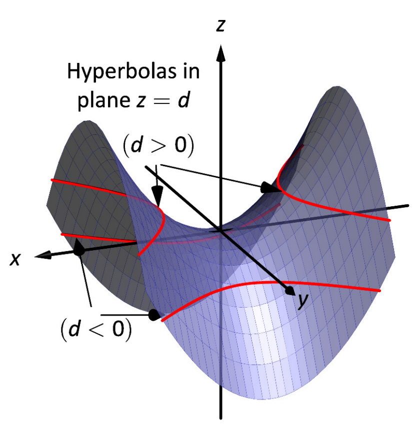
The one variable with a positive coefficient corresponds to the axis that the hyperboloid "opens" along. In the case illustrated, when $|d| < |c|$, there is no trace.

Hyperbolic Paraboloid, $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$



Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Hyperbola





 The parabolic traces will open along the axis of the one variable that is raised to the first power.

Example 11.1.7 Sketching quadric surfaces

Sketch the quadric surface defined by the given equation.

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$ 2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1.$ 3. $z = y^2 - x^2.$

Solution

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$:

   We first identify the quadric by pattern–matching with the equations given previously. Only two surfaces have equations where one variable is raised to the first power, the elliptic paraboloid and the hyperbolic paraboloid. In the latter case, the other variables have different signs, so we conclude that this describes a hyperbolic paraboloid. As the variable with the first power is y, we note the paraboloid opens along the y-axis.

   To make a decent sketch by hand, we need only draw a few traces. In this case, the traces $x = 0$ and $z = 0$ form parabolas that outline the shape.

   $x = 0$: The trace is the parabola $y = z^2/16$

   $z = 0$: The trace is the parabola $y = x^2/4$.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 11.1.14(a). This is enough to give an idea of what the paraboloid looks like. The surface is filled in in (b).

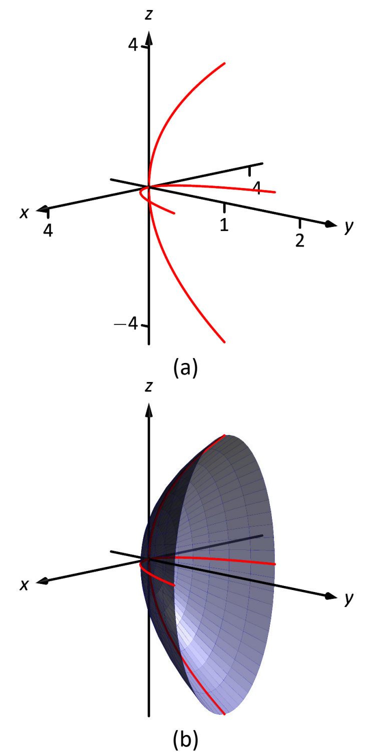


Figure 11.1.14: Sketching an elliptic paraboloid.

2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$:

   This is an ellipsoid. We can get a good idea of its shape by drawing the traces in the coordinate planes.

   $x = 0$: The trace is the ellipse $\dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$. The major axis is along the y–axis with length 6 (as $b = 3$, the length of the axis is 6); the minor axis is along the z-axis with length 4.

   $y = 0$: The trace is the ellipse $x^2 + \dfrac{z^2}{4} = 1$. The major axis is along the z-axis, and the minor axis has length 2 along the x-axis.

   $z = 0$: The trace is the ellipse $x^2 + \dfrac{y^2}{9} = 1$, with major axis along the y-axis.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 11.1.15(a). Filling in the surface gives Figure 11.1.15(b).

3. $z = y^2 - x^2$:

   This defines a hyperbolic paraboloid, very similar to the one shown in the gallery of quadric sections. Consider the traces in the $y - z$ and $x - z$ planes:

   $x = 0$: The trace is $z = y^2$, a parabola opening up in the $y - z$ plane.

   $y = 0$: The trace is $z = -x^2$, a parabola opening down in the $x - z$ plane.

   Sketching these two parabolas gives a sketch like that in Figure 11.1.16(a), and filling in the surface gives a sketch like (b).

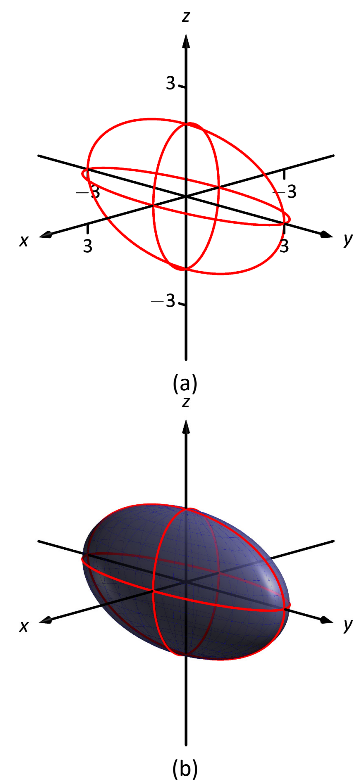
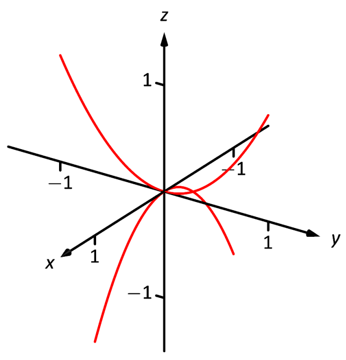


Figure 11.1.15: Sketching an ellipsoid.



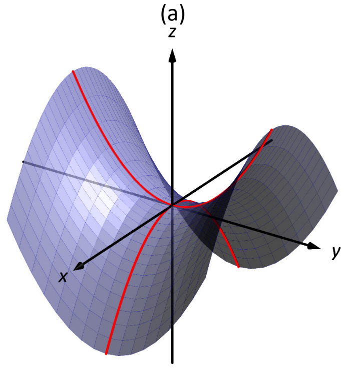


Figure 11.1.16: Sketching a hyperbolic paraboloid.

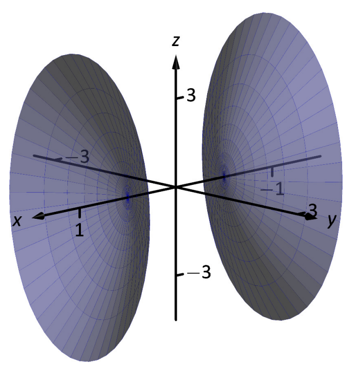


Figure 11.1.17: A possible equation of this quadric surface is found in Example 11.1.8.

Example 11.1.8 Identifying quadric surfaces

Consider the quadric surface shown in Figure 11.1.17. Which of the following equations best fits this surface?

(a) $x^2 - y^2 - \dfrac{z^2}{9} = 0$ (c) $z^2 - x^2 - y^2 = 1$

(b) $x^2 - y^2 - z^2 = 1$ (d) $4x^2 - y^2 - \dfrac{z^2}{9} = 1$

Solution The image clearly displays a hyperboloid of two sheets. The gallery informs us that the equation will have a form similar to $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

We can immediately eliminate option (a), as the constant in that equation is not 1.

The hyperboloid "opens" along the x-axis, meaning x must be the only variable with a positive coefficient, eliminating (c).

The hyperboloid is wider in the z-direction than in the y-direction, so we need an equation where $c > b$. This eliminates (b), leaving us with (d). We should verify that the equation given in (d), $4x^2 - y^2 - \frac{z^2}{9} = 1$, fits.

We already established that this equation describes a hyperboloid of two sheets that opens in the x-direction and is wider in the z-direction than in the y. Now note the coefficient of the x-term. Rewriting $4x^2$ in standard form, we have: $4x^2 = \dfrac{x^2}{(1/2)^2}$. Thus when $y = 0$ and $z = 0$, x must be $1/2$; i.e., each hyperboloid "starts" at $x = 1/2$. This matches our figure.

We conclude that $4x^2 - y^2 - \dfrac{z^2}{9} = 1$ best fits the graph.

Exercises 11.1

Terms and Concepts

1. Axes drawn in space must conform to the _________ _________ rule.

2. In the plane, the equation $x = 2$ defines a _________; in space, $x = 2$ defines a _________.

3. In the plane, the equation $y = x^2$ defines a _________; in space, $y = x^2$ defines a _________.

4. Which quadric surface looks like a Pringles® chip?

5. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the x-axis, what quadric surface is formed?

6. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the y-axis, what quadric surface is formed?

Problems

7. The points $A = (1, 4, 2)$, $B = (2, 6, 3)$ and $C = (4, 3, 1)$ form a triangle in space. Find the distances between each pair of points and determine if the triangle is a right triangle.

8. The points $A = (1, 1, 3)$, $B = (3, 2, 7)$, $C = (2, 0, 8)$ and $D = (0, -1, 4)$ form a quadrilateral $ABCD$ in space. Is this a parallelogram?

9. Find the center and radius of the sphere defined by $x^2 - 8x + y^2 + 2y + z^2 + 8 = 0$.

10. Find the center and radius of the sphere defined by $x^2 + y^2 + z^2 + 4x - 2y - 4z + 4 = 0$.

In Exercises 11 – 14, describe the region in space defined by the inequalities.

11. $x^2 + y^2 + z^2 < 1$

12. $0 \leq x \leq 3$

13. $x \geq 0,\ y \geq 0,\ z \geq 0$

14. $y \geq 3$

In Exercises 15 – 18, sketch the cylinder in space.

15. $z = x^3$

16. $y = \cos z$

17. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

18. $y = \dfrac{1}{x}$

In Exercises 19 – 22, give the equation of the surface of revolution described.

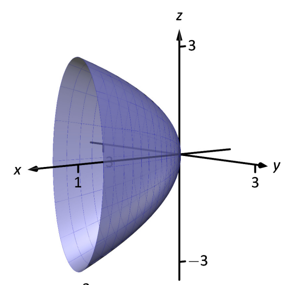
19. Revolve $z = \dfrac{1}{1 + y^2}$ about the y-axis.

20. Revolve $y = x^2$ about the x-axis.

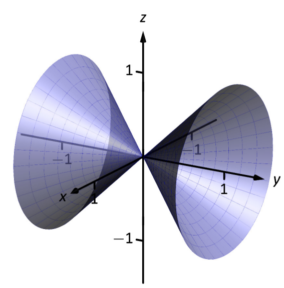
21. Revolve $z = x^2$ about the z-axis.

22. Revolve $z = 1/x$ about the z-axis.

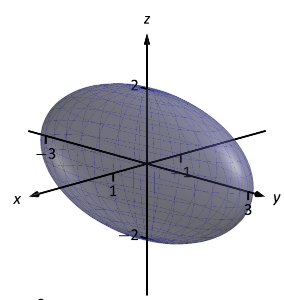
In Exercises 23 – 26, a quadric surface is sketched. Determine which of the given equations best fits the graph.

23. 

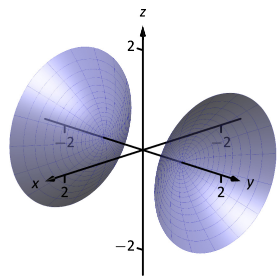
(a) $x = y^2 + \dfrac{z^2}{9}$ (b) $x = y^2 + \dfrac{z^2}{3}$

24. 

(a) $x^2 - y^2 - z^2 = 0$ (b) $x^2 - y^2 + z^2 = 0$

25. 

(a) $x^2 + \dfrac{y^2}{3} + \dfrac{z^2}{2} = 1$ (b) $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$

26. 

(a) $y^2 - x^2 - z^2 = 1$

(b) $y^2 + x^2 - z^2 = 1$

In Exercises 27 – 32, sketch the quadric surface.

27. $z - y^2 + x^2 = 0$

28. $z^2 = x^2 + \frac{y^2}{4}$

29. $x = -y^2 - z^2$

30. $16x^2 - 16y^2 - 16z^2 = 1$

31. $\frac{x^2}{9} - y^2 + \frac{z^2}{25} = 1$

32. $4x^2 + 2y^2 + z^2 = 4$

11.2 Introduction to Multivariable Functions

Definition 11.2.1 Function of Two Variables

Let D be a subset of $\mathbb{R}^2$. A **function f of two variables** is a rule that assigns each pair (x, y) in D a value $z = f(x, y)$ in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

Example 11.2.1 Understanding a function of two variables

Let $z = f(x, y) = x^2 - y$. Evaluate $f(1, 2)$, $f(2, 1)$, and $f(-2, 4)$; find the domain and range of f.

SOLUTION Using the definition $f(x, y) = x^2 - y$, we have:

$$\begin{aligned} f(1,2) &= 1^2 - 2 = -1 \\ f(2,1) &= 2^2 - 1 = 3 \\ f(-2,4) &= (-2)^2 - 4 = 0 \end{aligned}$$

The domain is not specified, so we take it to be all possible pairs in $\mathbb{R}^2$ for which f is defined. In this example, f is defined for *all* pairs (x, y), so the domain D of f is $\mathbb{R}^2$.

The output of f can be made as large or small as possible; any real number r can be the output. (In fact, given any real number r, $f(0, -r) = r$.) So the range R of f is $\mathbb{R}$.

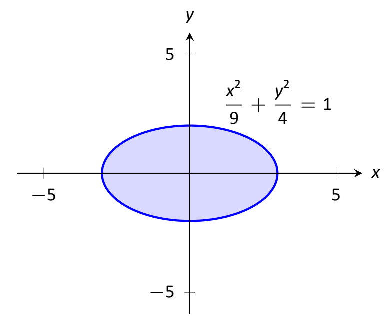


Figure 11.2.1: Illustrating the domain of $f(x, y)$ in Example 11.2.2.

Example 11.2.2 Understanding a function of two variables

Let $f(x, y) = \sqrt{1 - \dfrac{x^2}{9} - \dfrac{y^2}{4}}$. Find the domain and range of f.

SOLUTION The domain is all pairs (x, y) allowable as input in f. Because of the square–root, we need (x, y) such that $0 \le 1 - \frac{x^2}{9} - \frac{y^2}{4}$:

$$\begin{aligned} 0 &\le 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &\le 1 \end{aligned}$$

The above equation describes an ellipse and its interior as shown in Figure 11.2.1. We can represent the domain D graphically with the figure; in set notation, we can write $D = \{(x, y) | \frac{x^2}{9} + \frac{y^2}{4} \le 1\}$.

The range is the set of all possible output values. The square–root ensures that all output is ≥ 0. Since the x and y terms are squared, then subtracted, inside the square–root, the largest output value comes at $x = 0, y = 0$: $f(0, 0) = 1$. Thus the range R is the interval $[0, 1]$.

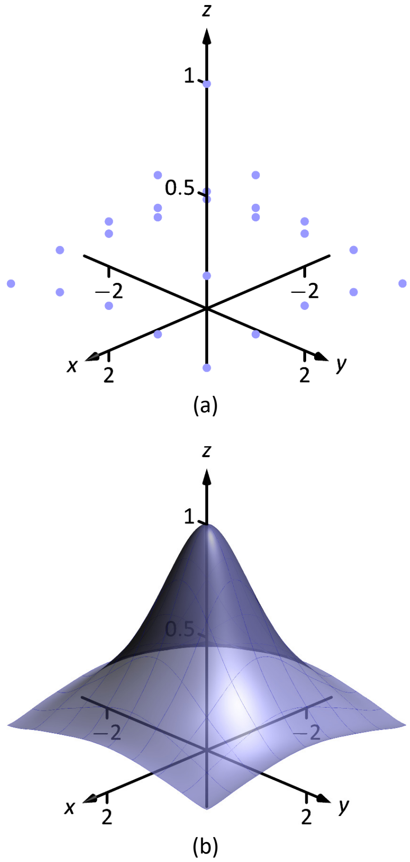


Figure 11.2.2: Graphing a function of two variables.

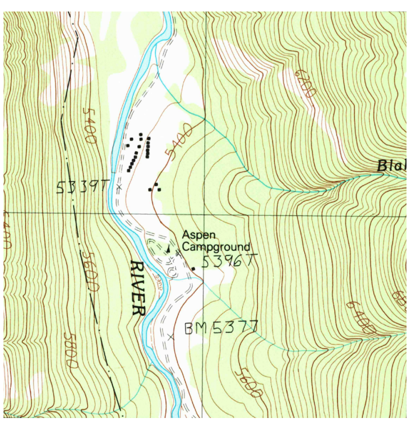


Figure 11.2.3: A topographical map displays elevation by drawing contour lines, along with the elevation is constant. Sample taken from the public domain USGS Digital Raster Graphics, `http://topmaps.usgs.gove/drg/`.

Graphing Functions of Two Variables

The **graph** of a function f of two variables is the set of all points $\big(x, y, f(x, y)\big)$ where (x, y) is in the domain of f. This creates a **surface** in space.

One can begin sketching a graph by plotting points, but this has limitations. Consider Figure 11.2.2(a) where 25 points have been plotted of

$$f(x, y) = \frac{1}{x^2 + y^2 + 1}.$$

More points have been plotted than one would reasonably want to do by hand, yet it is not clear at all what the graph of the function looks like. Technology allows us to plot lots of points, connect adjacent points with lines and add shading to create a graph like Figure 11.2.2b which does a far better job of illustrating the behavior of f.

While technology is readily available to help us graph functions of two variables, there is still a paper–and–pencil approach that is useful to understand and master as it, combined with high–quality graphics, gives one great insight into the behaviour of a function. This technique is known as sketching **level curves**.

Level Curves

It may be surprising to find that the problem of representing a three dimensional surface on paper is familiar to most people (they just don't realize it). Topographical maps, like the one shown in Figure 11.2.3, represent the surface of Earth by indicating points with the same elevation with **contour lines**. The elevations marked are equally spaced; in this example, each thin line indicates an elevation change in 50 ft increments and each thick line indicates a change of 200 ft. When lines are drawn close together, elevation changes rapidly (as one does not have to travel far to rise 50 ft). When lines are far apart, such as near "Aspen Campground," elevation changes more gradually as one has to walk farther to rise 50 ft.

Given a function $z = f(x, y)$, we can draw a "topographical map" of f by drawing **level curves** (or, contour lines). A level curve at $z = c$ is a curve in the x-y plane such that for all points (x, y) on the curve, $f(x, y) = c$.

When drawing level curves, it is important that the c values are spaced equally apart as that gives the best insight to how quickly the "elevation" is changing. Examples will help one understand this concept.

Example 11.2.3 Drawing Level Curves

Let $f(x, y) = \sqrt{1 - \dfrac{x^2}{9} - \dfrac{y^2}{4}}$. Find the level curves of f for $c = 0, 0.2, 0.4, 0.6, 0.8$ and 1.

Solution Consider first $c = 0$. The level curve for $c = 0$ is the set of all points (x, y) such that $0 = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$. Squaring both sides gives us

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

an ellipse centred at $(0, 0)$ with horizontal major axis of length 6 and minor axis of length 4. Thus for any point (x, y) on this curve, $f(x, y) = 0$.

Now consider the level curve for $c = 0.2$

$$\begin{aligned} 0.2 &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\ 0.04 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 0.96 \\ \frac{x^2}{8.64} + \frac{y^2}{3.84} &= 1. \end{aligned}$$

This is also an ellipse, where $a = \sqrt{8.64} \approx 2.94$ and $b = \sqrt{3.84} \approx 1.96$.

In general, for $z = c$, the level curve is:

$$\begin{aligned} c &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\ c^2 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 1 - c^2 \\ \frac{x^2}{9(1-c^2)} + \frac{y^2}{4(1-c^2)} &= 1, \end{aligned}$$

ellipses that are decreasing in size as c increases. A special case is when $c = 1$; there the ellipse is just the point $(0, 0)$.

The level curves are shown in Figure 11.2.4(a). Note how the level curves for $c = 0$ and $c = 0.2$ are very, very close together: this indicates that f is growing rapidly along those curves.

In Figure 11.2.4(b), the curves are drawn on a graph of f in space. Note how the elevations are evenly spaced. Near the level curves of $c = 0$ and $c = 0.2$ we can see that f indeed is growing quickly.

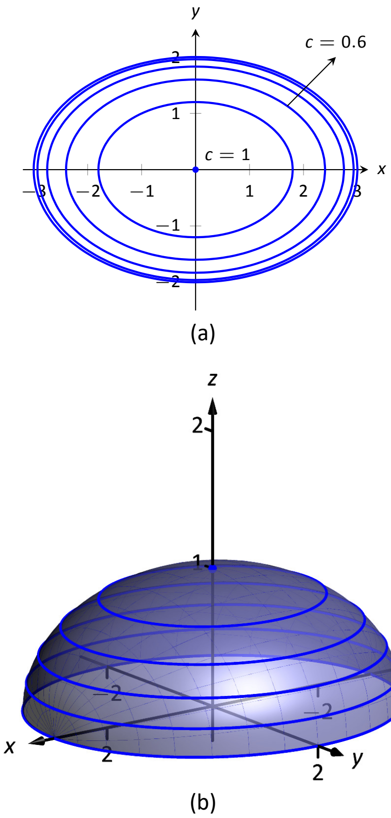


Figure 11.2.4: Graphing the level curves in Example 11.2.3.

Example 11.2.4 Analyzing Level Curves

Let $f(x, y) = \dfrac{x + y}{x^2 + y^2 + 1}$. Find the level curves for $z = c$.

Solution We begin by setting $f(x, y) = c$ for an arbitrary c and seeing if algebraic manipulation of the equation reveals anything significant.

$$\begin{aligned} \frac{x + y}{x^2 + y^2 + 1} &= c \\ x + y &= c(x^2 + y^2 + 1). \end{aligned}$$

We recognize this as a circle, though the center and radius are not yet clear. By completing the square, we can obtain:

$$\left(x - \frac{1}{2c}\right)^2 + \left(y - \frac{1}{2c}\right)^2 = \frac{1}{2c^2} - 1,$$

a circle centred at $\big(1/(2c), 1/(2c)\big)$ with radius $\sqrt{1/(2c^2) - 1}$, where $|c| < 1/\sqrt{2}$. The level curves for $c = \pm 0.2$, ± 0.4 and ± 0.6 are sketched in Figure 11.2.5(a). To help illustrate "elevation," we use thicker lines for c values near 0, and dashed lines indicate where $c < 0$.

There is one special level curve, when $c = 0$. The level curve in this situation is $x + y = 0$, the line $y = -x$.

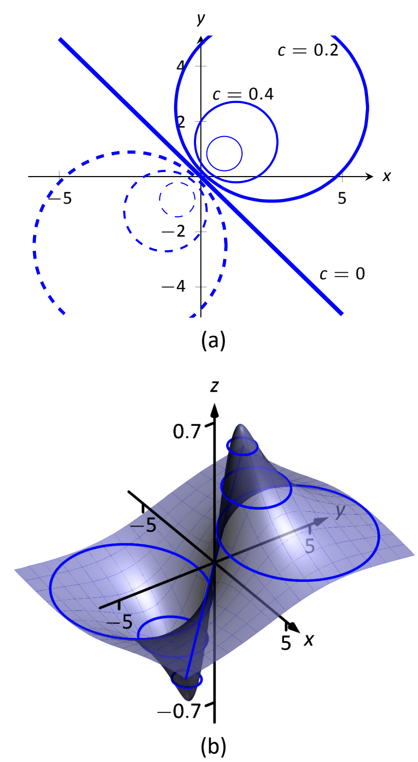


Figure 11.2.5: Graphing the level curves in Example 11.2.4.

In Figure 11.2.5(b) we see a graph of the surface. Note how the y-axis is pointing away from the viewer to more closely resemble the orientation of the level curves in (a).

Seeing the level curves helps us understand the graph. For instance, the graph does not make it clear that one can "walk" along the line $y = -x$ without elevation change, though the level curve does.

