Math 1010 Introduction to Calculus
University of Lethbridge

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Contributing Textbooks

Precalculus, Version $\lfloor \pi \rfloor = 3$
Carl Stitz and Jeff Zeager
www.stitz-zeager.com

APEX *Calculus*
Gregory Hartman et al
apexcalculus.com

This version of the text assembled and edited by Sean Fitzpatrick, University of Lethbridge, May, 2016.

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Preface

One of the challenges with a new course like Math 1010 is finding a suitable textbook for the course. This is made additionally difficult for a course that covers two topics – Precalculus and Calculus – that are usually offered as separate courses, with separate texts. I reviewed a number of commercially available options, but these all had two things in common: they did not quite meet our needs, and they were all very expensive (some were as much as $400).

Since writing a new textbook from scratch is a huge undertaking, requiring resources (like time) we simply did not have, I chose to explore non-commercial options. This took a bit of searching, since non-commercial texts, while inexpensive (or free), are of varying quality. Fortunately, there are some decent texts out there. Unfortunately, I couldn't find a single text that covered all of the material we need for Math 1010.

To get around this problem, I have selected two textbooks as our primary sources for the course. The first is *Precalculus*, version 3, by Carl Stitz and Jeff Zeager. The second is *APEX Calculus I*, version 3.0, by Hartman et al. Both texts have two very useful advantages. First, they're both free (as in beer): you can download either text in PDF format from the authors' web pages. Second, they're also *open source* texts (that is, free, as in speech). Both books are written using the LaTeXmarkup language, as is typical in mathematics publishing. What is not typical is that the authors of both texts make their source code freely available, allowing others (such as myself) to edit and customize the books as they see fit.

In the first iteration of this project (Fall 2015), I was only able to edit each text individually for length and content, resulting in two separate textbooks for Math 1010. This time around, I've had enough time to take the content of the Precalculus textbook and adapt its source code to be compatible with the formatting of the Calculus textbook, allowing me to produce a single textbook for all of Math 1010.

The book is very much a work in progress, and I will be editing it regularly. Feedback is always welcome.

Acknowledgements

First and foremost, I need to thank the authors of the two textbooks that provide the source material for this text. Without their hard work, and willingness to make their books (and the source code) freely available, it would not have been possible to create an affordable textbook for this course. You can find the original textbooks at their websites:

www.stitz-zeager.com, for the *Precalculus* textbook, by Stitz and Zeager, and

apexcalculus.com, for the APEX *Calculus* textbook, by Hartman et al.

I'd also like to thank Dave Morris for help with converting the graphics in the Precalculus textbook to work with the formatting code of the APEX text, and the other faculty members involved with this course — Alia Hamieh, David Kaminsky, and Nicole Wilson — for their input on the content of the text.

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1: The Real Numbers

1.1 Some Basic Set Theory Notions

While the authors would like nothing more than to delve quickly and deeply into the sheer excitement that is *Precalculus*, experience has taught us that a brief refresher on some basic notions is welcome, if not completely necessary, at this stage. To that end, we present a brief summary of 'set theory' and some of the associated vocabulary and notations we use in the text. Like all good Math books, we begin with a definition.

Definition 1 Set

A **set** is a well-defined collection of objects which are called the 'elements' of the set. Here, 'well-defined' means that it is possible to determine if something belongs to the collection or not, without prejudice.

For example, the collection of letters that make up the word "pronghorns" is well-defined and is a set, but the collection of the worst math teachers in the world is **not** well-defined, and so is **not** a set. In general, there are three ways to describe sets. They are

One thing that student evaluations teach us is that any given Mathematics instructor can be simultaneously the best and worst teacher ever, depending on who is completing the evaluation.

Key Idea 1 Ways to Describe Sets

1. **The Verbal Method:** Use a sentence to define a set.
2. **The Roster Method:** Begin with a left brace '$\{$', list each element of the set *only once* and then end with a right brace '$\}$'.
3. **The Set-Builder Method:** A combination of the verbal and roster methods using a "dummy variable" such as x.

For example, let S be the set described *verbally* as the set of letters that make up the word "pronghorns". A **roster** description of S would be $\{p, r, o, n, g, h, s\}$. Note that we listed 'r', 'o', and 'n' only once, even though they appear twice in "pronghorns." Also, the *order* of the elements doesn't matter, so $\{o, n, p, r, g, s, h\}$ is also a roster description of S. A **set-builder** description of S is:

$$\{x \,|\, x \text{ is a letter in the word "pronghorns".}\}$$

The way to read this is: 'The set of elements x <u>such that</u> x is a letter in the word "pronghorns."' In each of the above cases, we may use the familiar equals sign '$=$' and write $S = \{p, r, o, n, g, h, s\}$ or $S = \{x \,|\, x \text{ is a letter in the word "pronghorns".}\}$. Clearly r is in S and q is not in S. We express these sentiments mathematically by writing $r \in S$ and $q \notin S$.

More precisely, we have the following.

Definition 2 Notation for set inclusion

Let A be a set.

- If x is an element of A then we write $x \in A$ which is read 'x is in A'.
- If x is *not* an element of A then we write $x \notin A$ which is read 'x is not in A'.

Now let's consider the set $C = \{x \mid x \text{ is a consonant in the word "pronghorns"}\}$. A roster description of C is $C = \{p, r, n, g, h, s\}$. Note that by construction, every element of C is also in S. We express this relationship by stating that the set C is a **subset** of the set S, which is written in symbols as $C \subseteq S$. The more formal definition is given below.

Definition 3 Subset

Given sets A and B, we say that the set A is a **subset** of the set B and write '$A \subseteq B$' if every element in A is also an element of B.

Note that in our example above $C \subseteq S$, but not vice-versa, since $o \in S$ but $o \notin C$. Additionally, the set of vowels $V = \{a, e, i, o, u\}$, while it does have an element in common with S, is not a subset of S. (As an added note, S is not a subset of V, either.) We could, however, *build* a set which contains both S and V as subsets by gathering all of the elements in both S and V together into a single set, say $U = \{p, r, o, n, g, h, s, a, e, i, u\}$. Then $S \subseteq U$ and $V \subseteq U$. The set U we have built is called the **union** of the sets S and V and is denoted $S \cup V$. Furthermore, S and V aren't completely *different*[1] sets since they both contain the letter 'o.' The **intersection** of two sets is the set of elements (if any) the two sets have in common. In this case, the intersection of S and V is $\{o\}$, written $S \cap V = \{o\}$. We formalize these ideas below.

Definition 4 Intersection and Union

Suppose A and B are sets.

- The **intersection** of A and B is $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- The **union** of A and B is $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ (or both)}\}$

The key words in Definition 4 to focus on are the conjunctions: 'intersection' corresponds to 'and' meaning the elements have to be in *both* sets to be in the intersection, whereas 'union' corresponds to 'or' meaning the elements have to be in one set, or the other set (or both). In other words, to belong to the union of two sets an element must belong to *at least one* of them.

Returning to the sets C and V above, $C \cup V = \{p, r, n, g, h, s, a, e, i, o, u\}$. When it comes to their intersection, however, we run into a bit of notational

[1]Since the word 'different' could be ambiguous, mathematicians use the word *disjoint* to refer to two sets that have no elements in common.

awkwardness since C and V have no elements in common. While we could write $C \cap V = \{\}$, this sort of thing happens often enough that we give the set with no elements a name.

The full extent of the empty set's role will not be explored in this text, but it is of fundamental importance in Set Theory. In fact, the empty set can be used to generate numbers - mathematicians can create something from nothing! If you're interested, read about the von Neumann construction of the natural numbers or consider signing up for Math 2000.

Definition 5 Empty set

The **Empty Set** $\emptyset$ is the set which contains no elements. That is,

$$\emptyset = \{\} = \{x \,|\, x \neq x\}.$$

As promised, the empty set is the set containing no elements since no matter what 'x' is, '$x = x$.' Like the number '0,' the empty set plays a vital role in mathematics. We introduce it here more as a symbol of convenience as opposed to a contrivance. Using this new bit of notation, we have for the sets C and V above that $C \cap V = \emptyset$. A nice way to visualize relationships between sets and set operations is to draw a **Venn Diagram**. A Venn Diagram for the sets S, C and V is drawn in Figure 1.1.

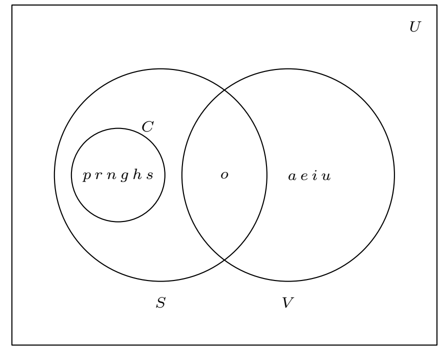
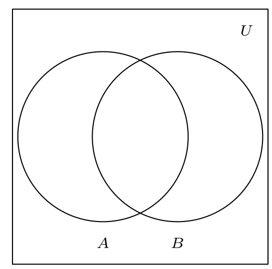


Figure 1.1: A Venn diagram for C, S, and V

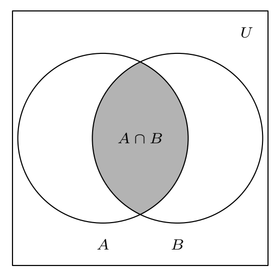
In Figure 1.1 we have three circles - one for each of the sets C, S and V. We visualize the area enclosed by each of these circles as the elements of each set. Here, we've spelled out the elements for definitiveness. Notice that the circle representing the set C is completely inside the circle representing S. This is a geometric way of showing that $C \subseteq S$. Also, notice that the circles representing S and V overlap on the letter 'o'. This common region is how we visualize $S \cap V$. Notice that since $C \cap V = \emptyset$, the circles which represent C and V have no overlap whatsoever.

All of these circles lie in a rectangle labelled U (for 'universal' set). A universal set contains all of the elements under discussion, so it could always be taken as the union of all of the sets in question, or an even larger set. In this case, we could take $U = S \cup V$ or U as the set of letters in the entire alphabet. The usual triptych of Venn Diagrams indicating generic sets A and B along with $A \cap B$ and $A \cup B$ is given below.

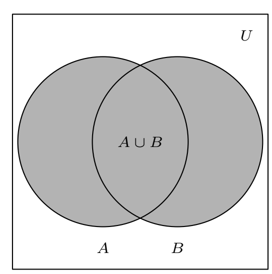
(The reader may well wonder if there is an ultimate universal set which contains *everything*. The short answer is 'no'. Our definition of a set turns out to be overly simplistic, but correcting this takes us well beyond the confines of this course. If you want the longer answer, you can begin by reading about Russell's Paradox on Wikipedia.)



Sets A and B.



$A \cap B$ is shaded.



$A \cup B$ is shaded.

Figure 1.2: Venn diagrams for intersection and union

1.1.1 Sets of Real Numbers

The playground for most of this text is the set of **Real Numbers**. Many quantities in the 'real world' can be quantified using real numbers: the temperature at a given time, the revenue generated by selling a certain number of products and the maximum population of Sasquatch which can inhabit a particular region are just three basic examples. A succinct, but nonetheless incomplete definition of a real number is given below.

Definition 6 The real numbers

A **real number** is any number which possesses a decimal representation. The set of real numbers is denoted by the character $\mathbb{R}$.

Certain subsets of the real numbers are worthy of note and are listed below. In more advanced courses like Analysis, you learn that the real numbers can be *constructed* from the rational numbers, which in turn can be constructed from the integers (which themselves come from the natural numbers, which in turn can be defined as sets...).

Definition 7 Sets of Numbers

1. The **Empty Set**: $\emptyset = \{\} = \{x \mid x \neq x\}$. This is the set with no elements. Like the number '0,' it plays a vital role in mathematics.
2. The **Natural Numbers**: $\mathbb{N} = \{1, 2, 3, \ldots\}$ The periods of ellipsis here indicate that the natural numbers contain 1, 2, 3, 'and so forth'.
3. The **Integers**: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
4. The **Rational Numbers**: $\mathbb{Q} = \left\{\frac{a}{b} \mid a \in \mathbb{Z} \text{ and } b \in \mathbb{Z}\right\}$. Rational numbers are the ratios of integers (provided the denominator is not zero!) It turns out that another way to describe the rational numbers is:

   $\mathbb{Q} = \{x \mid x \text{ possesses a repeating or terminating decimal representation.}\}$
5. The **Real Numbers**: $\mathbb{R} = \{x \mid x \text{ possesses a decimal representation.}\}$
6. The **Irrational Numbers**: $\mathbb{P} = \{x \mid x \text{ is a non-rational real number.}\}$ Said another way, an irrational number is a decimal which neither repeats nor terminates.
7. The **Complex Numbers**: $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}$ (We will not deal with complex numbers in Math 1010, although they usually make an appearance in Math 1410.)

An example of a number with a repeating decimal expansion is $a = 2.13234234234\ldots$. This is rational since $100a = 213.2342342342\ldots$, and $100000a = 213234.234234\ldots$ so $99900a = 100000a - 100a = 213021$. This gives us the rational expression $a = \dfrac{213021}{99900}$.

The classic example of an irrational number is the number π (See Section **??**), but numbers like $\sqrt{2}$ and $0.101001000100001\ldots$ are other fine representatives.

It is important to note that every natural number is a whole number is an integer. Each integer is a rational number (take $b = 1$ in the above definition for $\mathbb{Q}$) and the rational numbers are all real numbers, since they possess decimal representations (via long division!). If we take $b = 0$ in the above definition of $\mathbb{C}$, we see that every real number is a complex number. In this sense, the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are 'nested' like Matryoshka dolls. More formally, these sets form a subset chain: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. The reader is encouraged to sketch a Venn Diagram depicting $\mathbb{R}$ and all of the subsets mentioned above. It is time for an example.

Example 1 Sets of real numbers

1. Write a roster description for $P = \{2^n \mid n \in \mathbb{N}\}$ and $E = \{2n \mid n \in \mathbb{Z}\}$.
2. Write a verbal description for $S = \{x^2 \mid x \in \mathbb{R}\}$.
3. Let $A = \{-117, \frac{4}{5}, 0.202\overline{002}, 0.202002000200002\ldots\}$.

   Which elements of A are natural numbers? Rational numbers? Real numbers?

Solution

1. To find a roster description for these sets, we need to list their elements. Starting with $P = \{2^n \,|\, n \in \mathbb{N}\}$, we substitute natural number values n into the formula 2^n. For $n = 1$ we get $2^1 = 2$, for $n = 2$ we get $2^2 = 4$, for $n = 3$ we get $2^3 = 8$ and for $n = 4$ we get $2^4 = 16$. Hence P describes the powers of 2, so a roster description for P is $P = \{2, 4, 8, 16, \ldots\}$ where the '$\ldots$' indicates the that pattern continues.

   Proceeding in the same way, we generate elements in $E = \{2n \,|\, n \in \mathbb{Z}\}$ by plugging in integer values of n into the formula $2n$. Starting with $n = 0$ we obtain $2(0) = 0$. For $n = 1$ we get $2(1) = 2$, for $n = -1$ we get $2(-1) = -2$ for $n = 2$, we get $2(2) = 4$ and for $n = -2$ we get $2(-2) = -4$. As n moves through the integers, $2n$ produces all of the *even* integers. A roster description for E is $E = \{0, \pm 2, \pm 4, \ldots\}$.

This isn't the most *precise* way to describe this set - it's always dangerous to use '$\ldots$' since we assume that the pattern is clearly demonstrated and thus made evident to the reader. Formulas are more precise because the pattern is clear.

It shouldn't be too surprising that E is the set of all even integers, since an even integer is *defined* to be an integer multiple of 2.

2. One way to verbally describe S is to say that S is the 'set of all squares of real numbers'. While this isn't incorrect, we'd like to take this opportunity to delve a little deeper. What makes the set $S = \{x^2 \,|\, x \in \mathbb{R}\}$ a little trickier to wrangle than the sets P or E above is that the dummy variable here, x, runs through all *real* numbers. Unlike the natural numbers or the integers, the real numbers cannot be listed in any methodical way. Nevertheless, we can select some real numbers, square them and get a sense of what kind of numbers lie in S. For $x = -2$, $x^2 = (-2)^2 = 4$ so 4 is in S, as are $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ and $(\sqrt{117})^2 = 117$. Even things like $(-\pi)^2$ and $(0.101001000100001\ldots)^2$ are in S.

   So suppose $s \in S$. What can be said about s? We know there is some real number x so that $s = x^2$. Since $x^2 \geq 0$ for any real number x, we know $s \geq 0$. This tells us that everything in S is a non-negative real number. This begs the question: are <u>all</u> of the non-negative real numbers in S? Suppose n is a non-negative real number, that is, $n \geq 0$. If n were in S, there would be a real number x so that $x^2 = n$. As you may recall, we can solve $x^2 = n$ by 'extracting square roots': $x = \pm\sqrt{n}$. Since $n \geq 0$, $\sqrt{n}$ is a real number. Moreover, $(\sqrt{n})^2 = n$ so n is the square of a real number which means $n \in S$. Hence, S is the set of non-negative real numbers.

The fact that the real numbers cannot be listed is a nontrivial statement. Interested readers are directed to a discussion of Cantor's Diagonal Argument.

3. The set A contains no natural numbers. Clearly, $\frac{4}{5}$ is a rational number as is -117 (which can be written as $\frac{-117}{1}$). It's the last two numbers listed in A, $0.20\overline{2002}$ and $0.202002000200002\ldots$, that warrant some discussion. First, recall that the 'line' over the digits 2002 in $0.20\overline{2002}$ (called the vinculum) indicates that these digits repeat, so it is a rational number. As for the number $0.202002000200002\ldots$, the '$\ldots$' indicates the pattern of adding an extra '0' followed by a '2' is what defines this real number. Despite the fact there is a *pattern* to this decimal, this decimal is *not repeating*, so it is not a rational number - it is, in fact, an irrational number. All of the elements of A are real numbers, since all of them can be expressed as decimals (remember that $\frac{4}{5} = 0.8$). $\square$

4. The set $\mathbb{N} \cup \mathbb{Q} = \{x \,|\, x \in \mathbb{N} \text{ or } x \in \mathbb{Q}\}$ is the union of the set of natural numbers with the set of rational numbers. Since every natural number is a rational number, $\mathbb{N}$ doesn't contribute any new elements to $\mathbb{Q}$, so $\mathbb{N} \cup \mathbb{Q} = \mathbb{Q}$. For the set $\mathbb{Q} \cup \mathbb{P}$, we note that every real number is either rational or not, hence $\mathbb{Q} \cup \mathbb{P} = \mathbb{R}$, pretty much by the definition of the set $\mathbb{P}$.

As you may recall, we often visualize the set of real numbers $\mathbb{R}$ as a line where each point on the line corresponds to one and only one real number. Given two different real numbers a and b, we write $a < b$ if a is located to the left of b on the number line, as shown in Figure 1.3.

While this notion seems innocuous, it is worth pointing out that this convention is rooted in two deep properties of real numbers. The first property is that $\mathbb{R}$ is complete. This means that there are no 'holes' or 'gaps' in the real number line.[2] Another way to think about this is that if you choose any two distinct (different) real numbers, and look between them, you'll find a solid line segment (or interval) consisting of infinitely many real numbers.

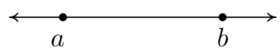


Figure 1.3: The real number line with two numbers a and b, where $a < b$.

The next result tells us what types of numbers we can expect to find.

Theorem 1 Density Property of $\mathbb{Q}$ and $\mathbb{P}$ in $\mathbb{R}$

Between any two distinct real numbers, there is at least one rational number and irrational number. It then follows that between any two distinct real numbers there will be infinitely many rational and irrational numbers.

The root word 'dense' here communicates the idea that rationals and irrationals are 'thoroughly mixed' into $\mathbb{R}$. The reader is encouraged to think about how one would find both a rational and an irrational number between, say, 0.9999 and 1. Once you've done that, ask yourself whether there is any difference between the numbers $0.\overline{9}$ and 1.

The Law of Trichotomy, strictly speaking, is an *axiom* of the real numbers: a basic requirement that we assume to be true. However, in any *construction* of the real, such as the method of Dedekind cuts, it is necessary to *prove* that the Law of Trichotomy is satisfied.

The second property $\mathbb{R}$ possesses that lets us view it as a line is that the set is totally ordered. This means that given any two real numbers a and b, either $a < b$, $a > b$ or $a = b$ which allows us to arrange the numbers from least (left) to greatest (right). You may have heard this property given as the 'Law of Trichotomy'.

Definition 8 Law of Trichotomy

If a and b are real numbers then **exactly one** of the following statements is true:

$a < b$ $\qquad$ $a > b$ $\qquad$ $a = b$

Segments of the real number line are called **intervals** of numbers. Below is a summary of the so-called **interval notation** associated with given sets of numbers. For intervals with finite endpoints, we list the left endpoint, then the right endpoint. We use square brackets, '[' or ']', if the endpoint is included in the interval and use a filled-in or 'closed' dot to indicate membership in the interval. Otherwise, we use parentheses, '(' or ')' and an 'open' circle to indicate that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbols $-\infty$ to indicate that the interval extends indefinitely to the left and ∞ to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use an appropriate arrow to indicate that the interval extends indefinitely in one (or both) directions.

[2]Alas, this intuitive feel for what it means to be 'complete' is as good as it gets at this level. Completeness does get a much more precise meaning later in courses like Analysis and Topology.

Definition 9 Interval Notation

Let a and b be real numbers with $a < b$.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid a < x < b\}$	(a, b)	a b
$\{x \mid a \leq x < b\}$	$[a, b)$	a b
$\{x \mid a < x \leq b\}$	$(a, b]$	a b
$\{x \mid a \leq x \leq b\}$	$[a, b]$	a b
$\{x \mid x < b\}$	$(-\infty, b)$	b
$\{x \mid x \leq b\}$	$(-\infty, b]$	b
$\{x \mid x > a\}$	(a, ∞)	a
$\{x \mid x \geq a\}$	$[a, \infty)$	a
$\mathbb{R}$	$(-\infty, \infty)$	

The importance of understanding interval notation in Calculus cannot be overstated. If you don't find yourself getting the hang of it through repeated use, you may need to take the time to just memorize this chart.

As you can glean from the table, for intervals with finite endpoints we start by writing 'left endpoint, right endpoint'. We use square brackets, '$[$' or '$]$', if the endpoint is included in the interval. This corresponds to a 'filled-in' or 'closed' dot on the number line to indicate that the number is included in the set. Otherwise, we use parentheses, '$($' or '$)$' that correspond to an 'open' circle which indicates that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbol $-\infty$ to indicate that the interval extends indefinitely to the left and the symbol ∞ to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use the appropriate arrow to indicate that the interval extends indefinitely in one or both directions.

Let's do a few examples to make sure we have the hang of the notation:

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid 1 \leq x < 3\}$	$[1, 3)$	1 3
$\{x \mid -1 \leq x \leq 4\}$	$[-1, 4]$	-1 4
$\{x \mid x \leq 5\}$	$(-\infty, 5]$	5
$\{x \mid x > -2\}$	$(-2, \infty)$	-2

We defined the intersection and union of arbitrary sets in Definition 4. Recall that the union of two sets consists of the totality of the elements in each of the sets, collected together. For example, if $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$, then $A \cap B = \{2\}$ and $A \cup B = \{1, 2, 3, 4, 6\}$. If $A = [-5, 3)$ and $B = (1, \infty)$, then we can find $A \cap B$ and $A \cup B$ graphically. To find $A \cap B$, we shade the overlap of the two and obtain $A \cap B = (1, 3)$. To find $A \cup B$, we shade each of A and B and describe the resulting shaded region to find $A \cup B = [-5, \infty)$.

While both intersection and union are important, we have more occasion to use union in this text than intersection, simply because most of the sets of real numbers we will be working with are either intervals or are unions of intervals, as the following example illustrates.

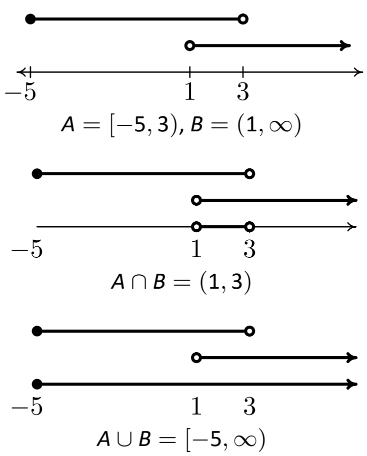


Figure 1.4: Union and intersection of intervals

Example 2 Expressing sets as unions of intervals

Express the following sets of numbers using interval notation.

1. $\{x \mid x \leq -2 \text{ or } x \geq 2\}$
2. $\{x \mid x \neq 3\}$
3. $\{x \mid x \neq \pm 3\}$
4. $\{x \mid -1 < x \leq 3 \text{ or } x = 5\}$

Solution

1. The best way to proceed here is to graph the set of numbers on the number line and glean the answer from it. The inequality $x \leq -2$ corresponds to the interval $(-\infty, -2]$ and the inequality $x \geq 2$ corresponds to the interval $[2, \infty)$. Since we are looking to describe the real numbers x in one of these *or* the other, we have $\{x \mid x \leq -2 \text{ or } x \geq 2\} = (-\infty, -2] \cup [2, \infty)$.

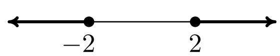


Figure 1.5: The set $(-\infty, -2] \cup [2, \infty)$

2. For the set $\{x \mid x \neq 3\}$, we shade the entire real number line except $x = 3$, where we leave an open circle. This divides the real number line into two intervals, $(-\infty, 3)$ and $(3, \infty)$. Since the values of x could be in either one of these intervals *or* the other, we have that $\{x \mid x \neq 3\} = (-\infty, 3) \cup (3, \infty)$

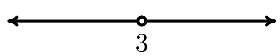


Figure 1.6: The set $(-\infty, 3) \cup (3, \infty)$

3. For the set $\{x \mid x \neq \pm 3\}$, we proceed as before and exclude both $x = 3$ and $x = -3$ from our set. This breaks the number line into *three* intervals, $(-\infty, -3)$, $(-3, 3)$ and $(3, \infty)$. Since the set describes real numbers which come from the first, second *or* third interval, we have $\{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

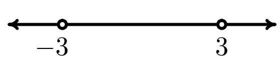


Figure 1.7: The set $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

4. Graphing the set $\{x \mid -1 < x \leq 3 \text{ or } x = 5\}$, we get one interval, $(-1, 3]$ along with a single number, or point, $\{5\}$. While we *could* express the latter as $[5, 5]$ (Can you see why?), we choose to write our answer as $\{x \mid -1 < x \leq 3 \text{ or } x = 5\} = (-1, 3] \cup \{5\}$.

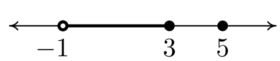
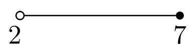
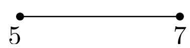
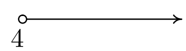


Figure 1.8: The set $(-1, 3] \cup \{5\}$

Exercises 1.1

Problems

1. Fill in the chart below:

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$		
	$[0,3)$	
		
$\{x \mid -5 < x \leq 0\}$		
	$(-3,3)$	
		
$\{x \mid x \leq 3\}$		
	$(-\infty,9)$	
		
$\{x \mid x \geq -3\}$		

In Exercises 2 – 7, find the indicated intersection or union and simplify if possible. Express your answers in interval notation.

2. $(-1,5] \cap [0,8)$

3. $(-1,1) \cup [0,6]$

4. $(-\infty,4] \cap (0,\infty)$

5. $(-\infty,0) \cap [1,5]$

6. $(-\infty,0) \cup [1,5]$

7. $(-\infty,5] \cap [5,8)$

In Exercises 8 – 19, write the set using interval notation.

8. $\{x \mid x \neq 5\}$

9. $\{x \mid x \neq -1\}$

10. $\{x \mid x \neq -3,\ 4\}$

11. $\{x \mid x \neq 0,\ 2\}$

12. $\{x \mid x \neq 2,\ -2\}$

13. $\{x \mid x \neq 0,\ \pm 4\}$

14. $\{x \mid x \leq -1 \text{ or } x \geq 1\}$

15. $\{x \mid x < 3 \text{ or } x \geq 2\}$

16. $\{x \mid x \leq -3 \text{ or } x > 0\}$

17. $\{x \mid x \leq 5 \text{ or } x = 6\}$

18. $\{x \mid x > 2 \text{ or } x = \pm 1\}$

19. $\{x \mid -3 < x < 3 \text{ or } x = 4\}$

1.2 Real Number Arithmetic

In this section we list the properties of real number arithmetic. This is meant to be a succinct, targeted review so we'll resist the temptation to wax poetic about these axioms and their subtleties and refer the interested reader to a more formal course in Abstract Algebra. There are two (primary) operations one can perform with real numbers: addition and multiplication.

Definition 10 Properties of Real Number Addition

- **Closure:** For all real numbers a and b, $a + b$ is also a real number.
- **Commutativity:** For all real numbers a and b, $a + b = b + a$.
- **Associativity:** For all real numbers a, b and c, $a + (b + c) = (a + b) + c$.
- **Identity:** There is a real number '0' so that for all real numbers a, $a + 0 = a$.
- **Inverse:** For all real numbers a, there is a real number $-a$ such that $a + (-a) = 0$.
- **Definition of Subtraction:** For all real numbers a and b, $a - b = a + (-b)$.

Next, we give real number multiplication a similar treatment. Recall that we may denote the product of two real numbers a and b a variety of ways: ab, $a \cdot b$, $a(b)$, $(a)(b)$ and so on. We'll refrain from using $a \times b$ for real number multiplication in this text.

Definition 11 Properties of Real Number Multiplication

- **Closure:** For all real numbers a and b, ab is also a real number.
- **Commutativity:** For all real numbers a and b, $ab = ba$.
- **Associativity:** For all real numbers a, b and c, $a(bc) = (ab)c$.
- **Identity:** There is a real number '1' so that for all real numbers a, $a \cdot 1 = a$.
- **Inverse:** For all real numbers $a \neq 0$, there is a real number $\dfrac{1}{a}$ such that $a\left(\dfrac{1}{a}\right) = 1$.
- **Definition of Division:** For all real numbers a and $b \neq 0$, $a \div b = \dfrac{a}{b} = a\left(\dfrac{1}{b}\right)$.

While most students (and some faculty) tend to skip over these properties or give them a cursory glance at best, it is important to realize that the properties stated above are what drive the symbolic manipulation for all of Algebra. When listing a tally of more than two numbers, $1+2+3$ for example, we don't need to specify the order in which those numbers are added. Notice though, try as we might, we can add only two numbers at a time and it is the associative property of addition which assures us that we could organize this sum as $(1+2)+3$ or $1+(2+3)$. This brings up a note about 'grouping symbols'. Recall that parentheses and brackets are used in order to specify which operations are to be performed first. In the absence of such grouping symbols, multiplication (and hence division) is given priority over addition (and hence subtraction). For example, $1 + 2 \cdot 3 = 1 + 6 = 7$, but $(1 + 2) \cdot 3 = 3 \cdot 3 = 9$. As you may recall, we can 'distribute' the 3 across the addition if we really wanted to do the multiplication first: $(1 + 2) \cdot 3 = 1 \cdot 3 + 2 \cdot 3 = 3 + 6 = 9$. More generally, we have the following.

Definition 12 The Distributive Property and Factoring

For all real numbers a, b and c:

- **Distributive Property:** $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$.
- **Factoring:** $ab + ac = a(b + c)$ and $ac + bc = (a + b)c$.

Warning: A common source of errors for beginning students is the misuse (that is, lack of use) of parentheses. When in doubt, more is better than less: redundant parentheses add clutter, but do not change meaning, whereas writing $2x + 1$ when you meant to write $2(x + 1)$ is almost guaranteed to cause you to make a mistake. (Even if you're able to proceed correctly in spite of your lack of proper notation, this is the sort of thing that will get you on your grader's bad side, so it's probably best to avoid the problem in the first place.)

It is worth pointing out that we didn't really need to list the Distributive Property both for $a(b+c)$ (distributing from the left) and $(a+b)c$ (distributing from the right), since the commutative property of multiplication gives us one from the other. Also, 'factoring' really is the same equation as the distributive property, just read from right to left. These are the first of many redundancies in this section, and they exist in this review section for one reason only - in our experience, many students *see* these things differently so we will list them as such.

It is hard to overstate the importance of the Distributive Property. For example, in the expression $5(2 + x)$, without knowing the value of x, we cannot perform the addition inside the parentheses first; we must rely on the distributive property here to get $5(2 + x) = 5 \cdot 2 + 5 \cdot x = 10 + 5x$. The Distributive Property is also responsible for combining 'like terms'. Why is $3x + 2x = 5x$? Because $3x + 2x = (3 + 2)x = 5x$.

We continue our review with summaries of other properties of arithmetic, each of which can be derived from the properties listed above. First up are properties of the additive identity 0.

The Zero Product Property drives most of the equation solving algorithms in Algebra because it allows us to take complicated equations and reduce them to simpler ones. For example, you may recall that one way to solve $x^2+x-6=0$ is by factoring the left hand side of this equation to get $(x-2)(x+3)=0$. From here, we apply the Zero Product Property and set each factor equal to zero. This yields $x-2=0$ or $x+3=0$ so $x=2$ or $x=-3$. This application to solving equations leads, in turn, to some deep and profound structure theorems in Chapter 4.

The expression $\frac{0}{0}$ is technically an 'indeterminate form' as opposed to being strictly 'undefined' meaning that with Calculus we can make some sense of it in certain situations. We'll talk more about this in Chapter **??**.

Theorem 2 Properties of Zero

Suppose a and b are real numbers.

- **Zero Product Property:** $ab=0$ if and only if $a=0$ or $b=0$ (or both)

  Note: This not only says that $0\cdot a=0$ for any real number a, it also says that the *only* way to get an answer of '0' when multiplying two real numbers is to have one (or both) of the numbers be '0' in the first place.

- **Zeros in Fractions:** If $a\neq 0$, $\dfrac{0}{a}=0\cdot\left(\dfrac{1}{a}\right)=0$.

  Note: The quantity $\dfrac{a}{0}$ is undefined.

We now continue with a review of arithmetic with fractions.

It's always worth remembering that division is the same as multiplication by the reciprocal. You'd be surprised how often this comes in handy.

Note: A common denominator is **not** required to **multiply** or **divide** fractions!

Note: A common denominator **is** required to **add or subtract** fractions!

Note: The *only* way to change the denominator is to multiply both it and the numerator by the same nonzero value because we are, in essence, multiplying the fraction by 1.

We reduce fractions by 'cancelling' common factors - this is really just reading the previous property 'from right to left'.**Caution:** We may only cancel common **factors** from both numerator and denominator.

Key Idea 2 Properties of Fractions

Suppose a, b, c and d are real numbers. Assume them to be nonzero whenever necessary; for example, when they appear in a denominator.

- **Identity Properties:** $a=\dfrac{a}{1}$ and $\dfrac{a}{a}=1$.
- **Fraction Equality:** $\dfrac{a}{b}=\dfrac{c}{d}$ if and only if $ad=bc$.
- **Multiplication of Fractions:** $\dfrac{a}{b}\cdot\dfrac{c}{d}=\dfrac{ac}{bd}$. In particular: $\dfrac{a}{b}\cdot c=\dfrac{a}{b}\cdot\dfrac{c}{1}=\dfrac{ac}{b}$
- **Division of Fractions:** $\dfrac{a}{b}\Big/\dfrac{c}{d}=\dfrac{a}{b}\cdot\dfrac{d}{c}=\dfrac{ad}{bc}$.

  In particular: $1\Big/\dfrac{a}{b}=\dfrac{b}{a}$ and $\dfrac{a}{b}/c=\dfrac{a}{b}\Big/\dfrac{c}{1}=\dfrac{a}{b}\cdot\dfrac{1}{c}=\dfrac{a}{bc}$
- **Addition and Subtraction of Fractions:** $\dfrac{a}{b}\pm\dfrac{c}{b}=\dfrac{a\pm c}{b}$.
- **Equivalent Fractions:** $\dfrac{a}{b}=\dfrac{ad}{bd}$, since $\dfrac{a}{b}=\dfrac{a}{b}\cdot 1=\dfrac{a}{b}\cdot\dfrac{d}{d}=\dfrac{ad}{bd}$
- **'Reducing' Fractions:** $\dfrac{a\cancel{d}}{b\cancel{d}}=\dfrac{a}{b}$, since $\dfrac{ad}{bd}=\dfrac{a}{b}\cdot\dfrac{d}{d}=\dfrac{a}{b}\cdot 1=\dfrac{a}{b}$.

  In particular, $\dfrac{ab}{b}=a$ since $\dfrac{ab}{b}=\dfrac{ab}{1\cdot b}=\dfrac{a\cancel{b}}{1\cdot\cancel{b}}=\dfrac{a}{1}=a$ and $\dfrac{b-a}{a-b}=\dfrac{(-1)\cancel{(a-b)}}{\cancel{(a-b)}}=-1$.

Next up is a review of the arithmetic of 'negatives'. On page 10 we first introduced the dash which we all recognize as the 'negative' symbol in terms of the additive inverse. For example, the number -3 (read 'negative 3') is defined so that $3+(-3)=0$. We then defined subtraction using the concept of the additive inverse again so that, for example, $5-3=5+(-3)$.

It might be junior high (elementary?) school material, but arithmetic with fractions is one of the most common sources of errors among university students. If you're not comfortable working with fractions, we strongly recommend seeing your instructor (or a tutor) to go over this material until you're completely confident that you understand it. Experience (and even formal educational studies) suggest that your success handling fractions corresponds pretty well with your overall success in passing your Mathematics courses.

Key Idea 3 Properties of Negatives

Given real numbers a and b we have the following.

- **Additive Inverse Properties:** $-a=(-1)a$ and $-(-a)=a$
- **Products of Negatives:** $(-a)(-b)=ab$.
- **Negatives and Products:** $-ab=-(ab)=(-a)b=a(-b)$.
- **Negatives and Fractions:** If b is nonzero, $-\dfrac{a}{b}=\dfrac{-a}{b}=\dfrac{a}{-b}$ and $\dfrac{-a}{-b}=\dfrac{a}{b}$.
- **'Distributing' Negatives:** $-(a+b)=-a-b$ and $-(a-b)=-a+b=b-a$.
- **'Factoring' Negatives:** $-a-b=-(a+b)$ and $b-a=-(a-b)$.

In this text we do not distinguish typographically between the dashes in the expressions '$5-3$' and '-3' even though they are mathematically quite different. In the expression '$5-3$,' the dash is a *binary* operation (that is, an operation requiring *two* numbers) whereas in '-3', the dash is a *unary* operation (that is, an operation requiring only one number). You might ask, 'Who cares?' Your calculator does - that's who! In the text we can write $-3-3=-6$ but that will not work in your calculator. Instead you'd need to type $^{-}3-3$ to get -6 where the first dash comes from the '$+/-$' key.

An important point here is that when we 'distribute' negatives, we do so across addition or subtraction only. This is because we are really distributing a factor of -1 across each of these terms: $-(a+b)=(-1)(a+b)=(-1)(a)+(-1)(b)=(-a)+(-b)=-a-b$. Negatives do not 'distribute' across multiplication: $-(2\cdot3)\neq(-2)\cdot(-3)$. Instead, $-(2\cdot3)=(-2)\cdot(3)=(2)\cdot(-3)=-6$. The same sort of thing goes for fractions: $-\frac{3}{5}$ can be written as $\frac{-3}{5}$ or $\frac{3}{-5}$, but not $\frac{-3}{-5}$. It's about time we did a few examples to see how these properties work in practice.

Example 3 Arithmetic with fractions

Perform the indicated operations and simplify. By 'simplify' here, we mean to have the final answer written in the form $\frac{a}{b}$ where a and b are integers which have no common factors. Said another way, we want $\frac{a}{b}$ in 'lowest terms'.

1. $\dfrac{1}{4}+\dfrac{6}{7}$
2. $\dfrac{5}{12}-\left(\dfrac{47}{30}-\dfrac{7}{3}\right)$
3. $\dfrac{\dfrac{12}{5}-\dfrac{7}{24}}{1+\left(\dfrac{12}{5}\right)\left(\dfrac{7}{24}\right)}$
4. $\dfrac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)}$
5. $\left(\dfrac{3}{5}\right)\left(\dfrac{5}{13}\right)-\left(\dfrac{4}{5}\right)\left(-\dfrac{12}{13}\right)$

Solution

1. It may seem silly to start with an example this basic but experience has taught us not to take much for granted. We start by finding the lowest common denominator and then we rewrite the fractions using that new denominator. Since 4 and 7 are **relatively prime**, meaning they have no

factors in common, the lowest common denominator is $4 \cdot 7 = 28$.

$$\begin{aligned} \frac{1}{4} + \frac{6}{7} &= \frac{1}{4} \cdot \frac{7}{7} + \frac{6}{7} \cdot \frac{4}{4} && \text{Equivalent Fractions} \\ &= \frac{7}{28} + \frac{24}{28} && \text{Multiplication of Fractions} \\ &= \frac{31}{28} && \text{Addition of Fractions} \end{aligned}$$

The result is in lowest terms because 31 and 28 are relatively prime so we're done.

We could have used $12 \cdot 30 \cdot 3 = 1080$ as our common denominator but then the numerators would become unnecessarily large. It's best to use the *lowest* common denominator.

2. We could begin with the subtraction in parentheses, namely $\frac{47}{30} - \frac{7}{3}$, and then subtract that result from $\frac{5}{12}$. It's easier, however, to first distribute the negative across the quantity in parentheses and then use the Associative Property to perform all of the addition and subtraction in one step. The lowest common denominator for all three fractions is 60.

$$\begin{aligned} \frac{5}{12} - \left(\frac{47}{30} - \frac{7}{3}\right) &= \frac{5}{12} - \frac{47}{30} + \frac{7}{3} && \text{Distribute the Negative} \\ &= \frac{5}{12} \cdot \frac{5}{5} - \frac{47}{30} \cdot \frac{2}{2} + \frac{7}{3} \cdot \frac{20}{20} && \text{Equivalent Fractions} \\ &= \frac{25}{60} - \frac{94}{60} + \frac{140}{60} && \text{Multiplication of Fractions} \\ &= \frac{71}{60} && \text{Addition and Subtraction of Fractions} \end{aligned}$$

The numerator and denominator are relatively prime so the fraction is in lowest terms and we have our final answer.

3. What we are asked to simplify in this problem is known as a 'complex' or 'compound' fraction. Simply put, we have fractions within a fraction. The longest division line (also called a 'vinculum') acts as a grouping symbol, quite literally dividing the compound fraction into a numerator (containing fractions) and a denominator (which in this case does not contain fractions):

$$\frac{\frac{12}{5} - \frac{7}{24}}{1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)} = \frac{\left(\frac{12}{5} - \frac{7}{24}\right)}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)}$$

The first step to simplifying a compound fraction like this one is to see if you can simplify the little fractions inside it. There are two ways to proceed. One is to simplify the numerator and denominator separately, and then use the fact that division is the same thing as multiplication by the reciprocal, as follows:

$$\begin{aligned}
\frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} &= \frac{\left(\frac{12}{5}\cdot\frac{24}{24}-\frac{7}{24}\cdot\frac{5}{5}\right)}{\left(1\cdot\frac{120}{120}+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} && \text{Equivalent Fractions}\\
&= \frac{288/120-35/120}{120/120+84/120} && \text{Multiplication of fractions}\\
&= \frac{253/120}{204/120} && \text{Addition and subtraction of fractions}\\
&= \frac{253}{\cancel{120}}\cdot\frac{\cancel{120}}{204} && \text{Division of fractions and cancellation}\\
&= \frac{253}{204}
\end{aligned}$$

Since $253 = 11 \cdot 23$ and $204 = 2 \cdot 2 \cdot 3 \cdot 17$ have no common factors our result is in lowest terms which means we are done.

While there is nothing wrong with the above approach, we can also use our Equivalent Fractions property to rid ourselves of the 'compound' nature of this fraction straight away. The idea is to multiply both the numerator and denominator by the lowest common denominator of each of the 'smaller' fractions - in this case, $24 \cdot 5 = 120$.

$$\begin{aligned}
\frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} &= \frac{\left(\frac{12}{5}-\frac{7}{24}\right)\cdot 120}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)\cdot 120} && \text{Equivalent Fractions}\\
&= \frac{\left(\frac{12}{5}\right)(120)-\left(\frac{7}{24}\right)(120)}{(1)(120)+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)(120)} && \text{Distributive Property}\\
&= \frac{\frac{12\cdot 120}{5}-\frac{7\cdot 120}{24}}{120+\frac{12\cdot 7\cdot 120}{5\cdot 24}} && \text{Multiply fractions}\\
&= \frac{\frac{12\cdot 24\cdot \cancel{5}}{\cancel{5}}-\frac{7\cdot 5\cdot \cancel{24}}{\cancel{24}}}{120+\frac{12\cdot 7\cdot \cancel{5}\cdot\cancel{24}}{\cancel{5}\cdot\cancel{24}}} && \text{Factor and cancel}\\
&= \frac{(12\cdot 24)-(7\cdot 5)}{120+(12\cdot 7)}\\
&= \frac{288-35}{120+84} = \frac{253}{204},
\end{aligned}$$

which is the same as we obtained above.

4. This fraction may look simpler than the one before it, but the negative signs and parentheses mean that we shouldn't get complacent. Again we note that the division line here acts as a grouping symbol. That is,

$$\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} = \frac{((2(2)+1)(-3-(-3))-5(4-7))}{(4-2(3))}$$

This means that we should simplify the numerator and denominator first, then perform the division last. We tend to what's in parentheses first, giving multiplication priority over addition and subtraction.

$$\begin{aligned}
\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} &= \frac{(4+1)(-3+3)-5(-3)}{4-6} \\
&= \frac{(5)(0)+15}{-2} \\
&= \frac{15}{-2} \\
&= -\frac{15}{2} && \text{Properties of Negatives}
\end{aligned}$$

Since $15 = 3 \cdot 5$ and 2 have no common factors, we are done.

5. In this problem, we have multiplication and subtraction. Multiplication takes precedence so we perform it first. Recall that to multiply fractions, we do *not* need to obtain common denominators; rather, we multiply the corresponding numerators together along with the corresponding denominators. Like the previous example, we have parentheses and negative signs for added fun!

$$\begin{aligned}
\left(\frac{3}{5}\right)\left(\frac{5}{13}\right)-\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) &= \frac{3\cdot 5}{5\cdot 13}-\frac{4\cdot(-12)}{5\cdot 13} && \text{Multiply fractions} \\
&= \frac{15}{65}-\frac{-48}{65} \\
&= \frac{15}{65}+\frac{48}{65} && \text{Properties of Negatives} \\
&= \frac{15+48}{65} && \text{Add numerators} \\
&= \frac{63}{65}
\end{aligned}$$

Since $64 = 3 \cdot 3 \cdot 7$ and $65 = 5 \cdot 13$ have no common factors, our answer $\frac{63}{65}$ is in lowest terms and we are done.

Of the issues discussed in the previous set of examples none causes students more trouble than simplifying compound fractions. We presented two different methods for simplifying them: one in which we simplified the overall numerator and denominator and then performed the division and one in which we removed the compound nature of the fraction at the very beginning. We encourage the reader to go back and use both methods on each of the compound fractions presented. Keep in mind that when a compound fraction is encountered in the rest of the text it will usually be simplified using only one method and we may not choose your favourite method. Feel free to use the other one in your notes.

Next, we review exponents and their properties. Recall that $2 \cdot 2 \cdot 2$ can be written as 2^3 because exponential notation expresses repeated multiplication. In the expression 2^3, 2 is called the **base** and 3 is called the **exponent**. In order to generalize exponents from natural numbers to the integers, and eventually to rational and real numbers, it is helpful to think of the exponent as a count of the number of factors of the base we are multiplying by 1. For instance,

$$2^3 = 1 \cdot (\text{three factors of two}) = 1 \cdot (2 \cdot 2 \cdot 2) = 8.$$

From this, it makes sense that

$$2^0 = 1 \cdot (\text{zero factors of two}) = 1.$$

What about 2^{-3}? The '$-$' in the exponent indicates that we are 'taking away' three factors of two, essentially dividing by three factors of two. So,

$$2^{-3} = 1 \div (\text{three factors of two}) = 1 \div (2 \cdot 2 \cdot 2) = \frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{8}.$$

We summarize the properties of integer exponents below.

Definition 13 Properties of Integer Exponents

Suppose a and b are nonzero real numbers and n and m are integers.

- **Product Rules:** $(ab)^n = a^n b^n$ and $a^n a^m = a^{n+m}$.
- **Quotient Rules:** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ and $\frac{a^n}{a^m} = a^{n-m}$.
- **Power Rule:** $(a^n)^m = a^{nm}$.
- **Negatives in Exponents:** $a^{-n} = \frac{1}{a^n}$.

  In particular, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$ and $\frac{1}{a^{-n}} = a^n$.
- **Zero Powers:** $a^0 = 1$.

  Note: The expression 0^0 is an indeterminate form.[a]
- **Powers of Zero:** For any *natural* number n, $0^n = 0$.

  Note: The expression 0^n for integers $n \leq 0$ is not defined.

[a]See the comment regarding '$\frac{0}{0}$' on page 12.

While it is important the state the Properties of Exponents, it is also equally important to take a moment to discuss one of the most common errors in Algebra. It is true that $(ab)^2 = a^2b^2$ (which some students refer to as 'distributing' the exponent to each factor) but you **cannot** do this sort of thing with addition. That is, in general, $(a+b)^2 \neq a^2 + b^2$. (For example, take $a = 3$ and $b = 4$.) The same goes for any other powers.

With exponents now in the mix, we can now state the Order of Operations Agreement.

Definition 14 Order of Operations Agreement

When evaluating an expression involving real numbers:

1. Evaluate any expressions in **p**arentheses (or other grouping symbols.)
2. Evaluate **e**xponents.
3. Evaluate **d**ivision and **m**ultiplication as you read from left to right.
4. Evaluate **a**ddition and **s**ubtraction as you read from left to right.

For example, $2+3\cdot 4^2 = 2+3\cdot 16 = 2+48 = 50$. Where students get into trouble is with things like -3^2. If we think of this as $0-3^2$, then it is clear that we evaluate the exponent first: $-3^2 = 0-3^2 = 0-9 = -9$. In general, we interpret $-a^n = -(a^n)$. If we want the 'negative' to also be raised to a power, we must write $(-a)^n$ instead. To summarize, $-3^2 = -9$ but $(-3)^2 = 9$.

Of course, many of the 'properties' we've stated in this section can be viewed as ways to circumvent the order of operations. We've already seen how the distributive property allows us to simplify $5(2+x)$ by performing the indicated multiplication **before** the addition that's in parentheses. Similarly, consider trying to evaluate $2^{30172}\cdot 2^{-30169}$. The Order of Operations Agreement demands that the exponents be dealt with first, however, trying to compute 2^{30172} is a challenge, even for a calculator. One of the Product Rules of Exponents, however, allow us to rewrite this product, essentially performing the multiplication first, to get: $2^{30172-30169} = 2^3 = 8$.

Example 4 Operations with exponents

Perform the indicated operations and simplify.

1. $\dfrac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2}$
2. $12(-5)(-5+3)^{-4}+6(-5)^2(-4)(-5+3)^{-5}$
3. $\dfrac{\left(\dfrac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\dfrac{5\cdot 3^{49}}{4^{34}}\right)}$
4. $\dfrac{2\left(\dfrac{5}{12}\right)^{-1}}{1-\left(\dfrac{5}{12}\right)^{-2}}$

Order of operations follows the "PEDMAS" rule some of you may have encountered.