Functions of Three Variables

We extend our study of multivariable functions to functions of three variables. (One can make a function of as many variables as one likes; we limit our study to three variables.)

Definition 11.2.2 Function of Three Variables

Let D be a subset of $\mathbb{R}^3$. A **function f of three variables** is a rule that assigns each triple (x, y, z) in D a value $w = f(x, y, z)$ in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

Note how this definition closely resembles that of Definition 11.2.1.

Example 11.2.5 Understanding a function of three variables

Let $f(x, y, z) = \dfrac{x^2 + z + 3\sin y}{x + 2y - z}$. Evaluate f at the point $(3, 0, 2)$ and find the domain and range of f.

Solution $$f(3, 0, 2) = \frac{3^2 + 2 + 3\sin 0}{3 + 2(0) - 2} = 11.$$

As the domain of f is not specified, we take it to be the set of all triples (x, y, z) for which $f(x, y, z)$ is defined. As we cannot divide by 0, we find the domain D is

$$D = \{(x, y, z) \mid x + 2y - z \neq 0\}.$$

We recognize that the set of all points in $\mathbb{R}^3$ that *are not* in D form a plane in space that passes through the origin (with normal vector $\langle 1, 2, -1\rangle$).

We determine the range R is $\mathbb{R}$; that is, all real numbers are possible outputs of f. There is no set way of establishing this. Rather, to get numbers near 0 we can let $y = 0$ and choose $z \approx -x^2$. To get numbers of arbitrarily large magnitude, we can let $z \approx x + 2y$.

Level Surfaces

It is very difficult to produce a meaningful graph of a function of three variables. A function of *one* variable is a *curve* drawn in *2* dimensions; a function of *two* variables is a *surface* drawn in *3* dimensions; a function of *three* variables is a *hypersurface* drawn in *4* dimensions.

There are a few techniques one can employ to try to "picture" a graph of three variables. One is an analogue of level curves: **level surfaces**. Given $w = f(x, y, z)$, the level surface at $w = c$ is the surface in space formed by all points (x, y, z) where $f(x, y, z) = c$.

Example 11.2.6 Finding level surfaces

If a point source S is radiating energy, the intensity I at a given point P in space is inversely proportional to the square of the distance between S and P. That is, when $S = (0,0,0)$, $I(x,y,z) = \dfrac{k}{x^2+y^2+z^2}$ for some constant k.

Let $k = 1$; find the level surfaces of I.

SOLUTION We can (mostly) answer this question using "common sense." If energy (say, in the form of light) is emanating from the origin, its intensity will be the same at all points equidistant from the origin. That is, at any point on the surface of a sphere centred at the origin, the intensity should be the same. Therefore, the level surfaces are spheres.

We now find this mathematically. The level surface at $I = c$ is defined by

$$c = \frac{1}{x^2+y^2+z^2}.$$

A small amount of algebra reveals

$$x^2+y^2+z^2 = \frac{1}{c}.$$

Given an intensity c, the level surface $I = c$ is a sphere of radius $1/\sqrt{c}$, centred at the origin.

Figure 11.2.6 gives a table of the radii of the spheres for given c values. Normally one would use equally spaced c values, but these values have been chosen purposefully. At a distance of 0.25 from the point source, the intensity is 16; to move to a point of half that intensity, one just moves out 0.1 to 0.35 – not much at all. To again halve the intensity, one moves 0.15, a little more than before.

Note how each time the intensity if halved, the distance required to move away grows. We conclude that the closer one is to the source, the more rapidly the intensity changes.

c	r
16.	0.25
8.	0.35
4.	0.5
2.	0.71
1.	1.
0.5	1.41
0.25	2.
0.125	2.83
0.0625	4.

Figure 11.2.6: A table of c values and the corresponding radius r of the spheres of constant value in Example 11.2.6.

In the next section we apply the concepts of limits to functions of two or more variables.

Exercises 11.2

Terms and Concepts

1. Give two examples (other than those given in the text) of "real world" functions that require more than one input.

2. The graph of a function of two variables is a ________.

3. Most people are familiar with the concept of level curves in the context of ________ maps.

4. T/F: Along a level curve, the output of a function does not change.

5. The analogue of a level curve for functions of three variables is a level ________.

6. What does it mean when level curves are close together? Far apart?

Problems

In Exercises 7 – 14, give the domain and range of the multivariable function.

7. $f(x, y) = x^2 + y^2 + 2$

8. $f(x, y) = x + 2y$

9. $f(x, y) = x - 2y$

10. $f(x, y) = \dfrac{1}{x + 2y}$

11. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

12. $f(x, y) = \sin x \cos y$

13. $f(x, y) = \sqrt{9 - x^2 - y^2}$

14. $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2 - 9}}$

In Exercises 15 – 22, describe in words and sketch the level curves for the function and given c values.

15. $f(x, y) = 3x - 2y$; $c = -2, 0, 2$

16. $f(x, y) = x^2 - y^2$; $c = -1, 0, 1$

17. $f(x, y) = x - y^2$; $c = -2, 0, 2$

18. $f(x, y) = \dfrac{1 - x^2 - y^2}{2y - 2x}$; $c = -2, 0, 2$

19. $f(x, y) = \dfrac{2x - 2y}{x^2 + y^2 + 1}$; $c = -1, 0, 1$

20. $f(x, y) = \dfrac{y - x^3 - 1}{x}$; $c = -3, -1, 0, 1, 3$

21. $f(x, y) = \sqrt{x^2 + 4y^2}$; $c = 1, 2, 3, 4$

22. $f(x, y) = x^2 + 4y^2$; $c = 1, 2, 3, 4$

In Exercises 23 – 26, give the domain and range of the functions of three variables.

23. $f(x, y, z) = \dfrac{x}{x + 2y - 4z}$

24. $f(x, y, z) = \dfrac{1}{1 - x^2 - y^2 - z^2}$

25. $f(x, y, z) = \sqrt{z - x^2 + y^2}$

26. $f(x, y, z) = z^2 \sin x \cos y$

In Exercises 27 – 30, describe the level surfaces of the given functions of three variables.

27. $f(x, y, z) = x^2 + y^2 + z^2$

28. $f(x, y, z) = z - x^2 + y^2$

29. $f(x, y, z) = \dfrac{x^2 + y^2}{z}$

30. $f(x, y, z) = \dfrac{z}{x - y}$

31. Compare the level curves of Exercises 21 and 22. How are they similar, and how are they different? Each surface is a quadric surface; describe how the level curves are consistent with what we know about each surface.

11.3 Limits and Continuity of Multivariable Functions

We continue with the pattern we have established in this text: after defining a new kind of function, we apply calculus ideas to it. The previous section defined functions of two and three variables; this section investigates what it means for these functions to be "continuous."

We begin with a series of definitions. We are used to "open intervals" such as $(1, 3)$, which represents the set of all x such that $1 < x < 3$, and "closed intervals" such as $[1, 3]$, which represents the set of all x such that $1 \leq x \leq 3$. We need analogous definitions for open and closed sets in the x-y plane.

Definition 11.3.1 Open Disk, Boundary and Interior Points, Open and Closed Sets, Bounded Sets

An **open disk** B in $\mathbb{R}^2$ centred at (x_0, y_0) with radius r is the set of all points (x, y) such that $\sqrt{(x-x_0)^2+(y-y_0)^2} < r$.

Let S be a set of points in $\mathbb{R}^2$. A point P in $\mathbb{R}^2$ is a **boundary point** of S if all open disks centred at P contain both points in S and points not in S.

A point P in S is an **interior point** of S if there is an open disk centred at P that contains only points in S.

A set S is **open** if every point in S is an interior point.

A set S is **closed** if it contains all of its boundary points.

A set S is **bounded** if there is an $M > 0$ such that the open disk, centred at the origin with radius M, contains S. A set that is not bounded is **unbounded**.

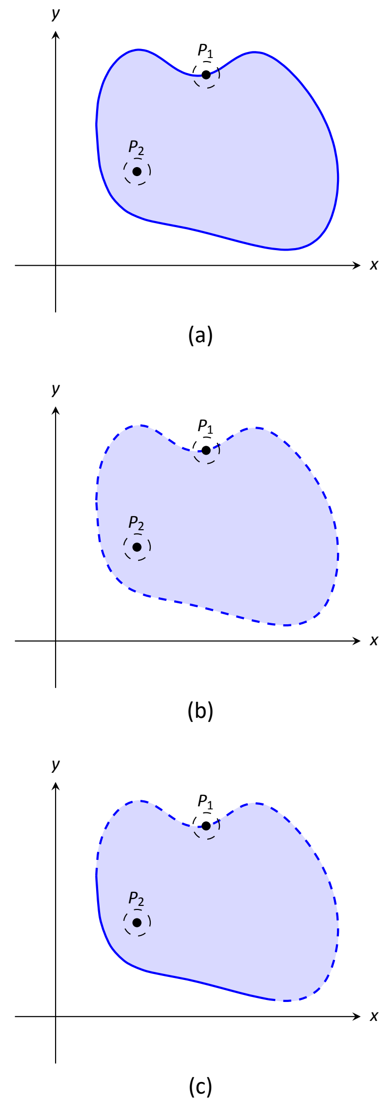


Figure 11.3.1: Illustrating open and closed sets in the x-y plane.

Figure 11.3.1 shows several sets in the x-y plane. In each set, point P_1 lies on the boundary of the set as all open disks centred there contain both points in, and not in, the set. In contrast, point P_2 is an interior point for there is an open disk centred there that lies entirely within the set.

The set depicted in Figure 11.3.1(a) is a closed set as it contains all of its boundary points. The set in (b) is open, for all of its points are interior points (or, equivalently, it does not contain any of its boundary points). The set in (c) is neither open nor closed as it contains some of its boundary points.

Example 11.3.1 Determining open/closed, bounded/unbounded

Determine if the domain of the function $f(x, y) = \sqrt{1 - x^2/9 - y^2/4}$ is open, closed, or neither, and if it is bounded.

Solution This domain of this function was found in Example 11.2.2 to be $D = \{(x, y) \mid \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$, the region *bounded* by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Since the region includes the boundary (indicated by the use of "$\leq$"), the set contains all of its boundary points and hence is closed. The region is bounded as a disk of radius 4, centred at the origin, contains D.

Example 11.3.2 Determining open/closed, bounded/unbounded

Determine if the domain of $f(x,y) = \frac{1}{x-y}$ is open, closed, or neither.

Solution As we cannot divide by 0, we find the domain to be $D = \{(x,y) \mid x - y \neq 0\}$. In other words, the domain is the set of all points (x,y) *not* on the line $y = x$.

The domain is sketched in Figure 11.3.2. Note how we can draw an open disk around any point in the domain that lies entirely inside the domain, and also note how the only boundary points of the domain are the points on the line $y = x$. We conclude the domain is an open set. The set is unbounded.

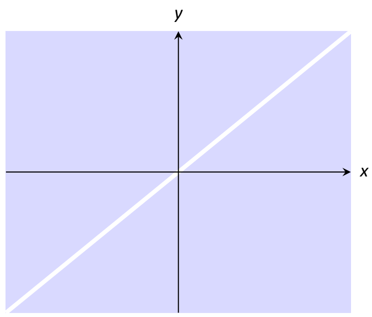


Figure 11.3.2: Sketching the domain of the function in Example 11.3.2.

Limits

Recall a pseudo–definition of the limit of a function of one variable: "$\lim_{x \to c} f(x) = L$" means that if x is "really close" to c, then $f(x)$ is "really close" to L. A similar pseudo–definition holds for functions of two variables. We'll say that

$$\text{“}\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L\text{”}$$

means "if the point (x,y) is really close to the point (x_0,y_0), then $f(x,y)$ is really close to L." The formal definition is given below.

Note: While our first limit definition was defined over an open interval, we now define limits over a set S in the plane (where S does not have to be open). As planar sets can be far more complicated than intervals, our definition adds the restriction ". . . where every open disk centred at P contains points in S other than P." In this text, all sets we'll consider will satisfy this condition and we won't bother to check; it is included in the definition for completeness.

Definition 11.3.2 Limit of a Function of Two Variables

Let S be a set containing $P = (x_0,y_0)$ where every open disk centred at P contains points in S other than P, let f be a function of two variables defined on S, except possibly at P, and let L be a real number. The **limit of** $f(x,y)$ **as** (x,y) **approaches** (x_0,y_0) **is** L, denoted

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all (x,y) in S, where $(x,y) \neq (x_0,y_0)$, if (x,y) is in the open disk centred at (x_0,y_0) with radius δ, then $|f(x,y) - L| < \varepsilon$.

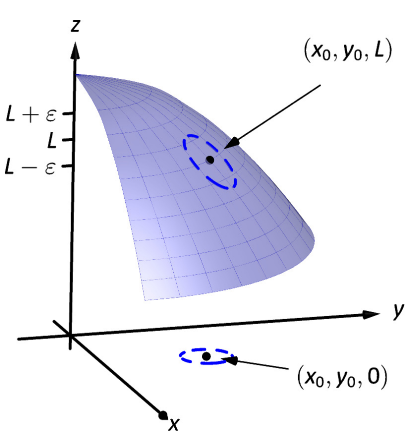


Figure 11.3.3: **Illustrating the definition of a limit.** The open disk in the x-y plane has radius δ. Let (x,y) be any point in this disk; $f(x,y)$ is within ε of L.

The concept behind Definition 11.3.2 is sketched in Figure 11.3.3. Given $\varepsilon > 0$, find $\delta > 0$ such that if (x,y) is any point in the open disk centred at (x_0,y_0) in the x-y plane with radius δ, then $f(x,y)$ should be within ε of L.

Computing limits using this definition is rather cumbersome. The following theorem allows us to evaluate limits much more easily.

Theorem 11.3.1 Basic Limit Properties of Functions of Two Variables

Let b, x_0, y_0, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = K.$$

The following limits hold.

1. Constants: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} b = b$
2. Identity $\displaystyle\lim_{(x,y)\to(x_0,y_0)} x = x_0;\qquad \lim_{(x,y)\to(x_0,y_0)} y = y_0$
3. Sums/Differences: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \big(f(x,y) \pm g(x,y)\big) = L \pm K$
4. Scalar Multiples: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} b\cdot f(x,y) = bL$
5. Products: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)\cdot g(x,y) = LK$
6. Quotients: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)/g(x,y) = L/K,\ (K\neq 0)$
7. Powers: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)^n = L^n$

This theorem, combined with Theorems 1.3.2 and 1.3.3 of Section 1.3, allows us to evaluate many limits.

Example 11.3.3 Evaluating a limit

Evaluate the following limits:

1. $\displaystyle\lim_{(x,y)\to(1,\pi)} \left(\frac{y}{x} + \cos(xy)\right)$
2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$

Solution

1. The aforementioned theorems allow us to simply evaluate $y/x + \cos(xy)$ when $x = 1$ and $y = \pi$. If an indeterminate form is returned, we must do more work to evaluate the limit; otherwise, the result is the limit. Therefore
$$\begin{aligned}\lim_{(x,y)\to(1,\pi)} \left(\frac{y}{x} + \cos(xy)\right) &= \frac{\pi}{1} + \cos\pi \\ &= \pi - 1.\end{aligned}$$
2. We attempt to evaluate the limit by substituting 0 in for x and y, but the result is the indeterminate form "0/0." To evaluate this limit, we must "do more work," but we have not yet learned what "kind" of work to do. Therefore we cannot yet evaluate this limit.

When dealing with functions of a single variable we also considered one–sided limits and stated

$$\lim_{x\to c} f(x) = L \quad \text{if, and only if,} \quad \lim_{x\to c^+} f(x) = L \quad \textbf{and} \quad \lim_{x\to c^-} f(x) = L.$$

That is, the limit is L if and only if $f(x)$ approaches L when x approaches c from **either** direction, the left or the right.

In the plane, there are infinitely many directions from which (x, y) might approach (x_0, y_0). In fact, we do not have to restrict ourselves to approaching (x_0, y_0) from a particular direction, but rather we can approach that point along a path that is not a straight line. It is possible to arrive at different limiting values by approaching (x_0, y_0) along different paths. If this happens, we say that $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist (this is analogous to the left and right hand limits of single variable functions not being equal).

Our theorems tell us that we can evaluate most limits quite simply, without worrying about paths. When indeterminate forms arise, the limit may or may not exist. If it does exist, it can be difficult to prove this as we need to show the same limiting value is obtained regardless of the path chosen. The case where the limit does not exist is often easier to deal with, for we can often pick two paths along which the limit is different.

Example 11.3.4 Showing limits do not exist

1. Show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$ does not exist by finding the limits along the lines $y = mx$.

2. Show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ does not exist by finding the limit along the path $y = -\sin x$.

Solution

1. Evaluating $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$ along the lines $y = mx$ means replace all y's with mx and evaluating the resulting limit:

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{3x(mx)}{x^2+(mx)^2} &= \lim_{x\to 0} \frac{3mx^2}{x^2(m^2+1)} \\ &= \lim_{x\to 0} \frac{3m}{m^2+1} \\ &= \frac{3m}{m^2+1}.\end{aligned}$$

While the limit exists for each choice of m, we get a *different* limit for each choice of m. That is, along different lines we get differing limiting values, meaning *the* limit does not exist.

2. Let $f(x, y) = \frac{\sin(xy)}{x+y}$. We are to show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist by finding the limit along the path $y = -\sin x$. First, however, consider the limits found along the lines $y = mx$ as done above.

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{\sin\big(x(mx)\big)}{x+mx} &= \lim_{x\to 0} \frac{\sin(mx^2)}{x(m+1)} \\ &= \lim_{x\to 0} \frac{\sin(mx^2)}{x} \cdot \frac{1}{m+1}.\end{aligned}$$

By applying L'Hospital's Rule, we can show this limit is 0 *except* when $m = -1$, that is, along the line $y = -x$. This line is not in the domain of f, so we have found the following fact: along every line $y = mx$ in the domain of f, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Now consider the limit along the path $y = -\sin x$:

$$\lim_{(x,-\sin x)\to(0,0)} \frac{\sin\big(-x\sin x\big)}{x - \sin x} = \lim_{x\to 0} \frac{\sin\big(-x\sin x\big)}{x - \sin x}$$

Now apply L'Hospital's Rule twice:

$$\begin{aligned}
&= \lim_{x\to 0} \frac{\cos\big(-x\sin x\big)(-\sin x - x\cos x)}{1-\cos x} \qquad (\text{“} = 0/0\text{”})\\
&= \lim_{x\to 0} \frac{-\sin\big(-x\sin x\big)(-\sin x - x\cos x)^2 + \cos\big(-x\sin x\big)(-2\cos x + x\sin x)}{\sin x}\\
&= \text{“}{-2}/0\text{”} \Rightarrow \text{the limit does not exist.}
\end{aligned}$$

Step back and consider what we have just discovered. Along any line $y = mx$ in the domain of the $f(x,y)$, the limit is 0. However, along the path $y = -\sin x$, which lies in the domain of $f(x,y)$ for all $x \neq 0$, the limit does not exist. Since the limit is not the same along every path to $(0,0)$, we say $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ does not exist.

Example 11.3.5 Finding a limit

Let $f(x,y) = \dfrac{5x^2y^2}{x^2+y^2}$. Find $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$.

Solution It is relatively easy to show that along any line $y = mx$, the limit is 0. This is not enough to prove that the limit exists, as demonstrated in the previous example, but it tells us that if the limit does exist then it must be 0.

To prove the limit is 0, we apply Definition 11.3.2. Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that if $\sqrt{(x-0)^2 + (y-0)^2} < \delta$, then $|f(x,y) - 0| < \varepsilon$.

Set $\delta < \sqrt{\varepsilon/5}$. Note that $\left|\dfrac{5y^2}{x^2+y^2}\right| < 5$ for all $(x,y) \neq (0,0)$, and that if $\sqrt{x^2+y^2} < \delta$, then $x^2 < \delta^2$.

Let $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2+y^2} < \delta$. Consider $|f(x,y) - 0|$:

$$\begin{aligned}
|f(x,y) - 0| &= \left|\frac{5x^2y^2}{x^2+y^2} - 0\right|\\
&= \left|x^2 \cdot \frac{5y^2}{x^2+y^2}\right|\\
&< \delta^2 \cdot 5\\
&< \frac{\varepsilon}{5} \cdot 5\\
&= \varepsilon.
\end{aligned}$$

Thus if $\sqrt{(x-0)^2 + (y-0)^2} < \delta$ then $|f(x,y) - 0| < \varepsilon$, which is what we wanted to show. Thus $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^2+y^2} = 0$.

Continuity

Definition 1.6.1 defines what it means for a function of one variable to be continuous. In brief, it meant that the graph of the function did not have breaks, holes, jumps, etc. We define continuity for functions of two variables in a similar way as we did for functions of one variable.

Definition 11.3.3 Continuous

Let a function $f(x,y)$ be defined on a set S containing the point (x_0,y_0).

1. f is **continuous** at (x_0,y_0) if $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.
2. f is **continuous on** S if f is continuous at all points in S. If f is continuous at all points in $\mathbb{R}^2$, we say that f is **continuous everywhere**.

Example 11.3.6 Continuity of a function of two variables

Let $f(x,y) = \begin{cases} \frac{\cos y \sin x}{x} & x \neq 0 \\ \cos y & x = 0 \end{cases}$. Is f continuous at $(0,0)$? Is f continuous everywhere?

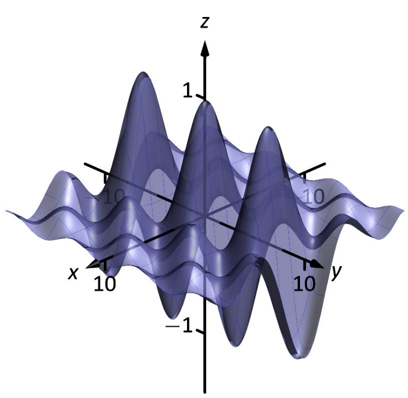


Figure 11.3.4: A graph of $f(x,y)$ in Example 11.3.6.

Solution To determine if f is continuous at $(0,0)$, we need to compare $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ to $f(0,0)$.

Applying the definition of f, we see that $f(0,0) = \cos 0 = 1$.

We now consider the limit $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$. Substituting 0 for x and y in $(\cos y \sin x)/x$ returns the indeterminate form "0/0", so we need to do more work to evaluate this limit.

Consider two related limits: $\displaystyle\lim_{(x,y)\to(0,0)} \cos y$ and $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin x}{x}$. The first limit does not contain x, and since $\cos y$ is continuous,

$$\lim_{(x,y)\to(0,0)} \cos y = \lim_{y\to 0} \cos y = \cos 0 = 1.$$

The second limit does not contain y. By Theorem 1.3.5 we can say

$$\lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Finally, Theorem 11.3.1 of this section states that we can combine these two limits as follows:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{\cos y \sin x}{x} &= \lim_{(x,y)\to(0,0)} (\cos y)\left(\frac{\sin x}{x}\right) \\
&= \left(\lim_{(x,y)\to(0,0)} \cos y\right)\left(\lim_{(x,y)\to(0,0)} \frac{\sin x}{x}\right) \\
&= (1)(1) \\
&= 1.
\end{aligned}$$

We have found that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos y \sin x}{x} = f(0,0)$, so f is continuous at $(0,0)$.

A similar analysis shows that f is continuous at all points in $\mathbb{R}^2$. As long as $x \neq 0$, we can evaluate the limit directly; when $x = 0$, a similar analysis shows that the limit is $\cos y$. Thus we can say that f is continuous everywhere. A graph of f is given in Figure 11.3.4. Notice how it has no breaks, jumps, etc.

The following theorem is very similar to Theorem 1.6.1, giving us ways to combine continuous functions to create other continuous functions.