Solution

1. We begin working inside parentheses then deal with the exponents before working through the other operations. As we saw in Example 3, the division here acts as a grouping symbol, so we save the division to the end.

$$\begin{aligned}\frac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2} &= \frac{(2)(8)-(4)^2}{(2)^2} = \frac{(2)(8)-16}{4}\\ &= \frac{16-16}{4} = \frac{0}{4} = 0\end{aligned}$$

2. As before, we simplify what's in the parentheses first, then work our way

through the exponents, multiplication, and finally, the addition.

$$
\begin{aligned}
12(-5)(-5+3)^{-4} + 6(-5)^2(-4)(-5+3)^{-5} \\
&= 12(-5)(-2)^{-4} + 6(-5)^2(-4)(-2)^{-5} \\
&= 12(-5)\left(\frac{1}{(-2)^4}\right) + 6(-5)^2(-4)\left(\frac{1}{(-2)^5}\right) \\
&= 12(-5)\left(\frac{1}{16}\right) + 6(25)(-4)\left(\frac{1}{-32}\right) \\
&= (-60)\left(\frac{1}{16}\right) + (-600)\left(\frac{1}{-32}\right) \\
&= \frac{-60}{16} + \left(\frac{-600}{-32}\right) \\
&= \frac{-15\cdot\cancel{4}}{4\cdot\cancel{4}} + \frac{-75\cdot\cancel{8}}{-4\cdot\cancel{8}} \\
&= \frac{-15}{4} + \frac{-75}{-4} \\
&= \frac{-15}{4} + \frac{75}{4} \\
&= \frac{-15+75}{4} \\
&= \frac{60}{4} \\
&= 15
\end{aligned}
$$

3. The Order of Operations Agreement mandates that we work within each set of parentheses first, giving precedence to the exponents, then the multiplication, and, finally the division. The trouble with this approach is that the exponents are so large that computation becomes a trifle unwieldy. What we observe, however, is that the bases of the exponential expressions, 3 and 4, occur in both the numerator and denominator of the compound fraction, giving us hope that we can use some of the Properties of Exponents (the Quotient Rule, in particular) to help us out. Our first step here is to invert and multiply. We see immediately that the 5's cancel after which we group the powers of 3 together and the powers of 4 together and apply the properties of exponents.

$$
\begin{aligned}
\frac{\left(\frac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\frac{5\cdot 3^{49}}{4^{34}}\right)} &= \frac{5\cdot 3^{51}}{4^{36}}\cdot\frac{4^{34}}{5\cdot 3^{49}} &&= \frac{\cancel{5}\cdot 3^{51}\cdot 4^{34}}{\cancel{5}\cdot 3^{49}\cdot 4^{36}} &&= \frac{3^{51}}{3^{49}}\cdot\frac{4^{34}}{4^{36}} \\
&= 3^{51-49}\cdot 4^{34-36} &&= 3^2\cdot 4^{-2} &&= 3^2\cdot\left(\frac{1}{4^2}\right) \\
&= 9\cdot\left(\frac{1}{16}\right) &&= \frac{9}{16}
\end{aligned}
$$

4. We have yet another instance of a compound fraction so our first order of business is to rid ourselves of the compound nature of the fraction like we did in Example 3. To do this, however, we need to tend to the exponents first so that we can determine what common denominator is needed to

simplify the fraction.

$$\frac{2\left(\frac{5}{12}\right)^{-1}}{1-\left(\frac{5}{12}\right)^{-2}} = \frac{2\left(\frac{12}{5}\right)}{1-\left(\frac{12}{5}\right)^{2}} = \frac{\left(\frac{24}{5}\right)}{1-\left(\frac{12^2}{5^2}\right)}$$

$$= \frac{\left(\frac{24}{5}\right)}{1-\left(\frac{144}{25}\right)} = \frac{\left(\frac{24}{5}\right)\cdot 25}{\left(1-\frac{144}{25}\right)\cdot 25}$$

$$= \frac{\left(\frac{24\cdot 5\cdot \cancel{5}}{\cancel{5}}\right)}{\left(1\cdot 25-\frac{144\cdot \cancel{25}}{\cancel{25}}\right)} = \frac{120}{25-144}$$

$$= \frac{120}{-119} = -\frac{120}{119}$$

Since 120 and 119 have no common factors, we are done.

It's important that you understand the difference between the statements $y = \sqrt{x}$ and $y^2 = x$. As we'll discuss in Chapter 2, the equation $y = \sqrt{x}$ defines y as a **function** of x, which means that for each value of $x \geq 0$ there is only one value of y such that $y = \sqrt{x}$. For example, $y = \sqrt{4}$ is equivalent to $y = 2$. On the other hand, there are **two** solutions to $y^2 = x$; namely, $y = \sqrt{x}$ and $y = -\sqrt{x}$. For example, the equation $y^2 = 4$ is equivalent to the two equations $y = 2$ and $y = -2$ (or, more concisely, $y = \pm 2$). Since these two equations are closely related, it's easy to mix them up. The main thing to remember is that $\sqrt{x}$ always denotes the *positive* square root of x.

We close our review of real number arithmetic with a discussion of roots and radical notation. Just as subtraction and division were defined in terms of the inverse of addition and multiplication, respectively, we define roots by undoing natural number exponents.

Definition 15 The principal n^{th} root

Let a be a real number and let n be a natural number. If n is odd, then the **principal n^{th} root** of a (denoted $\sqrt[n]{a}$) is the unique real number satisfying $\left(\sqrt[n]{a}\right)^n = a$. If n is even, $\sqrt[n]{a}$ is defined similarly provided $a \geq 0$ and $\sqrt[n]{a} \geq 0$. The number n is called the **index** of the root and the the number a is called the **radicand**. For $n = 2$, we write $\sqrt{a}$ instead of $\sqrt[2]{a}$.

The reasons for the added stipulations for even-indexed roots in Definition 15 can be found in the Properties of Negatives. First, for all real numbers, $x^{\text{even power}} \geq 0$, which means it is never negative. Thus if a is a *negative* real number, there are no real numbers x with $x^{\text{even power}} = a$. This is why if n is even, $\sqrt[n]{a}$ only exists if $a \geq 0$. The second restriction for even-indexed roots is that $\sqrt[n]{a} \geq 0$. This comes from the fact that $x^{\text{even power}} = (-x)^{\text{even power}}$, and we require $\sqrt[n]{a}$ to have just one value. So even though $2^4 = 16$ and $(-2)^4 = 16$, we require $\sqrt[4]{16} = 2$ and ignore -2.

Dealing with odd powers is much easier. For example, $x^3 = -8$ has one and only one real solution, namely $x = -2$, which means not only does $\sqrt[3]{-8}$ exist, there is only one choice, namely $\sqrt[3]{-8} = -2$. Of course, when it comes to solving $x^{5213} = -117$, it's not so clear that there is one and only one real solution, let alone that the solution is $\sqrt[5213]{-117}$. Such pills are easier to swallow once we've thought a bit about such equations graphically, (see Chapter 4) and ultimately, these things come from the completeness property of the real numbers mentioned earlier.

We list properties of radicals below as a 'theorem' since they can be justified using the properties of exponents.

Theorem 3 Properties of Radicals

Let a and b be real numbers and let m and n be natural numbers. If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

- **Product Rule:** $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
- **Quotient Rule:** $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, provided $b \neq 0$.
- **Power Rule:** $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

The proof of Theorem 3 is based on the definition of the principal n^{th} root and the Properties of Exponents. To establish the product rule, consider the following. If n is odd, then by definition $\sqrt[n]{ab}$ is the unique real number such that $(\sqrt[n]{ab})^n = ab$. Given that $(\sqrt[n]{a}\,\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab$ as well, it must be the case that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. If n is even, then $\sqrt[n]{ab}$ is the unique non-negative real number such that $(\sqrt[n]{ab})^n = ab$. Note that since n is even, $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are also non-negative thus $\sqrt[n]{a}\sqrt[n]{b} \geq 0$ as well. Proceeding as above, we find that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. The quotient rule is proved similarly and is left as an exercise. The power rule results from repeated application of the product rule, so long as $\sqrt[n]{a}$ is a real number to start with. We leave that as an exercise as well.

We pause here to point out one of the most common errors students make when working with radicals. Obviously $\sqrt{9} = 3$, $\sqrt{16} = 4$ and $\sqrt{9+16} = \sqrt{25} = 5$. Thus we can clearly see that $5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3 + 4 = 7$ because we all know that $5 \neq 7$. The authors urge you to **never consider 'distributing' roots or exponents**. It's wrong and no good will come of it because in general $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$.

Since radicals have properties inherited from exponents, they are often written as such. We define rational exponents in terms of radicals in the box below.

Definition 16 Rational exponents

Let a be a real number, let m be an integer and let n be a natural number.

- $a^{\frac{1}{n}} = \sqrt[n]{a}$ whenever $\sqrt[n]{a}$ is a real number. (If n is even we need $a \geq 0$.)
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ whenever $\sqrt[n]{a}$ is a real number.

Things get more complicated once complex numbers are involved. Fortunately (disappointingly?), that's not a can of worms we'll be opening in this course.

It would make life really nice if the rational exponents defined in Definition 16 had all of the same properties that integer exponents have as listed on page 17 - but they don't. Why not? Let's look at an example to see what goes wrong. Consider the Product Rule which says that $(ab)^n = a^n b^n$ and let $a = -16$, $b = -81$ and $n = \frac{1}{4}$. Plugging the values into the Product Rule yields the equation $((-16)(-81))^{1/4} = (-16)^{1/4}(-81)^{1/4}$. The left side of this equation is $1296^{1/4}$ which equals 6 but the right side is undefined because neither root is a real number. Would it help if, when it comes to even roots (as signified by even denominators in the fractional exponents), we ensure that everything they apply to is non-negative? That works for some of the rules - we leave it as an exercise to see which ones - but does not work for the Power Rule.

Consider the expression $\left(a^{2/3}\right)^{3/2}$. Applying the usual laws of exponents, we'd be tempted to simplify this as $\left(a^{2/3}\right)^{3/2} = a^{\frac{2}{3}\cdot\frac{3}{2}} = a^1 = a$. However, if we substitute $a = -1$ and apply Definition 16, we find $(-1)^{2/3} = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1$ so that $\left((-1)^{2/3}\right)^{3/2} = 1^{3/2} = \left(\sqrt{1}\right)^3 = 1^3 = 1$. Thus in this case we have $\left(a^{2/3}\right)^{3/2} \neq a$ even though all of the roots were defined. It is true, however, that $\left(a^{3/2}\right)^{2/3} = a$ and we leave this for the reader to show. The moral of the story is that when simplifying powers of rational exponents where the base is negative or worse, unknown, it's usually best to rewrite them as radicals.

Example 5 Combining operations

Perform the indicated operations and simplify.

1. $\dfrac{-(-4) - \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$

2. $\dfrac{2\left(\dfrac{\sqrt{3}}{3}\right)}{1 - \left(\dfrac{\sqrt{3}}{3}\right)^2}$

3. $(\sqrt[3]{-2} - \sqrt[3]{-54})^2$

4. $2\left(\dfrac{9}{4} - 3\right)^{1/3} + 2\left(\dfrac{9}{4}\right)\left(\dfrac{1}{3}\right)\left(\dfrac{9}{4} - 3\right)^{-2/3}$

Solution

1. We begin in the numerator and note that the radical here acts a grouping symbol,[3] so our first order of business is to simplify the radicand.

$$\begin{aligned}
\frac{-(-4) - \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)} &= \frac{-(-4) - \sqrt{16 - 4(2)(-3)}}{2(2)} \\
&= \frac{-(-4) - \sqrt{16 - 4(-6)}}{2(2)} \\
&= \frac{-(-4) - \sqrt{16 - (-24)}}{2(2)} \\
&= \frac{-(-4) - \sqrt{16 + 24}}{2(2)} \\
&= \frac{-(-4) - \sqrt{40}}{2(2)}
\end{aligned}$$

As you may recall, 40 can be factored using a perfect square as $40 = 4 \cdot 10$ so we use the product rule of radicals to write $\sqrt{40} = \sqrt{4 \cdot 10} =$

[3] The line extending horizontally from the square root symbol '$\sqrt{}$' is, you guessed it, another vinculum.

$\sqrt{4}\sqrt{10} = 2\sqrt{10}$. This lets us factor a '2' out of both terms in the numerator, eventually allowing us to cancel it with a factor of 2 in the denominator.

$$\begin{aligned}\frac{-(-4)-\sqrt{40}}{2(2)} &= \frac{-(-4)-2\sqrt{10}}{2(2)} &&= \frac{4-2\sqrt{10}}{2(2)}\\ &= \frac{2\cdot 2-2\sqrt{10}}{2(2)} &&= \frac{2(2-\sqrt{10})}{2(2)}\\ &= \frac{\cancel{2}(2-\sqrt{10})}{\cancel{2}(2)} &&= \frac{2-\sqrt{10}}{2}\end{aligned}$$

Since the numerator and denominator have no more common factors,[4] we are done.

2. Once again we have a compound fraction, so we first simplify the exponent in the denominator to see which factor we'll need to multiply by in order to clean up the fraction.

$$\begin{aligned}\frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{\sqrt{3}}{3}\right)^2} &= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{(\sqrt{3})^2}{3^2}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{3}{9}\right)}\\ &= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1\cdot\cancel{3}}{3\cdot\cancel{3}}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1}{3}\right)}\\ &= \frac{2\left(\frac{\sqrt{3}}{3}\right)\cdot 3}{\left(1-\left(\frac{1}{3}\right)\right)\cdot 3} &&= \frac{\frac{2\cdot\sqrt{3}\cdot\cancel{3}}{\cancel{3}}}{1\cdot 3-\frac{1\cdot\cancel{3}}{\cancel{3}}}\\ &= \frac{2\sqrt{3}}{3-1} &&= \frac{\cancel{2}\sqrt{3}}{\cancel{2}} = \sqrt{3}\end{aligned}$$

3. Working inside the parentheses, we first encounter $\sqrt[3]{-2}$. While the -2 isn't a perfect cube,[5] we may think of $-2 = (-1)(2)$. Since $(-1)^3 = -1$, -1 *is* a perfect cube, and we may write $\sqrt[3]{-2} = \sqrt[3]{(-1)(2)} = \sqrt[3]{-1}\sqrt[3]{2} = -\sqrt[3]{2}$. When it comes to $\sqrt[3]{54}$, we may write it as $\sqrt[3]{(-27)(2)} = \sqrt[3]{-27}\sqrt[3]{2} = -3\sqrt[3]{2}$. So,

$$\sqrt[3]{-2}-\sqrt[3]{-54} = -\sqrt[3]{2}-(-3\sqrt[3]{2}) = -\sqrt[3]{2}+3\sqrt[3]{2}.$$

At this stage, we can simplify $-\sqrt[3]{2}+3\sqrt[3]{2} = 2\sqrt[3]{2}$. You may remember this as being called 'combining like radicals,' but it is in fact just another application of the distributive property:

$$-\sqrt[3]{2}+3\sqrt[3]{2} = (-1)\sqrt[3]{2}+3\sqrt[3]{2} = (-1+3)\sqrt[3]{2} = 2\sqrt[3]{2}.$$

Putting all this together, we get:

$$\begin{aligned}(\sqrt[3]{-2}-\sqrt[3]{-54})^2 &= (-\sqrt[3]{2}+3\sqrt[3]{2})^2 &&= (2\sqrt[3]{2})^2\\ &= 2^2(\sqrt[3]{2})^2 = 4\sqrt[3]{2^2} &&= 4\sqrt[3]{4}\end{aligned}$$

Since there are no perfect integer cubes which are factors of 4 (apart from 1, of course), we are done.

[4] Do you see why we aren't 'cancelling' the remaining 2's?
[5] Of an integer, that is!

4. We start working in parentheses and get a common denominator to subtract the fractions:

$$\frac{9}{4} - 3 = \frac{9}{4} - \frac{3 \cdot 4}{1 \cdot 4} = \frac{9}{4} - \frac{12}{4} = \frac{-3}{4}$$

Since the denominators in the fractional exponents are odd, we can proceed using the properties of exponents:

$$\begin{aligned}
2\left(\frac{9}{4} - 3\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{9}{4} - 3\right)^{-2/3} \\
&= 2\left(\frac{-3}{4}\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{-3}{4}\right)^{-2/3} \\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{4}{-3}\right)^{2/3} \\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{(4)^{2/3}}{(-3)^{2/3}}\right) \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{2 \cdot 9 \cdot 1 \cdot 4^{2/3}}{4 \cdot 3 \cdot (-3)^{2/3}} \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{\cancel{2} \cdot 3 \cdot \cancel{3} \cdot 4^{2/3}}{2 \cdot \cancel{2} \cdot \cancel{3} \cdot (-3)^{2/3}} \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}}
\end{aligned}$$

At this point, we could start looking for common denominators but it turns out that these fractions reduce even further. Since $4 = 2^2$, $4^{1/3} = (2^2)^{1/3} = 2^{2/3}$. Similarly, $4^{2/3} = (2^2)^{2/3} = 2^{4/3}$. The expressions $(-3)^{1/3}$ and $(-3)^{2/3}$ contain negative bases so we proceed with caution and convert them back to radical notation to get: $(-3)^{1/3} = \sqrt[3]{-3} = -\sqrt[3]{3} = -3^{1/3}$ and $(-3)^{2/3} = (\sqrt[3]{-3})^2 = (-\sqrt[3]{3})^2 = (\sqrt[3]{3})^2 = 3^{2/3}$. Hence:

$$\begin{aligned}
\frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}} &= \frac{2 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3 \cdot 2^{4/3}}{2 \cdot 3^{2/3}} \\
&= \frac{2^1 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3^1 \cdot 2^{4/3}}{2^1 \cdot 3^{2/3}} \\
&= 2^{1-2/3} \cdot (-3^{1/3}) + 3^{1-2/3} \cdot 2^{4/3-1} \\
&= 2^{1/3} \cdot (-3^{1/3}) + 3^{1/3} \cdot 2^{1/3} \\
&= -2^{1/3} \cdot 3^{1/3} + 3^{1/3} \cdot 2^{1/3} \\
&= 0
\end{aligned}$$

Exercises 1.2

Problems

In Exercises 1 – 33, perform the indicated operations and simplify.

1. $5-2+3$

2. $5-(2+3)$

3. $\dfrac{2}{3}-\dfrac{4}{7}$

4. $\dfrac{3}{8}+\dfrac{5}{12}$

5. $\dfrac{5-3}{-2-4}$

6. $\dfrac{2(-3)}{3-(-3)}$

7. $\dfrac{2(3)-(4-1)}{2^2+1}$

8. $\dfrac{4-5.8}{2-2.1}$

9. $\dfrac{1-2(-3)}{5(-3)+7}$

10. $\dfrac{5(3)-7}{2(3)^2-3(3)-9}$

11. $\dfrac{2((-1)^2-1)}{((-1)^2+1)^2}$

12. $\dfrac{(-2)^2-(-2)-6}{(-2)^2-4}$

13. $\dfrac{3-\frac{4}{9}}{-2-(-3)}$

14. $\dfrac{\frac{2}{3}-\frac{4}{5}}{4-\frac{7}{10}}$

15. $\dfrac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2}$

16. $\dfrac{1-\left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}{1+\left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}$

17. $\left(\dfrac{2}{3}\right)^{-5}$

18. $3^{-1}-4^{-2}$

19. $\dfrac{1+2^{-3}}{3-4^{-1}}$

20. $\dfrac{3\cdot 5^{100}}{12\cdot 5^{98}}$

21. $\sqrt{3^2+4^2}$

22. $\sqrt{12}-\sqrt{75}$

23. $(-8)^{2/3}-9^{-3/2}$

24. $\left(-\frac{32}{9}\right)^{-3/5}$

25. $\sqrt{(3-4)^2+(5-2)^2}$

26. $\sqrt{(2-(-1))^2+\left(\frac{1}{2}-3\right)^2}$

27. $\sqrt{(\sqrt{5}-2\sqrt{5})^2+(\sqrt{18}-\sqrt{8})^2}$

28. $\dfrac{-12+\sqrt{18}}{21}$

29. $\dfrac{-2-\sqrt{(2)^2-4(3)(-1)}}{2(3)}$

30. $\dfrac{-(-4)+\sqrt{(-4)^2-4(1)(-1)}}{2(1)}$

31. $2(-5)(-5+1)^{-1}+(-5)^2(-1)(-5+1)^{-2}$

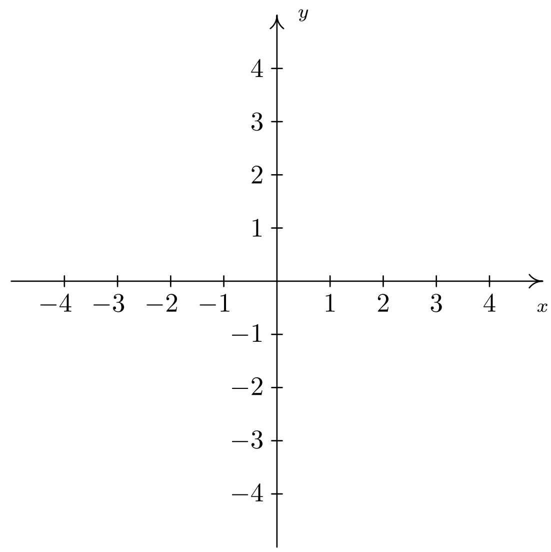
32. $3\sqrt{2(4)+1}+3(4)\left(\frac{1}{2}\right)(2(4)+1)^{-1/2}(2)$

33. $2(-7)\sqrt[3]{1-(-7)}+(-7)^2\left(\frac{1}{3}\right)(1-(-7))^{-2/3}(-1)$

1.3 The Cartesian Coordinate Plane

The Cartesian Plane is named in honour of René Descartes.

In order to visualize the pure excitement that is Precalculus, we need to unite Algebra and Geometry. Simply put, we must find a way to draw algebraic things. Let's start with possibly the greatest mathematical achievement of all time: the **Cartesian Coordinate Plane**. Imagine two real number lines crossing at a right angle at 0 as drawn below.

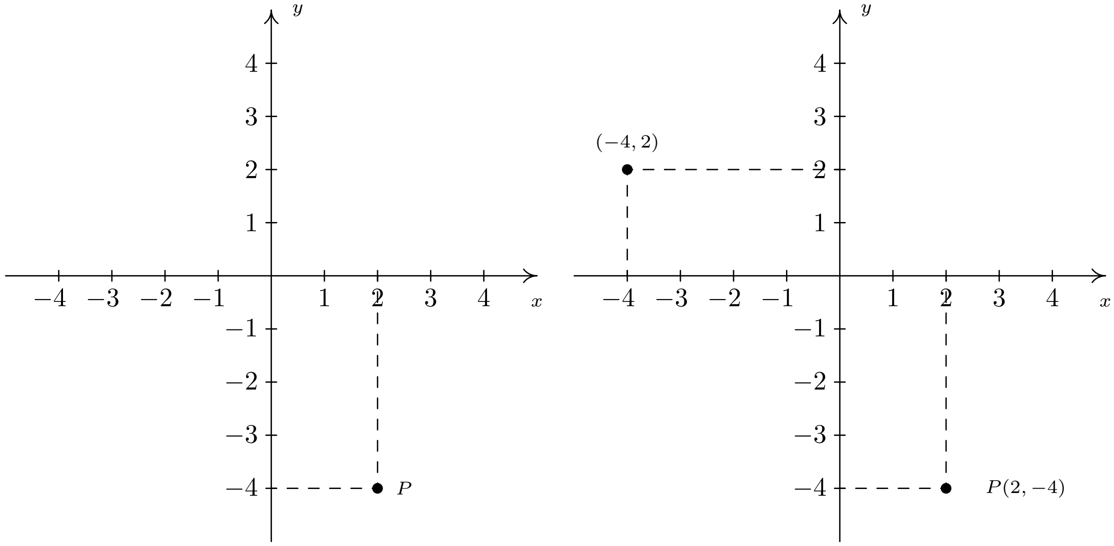


Usually extending off towards infinity is indicated by arrows, but here, the arrows are used to indicate the *direction* of increasing values of *x* and *y*.

The horizontal number line is usually called the ***x*-axis** while the vertical number line is usually called the ***y*-axis**. As with the usual number line, we imagine these axes extending off indefinitely in both directions. Having two number lines allows us to locate the positions of points off of the number lines as well as points on the lines themselves.

The names of the coordinates can vary depending on the context of the application. If, for example, the horizontal axis represented time we might choose to call it the *t*-axis. The first number in the ordered pair would then be the *t*-coordinate.

For example, consider the point *P* on the next page. To use the numbers on the axes to label this point, we imagine dropping a vertical line from the *x*-axis to *P* and extending a horizontal line from the *y*-axis to *P*. This process is sometimes called 'projecting' the point *P* to the *x*- (respectively *y*-) axis. We then describe the point *P* using the **ordered pair** $(2, -4)$. The first number in the ordered pair is called the **abscissa** or ***x*-coordinate** and the second is called the **ordinate** or ***y*-coordinate**. Taken together, the ordered pair $(2, -4)$ comprise the **Cartesian coordinates** of the point *P*. In practice, the distinction between a point and its coordinates is blurred; for example, we often speak of 'the point $(2, -4)$.' We can think of $(2, -4)$ as instructions on how to reach *P* from the **origin** $(0, 0)$ by moving 2 units to the right and 4 units downwards. Notice that the order in the ordered pair is important — if we wish to plot the point $(-4, 2)$, we would move to the left 4 units from the origin and then move upwards 2 units, as below on the right.



When we speak of the Cartesian Coordinate Plane, we mean the set of all possible ordered pairs (x, y) as x and y take values from the real numbers. Below is a summary of important facts about Cartesian coordinates.

Key Idea 4 Important Facts about the Cartesian Coordinate Plane

- (a, b) and (c, d) represent the same point in the plane if and only if $a = c$ and $b = d$.
- (x, y) lies on the x-axis if and only if $y = 0$.
- (x, y) lies on the y-axis if and only if $x = 0$.
- The origin is the point $(0, 0)$. It is the only point common to both axes.

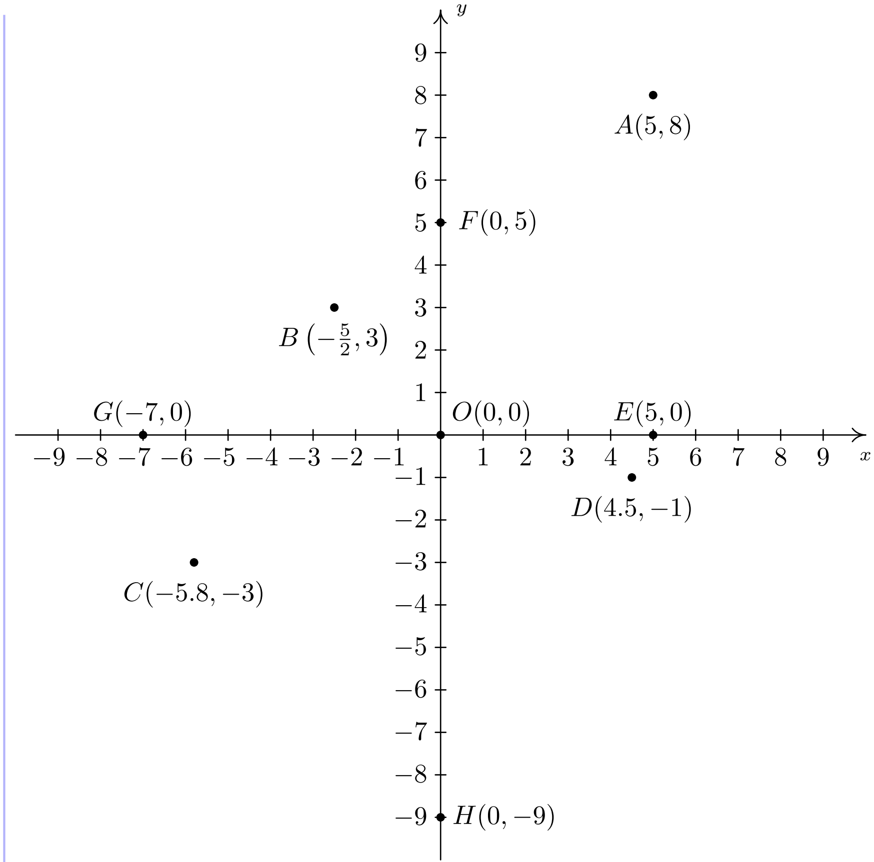
Cartesian coordinates are sometimes referred to as *rectangular coordinates*, to distinguish them from other coordinate systems such as *polar coordinates*. See Section **??** for more details.

Example 6 Plotting points in the Cartesian Plane

Plot the following points: $A(5, 8)$, $B\left(-\frac{5}{2}, 3\right)$, $C(-5.8, -3)$, $D(4.5, -1)$, $E(5, 0)$, $F(0, 5)$, $G(-7, 0)$, $H(0, -9)$, $O(0, 0)$.

The letter O is almost always reserved for the origin.

Solution To plot these points, we start at the origin and move to the right if the x-coordinate is positive; to the left if it is negative. Next, we move up if the y-coordinate is positive or down if it is negative. If the x-coordinate is 0, we start at the origin and move along the y-axis only. If the y-coordinate is 0 we move along the x-axis only.



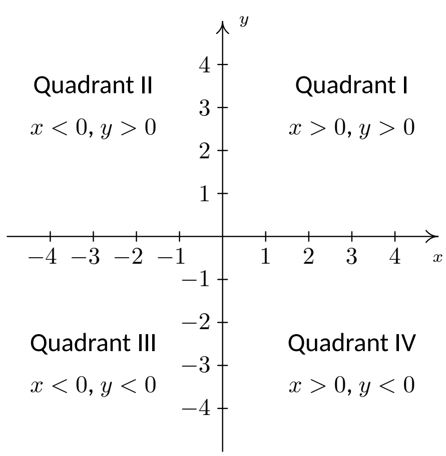


Figure 1.9: The four quadrants of the Cartesian plane

The axes divide the plane into four regions called **quadrants**. They are labelled with Roman numerals and proceed counterclockwise around the plane: see Figure 1.9.

For example, $(1, 2)$ lies in Quadrant I, $(-1, 2)$ in Quadrant II, $(-1, -2)$ in Quadrant III and $(1, -2)$ in Quadrant IV. If a point other than the origin happens to lie on the axes, we typically refer to that point as lying on the positive or negative *x*-axis (if $y = 0$) or on the positive or negative *y*-axis (if $x = 0$). For example, $(0, 4)$ lies on the positive *y*-axis whereas $(-117, 0)$ lies on the negative *x*-axis. Such points do not belong to any of the four quadrants.

One of the most important concepts in all of Mathematics is **symmetry**. There are many types of symmetry in Mathematics, but three of them can be discussed easily using Cartesian Coordinates.

Definition 17 Symmetry in the Cartesian Plane

Two points (a, b) and (c, d) in the plane are said to be

- **symmetric about the *x*-axis** if $a = c$ and $b = -d$
- **symmetric about the *y*-axis** if $a = -c$ and $b = d$
- **symmetric about the origin** if $a = -c$ and $b = -d$

In Figure 1.10, P and S are symmetric about the x-axis, as are Q and R; P and Q are symmetric about the y-axis, as are R and S; and P and R are symmetric about the origin, as are Q and S.

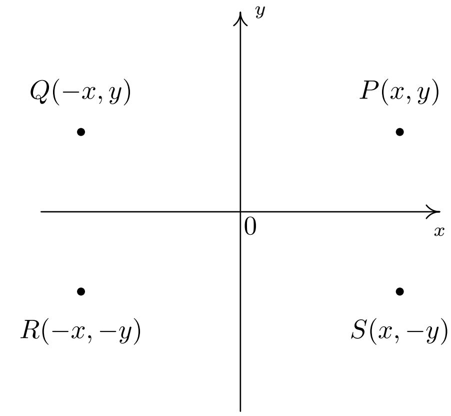


Figure 1.10: The three types of symmetry in the plane

Example 7 Finding points exhibiting symmetry

Let P be the point $(-2, 3)$. Find the points which are symmetric to P about the:

1. x-axis
2. y-axis
3. origin

Check your answer by plotting the points.

Solution The figure after Definition 17 gives us a good way to think about finding symmetric points in terms of taking the opposites of the x- and/or y-coordinates of $P(-2, 3)$.

1. To find the point symmetric about the x-axis, we replace the y-coordinate with its opposite to get $(-2, -3)$.
2. To find the point symmetric about the y-axis, we replace the x-coordinate with its opposite to get $(2, 3)$.
3. To find the point symmetric about the origin, we replace the x- and y-coordinates with their opposites to get $(2, -3)$.

   The points are plotted in Figure 1.11.

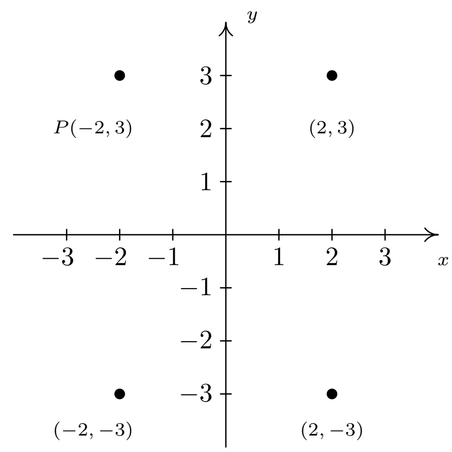


Figure 1.11: The point $P(-2, 3)$ and its three reflections

One way to visualize the processes in the previous example is with the concept of a **reflection**. If we start with our point $(-2, 3)$ and pretend that the x-axis is a mirror, then the reflection of $(-2, 3)$ across the x-axis would lie at $(-2, -3)$. If we pretend that the y-axis is a mirror, the reflection of $(-2, 3)$ across that axis would be $(2, 3)$. If we reflect across the x-axis and then the y-axis, we would go from $(-2, 3)$ to $(-2, -3)$ then to $(2, -3)$, and so we would end up at the point symmetric to $(-2, 3)$ about the origin. We summarize and generalize this process below.

Key Idea 5 Reflections in the Cartesian Plane

To reflect a point (x, y) about the:

- x-axis, replace y with $-y$.
- y-axis, replace x with $-x$.
- origin, replace x with $-x$ and y with $-y$.

1.3.1 Distance in the Plane

Another important concept in Geometry is the notion of length. If we are going to unite Algebra and Geometry using the Cartesian Plane, then we need to develop an algebraic understanding of what distance in the plane means. Suppose we have two points, $P(x_0, y_0)$ and $Q(x_1, y_1)$, in the plane. By the **distance** d between P and Q, we mean the length of the line segment joining P with Q. (Remember, given any two distinct points in the plane, there is a unique line containing both points.) Our goal now is to create an algebraic formula to compute the distance between these two points. Consider the generic situation in Figure 1.12.

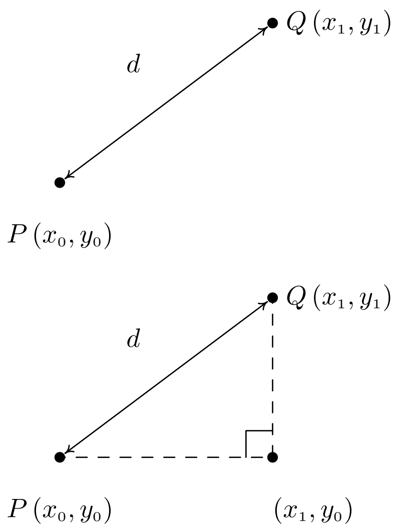


Figure 1.12: Distance between P and Q

With a little more imagination, we can envision a right triangle whose hypotenuse has length d as drawn above on the right. From the latter figure, we see that the lengths of the legs of the triangle are $|x_1 - x_0|$ and $|y_1 - y_0|$ so the Pythagorean Theorem gives us

$$|x_1 - x_0|^2 + |y_1 - y_0|^2 = d^2$$

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = d^2$$

(Do you remember why we can replace the absolute value notation with parentheses?) By extracting the square root of both sides of the second equation and using the fact that distance is never negative, we get

Key Idea 6 The Distance Formula

The distance d between the points $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

It is not always the case that the points P and Q lend themselves to constructing such a triangle. If the points P and Q are arranged vertically or horizontally, or describe the exact same point, we cannot use the above geometric argument to derive the distance formula. It is left to the reader in Exercise 16 to verify Equation 6 for these cases.

Example 8 Distance between two points

Find and simplify the distance between $P(-2, 3)$ and $Q(1, -3)$.

Solution

$$\begin{aligned} d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\ &= \sqrt{(1 - (-2))^2 + (-3 - 3)^2} \\ &= \sqrt{9 + 36} \\ &= 3\sqrt{5} \end{aligned}$$

So the distance is $3\sqrt{5}$.

Example 9 Finding points at a given distance

Find all of the points with x-coordinate 1 which are 4 units from the point $(3, 2)$.

Solution We shall soon see that the points we wish to find are on the line $x = 1$, but for now we'll just view them as points of the form $(1, y)$.

We require that the distance from $(3, 2)$ to $(1, y)$ be 4. The Distance Formula, Equation 6, yields

$$\begin{aligned}
d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
4 &= \sqrt{(1-3)^2 + (y-2)^2} \\
4 &= \sqrt{4 + (y-2)^2} \\
4^2 &= \left(\sqrt{4 + (y-2)^2}\right)^2 && \text{squaring both sides} \\
16 &= 4 + (y-2)^2 \\
12 &= (y-2)^2 \\
(y-2)^2 &= 12 \\
y - 2 &= \pm\sqrt{12} && \text{extracting the square root} \\
y - 2 &= \pm 2\sqrt{3} \\
y &= 2 \pm 2\sqrt{3}
\end{aligned}$$

We obtain two answers: $(1, 2+2\sqrt{3})$ and $(1, 2-2\sqrt{3})$. The reader is encouraged to think about why there are two answers.

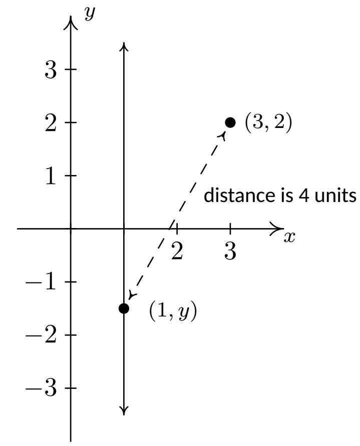


Figure 1.13: Diagram for Example 9

Related to finding the distance between two points is the problem of finding the **midpoint** of the line segment connecting two points. Given two points, $P(x_0, y_0)$ and $Q(x_1, y_1)$, the **midpoint** M of P and Q is defined to be the point on the line segment connecting P and Q whose distance from P is equal to its distance from Q.

If we think of reaching M by going 'halfway over' and 'halfway up' we get the following formula.

Key Idea 7 The Midpoint Formula

The midpoint M of the line segment connecting $P(x_0, y_0)$ and $Q(x_1, y_1)$ is:

$$M = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right)$$

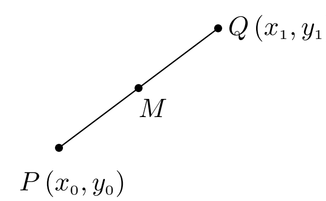


Figure 1.14: The midpoint of a line segment

If we let d denote the distance between P and Q, we leave it as Exercise 17 to show that the distance between P and M is $d/2$ which is the same as the distance between M and Q. This suffices to show that Equation **??** gives the coordinates of the midpoint.

Example 10 Finding the midpoint of a line segment

Find the midpoint of the line segment connecting $P(-2, 3)$ and $Q(1, -3)$.

Solution

$$\begin{aligned}
M &= \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right) \\
&= \left(\frac{(-2)+1}{2}, \frac{3+(-3)}{2}\right) = \left(-\frac{1}{2}, \frac{0}{2}\right) \\
&= \left(-\frac{1}{2}, 0\right)
\end{aligned}$$

The midpoint is $\left(-\frac{1}{2}, 0\right)$.

We close with a more abstract application of the Midpoint Formula. We will revisit the following example in Exercise 72 in Section 3.1.

Example 11 **An abstract midpoint problem**

If $a \neq b$, prove that the line $y = x$ equally divides the line segment with endpoints (a, b) and (b, a).

SOLUTION To prove the claim, we use Equation **??** to find the midpoint

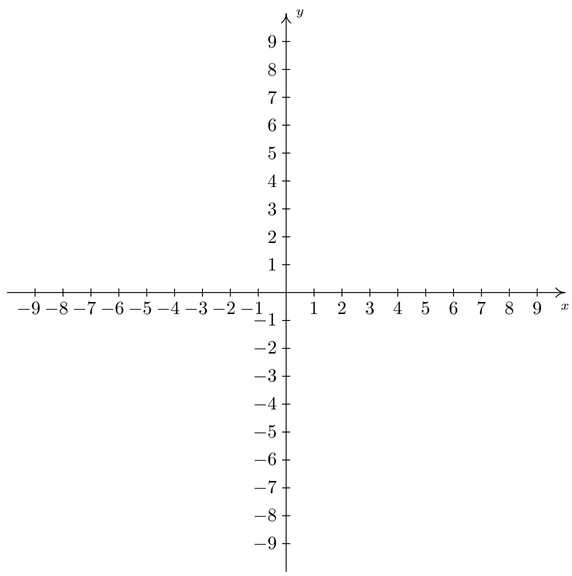
$$\begin{aligned} M &= \left(\frac{a+b}{2}, \frac{b+a}{2}\right) \\ &= \left(\frac{a+b}{2}, \frac{a+b}{2}\right) \end{aligned}$$

Since the x and y coordinates of this point are the same, we find that the midpoint lies on the line $y = x$, as required.

Exercises 1.3

Problems

1. Plot and label the points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$ and $H(7,5)$ in the Cartesian Coordinate Plane given below.



2. For each point given in Exercise 1 above
   - Identify the quadrant or axis in/on which the point lies.
   - Find the point symmetric to the given point about the x-axis.
   - Find the point symmetric to the given point about the y-axis.
   - Find the point symmetric to the given point about the origin.

In Exercises 3 – 10, find the distance d between the points and the midpoint M of the line segment which connects them.

3. $(1,2)$, $(-3,5)$
4. $(3,-10)$, $(-1,2)$
5. $\left(\frac{1}{2},4\right)$, $\left(\frac{3}{2},-1\right)$
6. $\left(-\frac{2}{3},\frac{3}{2}\right)$, $\left(\frac{7}{3},2\right)$
7. $\left(\frac{24}{5},\frac{6}{5}\right)$, $\left(-\frac{11}{5},-\frac{19}{5}\right)$.
8. $(\sqrt{2},\sqrt{3})$, $(-\sqrt{8},-\sqrt{12})$
9. $(2\sqrt{45},\sqrt{12})$, $(\sqrt{20},\sqrt{27})$.
10. $(0,0)$, (x,y)
11. Find all of the points of the form $(x,-1)$ which are 4 units from the point $(3,2)$.
12. Find all of the points on the y-axis which are 5 units from the point $(-5,3)$.
13. Find all of the points on the x-axis which are 2 units from the point $(-1,1)$.
14. Find all of the points of the form $(x,-x)$ which are 1 unit from the origin.
15. Let's assume for a moment that we are standing at the origin and the positive y-axis points due North while the positive x-axis points due East. Our Sasquatch-o-meter tells us that Sasquatch is 3 miles West and 4 miles South of our current position. What are the coordinates of his position? How far away is he from us? If he runs 7 miles due East what would his new position be?
16. Verify the Distance Formula 6 for the cases when:
    (a) The points are arranged vertically. (Hint: Use $P(a,y_0)$ and $Q(a,y_1)$.)
    (b) The points are arranged horizontally. (Hint: Use $P(x_0,b)$ and $Q(x_1,b)$.)
    (c) The points are actually the same point. (You shouldn't need a hint for this one.)
17. Verify the Midpoint Formula by showing the distance between $P(x_1,y_1)$ and M and the distance between M and $Q(x_2,y_2)$ are both half of the distance between P and Q.
18. Show that the points A, B and C below are the vertices of a right triangle.
    (a) $A(-3,2)$, $B(-6,4)$, and $C(1,8)$
    (b) $A(-3,1)$, $B(4,0)$ and $C(0,-3)$
19. Find a point $D(x,y)$ such that the points $A(-3,1)$, $B(4,0)$, $C(0,-3)$ and D are the corners of a square. Justify your answer.
20. Discuss with your classmates how many numbers are in the interval $(0,1)$.
21. The world is not flat. (There are those who disagree with this statement. Look them up on the Internet some time when you're bored.) Thus the Cartesian Plane cannot possibly be the end of the story. Discuss with your classmates how you would extend Cartesian Coordinates to represent the three dimensional world. What would the Distance and Midpoint formulas look like, assuming those concepts make sense at all?

1.4 Complex Numbers

We conclude our first chapter with a review the set of **Complex Numbers**. As you may recall, the complex numbers fill an algebraic gap left by the real numbers. There is no real number x with $x^2 = -1$, since for any real number $x^2 \geq 0$. However, we could formally extract square roots and write $x = \pm\sqrt{-1}$. We build the complex numbers by relabelling the quantity $\sqrt{-1}$ as i, the unfortunately misnamed **imaginary unit**.[6] The number i, while not a real number, is defined so that it plays along well with real numbers and acts very much like any other radical expression. For instance, $3(2i) = 6i$, $7i - 3i = 4i$, $(2-7i)+(3+4i) = 5-3i$, and so forth. The key properties which distinguish i from the real numbers are listed below.

Historically, the lack of solutions to the equation $x^2 = -1$ had nothing to do with the development of the complex numbers. Until the 19th century, equations such as $x^2 = -1$ would have been considered in the context of the analytic geometry of Descartes. The lack of solutions simply indicated that the graph $y = x^2$ did not intersect the line $y = -1$. The more remarkable case was that of *cubic* equations, of the form $x^3 = ax + b$. In this case a **real** solution is *guaranteed*, but there are cases where one needs **complex** numbers to find it! For details, see the excellent book *Visual Complex Analysis*, by Tristan Needham.

Definition 18 The imaginary unit

The imaginary unit i satisfies the two following properties:

1. $i^2 = -1$
2. If c is a real number with $c \geq 0$ then $\sqrt{-c} = i\sqrt{c}$

Note the use of the indefinite article 'a'. Whatever beast is chosen to be i, $-i$ is the other square root of -1.

Property 1 in Definition 18 establishes that i does act as a square root of -1, and property 2 establishes what we mean by the 'principal square root' of a negative real number. In property 2, it is important to remember the restriction on c. For example, it is perfectly acceptable to say $\sqrt{-4} = i\sqrt{4} = i(2) = 2i$. However, $\sqrt{-(-4)} \neq i\sqrt{-4}$, otherwise, we'd get

$$2 = \sqrt{4} = \sqrt{-(-4)} = i\sqrt{-4} = i(2i) = 2i^2 = 2(-1) = -2,$$

which is unacceptable. The moral of this story is that the general properties of radicals do not apply for even roots of negative quantities. With Definition 18 in place, we are now in position to define the **complex numbers**.

Definition 19 Complex number

A **complex number** is a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit. The set of complex numbers is denoted $\mathbb{C}$.

To use the language of Section 1.1.1, $\mathbb{R} \subseteq \mathbb{C}$.

Complex numbers include things you'd normally expect, like $3 + 2i$ and $\frac{2}{5} - i\sqrt{3}$. However, don't forget that a or b could be zero, which means numbers like $3i$ and 6 are also complex numbers. In other words, don't forget that the complex numbers *include* the real numbers, so 0 and $\pi - \sqrt{21}$ are both considered complex numbers. The arithmetic of complex numbers is as you would expect. The only things you need to remember are the two properties in Definition 18. The next example should help recall how these animals behave.

Example 12 Arithmetic with complex numbers

Perform the indicated operations.

[6]Some Technical Mathematics textbooks label it 'j'. While it carries the adjective 'imaginary', these numbers have essential real-world implications. For example, every electronic device owes its existence to the study of 'imaginary' numbers.

1. $(1-2i)-(3+4i)$
2. $(1-2i)(3+4i)$
3. $\dfrac{1-2i}{3-4i}$
4. $\sqrt{-3}\sqrt{-12}$
5. $\sqrt{(-3)(-12)}$
6. $(x-[1+2i])(x-[1-2i])$

Solution

1. As mentioned earlier, we treat expressions involving i as we would any other radical. We distribute and combine like terms:

$$\begin{aligned} (1-2i)-(3+4i) &= 1-2i-3-4i && \text{Distribute} \\ &= -2-6i && \text{Gather like terms} \end{aligned}$$

   Technically, we'd have to rewrite our answer $-2-6i$ as $(-2)+(-6)i$ to be (in the strictest sense) 'in the form $a+bi$'. That being said, even pedants have their limits, and we'll consider $-2-6i$ good enough.

2. Using the Distributive Property (a.k.a. F.O.I.L.), we get

$$\begin{aligned} (1-2i)(3+4i) &= (1)(3)+(1)(4i)-(2i)(3)-(2i)(4i) && \text{F.O.I.L.} \\ &= 3+4i-6i-8i^2 \\ &= 3-2i-8(-1) && i^2=-1 \\ &= 3-2i+8 \\ &= 11-2i \end{aligned}$$

3. How in the world are we supposed to simplify $\frac{1-2i}{3-4i}$? Well, we deal with the denominator $3-4i$ as we would any other denominator containing two terms, one of which is a square root: we and multiply both numerator and denominator by $3+4i$, the (complex) conjugate of $3-4i$. Doing so produces

$$\begin{aligned} \frac{1-2i}{3-4i} &= \frac{(1-2i)(3+4i)}{(3-4i)(3+4i)} && \text{Equivalent Fractions} \\ &= \frac{3+4i-6i-8i^2}{9-16i^2} && \text{F.O.I.L.} \\ &= \frac{3-2i-8(-1)}{9-16(-1)} && i^2=-1 \\ &= \frac{11-2i}{25} \\ &= \frac{11}{25}-\frac{2}{25}i \end{aligned}$$

4. We use property 2 of Definition 18 first, then apply the rules of radicals applicable to real numbers to get $\sqrt{-3}\sqrt{-12} = \left(i\sqrt{3}\right)\left(i\sqrt{12}\right) = i^2\sqrt{3\cdot 12} = -\sqrt{36} = -6$.

5. We adhere to the order of operations here and perform the multiplication before the radical to get $\sqrt{(-3)(-12)} = \sqrt{36} = 6$.

6. We can brute force multiply using the distributive property and see that

$$\begin{aligned}
(x - [1+2i])(x - [1-2i]) &= x^2 - x[1-2i] - x[1+2i] + [1-2i][1+2i] && \text{F.O.I.L.}\\
&= x^2 - x + 2ix - x - 2ix + 1 - 2i + 2i - 4i^2 && \text{Distribute}\\
&= x^2 - 2x + 1 - 4(-1) && \text{Gather like terms}\\
&= x^2 - 2x + 5 && i^2 = -1
\end{aligned}$$

This type of factoring will be revisited in Section 4.4.

In the previous example, we used the idea of a 'conjugate' to divide two complex numbers. (You may recall using conjugates to rationalize expressions involving square roots.) More generally, the **complex conjugate** of a complex number $a + bi$ is the number $a - bi$. The notation commonly used for complex conjugation is a 'bar': $\overline{a+bi} = a - bi$. For example, $\overline{3+2i} = 3 - 2i$ and $\overline{3-2i} = 3 + 2i$. To find $\overline{6}$, we note that $\overline{6} = \overline{6+0i} = 6 - 0i = 6$, so $\overline{6} = 6$. Similarly, $\overline{4i} = -4i$, since $\overline{4i} = \overline{0+4i} = 0 - 4i = -4i$. Note that $\overline{3+\sqrt{5}} = 3 + \sqrt{5}$, not $3 - \sqrt{5}$, since $\overline{3+\sqrt{5}} = \overline{3+\sqrt{5}+0i} = 3 + \sqrt{5} - 0i = 3 + \sqrt{5}$. Here, the conjugation specified by the 'bar' notation involves reversing the sign before $i = \sqrt{-1}$, not before $\sqrt{5}$. The properties of the conjugate are summarized in the following theorem.

Theorem 4 Properties of the Complex Conjugate

Let z and w be complex numbers.

- $\overline{\overline{z}} = z$
- $\overline{z+w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z}\,\overline{w}$
- $\overline{z^n} = (\overline{z})^n$, for any natural number n
- z is a real number if and only if $\overline{z} = z$.

Essentially, Theorem 4 says that complex conjugation works well with addition, multiplication and powers. The proofs of these properties can best be achieved by writing out $z = a + bi$ and $w = c + di$ for real numbers a, b, c and d. Next, we compute the left and right sides of each equation and verify that they are the same.

The proof of the first property is a very quick exercise. To prove the second property, we compare $\overline{z+w}$ with $\overline{z} + \overline{w}$. We have $\overline{z} + \overline{w} = \overline{a+bi} + \overline{c+di} = a - bi + c - di$. To find $\overline{z+w}$, we first compute

$$z + w = (a+bi) + (c+di) = (a+c) + (b+d)i$$

so

$$\overline{z+w} = \overline{(a+c)+(b+d)i} = (a+c) - (b+d)i = a + c - bi - di = a - bi + c - di = \overline{z} + \overline{w}$$

As such, we have established $\overline{z+w} = \overline{z} + \overline{w}$. The proof for multiplication works similarly. The proof that the conjugate works well with powers can be viewed as a repeated application of the product rule, and is best proved using a technique called Mathematical Induction. The last property is a characterization of real numbers. If z is real, then $z = a + 0i$, so $\overline{z} = a - 0i = a = z$. On the other hand, if $z = \overline{z}$, then $a + bi = a - bi$ which means $b = -b$ so $b = 0$. Hence, $z = a + 0i = a$ and is real.

Proof by Mathematical Induction is usually taught in Math 2000.

We now consider the problem of solving quadratic equations. Consider $x^2 - 2x + 5 = 0$. The discriminant $b^2 - 4ac = -16$ is negative, so we know by Theorem **??** there are no *real* solutions, since the Quadratic Formula would involve the term $\sqrt{-16}$. Complex numbers, however, are built just for such situations, so we can go ahead and apply the Quadratic Formula to get:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Example 13 **Finding complex solutions**

Find the complex solutions to the following equations.

1. $\dfrac{2x}{x+1} = x + 3$
2. $2t^4 = 9t^2 + 5$
3. $z^3 + 1 = 0$

Solution

Remember, all real numbers are complex numbers, so 'complex solutions' means both real and non-real answers.

1. Clearing fractions yields a quadratic equation so we collect all terms on one side and apply the Quadratic Formula.

$$\begin{aligned}
\frac{2x}{x+1} &= x+3 \\
2x &= (x+3)(x+1) && \text{Multiply by } (x+1) \text{ to clear denominators} \\
2x &= x^2 + x + 3x + 3 && \text{F.O.I.L.} \\
2x &= x^2 + 4x + 3 && \text{Gather like terms} \\
0 &= x^2 + 2x + 3 && \text{Subtract } 2x
\end{aligned}$$

From here, we apply the Quadratic Formula

$$\begin{aligned}
x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2(1)} && \text{Quadratic Formula} \\
&= \frac{-2 \pm \sqrt{-8}}{2} && \text{Simplify} \\
&= \frac{-2 \pm i\sqrt{8}}{2} && \text{Definition of } i \\
&= \frac{-2 \pm i2\sqrt{2}}{2} && \text{Product Rule for Radicals} \\
&= \frac{\cancel{2}(-1 \pm i\sqrt{2})}{\cancel{2}} && \text{Factor and reduce} \\
&= -1 \pm i\sqrt{2}
\end{aligned}$$

We get two answers: $x = -1 + i\sqrt{2}$ and its conjugate $x = -1 - i\sqrt{2}$. Checking both of these answers reviews all of the salient points about complex number arithmetic and is therefore strongly encouraged.