Theorem 11.3.2 Properties of Continuous Functions

Let f and g be continuous on a set S, let c be a real number, and let n be a positive integer. The following functions are continuous on S.

1. Sums/Differences: $f \pm g$
2. Constant Multiples: $c \cdot f$
3. Products: $f \cdot g$
4. Quotients: f/g (as longs as $g \neq 0$ on S)
5. Powers: f^n
6. Roots: $\sqrt[n]{f}$ (if n is even then $f \geq 0$ on S; if n is odd, then true for all values of f on S.)
7. Compositions: Adjust the definitions of f and g to: Let f be continuous on S, where the range of f on S is J, and let g be a single variable function that is continuous on J. Then $g \circ f$, i.e., $g(f(x, y))$, is continuous on S.

Example 11.3.7 Establishing continuity of a function

Let $f(x, y) = \sin(x^2 \cos y)$. Show f is continuous everywhere.

Solution We will apply both Theorems 1.6.1 and 11.3.2. Let $f_1(x, y) = x^2$. Since y is not actually used in the function, and polynomials are continuous (by Theorem 1.6.1), we conclude f_1 is continuous everywhere. A similar statement can be made about $f_2(x, y) = \cos y$. Part 3 of Theorem 11.3.2 states that $f_3 = f_1 \cdot f_2$ is continuous everywhere, and Part 7 of the theorem states the composition of sine with f_3 is continuous: that is, $\sin(f_3) = \sin(x^2 \cos y)$ is continuous everywhere.

Functions of Three Variables

The definitions and theorems given in this section can be extended in a natural way to definitions and theorems about functions of three (or more) variables. We cover the key concepts here; some terms from Definitions 11.3.1 and 11.3.3 are not redefined but their analogous meanings should be clear to the reader.

Definition 11.3.4 Open Balls, Limit, Continuous

1. An **open ball** in $\mathbb{R}^3$ centred at (x_0, y_0, z_0) with radius r is the set of all points (x, y, z) such that $\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = r$.

2. Let D be an open set in $\mathbb{R}^3$ containing (x_0, y_0, z_0) where every open ball centred at (x_0, y_0, z_0) contains points of D other than (x_0, y_0, z_0), and let $f(x, y, z)$ be a function of three variables defined on D, except possibly at (x_0, y_0, z_0). The **limit** of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0) is L, denoted
$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = L,$$
means that given any $\varepsilon > 0$, there is a $\delta > 0$ such that for all (x, y, z) in D, $(x, y, z) \neq (x_0, y_0, z_0)$, if (x, y, z) is in the open ball centred at (x_0, y_0, z_0) with radius δ, then $|f(x, y, z) - L| < \varepsilon$.

3. Let $f(x, y, z)$ be defined on a set D containing (x_0, y_0, z_0). f is **continuous** at (x_0, y_0, z_0) if $\displaystyle\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0)$; if f is continuous at all points in D, we say f is **continuous on** D.

These definitions can also be extended naturally to apply to functions of four or more variables. Theorem 11.3.2 also applies to function of three or more variables, allowing us to say that the function

$$f(x, y, z) = \frac{e^{x^2+y}\sqrt{y^2+z^2+3}}{\sin(xyz)+5}$$

is continuous everywhere.

When considering single variable functions, we studied limits, then continuity, then the derivative. In our current study of multivariable functions, we have studied limits and continuity. In the next section we study derivation, which takes on a slight twist as we are in a multivarible context.

Exercises 11.3

Terms and Concepts

1. Describe in your own words the difference between boundary and interior points of a set.

2. Use your own words to describe (informally) what $\lim_{(x,y)\to(1,2)} f(x,y) = 17$ means.

3. Give an example of a closed, bounded set.

4. Give an example of a closed, unbounded set.

5. Give an example of a open, bounded set.

6. Give an example of a open, unbounded set.

Problems

In Exercises 7 – 10, a set S is given.

(a) Give one boundary point and one interior point, when possible, of S.

(b) State whether S is open, closed, or neither.

(c) State whether S is bounded or unbounded.

7. $S = \left\{(x,y) \;\middle|\; \frac{(x-1)^2}{4} + \frac{(y-3)^2}{9} \leq 1\right\}$

8. $S = \left\{(x,y) \mid y \neq x^2\right\}$

9. $S = \left\{(x,y) \mid x^2 + y^2 = 1\right\}$

10. $S = \{(x,y) | y > \sin x\}$

In Exercises 11 – 14:

(a) Find the domain D of the given function.

(b) State whether D is an open or closed set.

(c) State whether D is bounded or unbounded.

11. $f(x,y) = \sqrt{9 - x^2 - y^2}$

12. $f(x,y) = \sqrt{y - x^2}$

13. $f(x,y) = \dfrac{1}{\sqrt{y - x^2}}$

14. $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

In Exercises 15 – 20, a limit is given. Evaluate the limit along the paths given, then state why these results show the given limit does not exist.

15. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$
    (a) Along the path $y = 0$.
    (b) Along the path $x = 0$.

16. $\lim_{(x,y)\to(0,0)} \dfrac{x + y}{x - y}$
    (a) Along the path $y = mx$.

17. $\lim_{(x,y)\to(0,0)} \dfrac{xy - y^2}{y^2 + x}$
    (a) Along the path $y = mx$.
    (b) Along the path $x = 0$.

18. $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2)}{y}$
    (a) Along the path $y = mx$.
    (b) Along the path $y = x^2$.

19. $\lim_{(x,y)\to(1,2)} \dfrac{x + y - 3}{x^2 - 1}$
    (a) Along the path $y = 2$.
    (b) Along the path $y = x + 1$.

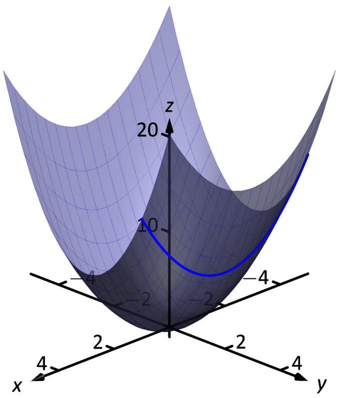
20. $\lim_{(x,y)\to(\pi,\pi/2)} \dfrac{\sin x}{\cos y}$
    (a) Along the path $x = \pi$.
    (b) Along the path $y = x - \pi/2$.

11.4 Partial Derivatives

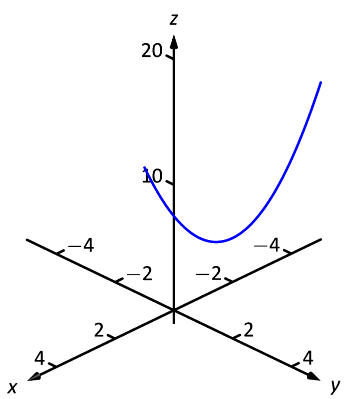
Let y be a function of x. We have studied in great detail the derivative of y with respect to x, that is, $\frac{dy}{dx}$, which measures the rate at which y changes with respect to x. Consider now $z = f(x, y)$. It makes sense to want to know how z changes with respect to x and/or y. This section begins our investigation into these rates of change.

Consider the function $z = f(x, y) = x^2 + 2y^2$, as graphed in Figure 11.4.1(a). By fixing $y = 2$, we focus our attention to all points on the surface where the y-value is 2, shown in both parts (a) and (b) of the figure. These points form a curve in space: $z = f(x, 2) = x^2 + 8$ which is a function of just one variable. We can take the derivative of z with respect to x along this curve and find equations of tangent lines, etc.

The key notion to extract from this example is: by treating y as constant (it does not vary) we can consider how z changes with respect to x. In a similar fashion, we can hold x constant and consider how z changes with respect to y. This is the underlying principle of **partial derivatives**. We state the formal, limit–based definition first, then show how to compute these partial derivatives without directly taking limits.



(a)



(b)

Figure 11.4.1: By fixing $y = 2$, the surface $f(x, y) = x^2 + 2y^2$ is a curve in space.

Definition 11.4.1 Partial Derivative

Let $z = f(x, y)$ be a continuous function on an open set S in $\mathbb{R}^2$.

1. The **partial derivative of f with respect to x** is:
$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

2. The **partial derivative of f with respect to y** is:
$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

Alternate notations for $f_x(x, y)$ include:
$$\frac{\partial}{\partial x} f(x, y),\quad \frac{\partial f}{\partial x},\quad \frac{\partial z}{\partial x},\quad \text{and } z_x,$$
with similar notations for $f_y(x, y)$. For ease of notation, $f_x(x, y)$ is often abbreviated f_x.

Example 11.4.1 Computing partial derivatives with the limit definition

Let $f(x, y) = x^2y + 2x + y^3$. Find $f_x(x, y)$ using the limit definition.

SOLUTION Using Definition 11.4.1, we have:

$$\begin{aligned}
f_x(x, y) &= \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h} \\
&= \lim_{h\to 0} \frac{(x + h)^2y + 2(x + h) + y^3 - (x^2y + 2x + y^3)}{h} \\
&= \lim_{h\to 0} \frac{x^2y + 2xhy + h^2y + 2x + 2h + y^3 - (x^2y + 2x + y^3)}{h} \\
&= \lim_{h\to 0} \frac{2xhy + h^2y + 2h}{h} \\
&= \lim_{h\to 0} 2xy + hy + 2 \\
&= 2xy + 2.
\end{aligned}$$

We have found $f_x(x, y) = 2xy + 2$.

Example 11.4.1 found a partial derivative using the formal, limit–based definition. Using limits is not necessary, though, as we can rely on our previous knowledge of derivatives to compute partial derivatives easily. When computing $f_x(x,y)$, we hold y fixed – it does not vary. Therefore we can compute the derivative with respect to x by treating y as a constant or coefficient.

Just as $\frac{d}{dx}\big(5x^2\big) = 10x$, we compute $\frac{\partial}{\partial x}\big(x^2y\big) = 2xy$. Here we are treating y as a coefficient.

Just as $\frac{d}{dx}\big(5^3\big) = 0$, we compute $\frac{\partial}{\partial x}\big(y^3\big) = 0$. Here we are treating y as a constant. More examples will help make this clear.

Example 11.4.2 Finding partial derivatives

Find $f_x(x,y)$ and $f_y(x,y)$ in each of the following.

1. $f(x,y) = x^3y^2 + 5y^2 - x + 7$
2. $f(x,y) = \cos(xy^2) + \sin x$
3. $f(x,y) = e^{x^2y^3}\sqrt{x^2+1}$

Solution

1. We have $f(x,y) = x^3y^2 + 5y^2 - x + 7$.
   Begin with $f_x(x,y)$. Keep y fixed, treating it as a constant or coefficient, as appropriate:
   $$f_x(x,y) = 3x^2y^2 - 1.$$
   Note how the $5y^2$ and 7 terms go to zero.

   To compute $f_y(x,y)$, we hold x fixed:
   $$f_y(x,y) = 2x^3y + 10y.$$
   Note how the $-x$ and 7 terms go to zero.

2. We have $f(x,y) = \cos(xy^2) + \sin x$.
   Begin with $f_x(x,y)$. We need to apply the Chain Rule with the cosine term; y^2 is the coefficient of the x-term inside the cosine function.
   $$f_x(x,y) = -\sin(xy^2)(y^2) + \cos x = -y^2\sin(xy^2) + \cos x.$$
   To find $f_y(x,y)$, note that x is the coefficient of the y^2 term inside of the cosine term; also note that since x is fixed, $\sin x$ is also fixed, and we treat it as a constant.
   $$f_y(x,y) = -\sin(xy^2)(2xy) = -2xy\sin(xy^2).$$

3. We have $f(x,y) = e^{x^2y^3}\sqrt{x^2+1}$.
   Beginning with $f_x(x,y)$, note how we need to apply the Product Rule.
   $$\begin{aligned} f_x(x,y) &= e^{x^2y^3}(2xy^3)\sqrt{x^2+1} + e^{x^2y^3}\frac{1}{2}\big(x^2+1\big)^{-1/2}(2x) \\ &= 2xy^3e^{x^2y^3}\sqrt{x^2+1} + \frac{xe^{x^2y^3}}{\sqrt{x^2+1}}. \end{aligned}$$
   Note that when finding $f_y(x,y)$ we do not have to apply the Product Rule; since $\sqrt{x^2+1}$ does not contain y, we treat it as fixed and hence becomes a coefficient of the $e^{x^2y^3}$ term.
   $$f_y(x,y) = e^{x^2y^3}(3x^2y^2)\sqrt{x^2+1} = 3x^2y^2e^{x^2y^3}\sqrt{x^2+1}.$$

We have shown *how* to compute a partial derivative, but it may still not be clear what a partial derivative *means*. Given $z = f(x, y)$, $f_x(x, y)$ measures the rate at which z changes as only x varies: y is held constant.

Imagine standing in a rolling meadow, then beginning to walk due east. Depending on your location, you might walk up, sharply down, or perhaps not change elevation at all. This is similar to measuring z_x: you are moving only east (in the "x"-direction) and not north/south at all. Going back to your original location, imagine now walking due north (in the "y"-direction). Perhaps walking due north does not change your elevation at all. This is analogous to $z_y = 0$: z does not change with respect to y. We can see that z_x and z_y do not have to be the same, or even similar, as it is easy to imagine circumstances where walking east means you walk downhill, though walking north makes you walk uphill.

The following example helps us visualize this more.

Example 11.4.3 Evaluating partial derivatives

Let $z = f(x, y) = -x^2 - \frac{1}{2}y^2 + xy + 10$. Find $f_x(2, 1)$ and $f_y(2, 1)$ and interpret their meaning.

Solution We begin by computing $f_x(x, y) = -2x + y$ and $f_y(x, y) = -y + x$. Thus

$$f_x(2,1) = -3 \quad \text{and} \quad f_y(2,1) = 1.$$

It is also useful to note that $f(2, 1) = 7.5$. What does each of these numbers mean?

Consider $f_x(2, 1) = -3$, along with Figure 11.4.2(a). If one "stands" on the surface at the point $(2, 1, 7.5)$ and moves parallel to the x-axis (i.e., only the x-value changes, not the y-value), then the instantaneous rate of change is -3. Increasing the x-value will decrease the z-value; decreasing the x-value will increase the z-value.

Now consider $f_y(2, 1) = 1$, illustrated in Figure 11.4.2(b). Moving along the curve drawn on the surface, i.e., parallel to the y-axis and not changing the x-values, increases the z-value instantaneously at a rate of 1. Increasing the y-value by 1 would increase the z-value by approximately 1.

Since the magnitude of f_x is greater than the magnitude of f_y at $(2, 1)$, it is "steeper" in the x-direction than in the y-direction.

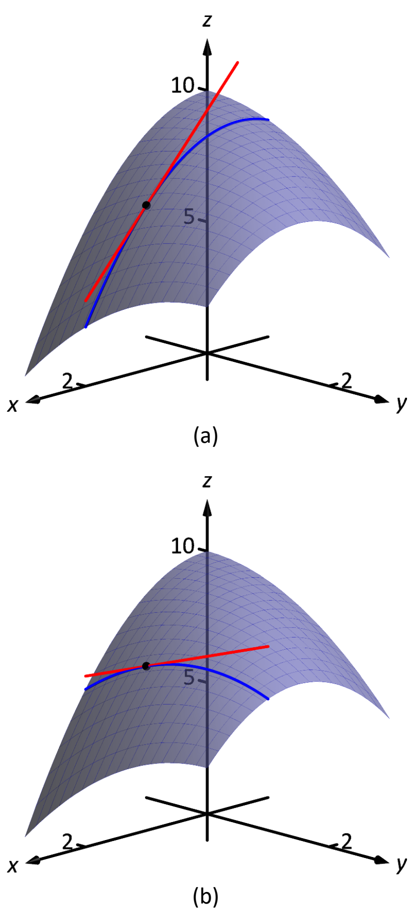


Figure 11.4.2: Illustrating the meaning of partial derivatives.

Second Partial Derivatives

Let $z = f(x, y)$. We have learned to find the partial derivatives $f_x(x, y)$ and $f_y(x, y)$, which are each functions of x and y. Therefore we can take partial derivatives of them, each with respect to x and y. We define these "second partials" along with the notation, give examples, then discuss their meaning.

Definition 11.4.2 Second Partial Derivative, Mixed Partial Derivative

Let $z = f(x, y)$ be continuous on an open set S.

1. The **second partial derivative of** f **with respect to** x **then** x is

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \left(f_x\right)_x = f_{xx}$$

2. The **second partial derivative of** f **with respect to** x **then** y is

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \left(f_x\right)_y = f_{xy}$$

Similar definitions hold for $\dfrac{\partial^2 f}{\partial y^2} = f_{yy}$ and $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}$.

The second partial derivatives f_{xy} and f_{yx} are **mixed partial derivatives**.

Note: The terms in Definition 11.4.2 all depend on limits, so each definition comes with the caveat "where the limit exists."

The notation of second partial derivatives gives some insight into the notation of the second derivative of a function of a single variable. If $y = f(x)$, then $f''(x) = \dfrac{d^2y}{dx^2}$. The "$d^2y$" portion means "take the derivative of y twice," while "dx^2" means "with respect to x both times." When we only know of functions of a single variable, this latter phrase seems silly: there is only one variable to take the derivative with respect to. Now that we understand functions of multiple variables, we see the importance of specifying which variables we are referring to.

Example 11.4.4 Second partial derivatives

For each of the following, find all six first and second partial derivatives. That is, find

$$f_x, \quad f_y, \quad f_{xx}, \quad f_{yy}, \quad f_{xy} \quad \text{and} \quad f_{yx}\,.$$

1. $f(x, y) = x^3y^2 + 2xy^3 + \cos x$
2. $f(x, y) = \dfrac{x^3}{y^2}$
3. $f(x, y) = e^x \sin(x^2y)$

Solution In each, we give f_x and f_y immediately and then spend time deriving the second partial derivatives.

1. $f(x, y) = x^3y^2 + 2xy^3 + \cos x$

   $f_x(x, y) = 3x^2y^2 + 2y^3 - \sin x$

$f_y(x,y) = 2x^3y + 6xy^2$

$f_{xx}(x,y) = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\big(3x^2y^2 + 2y^3 - \sin x\big) = 6xy^2 - \cos x$

$f_{yy}(x,y) = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\big(2x^3y + 6xy^2\big) = 2x^3 + 12xy$

$f_{xy}(x,y) = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\big(3x^2y^2 + 2y^3 - \sin x\big) = 6x^2y + 6y^2$

$f_{yx}(x,y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\big(2x^3y + 6xy^2\big) = 6x^2y + 6y^2$

2. $f(x,y) = \frac{x^3}{y^2} = x^3y^{-2}$

   $f_x(x,y) = \frac{3x^2}{y^2}$

   $f_y(x,y) = -\frac{2x^3}{y^3}$

   $f_{xx}(x,y) = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\big(\frac{3x^2}{y^2}\big) = \frac{6x}{y^2}$

   $f_{yy}(x,y) = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\big(-\frac{2x^3}{y^3}\big) = \frac{6x^3}{y^4}$

   $f_{xy}(x,y) = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\big(\frac{3x^2}{y^2}\big) = -\frac{6x^2}{y^3}$

   $f_{yx}(x,y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\big(-\frac{2x^3}{y^3}\big) = -\frac{6x^2}{y^3}$

3. $f(x,y) = e^x\sin(x^2y)$

   Because the following partial derivatives get rather long, we omit the extra notation and just give the results. In several cases, multiple applications of the Product and Chain Rules will be necessary, followed by some basic combination of like terms.

   $f_x(x,y) = e^x\sin(x^2y) + 2xye^x\cos(x^2y)$

   $f_y(x,y) = x^2e^x\cos(x^2y)$

   $f_{xx}(x,y) = e^x\sin(x^2y) + 4xye^x\cos(x^2y) + 2ye^x\cos(x^2y) - 4x^2y^2e^x\sin(x^2y)$

   $f_{yy}(x,y) = -x^4e^x\sin(x^2y)$

   $f_{xy}(x,y) = x^2e^x\cos(x^2y) + 2xe^x\cos(x^2y) - 2x^3ye^x\sin(x^2y)$

   $f_{yx}(x,y) = x^2e^x\cos(x^2y) + 2xe^x\cos(x^2y) - 2x^3ye^x\sin(x^2y)$

Notice how in each of the three functions in Example 11.4.4, $f_{xy} = f_{yx}$. Due to the complexity of the examples, this likely is not a coincidence. The following theorem states that it is not.

Theorem 11.4.1 Mixed Partial Derivatives

Let f be defined such that f_{xy} and f_{yx} are continuous on an open set S. Then for each point (x, y) in S, $f_{xy}(x, y) = f_{yx}(x, y)$.

Finding f_{xy} and f_{yx} independently and comparing the results provides a convenient way of checking our work.

Understanding Second Partial Derivatives

Now that we know *how* to find second partials, we investigate *what* they tell us.

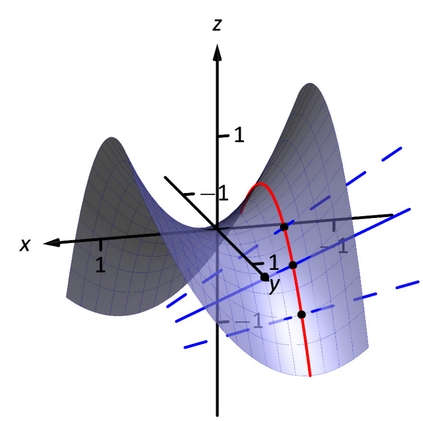
Again we refer back to a function $y = f(x)$ of a single variable. The second derivative of f is "the derivative of the derivative," or "the rate of change of the rate of change." The second derivative measures how much the derivative is changing. If $f''(x) < 0$, then the derivative is getting smaller (so the graph of f is concave down); if $f''(x) > 0$, then the derivative is growing, making the graph of f concave up.

Now consider $z = f(x, y)$. Similar statements can be made about f_{xx} and f_{yy} as could be made about $f''(x)$ above. When taking derivatives with respect to x twice, we measure how much f_x changes with respect to x. If $f_{xx}(x, y) < 0$, it means that as x increases, f_x decreases, and the graph of f will be concave down *in the x-direction*. Using the analogy of standing in the rolling meadow used earlier in this section, f_{xx} measures whether one's path is concave up/down when walking due east.

Similarly, f_{yy} measures the concavity in the y-direction. If $f_{yy}(x, y) > 0$, then f_y is increasing with respect to y and the graph of f will be concave up in the y-direction. Appealing to the rolling meadow analogy again, f_{yy} measures whether one's path is concave up/down when walking due north.

We now consider the mixed partials f_{xy} and f_{yx}. The mixed partial f_{xy} measures how much f_x changes with respect to y. Once again using the rolling meadow analogy, f_x measures the slope if one walks due east. Looking east, begin walking *north* (side–stepping). Is the path towards the east getting steeper? If so, $f_{xy} > 0$. Is the path towards the east not changing in steepness? If so, then $f_{xy} = 0$. A similar thing can be said about f_{yx}: consider the steepness of paths heading north while side–stepping to the east.

The following example examines these ideas with concrete numbers and graphs.



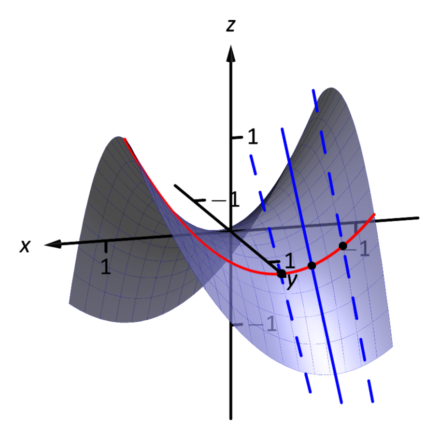


Figure 11.4.3: Understanding the second partial derivatives in Example 11.4.5.

Example 11.4.5 Understanding second partial derivatives

Let $z = x^2 - y^2 + xy$. Evaluate the 6 first and second partial derivatives at $(-1/2, 1/2)$ and interpret what each of these numbers mean.

Solution We find that:

$f_x(x, y) = 2x + y$, $f_y(x, y) = -2y + x$, $f_{xx}(x, y) = 2$, $f_{yy}(x, y) = -2$ and $f_{xy}(x, y) = f_{yx}(x, y) = 1$. Thus at $(-1/2, 1/2)$ we have

$$f_x(-1/2, 1/2) = -1/2, \qquad f_y(-1/2, 1/2) = -3/2.$$

The slope of the tangent line at $(-1/2, 1/2, -1/4)$ in the direction of x is $-1/2$:

if one moves from that point parallel to the x-axis, the instantaneous rate of change will be $-1/2$. The slope of the tangent line at this point in the direction of y is $-3/2$: if one moves from this point parallel to the y-axis, the instantaneous rate of change will be $-3/2$. These tangents lines are graphed in Figure 11.4.3(a) and (b), respectively, where the tangent lines are drawn in a solid line.

Now consider only Figure 11.4.3(a). Three directed tangent lines are drawn (two are dashed), each in the direction of x; that is, each has a slope determined by f_x. Note how as y increases, the slope of these lines get closer to 0. Since the slopes are all negative, getting closer to 0 means the *slopes are increasing.* The slopes given by f_x are increasing as y increases, meaning f_{xy} must be positive.

Since $f_{xy} = f_{yx}$, we also expect f_y to increase as x increases. Consider Figure 11.4.3(b) where again three directed tangent lines are drawn, this time each in the direction of y with slopes determined by f_y. As x increases, the slopes become less steep (closer to 0). Since these are negative slopes, this means the slopes are increasing.

Thus far we have a visual understanding of f_x, f_y, and $f_{xy} = f_{yx}$. We now interpret f_{xx} and f_{yy}. In Figure 11.4.3(a), we see a curve drawn where x is held constant at $x = -1/2$: only y varies. This curve is clearly concave down, corresponding to the fact that $f_{yy} < 0$. In part (b) of the figure, we see a similar curve where y is constant and only x varies. This curve is concave up, corresponding to the fact that $f_{xx} > 0$.

Partial Derivatives and Functions of Three Variables

The concepts underlying partial derivatives can be easily extend to more than two variables. We give some definitions and examples in the case of three variables and trust the reader can extend these definitions to more variables if needed.

Definition 11.4.3 Partial Derivatives with Three Variables

Let $w = f(x, y, z)$ be a continuous function on an open set S in $\mathbb{R}^3$.
The **partial derivative of** f **with respect to** x is:

$$f_x(x, y, z) = \lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}.$$

Similar definitions hold for $f_y(x, y, z)$ and $f_z(x, y, z)$.

By taking partial derivatives of partial derivatives, we can find second partial derivatives of f with respect to z then y, for instance, just as before.

Example 11.4.6 Partial derivatives of functions of three variables

For each of the following, find f_x, f_y, f_z, f_{xz}, f_{yz}, and f_{zz}.

1. $f(x, y, z) = x^2y^3z^4 + x^2y^2 + x^3z^3 + y^4z^4$
2. $f(x, y, z) = x\sin(yz)$

Solution

1. $f_x = 2xy^3z^4 + 2xy^2 + 3x^2z^3$; $\quad f_y = 3x^2y^2z^4 + 2x^2y + 4y^3z^4$;
   $f_z = 4x^2y^3z^3 + 3x^3z^2 + 4y^4z^3$; $\quad f_{xz} = 8xy^3z^3 + 9x^2z^2$;
   $f_{yz} = 12x^2y^2z^3 + 16y^3z^3$; $\quad f_{zz} = 12x^2y^3z^2 + 6x^3z + 12y^4z^2$

2. $f_x = \sin(yz)$; $f_y = xz\cos(yz)$; $f_z = xy\cos(yz)$;
$f_{xz} = y\cos(yz)$; $f_{yz} = x\cos(yz) - xyz\sin(yz)$; $f_{zz} = -xy^2\sin(xy)$

Higher Order Partial Derivatives

We can continue taking partial derivatives of partial derivatives of partial derivatives of ...; we do not have to stop with second partial derivatives. These higher order partial derivatives do not have a tidy graphical interpretation; nevertheless they are not hard to compute and worthy of some practice.

We do not formally define each higher order derivative, but rather give just a few examples of the notation.

$$f_{xyx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) \quad \text{and}$$
$$f_{xyz}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right).$$

Example 11.4.7 Higher order partial derivatives

1. Let $f(x,y) = x^2y^2 + \sin(xy)$. Find f_{xxy} and f_{yxx}.
2. Let $f(x,y,z) = x^3e^{xy} + \cos(z)$. Find f_{xyz}.

SOLUTION

1. To find f_{xxy}, we first find f_x, then f_{xx}, then f_{xxy}:

$$f_x = 2xy^2 + y\cos(xy) \qquad f_{xx} = 2y^2 - y^2\sin(xy)$$
$$f_{xxy} = 4y - 2y\sin(xy) - xy^2\cos(xy).$$

To find f_{yxx}, we first find f_y, then f_{yx}, then f_{yxx}:

$$f_y = 2x^2y + x\cos(xy) \qquad f_{yx} = 4xy + \cos(xy) - xy\sin(xy)$$
$$\begin{aligned} f_{yxx} &= 4y - y\sin(xy) - \big(y\sin(xy) + xy^2\cos(xy)\big) \\ &= 4y - 2y\sin(xy) - xy^2\cos(xy). \end{aligned}$$

Note how $f_{xxy} = f_{yxx}$.

2. To find f_{xyz}, we find f_x, then f_{xy}, then f_{xyz}:

$$f_x = 3x^2e^{xy} + x^3ye^{xy} \qquad f_{xy} = 3x^3e^{xy} + x^3e^{xy} + x^4ye^{xy} = 4x^3e^{xy} + x^4ye^{xy}$$
$$f_{xyz} = 0.$$

In the previous example we saw that $f_{xxy} = f_{yxx}$; this is not a coincidence. While we do not state this as a formal theorem, as long as each partial derivative is continuous, it does not matter the order in which the partial derivatives are taken. For instance, $f_{xxy} = f_{xyx} = f_{yxx}$.

This can be useful at times. Had we known this, the second part of Example 11.4.7 would have been much simpler to compute. Instead of computing f_{xyz}

in the x, y then z orders, we could have applied the z, then x then y order (as $f_{xyz} = f_{zxy}$). It is easy to see that $f_z = -\sin z$; then f_{zx} and f_{zxy} are clearly 0 as f_z does not contain an x or y.

A brief review of this section: partial derivatives measure the instantaneous rate of change of a multivariable function with respect to one variable. With $z = f(x, y)$, the partial derivatives f_x and f_y measure the instantaneous rate of change of z when moving parallel to the x- and y-axes, respectively. How do we measure the rate of change at a point when we do not move parallel to one of these axes? What if we move in the direction given by the vector $\langle 2, 1\rangle$? Can we measure that rate of change? The answer is, of course, yes, we can. This is

Exercises 11.4

Terms and Concepts

1. What is the difference between a constant and a coefficient?

2. Given a function $z = f(x, y)$, explain in your own words how to compute f_x.

3. In the mixed partial fraction f_{xy}, which is computed first, f_x or f_y?

4. In the mixed partial fraction $\frac{\partial^2 f}{\partial x \partial y}$, which is computed first, f_x or f_y?

Problems

In Exercises 5 – 8, evaluate $f_x(x, y)$ and $f_y(x, y)$ at the indicated point.

5. $f(x, y) = x^2y - x + 2y + 3$ at $(1, 2)$

6. $f(x, y) = x^3 - 3x + y^2 - 6y$ at $(-1, 3)$

7. $f(x, y) = \sin y \cos x$ at $(\pi/3, \pi/3)$

8. $f(x, y) = \ln(xy)$ at $(-2, -3)$

In Exercises 9 – 26, find f_x, f_y, f_{xx}, f_{yy}, f_{xy} and f_{yx}.

9. $f(x, y) = x^2y + 3x^2 + 4y - 5$

10. $f(x, y) = y^3 + 3xy^2 + 3x^2y + x^3$

11. $f(x, y) = \dfrac{x}{y}$

12. $f(x, y) = \dfrac{4}{xy}$

13. $f(x, y) = e^{x^2+y^2}$

14. $f(x, y) = e^{x+2y}$

15. $f(x, y) = \sin x \cos y$

16. $f(x, y) = (x + y)^3$

17. $f(x, y) = \cos(5xy^3)$

18. $f(x, y) = \sin(5x^2 + 2y^3)$

19. $f(x, y) = \sqrt{4xy^2 + 1}$

20. $f(x, y) = (2x + 5y)\sqrt{y}$

21. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

22. $f(x, y) = 5x - 17y$

23. $f(x, y) = 3x^2 + 1$

24. $f(x, y) = \ln(x^2 + y)$

25. $f(x, y) = \dfrac{\ln x}{4y}$

26. $f(x, y) = 5e^x \sin y + 9$

In Exercises 27 – 30, form a function $z = f(x, y)$ such that f_x and f_y match those given.

27. $f_x = \sin y + 1, \quad f_y = x \cos y$

28. $f_x = x + y, \quad f_y = x + y$

29. $f_x = 6xy - 4y^2, \quad f_y = 3x^2 - 8xy + 2$

30. $f_x = \dfrac{2x}{x^2 + y^2}, \quad f_y = \dfrac{2y}{x^2 + y^2}$

In Exercises 31 – 34, find f_x, f_y, f_z, f_{yz} and f_{zy}.

31. $f(x, y, z) = x^2e^{2y-3z}$

32. $f(x, y, z) = x^3y^2 + x^3z + y^2z$

33. $f(x, y, z) = \dfrac{3x}{7y^2z}$

34. $f(x, y, z) = \ln(xyz)$

11.5 Tangent Lines, Normal Lines, and Tangent Planes

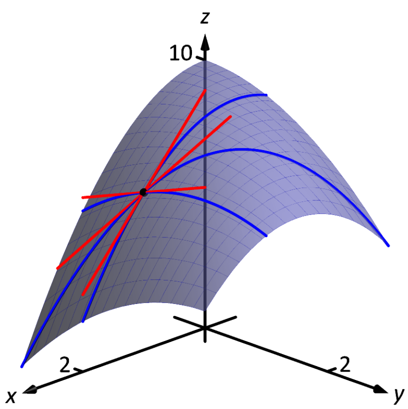


Figure 11.5.1: Showing various lines tangent to a surface.

Derivatives and tangent lines go hand–in–hand. Given $y = f(x)$, the line tangent to the graph of f at $x = x_0$ is the line through $\big(x_0, f(x_0)\big)$ with slope $f'(x_0)$; that is, the slope of the tangent line is the instantaneous rate of change of f at x_0.

When dealing with functions of two variables, the graph is no longer a curve but a surface. At a given point on the surface, it seems there are many lines that fit our intuition of being "tangent" to the surface.

In Figure 11.5.1 we see lines that are tangent to curves in space. Since each curve lies on a surface, it makes sense to say that the lines are also tangent to the surface. The next definition formally defines what it means to be "tangent to a surface."

Definition 11.5.1 Directional Tangent Line

Let $z = f(x, y)$ be differentiable on an open set S containing (x_0, y_0) and let $\vec{u} = \langle u_1, u_2\rangle$ be a unit vector.

1. The line ℓ_x through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle 1, 0, f_x(x_0, y_0)\rangle$ is the **tangent line to f in the direction of x at** (x_0, y_0).
2. The line ℓ_y through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle 0, 1, f_y(x_0, y_0)\rangle$ is the **tangent line to f in the direction of y at** (x_0, y_0).
3. The line $\ell_{\vec{u}}$ through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle u_1, u_2, D_{\vec{u}}f(x_0, y_0)\rangle$ is the **tangent line to f in the direction of $\vec{u}$ at** (x_0, y_0).

It is instructive to consider each of three directions given in the definition in terms of "slope." The direction of ℓ_x is $\langle 1, 0, f_x(x_0, y_0)\rangle$; that is, the "run" is one unit in the x-direction and the "rise" is $f_x(x_0, y_0)$ units in the z-direction. Note how the slope is just the partial derivative with respect to x. A similar statement can be made for ℓ_y. The direction of $\ell_{\vec{u}}$ is $\langle u_1, u_2, D_{\vec{u}}f(x_0, y_0)\rangle$; the "run" is one unit in the $\vec{u}$ direction (where $\vec{u}$ is a unit vector) and the "rise" is the directional derivative of z in that direction.

Definition 11.5.1 leads to the following parametric equations of directional tangent lines:

$$\ell_x(t) = \begin{cases} x = x_0 + t \\ y = y_0 \\ z = z_0 + f_x(x_0, y_0)t \end{cases}, \quad \ell_y(t) = \begin{cases} x = x_0 \\ y = y_0 + t \\ z = z_0 + f_y(x_0, y_0)t \end{cases} \quad \text{and} \quad \ell_{\vec{u}}(t) = \begin{cases} x = x_0 + u_1 t \\ y = y_0 + u_2 t \\ z = z_0 + D_{\vec{u}}f(x_0, y_0)t \end{cases}.$$

Example 11.5.1 Finding directional tangent lines

Find the lines tangent to the surface $z = \sin x \cos y$ at $(\pi/2, \pi/2)$ in the x and y directions and also in the direction of $\vec{v} = \langle -1, 1\rangle$.

Solution The partial derivatives with respect to x and y are:

$$\begin{aligned} f_x(x, y) &= \cos x \cos y \quad &\Rightarrow \quad f_x(\pi/2, \pi/2) = 0 \\ f_y(x, y) &= -\sin x \sin y \quad &\Rightarrow \quad f_y(\pi/2, \pi/2) = -1. \end{aligned}$$

At $(\pi/2, \pi/2)$, the z-value is 0.

Thus the parametric equations of the line tangent to f at $(\pi/2, \pi/2)$ in the directions of x and y are:

$$\ell_x(t) = \begin{cases} x = \pi/2 + t \\ y = \pi/2 \\ z = 0 \end{cases} \quad \text{and} \quad \ell_y(t) = \begin{cases} x = \pi/2 \\ y = \pi/2 + t \\ z = -t \end{cases}.$$

The two lines are shown with the surface in Figure 11.5.2(a). To find the equation of the tangent line in the direction of $\vec{v}$, we first find the unit vector in the direction of $\vec{v}$: $\vec{u} = \left\langle -1/\sqrt{2}, 1/\sqrt{2}\right\rangle$. The directional derivative at $(\pi/2, \pi, 2)$ in the direction of $\vec{u}$ is

$$D_{\vec{u}}\, f(\pi/2, \pi, 2) = \langle 0, -1\rangle \cdot \left\langle -1/\sqrt{2}, 1/\sqrt{2}\right\rangle = -1/\sqrt{2}.$$

Thus the directional tangent line is

$$\ell_{\vec{u}}(t) = \begin{cases} x = \pi/2 - t/\sqrt{2} \\ y = \pi/2 + t/\sqrt{2} \\ z = -t/\sqrt{2} \end{cases}.$$

The curve through $(\pi/2, \pi/2, 0)$ in the direction of $\vec{v}$ is shown in Figure 11.5.2(b) along with $\ell_{\vec{u}}(t)$.

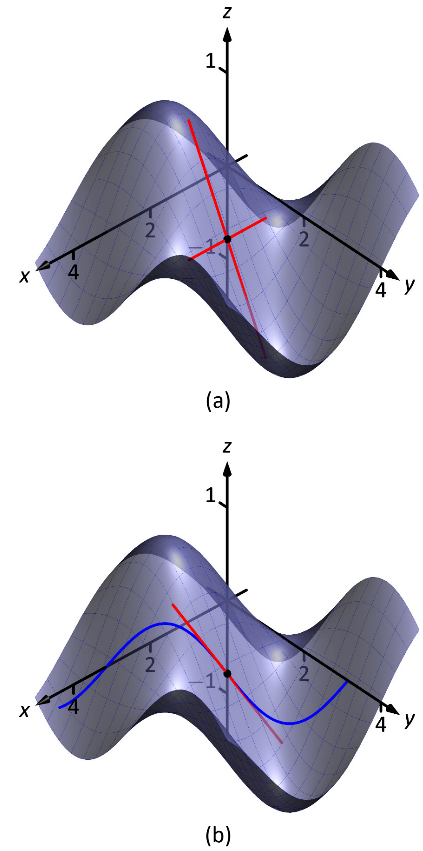


Figure 11.5.2: A surface and directional tangent lines in Example 11.5.1.

Example 11.5.2 Finding directional tangent lines

Let $f(x, y) = 4xy - x^4 - y^4$. Find the equations of *all* directional tangent lines to f at $(1, 1)$.

Solution First note that $f(1, 1) = 2$. We need to compute directional derivatives, so we need ∇f. We begin by computing partial derivatives.

$$f_x = 4y - 4x^3 \Rightarrow f_x(1, 1) = 0; \quad f_y = 4x - 4y^3 \Rightarrow f_y(1, 1) = 0.$$

Thus $\nabla f(1, 1) = \langle 0, 0\rangle$. Let $\vec{u} = \langle u_1, u_2\rangle$ be any unit vector. The directional derivative of f at $(1, 1)$ will be $D_{\vec{u}}\, f(1, 1) = \langle 0, 0\rangle \cdot \langle u_1, u_2\rangle = 0$. It does not matter what direction we choose; the directional derivative is always 0. Therefore

$$\ell_{\vec{u}}(t) = \begin{cases} x = 1 + u_1 t \\ y = 1 + u_2 t \\ z = 2 \end{cases}.$$

Figure 11.5.3 shows a graph of f and the point $(1, 1, 2)$. Note that this point comes at the top of a "hill," and therefore every tangent line through this point will have a "slope" of 0.

That is, consider any curve on the surface that goes through this point. Each curve will have a relative maximum at this point, hence its tangent line will have a slope of 0. The following section investigates the points on surfaces where all tangent lines have a slope of 0.

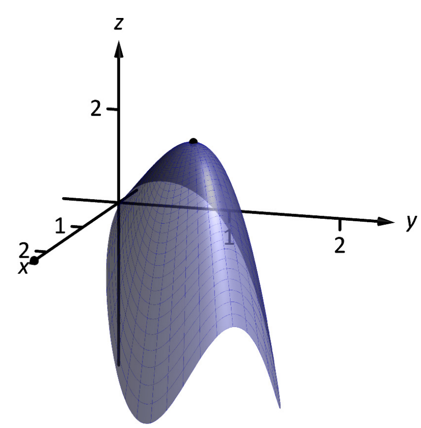


Figure 11.5.3: Graphing f in Example 11.5.2.

Normal Lines

When dealing with a function $y = f(x)$ of one variable, we stated that a line through $(c, f(c))$ was *tangent* to f if the line had a slope of $f'(c)$ and was *normal* (or, *perpendicular, orthogonal*) to f if it had a slope of $-1/f'(c)$. We extend the concept of normal, or orthogonal, to functions of two variables.

Let $z = f(x, y)$ be a differentiable function of two variables. By Definition 11.5.1, at (x_0, y_0), $\ell_x(t)$ is a line parallel to the vector $\vec{d_x} = \langle 1, 0, f_x(x_0, y_0)\rangle$ and $\ell_y(t)$ is a line parallel to $\vec{d_y} = \langle 0, 1, f_y(x_0, y_0)\rangle$. Since lines in these directions through $\big(x_0, y_0, f(x_0, y_0)\big)$ are *tangent* to the surface, a line through this point and orthogonal to these directions would be *orthogonal*, or *normal*, to the surface. We can use this direction to create a normal line.

The direction of the normal line is orthogonal to $\vec{d_x}$ and $\vec{d_y}$, hence the direction is parallel to $\vec{d_n} = \vec{d_x} \times \vec{d_y}$. It turns out this cross product has a very simple form:

$$\vec{d_x} \times \vec{d_y} = \langle 1, 0, f_x\rangle \times \langle 0, 1, f_y\rangle = \langle -f_x, -f_y, 1\rangle .$$

It is often more convenient to refer to the opposite of this direction, namely $\langle f_x, f_y, -1\rangle$. This leads to a definition.

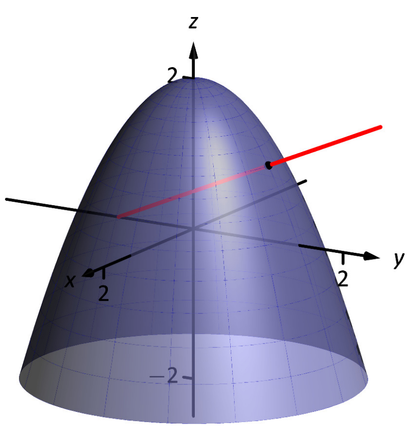


Figure 11.5.4: Graphing a surface with a normal line from Example 11.5.3.

Definition 11.5.2 Normal Line

Let $z = f(x, y)$ be differentiable on an open set S containing (x_0, y_0) where

$$a = f_x(x_0, y_0) \quad \text{and} \quad b = f_y(x_0, y_0)$$

are defined.

1. A nonzero vector parallel to $\vec{n} = \langle a, b, -1\rangle$ is **orthogonal to f at** $P = \big(x_0, y_0, f(x_0, y_0)\big)$.
2. The line ℓ_n through P with direction parallel to $\vec{n}$ is the **normal line to f at P**.

Thus the parametric equations of the normal line to a surface f at $\big(x_0, y_0, f(x_0, y_0)\big)$ is:

$$\ell_n(t) = \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = f(x_0, y_0) - t \end{cases} .$$

Example 11.5.3 Finding a normal line

Find the equation of the normal line to $z = -x^2 - y^2 + 2$ at $(0, 1)$.

Solution We find $z_x(x, y) = -2x$ and $z_y(x, y) = -2y$; at $(0, 1)$, we have $z_x = 0$ and $z_y = -2$. We take the direction of the normal line, following Definition 11.5.2, to be $\vec{n} = \langle 0, -2, -1\rangle$. The line with this direction going through the point $(0, 1, 1)$ is

$$\ell_n(t) = \begin{cases} x = 0 \\ y = -2t + 1 \\ z = -t + 1 \end{cases} \quad \text{or} \quad \ell_n(t) = \langle 0, -2, -1\rangle\, t + \langle 0, 1, 1\rangle .$$

The surface $z = -x^2 - y^2 + 2$, along with the found normal line, is graphed in Figure 11.5.4.

The direction of the normal line has many uses, one of which is the definition of the **tangent plane** which we define shortly. Another use is in measuring distances from the surface to a point. Given a point Q in space, it is a general geometric concept to define the distance from Q to the surface as being the length of the shortest line segment $\overline{PQ}$ over all points P on the surface. This, in turn, implies that $\overrightarrow{PQ}$ will be orthogonal to the surface at P. Therefore we can measure the distance from Q to the surface f by finding a point P on the surface such that $\overrightarrow{PQ}$ is parallel to the normal line to f at P.

Example 11.5.4 Finding the distance from a point to a surface

Let $f(x,y) = 2 - x^2 - y^2$ and let $Q = (2,2,2)$. Find the distance from Q to the surface defined by f.