2. Since we have three terms, and the exponent on one term ('4' on t^4) is exactly twice the exponent on the other ('2' on t^2), we have a Quadratic in Disguise. We proceed accordingly.

$$\begin{array}{rcll} 2t^4 & = & 9t^2 + 5 & \\ 2t^4 - 9t^2 - 5 & = & 0 & \text{Subtract } 9t^2 \text{ and } 5 \\ (2t^2+1)(t^2-5) & = & 0 & \text{Factor} \\ 2t^2 + 1 = 0 & \text{or} & t^2 = 5 & \text{Zero Product Property} \end{array}$$

From $2t^2 + 1 = 0$ we get $2t^2 = -1$, or $t^2 = -\frac{1}{2}$. We extract square roots as follows:

$$t = \pm\sqrt{-\frac{1}{2}} = \pm i\sqrt{\frac{1}{2}} = \pm i\frac{\sqrt{1}}{\sqrt{2}} = \pm i\frac{1}{\sqrt{2}} = \pm\frac{i\sqrt{2}}{2},$$

where we have rationalized the denominator per convention. From $t^2 = 5$, we get $t = \pm\sqrt{5}$. In total, we have four complex solutions - two real: $t = \pm\sqrt{5}$ and two non-real: $t = \pm\frac{i\sqrt{2}}{2}$.

3. To find the *real* solutions to $z^3 + 1 = 0$, we can subtract the 1 from both sides and extract cube roots: $z^3 = -1$, so $z = \sqrt[3]{-1} = -1$. It turns out there are two more non-real complex number solutions to this equation. To get at these, we factor:

$$\begin{array}{rcll} z^3 + 1 & = & 0 & \\ (z+1)(z^2 - z + 1) & = & 0 & \text{Factor (Sum of Two Cubes)} \\ z + 1 = 0 & \text{or} & z^2 - z + 1 = 0 & \end{array}$$

From $z + 1 = 0$, we get our real solution $z = -1$. From $z^2 - z + 1 = 0$, we apply the Quadratic Formula to get:

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

Thus we get *three* solutions to $z^3 + 1 = 0$ - one real: $z = -1$ and two non-real: $z = \frac{1 \pm i\sqrt{3}}{2}$. As always, the reader is encouraged to test their algebraic mettle and check these solutions.

It is no coincidence that the non-real solutions to the equations in Example 13 appear in complex conjugate pairs. Any time we use the Quadratic Formula to solve an equation with real coefficients, the answers will form a complex conjugate pair owing to the $\pm$ in the Quadratic Formula. This leads us to a generalization of Theorem **??** which we state on the next page.

Theorem 5 Discriminant Theorem

Given a Quadratic Equation $AX^2 + BX + C = 0$, where A, B and C are real numbers, let $D = B^2 - 4AC$ be the discriminant.

- If $D > 0$, there are two distinct real number solutions to the equation.
- If $D = 0$, there is one (repeated) real number solution.

  Note: 'Repeated' here comes from the fact that 'both' solutions $\frac{-B\pm 0}{2A}$ reduce to $-\frac{B}{2A}$.
- If $D < 0$, there are two non-real solutions which form a complex conjugate pair.

We will have much more to say about complex solutions to equations in Section 4.4 and we will revisit Theorem 5 then.

Exercises 1.4

Problems

In Exercises 1 – 10, use the given complex numbers z and w to find and simplify the following:

- $z+w$
- zw
- z^2
- $\dfrac{1}{z}$
- $\dfrac{z}{w}$
- $\dfrac{w}{z}$
- $\bar{z}$
- $z\bar{z}$
- $(\bar{z})^2$

1. $z=2+3i$, $w=4i$

2. $z=1+i$, $w=-i$

3. $z=i$, $w=-1+2i$

4. $z=4i$, $w=2-2i$

5. $z=3-5i$, $w=2+7i$

6. $z=-5+i$, $w=4+2i$

7. $z=\sqrt{2}-i\sqrt{2}$, $w=\sqrt{2}+i\sqrt{2}$

8. $z=1-i\sqrt{3}$, $w=-1-i\sqrt{3}$

9. $z=\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i$, $w=-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i$

10. $z=-\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i$, $w=-\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i$

In Exercises 11 – 20, simplify the quantity.

11. $\sqrt{-49}$

12. $\sqrt{-9}$

13. $\sqrt{-25}\sqrt{-4}$

14. $\sqrt{(-25)(-4)}$

15. $\sqrt{-9}\sqrt{-16}$

16. $\sqrt{(-9)(-16)}$

17. $\sqrt{-(-9)}$

18. $-\sqrt{(-9)}$

We know that $i^2=-1$ which means $i^3=i^2\cdot i=(-1)\cdot i=-i$ and $i^4=i^2\cdot i^2=(-1)(-1)=1$. In Exercises 19 – 28, use this information to simplify the given power of i.

19. i^5

20. i^6

21. i^7

22. i^8

23. i^{15}

24. i^{26}

25. i^{117}

26. i^{304}

In Exercises 27 – 36, find all complex solutions.

27. $3x^2+6=4x$

28. $15t^2+2t+5=3t(t^2+1)$

29. $3y^2+4=y^4$

30. $\dfrac{2}{1-w}=w$

31. $\dfrac{y}{3}-\dfrac{3}{y}=y$

32. $\dfrac{x^3}{2x-1}=\dfrac{x}{3}$

33. $x=\dfrac{2}{\sqrt{5}-x}$

34. $\dfrac{5y^4+1}{y^2-1}=3y^2$

35. $z^4=16$

36. Multiply and simplify: $\left(x-[3-i\sqrt{23}]\right)\left(x-[3+i\sqrt{23}]\right)$

2: Relations and Functions

2.1 Relations

From one point of view, all of Precalculus can be thought of as studying sets of points in the plane. With the Cartesian Plane now fresh in our memory we can discuss those sets in more detail and as usual, we begin with a definition.

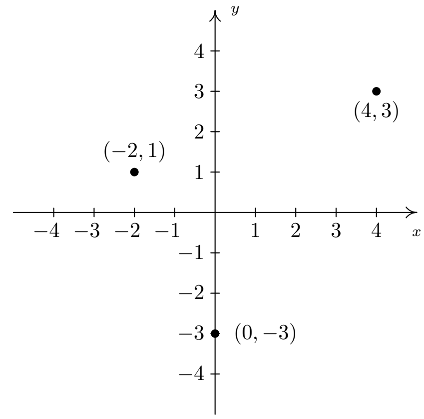


Figure 2.1: The graph of the relation $R = \{(-2,1),(4,3),(0,-3)\}$

Definition 20 Relations in the Cartesian Plane

A **relation** is a set of points in the plane.

Since relations are sets, we can describe them using the techniques presented in Section **??**. That is, we can describe a relation verbally, using the roster method, or using set-builder notation. Since the elements in a relation are points in the plane, we often try to describe the relation graphically or algebraically as well. Depending on the situation, one method may be easier or more convenient to use than another. As an example, consider the relation $R = \{(-2,1),(4,3),(0,-3)\}$. As written, R is described using the roster method. Since R consists of points in the plane, we follow our instinct and plot the points. Doing so produces the **graph** of R: see Figure 2.1.

In the following example, we graph a variety of relations.

Example 14 Graphing relations

Graph the following relations.

1. $A = \{(0,0),(-3,1),(4,2),(-3,2)\}$
2. $HLS_1 = \{(x,3) \mid -2 \le x \le 4\}$
3. $HLS_2 = \{(x,3) \mid -2 \le x < 4\}$
4. $V = \{(3,y) \mid y \text{ is a real number}\}$
5. $H = \{(x,y) \mid y = -2\}$
6. $R = \{(x,y) \mid 1 < y \le 3\}$

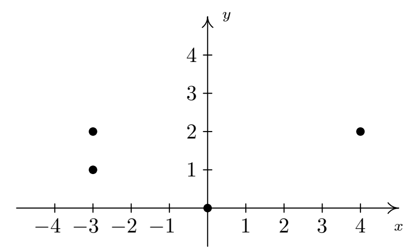


Figure 2.2: The graph of A

Solution

1. To graph A, we simply plot all of the points which belong to A, as shown below on the left.

2. Don't let the notation in this part fool you. The name of this relation is HLS_1, just like the name of the relation in number 1 was A. The letters and numbers are just part of its name, just like the numbers and letters of the phrase 'King George III' were part of George's name. In words, $\{(x,3) \mid -2 \le x \le 4\}$ reads 'the set of points $(x,3)$ such that $-2 \le x \le 4$.' All of these points have the same y-coordinate, 3, but the x-coordinate is allowed to vary between -2 and 4, inclusive. Some of the points which belong to HLS_1 include some friendly points like: $(-2,3)$, $(-1,3)$, $(0,3)$, $(1,3)$, $(2,3)$, $(3,3)$, and $(4,3)$. However, HLS_1 also contains the points $(0.829,3)$, $\left(-\frac{5}{6},3\right)$, $(\sqrt{\pi},3)$, and so on. It is impossible to list all of these points, which is why the variable x is used. Plotting several friendly representative points should convince you that HLS_1 describes the horizontal line segment from the point $(-2,3)$ up to and including the point $(4,3)$.

Listing the points in a line segment is *really* impossible. The interested reader is encouraged to research **countable** versus **uncountable** sets.

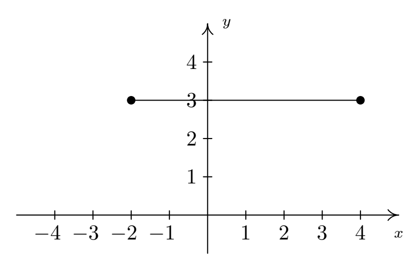
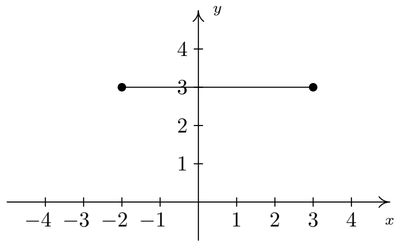


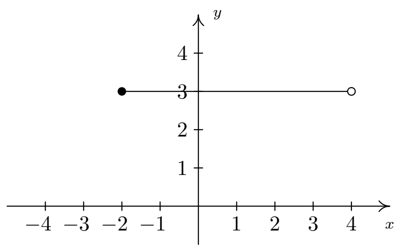
Figure 2.3: The graph of HLS_1

3. HLS_2 is hauntingly similar to HLS_1. In fact, the only difference between the two is that instead of '$-2 \le x \le 4$' we have '$-2 \le x < 4$'. This means that we still get a horizontal line segment which includes $(-2,3)$ and extends to $(4,3)$, but we do *not* include $(4,3)$ because of the strict

inequality $x < 4$. How do we denote this on our graph? It is a common mistake to make the graph start at $(-2, 3)$ end at $(3, 3)$ as pictured below on the left. The problem with this graph is that we are forgetting about the points like $(3.1, 3)$, $(3.5, 3)$, $(3.9, 3)$, $(3.99, 3)$, and so forth. There is no real number that comes 'immediately before' 4, so to describe the set of points we want, we draw the horizontal line segment starting at $(-2, 3)$ and draw an open circle at $(4, 3)$ as depicted below on the right.



This is NOT the correct graph of HLS_2



The graph of HLS_2

Figure 2.5: Getting the right graph for HLS_2

4. Next, we come to the relation *V*, described as the set of points $(3, y)$ such that *y* is a real number. All of these points have an *x*-coordinate of 3, but the *y*-coordinate is free to be whatever it wants to be, without restriction. Plotting a few 'friendly' points of *V* should convince you that all the points of *V* lie on the vertical line $x = 3$. Since there is no restriction on the *y*-coordinate, we put arrows on the end of the portion of the line we draw to indicate it extends indefinitely in both directions. The graph of *V* is below on the left.

5. Though written slightly differently, the relation $H = \{(x, y) \,|\, y = -2\}$ is similar to the relation *V* above in that only one of the coordinates, in this case the *y*-coordinate, is specified, leaving *x* to be 'free'. Plotting some representative points gives us the horizontal line $y = -2$.

6. For our last example, we turn to $R = \{(x, y) \,|\, 1 < y \leq 3\}$. As in the previous example, *x* is free to be whatever it likes. The value of *y*, on the other hand, while not completely free, is permitted to roam between 1 and 3 excluding 1, but including 3. After plotting some friendly elements of *R*, it should become clear that *R* consists of the region between the horizontal lines $y = 1$ and $y = 3$. Since *R* requires that the *y*-coordinates be greater than 1, but not equal to 1, we dash the line $y = 1$ to indicate that those points do not belong to *R*.

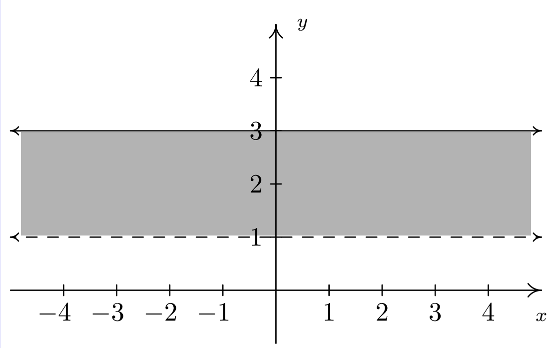


Figure 2.4: The graph of *R*

When we say you should plot some points in the relation *H*, the word 'some' is a relative term. It may take 5, 10, or 50 points until you see the pattern, depending on the relation.

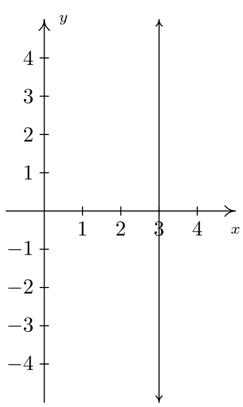


Figure 2.6: The graph of *V*

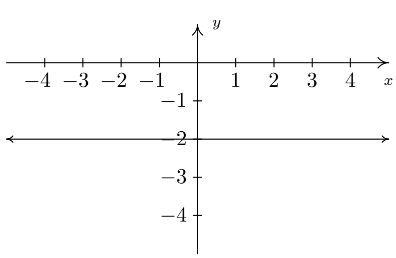


Figure 2.7: The graph of *H*

The relations *V* and *H* in the previous example lead us to our final way to describe relations: **algebraically**. We can more succinctly describe the points in *V* as those points which satisfy the equation '$x = 3$'. Most likely, you have seen equations like this before. Depending on the context, '$x = 3$' could mean we have solved an equation for *x* and arrived at the solution $x = 3$. In this case, however, '$x = 3$' describes a set of points in the plane whose *x*-coordinate is 3. Similarly, the relation *H* above can be described by the equation '$y = -2$'. At some point in your mathematical upbringing, you probably learned the following.

Key Idea 8 Equations of Vertical and Horizontal Lines

- The graph of the equation $x = a$ is a **vertical line** through $(a, 0)$.
- The graph of the equation $y = b$ is a **horizontal line** through $(0, b)$.

Given that the very simple equations $x = a$ and $y = b$ produced lines, it's natural to wonder what shapes other equations might yield. Thus our next objective is to study the graphs of equations in a more general setting as we continue to unite Algebra and Geometry.

2.1.1 Graphs of Equations

In this section, we delve more deeply into the connection between Algebra and Geometry by focusing on graphing relations described by equations. The main idea of this section is the following.

Key Idea 9 The Fundamental Graphing Principle

The graph of an equation is the set of points which satisfy the equation. That is, a point (x, y) is on the graph of an equation if and only if x and y satisfy the equation.

Here, 'x and y satisfy the equation' means 'x and y make the equation true'. It is at this point that we gain some insight into the word 'relation'. If the equation to be graphed contains both x and y, then the equation itself is what is relating the two variables. More specifically, in the next two examples, we consider the graph of the equation $x^2 + y^3 = 1$. Even though it is not specifically spelled out, what we are doing is graphing the relation $R = \{(x, y) \,|\, x^2 + y^3 = 1\}$. The points (x, y) we graph belong to the *relation* R and are necessarily *related* by the equation $x^2 + y^3 = 1$, since it is those pairs of x and y which make the equation true.

Example 15 Checking to see if a point lies on a graph
Determine whether or not $(2, -1)$ is on the graph of $x^2 + y^3 = 1$.

Solution We substitute $x = 2$ and $y = -1$ into the equation to see if the equation is satisfied.

$$\begin{aligned} (2)^2 + (-1)^3 &\stackrel{?}{=} 1 \\ 3 &\neq 1 \end{aligned}$$

Hence, $(2, -1)$ is **not** on the graph of $x^2 + y^3 = 1$. □

We could spend hours randomly guessing and checking to see if points are on the graph of the equation. A more systematic approach is outlined in the following example.

Example 16 **Determining points on a graph systematically**

Graph $x^2 + y^3 = 1$.

Solution To efficiently generate points on the graph of this equation, we first solve for y

$$\begin{aligned} x^2 + y^3 &= 1 \\ y^3 &= 1 - x^2 \\ \sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\ y &= \sqrt[3]{1 - x^2} \end{aligned}$$

We now substitute a value in for x, determine the corresponding value y, and plot the resulting point (x, y). For example, substituting $x = -3$ into the equation yields

$$y = \sqrt[3]{1 - x^2} = \sqrt[3]{1 - (-3)^2} = \sqrt[3]{-8} = -2,$$

so the point $(-3, -2)$ is on the graph. Continuing in this manner, we generate a table of points which are on the graph of the equation. These points are then plotted in the plane as shown in Figure 2.8.

x	y	(x, y)
-3	-2	$(-3, -2)$
-2	$-\sqrt[3]{3}$	$(-2, -\sqrt[3]{3})$
-1	0	$(-1, 0)$
0	1	$(0, 1)$
1	0	$(1, 0)$
2	$-\sqrt[3]{3}$	$(2, -\sqrt[3]{3})$
3	-2	$(3, -2)$

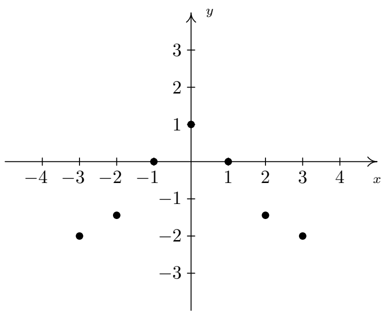


Figure 2.8: Points on the curve $x^2 + y^3 = 1$

Remember, these points constitute only a small sampling of the points on the graph of this equation. To get a better idea of the shape of the graph, we could plot more points until we feel comfortable 'connecting the dots'. Doing so would result in a curve similar to the one pictured in Figure 2.9.

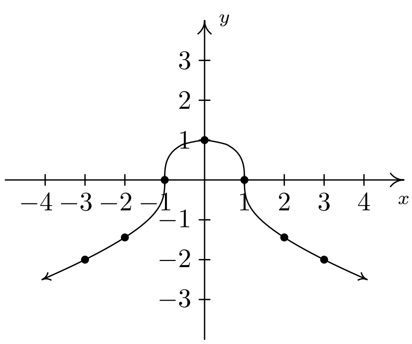


Figure 2.9: The completed graph of $x^2 + y^3 = 1$

Don't worry if you don't get all of the little bends and curves just right — Calculus is where the art of precise graphing takes center stage. For now, we will settle with our naive 'plug and plot' approach to graphing. If you feel like all of this tedious computation and plotting is beneath you, then you can try inputting the equation into a graphing calculator or an online tool such as Wolfram Alpha.

Of all of the points on the graph of an equation, the places where the graph crosses or touches the axes hold special significance. These are called the **intercepts** of the graph. Intercepts come in two distinct varieties: x-intercepts and y-intercepts. They are defined below.

Definition 21 ***x*- and *y*-intercepts**

Suppose the graph of an equation is given.

- A point on a graph which is also on the x-axis is called an ***x*-intercept** of the graph.
- A point on a graph which is also on the y-axis is called an ***y*-intercept** of the graph.

In our previous example the graph had two x-intercepts, $(-1, 0)$ and $(1, 0)$, and one y-intercept, $(0, 1)$. The graph of an equation can have any number of intercepts, including none at all! Since x-intercepts lie on the x-axis, we can find them by setting $y = 0$ in the equation. Similarly, since y-intercepts lie on the y-axis, we can find them by setting $x = 0$ in the equation. Keep in mind, intercepts are *points* and therefore must be written as ordered pairs. To summarize,

Key Idea 10 Finding the Intercepts of the Graph of an Equation

Given an equation involving x and y, we find the intercepts of the graph as follows:

- x-intercepts have the form $(x, 0)$; set $y = 0$ in the equation and solve for x.
- y-intercepts have the form $(0, y)$; set $x = 0$ in the equation and solve for y.

Another fact which you may have noticed about the graph in the previous example is that it seems to be symmetric about the y-axis. To actually prove this analytically, we assume (x, y) is a generic point on the graph of the equation. That is, we assume $x^2 + y^3 = 1$ is true. As we learned in Section 1.3, the point symmetric to (x, y) about the y-axis is $(-x, y)$. To show that the graph is symmetric about the y-axis, we need to show that $(-x, y)$ satisfies the equation $x^2 + y^3 = 1$, too. Substituting $(-x, y)$ into the equation gives

$$\begin{aligned} (-x)^2 + (y)^3 &\stackrel{?}{=} 1 \\ x^2 + y^3 &\stackrel{\checkmark}{=} 1 \end{aligned}$$

Since we are assuming the original equation $x^2 + y^3 = 1$ is true, we have shown that $(-x, y)$ satisfies the equation (since it leads to a true result) and hence is on the graph. In this way, we can check whether the graph of a given equation possesses any of the symmetries discussed in Section 1.3. We summarize the procedure in the following result.

Key Idea 11 Testing the Graph of an Equation for Symmetry

To test the graph of an equation for symmetry

- about the y-axis $-$ substitute $(-x, y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the y-axis.
- about the x-axis – substitute $(x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the x-axis.
- about the origin - substitute $(-x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the origin.

Intercepts and symmetry are two tools which can help us sketch the graph of an equation analytically, as demonstrated in the next example.

Example 17 Finding intercepts and testing for symmetry

Find the x- and y-intercepts (if any) of the graph of $(x - 2)^2 + y^2 = 1$. Test for symmetry. Plot additional points as needed to complete the graph.

Solution To look for x-intercepts, we set $y = 0$ and solve

$$\begin{aligned}
(x-2)^2+y^2 &= 1\\
(x-2)^2+0^2 &= 1\\
(x-2)^2 &= 1\\
\sqrt{(x-2)^2} &= \sqrt{1} \qquad \text{extract square roots}\\
x-2 &= \pm 1\\
x &= 2\pm 1\\
x &= 3,1
\end{aligned}$$

We get two answers for x which correspond to two x-intercepts: $(1,0)$ and $(3,0)$. Turning our attention to y-intercepts, we set $x=0$ and solve

$$\begin{aligned}
(x-2)^2+y^2 &= 1\\
(0-2)^2+y^2 &= 1\\
4+y^2 &= 1\\
y^2 &= -3
\end{aligned}$$

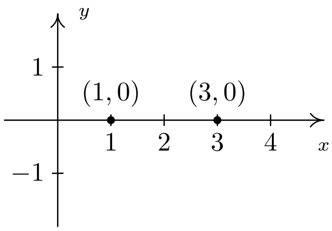


Figure 2.10: Plotting the data so far

Since there is no real number which squares to a negative number (Do you remember why?), we are forced to conclude that the graph has no y-intercepts. We plot our results so far in Figure 2.10.

Moving along to symmetry, we can immediately dismiss the possibility that the graph is symmetric about the y-axis or the origin. If the graph possessed either of these symmetries, then the fact that $(1,0)$ is on the graph would mean $(-1,0)$ would have to be on the graph. (Why?) Since $(-1,0)$ would be another x-intercept (and we've found all of these), the graph can't have y-axis or origin symmetry. The only symmetry left to test is symmetry about the x-axis. To that end, we substitute $(x,-y)$ into the equation and simplify

$$\begin{aligned}
(x-2)^2+y^2 &= 1\\
(x-2)^2+(-y)^2 &\stackrel{?}{=} 1\\
(x-2)^2+y^2 &\stackrel{\checkmark}{=} 1
\end{aligned}$$

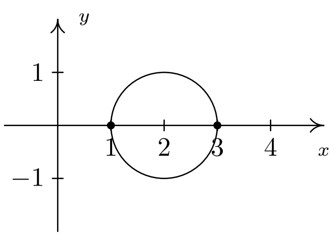


Figure 2.11: The final result

Since we have obtained our original equation, we know the graph is symmetric about the x-axis. This means we can cut our 'plug and plot' time in half: whatever happens below the x-axis is reflected above the x-axis, and vice-versa. Proceeding as we did in the previous example, we obtain the plot shown in Figure 2.11.

A couple of remarks are in order. First, it is entirely possible to choose a value for x which does not correspond to a point on the graph. For example, in the previous example, if we solve for y as is our custom, we get

$$y=\pm\sqrt{1-(x-2)^2}.$$

Upon substituting $x=0$ into the equation, we would obtain

$$y=\pm\sqrt{1-(0-2)^2}=\pm\sqrt{1-4}=\pm\sqrt{-3},$$

which is not a real number. This means there are no points on the graph with an x-coordinate of 0. When this happens, we move on and try another point. This is another drawback of the 'plug-and-plot' approach to graphing equations. Luckily, we will devote much of the remainder of this book to developing techniques which allow us to graph entire families of equations quickly. Second, it is instructive to show what would have happened had we tested the equation

By the end of this course, you'll be able to accurately graph a wide variety of equations, without the use of a calculator, if you can believe it!

in the last example for symmetry about the y-axis. Substituting $(-x, y)$ into the equation yields

$$\begin{aligned}
(x-2)^2 + y^2 &= 1 \\
(-x-2)^2 + y^2 &\stackrel{?}{=} 1 \\
((-1)(x+2))^2 + y^2 &\stackrel{?}{=} 1 \\
(x+2)^2 + y^2 &\stackrel{?}{=} 1.
\end{aligned}$$

This last equation does not *appear* to be equivalent to our original equation. However, to actually prove that the graph is not symmetric about the y-axis, we need to find a point (x, y) on the graph whose reflection $(-x, y)$ is not. Our x-intercept $(1, 0)$ fits this bill nicely, since if we substitute $(-1, 0)$ into the equation we get

$$\begin{aligned}
(x-2)^2 + y^2 &\stackrel{?}{=} 1 \\
(-1-2)^2 + 0^2 &\neq 1 \\
9 &\neq 1.
\end{aligned}$$

This proves that $(-1, 0)$ is not on the graph.

Exercises 2.1

Problems

In Exercises 1 – 21, graph the given relation.

1. $\{(-3, 9),\ (-2, 4),\ (-1, 1),\ (0, 0),\ (1, 1),\ (2, 4),\ (3, 9)\}$

2. $\{(-2, 0),\ (-1, 1),\ (-1, -1),\ (0, 2),\ (0, -2),\ (1, 3),\ (1, -3)\}$

3. $\{(m, 2m) \mid m = 0, \pm 1, \pm 2\}$

4. $\left\{\left(\frac{6}{k}, k\right) \mid k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\right\}$

5. $\left\{\left(n, 4 - n^2\right) \mid n = 0, \pm 1, \pm 2\right\}$

6. $\left\{\left(\sqrt{j}, j\right) \mid j = 0, 1, 4, 9\right\}$

7. $\{(x, -2) \mid x > -4\}$

8. $\{(x, 3) \mid x \leq 4\}$

9. $\{(-1, y) \mid y > 1\}$

10. $\{(2, y) \mid y \leq 5\}$

11. $\{(-2, y) \mid -3 < y \leq 4\}$

12. $\{(3, y) \mid -4 \leq y < 3\}$

13. $\{(x, 2) \mid -2 \leq x < 3\}$

14. $\{(x, -3) \mid -4 < x \leq 4\}$

15. $\{(x, y) \mid x > -2\}$

16. $\{(x, y) \mid x \leq 3\}$

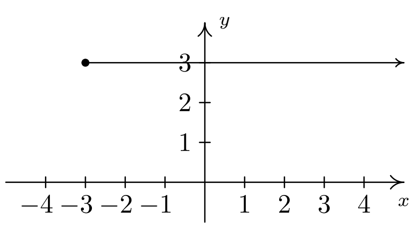
17. $\{(x, y) \mid y < 4\}$

18. $\{(x, y) \mid x \leq 3,\ y < 2\}$

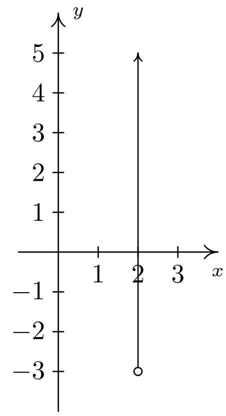
19. $\{(x, y) \mid x > 0,\ y < 4\}$

20. $\{(x, y) \mid -\sqrt{2} \leq x \leq \frac{2}{3},\ \pi < y \leq \frac{9}{2}\}$

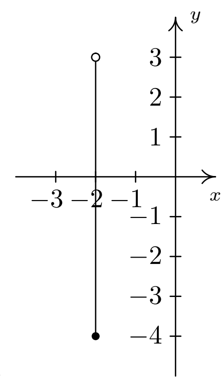
In Exercises 21 – 30, describe the given relation using either the roster or set-builder method.

21. 

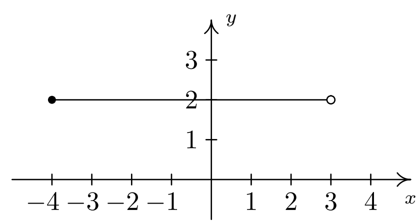
Relation A

22. 

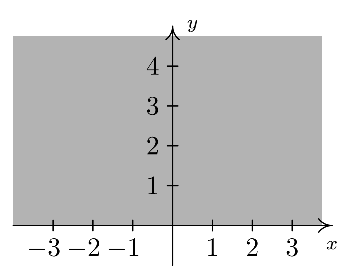
Relation B

23. 

Relation C

24. 

Relation D

25. 

Relation E

Relation J

In Exercises 31 – 36, graph the given line.

31. $x = -2$

32. $x = 3$

33. $y = 3$

34. $y = -2$

35. $x = 0$

36. $y = 0$

Some relations are fairly easy to describe in words or with the roster method but are rather difficult, if not impossible, to graph. For Exercises 37 – 40, discuss with your classmates how you might graph the given relation.

37. $\{(x, y) \mid x$ is an odd integer, and y is an even integer.$\}$

38. $\{(x, 1) \mid x$ is an irrational number $\}$

39. $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), \ldots\}$

40. $\{\ldots, (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), \ldots\}$

For each equation given in Exercises 41 – 52, (a) Find the *x* and *y* intercepts of the graph, if any exist; (b) Follow the procedure in Example 16 to create a table of sample points on the graph of the equation; (c) Plot the sample points and create a rough sketch of the graph of the equation; Test for symmetry. If the equation appears to fail any of the symmetry tests, find a point on the graph of the equation whose reflection fails to be on the graph as was done at the end of Example 17.

41. $y = x^2 + 1$

42. $y = x^2 - 2x - 8$

43. $y = x^3 - x$

44. $y = \frac{x^3}{4} - 3x$

45. $y = \sqrt{x - 2}$

46. $y = 2\sqrt{x + 4} - 2$

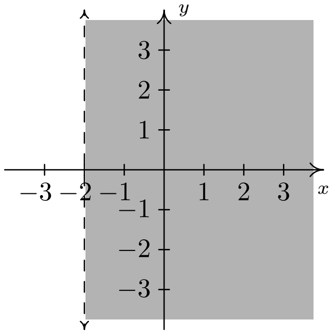
47. $3x - y = 7$

48. $3x - 2y = 10$

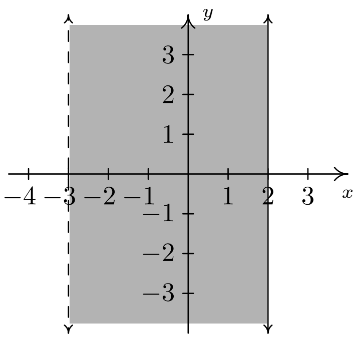
49. $(x + 2)^2 + y^2 = 16$

50. $x^2 - y^2 = 1$

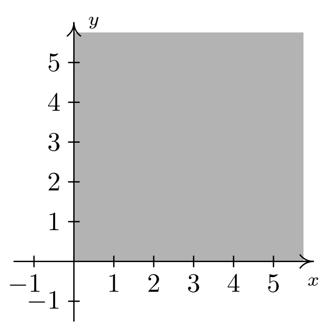
51. $4y^2 - 9x^2 = 36$

26. 

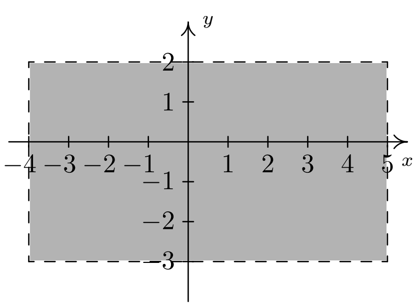
Relation F

27. 

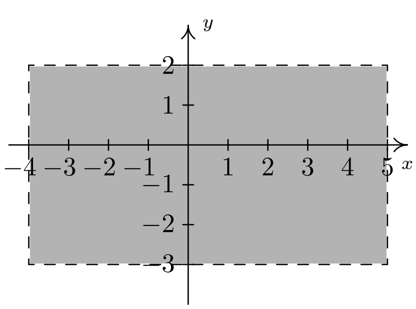
Relation G

28. 

Relation H

29. 

Relation I

30. 

52. $x^3y = -4$

53. With the help of your classmates, find examples of equations whose graphs possess

    - symmetry about the x-axis only
    - symmetry about the y-axis only
    - symmetry about the origin only
    - symmetry about the x-axis, y-axis, and origin

    Can you find an example of an equation whose graph possesses exactly *two* of the symmetries listed above? Why or why not?

2.2 Introduction to Functions

One of the core concepts in College Algebra is the **function**. There are many ways to describe a function and we begin by defining a function as a special kind of relation.

Definition 22 Function

A relation in which each x-coordinate is matched with only one y-coordinate is said to describe y as a **function** of x.

Example 18 Determining if a relation is a function

Which of the following relations describe y as a function of x?

1. $R_1 = \{(-2,1),(1,3),(1,4),(3,-1)\}$
2. $R_2 = \{(-2,1),(1,3),(2,3),(3,-1)\}$

We will have occasion later in the text to concern ourselves with the concept of x being a function of y. In this case, R_1 represents x as a function of y; R_2 does not.

Solution A quick scan of the points in R_1 reveals that the x-coordinate 1 is matched with two *different* y-coordinates: namely 3 and 4. Hence in R_1, y is not a function of x. On the other hand, every x-coordinate in R_2 occurs only once which means each x-coordinate has only one corresponding y-coordinate. So, R_2 does represent y as a function of x.

Note that in the previous example, the relation R_2 contained two different points with the same y-coordinates, namely $(1,3)$ and $(2,3)$. Remember, in order to say y is a function of x, we just need to ensure the same x-coordinate isn't used in more than one point.

To see what the function concept means geometrically, we graph R_1 and R_2 in the plane.

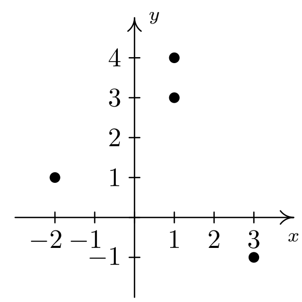


Figure 2.12: The graph of R_1

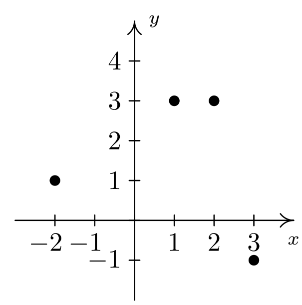


Figure 2.13: The graph of R_2

The fact that the x-coordinate 1 is matched with two different y-coordinates in R_1 presents itself graphically as the points $(1,3)$ and $(1,4)$ lying on the same vertical line, $x = 1$. If we turn our attention to the graph of R_2, we see that no two points of the relation lie on the same vertical line. We can generalize this idea as follows

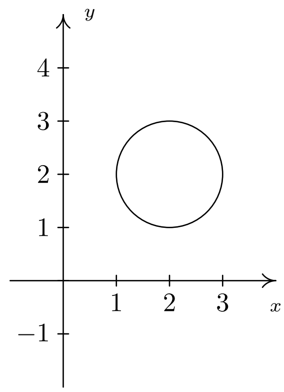
Theorem 6 The Vertical Line Test

A set of points in the plane represents y as a function of x if and only if no two points lie on the same vertical line.

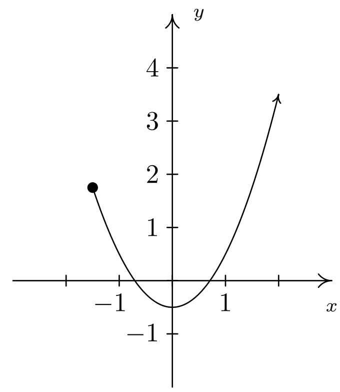
It is worth taking some time to meditate on the Vertical Line Test; it will check to see how well you understand the concept of 'function' as well as the concept of 'graph'.

Example 19 Using the Vertical Line Test

Use the Vertical Line Test to determine which of the following relations describes y as a function of x.



The graph of R



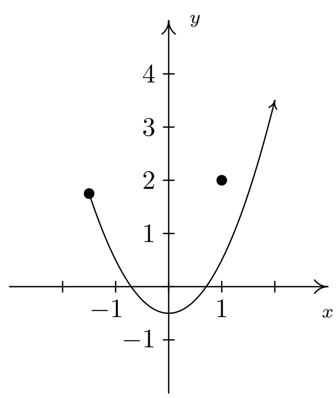
The graph of S

Solution Looking at the graph of R, we can easily imagine a vertical line crossing the graph more than once. Hence, R does not represent y as a function of x. However, in the graph of S, every vertical line crosses the graph at most once, so S does represent y as a function of x. □

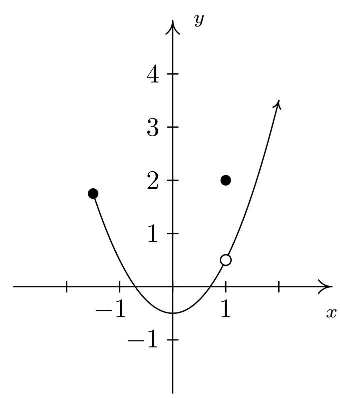
In the previous test, we say that the graph of the relation R **fails** the Vertical Line Test, whereas the graph of S **passes** the Vertical Line Test. Note that in the graph of R there are infinitely many vertical lines which cross the graph more than once. However, to fail the Vertical Line Test, all you need is one vertical line that fits the bill, as the next example illustrates.

Example 20 Using the Vertical Line Test

Use the Vertical Line Test to determine which of the following relations describes y as a function of x.



The graph of S_1



The graph of S_2

Solution Both S_1 and S_2 are slight modifications to the relation S in the previous example whose graph we determined passed the Vertical Line Test. In both S_1 and S_2, it is the addition of the point $(1, 2)$ which threatens to cause trouble. In S_1, there is a point on the curve with x-coordinate 1 just below $(1, 2)$, which means that both $(1, 2)$ and this point on the curve lie on the vertical line $x = 1$. (See the picture below and the left.) Hence, the graph of S_1 fails the Vertical Line Test, so y is not a function of x here. However, in S_2 notice that the point with x-coordinate 1 on the curve has been omitted, leaving an 'open circle' there. Hence, the vertical line $x = 1$ crosses the graph of S_2 only at the point $(1, 2)$. Indeed, any vertical line will cross the graph at most once, so we

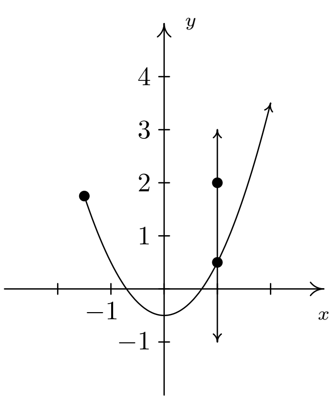


Figure 2.14: S_1 and the line $x = 1$

have that the graph of S_2 passes the Vertical Line Test. Thus it describes *y* as a function of *x*.

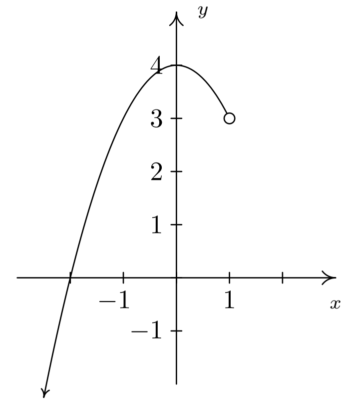


Figure 2.15: The graph of *G* for Example 21

Suppose a relation *F* describes *y* as a function of *x*. The sets of *x*- and *y*-coordinates are given special names which we define below.

> **Definition 23 Domain and range**
>
> Suppose *F* is a relation which describes *y* as a function of *x*.
>
> - The set of the *x*-coordinates of the points in *F* is called the **domain** of *F*.
> - The set of the *y*-coordinates of the points in *F* is called the **range** of *F*.

We demonstrate finding the domain and range of functions given to us either graphically or via the roster method in the following example.

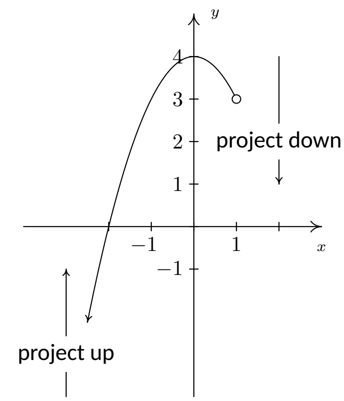


Figure 2.16: Projecting the graph onto the *x*-axis in Example 21

Example 21 Finding domain and range

Find the domain and range of the function $F = \{(-3, 2), (0, 1), (4, 2), (5, 2)\}$ and of the function *G* whose graph is given in Figure 2.15.

Solution The domain of *F* is the set of the *x*-coordinates of the points in *F*, namely $\{-3, 0, 4, 5\}$ and the range of *F* is the set of the *y*-coordinates, namely $\{1, 2\}$.

To determine the domain and range of *G*, we need to determine which *x* and *y* values occur as coordinates of points on the given graph. To find the domain, it may be helpful to imagine collapsing the curve to the *x*-axis and determining the portion of the *x*-axis that gets covered. This is called **projecting** the curve to the *x*-axis. Before we start projecting, we need to pay attention to two subtle notations on the graph: the arrowhead on the lower left corner of the graph indicates that the graph continues to curve downwards to the left forever more; and the open circle at $(1, 3)$ indicates that the point $(1, 3)$ isn't on the graph, but all points on the curve leading up to that point are.

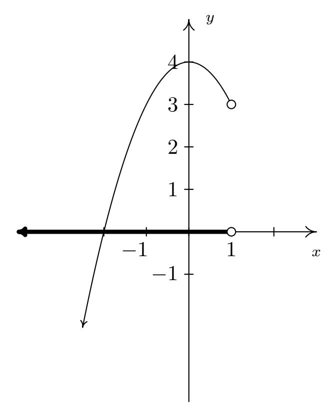


Figure 2.17: The domain of *G* in Example 21

We see from Figures 2.16 and 2.17 that if we project the graph of *G* to the *x*-axis, we get all real numbers less than 1. Using interval notation, we write the domain of *G* as $(-\infty, 1)$. To determine the range of *G*, we project the curve to the *y*-axis as follows:

Note that even though there is an open circle at $(1, 3)$, we still include the *y* value of 3 in our range, since the point $(-1, 3)$ is on the graph of *G*. Referring to Figures 2.18 and 2.19, we see that the range of *G* is all real numbers less than or equal to 4, or, in interval notation, $(-\infty, 4]$.

All functions are relations, but not all relations are functions. Thus the equations which described the relations in Section 2.1 may or may not describe *y* as a function of *x*. The algebraic representation of functions is possibly the most important way to view them so we need a process for determining whether or not an equation of a relation represents a function. (We delay the discussion of finding the domain of a function given algebraically until Section 2.3.)

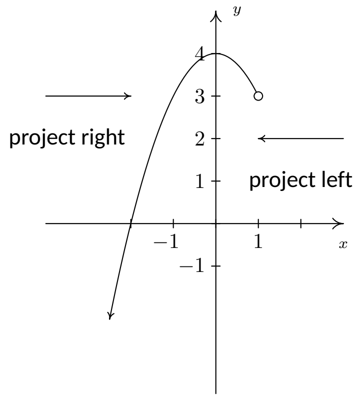


Figure 2.18: Projecting the graph onto the y-axis in Example 21

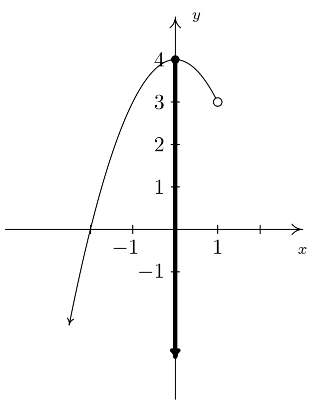


Figure 2.19: The range of G in Example 21

Example 22 Functions defined by equations

Determine which equations represent y as a function of x.

1. $x^3 + y^2 = 1$
2. $x^2 + y^3 = 1$
3. $x^2y = 1 - 3y$

Solution For each of these equations, we solve for y and determine whether each choice of x will determine only one corresponding value of y.

1.
$$\begin{aligned} x^3 + y^2 &= 1 \\ y^2 &= 1 - x^3 \\ \sqrt{y^2} &= \sqrt{1-x^3} \qquad \text{extract square roots} \\ y &= \pm\sqrt{1-x^3} \end{aligned}$$

If we substitute $x = 0$ into our equation for y, we get $y = \pm\sqrt{1-0^3} = \pm 1$, so that $(0, 1)$ and $(0, -1)$ are on the graph of this equation. Hence, this equation does not represent y as a function of x.

2.
$$\begin{aligned} x^2 + y^3 &= 1 \\ y^3 &= 1 - x^2 \\ \sqrt[3]{y^3} &= \sqrt[3]{1-x^2} \\ y &= \sqrt[3]{1-x^2} \end{aligned}$$

For every choice of x, the equation $y = \sqrt[3]{1-x^2}$ returns only **one** value of y. Hence, this equation describes y as a function of x.

3.
$$\begin{aligned} x^2y &= 1 - 3y \\ x^2y + 3y &= 1 \\ y\left(x^2+3\right) &= 1 \qquad \text{factor} \\ y &= \frac{1}{x^2+3} \end{aligned}$$

For each choice of x, there is only one value for y, so this equation describes y as a function of x.

Exercises 2.2

Problems

In Exercises 1 – 12, determine whether or not the relation represents *y* as a function of *x*. Find the domain and range of those relations which are functions.

1. $\{(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)\}$

2. $\{(-3,0),(1,6),(2,-3),(4,2),(-5,6),(4,-9),(6,2)\}$

3. $\{(-3,0),(-7,6),(5,5),(6,4),(4,9),(3,0)\}$

4. $\{(1,2),(4,4),(9,6),(16,8),(25,10),(36,12),\ldots\}$

5. $\{(x,y)\mid x \text{ is an odd integer, and } y \text{ is an even integer}\}$

6. $\{(x,1)\mid x \text{ is an irrational number}\}$

7. $\{(1,0),(2,1),(4,2),(8,3),(16,4),(32,5),\ldots\}$

8. $\{\ldots,(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9),\ldots\}$

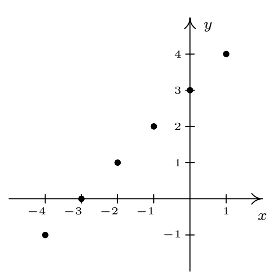
9. $\{(-2,y)\mid -3<y<4\}$

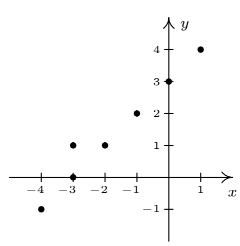
10. $\{(x,3)\mid -2\le x<4\}$

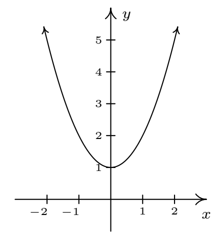
11. $\{\left(x,x^2\right)\mid x \text{ is a real number}\}$

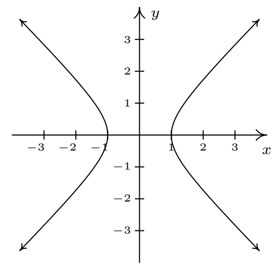
12. $\{\left(x^2,x\right)\mid x \text{ is a real number}\}$

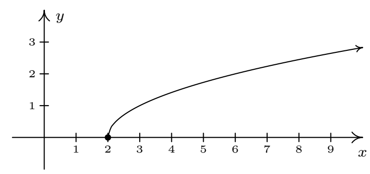
In Exercises 13 – 32, determine whether or not the relation represents *y* as a function of *x*. Find the domain and range of those relations which are functions.

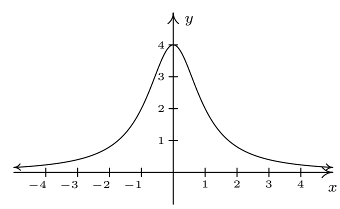
13. 

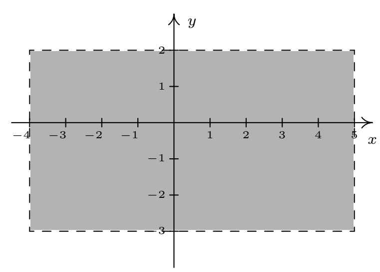
14. 

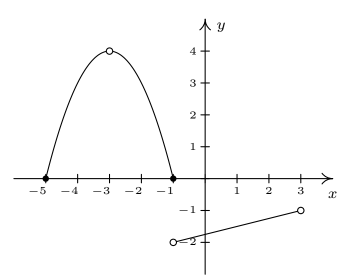
15. 

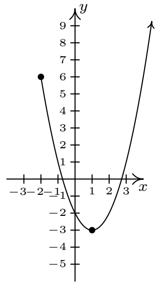
16. 

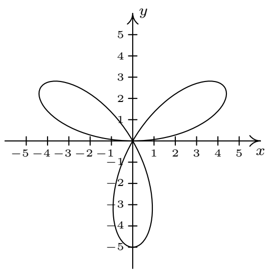
17. 

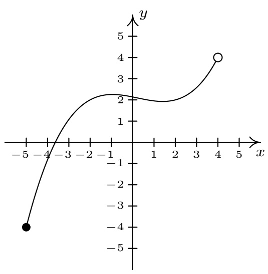
18. 

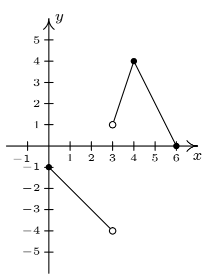
19. 

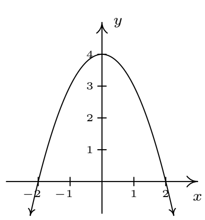
20. 

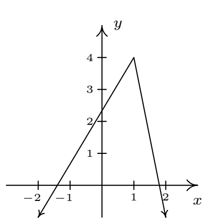
21. 

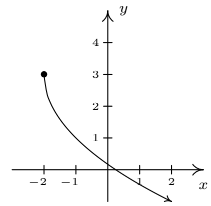
22. 

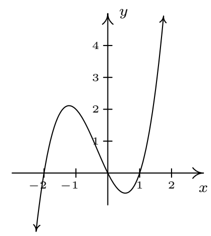
23. 

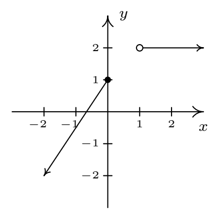
24. 

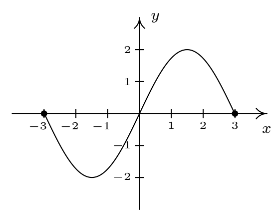
25. 

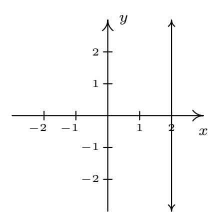
26. 

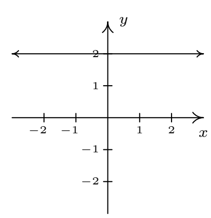
27. 

28. 

29. 

30. 

31. 

32. 

In Exercises 33 – 47, determine whether or not the equation represents *y* as a function of *x*.

33. $y = x^3 - x$

34. $y = \sqrt{x - 2}$

35. $x^3 y = -4$

36. $x^2 - y^2 = 1$

37. $y = \dfrac{x}{x^2 - 9}$

38. $x = -6$

39. $x = y^2 + 4$

40. $y = x^2 + 4$

41. $x^2 + y^2 = 4$

42. $y = \sqrt{4 - x^2}$

43. $x^2 - y^2 = 4$

44. $x^3 + y^3 = 4$

45. $2x + 3y = 4$

46. $2xy = 4$

47. $x^2 = y^2$

48. Explain why the population P of Sasquatch in a given area is a function of time t. What would be the range of this function?

49. Explain why the relation between your classmates and their email addresses may not be a function. What about phone numbers and Social Security Numbers?

Some relations are fairly easy to describe in words or with the roster method but are rather difficult, if not impossible, to graph. For Exercises 50 – 53, discuss with your classmates how you might graph the given relation.