Solution This surface is used in Example 11.5.2, so we know that at (x,y), the direction of the normal line will be $\vec{d}_n = \langle -2x, -2y, -1\rangle$. A point P on the surface will have coordinates $(x, y, 2-x^2-y^2)$, so $\overrightarrow{PQ} = \left\langle 2-x, 2-y, x^2+y^2\right\rangle$. To find where $\overrightarrow{PQ}$ is parallel to $\vec{d}_n$, we need to find x, y and c such that $c\overrightarrow{PQ} = \vec{d}_n$.

$$\begin{aligned} c\overrightarrow{PQ} &= \vec{d}_n \\ c\left\langle 2-x, 2-y, x^2+y^2\right\rangle &= \langle -2x, -2y, -1\rangle\,. \end{aligned}$$

This implies

$$\begin{aligned} c(2-x) &= -2x \\ c(2-y) &= -2y \\ c(x^2+y^2) &= -1 \end{aligned}$$

In each equation, we can solve for c:

$$c = \frac{-2x}{2-x} = \frac{-2y}{2-y} = \frac{-1}{x^2+y^2}.$$

The first two fractions imply $x = y$, and so the last fraction can be rewritten as $c = -1/(2x^2)$. Then

$$\begin{aligned} \frac{-2x}{2-x} &= \frac{-1}{2x^2} \\ -2x(2x^2) &= -1(2-x) \\ 4x^3 &= 2-x \\ 4x^3 + x - 2 &= 0. \end{aligned}$$

This last equation is a cubic, which is not difficult to solve with a numeric solver. We find that $x = 0.689$, hence $P = (0.689, 0.689, 1.051)$. We find the distance from Q to the surface of f is

$$\parallel \overrightarrow{PQ} \parallel = \sqrt{(2-0.689)^2 + (2-0.689)^2 + (2-1.051)^2} = 2.083.$$

We can take the concept of measuring the distance from a point to a surface to find a point Q a particular distance from a surface at a given point P on the surface.

Example 11.5.5 Finding a point a set distance from a surface

Let $f(x,y) = x - y^2 + 3$. Let $P = \big(2, 1, f(2,1)\big) = (2, 1, 4)$. Find points Q in space that are 4 units from the surface of f at P. That is, find Q such that $\parallel \overrightarrow{PQ} \parallel = 4$ and $\overrightarrow{PQ}$ is orthogonal to f at P.

SOLUTION We begin by finding partial derivatives:

$$\begin{aligned} f_x(x,y) &= 1 & \Rightarrow && f_x(2,1) &= 1 \\ f_y(x,y) &= -2y & \Rightarrow && f_y(2,1) &= -2 \end{aligned}$$

The vector $\vec{n} = \langle 1, -2, -1 \rangle$ is orthogonal to f at P. For reasons that will become more clear in a moment, we find the unit vector in the direction of $\vec{n}$:

$$\vec{u} = \frac{\vec{n}}{\parallel \vec{n} \parallel} = \left\langle 1/\sqrt{6}, -2/\sqrt{6}, -1/\sqrt{6} \right\rangle \approx \langle 0.408, -0.816, -0.408 \rangle.$$

Thus a the normal line to f at P can be written as

$$\ell_n(t) = \langle 2, 1, 4 \rangle + t \langle 0.408, -0.816, -0.408 \rangle.$$

An advantage of this parametrization of the line is that letting $t = t_0$ gives a point on the line that is $|t_0|$ units from P. (This is because the direction of the line is given in terms of a unit vector.) There are thus two points in space 4 units from P:

$$\begin{aligned} Q_1 &= \ell_n(4) & \qquad Q_2 &= \ell_n(-4) \\ &\approx \langle 3.63, -2.27, 2.37 \rangle & &\approx \langle 0.37, 4.27, 5.63 \rangle \end{aligned}$$

The surface is graphed along with points P, Q_1, Q_2 and a portion of the normal line to f at P.

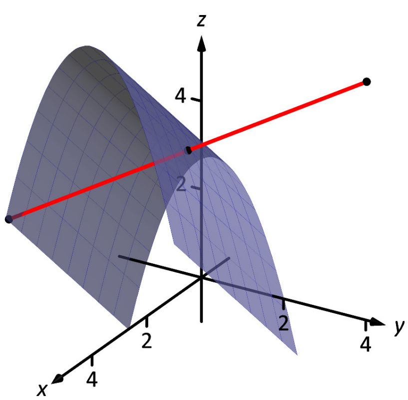


Figure 11.5.5: Graphing the surface in Example 11.5.5 along with points 4 units from the surface.

Tangent Planes

We can use the direction of the normal line to define a plane. With $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$ and $P = \big(x_0, y_0, f(x_0, y_0)\big)$, the vector $\vec{n} = \langle a, b, -1 \rangle$ is orthogonal to f at P. The plane through P with normal vector $\vec{n}$ is therefore **tangent** to f at P.

Definition 11.5.3 Tangent Plane

Let $z = f(x,y)$ be differentiable on an open set S containing (x_0, y_0), where $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$, $\vec{n} = \langle a, b, -1 \rangle$ and $P = \big(x_0, y_0, f(x_0, y_0)\big)$.

The plane through P with normal vector $\vec{n}$ is the **tangent plane to f at P**. The standard form of this plane is

$$a(x - x_0) + b(y - y_0) - \big(z - f(x_0, y_0)\big) = 0.$$

Example 11.5.6 Finding tangent planes

Find the equation of the tangent plane to $z = -x^2 - y^2 + 2$ at $(0, 1)$.

Solution Note that this is the same surface and point used in Example 11.5.3. There we found $\vec{n} = \langle 0, -2, -1\rangle$ and $P = (0, 1, 1)$. Therefore the equation of the tangent plane is

$$-2(y-1) - (z-1) = 0.$$

The surface $z = -x^2 - y^2 + 2$ and tangent plane are graphed in Figure 11.5.6.

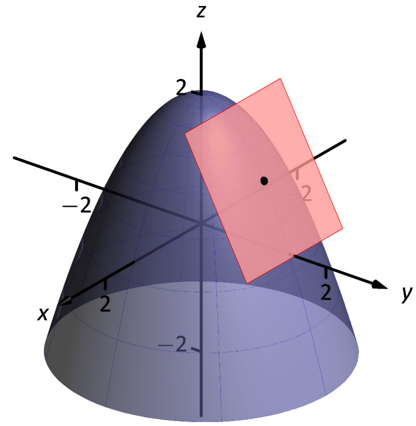


Figure 11.5.6: Graphing a surface with tangent plane from Example 11.5.6.

Example 11.5.7 Using the tangent plane to approximate function values

The point $(3, -1, 4)$ lies on the surface of an unknown differentiable function f where $f_x(3, -1) = 2$ and $f_y(3, -1) = -1/2$. Find the equation of the tangent plane to f at P, and use this to approximate the value of $f(2.9, -0.8)$.

Solution Knowing the partial derivatives at $(3, -1)$ allows us to form the normal vector to the tangent plane, $\vec{n} = \langle 2, -1/2, -1\rangle$. Thus the equation of the tangent line to f at P is:

$$2(x-3) - 1/2(y+1) - (z-4) = 0 \quad \Rightarrow \quad z = 2(x-3) - 1/2(y+1) + 4. \quad (11.1)$$

Just as tangent lines provide excellent approximations of curves near their point of intersection, tangent planes provide excellent approximations of surfaces near their point of intersection. So $f(2.9, -0.8) \approx z(2.9, -0.8) = 3.7$.

This is not a new method of approximation. Compare the right hand expression for z in Equation (11.1) to the total differential:

$$dz = f_x dx + f_y dy \quad \text{and} \quad z = \underbrace{\underbrace{2}_{f_x}\underbrace{(x-3)}_{dx} + \underbrace{-1/2}_{f_y}\underbrace{(y+1)}_{dy}}_{dz} + 4.$$

Thus the "new z-value" is the sum of the change in z (i.e., dz) and the old z-value (4). As mentioned when studying the total differential, it is not uncommon to know partial derivative information about a unknown function, and tangent planes are used to give accurate approximations of the function.

Tangent lines and planes to surfaces have many uses, including the study of instantaneous rates of changes and making approximations. Normal lines also have many uses. In this section we focused on using them to measure distances from a surface. Another interesting application is in computer graphics, where the effects of light on a surface are determined using normal vectors.

Exercises 11.5

Terms and Concepts

1. Explain how the vector $\vec{v} = \langle 1, 0, 3\rangle$ can be thought of as having a "slope" of 3.

2. Explain how the vector $\vec{v} = \langle 0.6, 0.8, -2\rangle$ can be thought of as having a "slope" of -2.

3. T/F: Let $z = f(x, y)$ be differentiable at P. If $\vec{n}$ is a normal vector to the tangent plane of f at P, then $\vec{n}$ is orthogonal to ℓ_x and ℓ_y at P.

4. Explain in your own words why we do not refer to *the* tangent line to a surface at a point, but rather to *directional* tangent line*s* to a surface at a point.

Problems

In Exercises 5 – 8, a function $z = f(x, y)$, a vector $\vec{v}$ and a point P are given. Give the parametric equations of the following directional tangent lines to f at P:

(a) $\ell_x(t)$

(b) $\ell_y(t)$

(c) $\ell_{\vec{u}}(t)$**, where $\vec{u}$ is the unit vector in the direction of $\vec{v}$.**

5. $f(x, y) = 2x^2y - 4xy^2$, $\vec{v} = \langle 1, 3\rangle$, $P = (2, 3)$.

6. $f(x, y) = 3\cos x \sin y$, $\vec{v} = \langle 1, 2\rangle$, $P = (\pi/3, \pi/6)$.

7. $f(x, y) = 3x - 5y$, $\vec{v} = \langle 1, 1\rangle$, $P = (4, 2)$.

8. $f(x, y) = x^2 - 2x - y^2 + 4y$, $\vec{v} = \langle 1, 1\rangle$, $P = (1, 2)$.

In Exercises 9 – 12, a function $z = f(x, y)$ and a point P are given. Find the equation of the normal line to f at P. Note: these are the same functions as in Exercises 5 – 8.

9. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

10. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

11. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

12. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

In Exercises 13 – 16, a function $z = f(x, y)$ and a point P are given. Find the two points that are 2 units from the surface f at P. Note: these are the same functions as in Exercises 5 – 8.

13. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

14. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

15. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

16. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

In Exercises 17 – 20, a function $z = f(x, y)$ and a point P are given. Find the equation of the tangent plane to f at P. Note: these are the same functions as in Exercises 5 – 8.

17. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

18. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

19. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

20. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

A: SOLUTIONS TO SELECTED PROBLEMS

Chapter 6

Section 6.1

1. Chain Rule.
3. $\frac{1}{8}(x^3-5)^8+C$
5. $\frac{1}{18}\left(x^2+1\right)^9+C$
7. $\frac{1}{2}\ln|2x+7|+C$
9. $\frac{2}{3}(x+3)^{3/2}-6(x+3)^{1/2}+C=\frac{2}{3}(x-6)\sqrt{x+3}+C$
11. $2e^{\sqrt{x}}+C$
13. $-\frac{1}{2x^2}-\frac{1}{x}+C$
15. $\frac{\sin^3(x)}{3}+C$
17. $-\frac{1}{6}\sin(3-6x)+C$
19. $\frac{1}{2}\ln|\sec(2x)+\tan(2x)|+C$
21. $\frac{\sin\left(x^2\right)}{2}+C$
23. The key is to rewrite $\cot x$ as $\cos x/\sin x$, and let $u=\sin x$.
25. $\frac{1}{3}e^{3x-1}+C$
27. $\frac{1}{2}e^{(x-1)^2}+C$
29. $\ln\left(e^x+1\right)+C$
31. $\frac{27^x}{\ln 27}+C$
33. $\frac{1}{2}\ln^2(x)+C$
35. $\frac{3}{2}(\ln x)^2+C$
37. $\frac{x^2}{2}+3x+\ln|x|+C$
39. $\frac{x^3}{3}-\frac{x^2}{2}+x-2\ln|x+1|+C$
41. $\frac{3}{2}x^2-8x+15\ln|x+1|+C$
43. $\sqrt{7}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right)+C$
45. $14\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$
47. $\frac{5}{4}\sec^{-1}(|x|/4)+C$
49. $\frac{\tan^{-1}\left(\frac{x-1}{\sqrt{7}}\right)}{\sqrt{7}}+C$
51. $3\sin^{-1}\left(\frac{x-4}{5}\right)+C$
53. $-\frac{1}{3(x^3+3)}+C$
55. $-\sqrt{1-x^2}+C$
57. $-\frac{2}{3}\cos^{\frac{3}{2}}(x)+C$
59. $\ln|x-5|+C$
61. $\frac{3x^2}{2}+\ln\left|x^2+3x+5\right|-5x+C$
63. $3\ln\left|3x^2+9x+7\right|+C$
65. $\frac{1}{18}\tan^{-1}\left(\frac{x^2}{9}\right)+C$
67. $\sec^{-1}(|2x|)+C$
69. $\frac{3}{2}\ln\left|x^2-2x+10\right|+\frac{1}{3}\tan^{-1}\left(\frac{x-1}{3}\right)+C$
71. $\frac{15}{2}\ln\left|x^2-10x+32\right|+x+\frac{41\tan^{-1}\left(\frac{x-5}{\sqrt{7}}\right)}{\sqrt{7}}+C$
73. $\frac{x^2}{2}+3\ln\left|x^2+4x+9\right|-4x+\frac{24\tan^{-1}\left(\frac{x+2}{\sqrt{5}}\right)}{\sqrt{5}}+C$
75. $\tan^{-1}(\sin(x))+C$
77. $3\sqrt{x^2-2x-6}+C$
79. $-\ln 2$
81. $2/3$
83. $(1-e)/2$
85. $\pi/2$

Section 6.2

1. T
3. Determining which functions in the integrand to set equal to "u" and which to set equal to "dv".
5. $\sin x-x\cos x+C$
7. $-x^2\cos x+2x\sin x+2\cos x+C$
9. $1/2e^{x^2}+C$
11. $-\frac{1}{2}xe^{-2x}-\frac{e^{-2x}}{4}+C$
13. $1/5e^{2x}(\sin x+2\cos x)+C$
15. $1/10e^{5x}(\sin(5x)+\cos(5x))+C$
17. $\sqrt{1-x^2}+x\sin^{-1}(x)+C$
19. $\frac{1}{2}x^2\tan^{-1}(x)-\frac{x}{2}+\frac{1}{2}\tan^{-1}(x)+C$
21. $\frac{1}{2}x^2\ln|x|-\frac{x^2}{4}+C$
23. $-\frac{x^2}{4}+\frac{1}{2}x^2\ln|x-1|-\frac{x}{2}-\frac{1}{2}\ln|x-1|+C$
25. $\frac{1}{3}x^3\ln|x|-\frac{x^3}{9}+C$
27. $(x+1)(\ln(x+1))^2-2(x+1)\ln(x+1)+2(x+1)+C$
29. $\ln|\sin(x)|-x\cot(x)+C$
31. $\frac{1}{3}(x^2-2)^{3/2}+C$
33. $x\sec x-\ln|\sec x+\tan x|+C$
35. $1/2x\big(\sin(\ln x)-\cos(\ln x)\big)+C$
37. $2\sin\left(\sqrt{x}\right)-2\sqrt{x}\cos\left(\sqrt{x}\right)+C$
39. $2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}+C$
41. π
43. 0
45. $1/2$
47. $\frac{3}{4e^2}-\frac{5}{4e^4}$
49. $1/5\left(e^{\pi}+e^{-\pi}\right)$

Section 6.3

1. F
3. F
5. $-\frac{1}{5}\cos^5(x)+C$
7. $\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x+C$
9. $\frac{1}{11}\sin^{11}x-\frac{2}{9}\sin^9 x+\frac{1}{7}\sin^7 x+C$
11. $\frac{x}{8}-\frac{1}{32}\sin(4x)+C$
13. $\frac{1}{2}\left(-\frac{1}{8}\cos(8x)-\frac{1}{2}\cos(2x)\right)+C$

15. $\frac{1}{2}\left(\frac{1}{4}\sin(4x)-\frac{1}{10}\sin(10x)\right)+C$

17. $\frac{1}{2}\left(\sin(x)+\frac{1}{3}\sin(3x)\right)+C$

19. $\frac{\tan^5(x)}{5}+C$

21. $\frac{\tan^6(x)}{6}+\frac{\tan^4(x)}{4}+C$

23. $\frac{\sec^5(x)}{5}-\frac{\sec^3(x)}{3}+C$

25. $\frac{1}{3}\tan^3 x-\tan x+x+C$

27. $\frac{1}{2}(\sec x\tan x-\ln|\sec x+\tan x|)+C$

29. $\frac{2}{5}$

31. $32/315$

33. $2/3$

35. $16/15$

Section 6.4

1. backwards

3. (a) $\tan^2\theta+1=\sec^2\theta$
   (b) $9\sec^2\theta$.

5. $\frac{1}{2}\left(x\sqrt{x^2+1}+\ln|\sqrt{x^2+1}+x|\right)+C$

7. $\frac{1}{2}\left(\sin^{-1}x+x\sqrt{1-x^2}\right)+C$

9. $\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln|x+\sqrt{x^2-1}|+C$

11. $x\sqrt{x^2+1/4}+\frac{1}{4}\ln|2\sqrt{x^2+1/4}+2x|+C=$
$\frac{1}{2}x\sqrt{4x^2+1}+\frac{1}{4}\ln|\sqrt{4x^2+1}+2x|+C$

13. $4\left(\frac{1}{2}x\sqrt{x^2-1/16}-\frac{1}{32}\ln|4x+4\sqrt{x^2-1/16}|\right)+C=$
$\frac{1}{2}x\sqrt{16x^2-1}-\frac{1}{8}\ln|4x+\sqrt{16x^2-1}|+C$

15. $3\sin^{-1}\left(\frac{x}{\sqrt{7}}\right)+C$ (Trig. Subst. is not needed)

17. $\sqrt{x^2-11}-\sqrt{11}\sec^{-1}(x/\sqrt{11})+C$

19. $\sqrt{x^2-3}+C$ (Trig. Subst. is not needed)

21. $-\frac{1}{\sqrt{x^2+9}}+C$ (Trig. Subst. is not needed)

23. $\frac{1}{18}\frac{x+2}{x^2+4x+13}+\frac{1}{54}\tan^{-1}\left(\frac{x+2}{2}\right)+C$

25. $\frac{1}{7}\left(-\frac{\sqrt{5-x^2}}{x}-\sin^{-1}(x/\sqrt{5})\right)+C$

27. $\pi/2$

29. $2\sqrt{2}+2\ln(1+\sqrt{2})$

31. $9\sin^{-1}(1/3)+\sqrt{8}$ Note: the new lower bound is $\theta=\sin^{-1}(-1/3)$ and the new upper bound is $\theta=\sin^{-1}(1/3)$. The final answer comes with recognizing that $\sin^{-1}(-1/3)=-\sin^{-1}(1/3)$ and that $\cos\left(\sin^{-1}(1/3)\right)=\cos\left(\sin^{-1}(-1/3)\right)=\sqrt{8}/3$.

Section 6.5

1. rational

3. $\frac{A}{x}+\frac{B}{x-3}$

5. $\frac{A}{x-\sqrt{7}}+\frac{B}{x+\sqrt{7}}$

7. $3\ln|x-2|+4\ln|x+5|+C$

9. $\frac{1}{3}(\ln|x+2|-\ln|x-2|)+C$

11. $\ln|x+5|-\frac{2}{x+5}+C$

13. $\frac{5}{x+1}+7\ln|x|+2\ln|x+1|+C$

15. $-\frac{1}{5}\ln|5x-1|+\frac{2}{3}\ln|3x-1|+\frac{3}{7}\ln|7x+3|+C$

17. $\frac{x^2}{2}+x+\frac{125}{9}\ln|x-5|+\frac{64}{9}\ln|x+4|-\frac{35}{2}+C$

19. $\frac{1}{6}\left(-\ln\left|x^2+2x+3\right|+2\ln|x|-\sqrt{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right)+C$

21. $\ln\left|3x^2+5x-1\right|+2\ln|x+1|+C$

23. $\frac{9}{10}\ln\left|x^2+9\right|+\frac{1}{5}\ln|x+1|-\frac{4}{15}\tan^{-1}\left(\frac{x}{3}\right)+C$

25. $3\left(\ln\left|x^2-2x+11\right|+\ln|x-9|\right)+3\sqrt{\frac{2}{5}}\tan^{-1}\left(\frac{x-1}{\sqrt{10}}\right)+C$

27. $\ln(2000/243)\approx 2.108$

29. $-\pi/4+\tan^{-1}3-\ln(11/9)\approx 0.263$

Section 6.6

1. The interval of integration is finite, and the integrand is continuous on that interval.

3. converges; could also state <10.

5. $p>1$

7. $e^5/2$

9. $1/3$

11. $1/\ln 2$

13. diverges

15. 1

17. diverges

19. diverges

21. diverges

23. 1

25. 0

27. $-1/4$

29. diverges

31. 1

33. $1/2$

35. diverges; Limit Comparison Test with $1/x$.

37. diverges; Limit Comparison Test with $1/x$.

39. converges; Direct Comparison Test with e^{-x}.

41. converges; Direct Comparison Test with $1/(x^2-1)$.

43. converges; Direct Comparison Test with $1/e^x$.

Chapter 7

Section 7.1

1. T

3. Answers will vary.

5. $16/3$

7. π

9. $2\sqrt{2}$

11. $4/3$

13. $4/3$

15. 8

17. $37/12$

19. On regions such as $[\pi/6,5\pi/6]$, the area is $3\sqrt{3}/2$. On regions such as $[-\pi/2,\pi/6]$, the area is $3\sqrt{3}/4$.

21. 1

23. $9/2$

25. $1/12(9-2\sqrt{2}) \approx 0.514$

27. 1

29. 4

31. 219,000 ft^2

Section 7.2

1. T

3. Recall that "dx" does not just "sit there;" it is multiplied by $A(x)$ and represents the thickness of a small slice of the solid. Therefore dx has units of in, giving $A(x)\,dx$ the units of in^3.

5. $48\pi\sqrt{3}/5$ units^3

7. $\pi^2/4$ units^3

9. $9\pi/2$ units^3

11. $\pi^2 - 2\pi$ units^3

13. (a) $\pi/2$
 (b) $5\pi/6$
 (c) $4\pi/5$
 (d) $8\pi/15$

15. (a) $4\pi/3$
 (b) $2\pi/3$
 (c) $4\pi/3$
 (d) $\pi/3$

17. (a) $\pi^2/2$
 (b) $\pi^2/2 - 4\pi\sinh^{-1}(1)$
 (c) $\pi^2/2 + 4\pi\sinh^{-1}(1)$

19. Placing the tip of the cone at the origin such that the x-axis runs through the center of the circular base, we have $A(x) = \pi x^2/4$. Thus the volume is $250\pi/3$ units^3.

21. Orient the cone such that the tip is at the origin and the x-axis is perpendicular to the base. The cross–sections of this cone are right, isosceles triangles with side length $2x/5$; thus the cross–sectional areas are $A(x) = 2x^2/25$, giving a volume of $80/3$ units^3.