50. $\{(x, y) \mid x$ is an odd integer, and y is an even integer.$\}$

51. $\{(x, 1) \mid x$ is an irrational number $\}$

52. $\{(1,0), (2,1), (4,2), (8,3), (16,4), (32,5), \ldots\}$

53. $\{\ldots, (-3,9), (-2,4), (-1,1), (0,0), (1,1), (2,4), (3,9), \ldots\}$

2.3 Function Notation

In Definition 22, we described a function as a special kind of relation $-$ one in which each x-coordinate is matched with only one y-coordinate. In this section, we focus more on the **process** by which the x is matched with the y. If we think of the domain of a function as a set of **inputs** and the range as a set of **outputs**, we can think of a function f as a process by which each input x is matched with only one output y. Since the output is completely determined by the input x and the process f, we symbolize the output with **function notation**: '$f(x)$', read 'f **of** x.' In other words, $f(x)$ is the output which results by applying the process f to the input x. In this case, the parentheses here do not indicate multiplication, as they do elsewhere in Algebra. This can cause confusion if the context is not clear, so you must read carefully. This relationship is typically visualized using a diagram similar to the one in Figure 2.20.

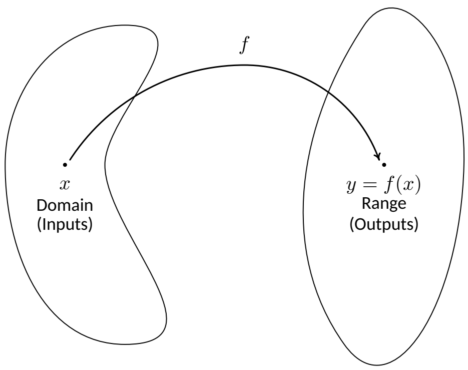


Figure 2.20: Graphical depiction of a function

The value of y is completely dependent on the choice of x. For this reason, x is often called the **independent variable**, or **argument** of f, whereas y is often called the **dependent variable**.

As we shall see, the process of a function f is usually described using an algebraic formula. For example, suppose a function f takes a real number and performs the following two steps, in sequence

1. Multiply by 3

2. Add 4

If we choose 5 as our input, in Step 1 we multiply by 3 to get $(5)(3) = 15$. In Step 2, we add 4 to our result from Step 1 which yields $15 + 4 = 19$. Using function notation, we would write $f(5) = 19$ to indicate that the result of applying the process f to the input 5 gives the output 19. In general, if we use x for the input, applying Step 1 produces $3x$. Following with Step 2 produces $3x + 4$ as our final output. Hence for an input x, we get the output $f(x) = 3x + 4$. Notice that to check our formula for the case $x = 5$, we replace the occurrence of x in the formula for $f(x)$ with 5 to get $f(5) = 3(5) + 4 = 15 + 4 = 19$, as required.

Example 23 Finding a forumula for a function

Suppose a function g is described by applying the following steps, in sequence

1. add 4

2. multiply by 3

Determine $g(5)$ and find an expression for $g(x)$.

Solution Starting with 5, Step 1 gives $5 + 4 = 9$. Continuing with Step 2, we get $(3)(9) = 27$. To find a formula for $g(x)$, we start with our input x. Step 1 produces $x + 4$. We now wish to multiply this entire quantity by 3, so we use a parentheses: $3(x + 4) = 3x + 12$. Hence, $g(x) = 3x + 12$. We can check our formula by replacing x with 5 to get $g(5) = 3(5) + 12 = 15 + 12 = 27\,\checkmark$.

Most of the functions we will encounter in Math 1010 will be described using formulas like the ones we developed for $f(x)$ and $g(x)$ above. Evaluating formulas using this function notation is a key skill for success in this and many other Math courses.

Example 24 Using function notation

Let $f(x) = -x^2 + 3x + 4$

1. Find and simplify the following.
   (a) $f(-1), f(0), f(2)$
   (b) $f(2x)$, $2f(x)$
   (c) $f(x+2), f(x)+2, f(x)+f(2)$
2. Solve $f(x) = 4$.

SOLUTION

1. (a) To find $f(-1)$, we replace every occurrence of x in the expression $f(x)$ with -1

$$\begin{aligned} f(-1) &= -(-1)^2 + 3(-1) + 4 \\ &= -(1) + (-3) + 4 \\ &= 0 \end{aligned}$$

Similarly, $f(0) = -(0)^2 + 3(0) + 4 = 4$, and $f(2) = -(2)^2 + 3(2) + 4 = -4 + 6 + 4 = 6$.

(b) To find $f(2x)$, we replace every occurrence of x with the quantity $2x$

$$\begin{aligned} f(2x) &= -(2x)^2 + 3(2x) + 4 \\ &= -(4x^2) + (6x) + 4 \\ &= -4x^2 + 6x + 4 \end{aligned}$$

The expression $2f(x)$ means we multiply the expression $f(x)$ by 2

$$\begin{aligned} 2f(x) &= 2\left(-x^2 + 3x + 4\right) \\ &= -2x^2 + 6x + 8 \end{aligned}$$

(c) To find $f(x+2)$, we replace every occurrence of x with the quantity $x + 2$

$$\begin{aligned} f(x+2) &= -(x+2)^2 + 3(x+2) + 4 \\ &= -\left(x^2 + 4x + 4\right) + (3x + 6) + 4 \\ &= -x^2 - 4x - 4 + 3x + 6 + 4 \\ &= -x^2 - x + 6 \end{aligned}$$

To find $f(x) + 2$, we add 2 to the expression for $f(x)$

$$\begin{aligned} f(x) + 2 &= \left(-x^2 + 3x + 4\right) + 2 \\ &= -x^2 + 3x + 6 \end{aligned}$$

From our work above, we see $f(2) = 6$ so that

$$\begin{aligned} f(x) + f(2) &= \left(-x^2 + 3x + 4\right) + 6 \\ &= -x^2 + 3x + 10 \end{aligned}$$

2. Since $f(x) = -x^2 + 3x + 4$, the equation $f(x) = 4$ is equivalent to $-x^2 + 3x + 4 = 4$. Solving we get $-x^2 + 3x = 0$, or $x(-x+3) = 0$. We get $x = 0$ or $x = 3$, and we can verify these answers by checking that $f(0) = 4$ and $f(3) = 4$.

A few notes about Example 24 are in order. First note the difference between the answers for $f(2x)$ and $2f(x)$. For $f(2x)$, we are multiplying the *input* by 2; for $2f(x)$, we are multiplying the *output* by 2. As we see, we get entirely different results. Along these lines, note that $f(x+2)$, $f(x)+2$ and $f(x)+f(2)$ are three *different* expressions as well. Even though function notation uses parentheses, as does multiplication, there is *no* general 'distributive property' of function notation. Finally, note the practice of using parentheses when substituting one algebraic expression into another; we highly recommend this practice as it will reduce careless errors.

Suppose now we wish to find $r(3)$ for $r(x) = \dfrac{2x}{x^2 - 9}$. Substitution gives

$$r(3) = \frac{2(3)}{(3)^2 - 9} = \frac{6}{0},$$

which is undefined. (Why is this, again?) The number 3 is not an allowable input to the function r; in other words, 3 is not in the domain of r. Which other real numbers are forbidden in this formula? We think back to arithmetic. The reason $r(3)$ is undefined is because substitution results in a division by 0. To determine which other numbers result in such a transgression, we set the denominator equal to 0 and solve

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ \sqrt{x^2} &= \sqrt{9} \quad \text{extract square roots} \\ x &= \pm 3 \end{aligned}$$

As long as we substitute numbers other than 3 and -3, the expression $r(x)$ is a real number. Hence, we write our domain in interval notation (see the Exercises for Section 1.3) as $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. When a formula for a function is given, we assume that the function is valid for all real numbers which make arithmetic sense when substituted into the formula. This set of numbers is often called the **implied domain** (or 'implicit domain') of the function. At this stage, there are only two mathematical sins we need to avoid: division by 0 and extracting even roots of negative numbers. The following example illustrates these concepts.

Example 25 Determining an implied domain

Find the domain of the following functions.

1. $g(x) = \sqrt{4 - 3x}$
2. $h(x) = \sqrt[5]{4 - 3x}$
3. $f(x) = \dfrac{2}{1 - \dfrac{4x}{x - 3}}$
4. $F(x) = \dfrac{\sqrt[4]{2x + 1}}{x^2 - 1}$
5. $r(t) = \dfrac{4}{6 - \sqrt{t + 3}}$
6. $I(x) = \dfrac{3x^2}{x}$

The 'radicand' is the expression 'inside' the radical.

Solution

1. The potential disaster for g is if the radicand is negative. To avoid this, we set $4 - 3x \geq 0$. From this, we get $3x \leq 4$ or $x \leq \frac{4}{3}$. What this shows is that as long as $x \leq \frac{4}{3}$, the expression $4 - 3x \geq 0$, and the formula $g(x)$ returns a real number. Our domain is $\left(-\infty, \frac{4}{3}\right]$.

2. The formula for $h(x)$ is hauntingly close to that of $g(x)$ with one key difference — whereas the expression for $g(x)$ includes an even indexed root (namely a square root), the formula for $h(x)$ involves an odd indexed root (the fifth root). Since odd roots of real numbers (even negative real numbers) are real numbers, there is no restriction on the inputs to h. Hence, the domain is $(-\infty, \infty)$.

3. In the expression for f, there are two denominators. We need to make sure neither of them is 0. To that end, we set each denominator equal to 0 and solve. For the 'small' denominator, we get $x - 3 = 0$ or $x = 3$. For the 'large' denominator

$$\begin{aligned} 1 - \frac{4x}{x-3} &= 0 \\ 1 &= \frac{4x}{x-3} \\ (1)(x-3) &= \left(\frac{4x}{\cancel{x-3}}\right)\cancel{(x-3)} \quad \text{clear denominators} \\ x - 3 &= 4x \\ -3 &= 3x \\ -1 &= x \end{aligned}$$

So we get two real numbers which make denominators 0, namely $x = -1$ and $x = 3$. Our domain is all real numbers except -1 and 3: $(-\infty, -1) \cup (-1, 3) \cup (3, \infty)$.

4. In finding the domain of F, we notice that we have two potentially hazardous issues: not only do we have a denominator, we have a fourth (even-indexed) root. Our strategy is to determine the restrictions imposed by each part and select the real numbers which satisfy both conditions. To satisfy the fourth root, we require $2x + 1 \geq 0$. From this we get $2x \geq -1$ or $x \geq -\frac{1}{2}$. Next, we round up the values of x which could cause trouble in the denominator by setting the denominator equal to 0. We get $x^2 - 1 = 0$, or $x = \pm 1$. Hence, in order for a real number x to be in the domain of F, $x \geq -\frac{1}{2}$ but $x \neq \pm 1$. In interval notation, this set is $\left[-\frac{1}{2}, 1\right) \cup (1, \infty)$.

5. Don't be put off by the 't' here. It is an independent variable representing a real number, just like x does, and is subject to the same restrictions. As in the previous problem, we have double danger here: we have a square root and a denominator. To satisfy the square root, we need a non-negative radicand so we set $t + 3 \geq 0$ to get $t \geq -3$. Setting the denominator equal to zero gives $6 - \sqrt{t+3} = 0$, or $\sqrt{t+3} = 6$. Squaring both sides gives $t + 3 = 36$, or $t = 33$. Since we squared both sides in the course of solving this equation, we need to check our answer. Sure enough, when $t = 33$, $6 - \sqrt{t+3} = 6 - \sqrt{36} = 0$, so $t = 33$ will cause problems in the denominator. At last we can find the domain of r: we need $t \geq -3$, but $t \neq 33$. Our final answer is $[-3, 33) \cup (33, \infty)$.

Squaring both sides of an equation can introduce *extraneous solutions*. Do you remember why? Consider squaring both sides to 'solve' $\sqrt{t+1} = -2$.

6. It's tempting to simplify $I(x) = \frac{3x^2}{x} = 3x$, and, since there are no longer any denominators, claim that there are no longer any restrictions. However, in simplifying $I(x)$, we are assuming $x \neq 0$, since $\frac{0}{0}$ is undefined. Proceeding as before, we find the domain of I to be all real numbers except 0: $(-\infty, 0) \cup (0, \infty)$.

It is worth reiterating the importance of finding the domain of a function *before* simplifying, as evidenced by the function I in the previous example. Even though the formula $I(x)$ simplifies to $3x$, it would be inaccurate to write $I(x) = 3x$ without adding the stipulation that $x \neq 0$. It would be analogous to not reporting taxable income or some other sin of omission.

2.3.1 Modelling with Functions

The importance of Mathematics to our society lies in its value to approximate, or **model** real-world phenomenon. Whether it be used to predict the high temperature on a given day, determine the hours of daylight on a given day, or predict population trends of various and sundry real and mythical beasts, Mathematics is second only to literacy in the importance humanity's development.

It is important to keep in mind that any time Mathematics is used to approximate reality, there are always limitations to the model. For example, suppose grapes are on sale at the local market for $\$1.50$ per pound. Then one pound of grapes costs $\$1.50$, two pounds of grapes cost $\$3.00$, and so forth. Suppose we want to develop a formula which relates the cost of buying grapes to the amount of grapes being purchased. Since these two quantities vary from situation to situation, we assign them variables. Let c denote the cost of the grapes and let g denote the amount of grapes purchased. To find the cost c of the grapes, we multiply the amount of grapes g by the price $\$1.50$ dollars per pound to get

$$c = 1.5g$$

In order for the units to be correct in the formula, g must be measured in *pounds* of grapes in which case the computed value of c is measured in *dollars*. Since we're interested in finding the cost c given an amount g, we think of g as the independent variable and c as the dependent variable. Using the language of function notation, we write

$$c(g) = 1.5g$$

where g is the amount of grapes purchased (in pounds) and $c(g)$ is the cost (in dollars). For example, $c(5)$ represents the cost, in dollars, to purchase 5 pounds of grapes. In this case, $c(5) = 1.5(5) = 7.5$, so it would cost $\$7.50$. If, on the other hand, we wanted to find the *amount* of grapes we can purchase for $\$5$, we would need to set $c(g) = 5$ and solve for g. In this case, $c(g) = 1.5g$, so solving $c(g) = 5$ is equivalent to solving $1.5g = 5$ Doing so gives $g = \frac{5}{1.5} = 3.\overline{3}$. This means we can purchase exactly $3.\overline{3}$ pounds of grapes for $\$5$. Of course, you would be hard-pressed to buy exactly $3.\overline{3}$ pounds of grapes, (you could get close... within a certain specified margin of error, perhaps) and this leads us to our next topic of discussion, the **applied domain**, or 'explicit domain' of a function.

Even though, mathematically, $c(g) = 1.5g$ has no domain restrictions (there are no denominators and no even-indexed radicals), there are certain values of g that don't make any physical sense. For example, $g = -1$ corresponds to 'purchasing' -1 pounds of grapes. (Maybe this means *returning* a pound of grapes?)

Also, unless the 'local market' mentioned is the State of California (or some other exporter of grapes), it also doesn't make much sense for $g = 500{,}000{,}000$, either. So the reality of the situation limits what g can be, and these limits determine the applied domain of g. Typically, an applied domain is stated explicitly. In this case, it would be common to see something like $c(g) = 1.5g$, $0 \leq g \leq 100$, meaning the number of pounds of grapes purchased is limited from 0 up to 100. The upper bound here, 100 may represent the inventory of the market, or some other limit as set by local policy or law. Even with this restriction, our model has its limitations. As we saw above, it is virtually impossible to buy exactly $3.\overline{3}$ pounds of grapes so that our cost is exactly \$5. In this case, being sensible shoppers, we would most likely 'round down' and purchase 3 pounds of grapes or however close the market scale can read to $3.\overline{3}$ without being over. It is time for a more sophisticated example.

Example 26 **Height of a model rocket**

The height h in feet of a model rocket above the ground t seconds after lift-off is given by

$$h(t) = \begin{cases} -5t^2 + 100t, & \text{if} \quad 0 \leq t \leq 20 \\ 0, & \text{if} \quad t > 20 \end{cases}$$

1. Find and interpret $h(10)$ and $h(60)$.

2. Solve $h(t) = 375$ and interpret your answers.

Solution

1. We first note that the independent variable here is t, chosen because it represents time. Secondly, the function is broken up into two rules: one formula for values of t between 0 and 20 inclusive, and another for values of t greater than 20. Since $t = 10$ satisfies the inequality $0 \leq t \leq 20$, we use the first formula listed, $h(t) = -5t^2 + 100t$, to find $h(10)$. We get $h(10) = -5(10)^2 + 100(10) = 500$. Since t represents the number of seconds since lift-off and $h(t)$ is the height above the ground in feet, the equation $h(10) = 500$ means that 10 seconds after lift-off, the model rocket is 500 feet above the ground. To find $h(60)$, we note that $t = 60$ satisfies $t > 20$, so we use the rule $h(t) = 0$. This function returns a value of 0 regardless of what value is substituted in for t, so $h(60) = 0$. This means that 60 seconds after lift-off, the rocket is 0 feet above the ground; in other words, a minute after lift-off, the rocket has already returned to Earth.

2. Since the function h is defined in pieces, we need to solve $h(t) = 375$ in pieces. For $0 \leq t \leq 20$, $h(t) = -5t^2 + 100t$, so for these values of t, we solve $-5t^2 + 100t = 375$. Rearranging terms, we get $5t^2 - 100t + 375 = 0$, and factoring gives $5(t-5)(t-15) = 0$. Our answers are $t = 5$ and $t = 15$, and since both of these values of t lie between 0 and 20, we keep both solutions. For $t > 20$, $h(t) = 0$, and in this case, there are no solutions to $0 = 375$. In terms of the model rocket, solving $h(t) = 375$ corresponds to finding when, if ever, the rocket reaches 375 feet above the ground. Our two answers, $t = 5$ and $t = 15$ correspond to the rocket reaching this altitude *twice* – once 5 seconds after launch, and again 15 seconds after launch.

The type of function in the previous example is called a **piecewise-defined** function, or 'piecewise' function for short. Many real-world phenomena, income tax formulas for example, are modelled by such functions.

By the way, if we wanted to avoid using a piecewise function in Example 26, we could have used $h(t) = -5t^2 + 100t$ on the explicit domain $0 \leq t \leq 20$ because after 20 seconds, the rocket is on the ground and stops moving. In many cases, though, piecewise functions are your only choice, so it's best to understand them well.

Mathematical modelling is not a one-section topic. It's not even a one-*course* topic as is evidenced by undergraduate and graduate courses in mathematical modelling being offered at many universities. Thus our goal in this section cannot possibly be to tell you the whole story. What we can do is get you started. As we study new classes of functions, we will see what phenomena they can be used to model. In that respect, mathematical modelling cannot be a topic in a book, but rather, must be a theme of the book. For now, we have you explore some very basic models in the Exercises because you need to crawl to walk to run. As we learn more about functions, we'll help you build your own models and get you on your way to applying Mathematics to your world.

Exercises 2.3

Problems

In Exercises 1 – 10, find an expression for $f(x)$ and state its domain.

1. f is a function that takes a real number x and performs the following three steps in the order given: (1) multiply by 2; (2) add 3; (3) divide by 4.

2. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) multiply by 2; (3) divide by 4.

3. f is a function that takes a real number x and performs the following three steps in the order given: (1) divide by 4; (2) add 3; (3) multiply by 2.

4. f is a function that takes a real number x and performs the following three steps in the order given: (1) multiply by 2; (2) add 3; (3) take the square root.

5. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) multiply by 2; (3) take the square root.

6. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) take the square root; (3) multiply by 2.

7. f is a function that takes a real number x and performs the following three steps in the order given: (1) take the square root; (2) subtract 13; (3) make the quantity the denominator of a fraction with numerator 4.

8. f is a function that takes a real number x and performs the following three steps in the order given: (1) subtract 13; (2) take the square root; (3) make the quantity the denominator of a fraction with numerator 4.

9. f is a function that takes a real number x and performs the following three steps in the order given: (1) take the square root; (2) make the quantity the denominator of a fraction with numerator 4; (3) subtract 13.

10. f is a function that takes a real number x and performs the following three steps in the order given: (1) make the quantity the denominator of a fraction with numerator 4; (2) take the square root; (3) subtract 13.

In Exercises 11 – 18, use the given function f to find and simplify the following:

- $f(3)$
- $f(-1)$
- $f\left(\frac{3}{2}\right)$
- $f(4x)$
- $4f(x)$
- $f(-x)$
- $f(x-4)$
- $f(x)-4$
- $f\left(x^2\right)$

11. $f(x) = 2x + 1$

12. $f(x) = 3 - 4x$

13. $f(x) = 2 - x^2$

14. $f(x) = x^2 - 3x + 2$

15. $f(x) = \dfrac{x}{x-1}$

16. $f(x) = \dfrac{2}{x^3}$

17. $f(x) = 6$

18. $f(x) = 0$

In Exercises 19 – 26, use the given function f to find and simplify the following:

- $f(2)$
- $f(-2)$
- $f(2a)$
- $2f(a)$
- $f(a+2)$
- $f(a) + f(2)$
- $f\left(\frac{2}{a}\right)$
- $\frac{f(a)}{2}$
- $f(a+h)$

19. $f(x) = 2x - 5$

20. $f(x) = 5 - 2x$

21. $f(x) = 2x^2 - 1$

22. $f(x) = 3x^2 + 3x - 2$

23. $f(x) = \sqrt{2x+1}$

24. $f(x) = 117$

25. $f(x) = \dfrac{x}{2}$

26. $f(x) = \dfrac{2}{x}$

In Exercises 27 – 34, use the given function f to find $f(0)$ and solve $f(x) = 0$.

27. $f(x) = 2x - 1$

28. $f(x) = 3 - \frac{2}{5}x$

29. $f(x) = 2x^2 - 6$

30. $f(x) = x^2 - x - 12$

31. $f(x) = \sqrt{x+4}$

32. $f(x) = \sqrt{1-2x}$

33. $f(x) = \dfrac{3}{4-x}$

34. $f(x) = \dfrac{3x^2-12x}{4-x^2}$

35. Let $f(x) = \begin{cases} x+5 & \text{if} \quad x \leq -3 \\ \sqrt{9-x^2} & \text{if} \quad -3 < x \leq 3 \\ -x+5 & \text{if} \quad x > 3 \end{cases}$ Compute the following function values.

(a) $f(-4)$
(b) $f(-3)$
(c) $f(3)$
(d) $f(3.001)$
(e) $f(-3.001)$
(f) $f(2)$

36. Let $f(x) = \begin{cases} x^2 & \text{if} \quad x \leq -1 \\ \sqrt{1-x^2} & \text{if} \quad -1 < x \leq 1 \\ x & \text{if} \quad x > 1 \end{cases}$ Compute the following function values.

(a) $f(4)$
(b) $f(-3)$
(c) $f(1)$
(d) $f(0)$
(e) $f(-1)$
(f) $f(-0.999)$

In Exercises 37 – 62, find the (implied) domain of the function.

37. $f(x) = x^4 - 13x^3 + 56x^2 - 19$

38. $f(x) = x^2 + 4$

39. $f(x) = \dfrac{x-2}{x+1}$

40. $f(x) = \dfrac{3x}{x^2+x-2}$

41. $f(x) = \dfrac{2x}{x^2+3}$

42. $f(x) = \dfrac{2x}{x^2-3}$

43. $f(x) = \dfrac{x+4}{x^2-36}$

44. $f(x) = \dfrac{x-2}{x-2}$

45. $f(x) = \sqrt{3-x}$

46. $f(x) = \sqrt{2x+5}$

47. $f(x) = 9x\sqrt{x+3}$

48. $f(x) = \dfrac{\sqrt{7-x}}{x^2+1}$

49. $f(x) = \sqrt{6x-2}$

50. $f(x) = \dfrac{6}{\sqrt{6x-2}}$

51. $f(x) = \sqrt[3]{6x-2}$

52. $f(x) = \dfrac{6}{4-\sqrt{6x-2}}$

53. $f(x) = \dfrac{\sqrt{6x-2}}{x^2-36}$

54. $f(x) = \dfrac{\sqrt[3]{6x-2}}{x^2+36}$

55. $s(t) = \dfrac{t}{t-8}$

56. $Q(r) = \dfrac{\sqrt{r}}{r-8}$

57. $b(\theta) = \dfrac{\theta}{\sqrt{\theta-8}}$

58. $A(x) = \sqrt{x-7} + \sqrt{9-x}$

59. $\alpha(y) = \sqrt[3]{\dfrac{y}{y-8}}$

60. $g(v) = \dfrac{1}{4-\dfrac{1}{v^2}}$

61. $T(t) = \dfrac{\sqrt{t}-8}{5-t}$

62. $u(w) = \dfrac{w-8}{5-\sqrt{w}}$

63. The area A enclosed by a square, in square inches, is a function of the length of one of its sides x, when measured in inches. This relation is expressed by the formula $A(x) = x^2$ for $x > 0$. Find $A(3)$ and solve $A(x) = 36$. Interpret your answers to each. Why is x restricted to $x > 0$?

64. The area A enclosed by a circle, in square meters, is a function of its radius r, when measured in meters. This relation is expressed by the formula $A(r) = \pi r^2$ for $r > 0$. Find $A(2)$ and solve $A(r) = 16\pi$. Interpret your answers to each. Why is r restricted to $r > 0$?

65. The volume V enclosed by a cube, in cubic centimeters, is a function of the length of one of its sides x, when measured in centimeters. This relation is expressed by the formula $V(x) = x^3$ for $x > 0$. Find $V(5)$ and solve $V(x) = 27$. Interpret your answers to each. Why is x restricted to $x > 0$?

66. The volume V enclosed by a sphere, in cubic feet, is a function of the radius of the sphere r, when measured in feet.

This relation is expressed by the formula $V(r) = \frac{4\pi}{3}r^3$ for $r > 0$. Find $V(3)$ and solve $V(r) = \frac{32\pi}{3}$. Interpret your answers to each. Why is r restricted to $r > 0$?

67. The volume V enclosed by a sphere, in cubic feet, is a function of the radius of the sphere r, when measured in feet. This relation is expressed by the formula $V(r) = \frac{4\pi}{3}r^3$ for $r > 0$. Find $V(3)$ and solve $V(r) = \frac{32\pi}{3}$. Interpret your answers to each. Why is r restricted to $r > 0$?

68. The height of an object dropped from the roof of an eight story building is modeled by: $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, h is the height of the object off the ground, in feet, t seconds after the object is dropped. Find $h(0)$ and solve $h(t) = 0$. Interpret your answers to each. Why is t restricted to $0 \leq t \leq 2$?

69. The temperature T in degrees Fahrenheit t hours after 6 AM is given by $T(t) = -\frac{1}{2}t^2 + 8t + 3$ for $0 \leq t \leq 12$. Find and interpret $T(0)$, $T(6)$ and $T(12)$.

70. The function $C(x) = x^2 - 10x + 27$ models the cost, in *hundreds* of dollars, to produce x *thousand* pens. Find and interpret $C(0)$, $C(2)$ and $C(5)$.

    (The value $C(0)$ is called the 'fixed' or 'start-up' cost. We'll revisit this concept on page 73.)

71. Using data from the Bureau of Transportation Statistics, the average fuel economy F in miles per gallon for passenger cars in the US can be modelled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Use your calculator to find $F(0)$, $F(14)$ and $F(28)$. Round your answers to two decimal places and interpret your answers to each.

72. The population of Sasquatch in Portage County can be modeled by the function $P(t) = \frac{150t}{t+15}$, where t represents the number of years since 1803. Find and interpret $P(0)$ and $P(205)$. Discuss with your classmates what the applied domain and range of P should be.

73. For n copies of the book *Me and my Sasquatch*, a print on-demand company charges $C(n)$ dollars, where $C(n)$ is determined by the formula

$$C(n) = \begin{cases} 15n & \text{if} \quad 1 \leq n \leq 25 \\ 13.50n & \text{if} \quad 25 < n \leq 50 \\ 12n & \text{if} \quad n > 50 \end{cases}$$

    (a) Find and interpret $C(20)$.
    (b) How much does it cost to order 50 copies of the book? What about 51 copies?
    (c) Your answer to 73b should get you thinking. Suppose a bookstore estimates it will sell 50 copies of the book. How many books can, in fact, be ordered for the same price as those 50 copies? (Round your answer to a whole number of books.)

74. An on-line comic book retailer charges shipping costs according to the following formula

$$S(n) = \begin{cases} 1.5n + 2.5 & \text{if} \quad 1 \leq n \leq 14 \\ 0 & \text{if} \quad n \geq 15 \end{cases}$$

    where n is the number of comic books purchased and $S(n)$ is the shipping cost in dollars.

    (a) What is the cost to ship 10 comic books?
    (b) What is the significance of the formula $S(n) = 0$ for $n \geq 15$?

75. The cost C (in dollars) to talk m minutes a month on a mobile phone plan is modeled by

$$C(m) = \begin{cases} 25 & \text{if} \quad 0 \leq m \leq 1000 \\ 25 + 0.1(m - 1000) & \text{if} \quad m > 1000 \end{cases}$$

    (a) How much does it cost to talk 750 minutes per month with this plan?
    (b) How much does it cost to talk 20 hours a month with this plan?
    (c) Explain the terms of the plan verbally.

76. In Section **??** we defined the set of **integers** as $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. The **greatest integer of** x, denoted by $\lfloor x \rfloor$, is defined to be the largest integer k with $k \leq x$.

    Note: The use of the letter $\mathbb{Z}$ for the integers is ostensibly because the German word *zahlen* means 'to count.'

    (a) Find $\lfloor 0.785 \rfloor$, $\lfloor 117 \rfloor$, $\lfloor -2.001 \rfloor$, and $\lfloor \pi + 6 \rfloor$
    (b) Discuss with your classmates how $\lfloor x \rfloor$ may be described as a piecewise defined function.

        HINT: There are infinitely many pieces!
    (c) Is $\lfloor a + b \rfloor = \lfloor a \rfloor + \lfloor b \rfloor$ always true? What if a or b is an integer? Test some values, make a conjecture, and explain your result.

77.

78. We have through our examples tried to convince you that, in general, $f(a+b) \neq f(a)+f(b)$. It has been our experience that students refuse to believe us so we'll try again with a different approach. With the help of your classmates, find a function f for which the following properties are always true.

    (a) $f(0) = f(-1 + 1) = f(-1) + f(1)$
    (b) $f(5) = f(2 + 3) = f(2) + f(3)$
    (c) $f(-6) = f(0 - 6) = f(0) - f(6)$
    (d) $f(a + b) = f(a) + f(b)$ regardless of what two numbers we give you for a and b.

    How many functions did you find that failed to satisfy the conditions above? Did $f(x) = x^2$ work? What about $f(x) = \sqrt{x}$ or $f(x) = 3x + 7$ or $f(x) = \dfrac{1}{x}$? Did you find an attribute common to those functions that did succeed? You should have, because there is only one extremely special family of functions that actually works here. Thus we return to our previous statement, **in general**, $f(a + b) \neq f(a) + f(b)$.

2.4 Function Arithmetic

In the previous section we used the newly defined function notation to make sense of expressions such as '$f(x) + 2$' and '$2f(x)$' for a given function f. It would seem natural, then, that functions should have their own arithmetic which is consistent with the arithmetic of real numbers. The following definitions allow us to add, subtract, multiply and divide functions using the arithmetic we already know for real numbers.

Recall that if x is in the domains of both f and g, then we can say that x is an element of the intersection of the two domains.

Definition 24 Function Arithmetic

Suppose f and g are functions and x is in both the domain of f and the domain of g.

- The **sum** of f and g, denoted $f + g$, is the function defined by the formula
$$(f + g)(x) = f(x) + g(x)$$
- The **difference** of f and g, denoted $f - g$, is the function defined by the formula
$$(f - g)(x) = f(x) - g(x)$$
- The **product** of f and g, denoted fg, is the function defined by the formula
$$(fg)(x) = f(x)g(x)$$
- The **quotient** of f and g, denoted $\dfrac{f}{g}$, is the function defined by the formula
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$
provided $g(x) \neq 0$.

In other words, to add two functions, we add their outputs; to subtract two functions, we subtract their outputs, and so on. Note that while the formula $(f+g)(x) = f(x)+g(x)$ looks suspiciously like some kind of distributive property, it is nothing of the sort; the addition on the left hand side of the equation is *function* addition, and we are using this equation to *define* the output of the new function $f + g$ as the sum of the real number outputs from f and g.

Example 27 Arithmetic with functions

Let $f(x) = 6x^2 - 2x$ and $g(x) = 3 - \dfrac{1}{x}$.

1. Find $(f + g)(-1)$
2. Find $(fg)(2)$
3. Find the domain of $g - f$ then find and simplify a formula for $(g - f)(x)$.
4. Find the domain of $\left(\frac{g}{f}\right)$ then find and simplify a formula for $\left(\frac{g}{f}\right)(x)$.

Solution

1. To find $(f + g)(-1)$ we first find $f(-1) = 8$ and $g(-1) = 4$. By definition, we have that $(f + g)(-1) = f(-1) + g(-1) = 8 + 4 = 12$.

2. To find $(fg)(2)$, we first need $f(2)$ and $g(2)$. Since $f(2) = 20$ and $g(2) = \frac{5}{2}$, our formula yields $(fg)(2) = f(2)g(2) = (20)\left(\frac{5}{2}\right) = 50$.

3. One method to find the domain of $g - f$ is to find the domain of g and of f separately, then find the intersection of these two sets. Owing to the denominator in the expression $g(x) = 3 - \frac{1}{x}$, we get that the domain of g is $(-\infty, 0) \cup (0, \infty)$. Since $f(x) = 6x^2 - 2x$ is valid for all real numbers, we have no further restrictions. Thus the domain of $g - f$ matches the domain of g, namely, $(-\infty, 0) \cup (0, \infty)$.

   A second method is to analyze the formula for $(g - f)(x)$ *before simplifying* and look for the usual domain issues. In this case,

$$(g - f)(x) = g(x) - f(x) = \left(3 - \frac{1}{x}\right) - \left(6x^2 - 2x\right),$$

   so we find, as before, the domain is $(-\infty, 0) \cup (0, \infty)$.

   Moving along, we need to simplify a formula for $(g - f)(x)$. In this case, we get common denominators and attempt to reduce the resulting fraction. Doing so, we get

$$\begin{aligned}
(g - f)(x) &= g(x) - f(x) \\
&= \left(3 - \frac{1}{x}\right) - \left(6x^2 - 2x\right) \\
&= 3 - \frac{1}{x} - 6x^2 + 2x \\
&= \frac{3x}{x} - \frac{1}{x} - \frac{6x^3}{x} + \frac{2x^2}{x} \qquad \text{get common denominators} \\
&= \frac{3x - 1 - 6x^3 - 2x^2}{x} \\
&= \frac{-6x^3 - 2x^2 + 3x - 1}{x}
\end{aligned}$$

4. As in the previous example, we have two ways to approach finding the domain of $\frac{g}{f}$. First, we can find the domain of g and f separately, and find the intersection of these two sets. In addition, since $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$, we are introducing a new denominator, namely $f(x)$, so we need to guard against this being 0 as well. Our previous work tells us that the domain of g is $(-\infty, 0) \cup (0, \infty)$ and the domain of f is $(-\infty, \infty)$. Setting $f(x) = 0$ gives $6x^2 - 2x = 0$ or $x = 0, \frac{1}{3}$. As a result, the domain of $\frac{g}{f}$ is all real numbers except $x = 0$ and $x = \frac{1}{3}$, or $(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$.

   Alternatively, we may proceed as above and analyze the expression $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$ *before* simplifying. In this case,

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{3 - \frac{1}{x}}{6x^2 - 2x}$$

   We see immediately from the 'little' denominator that $x \neq 0$. To keep the 'big' denominator away from 0, we solve $6x^2 - 2x = 0$ and get $x = 0$ or

$x = \frac{1}{3}$. Hence, as before, we find the domain of $\frac{g}{f}$ to be

$$(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right).$$

Next, we find and simplify a formula for $\left(\frac{g}{f}\right)(x)$.

$$\begin{aligned}
\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} \\
&= \frac{3 - \frac{1}{x}}{6x^2 - 2x} \\
&= \frac{3 - \frac{1}{x}}{6x^2 - 2x} \cdot \frac{x}{x} && \text{simplify compound fractions} \\
&= \frac{\left(3 - \frac{1}{x}\right) x}{(6x^2 - 2x)\, x} \\
&= \frac{3x - 1}{(6x^2 - 2x)\, x} \\
&= \frac{3x - 1}{2x^2(3x - 1)} && \text{factor} \\
&= \frac{\cancelto{1}{(3x - 1)}}{2x^2\cancel{(3x - 1)}} && \text{cancel} \\
&= \frac{1}{2x^2}
\end{aligned}$$

Please note the importance of finding the domain of a function *before* simplifying its expression. In number 4 in Example 27 above, had we waited to find the domain of $\frac{g}{f}$ until after simplifying, we'd just have the formula $\frac{1}{2x^2}$ to go by, and we would (incorrectly!) state the domain as $(-\infty, 0) \cup (0, \infty)$, since the other troublesome number, $x = \frac{1}{3}$, was cancelled away.

We'll see what cancelling factors means geometrically in Chapter **??**.

Next, we turn our attention to the **difference quotient** of a function.

Definition 25 Difference quotient of a function

Given a function f, the **difference quotient** of f is the expression

$$\frac{f(x+h) - f(x)}{h}$$

We will revisit this concept in Section 3.1, but for now, we use it as a way to practice function notation and function arithmetic. For reasons which will become clear in Calculus, 'simplifying' a difference quotient means rewriting it in a form where the 'h' in the definition of the difference quotient cancels from the denominator. Once that happens, we consider our work to be done.

Example 28 Computing difference quotients

Find and simplify the difference quotients for the following functions

1. $f(x) = x^2 - x - 2$
2. $g(x) = \dfrac{3}{2x+1}$
3. $r(x) = \sqrt{x}$

Solution

1. To find $f(x+h)$, we replace every occurrence of x in the formula $f(x) = x^2 - x - 2$ with the quantity $(x+h)$ to get

$$\begin{aligned} f(x+h) &= (x+h)^2 - (x+h) - 2 \\ &= x^2 + 2xh + h^2 - x - h - 2. \end{aligned}$$

So the difference quotient is

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\left(x^2 + 2xh + h^2 - x - h - 2\right) - \left(x^2 - x - 2\right)}{h} \\ &= \frac{x^2 + 2xh + h^2 - x - h - 2 - x^2 + x + 2}{h} \\ &= \frac{2xh + h^2 - h}{h} \\ &= \frac{h\,(2x + h - 1)}{h} && \text{factor} \\ &= \frac{\cancel{h}\,(2x + h - 1)}{\cancel{h}} && \text{cancel} \\ &= 2x + h - 1. \end{aligned}$$

2. To find $g(x+h)$, we replace every occurrence of x in the formula $g(x) = \frac{3}{2x+1}$ with the quantity $(x+h)$ to get

$$\begin{aligned} g(x+h) &= \frac{3}{2(x+h)+1} \\ &= \frac{3}{2x+2h+1}, \end{aligned}$$

which yields

$$\begin{aligned} \frac{g(x+h) - g(x)}{h} &= \frac{\frac{3}{2x+2h+1} - \frac{3}{2x+1}}{h} \\ &= \frac{\frac{3}{2x+2h+1} - \frac{3}{2x+1}}{h} \cdot \frac{(2x+2h+1)(2x+1)}{(2x+2h+1)(2x+1)} \end{aligned}$$

$$\begin{aligned}
&= \frac{3(2x+1)-3(2x+2h+1)}{h(2x+2h+1)(2x+1)} \\
&= \frac{6x+3-6x-6h-3}{h(2x+2h+1)(2x+1)} \\
&= \frac{-6h}{h(2x+2h+1)(2x+1)} \\
&= \frac{-6\cancel{h}}{\cancel{h}(2x+2h+1)(2x+1)} \\
&= \frac{-6}{(2x+2h+1)(2x+1)}.
\end{aligned}$$

Since we have managed to cancel the original 'h' from the denominator, we are done.

3. For $r(x) = \sqrt{x}$, we get $r(x+h) = \sqrt{x+h}$ so the difference quotient is

$$\frac{r(x+h)-r(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

In order to cancel the 'h' from the denominator, we rationalize the *numerator* by multiplying by its conjugate.

$$\begin{aligned}
\frac{r(x+h)-r(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \frac{\left(\sqrt{x+h}-\sqrt{x}\right)}{h}\cdot\frac{\left(\sqrt{x+h}+\sqrt{x}\right)}{\left(\sqrt{x+h}+\sqrt{x}\right)} \quad \text{Multiply by the conjugate.} \\
&= \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \quad \text{Difference of Squares.} \\
&= \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{\overset{1}{\cancel{h}}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{1}{\sqrt{x+h}+\sqrt{x}}
\end{aligned}$$

Since we have removed the original 'h' from the denominator, we are done.

As mentioned before, we will revisit difference quotients in Section 3.1 where we will explain them geometrically. For now, we want to move on to some classic applications of function arithmetic from Economics and for that, we need to think like an entrepreneur.

Suppose you are a manufacturer making a certain product. Let x be the **production level**, that is, the number of items produced in a given time period. It is customary to let $C(x)$ denote the function which calculates the total **cost** of producing the x items. The quantity $C(0)$, which represents the cost of producing no items, is called the **fixed** cost, and represents the amount of money required to begin production. Associated with the total cost $C(x)$ is cost per item, or **average cost**, denoted $\overline{C}(x)$ and read 'C-bar' of x. To compute $\overline{C}(x)$, we take the total cost $C(x)$ and divide by the number of items produced x to get

$$\overline{C}(x) = \frac{C(x)}{x}$$

On the retail end, we have the **price** p charged per item. To simplify the dialogue and computations in this text, we assume that *the number of items sold equals the number of items produced*. From a retail perspective, it seems natural to think of the number of items sold, x, as a function of the price charged, p. After all, the retailer can easily adjust the price to sell more product. In the language of functions, x would be the *dependent* variable and p would be the *independent* variable or, using function notation, we have a function $x(p)$. While we will adopt this convention later in the text, (see Example **??** in Section **??**) we will hold with tradition at this point and consider the price p as a function of the number of items sold, x. That is, we regard x as the independent variable and p as the dependent variable and speak of the **price-demand** function, $p(x)$. Hence, $p(x)$ returns the price charged per item when x items are produced and sold. Our next function to consider is the **revenue** function, $R(x)$. The function $R(x)$ computes the amount of money collected as a result of selling x items. Since $p(x)$ is the price charged per item, we have $R(x) = xp(x)$. Finally, the **profit** function, $P(x)$ calculates how much money is earned after the costs are paid. That is, $P(x) = (R - C)(x) = R(x) - C(x)$. We summarize all of these functions below.

Key Idea 12 Summary of Common Economic Functions

Suppose x represents the quantity of items produced and sold.

- The price-demand function $p(x)$ calculates the price per item.
- The revenue function $R(x)$ calculates the total money collected by selling x items at a price $p(x)$, $R(x) = x\,p(x)$.
- The cost function $C(x)$ calculates the cost to produce x items. The value $C(0)$ is called the fixed cost or start-up cost.
- The average cost function $\overline{C}(x) = \frac{C(x)}{x}$ calculates the cost per item when making x items. Here, we necessarily assume $x > 0$.
- The profit function $P(x)$ calculates the money earned after costs are paid when x items are produced and sold, $P(x) = (R - C)(x) = R(x) - C(x)$.

Example 29 Computing (and interpreting) cost and profit functions

Let x represent the number of dOpi media players ('dOpis' – pronounced 'dopeys' ...) produced and sold in a typical week. Suppose the cost, in dollars, to produce x dOpis is given by $C(x) = 100x + 2000$, for $x \geq 0$, and the price, in dollars per dOpi, is given by $p(x) = 450 - 15x$ for $0 \leq x \leq 30$.

1. Find and interpret $C(0)$.
2. Find and interpret $\overline{C}(10)$.
3. Find and interpret $p(0)$ and $p(20)$.
4. Solve $p(x) = 0$ and interpret the result.
5. Find and simplify expressions for the revenue function $R(x)$ and the profit function $P(x)$.
6. Find and interpret $R(0)$ and $P(0)$.
7. Solve $P(x) = 0$ and interpret the result.

Solution

1. We substitute $x = 0$ into the formula for $C(x)$ and get $C(0) = 100(0) + 2000 = 2000$. This means to produce 0 dOpis, it costs $\$2000$. In other words, the fixed (or start-up) costs are $\$2000$. The reader is encouraged to contemplate what sorts of expenses these might be.
2. Since $\overline{C}(x) = \frac{C(x)}{x}$, $\overline{C}(10) = \frac{C(10)}{10} = \frac{3000}{10} = 300$. This means when 10 dOpis are produced, the cost to manufacture them amounts to $\$300$ per dOpi.
3. Plugging $x = 0$ into the expression for $p(x)$ gives $p(0) = 450 - 15(0) = 450$. This means no dOpis are sold if the price is $\$450$ per dOpi. On the other hand, $p(20) = 450 - 15(20) = 150$ which means to sell 20 dOpis in a typical week, the price should be set at $\$150$ per dOpi.
4. Setting $p(x) = 0$ gives $450 - 15x = 0$. Solving gives $x = 30$. This means in order to sell 30 dOpis in a typical week, the price needs to be set to $\$0$. What's more, this means that even if dOpis were given away for free, the retailer would only be able to move 30 of them.
5. To find the revenue, we compute $R(x) = xp(x) = x(450 - 15x) = 450x - 15x^2$. Since the formula for $p(x)$ is valid only for $0 \leq x \leq 30$, our formula $R(x)$ is also restricted to $0 \leq x \leq 30$. For the profit, $P(x) = (R - C)(x) = R(x) - C(x)$. Using the given formula for $C(x)$ and the derived formula for $R(x)$, we get $P(x) = \left(450x - 15x^2\right) - (100x + 2000) = -15x^2 + 350x - 2000$. As before, the validity of this formula is for $0 \leq x \leq 30$ only.
6. We find $R(0) = 0$ which means if no dOpis are sold, we have no revenue, which makes sense. Turning to profit, $P(0) = -2000$ since $P(x) = R(x) - C(x)$ and $P(0) = R(0) - C(0) = -2000$. This means that if no dOpis are sold, more money ($\$2000$ to be exact!) was put into producing the dOpis than was recouped in sales. In number 1, we found the fixed costs to be $\$2000$, so it makes sense that if we sell no dOpis, we are out those start-up costs.
7. Setting $P(x) = 0$ gives $-15x^2 + 350x - 2000 = 0$. Factoring gives $-5(x - 10)(3x - 40) = 0$ so $x = 10$ or $x = \frac{40}{3}$. What do these values mean in the context of the problem? Since $P(x) = R(x) - C(x)$, solving $P(x) = 0$ is the same as solving $R(x) = C(x)$. This means that the solutions to $P(x) = 0$ are the production (and sales) figures for which the sales revenue exactly balances the total production costs. These are the so-called '**break even**' points. The solution $x = 10$ means 10 dOpis should be produced (and

sold) during the week to recoup the cost of production. For $x = \frac{40}{3} = 13.\overline{3}$, things are a bit more complicated. Even though $x = 13.\overline{3}$ satisfies $0 \leq x \leq 30$, and hence is in the domain of P, it doesn't make sense in the context of this problem to produce a fractional part of a dOpi. Evaluating $P(13) = 15$ and $P(14) = -40$, we see that producing and selling 13 dOpis per week makes a (slight) profit, whereas producing just one more puts us back into the red. While breaking even is nice, we ultimately would like to find what production level (and price) will result in the largest profit, and we'll do just that ...in Section 3.3.

Recall from Section 2.3.1 that in problems such as this, it is necessary to take the **applied domain** of the function into account.

Exercises 2.4

Problems

In Exercises 1 – 10, use the pair of functions f and g to find the following values if they exist:

- $(f+g)(2)$
- $(f-g)(-1)$
- $(g-f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

1. $f(x) = 3x + 1$ and $g(x) = 4 - x$
2. $f(x) = x^2$ and $g(x) = -2x + 1$
3. $f(x) = x^2 - x$ and $g(x) = 12 - x^2$
4. $f(x) = 2x^3$ and $g(x) = -x^2 - 2x - 3$
5. $f(x) = \sqrt{x+3}$ and $g(x) = 2x - 1$
6. $f(x) = \sqrt{4-x}$ and $g(x) = \sqrt{x+2}$
7. $f(x) = 2x$ and $g(x) = \dfrac{1}{2x+1}$
8. $f(x) = x^2$ and $g(x) = \dfrac{3}{2x-3}$
9. $f(x) = x^2$ and $g(x) = \dfrac{1}{x^2}$
10. $f(x) = x^2 + 1$ and $g(x) = \dfrac{1}{x^2+1}$

In Exercises 11 – 20, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f+g)(x)$
- $(f-g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

11. $f(x) = 2x + 1$ and $g(x) = x - 2$
12. $f(x) = 1 - 4x$ and $g(x) = 2x - 1$
13. $f(x) = x^2$ and $g(x) = 3x - 1$
14. $f(x) = x^2 - x$ and $g(x) = 7x$
15. $f(x) = x^2 - 4$ and $g(x) = 3x + 6$
16. $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$
17. $f(x) = \dfrac{x}{2}$ and $g(x) = \dfrac{2}{x}$
18. $f(x) = x - 1$ and $g(x) = \dfrac{1}{x-1}$
19. $f(x) = x$ and $g(x) = \sqrt{x+1}$
20. $f(x) = \sqrt{x-5}$ and $g(x) = f(x) = \sqrt{x-5}$

In Exercises 21 – 45, find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ for the given function.

21. $f(x) = 2x - 5$
22. $f(x) = -3x + 5$
23. $f(x) = 6$
24. $f(x) = 3x^2 - x$
25. $f(x) = -x^2 + 2x - 1$
26. $f(x) = 4x^2$
27. $f(x) = x - x^2$
28. $f(x) = x^3 + 1$
29. $f(x) = mx + b$ where $m \neq 0$
30. $f(x) = ax^2 + bx + c$ where $a \neq 0$
31. $f(x) = \dfrac{2}{x}$
32. $f(x) = \dfrac{3}{1-x}$
33. $f(x) = \dfrac{1}{x^2}$
34. $f(x) = \dfrac{2}{x+5}$
35. $f(x) = \dfrac{1}{4x-3}$
36. $f(x) = \dfrac{3x}{x+1}$
37. $f(x) = \dfrac{x}{x-9}$
38. $f(x) = \dfrac{x^2}{2x+1}$
39. $f(x) = \sqrt{x-9}$
40. $f(x) = \sqrt{2x+1}$
41. $f(x) = \sqrt{-4x+5}$
42. $f(x) = \sqrt{4-x}$

43. $f(x) = \sqrt{ax+b}$, where $a \neq 0$.

44. $f(x) = x\sqrt{x}$

45. $f(x) = \sqrt[3]{x}$. **HINT:** $(a-b)\left(a^2+ab+b^2\right) = a^3 - b^3$

In Exercises 46 – 50, $C(x)$ denotes the cost to produce x items and $p(x)$ denotes the price-demand function in the given economic scenario. In each Exercise, do the following:

- **Find and interpret $C(0)$.**
- **Find and interpret $\overline{C}(10)$.**
- **Find and interpret $p(5)$**
- **Find and simplify $R(x)$.**
- **Find and simplify $P(x)$.**
- **Solve $P(x) = 0$ and interpret.**

46. The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, $0 \leq x \leq 15$.

47. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x$, $0 \leq x \leq 35$.

48. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is $C(x) = 18x + 240$, $x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x$, $0 \leq x \leq 30$.

49. The daily cost, in dollars, to produce x Sasquatch Berry Pies $C(x) = 3x + 36$, $x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x$, $0 \leq x \leq 24$.

50. The monthly cost, in hundreds of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000$, $x \geq 0$ and the price-demand function, in hundreds of dollars per scooter, is $p(x) = 140 - 2x$, $0 \leq x \leq 70$.

In Exercises 51 – 64, let f be the function defined by

$$f = \{(-3,4), (-2,2), (-1,0), (0,1), (1,3), (2,4), (3,-1)\}$$

and let g be the function defined

$$g = \{(-3,-2), (-2,0), (-1,-4), (0,0), (1,-3), (2,1), (3,2)\}.$$

Compute the indicated value if it exists.

51. $(f+g)(-3)$

52. $(f-g)(2)$

53. $(fg)(-1)$

54. $(g+f)(1)$

55. $(g-f)(3)$

56. $(gf)(-3)$

57. $\left(\frac{f}{g}\right)(-2)$

58. $\left(\frac{f}{g}\right)(-1)$

59. $\left(\frac{f}{g}\right)(2)$

60. $\left(\frac{g}{f}\right)(-1)$

61. $\left(\frac{g}{f}\right)(3)$

62. $\left(\frac{g}{f}\right)(-3)$

2.5 Graphs of Functions

In Section 2.2 we defined a function as a special type of relation; one in which each x-coordinate was matched with only one y-coordinate. We spent most of our time in that section looking at functions graphically because they were, after all, just sets of points in the plane. Then in Section 2.3 we described a function as a process and defined the notation necessary to work with functions algebraically. So now it's time to look at functions graphically again, only this time we'll do so with the notation defined in Section 2.3. We start with what should not be a surprising connection.

Key Idea 13 The Fundamental Graphing Principle for Functions

The graph of a function f is the set of points which satisfy the equation $y = f(x)$. That is, the point (x, y) is on the graph of f if and only if $y = f(x)$.