Section 7.3

1. T

3. F

5. $9\pi/2$ units^3

7. $\pi^2 - 2\pi$ units^3

9. $48\pi\sqrt{3}/5$ units^3

11. $\pi^2/4$ units^3

13. (a) $4\pi/5$
 (b) $8\pi/15$
 (c) $\pi/2$
 (d) $5\pi/6$

15. (a) $4\pi/3$
 (b) $\pi/3$
 (c) $4\pi/3$
 (d) $2\pi/3$

17. (a) $2\pi(\sqrt{2}-1)$
 (b) $2\pi(1-\sqrt{2}+\sinh^{-1}(1))$

Section 7.4

1. T

3. $\sqrt{2}$

5. $4/3$

7. $109/2$

9. $12/5$

11. $-\ln(2-\sqrt{3}) \approx 1.31696$

13. $\int_0^1 \sqrt{1+4x^2}\,dx$

15. $\int_0^1 \sqrt{1+\frac{1}{4x}}\,dx$

17. $\int_{-1}^1 \sqrt{1+\frac{x^2}{1-x^2}}\,dx$

19. $\int_1^2 \sqrt{1+\frac{1}{x^4}}\,dx$

21. 1.4790

23. Simpson's Rule fails, as it requires one to divide by 0. However, recognize the answer should be the same as for $y = x^2$; why?

25. Simpson's Rule fails.

27. 1.4058

29. $2\pi\int_0^1 2x\sqrt{5}\,dx = 2\pi\sqrt{5}$

31. $2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx = \pi/27(10\sqrt{10}-1)$

33. $2\pi\int_0^1 \sqrt{1-x^2}\sqrt{1+x/(1-x^2)}\,dx = 4\pi$

Section 7.5

1. In SI units, it is one joule, i.e., one Newton–meter, or kg·m/s^2·m. In Imperial Units, it is ft–lb.

3. Smaller.

5. (a) 500 ft–lb
 (b) $100 - 50\sqrt{2} \approx 29.29$ ft–lb

7. (a) $\frac{1}{2}\cdot d\cdot l^2$ ft–lb
 (b) 75 %
 (c) $\ell(1-\sqrt{2}/2) \approx 0.2929\ell$

9. (a) 756 ft–lb
 (b) 60,000 ft–lb
 (c) Yes, for the cable accounts for about 1% of the total work.

11. 575 ft–lb

13. 0.05 J

15. 5/3 ft–lb

17. $f\cdot d/2$ J

19. 5 ft–lb

21. (a) 52,929.6 ft–lb
 (b) 18,525.3 ft–lb
 (c) When 3.83 ft of water have been pumped from the tank, leaving about 2.17 ft in the tank.

23. 212,135 ft–lb

25. 187,214 ft–lb

27. 4,917,150 J

Section 7.6

1. Answers will vary.

3. 499.2 lb
5. 6739.2 lb
7. 3920.7 lb
9. 2496 lb
11. 602.59 lb
13. (a) 2340 lb
    (b) 5625 lb
15. (a) 1597.44 lb
    (b) 3840 lb
17. (a) 56.42 lb
    (b) 135.62 lb
19. 5.1 ft

Chapter 8

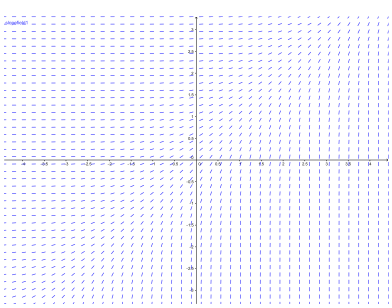
Section 8.1

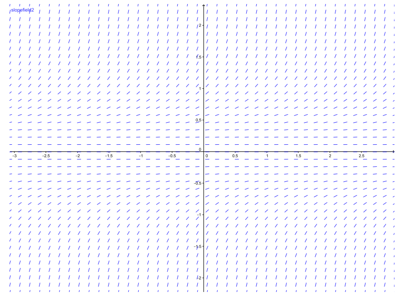
1. If $x = e^{4t}$, then $x' = 4e^{4t} = 4x$, $x'' = 16e^{4t} = 16x$, and $x''' = 64e^{4t} = 64x$. Thus $x''' - 12x'' + 48x' - 64x = 64x - 192x + 192x - 64x = 0$.
3. Yes: If $y = \sin t$ then $\dfrac{dy}{dt} = \cos t$ and $1 - \sin^2 t = \cos^2 t$.
5. Since $x(0) = Ce^0 = C$, we need $C = 100$. Verification is left to the student.
7. One option is $x(t) = 2\sin(t)$, since $x'(t) = 2\cos(t)$, and $(2\cos(t))^2 + (2\sin(t))^2 = 4$. There are other options.
9. Yes: any constant function will do the job.
11. Yes.
13. $C_1 = 100$, $C_2 = -90$

Section 8.2

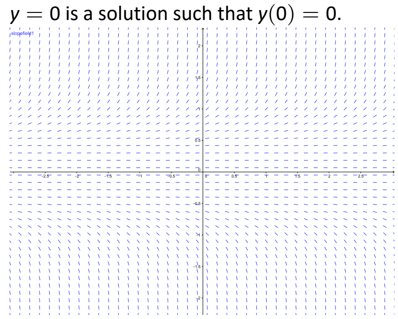
1. $y = \dfrac{1}{3}x^3 + \dfrac{1}{2}x^2 + \dfrac{13}{6}$
3. Equivalent answers: $y = \frac{1}{2}(\ln(1-x) - \ln(1+x))$ or $y = -\tanh^{-1}(x)$
5. Assuming $y \neq \pm 1$, we can write $\dfrac{dx}{dy} = \dfrac{1}{y^2 - 1}$, which gives $x = -\tanh^{-1}(y) + C$, so $\tanh^{-1}(y) = C - x$, and thus $y = \tanh(C - x)$. The condition $y(0) = 3$ gives $\tanh(C) = 3$, so $C = \tanh^{-1}(3)$.
7. Integrating once gives $y' = -\cos x + c$, and $y'(0) = 2$ implies $c = 3$. Integrating again gives $y = -\sin x + 3x + d$, and since $y(0) = 0$, $d = 0$.
9. Integrating gives $x(t) = \int_0^t \sin(u^2)\,du + \frac{1}{2}t^2 + 20$.
11. $x = (3t - 2)^{1/3}$
13. 170

Section 8.3

1. 

3. 

5. Yes, on both accounts. For $f(x, y) = y\sqrt{|x|}$, the function is continuous everywhere, and the partial derivative $\dfrac{\partial f}{\partial y}(x, y) = \sqrt{|x|}$ exists and is everywhere continuous.
7. We have to have $y \to \infty$ as $x \to$ *infty*: We begin at $(0, 0)$ and $f(0, 0) > 1$, so $y' > 1$. If we compare to the function $f(x) = x$, we have $y(0) = f(0)$ and $y'(x) > f'(x)$ for all x, which implies that $y(x) > f(x)$ for all x.
9. $y = 0$ is a solution such that $y(0) = 0$.



11. No, the equation is not defined at $(x, y) = (1, 0)$.

Section 8.4

1. $y^2 = x^2 + C$
3. $x = -\tanh(t^2/2)$
5. Notice that $xy + x + y + 1 = (x + 1)(y + 1)$. This gives $y = Ce^{x^2/2 + x} - 1$.
7. $y = \tan\left(\dfrac{\pi}{4} + \arctan x\right) = \dfrac{1 + x}{1 - x}$.
9. $y = \ln\left(\dfrac{x^2}{2} + e\right)$.
11. $y = \exp\left(\int_0^x e^{-t^2}\,dt\right)$.
13. $y = Ce^{x^2}$
15. $x^3 + x = t + 2$
17. $\sin(y) = -\cos(x) + C$

Section 8.5

In the exercises, feel free to leave answer as a definite integral if a closed form solution cannot be found. If you can find a closed form solution, you should give that.

1. The integrating factor is $r(x) = e^{x^2/2}$.
   The solution is $y = 1 + Ce^{-x^2/2}$.
3. The integrating factor is e^{x^3}.
   The solution is $y = e^{-x^3}(\sin x - x\cos x)$.

5. After rewriting as $y' + (x^2 + x)y = 3(x^2 + 1)$, we get the integrating factor $r(x) = e^{x^4/4 + x^2/2}$.
The solution is $y = 3e^{-x^4/4 - x^2/2} \int_0^x (t^2 + 1)e^{t^4/4 + t^2/2}\, dt$. (There is no closed form solution.)

7. (a) The general solution is

$$x(t) = \frac{\omega A_0}{\omega^2 - k^2}\left(\sin(\omega t) + \frac{k}{\omega}\cos(\omega t)\right) + Ce^{-kt}.$$

(We hope you haven't forgotten how to integrate by parts!)
(b)They won't. Since $k > 0$, the term that is determined by the initial conditions decays exponentially, so for $t >> 0$, there won't be much of a contribution from this term.

9. $k = 9/8$ grams per litre.

11. $y = 2e^{\cos(2x)+1} + 1$

13. $P(5) = 1000e^{2\times 5 - 0.05\times 5^2} = 1000e^{8.75} \approx 6.31 \times 10^6$

Section 8.6

1. $x(1) = 8.5$.

3. We get $y(1) \approx y_4 = 2.4414$.

5. Approximately: 1.0000, 1.2397, 1.3829

7. (a) 0, 0, 0
(b) $x = 0$ is a solution so errors are: 0, 0, 0.

Chapter 9

Section 9.1

1. Answers will vary.

3. Answers will vary.

5. $2, \frac{8}{3}, \frac{8}{3}, \frac{32}{15}, \frac{64}{45}$

7. $-\frac{1}{3}, -2, -\frac{81}{5}, -\frac{512}{3}, -\frac{15625}{7}$

9. $a_n = 3n + 1$

11. $a_n = 10 \cdot 2^{n-1}$

13. $1/7$

15. 0

17. diverges

19. converges to 0

21. diverges

23. converges to e

25. converges to 0

27. converges to 2

29. bounded

31. bounded

33. neither bounded above or below

35. monotonically increasing

37. never monotonic

39. Let $\{a_n\}$ be given such that $\lim\limits_{n\to\infty} |a_n| = 0$. By the definition of the limit of a sequence, given any $\varepsilon > 0$, there is a m such that for all $n > m$, $\big|\, |a_n| - 0 \big| < \varepsilon$. Since $\big|\, |a_n| - 0 \big| = |a_n - 0|$, this directly implies that for all $n > m$, $|a_n - 0| < \varepsilon$, meaning that $\lim\limits_{n\to\infty} a_n = 0$.

41. A sketch of one proof method:
Let any $\varepsilon > 0$ be given. Since $\{a_n\}$ and $\{b_n\}$ converge, there exists an $N > 0$ such that for all $n \geq N$, both a_n and b_n are within $\varepsilon/2$ of L; we can conclude that they are at most ε apart from each other. Since $a_n \leq c_n \leq b_n$, one can show that c_n is within ε of L, showing that $\{c_n\}$ also converges to L.

Section 9.2

1. Answers will vary.

3. One sequence is the sequence of terms $\{a_n\}$. The other is the sequence of n^{th} partial sums, $\{S_n\} = \{\sum_{i=1}^n a_i\}$.

5. F

7. (a) $-1, -\frac{1}{2}, -\frac{5}{6}, -\frac{7}{12}, -\frac{47}{60}$
(b) Plot omitted

9. (a) $-1, 0, -1, 0, -1$
(b) Plot omitted

11. (a) $1, \frac{3}{2}, \frac{5}{3}, \frac{41}{24}, \frac{103}{60}$
(b) Plot omitted

13. (a) $-0.9, -0.09, -0.819, -0.1629, -0.75339$
(b) Plot omitted

15. $\lim\limits_{n\to\infty} a_n = 3$; by Theorem 9.2.4 the series diverges.

17. $\lim\limits_{n\to\infty} a_n = \infty$; by Theorem 9.2.4 the series diverges.

19. $\lim\limits_{n\to\infty} a_n = 1/2$; by Theorem 9.2.4 the series diverges.

21. Converges; p-series with $p = 5$.

23. Diverges; geometric series with $r = 6/5$.

25. Diverges; fails n^{th} term test

27. F

29. Diverges; by Theorem 9.2.3 this is half the Harmonic Series, which diverges by growing without bound. "Half of growing without bound" is still growing without bound.

31. (a) $S_n = \frac{1-(1/4)^n}{3/4}$
(b) Converges to $4/3$.

33. (a) $S_n = \left(\frac{n(n+1)}{2}\right)^2$
(b) Diverges

35. (a) $S_n = 5\frac{1-1/2^n}{1/2}$
(b) Converges to 10.

37. (a) $S_n = \frac{1-(-1/3)^n}{4/3}$
(b) Converges to $3/4$.

39. (a) With partial fractions, $a_n = \frac{3}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$. Thus $S_n = \frac{3}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$.
(b) Converges to $9/4$

41. (a) $S_n = \ln\big(1/(n+1)\big)$
(b) Diverges (to $-\infty$).

43. (a) $a_n = \frac{1}{n(n+3)}$; using partial fractions, the resulting telescoping sum reduces to
$S_n = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$
(b) Converges to $11/18$.

45. (a) With partial fractions, $a_n = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$. Thus $S_n = \frac{1}{2}\left(3/2 - \frac{1}{n} - \frac{1}{n+1}\right)$.
(b) Converges to $3/4$.

47. (a) The n^{th} partial sum of the odd series is $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$. The n^{th} partial sum of the even series is $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}$. Each term of the even series is less than the corresponding term of the odd series, giving us our result.

(b) The n^{th} partial sum of the odd series is $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$. The n^{th} partial sum of 1 plus the even series is $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2(n-1)}$. Each term of the even series is now greater than or equal to the corresponding term of the odd series, with equality only on the first term. This gives us the result.

(c) If the odd series converges, the work done in (a) shows the even series converges also. (The sequence of the n^{th} partial sum of the even series is bounded and monotonically increasing.) Likewise, (b) shows that if the even series converges, the odd series will, too. Thus if either series converges, the other does.

Similarly, (a) and (b) can be used to show that if either series diverges, the other does, too.

(d) If both the even and odd series converge, then their sum would be a convergent series. This would imply that the Harmonic Series, their sum, is convergent. It is not. Hence each series diverges.

Section 9.3

1. continuous, positive and decreasing
3. The Integral Test (we do not have a continuous definition of $n!$ yet) and the Limit Comparison Test (same as above, hence we cannot take its derivative).
5. Converges
7. Diverges
9. Converges
11. Converges
13. Converges; compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, as $1/(n^2 + 3n - 5) \leq 1/n^2$ for all $n > 1$.
15. Diverges; compare to $\sum_{n=1}^{\infty} \frac{1}{n}$, as $1/n \leq \ln n/n$ for all $n \geq 3$.
17. Diverges; compare to $\sum_{n=1}^{\infty} \frac{1}{n}$. Since $n = \sqrt{n^2} > \sqrt{n^2 - 1}$, $1/n \leq 1/\sqrt{n^2 - 1}$ for all $n \geq 2$.
19. Diverges; compare to $\sum_{n=1}^{\infty} \frac{1}{n}$:
$$\frac{1}{n} = \frac{n^2}{n^3} < \frac{n^2 + n + 1}{n^3} < \frac{n^2 + n + 1}{n^3 - 5},$$
for all $n \geq 1$.
21. Diverges; compare to $\sum_{n=1}^{\infty} \frac{1}{n}$. Note that
$$\frac{n}{n^2 - 1} = \frac{n^2}{n^2 - 1} \cdot \frac{1}{n} > \frac{1}{n},$$
as $\frac{n^2}{n^2-1} > 1$, for all $n \geq 2$.
23. Converges; compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
25. Diverges; compare to $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.
27. Diverges; compare to $\sum_{n=1}^{\infty} \frac{1}{n}$.
29. Diverges; compare to $\sum_{n=1}^{\infty} \frac{1}{n}$. Just as $\lim_{n\to 0} \frac{\sin n}{n} = 1$, $\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$.
31. Converges; compare to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.
33. Converges; Integral Test
35. Diverges; the n^{th} Term Test and Direct Comparison Test can be used.
37. Converges; the Direct Comparison Test can be used with sequence $1/3^n$.
39. Diverges; the n^{th} Term Test can be used, along with the Integral Test.
41. (a) Converges; use Direct Comparison Test as $\frac{a_n}{n} < n$.

(b) Converges; since original series converges, we know $\lim_{n\to\infty} a_n = 0$. Thus for large n, $a_n a_{n+1} < a_n$.

(c) Converges; similar logic to part (b) so $(a_n)^2 < a_n$.

(d) May converge; certainly $na_n > a_n$ but that does not mean it does not converge.

(e) Does not converge, using logic from (b) and n^{th} Term Test.

Section 9.4

1. algebraic, or polynomial.
3. Integral Test, Limit Comparison Test, and Root Test
5. Converges
7. Converges
9. The Ratio Test is inconclusive; the p-Series Test states it diverges.
11. Converges
13. Converges; note the summation can be rewritten as $\sum_{n=1}^{\infty} \frac{2^n n!}{3^n n!}$, to which the Ratio Test or Geometric Series Test can be applied.
15. Converges
17. Converges
19. Diverges
21. Diverges. The Root Test is inconclusive, but the n^{th}-Term Test shows divergence. (The terms of the sequence approach e^2, not 0, as $n \to \infty$.)
23. Converges
25. Diverges; Limit Comparison Test with $1/n$.
27. Converges; Ratio Test or Limit Comparison Test with $1/3^n$.
29. Diverges; n^{th}-Term Test or Limit Comparison Test with 1.
31. Diverges; Direct Comparison Test with $1/n$
33. Converges; Root Test

Section 9.5

1. The signs of the terms do not alternate; in the given series, some terms are negative and the others positive, but they do not necessarily alternate.
3. Many examples exist; one common example is $a_n = (-1)^n/n$.
5. (a) converges

(b) converges (p-Series)

(c) absolute

7. (a) diverges (limit of terms is not 0)
   (b) diverges
   (c) n/a; diverges
9. (a) converges
   (b) diverges (Limit Comparison Test with $1/n$)
   (c) conditional
11. (a) diverges (limit of terms is not 0)
   (b) diverges
   (c) n/a; diverges
13. (a) diverges (terms oscillate between ± 1)
   (b) diverges
   (c) n/a; diverges
15. (a) converges
   (b) converges (Geometric Series with $r = 2/3$)
   (c) absolute
17. (a) converges
   (b) converges (Ratio Test)
   (c) absolute
19. (a) converges
   (b) diverges (p-Series Test with $p = 1/2$)
   (c) conditional
21. $S_5 = -1.1906$; $S_6 = -0.6767$;
   $$-1.1906 \leq \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} \leq -0.6767$$
23. $S_6 = 0.3681$; $S_7 = 0.3679$;
   $$0.3681 \leq \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \leq 0.3679$$
25. $n = 5$
27. Using the theorem, we find $n = 499$ guarantees the sum is within 0.001 of $\pi/4$. (Convergence is actually faster, as the sum is within ε of $\pi/24$ when $n \geq 249$.)

Section 9.6

1. 1
3. 5
5. $1 + 2x + 4x^2 + 8x^3 + 16x^4$
7. $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$
9. (a) $R = \infty$
   (b) $(-\infty, \infty)$
11. (a) $R = 1$
   (b) $(2, 4]$
13. (a) $R = 2$
   (b) $(-2, 2)$
15. (a) $R = 1/5$
   (b) $(4/5, 6/5)$
17. (a) $R = 1$
   (b) $(-1, 1)$
19. (a) $R = \infty$
   (b) $(-\infty, \infty)$
21. (a) $R = 1$
   (b) $[-1, 1]$
23. (a) $R = 0$
   (b) $x = 0$
25. (a) $f'(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}; \quad (-1, 1)$
   (b) $\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{n}{n+1} x^{n+1}; \quad (-1, 1)$
27. (a) $f'(x) = \sum_{n=1}^{\infty} \frac{n}{2^n} x^{n-1}; \quad (-2, 2)$
   (b) $\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{1}{(n+1)2^n} x^{n+1}; \quad [-2, 2)$
29. (a) $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!};$ $(-\infty, \infty)$
   (b) $\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}; \quad (-\infty, \infty)$
31. $1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$
33. $1 + x + x^2 + x^3 + x^4$
35. $0 + x + 0x^2 - \frac{1}{6}x^3 + 0x^4$

Section 9.7

1. The Maclaurin polynomial is a special case of Taylor polynomials. Taylor polynomials are centered at a specific x-value; when that x-value is 0, it is a Maclauring polynomial.
3. $p_2(x) = 6 + 3x - 4x^2$.
5. $p_3(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$
7. $p_5(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$
9. $p_4(x) = \frac{2x^4}{3} + \frac{4x^3}{3} + 2x^2 + 2x + 1$
11. $p_4(x) = x^4 - x^3 + x^2 - x + 1$
13. $p_4(x) = 1 + \frac{1}{2}(-1+x) - \frac{1}{8}(-1+x)^2 + \frac{1}{16}(-1+x)^3 - \frac{5}{128}(-1+x)^4$
15. $p_6(x) = \frac{1}{\sqrt{2}} - \frac{-\frac{\pi}{4}+x}{\sqrt{2}} - \frac{(-\frac{\pi}{4}+x)^2}{2\sqrt{2}} + \frac{(-\frac{\pi}{4}+x)^3}{6\sqrt{2}} + \frac{(-\frac{\pi}{4}+x)^4}{24\sqrt{2}} - \frac{(-\frac{\pi}{4}+x)^5}{120\sqrt{2}} - \frac{(-\frac{\pi}{4}+x)^6}{720\sqrt{2}}$
17. $p_5(x) = \frac{1}{2} - \frac{x-2}{4} + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4 - \frac{1}{64}(x-2)^5$
19. $p_3(x) = \frac{1}{2} + \frac{1+x}{2} + \frac{1}{4}(1+x)^2$
21. $p_3(x) = x - \frac{x^3}{6}$; $p_3(0.1) = 0.09983$. Error is bounded by $\pm\frac{1}{4!} \cdot 0.1^4 \approx \pm 0.000004167$.
23. $p_2(x) = 3 + \frac{1}{6}(-9+x) - \frac{1}{216}(-9+x)^2$; $p_2(10) = 3.16204$. The third derivative of $f(x) = \sqrt{x}$ is bounded on $(8, 11)$ by 0.003. Error is bounded by $\pm\frac{0.003}{3!} \cdot 1^3 = \pm 0.0005$.
25. The n^{th} derivative of $f(x) = e^x$ is bounded by 3 on intervals containing 0 and 1. Thus $|R_n(1)| \leq \frac{3}{(n+1)!} 1^{(n+1)}$. When $n = 7$, this is less than 0.0001.
27. The n^{th} derivative of $f(x) = \cos x$ is bounded by 1 on intervals containing 0 and $\pi/3$. Thus $|R_n(\pi/3)| \leq \frac{1}{(n+1)!}(\pi/3)^{(n+1)}$. When $n = 7$, this is less than 0.0001. Since the Maclaurin polynomial of $\cos x$ only uses even powers, we can actually just use $n = 6$.
29. The n^{th} term is $\frac{1}{n!}x^n$.
31. The n^{th} term is: when n even, 0; when n is odd, $\frac{(-1)^{(n-1)/2}}{n!}x^n$.

33. The n^{th} term is $(-1)^n x^n$.

35. $1+x+\dfrac{1}{2}x^2+\dfrac{1}{6}x^3+\dfrac{1}{24}x^4$

37. $1+2x-2x^2+4x^3-10x^4$

Section 9.8

1. A Taylor polynomial is a **polynomial**, containing a finite number of terms. A Taylor series is a **series**, the summation of an infinite number of terms.

3. All derivatives of e^x are e^x which evaluate to 1 at $x=0$.
The Taylor series starts $1+x+\frac{1}{2}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\cdots$;
the Taylor series is $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$

5. The n^{th} derivative of $1/(1-x)$ is $f^{(n)}(x)=(n)!/(1-x)^{n+1}$, which evaluates to $n!$ at $x=0$.
The Taylor series starts $1+x+x^2+x^3+\cdots$;
the Taylor series is $\displaystyle\sum_{n=0}^{\infty}x^n$

7. The Taylor series starts
$0-(x-\pi/2)+0x^2+\frac{1}{6}(x-\pi/2)^3+0x^4-\frac{1}{120}(x-\pi/2)^5$;
the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi/2)^{2n+1}}{(2n+1)!}$

9. $f^{(n)}(x)=(-1)^n e^{-x}$; at $x=0$, $f^{(n)}(0)=-1$ when n is odd and $f^{(n)}(0)=1$ when n is even.
The Taylor series starts $1-x+\frac{1}{2}x^2-\frac{1}{3!}x^3+\cdots$;
the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$.

11. $f^{(n)}(x)=(-1)^{n+1}\frac{n!}{(x+1)^{n+1}}$; at $x=1$, $f^{(n)}(1)=(-1)^{n+1}\frac{n!}{2^{n+1}}$
The Taylor series starts
$\frac{1}{2}+\frac{1}{4}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3\cdots$;
the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{2^{n+1}}$.

13. Given a value x, the magnitude of the error term $R_n(x)$ is bounded by
$$\left|R_n(x)\right|\le\frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x^{(n+1)}\right|,$$
where z is between 0 and x.
If $x>0$, then $z<x$ and $f^{(n+1)}(z)=e^z<e^x$. If $x<0$, then $x<z<0$ and $f^{(n+1)}(z)=e^z<1$. So given a fixed x value, let $M=\max\{e^x,1\}$; $f^{(n)}(z)<M$. This allows us to state
$$\left|R_n(x)\right|\le\frac{M}{(n+1)!}\left|x^{(n+1)}\right|.$$
For any x, $\displaystyle\lim_{n\to\infty}\frac{M}{(n+1)!}\left|x^{(n+1)}\right|=0$. Thus by the Squeeze Theorem, we conclude that $\displaystyle\lim_{n\to\infty}R_n(x)=0$ for all x, and hence
$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\quad\text{for all } x.$$

15. Given a value x, the magnitude of the error term $R_n(x)$ is bounded by
$$\left|R_n(x)\right|\le\frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(x-1)^{(n+1)}\right|,$$
where z is between 1 and x.
Note that $\left|f^{(n+1)}(x)\right|=\frac{n!}{x^{n+1}}$.
Per the statement of the problem, we only consider the case $1<x<2$.
If $1<x<2$, then $1<z<x$ and $f^{(n+1)}(z)=\frac{n!}{z^{n+1}}<n!$. Thus
$$\left|R_n(x)\right|\le\frac{n!}{(n+1)!}\left|(x-1)^{(n+1)}\right|=\frac{(x-1)^{n+1}}{n+1}<\frac{1}{n+1}.$$
Thus
$$\lim_{n\to\infty}\left|R_n(x)\right|<\lim_{n\to\infty}\frac{1}{n+1}=0,$$
hence
$$\ln x=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}\text{ on }(1,2).$$

17. Given $\cos x=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$,
$\cos(-x)=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(-x)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=\cos x$, as all powers in the series are even.

19. Given $\sin x=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$,
$\displaystyle\frac{d}{dx}\big(\sin x\big)=\frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)=$
$\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=\cos x$. (The summation still starts at $n=0$ as there was no constant term in the expansion of $\sin x$).

21. $1+\dfrac{x}{2}-\dfrac{x^2}{8}+\dfrac{x^3}{16}-\dfrac{5x^4}{128}$

23. $1+\dfrac{x}{3}-\dfrac{x^2}{9}+\dfrac{5x^3}{81}-\dfrac{10x^4}{243}$

25. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!}$.

27. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(2x+3)^{2n+1}}{(2n+1)!}$.

29. $x+x^2+\dfrac{x^3}{3}-\dfrac{x^5}{30}$

31. $\displaystyle\int_0^{\sqrt{\pi}}\sin\left(x^2\right)\,dx\approx\int_0^{\sqrt{\pi}}\left(x^2-\frac{x^6}{6}+\frac{x^{10}}{120}-\frac{x^{14}}{5040}\right)dx=$ 0.8877

Chapter 10

Section 10.1

1. When defining the conics as the intersections of a plane and a double napped cone, degenerate conics are created when the plane intersects the tips of the cones (usually taken as the origin). Nondegenerate conics are formed when this plane does not contain the origin.

3. Hyperbola

5. With a horizontal transverse axis, the x^2 term has a positive coefficient; with a vertical transverse axis, the y^2 term has a positive coefficient.

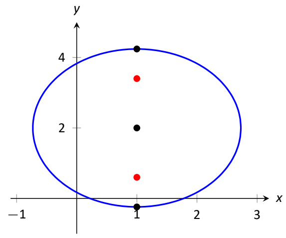
7. $y=\frac{1}{2}(x-3)^2+\frac{3}{2}$

9. $x=-\frac{1}{4}(y-5)^2+2$

11. $y=-\frac{1}{4}(x-1)^2+2$

13. $y=4x^2$

15. focus: $(0,1)$; directrix: $y=-1$. The point P is 2 units from each.

17. 

19. $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$; foci at $(-1 \pm \sqrt{5}, 2)$; $e = \sqrt{5}/3$

21. $\frac{x^2}{9} + \frac{y^2}{5} = 1$

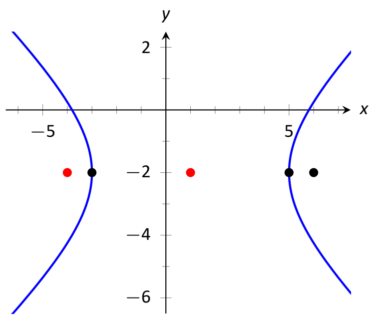
23. $\frac{(x-2)^2}{45} + \frac{y^2}{49} = 1$

25. $\frac{(x-1)^2}{2} + (y-2)^2 = 1$

27. $\frac{x^2}{4} + \frac{(y-3)^2}{6} = 1$

29. $x^2 - \frac{y^2}{3} = 1$

31. $\frac{(y-3)^2}{4} - \frac{(x-1)^2}{9} = 1$

33. 

35. $\frac{x^2}{4} - \frac{y^2}{5} = 1$

37. $\frac{(x-3)^2}{16} - \frac{(y-3)^2}{9} = 1$

39. $\frac{x^2}{4} - \frac{y^2}{3} = 1$

41. $(y-2)^2 - \frac{x^2}{10} = 1$

43. (a) $c = \sqrt{12-4} = 2\sqrt{2}$.

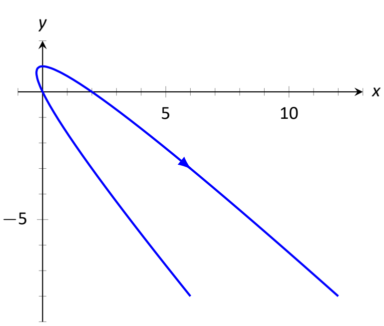
    (b) The sum of distances for each point is $2\sqrt{12} \approx 6.9282$.

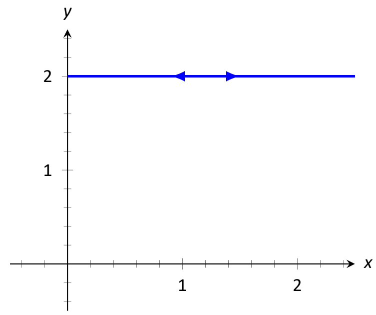
45. The sound originated from a point approximately 31m to the left of *B* and 1340m above it.

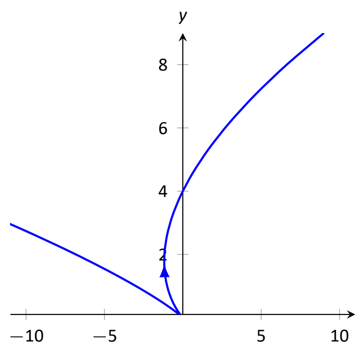
Section 10.2

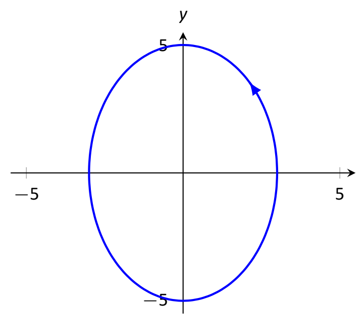
1. T

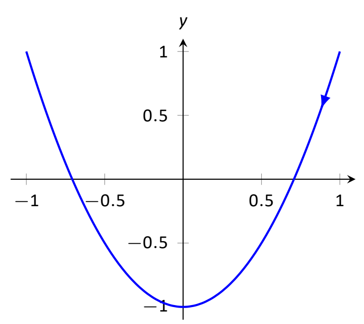
3. rectangular

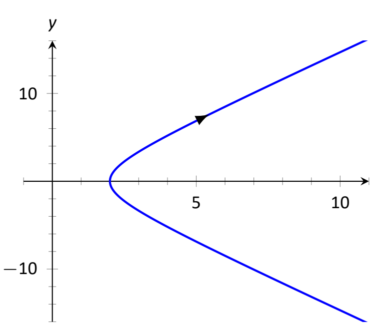
5. 

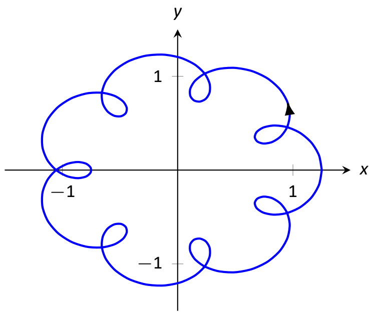
7. 

9. 

11. 

13. 

15. 

17. 

19. (a) Traces the parabola $y = x^2$, moves from left to right.
    (b) Traces the parabola $y = x^2$, but only from $-1 \leq x \leq 1$; traces this portion back and forth infinitely.
    (c) Traces the parabola $y = x^2$, but only for $0 < x$. Moves left to right.
    (d) Traces the parabola $y = x^2$, moves from right to left.

21. $y = -1.5x + 8.5$

23. $\frac{(x-1)^2}{16} + \frac{(y+2)^2}{9} = 1$

25. $y = 2x + 3$

27. $y = e^{2x} - 1$

29. $x^2 - y^2 = 1$

31. $y = \frac{b}{a}(x - x_0) + y_0$; line through (x_0, y_0) with slope b/a.

33. $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$; ellipse centered at (h, k) with horizontal axis of length $2a$ and vertical axis of length $2b$.

35. $x = (t + 11)/6$, $y = (t^2 - 97)/12$. At $t = 1$, $x = 2$, $y = -8$.
    $y' = 6x - 11$; when $x = 2$, $y' = 1$.

37. $x = \cos^{-1} t$, $y = \sqrt{1 - t^2}$. At $t = 1$, $x = 0$, $y = 0$.
    $y' = \cos x$; when $x = 0$, $y' = 1$.

39. $t = \pm 1$

41. $t = \pi/2, 3\pi/2$

43. $t = -1$

45. $t = \ldots \pi/2,\ 3\pi/2,\ 5\pi/2,\ \ldots$

47. $x = 4t$, $y = -16t^2 + 64t$

49. $x = 10t$, $y = -16t^2 + 320t$

51. $x = 3\cos(2\pi t) + 1$, $y = 3\sin(2\pi t) + 1$; other answers possible

53. $x = 5\cos t$, $y = \sqrt{24}\sin t$; other answers possible

55. $x = 2\tan t$, $y = \pm 6\sec t$; other answers possible

Section 10.3

1. F

3. F

5. (a) $\frac{dy}{dx} = 2t$
   (b) Tangent line: $y = 2(x - 1) + 1$; normal line: $y = -1/2(x - 1) + 1$

7. (a) $\frac{dy}{dx} = \frac{2t+1}{2t-1}$
   (b) Tangent line: $y = 3x + 2$; normal line: $y = -1/3x + 2$

9. (a) $\frac{dy}{dx} = \csc t$
   (b) $t = \pi/4$: Tangent line: $y = \sqrt{2}(x - \sqrt{2}) + 1$; normal line: $y = -1/\sqrt{2}(x - \sqrt{2}) + 1$

11. (a) $\frac{dy}{dx} = \frac{\cos t\sin(2t) + \sin t\cos(2t)}{-\sin t\sin(2t) + 2\cos t\cos(2t)}$
    (b) Tangent line: $y = x - \sqrt{2}$; normal line: $y = -x - \sqrt{2}$

13. $t = 0$

15. $t = -1/2$

17. The graph does not have a horizontal tangent line.

19. The solution is non-trivial; use identities $\sin(2t) = 2\sin t\cos t$ and $\cos(2t) = \cos^2 t - \sin^2 t$ to rewrite $g'(t) = 2\sin t(2\cos^2 t - \sin^2 t)$. On $[0, 2\pi]$, $\sin t = 0$ when $t = 0, \pi, 2\pi$, and $2\cos^2 t - \sin^2 t = 0$ when $t = \tan^{-1}(\sqrt{2}),\ \pi \pm \tan^{-1}(\sqrt{2}),\ 2\pi - \tan^{-1}(\sqrt{2})$.

21. $t_0 = 0$; $\lim_{t\to 0} \frac{dy}{dx} = 0$.

23. $t_0 = 1$; $\lim_{t\to 1} \frac{dy}{dx} = \infty$.

25. $\frac{d^2y}{dx^2} = 2$; always concave up

27. $\frac{d^2y}{dx^2} = -\frac{4}{(2t-1)^3}$; concave up on $(-\infty, 1/2)$; concave down on $(1/2, \infty)$.

29. $\frac{d^2y}{dx^2} = -\cot^3 t$; concave up on $(-\infty, 0)$; concave down on $(0, \infty)$.

31. $\frac{d^2y}{dx^2} = \frac{4(13+3\cos(4t))}{(\cos t+3\cos(3t))^3}$, obtained with a computer algebra system; concave up on $\left(-\tan^{-1}(\sqrt{2}/2), \tan^{-1}(\sqrt{2}/2)\right)$, concave down on $\left(-\pi/2, -\tan^{-1}(\sqrt{2}/2)\right) \cup \left(\tan^{-1}(\sqrt{2}/2), \pi/2\right)$

33. $L = 6\pi$

35. $L = 2\sqrt{34}$

37. $L \approx 2.4416$ (actual value: $L = 2.42211$)

39. $L \approx 4.19216$ (actual value: $L = 4.18308$)

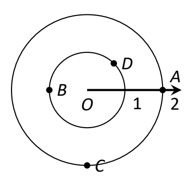
41. The answer is 16π for both (of course), but the integrals are different.

43. $SA \approx 8.50101$ (actual value $SA = 8.02851$)

Section 10.4

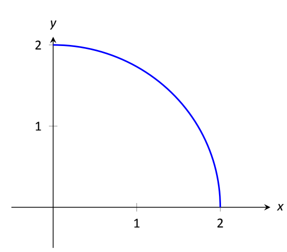
1. Answers will vary.

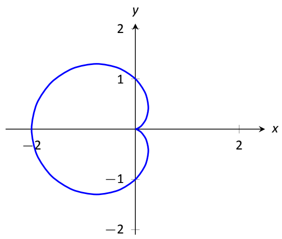
3. T

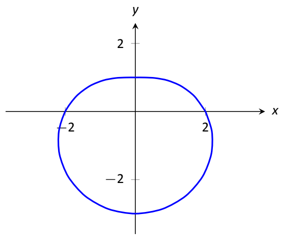
5. 

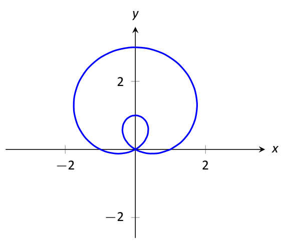
7. $A = P(2.5, \pi/4)$ and $P(-2.5, 5\pi/4)$;
   $B = P(-1, 5\pi/6)$ and $P(1, 11\pi/6)$;
   $C = P(3, 4\pi/3)$ and $P(-3, \pi/3)$;
   $D = P(1.5, 2\pi/3)$ and $P(-1.5, 5\pi/3)$;

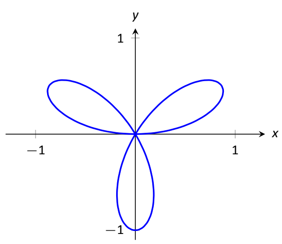
9. $A = (\sqrt{2}, \sqrt{2})$
   $B = (\sqrt{2}, -\sqrt{2})$
   $C = P(\sqrt{5}, -0.46)$
   $D = P(\sqrt{5}, 2.68)$

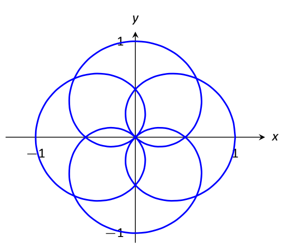
11. 

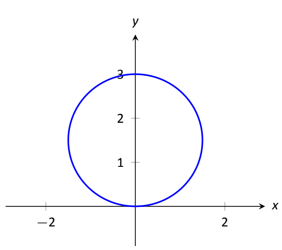
13. 

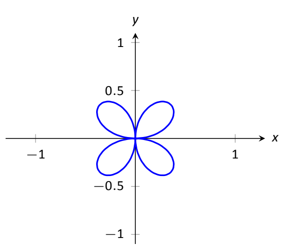
15. 

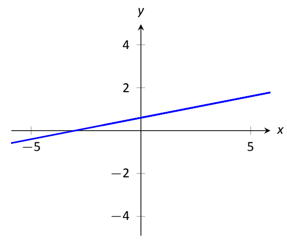
17. 

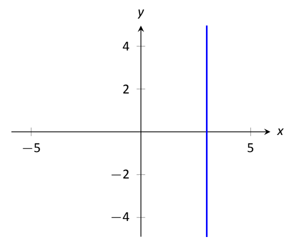
19. 

21. 

23. 

25. 

27. 

29. 

31. $(x-3)^2+y^2=3$

33. $(x-1/2)^2+(y-1/2)^2=1/2$

35. $x=3$

37. $x^4+x^2y^2-y^2=0$

39. $x^2+y^2=4$

41. $\theta=\pi/4$

43. $r=5\sec\theta$

45. $r=\cos\theta/\sin^2\theta$

47. $r=\sqrt{7}$

49. $P(\sqrt{3}/2,\pi/6)$, $P(0,\pi/2)$, $P(-\sqrt{3}/2,5\pi/6)$

51. $P(0,0)=P(0,\pi/2)$, $P(\sqrt{2},\pi/4)$

53. $P(\sqrt{2}/2,\pi/12)$, $P(-\sqrt{2}/2,5\pi/12)$, $P(\sqrt{2}/2,3\pi/4)$

55. For all points, $r=1$; $\theta=\pi/12,\ 5\pi/12,\ 7\pi/12,\ 11\pi/12,\ 13\pi/12,\ 17\pi/12,\ 19\pi/12,\ 23\pi/12.$

57. Answers will vary. If m and n do not have any common factors, then an interval of $2n\pi$ is needed to sketch the entire graph.

Section 10.5

1. Using $x=r\cos\theta$ and $y=r\sin\theta$, we can write $x=f(\theta)\cos\theta$, $y=f(\theta)\sin\theta$.

3. (a) $\frac{dy}{dx}=-\cot\theta$
   (b) tangent line: $y=-(x-\sqrt{2}/2)+\sqrt{2}/2$; normal line: $y=x$

5. (a) $\frac{dy}{dx}=\frac{\cos\theta(1+2\sin\theta)}{\cos^2\theta-\sin\theta(1+\sin\theta)}$
   (b) tangent line: $x=3\sqrt{3}/4$; normal line: $y=3/4$

7. (a) $\frac{dy}{dx}=\frac{\theta\cos\theta+\sin\theta}{\cos\theta-\theta\sin\theta}$
   (b) tangent line: $y=-2/\pi x+\pi/2$; normal line: $y=\pi/2x+\pi/2$

9. (a) $\frac{dy}{dx}=\frac{4\sin(\theta)\cos(4\theta)+\sin(4\theta)\cos(\theta)}{4\cos(\theta)\cos(4\theta)-\sin(\theta)\sin(4\theta)}$
   (b) tangent line: $y=5\sqrt{3}(x+\sqrt{3}/4)-3/4$; normal line: $y=-1/5\sqrt{3}(x+\sqrt{3}/4)-3/4$

11. horizontal: $\theta=\pi/2,3\pi/2$;
    vertical: $\theta=0,\pi,2\pi$

13. horizontal: $\theta=\tan^{-1}(1/\sqrt{5}),\ \pi/2,\ \pi-\tan^{-1}(1/\sqrt{5}),\ \pi+\tan^{-1}(1/\sqrt{5}),\ 3\pi/2,\ 2\pi-\tan^{-1}(1/\sqrt{5})$;
    vertical: $\theta=0,\ \tan^{-1}(\sqrt{5}),\ \pi-\tan^{-1}(\sqrt{5}),\ \pi,\ \pi+\tan^{-1}(\sqrt{5}),\ 2\pi-\tan^{-1}(\sqrt{5})$

15. In polar: $\theta=0\ \cong\ \theta=\pi$
    In rectangular: $y=0$

17. area = 4π

19. area = $\pi/12$

21. area = $3\pi/2$

23. area = $2\pi + 3\sqrt{3}/2$

25. area = 1

27. area = $\frac{1}{32}(4\pi - 3\sqrt{3})$

29. 4π

31. area = $\sqrt{2}\pi$

33. $L \approx 2.2592$; (actual value $L = 2.22748$)

35. $SA = 16\pi$

37. $SA = 32\pi/5$

39. $SA = 36\pi$

Chapter 11

Section 11.1

1. right hand

3. curve (a parabola); surface (a cylinder)

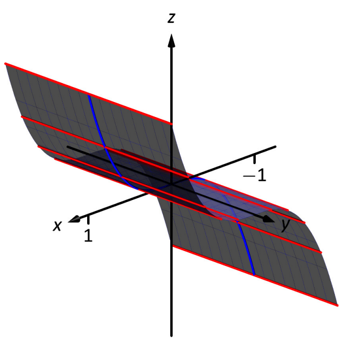
5. a hyperboloid of two sheets

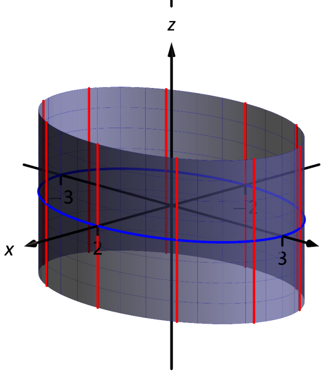
7. $\|\overline{AB}\| = \sqrt{6}$; $\|\overline{BC}\| = \sqrt{17}$; $\|\overline{AC}\| = \sqrt{11}$. Yes, it is a right triangle as $\|\overline{AB}\|^2 + \|\overline{AC}\|^2 = \|\overline{BC}\|^2$.

9. Center at $(4, -1, 0)$; radius = 3

11. Interior of a sphere with radius 1 centered at the origin.

13. The first octant of space; all points (x, y, z) where each of x, y and z are non-negative. (Analogous to the first quadrant in the plane.)

15. 

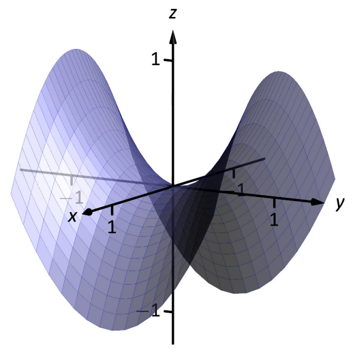
17. 