Example 30 Graphing a function

Graph $f(x) = x^2 - x - 6$.

x	$f(x)$	$(x, f(x))$
-3	6	$(-3, 6)$
-2	0	$(-2, 0)$
-1	-4	$(-1, -4)$
0	-6	$(0, -6)$
1	-6	$(1, -6)$
2	-4	$(2, -4)$
3	0	$(3, 0)$
4	6	$(4, 6)$

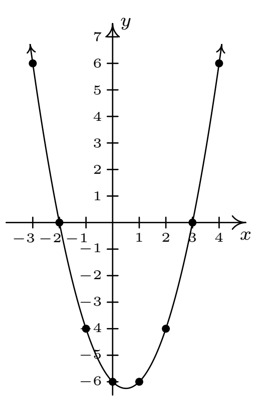


Figure 2.21: Graphing the function $f(x) = x^2 - x - 6$

Solution To graph f, we graph the equation $y = f(x)$. To this end, we use the techniques outlined in Section 2.1.1. Specifically, we check for intercepts, test for symmetry, and plot additional points as needed. To find the x-intercepts, we set $y = 0$. Since $y = f(x)$, this means $f(x) = 0$.

$$\begin{array}{rcll} f(x) & = & x^2 - x - 6 & \\ 0 & = & x^2 - x - 6 & \\ 0 & = & (x-3)(x+2) & \text{factor} \\ x - 3 = 0 & \text{or} & x + 2 = 0 & \\ x & = & -2, 3 & \end{array}$$

So we get $(-2, 0)$ and $(3, 0)$ as x-intercepts. To find the y-intercept, we set $x = 0$. Using function notation, this is the same as finding $f(0)$ and $f(0) = 0^2 - 0 - 6 = -6$. Thus the y-intercept is $(0, -6)$. As far as symmetry is concerned, we can tell from the intercepts that the graph possesses none of the three symmetries discussed thus far. (You should verify this.) We can make a table analogous to the ones we made in Section 2.1.1, plot the points and connect the dots in a somewhat pleasing fashion to get the graph shown in Figure 2.21.

Graphing piecewise-defined functions is a bit more of a challenge.

Example 31 Graphing a piecewise-defined function

Graph: $f(x) = \begin{cases} 4 - x^2 & \text{if } x < 1 \\ x - 3, & \text{if } x \geq 1 \end{cases}$

Solution We proceed as before – finding intercepts, testing for symmetry and then plotting additional points as needed. To find the x-intercepts, as before, we set $f(x) = 0$. The twist is that we have two formulas for $f(x)$. For $x < 1$, we use the formula $f(x) = 4 - x^2$. Setting $f(x) = 0$ gives $0 = 4 - x^2$, so that $x = \pm 2$. However, of these two answers, only $x = -2$ fits in the domain $x < 1$ for this piece. This means the only x-intercept for the $x < 1$ region of the x-axis is $(-2, 0)$. For $x \geq 1$, $f(x) = x - 3$. Setting $f(x) = 0$ gives $0 = x - 3$, or $x = 3$. Since $x = 3$ satisfies the inequality $x \geq 1$, we get $(3, 0)$ as another

x-intercept. Next, we seek the y-intercept. Notice that $x = 0$ falls in the domain $x < 1$. Thus $f(0) = 4 - 0^2 = 4$ yields the y-intercept $(0, 4)$. As far as symmetry is concerned, you can check that the equation $y = 4 - x^2$ is symmetric about the y-axis; unfortunately, this equation (and its symmetry) is valid only for $x < 1$. You can also verify $y = x - 3$ possesses none of the symmetries discussed in the Section 2.1.1. When plotting additional points, it is important to keep in mind the restrictions on x for each piece of the function. The sticking point for this function is $x = 1$, since this is where the equations change. When $x = 1$, we use the formula $f(x) = x - 3$, so the point on the graph $(1, f(1))$ is $(1, -2)$. However, for all values less than 1, we use the formula $f(x) = 4 - x^2$. As we have discussed earlier in Section 2.1, there is no real number which immediately precedes $x = 1$ on the number line. Thus for the values $x = 0.9$, $x = 0.99$, $x = 0.999$, and so on, we find the corresponding y values using the formula $f(x) = 4 - x^2$. Making a table as before, we see that as the x values sneak up to $x = 1$ in this fashion, the $f(x)$ values inch closer and closer to $4 - 1^2 = 3$. To indicate this graphically, we use an open circle at the point $(1, 3)$. Putting all of this information together and plotting additional points, we get the result in Figure 2.22.

x	$f(x)$	$(x, f(x))$
0.9	3.19	$(0.9, 3.19)$
0.99	≈ 3.02	$(0.99, 3.02)$
0.999	≈ 3.002	$(0.999, 3.002)$

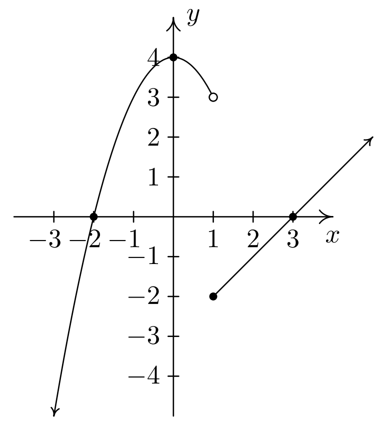


Figure 2.22: The graph of $f(x)$ from Example 31

In the previous two examples, the x-coordinates of the x-intercepts of the graph of $y = f(x)$ were found by solving $f(x) = 0$. For this reason, they are called the **zeros** of f.

Definition 26 Zeros of a function

The **zeros** of a function f are the solutions to the equation $f(x) = 0$. In other words, x is a zero of f if and only if $(x, 0)$ is an x-intercept of the graph of $y = f(x)$.

Of the three symmetries discussed in Section 2.1.1, only two are of significance to functions: symmetry about the y-axis and symmetry about the origin. Recall that we can test whether the graph of an equation is symmetric about the y-axis by replacing x with $-x$ and checking to see if an equivalent equation results. If we are graphing the equation $y = f(x)$, substituting $-x$ for x results in the equation $y = f(-x)$. In order for this equation to be equivalent to the original equation $y = f(x)$ we need $f(-x) = f(x)$. In a similar fashion, we recall that to test an equation's graph for symmetry about the origin, we replace x and y with $-x$ and $-y$, respectively. Doing this substitution in the equation $y = f(x)$ results in $-y = f(-x)$. Solving the latter equation for y gives $y = -f(-x)$. In order for this equation to be equivalent to the original equation $y = f(x)$ we need $-f(-x) = f(x)$, or, equivalently, $f(-x) = -f(x)$. These results are summarized below.

Key Idea 14 Testing the Graph of a Function for Symmetry

The graph of a function f is symmetric

- about the y-axis if and only if $f(-x) = f(x)$ for all x in the domain of f.
- about the origin if and only if $f(-x) = -f(x)$ for all x in the domain of f.

Note that for graphs of functions, we don't bother to discuss symmetry about the x-axis. Why do you suppose this is?

For reasons which won't become clear until we study polynomials, we call a function **even** if its graph is symmetric about the y-axis or **odd** if its graph is symmetric about the origin. Apart from a very specialized family of functions which are both even and odd, (any ideas?) functions fall into one of three distinct categories: even, odd, or neither even nor odd.

A good resource when you need to quickly check something like the graph of a function is Wolfram Alpha.
If you want a good (and free!) program you can run locally on a computer or tablet, we recommend trying Geogebra. It's free to download, works on all major operating systems, and it's pretty easy to figure out the basics.

Example 32 Even and odd functions

Determine analytically if the following functions are even, odd, or neither even nor odd. Verify your result with a graphing calculator or computer software.

1. $f(x) = \dfrac{5}{2-x^2}$
2. $g(x) = \dfrac{5x}{2-x^2}$
3. $h(x) = \dfrac{5x}{2-x^3}$
4. $i(x) = \dfrac{5x}{2x-x^3}$
5. $j(x) = x^2 - \dfrac{x}{100} - 1$
6. $p(x) = \begin{cases} rclx+3 & \text{if } x<0 \\ -x+3, & \text{if } x \geq 0 \end{cases}.$

Solution The first step in all of these problems is to replace x with $-x$ and simplify.

1.

$$\begin{aligned} f(x) &= \frac{5}{2-x^2} \\ f(-x) &= \frac{5}{2-(-x)^2} \\ f(-x) &= \frac{5}{2-x^2} \\ f(-x) &= f(x) \end{aligned}$$

Hence, f is **even**. A plot of $f(x)$ using GeoGebra is given in Figure 2.23.

This suggests that the graph of f is symmetric about the y-axis, as expected.

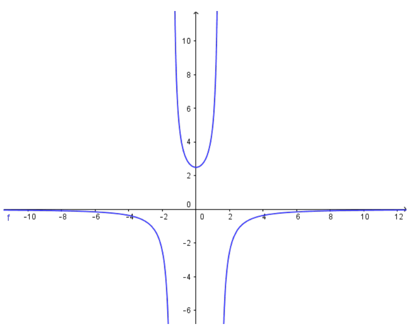


Figure 2.23: The graph of $f(x)$ in Example 32

While the plot provided by the software can provide us with visual evidence that a function is even or odd, this evidence is never conclusive. The only way to know for sure is to check analytically using the definitions of even and odd functions.

2.

$$\begin{aligned} g(x) &= \frac{5x}{2-x^2} \\ g(-x) &= \frac{5(-x)}{2-(-x)^2} \\ g(-x) &= \frac{-5x}{2-x^2} \end{aligned}$$

It doesn't appear that $g(-x)$ is equivalent to $g(x)$. To prove this, we check with an x value. After some trial and error, we see that $g(1) = 5$ whereas $g(-1) = -5$. This proves that g is not even, but it doesn't rule out the possibility that g is odd. (Why not?) To check if g is odd, we compare $g(-x)$ with $-g(x)$

$$\begin{aligned} -g(x) &= -\frac{5x}{2-x^2} \\ &= \frac{-5x}{2-x^2} \\ -g(x) &= g(-x) \end{aligned}$$

Hence, g is odd: see Figure 2.24.

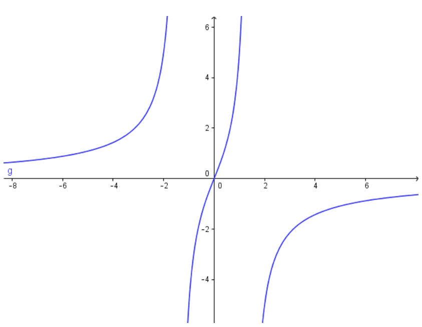


Figure 2.24: The graph of $g(x)$ in Example 32

3.

$$\begin{aligned} h(x) &= \frac{5x}{2-x^3} \\ h(-x) &= \frac{5(-x)}{2-(-x)^3} \\ h(-x) &= \frac{-5x}{2+x^3} \end{aligned}$$

Once again, $h(-x)$ doesn't appear to be equivalent to $h(x)$. We check with an x value, for example, $h(1) = 5$ but $h(-1) = -\frac{5}{3}$. This proves that h is not even and it also shows h is not odd. (Why?)

In Figure 2.25, the graph of h appears to be neither symmetric about the y-axis nor the origin.

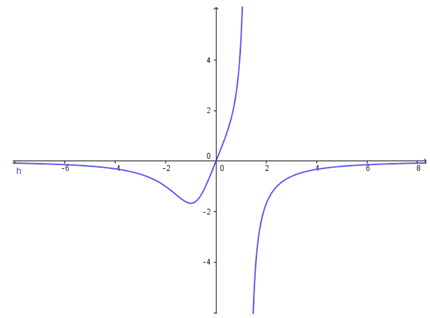


Figure 2.25: The graph of $h(x)$ in Example 32

4.

$$\begin{aligned} i(x) &= \frac{5x}{2x-x^3} \\ i(-x) &= \frac{5(-x)}{2(-x)-(-x)^3} \\ i(-x) &= \frac{-5x}{-2x+x^3} \end{aligned}$$

The expression $i(-x)$ doesn't appear to be equivalent to $i(x)$. However, after checking some x values, for example $x = 1$ yields $i(1) = 5$ and $i(-1) = 5$, it appears that $i(-x)$ does, in fact, equal $i(x)$. However, while this suggests i is even, it doesn't prove it. (It does, however, prove i is not odd.) To prove $i(-x) = i(x)$, we need to manipulate our expressions for $i(x)$ and $i(-x)$ and show that they are equivalent. A clue as to how to proceed is in the numerators: in the formula for $i(x)$, the numerator is $5x$ and in $i(-x)$ the numerator is $-5x$. To re-write $i(x)$ with a numerator of $-5x$, we need to multiply its numerator by -1. To keep the value of the fraction the same, we need to multiply the denominator by -1 as well. Thus

$$\begin{aligned} i(x) &= \frac{5x}{2x-x^3} \\ &= \frac{(-1)5x}{(-1)\left(2x-x^3\right)} \\ &= \frac{-5x}{-2x+x^3} \end{aligned}$$

Hence, $i(x) = i(-x)$, so i is even. See Figure 2.26 for the graph.

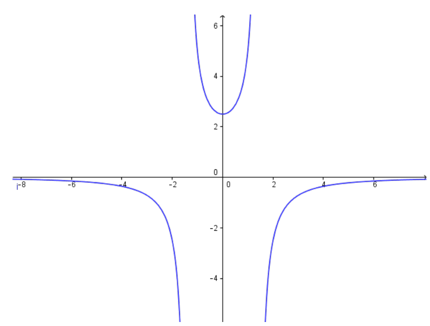


Figure 2.26: The graph of $i(x)$ in Example 32

5.

$$\begin{aligned} j(x) &= x^2 - \frac{x}{100} - 1 \\ j(-x) &= (-x)^2 - \frac{-x}{100} - 1 \\ j(-x) &= x^2 + \frac{x}{100} - 1 \end{aligned}$$

The expression for $j(-x)$ doesn't seem to be equivalent to $j(x)$, so we check using $x = 1$ to get $j(1) = -\frac{1}{100}$ and $j(-1) = \frac{1}{100}$. This rules out j being even. However, it doesn't rule out j being odd. Examining $-j(x)$ gives

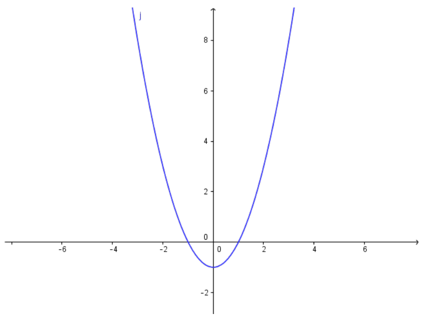


Figure 2.27: The graph of $j(x)$ in Example 32

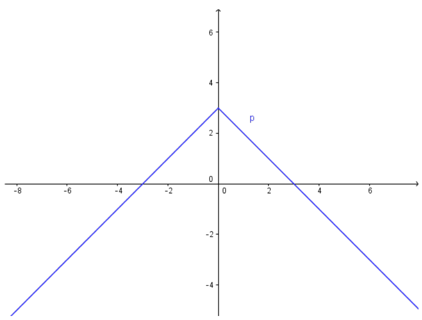


Figure 2.28: The graph of $p(x)$ in Example 32

$$\begin{aligned} j(x) &= x^2 - \frac{x}{100} - 1 \\ -j(x) &= -\left(x^2 - \frac{x}{100} - 1\right) \\ -j(x) &= -x^2 + \frac{x}{100} + 1 \end{aligned}$$

The expression $-j(x)$ doesn't seem to match $j(-x)$ either. Testing $x = 2$ gives $j(2) = \frac{149}{50}$ and $j(-2) = \frac{151}{50}$, so j is not odd, either.

Notice in Figure 2.27 that the computer plot seems to suggests that the graph of j is symmetric about the y-axis which would imply that j is even. However, we have proven that is not the case. The problem is that the effect of the $x/100$ term is so small, our eyes don't detect it in the graph.

6. Testing the graph of $y = p(x)$ for symmetry is complicated by the fact $p(x)$ is a piecewise-defined function. As always, we handle this by checking the condition for symmetry by checking it on each piece of the domain. We first consider the case when $x < 0$ and set about finding the correct expression for $p(-x)$. Even though $p(x) = x+3$ for $x < 0$, $p(-x) \neq -x+3$ here. The reason for this is that since $x < 0$, $-x > 0$ which means to find $p(-x)$, we need to use the *other* formula for $p(x)$, namely $p(x) = -x+3$. Hence, for $x < 0$, $p(-x) = -(-x) + 3 = x + 3 = p(x)$. For $x \geq 0$, $p(x) = -x+3$ and we have two cases. If $x > 0$, then $-x < 0$ so $p(-x) = (-x) + 3 = -x + 3 = p(x)$. If $x = 0$, then $p(0) = 3 = p(-0)$. Hence, in all cases, $p(-x) = p(x)$, so p is even. Since $p(0) = 3$ but $p(-0) = p(0) = 3 \neq -3$, we also have p is not odd.

   In Figure 2.28, we see that the graph appears to be symmetric about the y-axis.

There are two lessons to be learned from the last example. The first is that sampling function values at particular x values is not enough to prove that a function is even or odd $-$ despite the fact that $j(-1) = -j(1)$, j turned out not to be odd. Secondly, while the calculator may *suggest* mathematical truths, it is the Algebra which *proves* mathematical truths. (Or, in other words, don't rely too heavily on the machine!)

2.5.1 General Function Behaviour

The last topic we wish to address in this section is general function behaviour. As you shall see in the next several chapters, each family of functions has its own unique attributes and we will study them all in great detail. The purpose of this section's discussion, then, is to lay the foundation for that further study by investigating aspects of function behaviour which apply to all functions. To start, we will examine the concepts of **increasing**, **decreasing** and **constant**. Before defining the concepts algebraically, it is instructive to first look at them graphically. Consider the graph of the function f in Figure 2.29.

Reading from left to right, the graph 'starts' at the point $(-4, -3)$ and 'ends' at the point $(6, 5.5)$. If we imagine walking from left to right on the graph, between $(-4, -3)$ and $(-2, 4.5)$, we are walking 'uphill'; then between $(-2, 4.5)$ and $(3, -8)$, we are walking 'downhill'; and between $(3, -8)$ and $(4, -6)$, we are walking 'uphill' once more. From $(4, -6)$ to $(5, -6)$, we 'level off', and then resume walking 'uphill' from $(5,-6)$ to $(6,5.5)$. In other words, for the x values between -4 and -2 (inclusive), the y-coordinates on the graph are getting larger, or **increasing**, as we move from left to right. Since $y = f(x)$, the y values on the graph are the function values, and we say that the function f is **increasing** on the interval $[-4,-2]$. Analogously, we say that f is **decreasing** on the interval $[-2,3]$ increasing once more on the interval $[3,4]$, **constant** on $[4,5]$, and finally increasing once again on $[5,6]$. It is extremely important to notice that the behaviour (increasing, decreasing or constant) occurs on an interval on the x-axis. When we say that the function f is increasing on $[-4,-2]$ we do not mention the actual y values that f attains along the way. Thus, we report *where* the behaviour occurs, not to what extent the behaviour occurs. Also notice that we do not say that a function is increasing, decreasing or constant at a single x value. In fact, we would run into serious trouble in our previous example if we tried to do so because $x = -2$ is contained in an interval on which f was increasing and one on which it is decreasing. (There's more on this issue – and many others – in the Exercises.)

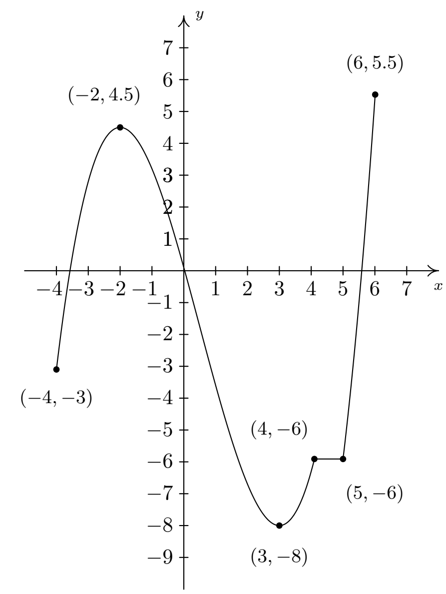


Figure 2.29: The graph $y = f(x)$

We're now ready for the more formal algebraic definitions of what it means for a function to be increasing, decreasing or constant.

Definition 27 Increasing, decreasing, and constant functions

Suppose f is a function defined on an interval I. We say f is:

- **increasing** on I if and only if $f(a) < f(b)$ for all real numbers a, b in I with $a < b$.
- **decreasing** on I if and only if $f(a) > f(b)$ for all real numbers a, b in I with $a < b$.
- **constant** on I if and only if $f(a) = f(b)$ for all real numbers a, b in I.

The notions of how quickly or how slowly a function increases or decreases are explored in Calculus.

It is worth taking some time to see that the algebraic descriptions of increasing, decreasing and constant as stated in Definition 27 agree with our graphical descriptions given earlier. You should look back through the examples and exercise sets in previous sections where graphs were given to see if you can determine the intervals on which the functions are increasing, decreasing or constant. Can you find an example of a function for which none of the concepts in Definition 27 apply?

Now let's turn our attention to a few of the points on the graph. Clearly the point $(-2,4.5)$ does not have the largest y value of all of the points on the graph of f — indeed that honour goes to $(6,5.5)$ — but $(-2,4.5)$ should get some sort of consolation prize for being 'the top of the hill' between $x = -4$ and $x = 3$. We say that the function f has a **local maximum** (or **relative maximum**) at the point $(-2,4.5)$, because the y-coordinate 4.5 is the largest y-value (hence, function value) on the curve 'near' $x = -2$. Similarly, we say that the function f has a **local minimum** (or **relative minimum**) at the point $(3,-8)$, since the y-coordinate -8 is the smallest function value near $x = 3$. Although it is tempting to say that local extrema occur when the function changes from increasing to decreasing or vice versa, it is not a precise enough way to define the concepts for the needs of Calculus. At the risk of being pedantic, we will present the traditional definitions and thoroughly vet the pathologies they induce in the Exercises. We have one

Typically, in (pre)calculus, whenever you're told that something occurs 'near' a given point, you should read this as 'on some open interval I containing that point'.

last observation to make before we proceed to the algebraic definitions and look at a fairly tame, yet helpful, example.

If we look at the entire graph, we see that the largest y value (the largest function value) is 5.5 at $x = 6$. In this case, we say the **maximum** (often called the 'absolute' or 'global' maximum) of f is 5.5; similarly, the **minimum** (again, 'absolute' or 'global' minimum can be used.) of f is -8.

'Maxima' is the plural of 'maximum' and 'mimima' is the plural of 'minimum'. 'Extrema' is the plural of 'extremum' which combines maximum and minimum.

We formalize these concepts in the following definitions.

Definition 28 Local maximum and minimum

Suppose f is a function with $f(a) = b$.

- We say f has a **local maximum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \geq f(x)$ for all x in I. The value $f(a) = b$ is called 'a local maximum value of f' in this case.
- We say f has a **local minimum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \leq f(x)$ for all x in I. The value $f(a) = b$ is called 'a local minimum value of f' in this case.
- The value b is called the **maximum** of f if $b \geq f(x)$ for all x in the domain of f.
- The value b is called the **minimum** of f if $b \leq f(x)$ for all x in the domain of f.

It's important to note that not every function will have all of these features. Indeed, it is possible to have a function with no local or absolute extrema at all! (Any ideas of what such a function's graph would have to look like?) We shall see examples of functions in the Exercises which have one or two, but not all, of these features, some that have instances of each type of extremum and some functions that seem to defy common sense. In all cases, though, we shall adhere to the algebraic definitions above as we explore the wonderful diversity of graphs that functions provide us.

Here is the 'tame' example which was promised earlier. It summarizes all of the concepts presented in this section as well as some from previous sections so you should spend some time thinking deeply about it before proceeding to the Exercises.

Example 33 A 'tame' example

Given the graph of $y = f(x)$ in Figure **??**, answer all of the following questions.

1. Find the domain of f.
2. Find the range of f.
3. List the x-intercepts, if any exist.
4. List the y-intercepts, if any exist.
5. Find the zeros of f.
6. Solve $f(x) < 0$.

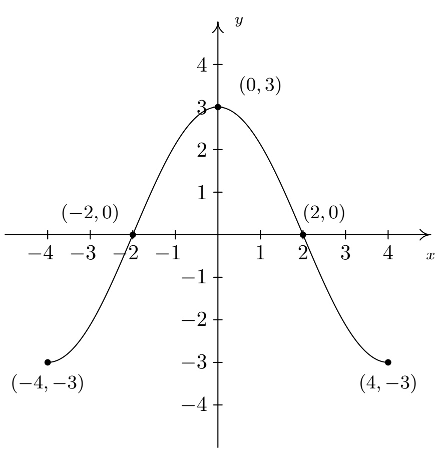


Figure 2.30: The graph for Example 33

7. Determine $f(2)$.
8. Solve $f(x) = -3$.
9. Find the number of solutions to $f(x) = 1$.
10. Does f appear to be even, odd, or neither?
11. List the intervals on which f is increasing.
12. List the intervals on which f is decreasing.
13. List the local maximums, if any exist.
14. List the local minimums, if any exist.
15. Find the maximum, if it exists.
16. Find the minimum, if it exists.

Solution

1. To find the domain of f, we proceed as in Section 2.2. By projecting the graph to the x-axis, we see that the portion of the x-axis which corresponds to a point on the graph is everything from -4 to 4, inclusive. Hence, the domain is $[-4, 4]$.
2. To find the range, we project the graph to the y-axis. We see that the y values from -3 to 3, inclusive, constitute the range of f. Hence, our answer is $[-3, 3]$.
3. The x-intercepts are the points on the graph with y-coordinate 0, namely $(-2, 0)$ and $(2, 0)$.
4. The y-intercept is the point on the graph with x-coordinate 0, namely $(0, 3)$.
5. The zeros of f are the x-coordinates of the x-intercepts of the graph of $y = f(x)$ which are $x = -2, 2$.
6. To solve $f(x) < 0$, we look for the x values of the points on the graph where the y-coordinate is less than 0. Graphically, we are looking for where the graph is below the x-axis. This happens for the x values from -4 to -2 and again from 2 to 4. So our answer is $[-4, -2) \cup (2, 4]$.
7. Since the graph of f is the graph of the equation $y = f(x)$, $f(2)$ is the y-coordinate of the point which corresponds to $x = 2$. Since the point $(2, 0)$ is on the graph, we have $f(2) = 0$.
8. To solve $f(x) = -3$, we look where $y = f(x) = -3$. We find two points with a y-coordinate of -3, namely $(-4, -3)$ and $(4, -3)$. Hence, the solutions to $f(x) = -3$ are $x = \pm 4$.
9. As in the previous problem, to solve $f(x) = 1$, we look for points on the graph where the y-coordinate is 1. Even though these points aren't specified, we see that the curve has two points with a y value of 1, as seen in the graph below. That means there are two solutions to $f(x) = 1$: see Figure 2.31.
10. The graph appears to be symmetric about the y-axis. This suggests (but does not prove) that f is even.
11. As we move from left to right, the graph rises from $(-4, -3)$ to $(0, 3)$. This means f is increasing on the interval $[-4, 0]$. (Remember, the answer here is an interval on the x-axis.)

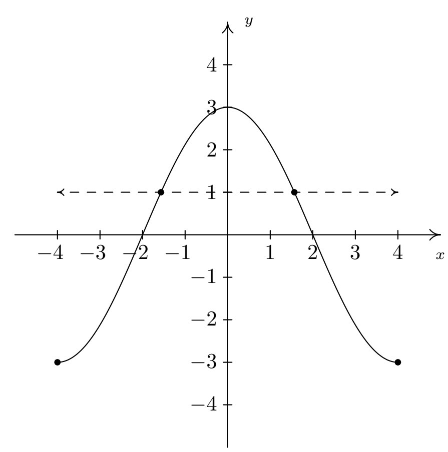


Figure 2.31: Solving $f(x) = 1$ in Example 33

12. As we move from left to right, the graph falls from $(0, 3)$ to $(4, -3)$. This means f is decreasing on the interval $[0, 4]$. (Remember, the answer here is an interval on the x-axis.)

13. The function has its only local maximum at $(0, 3)$ so $f(0) = 3$ is the local minimum value.

14. There are no local minimums. Why don't $(-4, -3)$ and $(4, -3)$ count? Let's consider the point $(-4, -3)$ for a moment. Recall that, in the definition of local minimum, there needs to be an open interval I which contains $x = -4$ such that $f(-4) < f(x)$ for all x in I different from -4. But if we put an open interval around $x = -4$ a portion of that interval will lie outside of the domain of f. Because we are unable to satisfy the requirements of the definition for a local minimum, we cannot claim that f has one at $(-4, -3)$. The point $(4, -3)$ fails for the same reason – no open interval around $x = 4$ stays within the domain of f.

15. The maximum value of f is the largest y-coordinate which is 3.

16. The minimum value of f is the smallest y-coordinate which is -3.

In general, the problem of finding maximum and minimum values, requires the techniques of Calculus. We will explore this in Chapter **??**. In the meantime, we'll have to rely on technology to assist us. Most graphing calculators and many mathematics software programs have 'Minimum' and 'Maximum' features which can be used to approximate these values, as we now demonstrate.

Example 34 Using the computer to find maxima and minima

Let $f(x) = \dfrac{15x}{x^2 + 3}$. Use the computer or a graphing calculator to approximate the intervals on which f is increasing and those on which it is decreasing. Approximate all extrema.

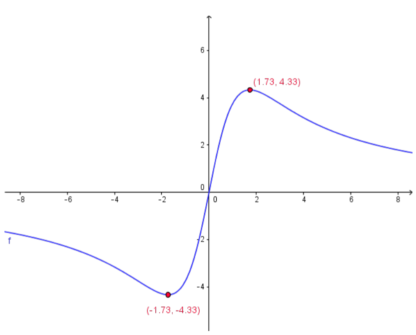


Figure 2.32: The local maximum and minimum of $f(x) = \dfrac{15x}{x^2 + 3}$ in Example 34

Solution Using GeoGebra, we enter `f(x) = 15x/(x^2+3)` to plot the graph of f. The command `Max[f,-3,3]` then calculates the maximum value of f on the interval $[-3, 3]$. Similarly, `Min[f,-3,3]` gives the minimum value of f on the interval $[-3, 3]$. The graph of f, together with the local maximum and local minimum, are plotted in Figure 2.32.

To two decimal places, f appears to have its only local minimum at $(-1.73, -4.33)$ and its only local maximum at $(1.73, 4.33)$. Given the symmetry about the origin suggested by the graph, the relation between these points shouldn't be too surprising. The function appears to be increasing on $[-1.73, 1.73]$ and decreasing on $(-\infty, -1.73] \cup [1.73, \infty)$. This makes -4.33 the (absolute) minimum and 4.33 the (absolute) maximum.

Example 35 Minimizing distance from a graph to the origin

Find the points on the graph of $y = (x - 3)^2$ which are closest to the origin. Round your answers to two decimal places.

Solution Suppose a point (x, y) is on the graph of $y = (x - 3)^2$. Its distance to the origin $(0, 0)$ is given by

$$\begin{aligned} d &= \sqrt{(x-0)^2+(y-0)^2} \\ &= \sqrt{x^2+y^2} \\ &= \sqrt{x^2+\left[(x-3)^2\right]^2} && \text{Since } y=(x-3)^2 \\ &= \sqrt{x^2+(x-3)^4} \end{aligned}$$

Given a value for x, the formula $d = \sqrt{x^2+(x-3)^4}$ is the distance from $(0,0)$ to the point (x,y) on the curve $y=(x-3)^2$. What we have defined, then, is a function $d(x)$ which we wish to minimize over all values of x. To accomplish this task analytically would require Calculus so as we've mentioned before, we can use a graphing calculator to find an approximate solution. Using Geogebra, we enter the function $d(x)$ as shown below and graph.

Using the Minimum feature, we see above on the right that the (absolute) minimum occurs near $x=2$. Rounding to two decimal places, we get that the minimum distance occurs when $x=2.00$. To find the y value on the parabola associated with $x=2.00$, we substitute 2.00 into the equation to get $y=(x-3)^2=(2.00-3)^2=1.00$. So, our final answer is $(2.00,1.00)$.

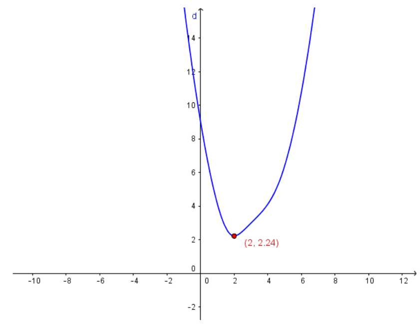


Figure 2.33: Minimizing $d(x)$ in Example 35

It seems silly to list a final answer as $(2.00,1.00)$. Indeed, Calculus confirms that the *exact* answer to this problem is, in fact, $(2,1)$. As you are well aware by now, the authors are overly pedantic, and as such, use the decimal places to remind the reader that *any* result garnered from a calculator in this fashion is an approximation, and should be treated as such. (What does the y value calculated by GeoGebra in Figure 2.33 mean in this problem?)

Exercises 2.5

Problems

In Exercises 1 – 12, sketch the graph of the given function. State the domain of the function, identify any intercepts and test for symmetry.

1. $f(x) = 2 - x$

2. $f(x) = \dfrac{x-2}{3}$

3. $f(x) = x^2 + 1$

4. $f(x) = 4 - x^2$

5. $f(x) = 2$

6. $f(x) = x^3$

7. $f(x) = x(x-1)(x+2)$

8. $f(x) = \sqrt{x-2}$

9. $f(x) = \sqrt{5-x}$

10. $f(x) = 3 - 2\sqrt{x+2}$

11. $f(x) = \sqrt[3]{x}$

12. $f(x) = \dfrac{1}{x^2+1}$

In Exercises 13 – 20, sketch the graph of the given piecewise-defined function.

13. $f(x) = \begin{cases} 4-x & \text{if} \quad x \le 3 \\ 2 & \text{if} \quad x > 3 \end{cases}$

14. $f(x) = \begin{cases} x^2 & \text{if} \quad x \le 0 \\ 2x & \text{if} \quad x > 0 \end{cases}$

15. $f(x) = \begin{cases} -3 & \text{if} \quad x < 0 \\ 2x-3 & \text{if} \quad 0 \le x \le 3 \\ 3 & \text{if} \quad x > 3 \end{cases}$

16. $f(x) = \begin{cases} x^2-4 & \text{if} \quad x \le -2 \\ 4-x^2 & \text{if} \quad -2 < x < 2 \\ x^2-4 & \text{if} \quad x \ge 2 \end{cases}$

17. $f(x) = \begin{cases} -2x-4 & \text{if} \quad x < 0 \\ 3x & \text{if} \quad x \ge 0 \end{cases}$

18. $f(x) = \begin{cases} \sqrt{x+4} & \text{if} \quad -4 \le x < 5 \\ \sqrt{x-1} & \text{if} \quad x \ge 5 \end{cases}$

19. $f(x) = \begin{cases} x^2 & \text{if} \quad x \le -2 \\ 3-x & \text{if} \quad -2 < x < 2 \\ 4 & \text{if} \quad x \ge 2 \end{cases}$

20. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad -6 < x < -1 \\ x & \text{if} \quad -1 < x < 1 \\ \sqrt{x} & \text{if} \quad 1 < x < 9 \end{cases}$

In Exercises 21 – 41, determine analytically if the following functions are even, odd or neither.

21. $f(x) = 7x$

22. $f(x) = 7x + 2$

23. $f(x) = 7$

24. $f(x) = 3x^2 - 4$

25. $f(x) = 4 - x^2$

26. $f(x) = x^2 - x - 6$

27. $f(x) = 2x^3 - x$

28. $f(x) = -x^5 + 2x^3 - x$

29. $f(x) = x^6 - x^4 + x^2 + 9$

30. $f(x) = x^3 + x^2 + x + 1$

31. $f(x) = \sqrt{1-x}$

32. $f(x) = \sqrt{1-x^2}$

33. $f(x) = 0$

34. $f(x) = \sqrt[3]{x^2}$

35. $f(x) = \sqrt[3]{x^2}$

36. $f(x) = \dfrac{3}{x^2}$

37. $f(x) = \dfrac{2x-1}{x+1}$

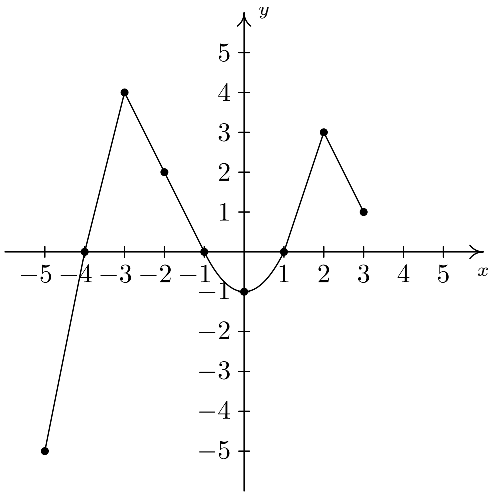
38. $f(x) = \dfrac{3x}{x^2+1}$

39. $f(x) = \dfrac{x^2-3}{x-4x^3}$

40. $f(x) = \dfrac{9}{\sqrt{4-x^2}}$

41. $f(x) = \dfrac{\sqrt[3]{x^3+x}}{5x}$

In Exercises 42 – 57, use the graph of $y = f(x)$ given below to answer the question.



42. Find the domain of f.

43. Find the range of f.

44. Determine $f(-2)$.

45. Solve $f(x) = 4$.

46. List the x-intercepts, if any exist.

47. List the y-intercepts, if any exist.

48. Find the zeros of f.

49. Solve $f(x) \geq 0$.

50. Find the number of solutions to $f(x) = 1$.

51. Does f appear to be even, odd, or neither?

52. List the intervals where f is increasing.

53. List the intervals where f is decreasing.

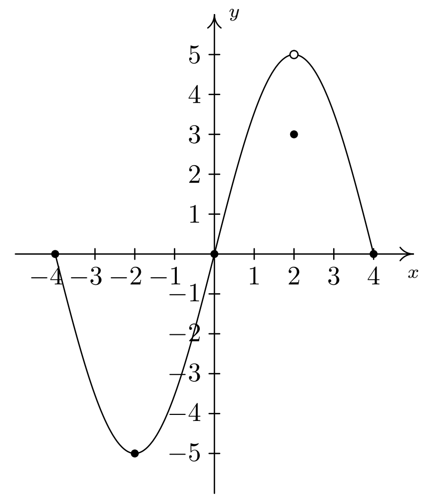
54. List the local maximums, if any exist.

55. List the local minimums, if any exist.

56. Find the maximum, if it exists.

57. Find the minimum, if it exists.

In Exercises 58 – 73, use the graph of $y = f(x)$ given below to answer the question.



58. Find the domain of f.

59. Find the range of f.

60. Determine $f(2)$.

61. Solve $f(x) = -5$.

62. List the x-intercepts, if any exist.

63. List the y-intercepts, if any exist.

64. Find the zeros of f.

65. Solve $f(x) \leq 0$.

66. Find the number of solutions to $f(x) = 3$.

67. Does f appear to be even, odd, or neither?

68. List the intervals where f is increasing.

69. List the intervals where f is decreasing.

70. List the local maximums, if any exist.

71. List the local minimums, if any exist.

72. Find the maximum, if it exists.

73. Find the minimum, if it exists.

In Exercises 74 – 77, use a graphing calculator or software (such as GeoGebra) to approximate the local and absolute extrema of the given function. Approximate the intervals on which the function is increasing and those on which it is decreasing. Round your answers to two decimal places.

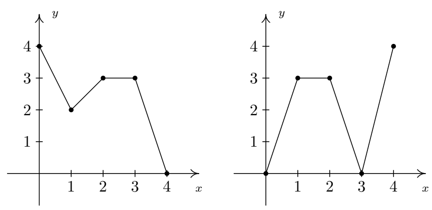
74. $f(x) = x^4 - 3x^3 - 24x^2 + 28x + 48$

75. $f(x) = x^{2/3}(x - 4)$

76. $f(x) = \sqrt{9 - x^2}$

77. $f(x) = x\sqrt{9 - x^2}$

In Exercises 78 – 87, use the graphs of $y = f(x)$ and $y = g(x)$ below to find the function value.



$y = f(x)$ $y = g(x)$

78. $(f + g)(0)$

79. $(f + g)(1)$

80. $(f - g)(1)$

81. $(g - f)(2)$

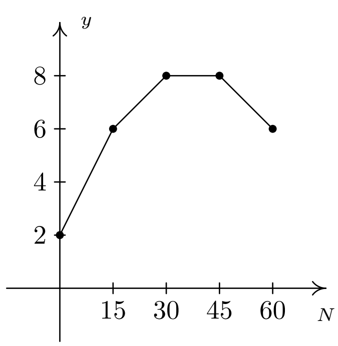
82. $(fg)(2)$

83. $(fg)(1)$

84. $\left(\dfrac{f}{g}\right)(4)$

85. $\left(\dfrac{g}{f}\right)(2)$

The graph below represents the height h of a Sasquatch (in feet) as a function of its age N in years. Use it to answer the questions in Exercises 86 – 90.



86. Find and interpret $h(0)$.

87. How tall is the Sasquatch when she is 15 years old?

88. Solve $h(N) = 6$ and interpret.

89. List the interval over which h is constant and interpret your answer.

90. List the interval over which h is decreasing and interpret your answer.

For Exercises 91 – 93, let $f(x) = \lfloor x \rfloor$ be the greatest integer function as defined in Exercise 76 in Section 2.3.

91. Graph $y = f(x)$. Be careful to correctly describe the behaviour of the graph near the integers.

92. Is f even, odd, or neither? Explain.

93. Discuss with your classmates which points on the graph are local minimums, local maximums or both. Is f ever increasing? Decreasing? Constant?

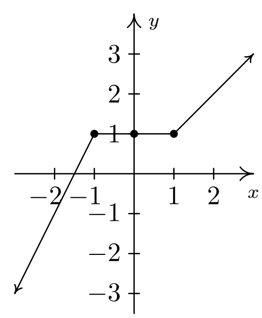
94. In Exercise 72 in Section 2.3, we saw that the population of Sasquatch in Portage County could be modeled by the function $P(t) = \dfrac{150t}{t + 15}$, where $t = 0$ represents the year 1803. Use your graphing calculator to analyze the general function behaviour of P. Will there ever be a time when 200 Sasquatch roam Portage County?

95. Suppose f and g are both even functions. What can be said about the functions $f + g$, $f - g$, fg and $\frac{f}{g}$? What if f and g are both odd? What if f is even but g is odd?

96. One of the most important aspects of the Cartesian Coordinate Plane is its ability to put Algebra into geometric terms and Geometry into algebraic terms. We've spent most of this chapter looking at this very phenomenon and now you should spend some time with your classmates reviewing what we've done. What major results do we have that tie Algebra and Geometry together? What concepts from Geometry have we not yet described algebraically? What topics from Intermediate Algebra have we not yet discussed geometrically?

It's now time to "thoroughly vet the pathologies induced" by the precise definitions of local maximum and local minimum. You and your classmates should carefully discuss Exercises 97 – 99. You will need to refer back to Definition 27 (Increasing, Decreasing and Constant) and Definition 28 (Maximum and Minimum) during the discussion.

97. Consider the graph of the function f given below.



   (a) Show that f has a local maximum but not a local minimum at the point $(-1, 1)$.

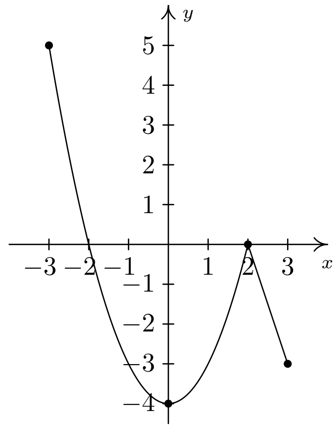
   (b) Show that f has a local minimum but not a local maximum at the point $(1, 1)$.

   (c) Show that f has a local maximum AND a local minimum at the point $(0, 1)$.

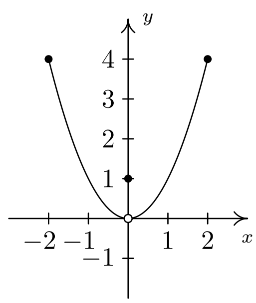
   (d) Show that f is constant on the interval $[-1, 1]$ and thus has both a local maximum AND a local minimum at every point $(x, f(x))$ where $-1 < x < 1$.

98. Using Example 33 as a guide, show that the function g whose graph is given below does <u>not</u> have a local maximum at $(-3, 5)$ nor does it have a local minimum at $(3, -3)$. Find

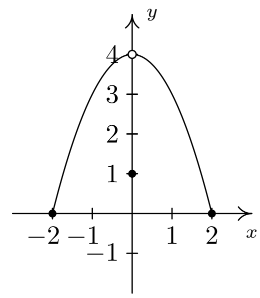
its extrema, both local and absolute. What's unique about the point $(0, -4)$ on this graph? Also find the intervals on which *g* is increasing and those on which *g* is decreasing.



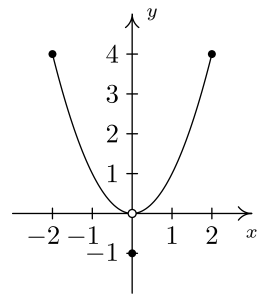
99. We said earlier in the section that it is not good enough to say local extrema exist where a function changes from increasing to decreasing or vice versa. As a previous exercise showed, we could have local extrema when a function is constant so now we need to examine some functions whose graphs do indeed change direction. Consider the functions graphed below. Notice that all four of them change direction at an open circle on the graph. Examine each for local extrema. What is the effect of placing the "dot" on the *y*-axis above or below the open circle? What could you say if no function value were assigned to $x = 0$?



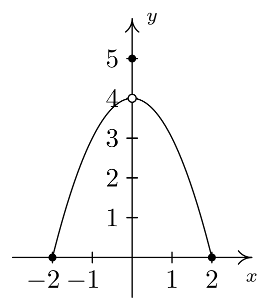
(a)



(b)



(c)



(d)

2.6 Transformations

In this section, we study how the graphs of functions change, or **transform**, when certain specialized modifications are made to their formulas. The transformations we will study fall into three broad categories: shifts, reflections and scalings, and we will present them in that order. Suppose that Figure 2.35 the complete graph of a function f.

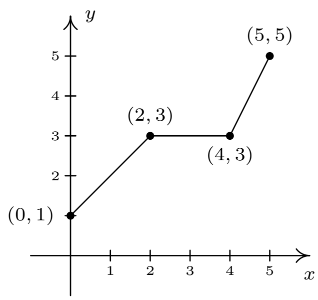


Figure 2.35: The graph of a function f

The Fundamental Graphing Principle for Functions says that for a point (a, b) to be on the graph, $f(a) = b$. In particular, we know $f(0) = 1$, $f(2) = 3$, $f(4) = 3$ and $f(5) = 5$. Suppose we wanted to graph the function defined by the formula $g(x) = f(x) + 2$. Let's take a minute to remind ourselves of what g is doing. We start with an input x to the function f and we obtain the output $f(x)$. The function g takes the output $f(x)$ and adds 2 to it. In order to graph g, we need to graph the points $(x, g(x))$. How are we to find the values for $g(x)$ without a formula for $f(x)$? The answer is that we don't need a *formula* for $f(x)$, we just need the *values* of $f(x)$. The values of $f(x)$ are the y values on the graph of $y = f(x)$. For example, using the points indicated on the graph of f, we can make the following table.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x) + 2$	$(x, g(x))$
0	$(0, 1)$	1	3	$(0, 3)$
2	$(2, 3)$	3	5	$(2, 5)$
4	$(4, 3)$	3	5	$(4, 5)$
5	$(5, 5)$	5	7	$(5, 7)$

In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$, so $g(a) = f(a) + 2 = b + 2$. Hence, $(a, b + 2)$ is on the graph of g. In other words, to obtain the graph of g, we add 2 to the y-coordinate of each point on the graph of f. Geometrically, adding 2 to the y-coordinate of a point moves the point 2 units above its previous location. Adding 2 to every y-coordinate on a graph *en masse* is usually described as 'shifting the graph up 2 units'. Notice that the graph retains the same basic shape as before, it is just 2 units above its original location. In other words, we connect the four points we moved in the same manner in which they were connected before: see Figure 2.34.

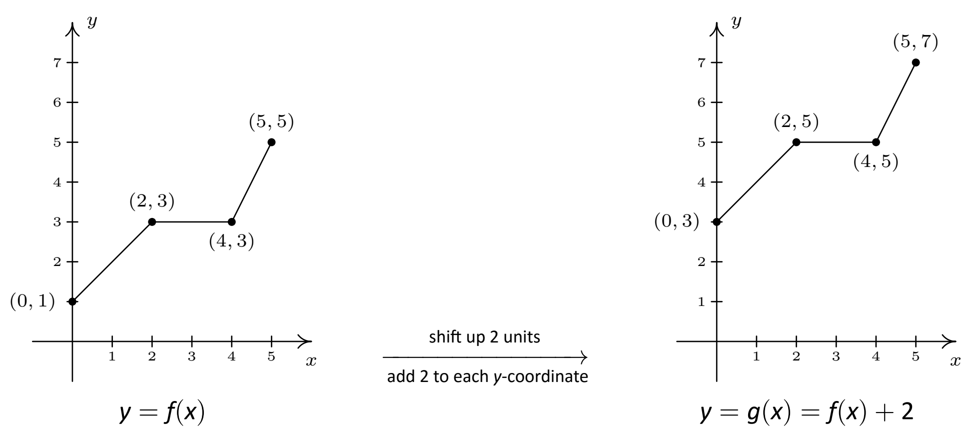


Figure 2.34: Shifting the graph of f up by 2 units

You'll note that the domain of f and the domain of g are the same, namely $[0, 5]$, but that the range of f is $[1, 5]$ while the range of g is $[3, 7]$. In general, shifting a function vertically like this will leave the domain unchanged, but could very well affect the range. You can easily imagine what would happen if we wanted to graph the function $j(x) = f(x) - 2$. Instead of adding 2 to each of the y-coordinates on the graph of f, we'd be subtracting 2. Geometrically, we would be moving the graph down 2 units. We leave it to the reader to verify that the domain of j is the same as f, but the range of j is $[-1, 3]$. What we have discussed is generalized in the following theorem.

Theorem 7 Vertical Shifts

Suppose f is a function and k is a positive number.

- To graph $y = f(x) + k$, shift the graph of $y = f(x)$ up k units by adding k to the y-coordinates of the points on the graph of f.
- To graph $y = f(x) - k$, shift the graph of $y = f(x)$ down k units by subtracting k from the y-coordinates of the points on the graph of f.

The key to understanding Theorem 7 and, indeed, all of the theorems in this section comes from an understanding of the Fundamental Graphing Principle for Functions. If (a, b) is on the graph of f, then $f(a) = b$. Substituting $x = a$ into the equation $y = f(x) + k$ gives $y = f(a) + k = b + k$. Hence, $(a, b + k)$ is on the graph of $y = f(x) + k$, and we have the result. In the language of 'inputs' and 'outputs', Theorem 7 can be paraphrased as "Adding to, or subtracting from, the *output* of a function causes the graph to shift up or down, respectively." So what happens if we add to or subtract from the *input* of the function?

Keeping with the graph of $y = f(x)$ above, suppose we wanted to graph $g(x) = f(x + 2)$. In other words, we are looking to see what happens when we add 2 to the input of the function.[1] Let's try to generate a table of values of g based on those we know for f. We quickly find that we run into some difficulties.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x+2)$	$(x, g(x))$
0	$(0, 1)$	1	$f(0 + 2) = f(2) = 3$	$(0, 3)$
2	$(2, 3)$	3	$f(2 + 2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$f(4 + 2) = f(6) = ?$	
5	$(5, 5)$	5	$f(5 + 2) = f(7) = ?$	

When we substitute $x = 4$ into the formula $g(x) = f(x + 2)$, we are asked to find $f(4 + 2) = f(6)$ which doesn't exist because the domain of f is only $[0, 5]$. The same thing happens when we attempt to find $g(5)$. What we need here is a new strategy. We know, for instance, $f(0) = 1$. To determine the corresponding point on the graph of g, we need to figure out what value of x we must substitute into $g(x) = f(x+2)$ so that the quantity $x+2$, works out to be 0. Solving $x+2 = 0$ gives $x = -2$, and $g(-2) = f((-2) + 2) = f(0) = 1$ so $(-2, 1)$ is on the graph of g. To use the fact $f(2) = 3$, we set $x + 2 = 2$ to get $x = 0$. Substituting gives

[1] We have spent a lot of time in this text showing you that $f(x + 2)$ and $f(x) + 2$ are, in general, wildly different algebraic animals. We will see momentarily that their geometry is also dramatically different.

$g(0) = f(0+2) = f(2) = 3$. Continuing in this fashion, we get

x	$x+2$	$g(x) = f(x+2)$	$(x, g(x))$
-2	0	$g(-2) = f(0) = 1$	$(-2, 1)$
0	2	$g(0) = f(2) = 3$	$(0, 3)$
2	4	$g(2) = f(4) = 3$	$(2, 3)$
3	5	$g(3) = f(5) = 5$	$(3, 5)$

In summary, the points $(0, 1)$, $(2, 3)$, $(4, 3)$ and $(5, 5)$ on the graph of $y = f(x)$ give rise to the points $(-2, 1)$, $(0, 3)$, $(2, 3)$ and $(3, 5)$ on the graph of $y = g(x)$, respectively. In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$. Solving $x + 2 = a$ gives $x = a - 2$ so that $g(a-2) = f((a-2)+2) = f(a) = b$. As such, $(a - 2, b)$ is on the graph of $y = g(x)$. The point $(a - 2, b)$ is exactly 2 units to the *left* of the point (a, b) so the graph of $y = g(x)$ is obtained by shifting the graph $y = f(x)$ to the left 2 units, as pictured below.