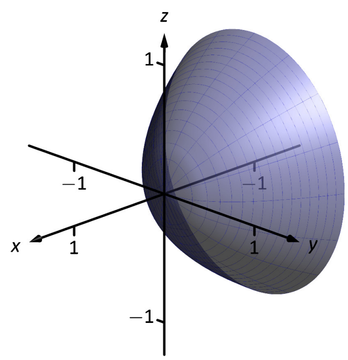
19. $x^2 + z^2 = \frac{1}{(1+y^2)^2}$

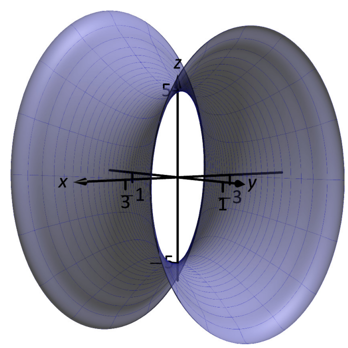
21. $z = (\sqrt{x^2+y^2})^2 = x^2 + y^2$

23. (a) $x = y^2 + \dfrac{z^2}{9}$

25. (b) $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$

27. 

29. 

31. 

Section 11.2

1. Answers will vary.

3. topographical

5. surface

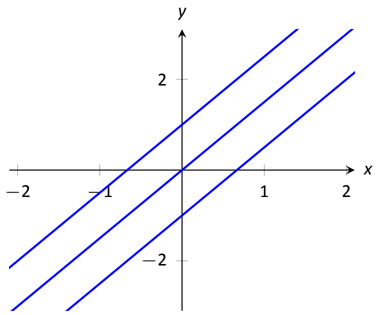
7. domain: $\mathbb{R}^2$
range: $z \geq 2$

9. domain: $\mathbb{R}^2$
range: $\mathbb{R}$

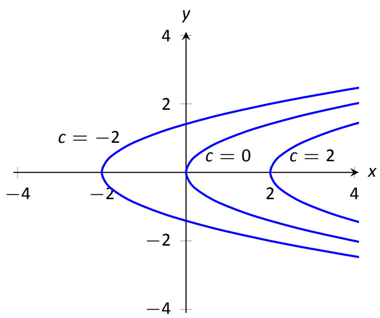
11. domain: $\mathbb{R}^2$
range: $0 < z \leq 1$

13. domain: $\{(x, y) \mid x^2 + y^2 \leq 9\}$, i.e., the domain is the circle and interior of a circle centered at the origin with radius 3.
range: $0 \leq z \leq 3$

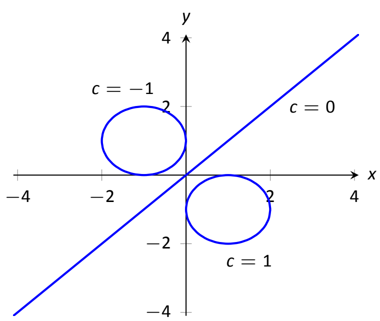
15. Level curves are lines $y = (3/2)x - c/2$.



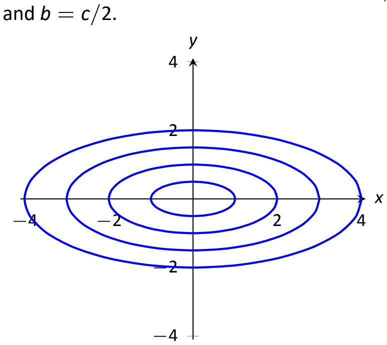
17. Level curves are parabolas $x = y^2 + c$.



19. When $c \neq 0$, the level curves are circles, centered at $(1/c, -1/c)$ with radius $\sqrt{2/c^2 - 1}$. When $c = 0$, the level curve is the line $y = x$.



21. Level curves are ellipses of the form $\frac{x^2}{c^2} + \frac{y^2}{c^2/4} = 1$, i.e., $a = c$ and $b = c/2$.



23. domain: $x + 2y - 4z \neq 0$; the set of points in $\mathbb{R}^3$ NOT in the domain form a plane through the origin.
range: $\mathbb{R}$

25. domain: $z \geq x^2 - y^2$; the set of points in $\mathbb{R}^3$ above (and including) the hyperbolic paraboloid $z = x^2 - y^2$.
range: $[0, \infty)$

27. The level surfaces are spheres, centered at the origin, with radius $\sqrt{c}$.

29. The level surfaces are paraboloids of the form $z = \frac{x^2}{c} + \frac{y^2}{c}$; the larger c, the "wider" the paraboloid.

31. The level curves for each surface are similar; for $z = \sqrt{x^2 + 4y^2}$ the level curves are ellipses of the form $\frac{x^2}{c^2} + \frac{y^2}{c^2/4} = 1$, i.e., $a = c$ and $b = c/2$; whereas for $z = x^2 + 4y^2$ the level curves are ellipses of the form $\frac{x^2}{c} + \frac{y^2}{c/4} = 1$, i.e., $a = \sqrt{c}$ and $b = \sqrt{c}/2$. The first set of ellipses are spaced evenly apart, meaning the function grows at a constant rate; the second set of ellipses are more closely spaced together as c grows, meaning the function grows faster and faster as c increases.
The function $z = \sqrt{x^2 + 4y^2}$ can be rewritten as $z^2 = x^2 + 4y^2$, an elliptic cone; the function $z = x^2 + 4y^2$ is a paraboloid, each matching the description above.

Section 11.3

1. Answers will vary.

3. Answers will vary.
One possible answer: $\{(x, y) | x^2 + y^2 \leq 1\}$

5. Answers will vary.
One possible answer: $\{(x, y) | x^2 + y^2 < 1\}$

7. (a) Answers will vary.
interior point: $(1, 3)$
boundary point: $(3, 3)$
(b) S is a closed set
(c) S is bounded

9. (a) Answers will vary.
interior point: none
boundary point: $(0, -1)$
(b) S is a closed set, consisting only of boundary points
(c) S is bounded

11. (a) $D = \{(x, y) \,|\, 9 - x^2 - y^2 \geq 0\}$.
(b) D is a closed set.
(c) D is bounded.

13. (a) $D = \{(x, y) \,|\, y > x^2\}$.
(b) D is an open set.
(c) D is unbounded.

15. (a) Along $y = 0$, the limit is 1.
(b) Along $x = 0$, the limit is -1.
Since the above limits are not equal, the limit does not exist.

17. (a) Along $y = mx$, the limit is $\dfrac{mx(1 - m)}{m^2x + 1} = 0$ for all m.
(b) Along $x = 0$, the limit is -1.
Since the above limits are not equal, the limit does not exist.

19. (a) Along $y = 2$, the limit is:
$$\begin{aligned} \lim_{(x,y)\to(1,2)} \frac{x + y - 3}{x^2 - 1} &= \lim_{x\to 1} \frac{x - 1}{x^2 - 1} \\ &= \lim_{x\to 1} \frac{1}{x + 1} \\ &= 1/2. \end{aligned}$$
(b) Along $y = x + 1$, the limit is:
$$\begin{aligned} \lim_{(x,y)\to(1,2)} \frac{x + y - 3}{x^2 - 1} &= \lim_{x\to 1} \frac{2(x - 1)}{x^2 - 1} \\ &= \lim_{x\to 1} \frac{2}{x + 1} \\ &= 1. \end{aligned}$$
Since the limits along the lines $y = 2$ and $y = x + 1$ differ, the overall limit does not exist.

Section 11.4

1. A constant is a number that is added or subtracted in an expression; a coefficient is a number that is being multiplied by a nonconstant function.

3. f_x

5. $f_x = 2xy - 1, f_y = x^2 + 2$
$f_x(1,2) = 3, f_y(1,2) = 3$

7. $f_x = -\sin x \sin y, f_y = \cos x \cos y$
$f_x(\pi/3, \pi/3) = -3/4, f_y(\pi/3, \pi/3) = 1/4$

9. $f_x = 2xy + 6x, f_y = x^2 + 4$
$f_{xx} = 2y + 6, f_{yy} = 0$
$f_{xy} = 2x, f_{yx} = 2x$

11. $f_x = 1/y, f_y = -x/y^2$
$f_{xx} = 0, f_{yy} = 2x/y^3$
$f_{xy} = -1/y^2, f_{yx} = -1/y^2$

13. $f_x = 2xe^{x^2+y^2}, f_y = 2ye^{x^2+y^2}$
$f_{xx} = 2e^{x^2+y^2} + 4x^2e^{x^2+y^2}, f_{yy} = 2e^{x^2+y^2} + 4y^2e^{x^2+y^2}$
$f_{xy} = 4xye^{x^2+y^2}, f_{yx} = 4xye^{x^2+y^2}$

15. $f_x = \cos x \cos y, f_y = -\sin x \sin y$
$f_{xx} = -\sin x \cos y, f_{yy} = -\sin x \cos y$
$f_{xy} = -\sin y \cos x, f_{yx} = -\sin y \cos x$

17. $f_x = -5y^3 \sin(5xy^3), f_y = -15xy^2 \sin(5xy^3)$
$f_{xx} = -25y^6 \cos(5xy^3),$
$f_{yy} = -225x^2y^4 \cos(5xy^3) - 30xy \sin(5xy^3)$
$f_{xy} = -75xy^5 \cos(5xy^3) - 15y^2 \sin(5xy^3),$
$f_{yx} = -75xy^5 \cos(5xy^3) - 15y^2 \sin(5xy^3)$

19. $f_x = \frac{2y^2}{\sqrt{4xy^2+1}}, f_y = \frac{4xy}{\sqrt{4xy^2+1}}$
$f_{xx} = -\frac{4y^4}{\sqrt{4xy^2+1}^3}, f_{yy} = -\frac{16x^2y^2}{\sqrt{4xy^2+1}^3} + \frac{4x}{\sqrt{4xy^2+1}}$
$f_{xy} = -\frac{8xy^3}{\sqrt{4xy^2+1}^3} + \frac{4y}{\sqrt{4xy^2+1}}, f_{yx} = -\frac{8xy^3}{\sqrt{4xy^2+1}^3} + \frac{4y}{\sqrt{4xy^2+1}}$

21. $f_x = -\frac{2x}{(x^2+y^2+1)^2}, f_y = -\frac{2y}{(x^2+y^2+1)^2}$
$f_{xx} = \frac{8x^2}{(x^2+y^2+1)^3} - \frac{2}{(x^2+y^2+1)^2}, f_{yy} = \frac{8y^2}{(x^2+y^2+1)^3} - \frac{2}{(x^2+y^2+1)^2}$
$f_{xy} = \frac{8xy}{(x^2+y^2+1)^3}, f_{yx} = \frac{8xy}{(x^2+y^2+1)^3}$

23. $f_x = 6x, f_y = 0$
$f_{xx} = 6, f_{yy} = 0$
$f_{xy} = 0, f_{yx} = 0$

25. $f_x = \frac{1}{4xy}, f_y = -\frac{\ln x}{4y^2}$
$f_{xx} = -\frac{1}{4x^2y}, f_{yy} = \frac{\ln x}{2y^3}$
$f_{xy} = -\frac{1}{4xy^2}, f_{yx} = -\frac{1}{4xy^2}$

27. $f(x,y) = x \sin y + x + C$, where C is any constant.

29. $f(x,y) = 3x^2y - 4xy^2 + 2y + C$, where C is any constant.

31. $f_x = 2xe^{2y-3z}, f_y = 2x^2e^{2y-3z}, f_z = -3x^2e^{2y-3z}$
$f_{yz} = -6x^2e^{2y-3z}, f_{zy} = -6x^2e^{2y-3z}$

33. $f_x = \frac{3}{7y^2z}, f_y = -\frac{6x}{7y^3z}, f_z = -\frac{3x}{7y^2z^2}$
$f_{yz} = \frac{6x}{7y^3z^2}, f_{zy} = \frac{6x}{7y^3z^2}$

Section 11.5

1. Answers will vary. The displacement of the vector is one unit in the x-direction and 3 units in the z-direction, with no change in y. Thus along a line parallel to $\vec{v}$, the change in z is 3 times the change in x – i.e., a "slope" of 3. Specifically, the line in the x-z plane parallel to z has a slope of 3.

3. T

5. (a) $\ell_x(t) = \begin{cases} x = 2 + t \\ y = 3 \\ z = -48 - 12t \end{cases}$

(b) $\ell_y(t) = \begin{cases} x = 2 \\ y = 3 + t \\ z = -48 - 40t \end{cases}$

(c) $\ell_{\vec{u}}(t) = \begin{cases} x = 2 + t/\sqrt{10} \\ y = 3 + 3t/\sqrt{10} \\ z = -48 - 66\sqrt{2/5}t \end{cases}$

7. (a) $\ell_x(t) = \begin{cases} x = 4 + t \\ y = 2 \\ z = 2 + 3t \end{cases}$

(b) $\ell_y(t) = \begin{cases} x = 4 \\ y = 2 + t \\ z = 2 - 5t \end{cases}$

(c) $\ell_{\vec{u}}(t) = \begin{cases} x = 4 + t/\sqrt{2} \\ y = 2 + t/\sqrt{2} \\ z = 2 - \sqrt{2}t \end{cases}$

9. $\ell_{\vec{n}}(t) = \begin{cases} x = 2 - 12t \\ y = 3 - 40t \\ z = -48 - t \end{cases}$

11. $\ell_{\vec{n}}(t) = \begin{cases} x = 4 + 3t \\ y = 2 - 5t \\ z = 2 - t \end{cases}$

13. $(1.425, 1.085, -48.078), (2.575, 4.915, -47.952)$

15. $(5.014, 0.31, 1.662)$ and $(2.986, 3.690, 2.338)$

17. $-12(x-2) - 40(y-3) - (z+48) = 0$

19. $3(x-4) - 5(y-2) - (z-2) = 0$ (Note that this tangent plane is the same as the original function, a plane.)

Index

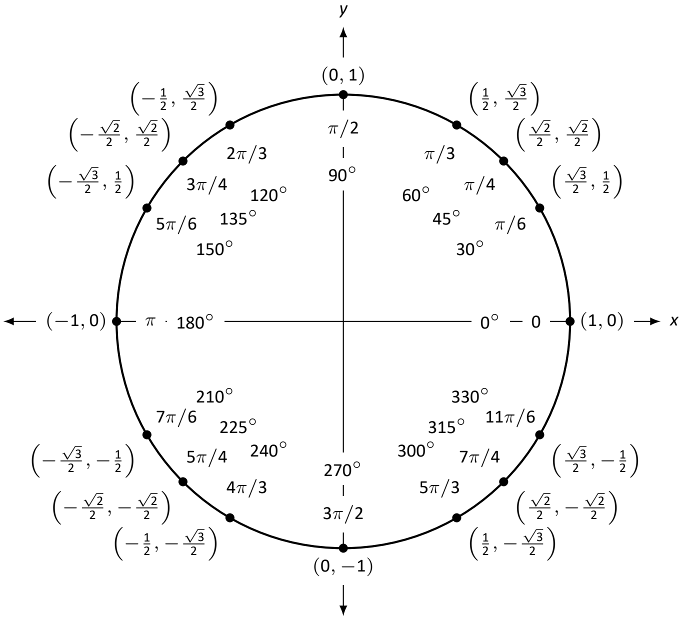
Differentiation Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$
8. $\frac{d}{dx}(x^n) = nx^{n-1}$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}(\sin x) = \cos x$
14. $\frac{d}{dx}(\cos x) = -\sin x$
15. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
16. $\frac{d}{dx}(\sec x) = \sec x \tan x$
17. $\frac{d}{dx}(\tan x) = \sec^2 x$
18. $\frac{d}{dx}(\cot x) = -\csc^2 x$
19. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}}$
22. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$
23. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$
25. $\frac{d}{dx}(\cosh x) = \sinh x$
26. $\frac{d}{dx}(\sinh x) = \cosh x$
27. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
29. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
30. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
31. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$
32. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$
34. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$
35. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$
36. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$

Integration Rules

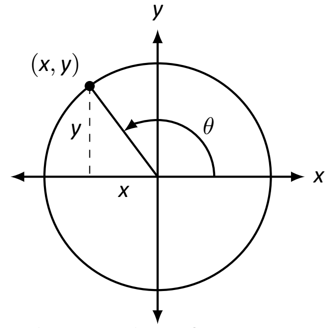
1. $\int c \cdot f(x)\, dx = c \int f(x)\, dx$
2. $\int f(x) \pm g(x)\, dx =$
   $\int f(x)\, dx \pm \int g(x)\, dx$
3. $\int 0\, dx = C$
4. $\int 1\, dx = x + C$
5. $\int x^n\, dx = \frac{1}{n+1}x^{n+1} + C,\ n \neq -1$
   $n \neq -1$
6. $\int e^x\, dx = e^x + C$
7. $\int a^x\, dx = \frac{1}{\ln a} \cdot a^x + C$
8. $\int \frac{1}{x}\, dx = \ln|x| + C$
9. $\int \cos x\, dx = \sin x + C$
10. $\int \sin x\, dx = -\cos x + C$
11. $\int \tan x\, dx = -\ln|\cos x| + C$
12. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
13. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$
14. $\int \cot x\, dx = \ln|\sin x| + C$
15. $\int \sec^2 x\, dx = \tan x + C$
16. $\int \csc^2 x\, dx = -\cot x + C$
17. $\int \sec x \tan x\, dx = \sec x + C$
18. $\int \csc x \cot x\, dx = -\csc x + C$
19. $\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
20. $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
21. $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
22. $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
23. $\int \frac{1}{x\sqrt{x^2-a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$
24. $\int \cosh x\, dx = \sinh x + C$
25. $\int \sinh x\, dx = \cosh x + C$
26. $\int \tanh x\, dx = \ln(\cosh x) + C$
27. $\int \coth x\, dx = \ln|\sinh x| + C$
28. $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$
29. $\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$
30. $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2}\ln\left|\frac{a+x}{a-x}\right| + C$
31. $\int \frac{1}{x\sqrt{a^2-x^2}}\, dx = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$
32. $\int \frac{1}{x\sqrt{x^2+a^2}}\, dx = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

The Unit Circle



Definitions of the Trigonometric Functions

Unit Circle Definition

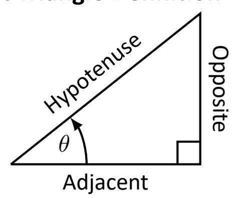


$\sin\theta = y \qquad \cos\theta = x$

$\csc\theta = \dfrac{1}{y} \qquad \sec\theta = \dfrac{1}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$

Right Triangle Definition



$\sin\theta = \dfrac{\text{O}}{\text{H}} \qquad \csc\theta = \dfrac{\text{H}}{\text{O}}$

$\cos\theta = \dfrac{\text{A}}{\text{H}} \qquad \sec\theta = \dfrac{\text{H}}{\text{A}}$

$\tan\theta = \dfrac{\text{O}}{\text{A}} \qquad \cot\theta = \dfrac{\text{A}}{\text{O}}$

Common Trigonometric Identities

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Cofunction Identities

$\sin\left(\dfrac{\pi}{2} - x\right) = \cos x \qquad \csc\left(\dfrac{\pi}{2} - x\right) = \sec x$

$\cos\left(\dfrac{\pi}{2} - x\right) = \sin x \qquad \sec\left(\dfrac{\pi}{2} - x\right) = \csc x$

$\tan\left(\dfrac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\dfrac{\pi}{2} - x\right) = \tan x$

Double Angle Formulas

$$\sin 2x = 2\sin x\cos x$$

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x\end{aligned}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Sum to Product Formulas

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Power–Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Even/Odd Identities

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\csc(-x) = -\csc x$

$\sec(-x) = \sec x$

$\cot(-x) = -\cot x$

Product to Sum Formulas

$$\sin x\sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x\cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\sin x\cos y = \frac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$$

Angle Sum/Difference Formulas

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

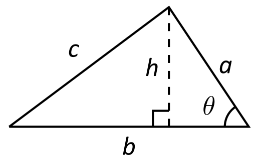
Areas and Volumes

Triangles

$h = a \sin\theta$

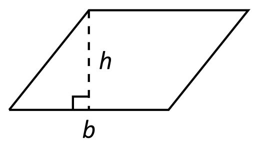
Area = $\frac{1}{2}bh$

Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos\theta$



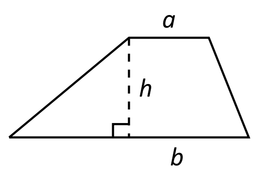
Parallelograms

Area = bh



Trapezoids

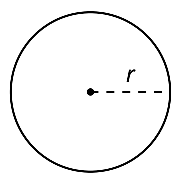
Area = $\frac{1}{2}(a+b)h$



Circles

Area = πr^2

Circumference = $2\pi r$

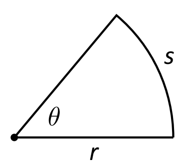


Sectors of Circles

θ in radians

Area = $\frac{1}{2}\theta r^2$

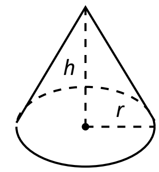
$s = r\theta$



Right Circular Cone

Volume = $\frac{1}{3}\pi r^2 h$

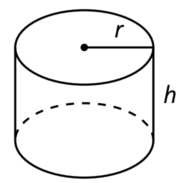
Surface Area =
$\pi r\sqrt{r^2+h^2} + \pi r^2$



Right Circular Cylinder

Volume = $\pi r^2 h$

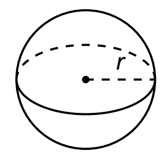
Surface Area =
$2\pi rh + 2\pi r^2$



Sphere

Volume = $\frac{4}{3}\pi r^3$

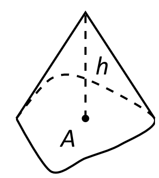
Surface Area =$4\pi r^2$



General Cone

Area of Base = A

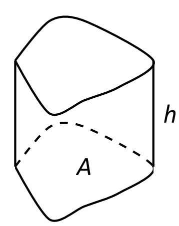
Volume = $\frac{1}{3}Ah$



General Right Cylinder

Area of Base = A

Volume = Ah



Algebra

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, and $0 \le b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors

$x^2 - a^2 = (x-a)(x+a)$ $\qquad$ $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x+a)(x^2 - ax + a^2)$ $\qquad$ $x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$

$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$

$(x-y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$

Binomial Theorem

$(x+y)^2 = x^2 + 2xy + y^2$ $\qquad$ $(x-y)^2 = x^2 - 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ $\qquad$ $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ $\qquad$ $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$acx^3 + adx^2 + bcx + bd = ax^2(cs + d) + b(cx + d) = (ax^2 + b)(cx + d)$

Arithmetic Operations

$$ab + ac = a(b+c) \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc} \qquad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{a-b}{c-d} = \frac{b-a}{d-c} \qquad \frac{ab + ac}{a} = b + c$$

Exponents and Radicals

$$a^0 = 1,\ \ a \ne 0 \qquad (ab)^x = a^x b^x \qquad a^x a^y = a^{x+y} \qquad \sqrt{a} = a^{1/2} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \sqrt[n]{a} = a^{1/n}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \sqrt[n]{a^m} = a^{m/n} \qquad a^{-x} = \frac{1}{a^x} \qquad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \qquad (a^x)^y = a^{xy} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Additional Formulas

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \ldots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right]$

Arc Length:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\;\;dx$$

Surface of Revolution:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\;\;dx$$

(where $f(x) \geq 0$)

$$S = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\;\;dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:

$$W = \int_a^b F(x)\;\;dx$$

Force Exerted by a Fluid:

$$F = \int_a^b w\,d(y)\,\ell(y)\;\;dy$$

Taylor Series Expansion for $f(x)$:

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
*n*th-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\|r\| < 1$	$\|r\| \geq 1$	$\text{Sum} = \frac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		$\text{Sum} = \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\, dn$ is convergent	$\int_1^{\infty} a(n)\, dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} a_n/b_n > 0$	Also diverges if $\lim_{n\to\infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} (a_n)^{1/n} < 1$	$\lim_{n\to\infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} (a_n)^{1/n} = \infty$