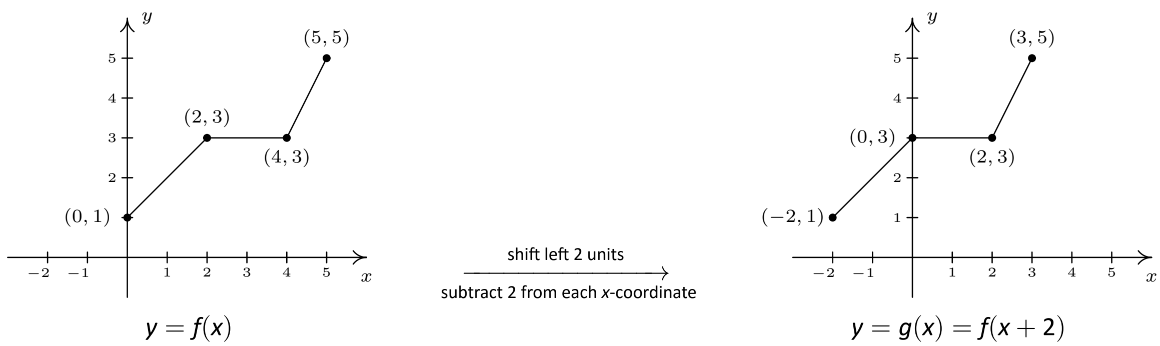


Figure 2.36: Shifting the graph of f left by 2 units

Note that while the ranges of f and g are the same, the domain of g is $[-2, 3]$ whereas the domain of f is $[0, 5]$. In general, when we shift the graph horizontally, the range will remain the same, but the domain could change. If we set out to graph $j(x) = f(x - 2)$, we would find ourselves *adding* 2 to all of the x values of the points on the graph of $y = f(x)$ to effect a shift to the *right* 2 units. Generalizing these notions produces the following result.

Theorem 8 Horizontal Shifts

Suppose f is a function and h is a positive number.

- To graph $y = f(x + h)$, shift the graph of $y = f(x)$ left h units by subtracting h from the x-coordinates of the points on the graph of f.
- To graph $y = f(x - h)$, shift the graph of $y = f(x)$ right h units by adding h to the x-coordinates of the points on the graph of f.

In other words, Theorem 8 says that adding to or subtracting from the *input* to a function amounts to shifting the graph left or right, respectively. Theorems 7 and 8 present a theme which will run common throughout the section: changes to the outputs from a function affect the y-coordinates of the graph, resulting in some kind of vertical change; changes to the inputs to a function affect the x-coordinates of the graph, resulting in some kind of horizontal change.

Example 36 **Transforming with vertical and horizontal shifts**

1. Graph $f(x) = \sqrt{x}$. Plot at least three points.
2. Use your graph in 1 to graph $g(x) = \sqrt{x} - 1$.
3. Use your graph in 1 to graph $j(x) = \sqrt{x - 1}$.
4. Use your graph in 1 to graph $m(x) = \sqrt{x + 3} - 2$.

SOLUTION

1. Owing to the square root, the domain of f is $x \geq 0$, or $[0, \infty)$. We choose perfect squares to build our table and graph below. From the graph we verify the domain of f is $[0, \infty)$ and the range of f is also $[0, \infty)$. The original function is plotted in Figure 2.37

x	$f(x)$	$(x, f(x))$
0	0	$(0, 0)$
1	1	$(1, 1)$
4	2	$(4, 2)$

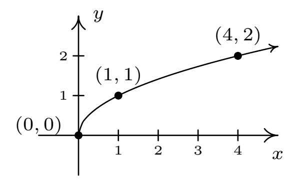


Figure 2.37: The graph $y = f(x) = \sqrt{x}$

2. The domain of g is the same as the domain of f, since the only condition on both functions is that $x \geq 0$. If we compare the formula for $g(x)$ with $f(x)$, we see that $g(x) = f(x) - 1$. In other words, we have subtracted 1 from the output of the function f. By Theorem 7, we know that in order to graph g, we shift the graph of f down one unit by subtracting 1 from each of the y-coordinates of the points on the graph of f. Applying this to the three points we have specified on the graph, we move $(0, 0)$ to $(0, -1)$, $(1, 1)$ to $(1, 0)$, and $(4, 2)$ to $(4, 1)$. The rest of the points follow suit, and we connect them with the same basic shape as before. We confirm the domain of g is $[0, \infty)$ and find the range of g to be $[-1, \infty)$. The graph of g is given in Figure 2.38.

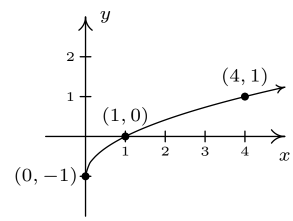


Figure 2.38: Graphing $g(x) = \sqrt{x} - 1$

3. Solving $x - 1 \geq 0$ gives $x \geq 1$, so the domain of j is $[1, \infty)$. To graph j, we note that $j(x) = f(x - 1)$. In other words, we are subtracting 1 from the *input* of f. According to Theorem 8, this induces a shift to the right of the graph of f. We add 1 to the x-coordinates of the points on the graph of f and get the result below. The graph reaffirms that the domain of j is $[1, \infty)$ and tells us that the range of j is $[0, \infty)$.

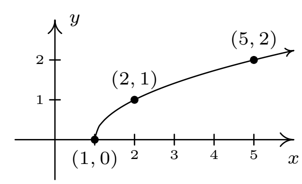


Figure 2.39: Graphing $j(x) = \sqrt{x} - 1$

4. To find the domain of m, we solve $x + 3 \geq 0$ and get $[-3, \infty)$. Comparing the formulas of $f(x)$ and $m(x)$, we have $m(x) = f(x + 3) - 2$. We have 3 being added to an input, indicating a horizontal shift, and 2 being subtracted from an output, indicating a vertical shift. We leave it to the reader to verify that, in this particular case, the order in which we perform these transformations is immaterial; we will arrive at the same graph regardless as to which transformation we apply first. (We shall see in the next example that order is generally important when applying more than one transformation to a graph.) We follow the convention 'inputs first', and to that end we first tackle the horizontal shift. Letting $m_1(x) = f(x + 3)$ denote this intermediate step, Theorem 8 tells us that the graph of $y = m_1(x)$ is the graph of f shifted to the left 3 units. Hence, we subtract 3 from each of the x-coordinates of the points on the graph of f.

   Since $m(x) = f(x+3) - 2$ and $f(x+3) = m_1(x)$, we have $m(x) = m_1(x) - 2$. We can apply Theorem 7 and obtain the graph of m by subtracting 2 from the y-coordinates of each of the points on the graph of $m_1(x)$. The graph verifies that the domain of m is $[-3, \infty)$ and we find the range of m to be $[-2, \infty)$.

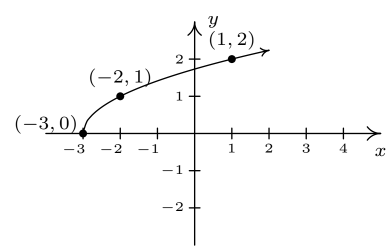


Figure 2.40: Graphing $m_1(x) = \sqrt{x + 3}$

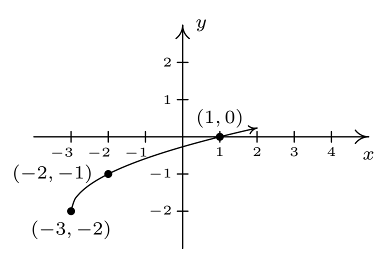


Figure 2.41: Graphing $m(x) = \sqrt{x + 3} - 2$

Keep in mind that we can check our answer to any of these kinds of problems by showing that any of the points we've moved lie on the graph of our final answer. For example, we can check that $(-3,-2)$ is on the graph of m by computing $m(-3) = \sqrt{(-3)+3} - 2 = \sqrt{0} - 2 = -2$ ✓

We now turn our attention to reflections. We know from Section 1.3 that to reflect a point (x, y) across the x-axis, we replace y with $-y$. If (x, y) is on the graph of f, then $y = f(x)$, so replacing y with $-y$ is the same as replacing $f(x)$ with $-f(x)$. Hence, the graph of $y = -f(x)$ is the graph of f reflected across the x-axis. Similarly, the graph of $y = f(-x)$ is the graph of f reflected across the y-axis. Returning to the language of inputs and outputs, multiplying the output from a function by -1 reflects its graph across the x-axis, while multiplying the input to a function by -1 reflects the graph across the y-axis.[2]

Theorem 9 Reflections

Suppose f is a function.

- To graph $y = -f(x)$, reflect the graph of $y = f(x)$ across the x-axis by multiplying the y-coordinates of the points on the graph of f by -1.
- To graph $y = f(-x)$, reflect the graph of $y = f(x)$ across the y-axis by multiplying the x-coordinates of the points on the graph of f by -1.

Applying Theorem 9 to the graph of $y = f(x)$ given at the beginning of the section, we can graph $y = -f(x)$ by reflecting the graph of f about the x-axis

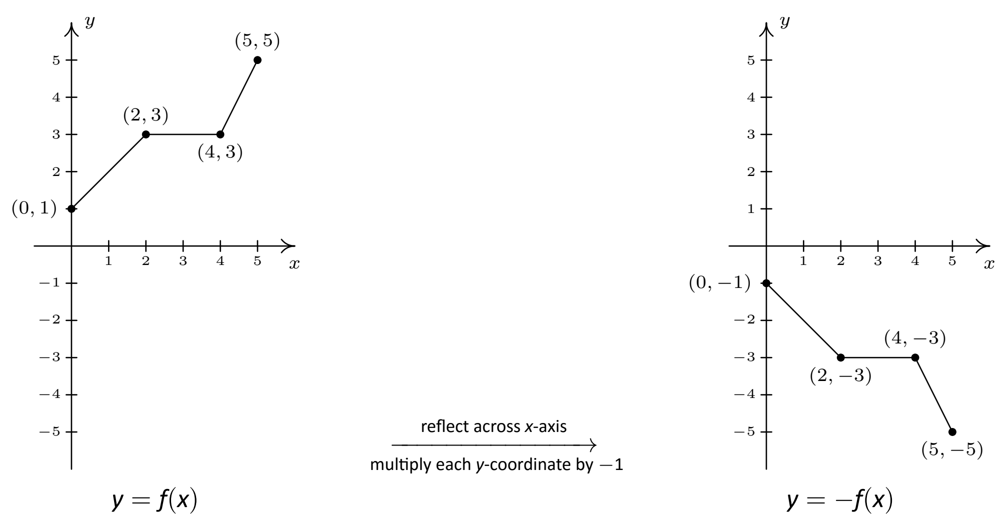


Figure 2.42: Reflecting the graph of f across the x-axis

By reflecting the graph of f across the y-axis, we obtain the graph of $y = f(-x)$.

[2]The expressions $-f(x)$ and $f(-x)$ should look familiar - they are the quantities we used in Section 2.5 to test if a function was even, odd or neither. The interested reader is invited to explore the role of reflections and symmetry of functions. What happens if you reflect an even function across the y-axis? What happens if you reflect an odd function across the y-axis? What about the x-axis?

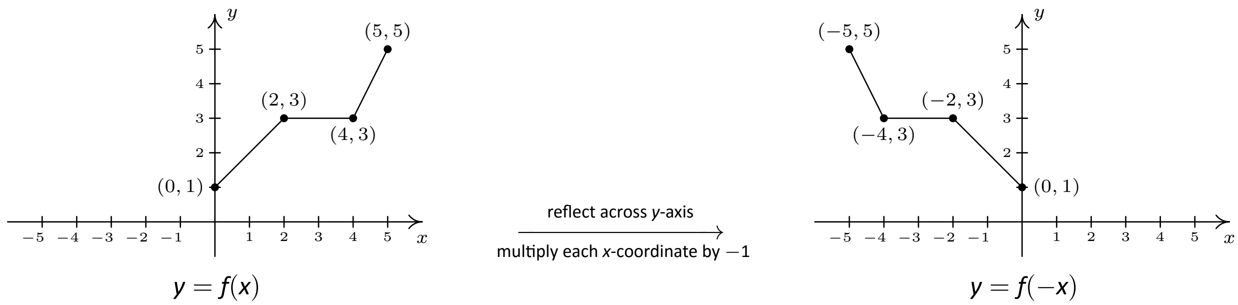


Figure 2.43: Reflecting the graph of f across the y-axis

With the addition of reflections, it is now more important than ever to consider the order of transformations, as the next example illustrates.

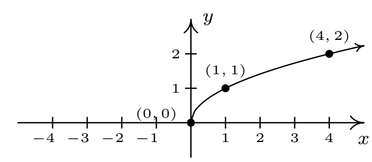


Figure 2.44: The graph $y = f(x)$ from Example 36

Example 37 Graphing reflections

Let $f(x) = \sqrt{x}$. Use the graph of f from Example 36 to graph the following functions. Also, state their domains and ranges.

1. $g(x) = \sqrt{-x}$
2. $j(x) = \sqrt{3-x}$
3. $m(x) = 3 - \sqrt{x}$

Solution

1. The mere sight of $\sqrt{-x}$ usually causes alarm, if not panic. When we discussed domains in Section 2.3, we clearly banished negatives from the radicands of even roots. However, we must remember that x is a variable, and as such, the quantity $-x$ isn't always negative. For example, if $x = -4$, $-x = 4$, thus $\sqrt{-x} = \sqrt{-(-4)} = 2$ is perfectly well-defined. To find the domain analytically, we set $-x \geq 0$ which gives $x \leq 0$, so that the domain of g is $(-\infty, 0]$. Since $g(x) = f(-x)$, Theorem 9 tells us that the graph of g is the reflection of the graph of f across the y-axis. We accomplish this by multiplying each x-coordinate on the graph of f by -1, so that the points $(0, 0)$, $(1, 1)$, and $(4, 2)$ move to $(0, 0)$, $(-1, 1)$, and $(-4, 2)$, respectively. Graphically, we see that the domain of g is $(-\infty, 0]$ and the range of g is the same as the range of f, namely $[0, \infty)$.

   If we had done the reflection first, then $j_1(x) = f(-x)$. Following this by a shift left would give us $j(x) = j_1(x+3) = f(-(x+3)) = f(-x-3) = \sqrt{-x-3}$ which isn't what we want. However, if we did the reflection first and followed it by a shift to the right 3 units, we would have arrived at the function $j(x)$. We leave it to the reader to verify the details.

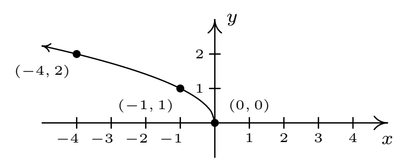


Figure 2.45: Reflecting $y = f(x)$ across the y-axis to obtain the graph of $g(x) = \sqrt{-x}$

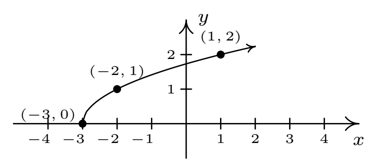


Figure 2.46: The intermediate function $j_1(x) = f(x+3)$

2. To determine the domain of $j(x) = \sqrt{3-x}$, we solve $3 - x \geq 0$ and get $x \leq 3$, or $(-\infty, 3]$. To determine which transformations we need to apply to the graph of f to obtain the graph of j, we rewrite $j(x) = \sqrt{-x+3} = f(-x+3)$. Comparing this formula with $f(x) = \sqrt{x}$, we see that not only are we multiplying the input x by -1, which results in a reflection across the y-axis, but also we are adding 3, which indicates a horizontal shift to the left. Does it matter in which order we do the transformations? If so, which order is the correct order? Let's consider the point $(4, 2)$ on the graph of f. We refer to the discussion leading up to Theorem 8. We know $f(4) = 2$ and wish to find the point on $y = j(x) = f(-x+3)$ which corresponds to $(4, 2)$. We set $-x+3 = 4$ and solve. Our first step is to subtract 3 from both sides to get $-x = 1$. Subtracting 3 from the x-coordinate 4 is shifting the point $(4, 2)$ to the left. From $-x = 1$, we then multiply[3]

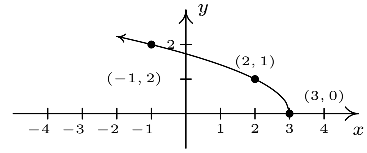


Figure 2.47: Reflecting $y = j_1(x)$ across the y-axis to obtain the graph of $j(x) = \sqrt{3-x}$

[3] Or divide - it amounts to the same thing.

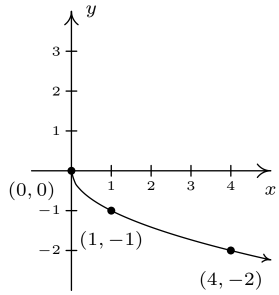


Figure 2.48: Reflecting $y = f(x)$ across the x-axis to obtain the graph of $m_1(x) = -\sqrt{x}$

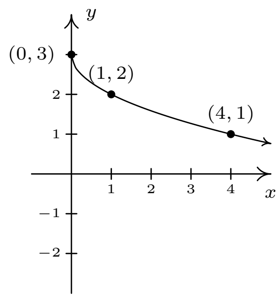
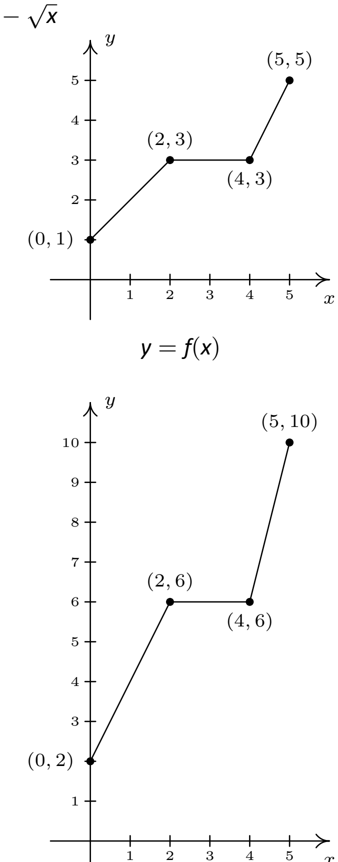


Figure 2.49: Shifting $y = m_1(x)$ up by three units to obtain the graph of $m(x) = 3 - \sqrt{x}$



$y = f(x)$

$y = 2f(x) = g(x)$

Figure 2.50: Graphing $g(x) = 2f(x)$

both sides by -1 to get $x = -1$. Multiplying the x-coordinate by -1 corresponds to reflecting the point about the y-axis. Hence, we perform the horizontal shift first, then follow it with the reflection about the y-axis. Starting with $f(x) = \sqrt{x}$, we let $j_1(x)$ be the intermediate function which shifts the graph of f 3 units to the left, $j_1(x) = f(x+3)$.

To obtain the function j, we reflect the graph of j_1 about y-axis. Theorem 9 tells us we have $j(x) = j_1(-x)$. Putting it all together, we have $j(x) = j_1(-x) = f(-x+3) = \sqrt{-x+3}$, which is what we want. From the graph, we confirm the domain of j is $(-\infty, 3]$ and we get that the range is $[0, \infty)$.

3. The domain of m works out to be the domain of f, $[0, \infty)$. Rewriting $m(x) = -\sqrt{x} + 3$, we see $m(x) = -f(x) + 3$. Since we are multiplying the output of f by -1 and then adding 3, we once again have two transformations to deal with: a reflection across the x-axis and a vertical shift. To determine the correct order in which to apply the transformations, we imagine trying to determine the point on the graph of m which corresponds to $(4, 2)$ on the graph of f. Since in the formula for $m(x)$, the input to f is just x, we substitute to find $m(4) = -f(4) + 3 = -2 + 3 = 1$. Hence, $(4, 1)$ is the corresponding point on the graph of m. If we closely examine the arithmetic, we see that we first multiply $f(4)$ by -1, which corresponds to the reflection across the x-axis, and then we add 3, which corresponds to the vertical shift. If we define an intermediate function $m_1(x) = -f(x)$ to take care of the reflection, we get the graph in Figure 2.48.

   To shift the graph of m_1 up 3 units, we set $m(x) = m_1(x) + 3$. Since $m_1(x) = -f(x)$, when we put it all together, we get $m(x) = m_1(x) + 3 = -f(x) + 3 = -\sqrt{x} + 3$. We see from the graph that the range of m is $(-\infty, 3]$.

We now turn our attention to our last class of transformations known as **scalings**. A thorough discussion of scalings can get complicated because they are not as straight-forward as the previous transformations. A quick review of what we've covered so far, namely vertical shifts, horizontal shifts and reflections, will show you why those transformations are known as **rigid transformations**. Simply put, they do not change the *shape* of the graph, only its position and orientation in the plane. If, however, we wanted to make a new graph twice as tall as a given graph, or one-third as wide, we would be changing the shape of the graph. This type of transformation is called **non-rigid** for obvious reasons. Not only will it be important for us to differentiate between modifying inputs versus outputs, we must also pay close attention to the magnitude of the changes we make. As you will see shortly, the Mathematics turns out to be easier than the associated grammar.

Suppose we wish to graph the function $g(x) = 2f(x)$ where $f(x)$ is the function whose graph is given in Figure 2.35 the beginning of the section. From its graph, we can build a table of values for g as before:

x	$(x, f(x))$	$f(x)$	$g(x) = 2f(x)$	$(x, g(x))$
0	$(0, 1)$	1	2	$(0, 2)$
2	$(2, 3)$	3	6	$(2, 6)$
4	$(4, 3)$	3	6	$(4, 6)$
5	$(5, 5)$	5	10	$(5, 10)$

In general, if (a, b) is on the graph of f, then $f(a) = b$ so that $g(a) = 2f(a) = 2b$ puts $(a, 2b)$ on the graph of g. In other words, to obtain the graph of g, we multiply all of the y-coordinates of the points on the graph of f by 2. Multiplying all of the y-coordinates of all of the points on the graph of f by 2 causes what is known as a 'vertical scaling (or 'vertical stretching', or 'vertical expansion' or 'vertical dilation') by a factor of 2', and the results are given in Figure 2.50

If we wish to graph $y = \frac{1}{2}f(x)$, we multiply the all of the y-coordinates of the points on the graph of f by $\frac{1}{2}$. This creates a 'vertical scaling by a factor of $\frac{1}{2}$' (also called 'vertical shrinking', 'vertical compression' or 'vertical contraction' by a factor of 2) as seen in Figure 2.51

These results are generalized in the following theorem.

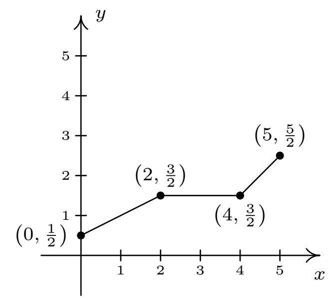


Figure 2.51: Vertical scaling by $\frac{1}{2}$

Theorem 10 Vertical Scalings

Suppose f is a function and $a > 0$. To graph $y = af(x)$, multiply all of the y-coordinates of the points on the graph of f by a. We say the graph of f has been vertically scaled by a factor of a.

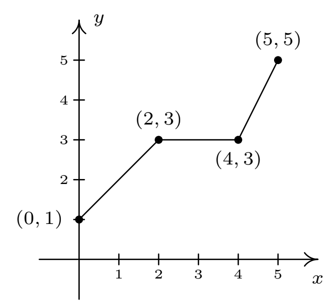
- If $a > 1$, we say the graph of f has undergone a vertical stretching (expansion, dilation) by a factor of a.
- If $0 < a < 1$, we say the graph of f has undergone a vertical shrinking (compression, contraction) by a factor of $\frac{1}{a}$.

A few remarks about Theorem 10 are in order. First, a note about the verbiage. To the authors, the words 'stretching', 'expansion', and 'dilation' all indicate something getting bigger. Hence, 'stretched by a factor of 2' makes sense if we are scaling something by multiplying it by 2. Similarly, we believe words like 'shrinking', 'compression' and 'contraction' all indicate something getting smaller, so if we scale something by a factor of $\frac{1}{2}$, we would say it 'shrinks by a factor of 2' - not 'shrinks by a factor of $\frac{1}{2}$'. This is why we have written the descriptions 'stretching by a factor of a' and 'shrinking by a factor of $\frac{1}{a}$' in the statement of the theorem. Second, in terms of inputs and outputs, Theorem 10 says multiplying the *outputs* from a function by positive number a causes the graph to be vertically scaled by a factor of a. It is natural to ask what would happen if we multiply the *inputs* of a function by a positive number. This leads us to our last transformation of the section.

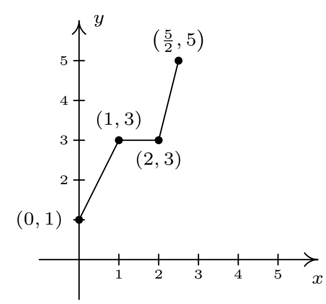
Referring to the graph of f given at the beginning of this section, suppose we want to graph $g(x) = f(2x)$. In other words, we are looking to see what effect multiplying the inputs to f by 2 has on its graph. If we attempt to build a table directly, we quickly run into the same problem we had in our discussion leading up to Theorem 8, as seen in the table below.

x	$(x, f(x))$	$f(x)$	$g(x) = f(2x)$	$(x, g(x))$
0	$(0, 1)$	1	$f(2 \cdot 0) = f(0) = 1$	$(0, 1)$
2	$(2, 3)$	3	$f(2 \cdot 2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$f(2 \cdot 4) = f(8) = ?$	
5	$(5, 5)$	5	$f(2 \cdot 5) = f(10) = ?$	

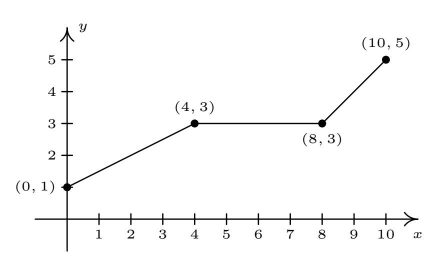
We solve this problem in the same way we solved this problem before. For example, if we want to determine the point on g which corresponds to the point $(2, 3)$ on the graph of f, we set $2x = 2$ so that $x = 1$. Substituting $x = 1$ into $g(x)$, we obtain $g(1) = f(2 \cdot 1) = f(2) = 3$, so that $(1, 3)$ is on the graph of g. Continuing in this fashion, we can complete our table as follows:



The graph $y = f(x)$ from Figure 2.35



The graph $y = g(x) = f(2x)$



The graph $y = h(x) = f(\frac{1}{2}x)$

Figure 2.52: The effect of horizontal scaling on a graph

x	$2x$	$g(x) = f(2x)$	$(x, g(x))$
0	0	$g(0) = f(0) = 1$	$(0, 0)$
1	2	$g(1) = f(2) = 3$	$(1, 3)$
2	4	$g(2) = f(4) = 3$	$(2, 3)$
$\frac{5}{2}$	5	$g\left(\frac{5}{2}\right) = f(5) = 5$	$\left(\frac{5}{2}, 5\right)$

In general, if (a, b) is on the graph of f, then $f(a) = b$. Hence $g\left(\frac{a}{2}\right) = f\left(2 \cdot \frac{a}{2}\right) = f(a) = b$ so that $\left(\frac{a}{2}, b\right)$ is on the graph of g. In other words, to graph g we divide the x-coordinates of the points on the graph of f by 2. This results in a horizontal scaling by a factor of $\frac{1}{2}$ (also called 'horizontal shrinking', 'horizontal compression' or 'horizontal contraction' by a factor of 2).

If, on the other hand, we wish to graph $y = f\left(\frac{1}{2}x\right)$, we end up multiplying the x-coordinates of the points on the graph of f by 2 which results in a horizontal scaling[4] by a factor of 2. The effect of both horizontal scalings is shown in Figure 2.52.

We have the following theorem.

Theorem 11 Horizontal Scalings.

Suppose f is a function and $b > 0$. To graph $y = f(bx)$, divide all of the x-coordinates of the points on the graph of f by b. We say the graph of f has been horizontally scaled by a factor of $\frac{1}{b}$.

- If $0 < b < 1$, we say the graph of f has undergone a horizontal stretching (expansion, dilation) by a factor of $\frac{1}{b}$.
- If $b > 1$, we say the graph of f has undergone a horizontal shrinking (compression, contraction) by a factor of b.

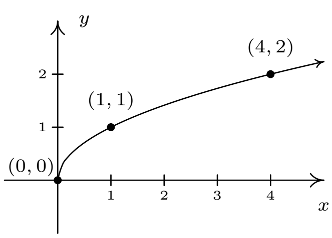


Figure 2.53: The graph $y = \sqrt{x}$

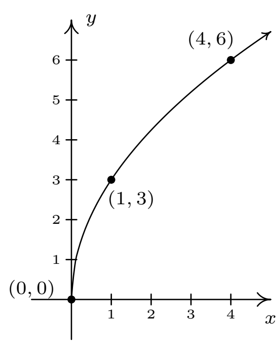


Figure 2.54: The graph $y = g(x) = 3\sqrt{x}$

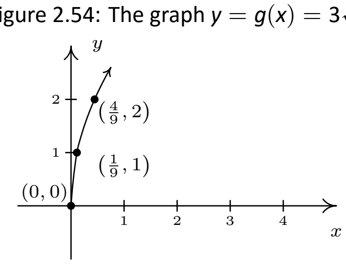


Figure 2.55: The graph $y = j(x) = \sqrt{9x}$

Theorem 11 tells us that if we multiply the input to a function by b, the resulting graph is scaled horizontally by a factor of $\frac{1}{b}$ since the x-values are divided by b to produce corresponding points on the graph of $y = f(bx)$. The next example explores how vertical and horizontal scalings sometimes interact with each other and with the other transformations introduced in this section.

Example 38 Applying vertical and horizontal scalings

Let $f(x) = \sqrt{x}$. Use the graph of f from Example 36 (see Figure 2.53) to graph the following functions. Also, state their domains and ranges.

1. $g(x) = 3\sqrt{x}$
2. $j(x) = \sqrt{9x}$
3. $m(x) = 1 - \sqrt{\frac{x+3}{2}}$

Solution

1. First we note that the domain of g is $[0, \infty)$ for the usual reason. Next, we have $g(x) = 3f(x)$ so by Theorem 10, we obtain the graph of g by multiplying all of the y-coordinates of the points on the graph of f by 3. The result is a vertical scaling of the graph of f by a factor of 3. We find the range of g is also $[0, \infty)$. The graph of g is given in Figure 2.54.

[4] Also called 'horizontal stretching', 'horizontal expansion' or 'horizontal dilation' by a factor of 2.

2. To determine the domain of j, we solve $9x \geq 0$ to find $x \geq 0$. Our domain is once again $[0, \infty)$. We recognize $j(x) = f(9x)$ and by Theorem 11, we obtain the graph of j by dividing the x-coordinates of the points on the graph of f by 9. From the graph in Figure 2.55, we see the range of j is also $[0, \infty)$.

3. Solving $\frac{x+3}{2} \geq 0$ gives $x \geq -3$, so the domain of m is $[-3, \infty)$. To take advantage of what we know of transformations, we rewrite $m(x) = -\sqrt{\frac{1}{2}x + \frac{3}{2}} + 1$, or $m(x) = -f\left(\frac{1}{2}x + \frac{3}{2}\right) + 1$. Focusing on the inputs first, we note that the input to f in the formula for $m(x)$ is $\frac{1}{2}x + \frac{3}{2}$. Multiplying the x by $\frac{1}{2}$ corresponds to a horizontal stretching by a factor of 2, and adding the $\frac{3}{2}$ corresponds to a shift to the left by $\frac{3}{2}$. As before, we resolve which to perform first by thinking about how we would find the point on m corresponding to a point on f, in this case, $(4, 2)$. To use $f(4) = 2$, we solve $\frac{1}{2}x + \frac{3}{2} = 4$. Our first step is to subtract the $\frac{3}{2}$ (the horizontal shift) to obtain $\frac{1}{2}x = \frac{5}{2}$. Next, we multiply by 2 (the horizontal stretching) and obtain $x = 5$. We define two intermediate functions to handle first the shift, then the stretching. In accordance with Theorem 8, $m_1(x) = f\left(x + \frac{3}{2}\right) = \sqrt{x + \frac{3}{2}}$ will shift the graph of f to the left $\frac{3}{2}$ units: see Figure 2.56

   Next, $m_2(x) = m_1\left(\frac{1}{2}x\right) = \sqrt{\frac{1}{2}x + \frac{3}{2}}$ will, according to Theorem 11, horizontally stretch the graph of m_1 by a factor of 2: see Figure 2.57

   We now examine what's happening to the outputs. From $m(x) = -f\left(\frac{1}{2}x + \frac{3}{2}\right) + 1$, we see that the output from f is being multiplied by -1 (a reflection about the x-axis) and then a 1 is added (a vertical shift up 1). As before, we can determine the correct order by looking at how the point $(4, 2)$ is moved. We already know that to make use of the equation $f(4) = 2$, we need to substitute $x = 5$. We get $m(5) = -f\left(\frac{1}{2}(5) + \frac{3}{2}\right) + 1 = -f(4) + 1 = -2 + 1 = -1$. We see that $f(4)$ (the output from f) is first multiplied by -1 then the 1 is added meaning we first reflect the graph about the x-axis then shift up 1. Theorem 9 tells us $m_3(x) = -m_2(x)$ will handle the reflection.

   Finally, to handle the vertical shift, Theorem 7 gives $m(x) = m_3(x) + 1$, and we see that the range of m is $(-\infty, 1]$. The graph of m is given in Figure 2.59.

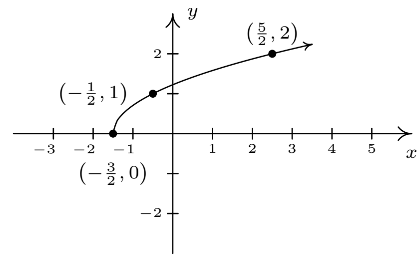


Figure 2.56: The graph $y = m_1(x) = f(x + \frac{3}{2}) = \sqrt{x + \frac{3}{2}}$

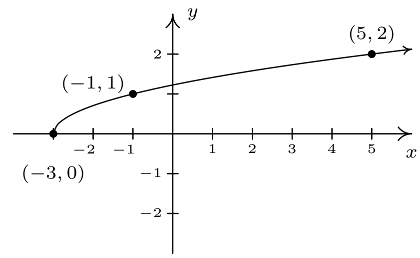


Figure 2.57: The graph $y = m_2(x) = m_1(\frac{1}{2}x) = \sqrt{\frac{1}{2}x + \frac{3}{2}}$

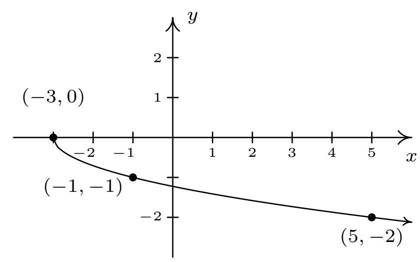


Figure 2.58: The graph $y = m_3(x) = -m_2(x) = -\sqrt{\frac{1}{2}x + \frac{3}{2}}$

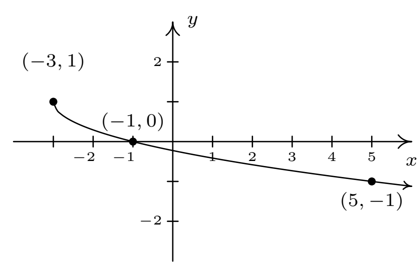


Figure 2.59: The graph $y = m(x) = m_3(x) + 1 = -\sqrt{\frac{1}{2}x + \frac{3}{2}}$

Some comments about Example 38 are in order. First, recalling the properties of radicals from Intermediate Algebra, we know that the functions g and j are the same, since j and g have the same domains and $j(x) = \sqrt{9x} = \sqrt{9}\sqrt{x} = 3\sqrt{x} = g(x)$. (We invite the reader to verify that all of the points we plotted on the graph of g lie on the graph of j and vice-versa.) Hence, for $f(x) = \sqrt{x}$, a vertical stretch by a factor of 3 and a horizontal shrinking by a factor of 9 result in the same transformation. While this kind of phenomenon is not universal, it happens commonly enough with some of the families of functions studied in College Algebra that it is worthy of note. Secondly, to graph the function m, we applied a series of four transformations. While it would have been easier on the authors to simply inform the reader of which steps to take, we have strived to explain why the order in which the transformations were applied made sense. We generalize the procedure in the theorem below.

Theorem 12 Transformations

Suppose f is a function. If $A \neq 0$ and $B \neq 0$, then to graph

$$g(x) = Af(Bx + H) + K$$

1. Subtract H from each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the left if $H > 0$ or right if $H < 0$.
2. Divide the x-coordinates of the points on the graph obtained in Step 1 by B. This results in a horizontal scaling, but may also include a reflection about the y-axis if $B < 0$.
3. Multiply the y-coordinates of the points on the graph obtained in Step 2 by A. This results in a vertical scaling, but may also include a reflection about the x-axis if $A < 0$.
4. Add K to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $K > 0$ or down if $K < 0$.

Theorem 12 can be established by generalizing the techniques developed in this section. Suppose (a, b) is on the graph of f. Then $f(a) = b$, and to make good use of this fact, we set $Bx + H = a$ and solve. We first subtract the H (causing the horizontal shift) and then divide by B. If B is a positive number, this induces only a horizontal scaling by a factor of $\frac{1}{B}$. If $B < 0$, then we have a factor of -1 in play, and dividing by it induces a reflection about the y-axis. So we have $x = \frac{a-H}{B}$ as the input to g which corresponds to the input $x = a$ to f. We now evaluate $g\left(\frac{a-H}{B}\right) = Af\left(B \cdot \frac{a-H}{B} + H\right) + K = Af(a) + K = Ab + K$. We notice that the output from f is first multiplied by A. As with the constant B, if $A > 0$, this induces only a vertical scaling. If $A < 0$, then the -1 induces a reflection across the x-axis. Finally, we add K to the result, which is our vertical shift. A less precise, but more intuitive way to paraphrase Theorem 12 is to think of the quantity $Bx + H$ is the 'inside' of the function f. What's happening inside f affects the inputs or x-coordinates of the points on the graph of f. To find the x-coordinates of the corresponding points on g, we undo what has been done to x in the same way we would solve an equation. What's happening to the output can be thought of as things happening 'outside' the function, f. Things happening outside affect the outputs or y-coordinates of the points on the graph of f. Here, we follow the usual order of operations agreement: we first multiply by A then add K to find the corresponding y-coordinates on the graph of g.

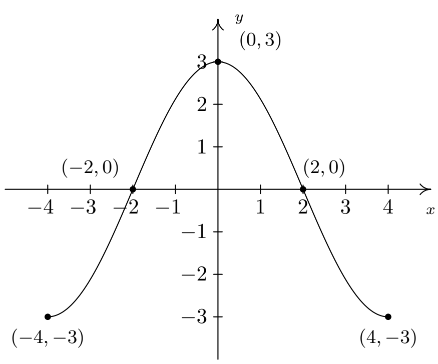


Figure 2.60: The graph $y = f(x)$ for Example 39

Example 39 Graphing a general transformation

The complete graph of $y = f(x)$ is shown in Figure 2.60. Use it to graph $g(x) = \frac{4-3f(1-2x)}{2}$.

Solution We use Theorem 12 to track the five 'key points' $(-4, -3)$, $(-2, 0)$, $(0, 3)$, $(2, 0)$ and $(4, -3)$ indicated on the graph of f to their new locations. We first rewrite $g(x)$ in the form presented in Theorem 12, $g(x) = -\frac{3}{2}f(-2x + 1) + 2$. We set $-2x + 1$ equal to the x-coordinates of the key points and solve. For example, solving $-2x + 1 = -4$, we first subtract 1 to get $-2x = -5$ then divide by -2 to get $x = \frac{5}{2}$. Subtracting the 1 is a horizontal shift to the left 1 unit. Dividing by -2 can be thought of as a two step process:

dividing by 2 which compresses the graph horizontally by a factor of 2 followed by dividing (multiplying) by -1 which causes a reflection across the y-axis. We summarize the results in the table in Figure 2.62

Next, we take each of the x values and substitute them into $g(x) = -\frac{3}{2}f(-2x+1)+2$ to get the corresponding y-values. Substituting $x = \frac{5}{2}$, and using the fact that $f(-4) = -3$, we get

$$g\left(\frac{5}{2}\right) = -\frac{3}{2}f\left(-2\left(\frac{5}{2}\right)+1\right)+2 = -\frac{3}{2}f(-4)+2 = -\frac{3}{2}(-3)+2 = \frac{9}{2}+2 = \frac{13}{2}$$

$(a, f(a))$	$-2x+1=a$	x
$(-4,-3)$	$-2x+1=-4$	$x=\frac{5}{2}$
$(-2,0)$	$-2x+1=-2$	$x=\frac{3}{2}$
$(0,3)$	$-2x+1=0$	$x=\frac{1}{2}$
$(2,0)$	$-2x+1=2$	$x=-\frac{1}{2}$
$(4,-3)$	$-2x+1=4$	$x=-\frac{3}{2}$

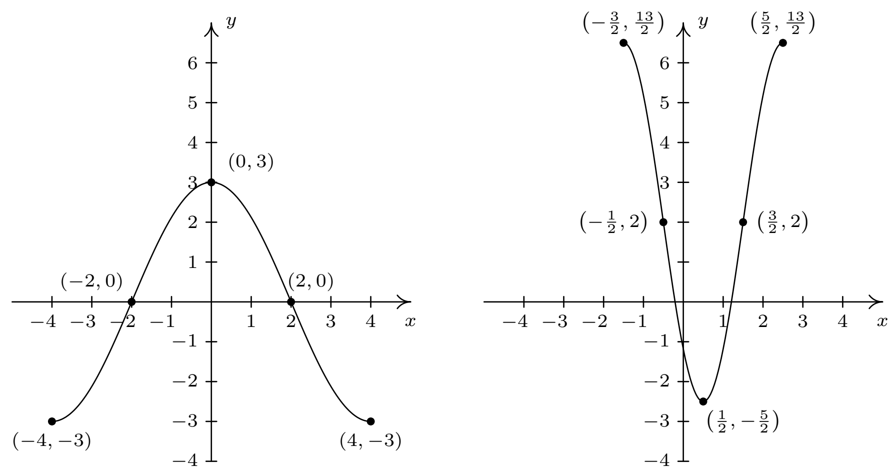
Figure 2.62: Tracking the x coordinates of transformed points

We see that the output from f is first multiplied by $-\frac{3}{2}$. Thinking of this as a two step process, multiplying by $\frac{3}{2}$ then by -1, we have a vertical stretching by a factor of $\frac{3}{2}$ followed by a reflection across the x-axis. Adding 2 results in a vertical shift up 2 units. Continuing in this manner, we get the table in Figure 2.63.

x	$g(x)$	$(x, g(x))$
$\frac{5}{2}$	$\frac{13}{2}$	$\left(\frac{5}{2},\frac{13}{2}\right)$
$\frac{3}{2}$	2	$\left(\frac{3}{2},2\right)$
$\frac{1}{2}$	$-\frac{5}{2}$	$\left(\frac{1}{2},-\frac{5}{2}\right)$
$-\frac{1}{2}$	2	$\left(-\frac{1}{2},2\right)$
$-\frac{3}{2}$	$\frac{13}{2}$	$\left(-\frac{3}{2},\frac{13}{2}\right)$

Figure 2.63: Getting the corresponding y coordinates

To graph g, we plot each of the points in the table above and connect them in the same order and fashion as the points to which they correspond. Plotting f and g side-by-side gives



$y = f(x)$ $\qquad$ $y = g(x) = g(x) = -\frac{3}{2}f(-2x+1)+2$

Figure 2.61: Determining the graph of $g(x) = -\frac{3}{2}f(-2x+1)+2$

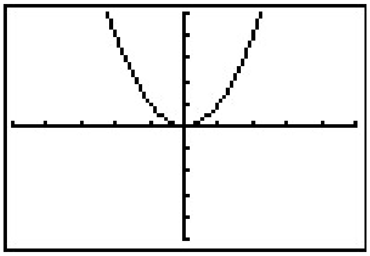
The reader is strongly encouraged to graph the series of functions which shows the gradual transformation of the graph of f into the graph of g. (You really should do this once in your life.) We have outlined the sequence of transformations in the above exposition; all that remains is to plot the five intermediate stages.

Our last example turns the tables and asks for the formula of a function given a desired sequence of transformations. If nothing else, it is a good review of function notation.

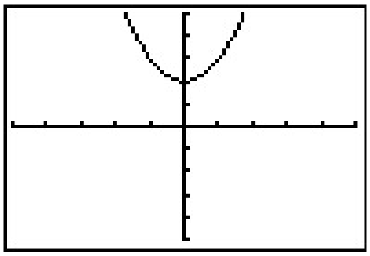
Example 40 Determining the formula for a transformed function

Let $f(x) = x^2$. Find and simplify the formula of the function $g(x)$ whose graph is the result of f undergoing the following sequence of transformations. Check your answer using a graphing calculator.

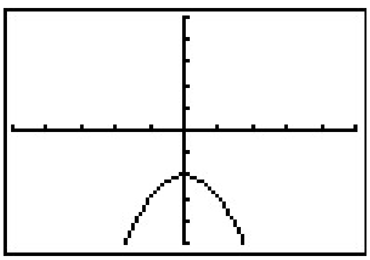
1. Vertical shift up 2 units
2. Reflection across the x-axis
3. Horizontal shift right 1 unit
4. Horizontal stretching by a factor of 2



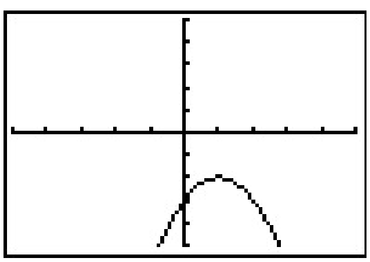
$y = f(x) = x^2$



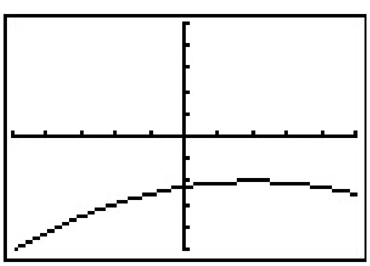
$y = g_1(x) = f(x) + 2 = x^2 + 2$
(Shift up by 2)



$y = g_2(x) = -g_1(x) = -x^2 - 2$
(Reflect across x-axis)



$y = g_3(x) = g_2(x - 1) = -x^2 + 2x - 3$
(Shift right one unit)

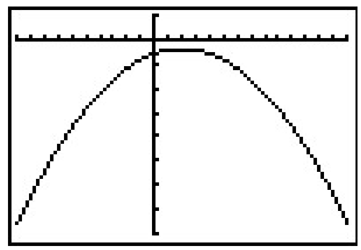


$y = g(x) = g_3((\frac{1}{2}x) = -\frac{1}{4}x^2 + x - 3$
(Horizontal stretch by a factor of 2)

Figure 2.64: The sequence of transformations in Example 40

Solution We build up to a formula for $g(x)$ using intermediate functions as we've seen in previous examples. We let g_1 take care of our first step. Theorem 7 tells us $g_1(x) = f(x) + 2 = x^2 + 2$. Next, we reflect the graph of g_1 about the x-axis using Theorem 9: $g_2(x) = -g_1(x) = -\left(x^2 + 2\right) = -x^2 - 2$. We shift the graph to the right 1 unit, according to Theorem 8, by setting $g_3(x) = g_2(x-1) = -(x-1)^2 - 2 = -x^2 + 2x - 3$. Finally, we induce a horizontal stretch by a factor of 2 using Theorem 11 to get $g(x) = g_3\left(\frac{1}{2}x\right) = -\left(\frac{1}{2}x\right)^2 + 2\left(\frac{1}{2}x\right) - 3$ which yields $g(x) = -\frac{1}{4}x^2 + x - 3$. We use the calculator to graph the stages below to confirm our result.

We have kept the viewing window the same in all of the graphs above. This had the undesirable consequence of making the last graph look 'incomplete' in that we cannot see the original shape of $f(x) = x^2$. Altering the viewing window results in a more complete graph of the transformed function shown below:



This example brings our first chapter to a close. In the chapters which lie ahead, be on the lookout for the concepts developed here to resurface as we study different families of functions.

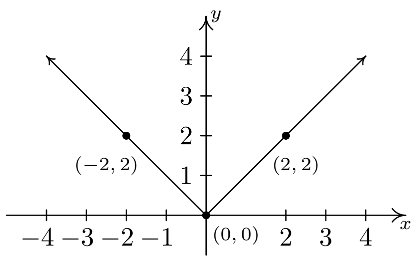
Exercises 2.6

Problems

Suppose $(2, -3)$ is on the graph of $y = f(x)$. In Exercises 1 – 20, use Theorem 12 to find a point on the graph of the given transformed function.

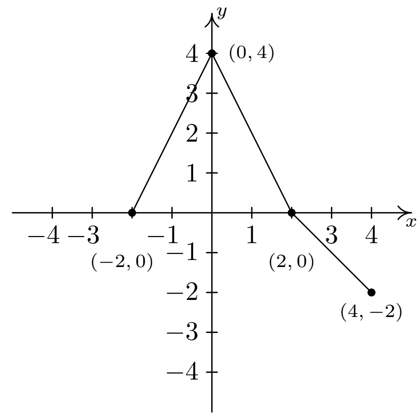
1. $y = f(x) + 3$
2. $y = f(x + 3)$
3. $y = f(x) - 1$
4. $y = f(x - 1)$
5. $y = 3f(x)$
6. $y = f(3x)$
7. $y = -f(x)$
8. $y = f(-x)$
9. $y = f(x - 3) + 1$
10. $y = 2f(x + 1)$
11. $y = 10 - f(x)$
12. $y = 3f(2x) - 1$
13. $y = \frac{1}{2}f(4 - x)$
14. $y = 5f(2x + 1) + 3$
15. $y = 2f(1 - x) - 1$
16. $y = f\left(\dfrac{7 - 2x}{4}\right)$
17. $y = \dfrac{f(3x) - 1}{2}$
18. $y = \dfrac{4 - f(3x - 1)}{7}$

The complete graph of $y = f(x)$ is given below. In Exercises 19 – 27, use it and Theorem 12 to graph the given transformed function.



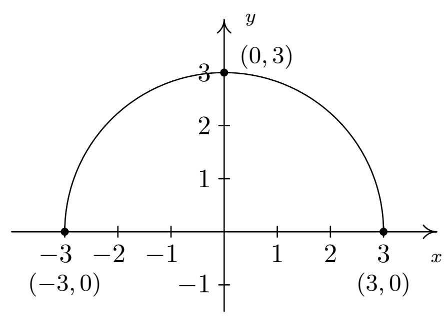
19. $y = f(x) + 1$
20. $y = f(x) - 2$
21. $y = f(x + 1)$
22. $y = f(x - 2)$
23. $y = 2f(x)$
24. $y = f(2x)$
25. $y = 2 - f(x)$
26. $y = f(2 - x)$
27. $y = 2 - f(2 - x)$

The complete graph of $y = f(x)$ is given below. In Exercises 28 – 36, use it and Theorem 12 to graph the given transformed function.



28. $y = f(x) - 1$
29. $y = f(x + 1)$
30. $y = \frac{1}{2}f(x)$
31. $y = f(2x)$
32. $y = -f(x)$
33. $y = f(-x)$
34. $y = f(x + 1) - 1$
35. $y = 1 - f(x)$
36. $y = \frac{1}{2}f(x + 1) - 1$

The complete graph of $y = f(x)$ is given below. In Exercises 37 – 48, use it and Theorem 12 to graph the given transformed function.



37. $g(x) = f(x) + 3$
38. $h(x) = f(x) - \frac{1}{2}$
39. $j(x) = f\left(x - \frac{2}{3}\right)$
40. $a(x) = f(x + 4)$

41. $b(x) = f(x+1) - 1$

42. $c(x) = \frac{3}{5}f(x)$

43. $d(x) = -2f(x)$

44. $k(x) = f\left(\frac{2}{3}x\right)$

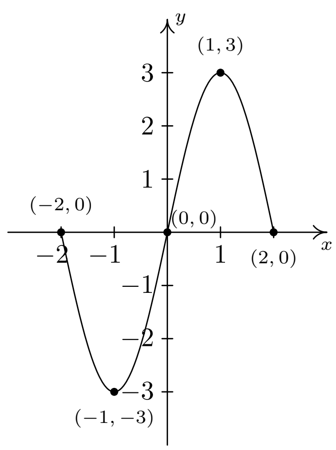
45. $m(x) = -\frac{1}{4}f(3x)$

46. $n(x) = 4f(x-3) - 6$

47. $p(x) = 4 + f(1-2x)$

48. $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3$

The complete graph of $y = S(x)$ is given below. The purpose of Exercises 49 – 52 is to graph $y = \frac{1}{2}S(-x+1)+1$ by graphing each transformation, one step at a time.



49. $y = S_1(x) = S(x+1)$

50. $y = S_2(x) = S_1(-x) = S(-x+1)$

51. $y = S_3(x) = \frac{1}{2}S_2(x) = \frac{1}{2}S(-x+1)$

52. $y = S_4(x) = S_3(x) + 1 = \frac{1}{2}S(-x+1) + 1$

Let $f(x) = \sqrt{x}$. In Exercises 53 – 62, find a formula for a function g whose graph is obtained from f from the given sequence of transformations.

53. (1) shift right 2 units; (2) shift down 3 units

54. (1) shift down 3 units; (2) shift right 2 units

55. (1) reflect across the x-axis; (2) shift up 1 unit

56. (1) shift up 1 unit; (2) reflect across the x-axis

57. (1) shift left 1 unit; (2) reflect across the y-axis; (3) shift up 2 units

58. (1) reflect across the y-axis; (2) shift left 1 unit; (3) shift up 2 units

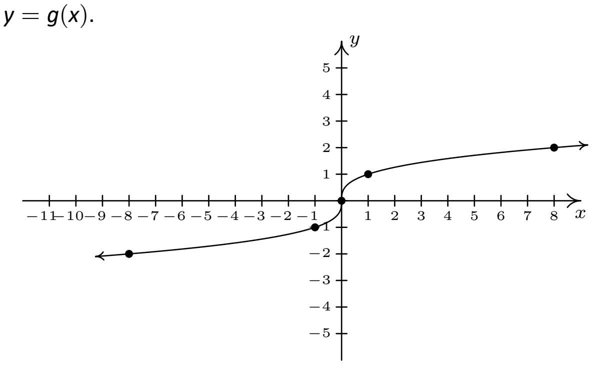
59. (1) shift left 3 units; (2) vertical stretch by a factor of 2; (3) shift down 4 units

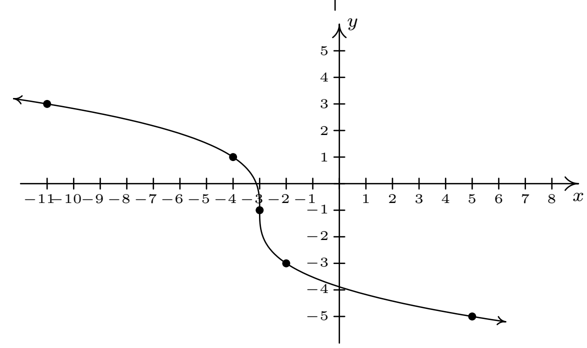
60. (1) shift left 3 units; (2) shift down 4 units; (3) vertical stretch by a factor of 2

61. (1) shift right 3 units; (2) horizontal shrink by a factor of 2; (3) shift up 1 unit

62. (1) horizontal shrink by a factor of 2; (2) shift right 3 units; (3) shift up 1 unit

63. The graph of $y = f(x) = \sqrt[3]{x}$ is given immediately below, and the graph of $y = g(x)$ is given below that of $y = f(x)$. Find a formula for g based on transformations of the graph of f. Check your answer by confirming that the points shown on the graph of g satisfy the equation $y = g(x)$.





64. For many common functions, the properties of Algebra make a horizontal scaling the same as a vertical scaling by (possibly) a different factor. For example, we stated earlier that $\sqrt{9x} = 3\sqrt{x}$. With the help of your classmates, find the equivalent vertical scaling produced by the horizontal scalings $y = (2x)^3$, $y = |5x|$, $y = \sqrt[3]{27x}$ and $y = \left(\frac{1}{2}x\right)^2$. What about $y = (-2x)^3$, $y = |-5x|$, $y = \sqrt[3]{-27x}$ and $y = \left(-\frac{1}{2}x\right)^2$?

65. We mentioned earlier in the section that, in general, the order in which transformations are applied matters, yet in our first example with two transformations the order did not matter. (You could perform the shift to the left followed by the shift down or you could shift down and then left to achieve the same result.) With the help of your classmates,

determine the situations in which order does matter and those in which it does not.

66. What happens if you reflect an even function across the y-axis?

67. What happens if you reflect an odd function across the y-axis?

68. What happens if you reflect an even function across the x-axis?

69. What happens if you reflect an odd function across the x-axis?

70. How would you describe symmetry about the origin in terms of reflections?

71. As we saw in Example 40, the viewing window on the graphing calculator affects how we see the transformations done to a graph. Using two different calculators, find viewing windows so that $f(x) = x^2$ on the one calculator looks like $g(x) = 3x^2$ on the other.

3: Linear and Quadratic Functions

3.1 Linear Functions

We now begin the study of families of functions. Our first family, linear functions, are old friends as we shall soon see. Recall from Geometry that two distinct points in the plane determine a unique line containing those points, as indicated in Figure 3.1.

To give a sense of the 'steepness' of the line, we recall that we can compute the **slope** of the line using the formula below.

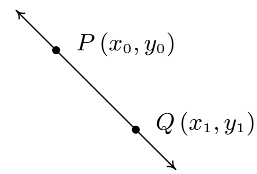


Figure 3.1: The line between two points P and Q

Definition 29 Slope

The **slope** m of the line containing the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is:

$$m = \frac{y_1 - y_0}{x_1 - x_0},$$

provided $x_1 \neq x_0$.

A couple of notes about Definition 29 are in order. First, don't ask why we use the letter 'm' to represent slope. There are many explanations out there, but apparently no one really knows for sure. Secondly, the stipulation $x_1 \neq x_0$ ensures that we aren't trying to divide by zero. The reader is invited to pause to think about what is happening geometrically; the anxious reader can skip along to the next example.

See www.mathforum.org or www.mathworld.wolfram.com for discussions on the use of the letter m to indicate slope.

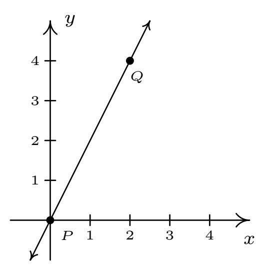
Example 41 Finding the slope of a line

Find the slope of the line containing the following pairs of points, if it exists. Plot each pair of points and the line containing them.

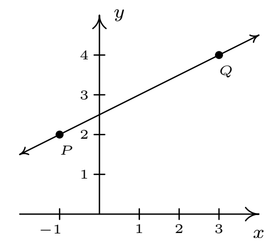
1. $P(0,0)$, $Q(2,4)$
2. $P(-1,2)$, $Q(3,4)$
3. $P(-2,3)$, $Q(2,-3)$
4. $P(-3,2)$, $Q(4,2)$
5. $P(2,3)$, $Q(2,-1)$
6. $P(2,3)$, $Q(2.1,-1)$

Solution In each of these examples, we apply the slope formula, from Definition 29.

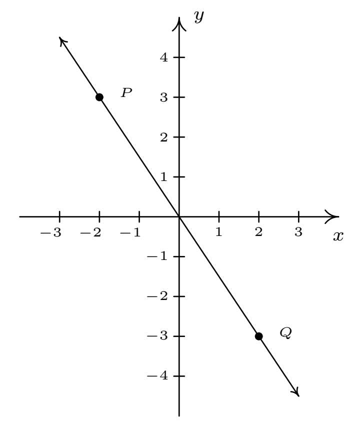
1. $m = \dfrac{4-0}{2-0} = \dfrac{4}{2} = 2$



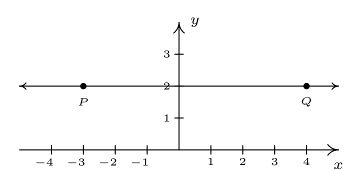
2. $m = \dfrac{4-2}{3-(-1)} = \dfrac{2}{4} = \dfrac{1}{2}$



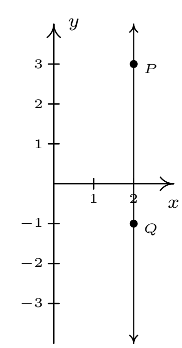
3. $m = \dfrac{-3-3}{2-(-2)} = \dfrac{-6}{4} = -\dfrac{3}{2}$



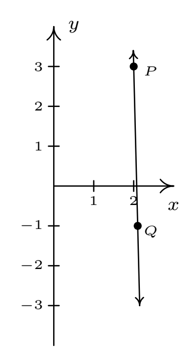
4. $m = \dfrac{2-2}{4-(-3)} = \dfrac{0}{7} = 0$



5. $m = \dfrac{-1-3}{2-2} = \dfrac{-4}{0}$, which is undefined



6. $m = \dfrac{-1-3}{2.1-2} = \dfrac{-4}{0.1} = -40$



A few comments about Example 41 are in order. First, for reasons which will be made clear soon, if the slope is positive then the resulting line is said to be increasing. If it is negative, we say the line is decreasing. A slope of 0 results in a horizontal line which we say is constant, and an undefined slope results in a vertical line. Second, the larger the slope is in absolute value, the steeper the line. You may recall from high school that slope can be described as the ratio '$\frac{\text{rise}}{\text{run}}$'. For example, in the second part of Example 41, we found the slope to be $\frac{1}{2}$. We can interpret this as a rise of 1 unit upward for every 2 units to the right we travel along the line, as shown in Figure 3.2.

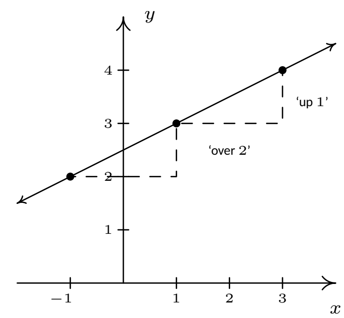


Figure 3.2: Slope as "rise over run"

Using more formal notation, given points (x_0, y_0) and (x_1, y_1), we use the Greek letter delta 'Δ' to write $\Delta y = y_1 - y_0$ and $\Delta x = x_1 - x_0$. In most scientific circles, the symbol Δ means 'change in'.

Hence, we may write

$$m = \frac{\Delta y}{\Delta x},$$

which describes the slope as the **rate of change** of y with respect to x. Rates of change abound in the 'real world', as the next example illustrates.

Example 42 Temperature rate of change

Suppose that two separate temperature readings were taken at the ranger station on the top of Mt. Sasquatch: at 6 AM the temperature was 2°C and at 10 AM it was 8°C.

1. Find the slope of the line containing the points $(6, 2)$ and $(10, 8)$.
2. Interpret your answer to the first part in terms of temperature and time.
3. Predict the temperature at noon.

Solution

1. For the slope, we have $m = \frac{8-2}{10-6} = \frac{6}{4} = \frac{3}{2}$.
2. Since the values in the numerator correspond to the temperatures in °C, and the values in the denominator correspond to time in hours, we can interpret the slope as $\frac{3}{2} = \frac{3^\circ\text{ C}}{2\text{ hour}}$, or 1.5°C per hour. Since the slope is positive, we know this corresponds to an increasing line. Hence, the temperature is increasing at a rate of 1.5°C per hour.
3. Noon is two hours after 10 AM. Assuming a temperature increase of 1.5°C per hour, in two hours the temperature should rise 3°C. Since the temperature at 10 AM is 82°C, we would expect the temperature at noon to be $8 + 3 = 11$°C.

Now it may well happen that in the previous scenario, at noon the temperature is only 10°C. This doesn't mean our calculations are incorrect, rather, it means that the temperature change throughout the day isn't a constant 1.5°C per hour. As discussed in Section 2.3.1, mathematical models are just that: models. The predictions we get out of the models may be mathematically accurate, but may not resemble what happens in the real world.

In Section 2.1, we discussed the equations of vertical and horizontal lines. Using the concept of slope, we can develop equations for the other varieties of lines. Suppose a line has a slope of m and contains the point (x_0, y_0). Suppose (x, y) is another point on the line, as indicated in Figure 3.3.

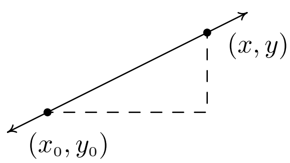


Figure 3.3: Deriving the point-slope formula

Definition 29 yields

$$\begin{aligned} m &= \frac{y - y_0}{x - x_0} \\ m(x - x_0) &= y - y_0 \\ y - y_0 &= m(x - x_0) \end{aligned}$$

We have just derived the **point-slope form** of a line.

Key Idea 15 The point-slope form of a line

The **point-slope form** of the equation of a line with slope m containing the point (x_0, y_0) is the equation $y - y_0 = m(x - x_0)$.

Example 43 Using the point-slope form

Write the equation of the line containing the points $(-1, 3)$ and $(2, 1)$.

Solution In order to use Key Idea 15 we need to find the slope of the line in question so we use Definition 29 to get $m = \frac{\Delta y}{\Delta x} = \frac{1-3}{2-(-1)} = -\frac{2}{3}$. We are spoiled for choice for a point (x_0, y_0). We'll use $(-1, 3)$ and leave it to the reader to check that using $(2, 1)$ results in the same equation. Substituting into the point-slope form of the line, we get

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 3 &= -\frac{2}{3}(x - (-1)) \\ y - 3 &= -\frac{2}{3}(x + 1) \\ y - 3 &= -\frac{2}{3}x - \frac{2}{3} \\ y &= -\frac{2}{3}x + \frac{7}{3}. \end{aligned}$$

We can check our answer by showing that both $(-1, 3)$ and $(2, 1)$ are on the graph of $y = -\frac{2}{3}x + \frac{7}{3}$ algebraically, as we did in Section 2.1.1.

In simplifying the equation of the line in the previous example, we produced another form of a line, the **slope-intercept form**. This is the familiar $y = mx + b$ form you have probably seen in high school. The 'intercept' in 'slope-intercept' comes from the fact that if we set $x = 0$, we get $y = b$. In other words, the y-intercept of the line $y = mx + b$ is $(0, b)$.

Key Idea 16 Slope intercept form of a line

The **slope-intercept form** of the line with slope m and y-intercept $(0, b)$ is the equation $y = mx + b$.

Note that if we have slope $m = 0$, we get the equation $y = b$ which matches our formula for a horizontal line given in Section 2.1. The formula given in Key Idea 16 can be used to describe all lines except vertical lines. All lines except vertical lines are functions (Why is this?) so we have finally reached a good point to introduce **linear functions**.

Definition 30 Linear function

A **linear function** is a function of the form

$$f(x) = mx + b,$$

where m and b are real numbers with $m \neq 0$. The domain of a linear function is $(-\infty, \infty)$.

For the case $m = 0$, we get $f(x) = b$. These are given their own classification.

Definition 31 Constant function

A **constant function** is a function of the form

$$f(x) = b,$$

where b is real number. The domain of a constant function is $(-\infty, \infty)$.

Recall that to graph a function, f, we graph the equation $y = f(x)$. Hence, the graph of a linear function is a line with slope m and y-intercept $(0, b)$; the graph of a constant function is a horizontal line (a line with slope $m = 0$) and a y-intercept of $(0, b)$. Now think back to Section **??**, specifically Definition 27 concerning increasing, decreasing and constant functions. A line with positive slope was called an increasing line because a linear function with $m > 0$ is an increasing function. Similarly, a line with a negative slope was called a decreasing line because a linear function with $m < 0$ is a decreasing function. And horizontal lines were called constant because, well, we hope you've already made the connection.

Example 44 Graphing linear functions

Graph the following functions. Identify the slope and y-intercept.

1. $f(x) = 3$
2. $f(x) = 3x - 1$
3. $f(x) = \dfrac{3 - 2x}{4}$
4. $f(x) = \dfrac{x^2 - 4}{x - 2}$

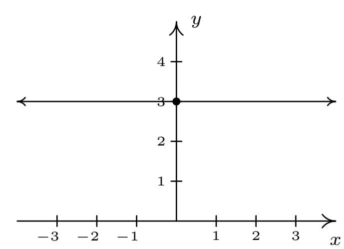


Figure 3.4: The graph of $f(x) = 3$

Solution

1. To graph $f(x) = 3$, we graph $y = 3$. This is a horizontal line ($m = 0$) through $(0, 3)$: see Figure 3.4.
2. The graph of $f(x) = 3x - 1$ is the graph of the line $y = 3x - 1$. Comparison of this equation with Equation 16 yields $m = 3$ and $b = -1$. Hence, our slope is 3 and our y-intercept is $(0, -1)$. To get another point on the line, we can plot $(1, f(1)) = (1, 2)$. Constructing the line through these points gives us Figure 3.5.
3. At first glance, the function $f(x) = \dfrac{3 - 2x}{4}$ does not fit the form in Definition 30 but after some rearranging we get $f(x) = \frac{3-2x}{4} = \frac{3}{4} - \frac{2x}{4} =$

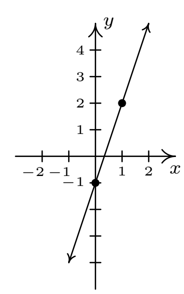


Figure 3.5: The graph of $f(x) = 3x - 1$

$-\frac{1}{2}x+\frac{3}{4}$. We identify $m=-\frac{1}{2}$ and $b=\frac{3}{4}$. Hence, our graph is a line with a slope of $-\frac{1}{2}$ and a y-intercept of $\left(0,\frac{3}{4}\right)$. Plotting an additional point, we can choose $(1, f(1))$ to get $\left(1,\frac{1}{4}\right)$: see Figure 3.6.

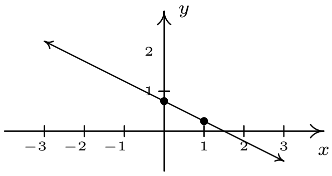


Figure 3.6: The graph of $f(x)=\dfrac{3-2x}{4}$

4. If we simplify the expression for f, we get

$$f(x)=\frac{x^2-4}{x-2}=\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}=x+2.$$

If we were to state $f(x)=x+2$, we would be committing a sin of omission. Remember, to find the domain of a function, we do so **before** we simplify! In this case, f has big problems when $x=2$, and as such, the domain of f is $(-\infty,2)\cup(2,\infty)$. To indicate this, we write $f(x)=x+2$, $x\neq 2$. So, except at $x=2$, we graph the line $y=x+2$. The slope $m=1$ and the y-intercept is $(0,2)$. A second point on the graph is $(1, f(1))=(1,3)$. Since our function f is not defined at $x=2$, we put an open circle at the point that would be on the line $y=x+2$ when $x=2$, namely $(2,4)$, as shown in Figure 3.7.

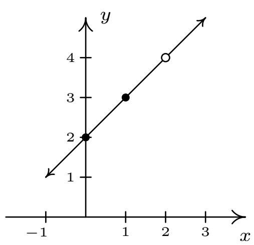


Figure 3.7: The graph of $f(x)=\dfrac{x^2-4}{x-2}$

The last two functions in the previous example showcase some of the difficulty in defining a linear function using the phrase 'of the form' as in Definition 30, since some algebraic manipulations may be needed to rewrite a given function to match 'the form'. Keep in mind that the domains of linear and constant functions are all real numbers $(-\infty,\infty)$, so while $f(x)=\frac{x^2-4}{x-2}$ simplified to a formula $f(x)=x+2$, f is not considered a linear function since its domain excludes $x=2$. However, we would consider

$$f(x)=\frac{2x^2+2}{x^2+1}$$

to be a constant function since its domain is all real numbers (Can you tell us why?) and

$$f(x)=\frac{2x^2+2}{x^2+1}=\frac{2\cancel{(x^2+1)}}{\cancel{(x^2+1)}}=2$$

The following example uses linear functions to model some basic economic relationships.

The similarity of this name to PortaJohn is deliberate.

Example 45 Pricing for a game system

The cost C, in dollars, to produce x PortaBoy game systems for a local retailer is given by $C(x)=80x+150$ for $x\geq 0$.

1. Find and interpret $C(10)$.
2. How many PortaBoys can be produced for $\$15{,}000$?
3. Explain the significance of the restriction on the domain, $x\geq 0$.
4. Find and interpret $C(0)$.
5. Find and interpret the slope of the graph of $y=C(x)$.

Solution

1. To find $C(10)$, we replace every occurrence of x with 10 in the formula for $C(x)$ to get $C(10) = 80(10) + 150 = 950$. Since x represents the number of PortaBoys produced, and $C(x)$ represents the cost, in dollars, $C(10) = 950$ means it costs $\$950$ to produce 10 PortaBoys for the local retailer.

2. To find how many PortaBoys can be produced for $\$15{,}000$, we solve $C(x) = 15000$, or $80x+150 = 15000$. Solving, we get $x = \frac{14850}{80} = 185.625$. Since we can only produce a whole number amount of PortaBoys, we can produce 185 PortaBoys for $\$15{,}000$.

3. The restriction $x \geq 0$ is the applied domain, as discussed in Section 2.3.1. In this context, x represents the number of PortaBoys produced. It makes no sense to produce a negative quantity of game systems.

4. We find $C(0) = 80(0) + 150 = 150$. This means it costs $\$150$ to produce 0 PortaBoys. As mentioned on page 73, this is the fixed, or start-up cost of this venture.

5. If we were to graph $y = C(x)$, we would be graphing the portion of the line $y = 80x + 150$ for $x \geq 0$. We recognize the slope, $m = 80$. Like any slope, we can interpret this as a rate of change. Here, $C(x)$ is the cost in dollars, while x measures the number of PortaBoys so

$$m = \frac{\Delta y}{\Delta x} = \frac{\Delta C}{\Delta x} = 80 = \frac{80}{1} = \frac{\$80}{1 \text{ PortaBoy}}.$$

   In other words, the cost is increasing at a rate of $\$80$ per PortaBoy produced. This is often called the **variable cost** for this venture.

Actually, it makes no sense to produce a fractional part of a game system, either, as we saw in the previous part of this example. This absurdity, however, seems quite forgiveable in some textbooks but not to us.

The next example asks us to find a linear function to model a related economic problem.

Example 46 Modelling demand

The local retailer in Example 45 has determined that the number x of PortaBoy game systems sold in a week is related to the price p in dollars of each system. When the price was $\$220$, 20 game systems were sold in a week. When the systems went on sale the following week, 40 systems were sold at $\$190$ a piece.

1. Find a linear function which fits this data. Use the weekly sales x as the independent variable and the price p as the dependent variable.

2. Find a suitable applied domain.

3. Interpret the slope.

4. If the retailer wants to sell 150 PortaBoys next week, what should the price be?

5. What would the weekly sales be if the price were set at $\$150$ per system?

Solution

1. We recall from Section 2.3 the meaning of 'independent' and 'dependent' variable. Since x is to be the independent variable, and p the dependent variable, we treat x as the input variable and p as the output variable. Hence, we are looking for a function of the form $p(x) = mx + b$. To

determine m and b, we use the fact that 20 PortaBoys were sold during the week when the price was 220 dollars and 40 units were sold when the price was 190 dollars. Using function notation, these two facts can be translated as $p(20) = 220$ and $p(40) = 190$. Since m represents the rate of change of p with respect to x, we have

$$m = \frac{\Delta p}{\Delta x} = \frac{190 - 220}{40 - 20} = \frac{-30}{20} = -1.5.$$

We now have determined $p(x) = -1.5x + b$. To determine b, we can use our given data again. Using $p(20) = 220$, we substitute $x = 20$ into $p(x) = 1.5x + b$ and set the result equal to 220: $-1.5(20) + b = 220$. Solving, we get $b = 250$. Hence, we get $p(x) = -1.5x + 250$. We can check our formula by computing $p(20)$ and $p(40)$ to see if we get 220 and 190, respectively. You may recall from page 73 that the function $p(x)$ is called the price-demand (or simply demand) function for this venture.

2. To determine the applied domain, we look at the physical constraints of the problem. Certainly, we can't sell a negative number of PortaBoys, so $x \geq 0$. However, we also note that the slope of this linear function is negative, and as such, the price is decreasing as more units are sold. Thus another constraint on the price is $p(x) \geq 0$. Solving $-1.5x + 250 \geq 0$ results in $-1.5x \geq -250$ or $x \leq \frac{500}{3} = 166.\overline{6}$. Since x represents the number of PortaBoys sold in a week, we round down to 166. As a result, a reasonable applied domain for p is $[0, 166]$.

3. The slope $m = -1.5$, once again, represents the rate of change of the price of a system with respect to weekly sales of PortaBoys. Since the slope is negative, we have that the price is decreasing at a rate of $\$1.50$ per PortaBoy sold. (Said differently, you can sell one more PortaBoy for every $\$1.50$ drop in price.)

4. To determine the price which will move 150 PortaBoys, we find $p(150) = -1.5(150) + 250 = 25$. That is, the price would have to be $\$25$.

5. If the price of a PortaBoy were set at $\$150$, we have $p(x) = 150$, or, $-1.5x + 250 = 150$. Solving, we get $-1.5x = -100$ or $x = 66.\overline{6}$. This means you would be able to sell 66 PortaBoys a week if the price were $\$150$ per system.

Not all real-world phenomena can be modelled using linear functions. Nevertheless, it is possible to use the concept of slope to help analyze non-linear functions using the following.

Definition 32 Average rate of change

Let f be a function defined on the interval $[a, b]$. The **average rate of change** of f over $[a, b]$ is defined as:

$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Geometrically, if we have the graph of $y = f(x)$, the average rate of change over $[a, b]$ is the slope of the line which connects $(a, f(a))$ and $(b, f(b))$. This is called the **secant line** through these points. For that reason, some textbooks use the notation m_{sec} for the average rate of change of a function. Note that for a linear function $m = m_{\text{sec}}$, or in other words, its rate of change over an interval is the same as its average rate of change.

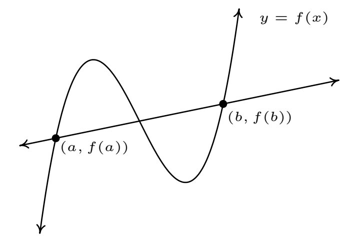


Figure 3.8: The graph of $y = f(x)$ and its secant line through $(a, f(a))$ and $(b, f(b))$

The interested reader may question the adjective 'average' in the phrase 'average rate of change'. In the figure above, we can see that the function changes wildly on $[a, b]$, yet the slope of the secant line only captures a snapshot of the action at a and b. This situation is entirely analogous to the average speed on a trip. Suppose it takes you 2 hours to travel 100 kilometres. Your average speed is $\frac{100\,\text{km}}{2\,\text{h}} = 50\,\text{km/h}$. However, it is entirely possible that at the start of your journey, you travelled 25 kilometres per hour, then sped up to 65 kilometres per hour, and so forth. The average rate of change is akin to your average speed on the trip. Your speedometer measures your speed at any one instant along the trip, your **instantaneous rate of change**, and this is one of the central themes of Calculus.

When interpreting rates of change, we interpret them the same way we did slopes. In the context of functions, it may be helpful to think of the average rate of change as:

$$\frac{\text{change in outputs}}{\text{change in inputs}}$$

Example 47 **A non-linear revenue model**

Recall from page 73, the revenue from selling x units at a price p per unit is given by the formula $R = xp$. Suppose we are in the scenario of Examples 45 and 46.

1. Find and simplify an expression for the weekly revenue $R(x)$ as a function of weekly sales x.

2. Find and interpret the average rate of change of $R(x)$ over the interval $[0, 50]$.

3. Find and interpret the average rate of change of $R(x)$ as x changes from 50 to 100 and compare that to your result in part 2.

4. Find and interpret the average rate of change of weekly revenue as weekly sales increase from 100 PortaBoys to 150 PortaBoys.

Solution

1. Since $R = xp$, we substitute $p(x) = -1.5x + 250$ from Example 46 to get $R(x) = x(-1.5x + 250) = -1.5x^2 + 250x$. Since we determined the price-demand function $p(x)$ is restricted to $0 \leq x \leq 166$, $R(x)$ is restricted to these values of x as well.

2. Using Definition 32, we get that the average rate of change is

$$\frac{\Delta R}{\Delta x} = \frac{R(50) - R(0)}{50 - 0} = \frac{8750 - 0}{50 - 0} = 175.$$

   Interpreting this slope as we have in similar situations, we conclude that for every additional PortaBoy sold during a given week, the weekly revenue increases \$175.

3. The wording of this part is slightly different than that in Definition 32, but its meaning is to find the average rate of change of R over the interval $[50, 100]$. To find this rate of change, we compute

$$\frac{\Delta R}{\Delta x} = \frac{R(100) - R(50)}{100 - 50} = \frac{10000 - 8750}{50} = 25.$$

In other words, for each additional PortaBoy sold, the revenue increases by $\$25$. Note that while the revenue is still increasing by selling more game systems, we aren't getting as much of an increase as we did in part 2 of this example. (Can you think of why this would happen?)

4. Translating the English to the mathematics, we are being asked to find the average rate of change of R over the interval $[100, 150]$. We find

$$\frac{\Delta R}{\Delta x} = \frac{R(150) - R(100)}{150 - 100} = \frac{3750 - 10000}{50} = -125.$$

This means that we are losing $\$125$ dollars of weekly revenue for each additional PortaBoy sold. (Can you think why this is possible?)

We close this section with a new look at difference quotients which were first introduced in Section 2.3. If we wish to compute the average rate of change of a function f over the interval $[x, x + h]$, then we would have

$$\frac{\Delta f}{\Delta x} = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}$$

As we have indicated, the rate of change of a function (average or otherwise) is of great importance in Calculus. (So we are not torturing you with these for nothing.) Also, we have the geometric interpretation of difference quotients which was promised to you back on page 72 – a difference quotient yields the slope of a secant line.

Exercises 3.1

Problems

In Exercises 1 – 11, find both the point-slope form and the slope-intercept form of the line with the given slope which passes through the given point.

1. $m = 3, \quad P(3, -1)$

2. $m = -2, \quad P(-5, 8)$

3. $m = -1, \quad P(-7, -1)$

4. $m = \frac{2}{3}, \quad P(-2, 1)$

5. $m = \frac{2}{3}, \quad P(-2, 1)$

6. $m = \frac{1}{7}, \quad P(-1, 4)$

7. $m = 0, \quad P(3, 117)$

8. $m = -\sqrt{2}, \quad P(0, -3)$

9. $m = -5, \quad P(\sqrt{3}, 2\sqrt{3})$

10. $m = 678, \quad P(-1, -12)$

In Exercises 11 – 21, find the slope-intercept form of the line which passes through the given points.

11. $P(0, 0), \ Q(-3, 5)$

12. $P(-1, -2), \ Q(3, -2)$

13. $P(5, 0), \ Q(0, -8)$

14. $P(3, -5), \ Q(7, 4)$

15. $P(-1, 5), \ Q(7, 5)$

16. $P(4, -8), \ Q(5, -8)$

17. $P\left(\frac{1}{2}, \frac{3}{4}\right), \ Q\left(\frac{5}{2}, -\frac{7}{4}\right)$

18. $P\left(\frac{2}{3}, \frac{7}{2}\right), \ Q\left(-\frac{1}{3}, \frac{3}{2}\right)$

19. $P\left(\sqrt{2}, -\sqrt{2}\right), \ Q\left(-\sqrt{2}, \sqrt{2}\right)$

20. $P\left(-\sqrt{3}, -1\right), \ Q\left(\sqrt{3}, 1\right)$

In Exercises 21 – 27, graph the function. Find the slope, *y*-intercept and *x*-intercept, if any exist.

21. $f(x) = 2x - 1$

22. $f(x) = 3 - x$

23. $f(x) = 3$

24. $f(x) = 0$

25. $f(x) = \frac{2}{3}x + \frac{1}{3}$

26. $f(x) = \dfrac{1 - x}{2}$

27. Find all of the points on the line $y = 2x + 1$ which are 4 units from the point $(-1, 3)$.

28. Jeff can walk comfortably at 3 miles per hour. Find a linear function d that represents the total distance Jeff can walk in t hours, assuming he doesn't take any breaks.

29. Carl can stuff 6 envelopes per *minute*. Find a linear function E that represents the total number of envelopes Carl can stuff after *t hours*, assuming he doesn't take any breaks.

30. A landscaping company charges $\$45$ per cubic yard of mulch plus a delivery charge of $\$20$. Find a linear function which computes the total cost C (in dollars) to deliver x cubic yards of mulch.

31. A plumber charges $\$50$ for a service call plus $\$80$ per hour. If she spends no longer than 8 hours a day at any one site, find a linear function that represents her total daily charges C (in dollars) as a function of time t (in hours) spent at any one given location.

32. A salesperson is paid \$200 per week plus 5% commission on her weekly sales of x dollars. Find a linear function that represents her total weekly pay, W (in dollars) in terms of x. What must her weekly sales be in order for her to earn \$475.00 for the week?

33. An on-demand publisher charges $\$22.50$ to print a 600 page book and $\$15.50$ to print a 400 page book. Find a linear function which models the cost of a book C as a function of the number of pages p. Interpret the slope of the linear function and find and interpret $C(0)$.

34. The Topology Taxi Company charges $\$2.50$ for the first fifth of a mile and $\$0.45$ for each additional fifth of a mile. Find a linear function which models the taxi fare F as a function of the number of miles driven, m. Interpret the slope of the linear function and find and interpret $F(0)$.

35. Water freezes at 0° Celsius and 32° Fahrenheit and it boils at 100°C and 212°F.

    (a) Find a linear function F that expresses temperature in the Fahrenheit scale in terms of degrees Celsius. Use this function to convert 20°C into Fahrenheit.

    (b) Find a linear function C that expresses temperature in the Celsius scale in terms of degrees Fahrenheit. Use this function to convert 110°F into Celsius.

    (c) Is there a temperature n such that $F(n) = C(n)$?

36. Legend has it that a bull Sasquatch in rut will howl approximately 9 times per hour when it is $40^{\circ}F$ outside and only 5 times per hour if it's $70^{\circ}F$. Assuming that the number of howls per hour, N, can be represented by a linear function of temperature Fahrenheit, find the number of howls per hour he'll make when it's only $20^{\circ}F$ outside. What is the applied domain of this function? Why?

37. Economic forces beyond anyone's control have changed the cost function for PortaBoys to $C(x) = 105x + 175$. Rework Example 45 with this new cost function.

38. In response to the economic forces in Exercise 37 above, the local retailer sets the selling price of a PortaBoy at \$250. Remarkably, 30 units were sold each week. When the systems went on sale for \$220, 40 units per week were sold. Rework Examples 46 and 47 with this new data. What difficulties do you encounter?

39. A local pizza store offers medium two-topping pizzas delivered for \$6.00 per pizza plus a \$1.50 delivery charge per order. On weekends, the store runs a 'game day' special: if six or more medium two-topping pizzas are ordered, they are \$5.50 each with no delivery charge. Write a piecewise-defined linear function which calculates the cost C (in dollars) of p medium two-topping pizzas delivered during a weekend.

40. A restaurant offers a buffet which costs \$15 per person. For parties of 10 or more people, a group discount applies, and the cost is \$12.50 per person. Write a piecewise-defined linear function which calculates the total bill T of a party of n people who all choose the buffet.

41. A mobile plan charges a base monthly rate of \$10 for the first 500 minutes of air time plus a charge of 15¢ for each additional minute. Write a piecewise-defined linear function which calculates the monthly cost C (in dollars) for using m minutes of air time.

    HINT: You may want to revisit Exercise 75 in Section 2.3

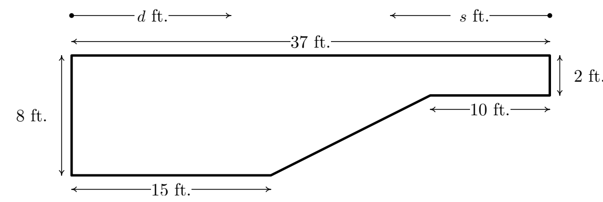
42. The local pet shop charges 12¢ per cricket up to 100 crickets, and 10¢ per cricket thereafter. Write a piecewise-defined linear function which calculates the price P, in dollars, of purchasing c crickets.

43. The cross-section of a swimming pool is below. Write a piecewise-defined linear function which describes the depth of the pool, D (in feet) as a function of:

    (a) the distance (in feet) from the edge of the shallow end of the pool, d.

    (b) the distance (in feet) from the edge of the deep end of the pool, s.

    (c) Graph each of the functions in (a) and (b). Discuss with your classmates how to transform one into the other and how they relate to the diagram of the pool.



In Exercises 44 – 50, compute the average rate of change of the function over the specified interval.

44. $f(x) = x^3,\ [-1, 2]$

45. $f(x) = \dfrac{1}{x},\ [1, 5]$

46. $f(x) = \sqrt{x},\ [0, 16]$

47. $f(x) = x^2,\ [-3, 3]$

48. $f(x) = \dfrac{x+4}{x-3},\ [5, 7]$

49. $f(x) = 3x^2 + 2x - 7,\ [-4, 2]$

In Exercises 50 – 54, compute the average rate of change of the given function over the interval $[x, x+h]$. Here we assume $[x, x+h]$ is in the domain of the function.

50. $f(x) = x^3$

51. $f(x) = \dfrac{1}{x}$

52. $f(x) = \dfrac{x+4}{x-3}$

53. $f(x) = 3x^2 + 2x - 7$

54. Using data from Bureau of Transportation Statistics, the average fuel economy F in miles per gallon for passenger cars in the US can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Find and interpret the average rate of change of F over the interval $[0, 28]$.

55. The temperature T in degrees Fahrenheit t hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

    (a) Find and interpret $T(4)$, $T(8)$ and $T(12)$.

    (b) Find and interpret the average rate of change of T over the interval $[4, 8]$.

    (c) Find and interpret the average rate of change of T from $t = 8$ to $t = 12$.

    (d) Find and interpret the average rate of temperature change between 10 AM and 6 PM.

56. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in *hundreds*, to produce x *thousand* pens. Find and interpret the average rate of change as production is increased from making 3000 to 5000 pens.

57. With the help of your classmates find several other "real-world" examples of rates of change that are used to describe non-linear phenomena.

58. With the help of your classmates find several other "real-world" examples of rates of change that are used to describe non-linear phenomena.

(Parallel Lines) Recall from high school that parallel lines have the same slope. (Please note that two vertical lines are also parallel to one another even though they have an undefined slope.) In Exercises 59 – 65, you are given a line and a point which is not on that line. Find the line parallel to the given line which passes through the given point.

59. $y = 3x + 2,\ P(0, 0)$

60. $y = -6x + 5,\ P(3, 2)$

61. $y = \frac{2}{3}x - 7,\ P(6, 0)$

62. $y = \dfrac{4 - x}{3},\ P(1, -1)$

63. $y = 6,\ P(3, -2)$

64. $x = 1,\ P(-5, 0)$

(Perpendicular Lines) Recall from high school that two non-vertical lines are perpendicular if and only if they have negative reciprocal slopes. That is to say, if one line has slope m_1 and the other has slope m_2 then $m_1 \cdot m_2 = -1$. (You will be guided through a proof of this result in Exercise 71.) Please note that a horizontal line is perpendicular to a vertical line and vice versa, so we assume $m_1 \neq 0$ and $m_2 \neq 0$. In Exercises 65 – 71, you are given a line and a point which is not on that line. Find the line perpendicular to the given line which passes through the given point.

65. $y = \frac{1}{3}x + 2,\ P(0, 0)$

66. $y = -6x + 5,\ P(3, 2)$

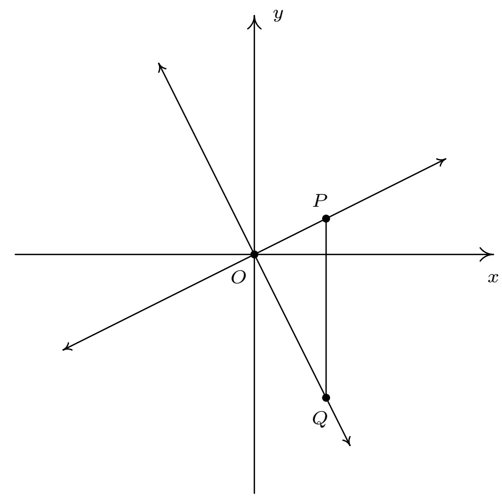
67. $y = \frac{2}{3}x - 7,\ P(6, 0)$

68. $y = \dfrac{4 - x}{3},\ P(1, -1)$

69. $y = 6,\ P(3, -2)$

70. $x = 1,\ P(-5, 0)$

71. We shall now prove that $y = m_1x + b_1$ is perpendicular to $y = m_2x + b_2$ if and only if $m_1 \cdot m_2 = -1$. To make our lives easier we shall assume that $m_1 > 0$ and $m_2 < 0$. We can also "move" the lines so that their point of intersection is the origin without messing things up, so we'll assume $b_1 = b_2 = 0$. (Take a moment with your classmates to discuss why this is okay.) Graphing the lines and plotting the points $O(0, 0)$, $P(1, m_1)$ and $Q(1, m_2)$ gives us the following set up.



The line $y = m_1x$ will be perpendicular to the line $y = m_2x$ if and only if $\triangle OPQ$ is a right triangle. Let d_1 be the distance from O to P, let d_2 be the distance from O to Q and let d_3 be the distance from P to Q. Use the Pythagorean Theorem to show that $\triangle OPQ$ is a right triangle if and only if $m_1 \cdot m_2 = -1$ by showing $d_1^2 + d_2^2 = d_3^2$ if and only if $m_1 \cdot m_2 = -1$.

72. Show that if $a \neq b$, the line containing the points (a, b) and (b, a) is perpendicular to the line $y = x$. (Coupled with the result from Example 11 on page 31, we have now shown that the line $y = x$ is a *perpendicular* bisector of the line segment connecting (a, b) and (b, a). This means the points (a, b) and (b, a) are symmetric about the line $y = x$. We will revisit this symmetry in section **??**.)

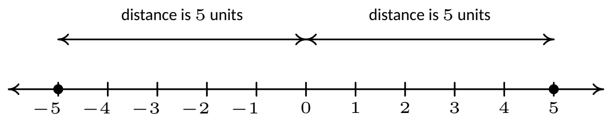
73. The function defined by $I(x) = x$ is called the Identity Function.

    (a) Discuss with your classmates why this name makes sense.

    (b) Show that the point-slope form of a line (Equation 15) can be obtained from I using a sequence of the transformations defined in Section 2.6.

3.2 Absolute Value Functions

There are a few ways to describe what is meant by the absolute value $|x|$ of a real number x. You may have been taught that $|x|$ is the distance from the real number x to 0 on the number line. So, for example, $|5| = 5$ and $|-5| = 5$, since each is 5 units from 0 on the number line.



Another way to define absolute value is by the equation $|x| = \sqrt{x^2}$. Using this definition, we have $|5| = \sqrt{(5)^2} = \sqrt{25} = 5$ and $|-5| = \sqrt{(-5)^2} = \sqrt{25} = 5$. The long and short of both of these procedures is that $|x|$ takes negative real numbers and assigns them to their positive counterparts while it leaves positive numbers alone. This last description is the one we shall adopt, and is summarized in the following definition.

Definition 33 Absolute value function

The **absolute value** of a real number x, denoted $|x|$, is given by

$$|x| = \begin{cases} -x, & \text{if} \quad x < 0 \\ x, & \text{if} \quad x \geq 0 \end{cases}$$

In Definition 33, we define $|x|$ using a piecewise-defined function. (See page 64 in Section 2.3.) To check that this definition agrees with what we previously understood as absolute value, note that since $5 \geq 0$, to find $|5|$ we use the rule $|x| = x$, so $|5| = 5$. Similarly, since $-5 < 0$, we use the rule $|x| = -x$, so that $|-5| = -(-5) = 5$. This is one of the times when it's best to interpret the expression '$-x$' as 'the opposite of x' as opposed to 'negative x'. Before we begin studying absolute value functions, we remind ourselves of the properties of absolute value.

Theorem 13 Properties of Absolute Value

Let a, b and x be real numbers and let n be an integer. Then

- **Product Rule:** $|ab| = |a||b|$
- **Power Rule:** $|a^n| = |a|^n$ whenever a^n is defined
- **Quotient Rule:** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$, provided $b \neq 0$

Equality Properties:

- $|x| = 0$ if and only if $x = 0$.
- For $c > 0$, $|x| = c$ if and only if $x = c$ or $-x = c$.
- For $c < 0$, $|x| = c$ has no solution.

The proofs of the Product and Quotient Rules in Theorem 13 boil down to checking four cases: when both a and b are positive; when they are both negative; when one is positive and the other is negative; and when one or both are zero.

For example, suppose we wish to show that $|ab| = |a||b|$. We need to show that this equation is true for all real numbers a and b. If a and b are both positive, then so is ab. Hence, $|a| = a$, $|b| = b$ and $|ab| = ab$. Hence, the equation $|ab| = |a||b|$ is the same as $ab = ab$ which is true. If both a and b are negative, then ab is positive. Hence, $|a| = -a$, $|b| = -b$ and $|ab| = ab$. The equation $|ab| = |a||b|$ becomes $ab = (-a)(-b)$, which is true. Suppose a is positive and b is negative. Then ab is negative, and we have $|ab| = -ab$, $|a| = a$ and $|b| = -b$. The equation $|ab| = |a||b|$ reduces to $-ab = a(-b)$ which is true. A symmetric argument shows the equation $|ab| = |a||b|$ holds when a is negative and b is positive. Finally, if either a or b (or both) are zero, then both sides of $|ab| = |a||b|$ are zero, so the equation holds in this case, too. All of this rhetoric has shown that the equation $|ab| = |a||b|$ holds true in all cases.

The proof of the Quotient Rule is very similar, with the exception that $b \neq 0$. The Power Rule can be shown by repeated application of the Product Rule. The 'Equality Properties' can be proved using Definition 33 and by looking at the cases when $x \geq 0$, in which case $|x| = x$, or when $x < 0$, in which case $|x| = -x$. For example, if $c > 0$, and $|x| = c$, then if $x \geq 0$, we have $x = |x| = c$. If, on the other hand, $x < 0$, then $-x = |x| = c$, so $x = -c$. The remaining properties are proved similarly and are left for the Exercises. Our first example reviews how to solve basic equations involving absolute value using the properties listed in Theorem 13.

Example 48 Solving equations with absolute values

Solve each of the following equations.

1. $|3x - 1| = 6$
2. $3 - |x + 5| = 1$
3. $3|2x+1| - 5 = 0$
4. $4 - |5x + 3| = 5$
5. $|x| = x^2 - 6$
6. $|x - 2| + 1 = x$

Solution

1. The equation $|3x-1| = 6$ is of the form $|x| = c$ for $c > 0$, so by the Equality Properties, $|3x-1| = 6$ is equivalent to $3x-1 = 6$ or $3x-1 = -6$. Solving the former, we arrive at $x = \frac{7}{3}$, and solving the latter, we get $x = -\frac{5}{3}$. We may check both of these solutions by substituting them into the original equation and showing that the arithmetic works out.

2. To use the Equality Properties to solve $3 - |x + 5| = 1$, we first isolate the absolute value.

$$\begin{aligned} 3 - |x+5| &= 1 \\ -|x+5| &= -2 \qquad \text{subtract 3} \\ |x+5| &= 2 \qquad \text{divide by } -1 \end{aligned}$$

From the Equality Properties, we have $x + 5 = 2$ or $x + 5 = -2$, and get our solutions to be $x = -3$ or $x = -7$. We leave it to the reader to check both answers in the original equation.

3. As in the previous example, we first isolate the absolute value in the equation $3|2x+1|-5=0$ and get $|2x+1|=\frac{5}{3}$. Using the Equality Properties, we have $2x+1=\frac{5}{3}$ or $2x+1=-\frac{5}{3}$. Solving the former gives $x=\frac{1}{3}$ and solving the latter gives $x=-\frac{4}{3}$. As usual, we may substitute both answers in the original equation to check.

4. Upon isolating the absolute value in the equation $4-|5x+3|=5$, we get $|5x+3|=-1$. At this point, we know there cannot be any real solution, since, by definition, the absolute value of *anything* is never negative. We are done.

5. The equation $|x|=x^2-6$ presents us with some difficulty, since x appears both inside and outside of the absolute value. Moreover, there are values of x for which x^2-6 is positive, negative and zero, so we cannot use the Equality Properties without the risk of introducing extraneous solutions, or worse, losing solutions. For this reason, we break equations like this into cases by rewriting the term in absolute values, $|x|$, using Definition 33. For $x<0$, $|x|=-x$, so for $x<0$, the equation $|x|=x^2-6$ is equivalent to $-x=x^2-6$. Rearranging this gives us $x^2+x-6=0$, or $(x+3)(x-2)=0$. We get $x=-3$ or $x=2$. Since only $x=-3$ satisfies $x<0$, this is the answer we keep. For $x\geq 0$, $|x|=x$, so the equation $|x|=x^2-6$ becomes $x=x^2-6$. From this, we get $x^2-x-6=0$ or $(x-3)(x+2)=0$. Our solutions are $x=3$ or $x=-2$, and since only $x=3$ satisfies $x\geq 0$, this is the one we keep. Hence, our two solutions to $|x|=x^2-6$ are $x=-3$ and $x=3$.

6. To solve $|x-2|+1=x$, we first isolate the absolute value and get $|x-2|=x-1$. Since we see x both inside and outside of the absolute value, we break the equation into cases. The term with absolute values here is $|x-2|$, so we replace 'x' with the quantity '$(x-2)$' in Definition 33 to get

$$|x-2|=\begin{cases} -(x-2), & \text{if} \quad (x-2)<0 \\ (x-2), & \text{if} \quad (x-2)\geq 0 \end{cases}$$

Simplifying yields

$$|x-2|=\begin{cases} -x+2, & \text{if} \quad x<2 \\ x-2, & \text{if} \quad x\geq 2 \end{cases}$$

So, for $x<2$, $|x-2|=-x+2$ and our equation $|x-2|=x-1$ becomes $-x+2=x-1$, which gives $x=\frac{3}{2}$. Since this solution satisfies $x<2$, we keep it. Next, for $x\geq 2$, $|x-2|=x-2$, so the equation $|x-2|=x-1$ becomes $x-2=x-1$. Here, the equation reduces to $-2=-1$, which signifies we have no solutions here. Hence, our only solution is $x=\frac{3}{2}$.

Next, we turn our attention to graphing absolute value functions. Our strategy in the next example is to make liberal use of Definition 33 along with what we know about graphing linear functions (from Section 3.1) and piecewise-defined functions (from Section 2.3).

Example 49 **Graphing absolute value functions**

Graph each of the following functions.

1. $f(x) = |x|$
2. $g(x) = |x-3|$
3. $h(x) = |x| - 3$
4. $i(x) = 4 - 2|3x+1|$

Find the zeros of each function and the x- and y-intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, list the intervals on which the function is increasing, decreasing or constant, and find the relative and absolute extrema, if they exist.

Solution

1. To find the zeros of f, we set $f(x) = 0$. We get $|x| = 0$, which, by Theorem 13 gives us $x = 0$. Since the zeros of f are the x-coordinates of the x-intercepts of the graph of $y = f(x)$, we get $(0,0)$ as our only x-intercept. To find the y-intercept, we set $x = 0$, and find $y = f(0) = 0$, so that $(0,0)$ is our y-intercept as well. Using Definition 33, we get

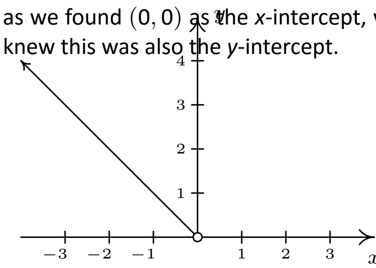
Since functions can have at most one y-intercept (Do you know why?), as soon as we found $(0,0)$ as the x-intercept, we knew this was also the y-intercept.

$$f(x) = |x| = \begin{cases} -x, & \text{if} \quad x < 0 \\ x, & \text{if} \quad x \geq 0 \end{cases}$$

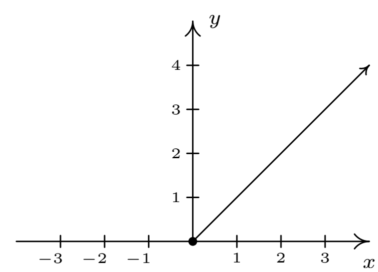
Hence, for $x < 0$, we are graphing the line $y = -x$; for $x \geq 0$, we have the line $y = x$. Proceeding as we did in Section 2.5, we get the first two graphs in Figure 3.9.

Notice that we have an 'open circle' at $(0,0)$ in the graph when $x < 0$. As we have seen before, this is due to the fact that the points on $y = -x$ approach $(0,0)$ as the x-values approach 0. Since x is required to be strictly less than zero on this stretch, the open circle is drawn at the origin. However, notice that when $x \geq 0$, we get to fill in the point at $(0,0)$, which effectively 'plugs' the hole indicated by the open circle. Thus our final result is the graph at the bottom of Figure 3.9.

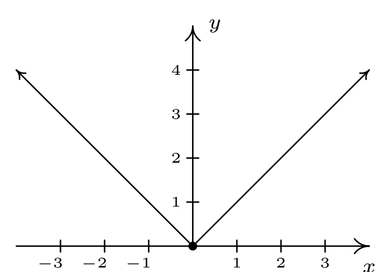
By projecting the graph to the x-axis, we see that the domain is $(-\infty, \infty)$. Projecting to the y-axis gives us the range $[0, \infty)$. The function is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$. The relative minimum value of f is the same as the absolute minimum, namely 0 which occurs at $(0,0)$. There is no relative maximum value of f. There is also no absolute maximum value of f, since the y values on the graph extend infinitely upwards.



$f(x) = |x|, x < 0$



$f(x) = |x|, x \geq 0$



$f(x) = |x|$

Figure 3.9: Constructing the graph of $f(x) = |x|$

2. To find the zeros of g, we set $g(x) = |x-3| = 0$. By Theorem 13, we get $x - 3 = 0$ so that $x = 3$. Hence, the x-intercept is $(3,0)$. To find our y-intercept, we set $x = 0$ so that $y = g(0) = |0-3| = 3$, which yields $(0,3)$ as our y-intercept. To graph $g(x) = |x-3|$, we use Definition 33 to rewrite g as

$$g(x) = |x-3| = \begin{cases} -(x-3), & \text{if} \quad (x-3) < 0 \\ (x-3), & \text{if} \quad (x-3) \geq 0 \end{cases}$$

Simplifying, we get

$$g(x) = \begin{cases} -x+3, & \text{if} \quad x < 3 \\ x-3, & \text{if} \quad x \geq 3 \end{cases}$$

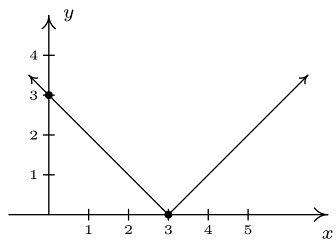


Figure 3.10: $g(x) = |x - 3|$

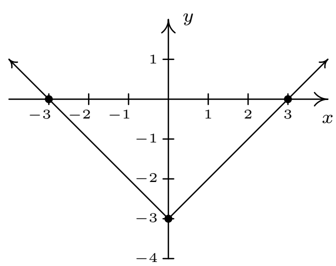


Figure 3.11: $h(x) = |x| - 3$

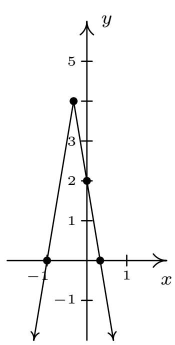


Figure 3.12: $i(x) = 4 - 2|3x + 1|$

As before, the open circle we introduce at $(3, 0)$ from the graph of $y = -x+3$ is filled by the point $(3, 0)$ from the line $y = x-3$. We determine the domain as $(-\infty, \infty)$ and the range as $[0, \infty)$. The function g is increasing on $[3, \infty)$ and decreasing on $(-\infty, 3]$. The relative and absolute minimum value of g is 0 which occurs at $(3, 0)$. As before, there is no relative or absolute maximum value of g.

3. Setting $h(x) = 0$ to look for zeros gives $|x| - 3 = 0$. As in Example 48, we isolate the absolute value to get $|x| = 3$ so that $x = 3$ or $x = -3$. As a result, we have a pair of x-intercepts: $(-3, 0)$ and $(3, 0)$. Setting $x = 0$ gives $y = h(0) = |0| - 3 = -3$, so our y-intercept is $(0, -3)$. As before, we rewrite the absolute value in h to get

$$h(x) = \begin{cases} -x - 3, & \text{if} \quad x < 0 \\ x - 3, & \text{if} \quad x \geq 0 \end{cases}$$

Once again, the open circle at $(0, -3)$ from one piece of the graph of h is filled by the point $(0, -3)$ from the other piece of h. From the graph, we determine the domain of h is $(-\infty, \infty)$ and the range is $[-3, \infty)$. On $[0, \infty)$, h is increasing; on $(-\infty, 0]$ it is decreasing. The relative minimum occurs at the point $(0, -3)$ on the graph, and we see -3 is both the relative and absolute minimum value of h. Also, h has no relative or absolute maximum value.

4. As before, we set $i(x) = 0$ to find the zeros of i and get $4 - 2|3x + 1| = 0$. Isolating the absolute value term gives $|3x + 1| = 2$, so either $3x + 1 = 2$ or $3x + 1 = -2$. We get $x = \frac{1}{3}$ or $x = -1$, so our x-intercepts are $\left(\frac{1}{3}, 0\right)$ and $(-1, 0)$. Substituting $x = 0$ gives $y = i(0) = 4 - 2|3(0) + 1| = 2$, for a y-intercept of $(0, 2)$. Rewriting the formula for $i(x)$ without absolute values gives

$$\begin{aligned} i(x) &= \begin{cases} 4 - 2(-(3x + 1)), & \text{if} \quad (3x + 1) < 0 \\ 4 - 2(3x + 1), & \text{if} \quad (3x + 1) \geq 0 \end{cases} \\ &= \begin{cases} 6x + 6, & \text{if} \quad x < -\frac{1}{3} \\ -6x + 2, & \text{if} \quad x \geq -\frac{1}{3} \end{cases} \end{aligned}$$

The usual analysis near the trouble spot $x = -\frac{1}{3}$ gives that the 'corner' of this graph is $\left(-\frac{1}{3}, 4\right)$, and we get the distinctive '$\vee$' shape: see Figure 3.12.

The domain of i is $(-\infty, \infty)$ while the range is $(-\infty, 4]$. The function i is increasing on $\left(-\infty, -\frac{1}{3}\right]$ and decreasing on $\left[-\frac{1}{3}, \infty\right)$. The relative maximum occurs at the point $\left(-\frac{1}{3}, 4\right)$ and the relative and absolute maximum value of i is 4. Since the graph of i extends downwards forever more, there is no absolute minimum value. As we can see from the graph, there is no relative minimum, either.

Note that all of the functions in the previous example bear the characteristic '$\vee$' shape of the graph of $y = |x|$. We could have graphed the functions g, h and i in Example 49 starting with the graph of $f(x) = |x|$ and applying transformations as in Section 2.6 as our next example illustrates.

Example 50 Graphing using transformations

Graph the following functions starting with the graph of $f(x) = |x|$ and using transformations.

1. $g(x) = |x - 3|$
2. $h(x) = |x| - 3$
3. $i(x) = 4 - 2|3x + 1|$

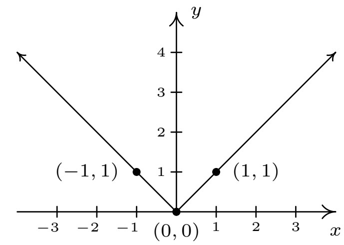


Figure 3.13: $f(x) = |x|$ with three labelled points

Solution We begin by graphing $f(x) = |x|$ and labelling three points, $(-1, 1)$, $(0, 0)$ and $(1, 1)$, as in Figure 3.13

1. Since $g(x) = |x - 3| = f(x - 3)$, Theorem 12 tells us to *add* 3 to each of the x-values of the points on the graph of $y = f(x)$ to obtain the graph of $y = g(x)$. This shifts the graph of $y = f(x)$ to the *right* 3 units and moves the point $(-1, 1)$ to $(2, 1)$, $(0, 0)$ to $(3, 0)$ and $(1, 1)$ to $(4, 1)$. Connecting these points in the classic '$\vee$' fashion produces the graph of $y = g(x)$ in Figure 3.14.

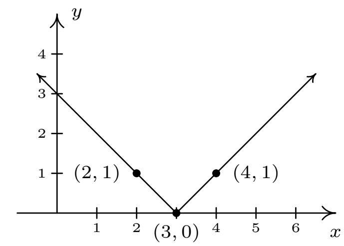


Figure 3.14: $g(x) = |x - 3| = f(x - 3)$

2. For $h(x) = |x| - 3 = f(x) - 3$, Theorem 12 tells us to *subtract* 3 from each of the y-values of the points on the graph of $y = f(x)$ to obtain the graph of $y = h(x)$. This shifts the graph of $y = f(x)$ *down* 3 units and moves $(-1, 1)$ to $(-1, -2)$, $(0, 0)$ to $(0, -3)$ and $(1, 1)$ to $(1, -2)$. Connecting these points with the '$\vee$' shape produces our graph of $y = h(x)$: see Figure 3.15.

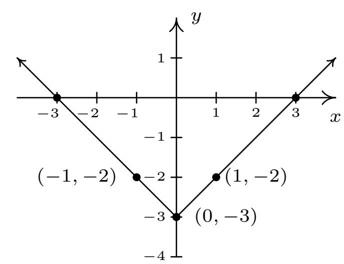


Figure 3.15: $h(x) = |x| - 3 = f(x) - 3$

3. We re-write $i(x) = 4 - 2|3x + 1| = 4 - 2f(3x + 1) = -2f(3x + 1) + 4$ and apply Theorem 12. First, we take care of the changes on the 'inside' of the absolute value. Instead of $|x|$, we have $|3x + 1|$, so, in accordance with Theorem 12, we first *subtract* 1 from each of the x-values of points on the graph of $y = f(x)$, then *divide* each of those new values by 3. This effects a horizontal shift *left* 1 unit followed by a horizontal *shrink* by a factor of 3. These transformations move $(-1, 1)$ to $\left(-\frac{2}{3}, 1\right)$, $(0, 0)$ to $\left(-\frac{1}{3}, 0\right)$ and $(1, 1)$ to $(0, 1)$. Next, we take care of what's happening 'outside of' the absolute value. Theorem 12 instructs us to first *multiply* each y-value of these new points by -2 then *add* 4. Geometrically, this corresponds to a vertical *stretch* by a factor of 2, a reflection across the x-axis and finally, a vertical shift *up* 4 units. These transformations move $\left(-\frac{2}{3}, 1\right)$ to $\left(-\frac{2}{3}, 2\right)$, $\left(-\frac{1}{3}, 0\right)$ to $\left(-\frac{1}{3}, 4\right)$, and $(0, 1)$ to $(0, 2)$. Connecting these points with the usual '$\vee$' shape produces our graph of $y = i(x)$.

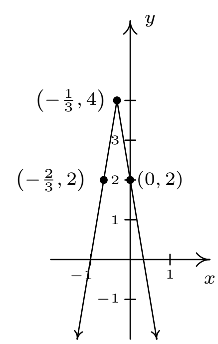


Figure 3.16: $i(x) = 4 - 2|3x + 1| = -2f(3x + 1) + 4$

While the methods in Section 2.6 can be used to graph an entire family of absolute value functions, not all functions involving absolute values posses the characteristic '$\vee$' shape. As the next example illustrates, often there is no substitute for appealing directly to the definition.

Example 51 A more complicated example

Graph each of the following functions. Find the zeros of each function and the x- and y-intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, list the intervals on which the function is increasing, decreasing or constant, and find the relative and absolute extrema, if they exist.

1. $f(x) = \dfrac{|x|}{x}$

2. $g(x) = |x+2| - |x-3| + 1$

Solution

1. We first note that, due to the fraction in the formula of $f(x)$, $x \neq 0$. Thus the domain is $(-\infty, 0) \cup (0, \infty)$. To find the zeros of f, we set $f(x) = \frac{|x|}{x} = 0$. This last equation implies $|x| = 0$, which, from Theorem 13, implies $x = 0$. However, $x = 0$ is not in the domain of f, which means we have, in fact, no x-intercepts. We have no y-intercepts either, since $f(0)$ is undefined. Re-writing the absolute value in the function gives

$$f(x) = \begin{cases} \dfrac{-x}{x}, & \text{if} \quad x < 0 \\ \dfrac{x}{x}, & \text{if} \quad x > 0 \end{cases} = \begin{cases} -1, & \text{if} \quad x < 0 \\ 1, & \text{if} \quad x > 0 \end{cases}$$

To graph this function, we graph two horizontal lines: $y = -1$ for $x < 0$ and $y = 1$ for $x > 0$. We have open circles at $(0, -1)$ and $(0, 1)$ (Can you explain why?) so we get the graph in figure 3.17.

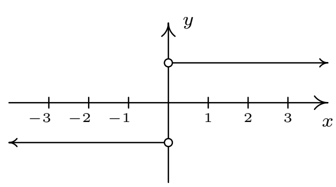


Figure 3.17: $f(x) = \dfrac{|x|}{x}$

As we found earlier, the domain is $(-\infty, 0) \cup (0, \infty)$. The range consists of just two y-values: $\{-1, 1\}$. The function f is constant on $(-\infty, 0)$ and $(0, \infty)$. The local minimum value of f is the absolute minimum value of f, namely -1; the local maximum and absolute maximum values for f also coincide — they both are 1. Every point on the graph of f is simultaneously a relative maximum and a relative minimum. (Can you remember why in light of Definition 28? This was explored in the Exercises in Section **??**.)

2. To find the zeros of g, we set $g(x) = 0$. The result is $|x+2| - |x-3| + 1 = 0$. Attempting to isolate the absolute value term is complicated by the fact that there are **two** terms with absolute values. In this case, it easier to proceed using cases by re-writing the function g with two separate applications of Definition 33 to remove each instance of the absolute values, one at a time. In the first round we get

$$\begin{aligned} g(x) &= \begin{cases} -(x+2) - |x-3| + 1, & \text{if} \quad (x+2) < 0 \\ (x+2) - |x-3| + 1, & \text{if} \quad (x+2) \geq 0 \end{cases} \\ &= \begin{cases} -x - 1 - |x-3|, & \text{if} \quad x < -2 \\ x + 3 - |x-3|, & \text{if} \quad x \geq -2 \end{cases} \end{aligned}$$

Given that

$$|x-3| = \begin{cases} -(x-3), & \text{if} \quad (x-3) < 0 \\ x-3, & \text{if} \quad (x-3) \geq 0 \end{cases} = \begin{cases} -x+3, & \text{if} \quad x < 3 \\ x-3, & \text{if} \quad x \geq 3 \end{cases},$$

we need to break up the domain again at $x = 3$. Note that if $x < -2$, then $x < 3$, so we replace $|x-3|$ with $-x+3$ for that part of the domain, too. Our completed revision of the form of g yields

$$g(x) = \begin{cases} -x-1-(-x+3), & \text{if} \quad x < -2 \\ x+3-(-x+3), & \text{if} \quad x \geq -2 \text{ and } x < 3 \\ x+3-(x-3), & \text{if} \quad x \geq 3 \end{cases}$$

$$= \begin{cases} -4, & \text{if} \quad x < -2 \\ 2x, & \text{if} \quad -2 \leq x < 3 \\ 6, & \text{if} \quad x \geq 3 \end{cases}$$

To solve $g(x) = 0$, we see that the only piece which contains a variable is $g(x) = 2x$ for $-2 \leq x < 3$. Solving $2x = 0$ gives $x = 0$. Since $x = 0$ is in the interval $[-2, 3)$, we keep this solution and have $(0, 0)$ as our only x-intercept. Accordingly, the y-intercept is also $(0, 0)$. To graph g, we start with $x < -2$ and graph the horizontal line $y = -4$ with an open circle at $(-2, -4)$. For $-2 \leq x < 3$, we graph the line $y = 2x$ and the point $(-2, -4)$ patches the hole left by the previous piece. An open circle at $(3, 6)$ completes the graph of this part. Finally, we graph the horizontal line $y = 6$ for $x \geq 3$, and the point $(3, 6)$ fills in the open circle left by the previous part of the graph. The finished graph is given in Figure 3.18

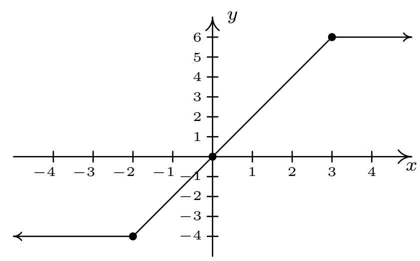


Figure 3.18: $g(x) = |x + 2| - |x - 3| + 1$

The domain of g is all real numbers, $(-\infty, \infty)$, and the range of g is all real numbers between -4 and 6 inclusive, $[-4, 6]$. The function is increasing on $[-2, 3]$ and constant on $(-\infty, -2]$ and $[3, \infty)$. The relative minimum value of f is -4 which matches the absolute minimum. The relative and absolute maximum values also coincide at 6. Every point on the graph of $y = g(x)$ for $x < -2$ and $x > 3$ yields both a relative minimum and relative maximum. The point $(-2, -4)$, however, gives only a relative minimum and the point $(3, 6)$ yields only a relative maximum. (Recall the Exercises in Section **??** which dealt with constant functions.)

Many of the applications that the authors are aware of involving absolute values also involve absolute value inequalities. For that reason, we save our discussion of applications for Section 3.4.

Exercises 3.2

Problems

In Exercises 1 – 16, solve the equation.

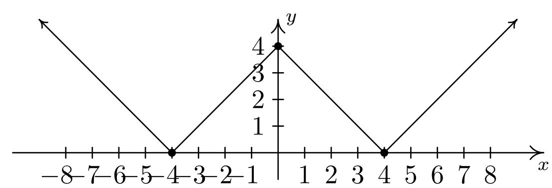
1. $|x| = 6$
2. $|3x - 1| = 10$
3. $|4 - x| = 7$
4. $4 - |x| = 3$
5. $2|5x + 1| - 3 = 0$
6. $|7x - 1| + 2 = 0$
7. $\dfrac{5 - |x|}{2} = 1$
8. $\frac{2}{3}|5 - 2x| - \frac{1}{2} = 5$
9. $|x| = x + 3$
10. $|2x - 1| = x + 1$
11. $4 - |x| = 2x + 1$
12. $|x - 4| = x - 5$
13. $|x| = x^2$
14. $|x| = 12 - x^2$
15. $|x^2 - 1| = 3$

Prove that if $|f(x)| = |g(x)|$ then either $f(x) = g(x)$ or $f(x) = -g(x)$. Use that result to solve the equations in Exercises 16 – 22.

16. $|3x - 2| = |2x + 7|$
17. $|3x + 1| = |4x|$
18. $|1 - 2x| = |x + 1|$
19. $|4 - x| - |x + 2| = 0$
20. $|2 - 5x| = 5|x + 1|$
21. $3|x - 1| = 2|x + 1|$

In Exercises 22 – 34, graph the function. Find the zeros of each function and the x- and y-intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, list the intervals on which the function is increasing, decreasing or constant, and find the relative and absolute extrema, if they exist.

22. $f(x) = |x + 4|$
23. $f(x) = |x| + 4$
24. $f(x) = |4x|$
25. $f(x) = -3|x|$
26. $f(x) = 3|x + 4| - 4$
27. $f(x) = \frac{1}{3}|2x - 1|$
28. $f(x) = \dfrac{|x + 4|}{x + 4}$
29. $f(x) = \dfrac{|2 - x|}{2 - x}$
30. $f(x) = x + |x| - 3$
31. $f(x) = |x + 2| - x$
32. $f(x) = |x + 2| - |x|$
33. $f(x) = |x + 4| + |x - 2|$
34. With the help of your classmates, find an absolute value function whose graph is given below.



35. With help from your classmates, prove the second, third and fifth parts of Theorem 13.
36. Prove **The Triangle Inequality:** For all real numbers a and b, $|a + b| \leq |a| + |b|$.

3.3 Quadratic Functions

You may recall studying quadratic equations in high school. In this section, we review those equations in the context of our next family of functions: the quadratic functions.

Definition 34 Quadratic function

A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a, b and c are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

The most basic quadratic function is $f(x) = x^2$, whose graph appears below. Its shape should look familiar from high school – it is called a **parabola**. The point $(0, 0)$ is called the **vertex** of the parabola. In this case, the vertex is a relative minimum and is also the where the absolute minimum value of f can be found.

Much like many of the absolute value functions in Section 3.2, knowing the graph of $f(x) = x^2$ enables us to graph an entire family of quadratic functions using transformations.

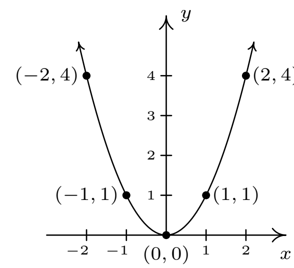


Figure 3.19: The graph of the basic quadratic function $f(x) = x^2$

Example 52 Graphics quadratic functions

Graph the following functions starting with the graph of $f(x) = x^2$ and using transformations. Find the vertex, state the range and find the x- and y-intercepts, if any exist.

1. $g(x) = (x + 2)^2 - 3$
2. $h(x) = -2(x - 3)^2 + 1$

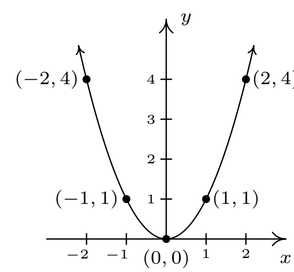


Figure 3.20: The graph $y = x^2$ with points labelled

Solution

1. Since $g(x) = (x+2)^2 - 3 = f(x+2) - 3$, Theorem 12 instructs us to first *subtract* 2 from each of the x-values of the points on $y = f(x)$. This shifts the graph of $y = f(x)$ to the *left* 2 units and moves $(-2, 4)$ to $(-4, 4)$, $(-1, 1)$ to $(-3, 1)$, $(0, 0)$ to $(-2, 0)$, $(1, 1)$ to $(-1, 1)$ and $(2, 4)$ to $(0, 4)$. Next, we *subtract* 3 from each of the y-values of these new points. This moves the graph *down* 3 units and moves $(-4, 4)$ to $(-4, 1)$, $(-3, 1)$ to $(-3, -2)$, $(-2, 0)$ to $(-2, 3)$, $(-1, 1)$ to $(-1, -2)$ and $(0, 4)$ to $(0, 1)$. We connect the dots in parabolic fashion to get the graph in Figure 3.21.

   From the graph, we see that the vertex has moved from $(0, 0)$ on the graph of $y = f(x)$ to $(-2, -3)$ on the graph of $y = g(x)$. This sets $[-3, \infty)$ as the range of g. We see that the graph of $y = g(x)$ crosses the x-axis twice, so we expect two x-intercepts. To find these, we set $y = g(x) = 0$ and solve. Doing so yields the equation $(x+2)^2 - 3 = 0$, or $(x+2)^2 = 3$. Extracting square roots gives $x + 2 = \pm\sqrt{3}$, or $x = -2 \pm \sqrt{3}$. Our x-intercepts are $(-2 - \sqrt{3}, 0) \approx (-3.73, 0)$ and $(-2 + \sqrt{3}, 0) \approx (-0.27, 0)$. The y-intercept of the graph, $(0, 1)$ was one of the points we originally plotted, so we are done.

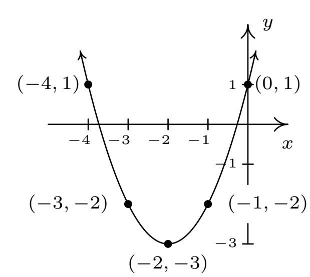


Figure 3.21: $g(x) = f(x + 2) - 3 = (x + 2)^2 - 3$

2. Following Theorem 12 once more, to graph $h(x) = -2(x - 3)^2 + 1 = -2f(x - 3) + 1$, we first start by *adding* 3 to each of the x-values of the

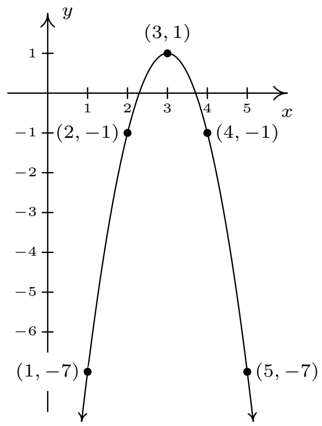


Figure 3.22: $h(x) = -2f(x-3)+1 = -2(x-3)^2+1$

points on the graph of $y = f(x)$. This effects a horizontal shift *right* 3 units and moves $(-2,4)$ to $(1,4)$, $(-1,1)$ to $(2,1)$, $(0,0)$ to $(3,0)$, $(1,1)$ to $(4,1)$ and $(2,4)$ to $(5,4)$. Next, we *multiply* each of our y-values first by -2 and then *add* 1 to that result. Geometrically, this is a vertical *stretch* by a factor of 2, followed by a reflection about the x-axis, followed by a vertical shift *up* 1 unit. This moves $(1,4)$ to $(1,-7)$, $(2,1)$ to $(2,-1)$, $(3,0)$ to $(3,1)$, $(4,1)$ to $(4,-1)$ and $(5,4)$ to $(5,-7)$, giving us the graph in Figure 3.22.

The vertex is $(3,1)$ which makes the range of h $(-\infty, 1]$. From our graph, we know that there are two x-intercepts, so we set $y = h(x) = 0$ and solve. We get $-2(x-3)^2+1 = 0$ which gives $(x-3)^2 = \frac{1}{2}$. Extracting square roots (and rationalizing denominators!) gives $x - 3 = \pm\frac{\sqrt{2}}{2}$, so that when we add 3 to each side, (and get common denominators!) we get $x = \frac{6\pm\sqrt{2}}{2}$. Hence, our x-intercepts are $\left(\frac{6-\sqrt{2}}{2}, 0\right) \approx (2.29, 0)$ and $\left(\frac{6+\sqrt{2}}{2}, 0\right) \approx (3.71, 0)$. Although our graph doesn't show it, there is a y-intercept which can be found by setting $x = 0$. With $h(0) = -2(0-3)^2+1 = -17$, we have that our y-intercept is $(0,-17)$.

A few remarks about Example 52 are in order. First note that neither the formula given for $g(x)$ nor the one given for $h(x)$ match the form given in Definition 34. We could, of course, convert both $g(x)$ and $h(x)$ into that form by expanding and collecting like terms. Doing so, we find $g(x) = (x+2)^2-3 = x^2+4x+1$ and $h(x) = -2(x-3)^2+1 = -2x^2+12x-17$. While these 'simplified' formulas for $g(x)$ and $h(x)$ satisfy Definition 34, they do not lend themselves to graphing easily. For that reason, the form of g and h presented in Example 53 is given a special name, which we list below, along with the form presented in Definition 34.

Definition 35 Standard and General Form of Quadratic Functions

Suppose f is a quadratic function.

- The **general form** of the quadratic function f is $f(x) = ax^2+bx+c$, where a, b and c are real numbers with $a \neq 0$.
- The **standard form** of the quadratic function f is $f(x) = a(x-h)^2+k$, where a, h and k are real numbers with $a \neq 0$.

It is important to note at this stage that we have no guarantees that *every* quadratic function can be written in standard form. This is actually true, and we prove this later in the exposition, but for now we celebrate the advantages of the standard form, starting with the following theorem.

Theorem 14 Vertex Formula for Quadratics in Standard Form

For the quadratic function $f(x) = a(x-h)^2+k$, where a, h and k are real numbers with $a \neq 0$, the vertex of the graph of $y = f(x)$ is (h,k).

We can readily verify the formula given Theorem 14 with the two functions given in Example 52. After a (slight) rewrite, $g(x) = (x+2)^2 - 3 = (x-(-2))^2 + (-3)$, and we identify $h = -2$ and $k = -3$. Sure enough, we found the vertex of the graph of $y = g(x)$ to be $(-2, -3)$. For $h(x) = -2(x-3)^2 + 1$, no rewrite is needed. We can directly identify $h = 3$ and $k = 1$ and, sure enough, we found the vertex of the graph of $y = h(x)$ to be $(3, 1)$.

To see why the formula in Theorem 14 produces the vertex, consider the graph of the equation $y = a(x-h)^2 + k$. When we substitute $x = h$, we get $y = k$, so (h, k) is on the graph. If $x \neq h$, then $x - h \neq 0$ so $(x-h)^2$ is a positive number. If $a > 0$, then $a(x-h)^2$ is positive, thus $y = a(x-h)^2 + k$ is always a number larger than k. This means that when $a > 0$, (h, k) is the lowest point on the graph and thus the parabola must open upwards, making (h, k) the vertex. A similar argument shows that if $a < 0$, (h, k) is the highest point on the graph, so the parabola opens downwards, and (h, k) is also the vertex in this case.

Alternatively, we can apply the machinery in Section 2.6. Since the vertex of $y = x^2$ is $(0, 0)$, we can determine the vertex of $y = a(x-h)^2 + k$ by determining the final destination of $(0, 0)$ as it is moved through each transformation. To obtain the formula $f(x) = a(x-h)^2 + k$, we start with $g(x) = x^2$ and first define $g_1(x) = ag(x) = ax^2$. This is results in a vertical scaling and/or reflection. (Just a scaling if $a > 0$. If $a < 0$, there is a reflection involved.) Since we multiply the output by a, we multiply the y-coordinates on the graph of g by a, so the point $(0, 0)$ remains $(0, 0)$ and remains the vertex. Next, we define $g_2(x) = g_1(x-h) = a(x-h)^2$. This induces a horizontal shift right or left h units (right if $h > 0$, left if $h < 0$.) moves the vertex, in either case, to $(h, 0)$. Finally, $f(x) = g_2(x) + k = a(x-h)^2 + k$ which effects a vertical shift up or down k units (up if $k > 0$, down if $k < 0$) resulting in the vertex moving from $(h, 0)$ to (h, k).

In addition to verifying Theorem 14, the arguments in the two preceding paragraphs have also shown us the role of the number a in the graphs of quadratic functions. The graph of $y = a(x-h)^2 + k$ is a parabola 'opening upwards' if $a > 0$, and 'opening downwards' if $a < 0$. Moreover, the symmetry enjoyed by the graph of $y = x^2$ about the y-axis is translated to a symmetry about the vertical line $x = h$ which is the vertical line through the vertex. (You should use transformations to verify this!) This line is called the **axis of symmetry** of the parabola and is shown as the dashed line in Figure 3.23

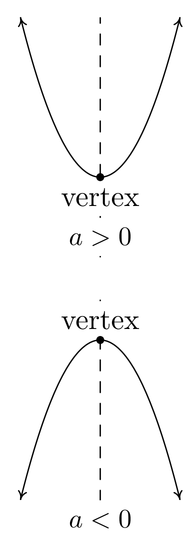


Figure 3.23: The axis of symmetry of a parabola

Without a doubt, the standard form of a quadratic function, coupled with the machinery in Section 2.6, allows us to list the attributes of the graphs of such functions quickly and elegantly. What remains to be shown, however, is the fact that every quadratic function *can be written* in standard form. To convert a quadratic function given in general form into standard form, we employ the ancient rite of 'Completing the Square'. We remind the reader how this is done in our next example.

Example 53 Converting from general to standard form

Convert the functions below from general form to standard form. Find the vertex, axis of symmetry and any x- or y-intercepts. Graph each function and determine its range.

1. $f(x) = x^2 - 4x + 3$.
2. $g(x) = 6 - x - x^2$

Solution

1. To convert from general form to standard form, we complete the square. First, we verify that the coefficient of x^2 is 1. Next, we find the coefficient

of x, in this case -4, and take half of it to get $\frac{1}{2}(-4) = -2$. This tells us that our target perfect square quantity is $(x-2)^2$. To get an expression equivalent to $(x-2)^2$, we need to add $(-2)^2 = 4$ to the $x^2 - 4x$ to create a perfect square trinomial, but to keep the balance, we must also subtract it. We collect the terms which create the perfect square and gather the remaining constant terms. Putting it all together, we get

If you forget why we do what we do to complete the square, start with $a(x-h)^2 + k$, multiply it out, step by step, and then reverse the process.

$$\begin{aligned}
f(x) &= x^2 - 4x + 3 && \text{(Compute } \tfrac{1}{2}(-4) = -2.\text{)}\\
&= \left(x^2 - 4x + \underline{4} - \underline{4}\right) + 3 \\
& && \text{(Add and subtract } (-2)^2 = 4 \text{ to } (x^2 + 4x).\text{)}\\
&= \left(x^2 - 4x + 4\right) - 4 + 3 && \text{(Group the perfect square trinomial.)}\\
&= (x-2)^2 - 1 && \text{(Factor the perfect square trinomial.)}
\end{aligned}$$

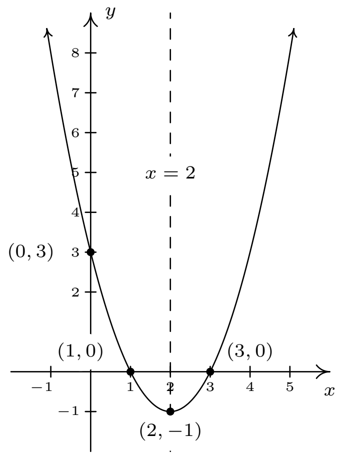


Figure 3.24: $f(x) = x^2 - 4x + 3$

Of course, we can always check our answer by multiplying out $f(x) = (x-2)^2 - 1$ to see that it simplifies to $f(x) = x^2 - 4x - 1$. In the form $f(x) = (x-2)^2 - 1$, we readily find the vertex to be $(2, -1)$ which makes the axis of symmetry $x = 2$. To find the x-intercepts, we set $y = f(x) = 0$. We are spoiled for choice, since we have *two* formulas for $f(x)$. Since we recognize $f(x) = x^2 - 4x + 3$ to be easily factorable, (experience pays off, here!) we proceed to solve $x^2 - 4x + 3 = 0$. Factoring gives $(x-3)(x-1) = 0$ so that $x = 3$ or $x = 1$. The x-intercepts are then $(1, 0)$ and $(3, 0)$. To find the y-intercept, we set $x = 0$. Once again, the general form $f(x) = x^2 - 4x + 3$ is easiest to work with here, and we find $y = f(0) = 3$. Hence, the y-intercept is $(0, 3)$. With the vertex, axis of symmetry and the intercepts, we get a pretty good graph without the need to plot additional points. We see that the range of f is $[-1, \infty)$ and we are done. The graph of f is given in Figure 3.24.

2. To get started, we rewrite $g(x) = 6 - x - x^2 = -x^2 - x + 6$ and note that the coefficient of x^2 is -1, not 1. This means our first step is to factor out the (-1) from both the x^2 and x terms. We then follow the completing the square recipe as above.

$$\begin{aligned}
g(x) &= -x^2 - x + 6 \\
&= (-1)\left(x^2 + x\right) + 6 \quad \text{(Factor the coefficient of } x^2 \text{ from } x^2 \text{ and } x.\text{)}\\
&= (-1)\left(x^2 + x + \underline{\frac{1}{4}} - \underline{\frac{1}{4}}\right) + 6 \\
&= (-1)\left(x^2 + x + \frac{1}{4}\right) + (-1)\left(-\frac{1}{4}\right) + 6 \\
& \qquad\qquad\qquad\qquad \text{(Group the perfect square trinomial.)}\\
&= -\left(x + \frac{1}{2}\right)^2 + \frac{25}{4}
\end{aligned}$$

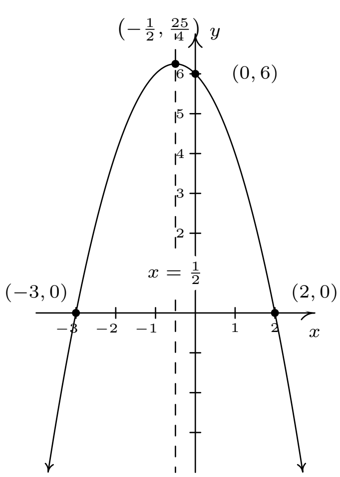


Figure 3.25: $g(x) = 6 - x - x^2$

From $g(x) = -\left(x + \frac{1}{2}\right)^2 + \frac{25}{4}$, we get the vertex to be $\left(-\frac{1}{2}, \frac{25}{4}\right)$ and the axis of symmetry to be $x = -\frac{1}{2}$. To get the x-intercepts, we opt to set the given formula $g(x) = 6 - x - x^2 = 0$. Solving, we get $x = -3$ and $x = 2$, so the x-intercepts are $(-3, 0)$ and $(2, 0)$. Setting $x = 0$, we find $g(0) = 6$, so the y-intercept is $(0, 6)$. Plotting these points gives us the graph in Figure 3.25. We see that the range of g is $\left(-\infty, \frac{25}{4}\right]$.

With Example 53 fresh in our minds, we are now in a position to show that every quadratic function can be written in standard form. We begin with $f(x) = ax^2 + bx + c$, assume $a \neq 0$, and complete the square in *complete* generality.

$$\begin{aligned}
f(x) &= ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c && \text{(Factor out coefficient of } x^2 \text{ from } x^2 \text{ and } x.) \\
&= a\left(x^2 + \frac{b}{a}x + \underline{\frac{b^2}{4a^2}} - \underline{\frac{b^2}{4a^2}}\right) + c \\
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - a\left(\frac{b^2}{4a^2}\right) + c \\
& && \text{(Group the perfect square trinomial.)} \\
&= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} && \text{(Factor and get a common denominator.)}
\end{aligned}$$

Comparing this last expression with the standard form, we identify $(x - h)$ with $\left(x + \frac{b}{2a}\right)$ so that $h = -\frac{b}{2a}$. Instead of memorizing the value $k = \frac{4ac-b^2}{4a}$, we see that $f\left(-\frac{b}{2a}\right) = \frac{4ac-b^2}{4a}$. As such, we have derived a vertex formula for the general form. We summarize both vertex formulas in the box at the top of the next page.

Theorem 15 Vertex Formulas for Quadratic Functions

Suppose a, b, c, h and k are real numbers with $a \neq 0$.

- If $f(x) = a(x - h)^2 + k$, the vertex of the graph of $y = f(x)$ is the point (h, k).
- If $f(x) = ax^2 + bx + c$, the vertex of the graph of $y = f(x)$ is the point $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

There are two more results which can be gleaned from the completed-square form of the general form of a quadratic function,

$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

We have seen that the number a in the standard form of a quadratic function determines whether the parabola opens upwards (if $a > 0$) or downwards (if $a < 0$). We see here that this number a is none other than the coefficient of x^2 in the general form of the quadratic function. In other words, it is the coefficient of x^2 alone which determines this behavior – a result that is generalized in Section 4.1. The second treasure is a re-discovery of the **quadratic formula**.

Theorem 16 The Quadratic Formula

If a, b and c are real numbers with $a \neq 0$, then the solutions to $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Assuming the conditions of Equation 16, the solutions to $ax^2 + bx + c = 0$ are precisely the zeros of $f(x) = ax^2 + bx + c$. Since

$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

the equation $ax^2 + bx + c = 0$ is equivalent to

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0.$$

Solving gives

$$\begin{aligned}
a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} &= 0 \\
a\left(x + \frac{b}{2a}\right)^2 &= -\frac{4ac - b^2}{4a} \\
\frac{1}{a}\left[a\left(x + \frac{b}{2a}\right)^2\right] &= \frac{1}{a}\left(\frac{b^2 - 4ac}{4a}\right) \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{extract square roots} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

In our discussions of domain, we were warned against having negative numbers underneath the square root. Given that $\sqrt{b^2 - 4ac}$ is part of the Quadratic Formula, we will need to pay special attention to the radicand $b^2 - 4ac$. It turns out that the quantity $b^2 - 4ac$ plays a critical role in determining the nature of the solutions to a quadratic equation. It is given a special name.

Definition 36 Discriminant

If a, b and c are real numbers with $a \neq 0$, then the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ is the quantity $b^2 - 4ac$.

The discriminant 'discriminates' between the kinds of solutions we get from a quadratic equation. These cases, and their relation to the discriminant, are summarized below.

Theorem 17 Discriminant Trichotomy

Let a, b and c be real numbers with $a \neq 0$.

- If $b^2 - 4ac < 0$, the equation $ax^2 + bx + c = 0$ has no real solutions.
- If $b^2 - 4ac = 0$, the equation $ax^2 + bx + c = 0$ has exactly one real solution.
- If $b^2 - 4ac > 0$, the equation $ax^2 + bx + c = 0$ has exactly two real solutions.

The proof of Theorem 17 stems from the position of the discriminant in the quadratic equation, and is left as a good mental exercise for the reader. The next example exploits the fruits of all of our labor in this section thus far.

Example 54 Computing and maximizing profit

Recall that the profit (defined on page 73) for a product is defined by the equation Profit = Revenue − Cost, or $P(x) = R(x) - C(x)$. In Example 47 the weekly revenue, in dollars, made by selling x PortaBoy Game Systems was found to be $R(x) = -1.5x^2 + 250x$ with the restriction (carried over from the price-demand function) that $0 \leq x \leq 166$. The cost, in dollars, to produce x PortaBoy Game Systems is given in Example 45 as $C(x) = 80x + 150$ for $x \geq 0$.

1. Determine the weekly profit function $P(x)$.
2. Graph $y = P(x)$. Include the x- and y-intercepts as well as the vertex and axis of symmetry.
3. Interpret the zeros of P.
4. Interpret the vertex of the graph of $y = P(x)$.
5. Recall that the weekly price-demand equation for PortaBoys is $p(x) = -1.5x + 250$, where $p(x)$ is the price per PortaBoy, in dollars, and x is the weekly sales. What should the price per system be in order to maximize profit?

Solution

1. To find the profit function $P(x)$, we subtract

   $$P(x) = R(x) - C(x) = \left(-1.5x^2 + 250x\right) - (80x + 150) = -1.5x^2 + 170x - 150.$$

   Since the revenue function is valid when $0 \leq x \leq 166$, P is also restricted to these values.

2. To find the x-intercepts, we set $P(x) = 0$ and solve $-1.5x^2 + 170x - 150 = 0$. The mere thought of trying to factor the left hand side of this equation could do serious psychological damage, so we resort to the quadratic formula, Equation 16. Identifying $a = -1.5$, $b = 170$, and $c = -150$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-170 \pm \sqrt{170^2-4(-1.5)(-150)}}{2(-1.5)} \\ &= \frac{-170 \pm \sqrt{28000}}{-3} \\ &= \frac{170 \pm 20\sqrt{70}}{3} \end{aligned}$$

We get two x-intercepts: $\left(\frac{170-20\sqrt{70}}{3}, 0\right)$ and $\left(\frac{170+20\sqrt{70}}{3}, 0\right)$. To find the y-intercept, we set $x = 0$ and find $y = P(0) = -150$ for a y-intercept of $(0, -150)$. To find the vertex, we use the fact that $P(x) = -1.5x^2 + 170x - 150$ is in the general form of a quadratic function and appeal to Equation 15. Substituting $a = -1.5$ and $b = 170$, we get $x = -\frac{170}{2(-1.5)} = \frac{170}{3}$. To find the y-coordinate of the vertex, we compute $P\left(\frac{170}{3}\right) = \frac{14000}{3}$ and find that our vertex is $\left(\frac{170}{3}, \frac{14000}{3}\right)$. The axis of symmetry is the vertical line passing through the vertex so it is the line $x = \frac{170}{3}$. To sketch a reasonable graph, we approximate the x-intercepts, $(0.89, 0)$ and $(112.44, 0)$, and the vertex, $(56.67, 4666.67)$. (Note that in order to get the x-intercepts and the vertex to show up in the same picture, we had to scale the x-axis differently than the y-axis in Figure 3.26. This results in the left-hand x-intercept and the y-intercept being uncomfortably close to each other and to the origin in the picture.)

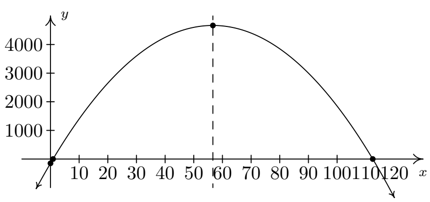


Figure 3.26: The graph of the profit function $P(x)$

3. The zeros of P are the solutions to $P(x) = 0$, which we have found to be approximately 0.89 and 112.44. As we saw in Example 29, these are the 'break-even' points of the profit function, where enough product is sold to recover the cost spent to make the product. More importantly, we see from the graph that as long as x is between 0.89 and 112.44, the graph $y = P(x)$ is above the x-axis, meaning $y = P(x) > 0$ there. This means that for these values of x, a profit is being made. Since x represents the weekly sales of PortaBoy Game Systems, we round the zeros to positive integers and have that as long as 1, but no more than 112 game systems are sold weekly, the retailer will make a profit.

4. From the graph, we see that the maximum value of P occurs at the vertex, which is approximately $(56.67, 4666.67)$. As above, x represents the weekly sales of PortaBoy systems, so we can't sell 56.67 game systems. Comparing $P(56) = 4666$ and $P(57) = 4666.5$, we conclude that we will make a maximum profit of $\$4666.50$ if we sell 57 game systems.

5. In the previous part, we found that we need to sell 57 PortaBoys per week to maximize profit. To find the price per PortaBoy, we substitute $x = 57$ into the price-demand function to get $p(57) = -1.5(57) + 250 = 164.5$. The price should be set at $\$164.50$.

Our next example is another classic application of quadratic functions.

Example 55 Optimizing pasture dimensions

Much to Donnie's surprise and delight, he inherits a large parcel of land in Ashtabula County from one of his (e)strange(d) relatives. The time is finally right for him to pursue his dream of farming alpaca. He wishes to build a rectangular pasture, and estimates that he has enough money for 200 linear feet of fencing material. If he makes the pasture adjacent to a stream (so no fencing is required on that side), what are the dimensions of the pasture which maximize the area? What is the maximum area? If an average alpaca needs 25 square feet of grazing area, how many alpaca can Donnie keep in his pasture?

SOLUTION It is always helpful to sketch the problem situation, so we do so in Figure 3.27.

We are tasked to find the dimensions of the pasture which would give a maximum area. We let w denote the width of the pasture and we let l denote the length of the pasture. Since the units given to us in the statement of the problem are feet, we assume w and l are measured in feet. The area of the pasture, which we'll call A, is related to w and l by the equation $A = wl$. Since w and l are both measured in feet, A has units of feet^2, or square feet. We are given the total amount of fencing available is 200 feet, which means $w + l + w = 200$, or, $l + 2w = 200$. We now have two equations, $A = wl$ and $l + 2w = 200$. In order to use the tools given to us in this section to *maximize* A, we need to use the information given to write A as a function of just *one* variable, either w or l. This is where we use the equation $l + 2w = 200$. Solving for l, we find $l = 200 - 2w$, and we substitute this into our equation for A. We get $A = wl = w(200 - 2w) = 200w - 2w^2$. We now have A as a function of w, $A(w) = 200w - 2w^2 = -2w^2 + 200w$.

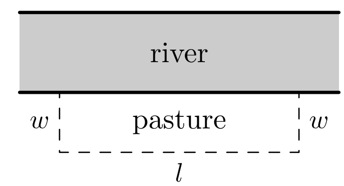


Figure 3.27: A diagram of pasture dimensions

Before we go any further, we need to find the applied domain of A so that we know what values of w make sense in this problem situation. (Donnie would be very upset if, for example, we told him the width of the pasture needs to be -50 feet.) Since w represents the width of the pasture, $w > 0$. Likewise, l represents the length of the pasture, so $l = 200 - 2w > 0$. Solving this latter inequality, we find $w < 100$. Hence, the function we wish to maximize is $A(w) = -2w^2 + 200w$ for $0 < w < 100$. Since A is a quadratic function (of w), we know that the graph of $y = A(w)$ is a parabola. Since the coefficient of w^2 is -2, we know that this parabola opens downwards. This means that there is a maximum value to be found, and we know it occurs at the vertex. Using the vertex formula, we find $w = -\frac{200}{2(-2)} = 50$, and $A(50) = -2(50)^2 + 200(50) = 5000$. Since $w = 50$ lies in the applied domain, $0 < w < 100$, we have that the area of the pasture is maximized when the width is 50 feet. To find the length, we use $l = 200 - 2w$ and find $l = 200 - 2(50) = 100$, so the length of the pasture is 100 feet. The maximum area is $A(50) = 5000$, or 5000 square feet. If an average alpaca requires 25 square feet of pasture, Donnie can raise $\frac{5000}{25} = 200$ average alpaca.

We conclude this section with the graph of a more complicated absolute value function.

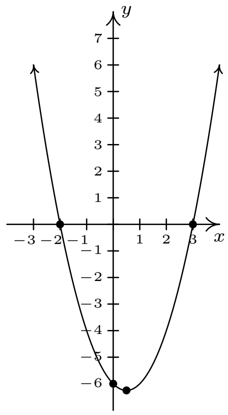
Example 56 Graphing the absolute value of a quadratic function

Graph $f(x) = |x^2 - x - 6|$.

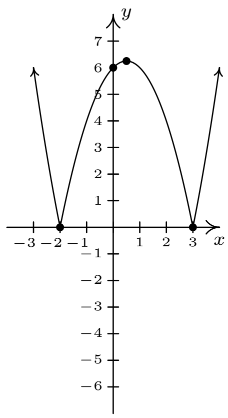
SOLUTION Using the definition of absolute value, Definition 33, we have

$$f(x) = \begin{cases} -\left(x^2 - x - 6\right), & \text{if} \quad x^2 - x - 6 < 0 \\ x^2 - x - 6, & \text{if} \quad x^2 - x - 6 \geq 0 \end{cases}$$

The trouble is that we have yet to develop any analytic techniques to solve nonlinear inequalities such as $x^2-x-6<0$. You won't have to wait long; this is one of the main topics of Section 3.4. Nevertheless, we can attack this problem graphically. To that end, we graph $y=g(x)=x^2-x-6$ using the intercepts and the vertex. To find the x-intercepts, we solve $x^2-x-6=0$. Factoring gives $(x-3)(x+2)=0$ so $x=-2$ or $x=3$. Hence, $(-2,0)$ and $(3,0)$ are x-intercepts. The y-intercept $(0,-6)$ is found by setting $x=0$. To plot the vertex, we find $x=-\frac{b}{2a}=-\frac{-1}{2(1)}=\frac{1}{2}$, and $y=\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)-6=-\frac{25}{4}=-6.25$. Plotting, we get the parabola seen below on the left. To obtain points on the graph of $y=f(x)=|x^2-x-6|$, we can take points on the graph of $g(x)=x^2-x-6$ and apply the absolute value to each of the y values on the parabola. We see from the graph of g that for $x\leq -2$ or $x\geq 3$, the y values on the parabola are greater than or equal to zero (since the graph is on or above the x-axis), so the absolute value leaves these portions of the graph alone. For x between -2 and 3, however, the y values on the parabola are negative. For example, the point $(0,-6)$ on $y=x^2-x-6$ would result in the point $(0,|-6|)=(0,-(-6))=(0,6)$ on the graph of $f(x)=|x^2-x-6|$. Proceeding in this manner for all points with x-coordinates between -2 and 3 results in the graph seen at the bottom of Figure 3.28.



$y=x^2-x-6$



$y=|x^2-x-6$

Figure 3.28: Obtaining the graph of $f(x)=|x^2-x-6|$

If we take a step back and look at the graphs of g and f in the last example, we notice that to obtain the graph of f from the graph of g, we reflect a *portion* of the graph of g about the x-axis. We can see this analytically by substituting $g(x)=x^2-x-6$ into the formula for $f(x)$ and calling to mind Theorem 9 from Section 2.6.

$$f(x)=\begin{cases}-g(x), & \text{if } g(x)<0\\ g(x), & \text{if } g(x)\geq 0\end{cases}$$

The function f is defined so that when $g(x)$ is negative (i.e., when its graph is below the x-axis), the graph of f is its refection across the x-axis. This is a general template to graph functions of the form $f(x)=|g(x)|$. From this perspective, the graph of $f(x)=|x|$ can be obtained by reflecting the portion of the line $g(x)=x$ which is below the x-axis back above the x-axis creating the characteristic '$\vee$' shape.

Exercises 3.3

Problems

In Exercises 1 – 10, graph the quadratic function. Find the x- and y-intercepts of each graph, if any exist. If it is given in general form, convert it into standard form; if it is given in standard form, convert it into general form. Find the domain and range of the function and list the intervals on which the function is increasing or decreasing. Identify the vertex and the axis of symmetry and determine whether the vertex yields a relative and absolute maximum or minimum.

1. $f(x) = x^2 + 2$
2. $f(x) = -(x+2)^2$
3. $f(x) = x^2 - 2x - 8$
4. $f(x) = -2(x+1)^2 + 4$
5. $f(x) = 2x^2 - 4x - 1$
6. $f(x) = -3x^2 + 4x - 7$
7. $f(x) = x^2 + x + 1$
8. $f(x) = -3x^2 + 5x + 4$
9. $f(x) = x^2 - \frac{1}{100}x - 1$[1]

In Exercises 10 – 15, the cost and price-demand functions are given for different scenarios. For each scenario,

- **Find the profit function $P(x)$.**
- **Find the number of items which need to be sold in order to maximize profit.**
- **Find the maximum profit.**
- **Find the price to charge per item in order to maximize profit.**
- **Find and interpret break-even points.**

10. The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, $0 \leq x \leq 15$.

11. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x$, $0 \leq x \leq 35$.

12. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is $C(x) = 18x + 240$, $x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x$, $0 \leq x \leq 30$.

13. The daily cost, in dollars, to produce x Sasquatch Berry Pies is $C(x) = 3x + 36$, $x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x$, $0 \leq x \leq 24$.

14. The monthly cost, in *hundreds* of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000$, $x \geq 0$ and the price-demand function, in *hundreds* of dollars per scooter, is $p(x) = 140 - 2x$, $0 \leq x \leq 70$.

15. The International Silver Strings Submarine Band holds a bake sale each year to fund their trip to the National Sasquatch Convention. It has been determined that the cost in dollars of baking x cookies is $C(x) = 0.1x + 25$ and that the demand function for their cookies is $p = 10 - .01x$. How many cookies should they bake in order to maximize their profit?

16. Using data from Bureau of Transportation Statistics, the average fuel economy F in miles per gallon for passenger cars in the US can be modelled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Find and interpret the coordinates of the vertex of the graph of $y = F(t)$.

17. The temperature T, in degrees Fahrenheit, t hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

What is the warmest temperature of the day? When does this happen?

18. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in *hundreds*, to produce x *thousand* pens. How many pens should be produced to minimize the cost? What is this minimum cost?

19. Skippy wishes to plant a vegetable garden along one side of his house. In his garage, he found 32 linear feet of fencing. Since one side of the garden will border the house, Skippy doesn't need fencing along that side. What are the dimensions of the garden which will maximize the area of the garden? What is the maximum area of the garden?

20. In the situation of Example 55, Donnie has a nightmare that one of his alpaca herd fell into the river and drowned. To avoid this, he wants to move his rectangular pasture *away* from the river. This means that all four sides of the pasture require fencing. If the total amount of fencing available is still 200 linear feet, what dimensions maximize the area of the pasture now? What is the maximum area? Assuming an average alpaca requires 25 square feet of pasture, how many alpaca can he raise now?

21. What is the largest rectangular area one can enclose with 14 inches of string?

[1] We have already seen the graph of this function. It was used as an example in Section 2.5 to show how the graphing calculator can be misleading.

22. The height of an object dropped from the roof of an eight story building is modelled by $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, h is the height of the object off the ground, in feet, t seconds after the object is dropped. How long before the object hits the ground?

23. The height h in feet of a model rocket above the ground t seconds after lift-off is given by $h(t) = -5t^2 + 100t$, for $0 \leq t \leq 20$. When does the rocket reach its maximum height above the ground? What is its maximum height?

24. Carl's friend Jason participates in the Highland Games. In one event, the hammer throw, the height h in feet of the hammer above the ground t seconds after Jason lets it go is modeled by $h(t) = -16t^2 + 22.08t + 6$. What is the hammer's maximum height? What is the hammer's total time in the air? Round your answers to two decimal places.

25. Assuming no air resistance or forces other than the Earth's gravity, the height above the ground at time t of a falling object is given by $s(t) = -4.9t^2 + v_0t + s_0$ where s is in meters, t is in seconds, v_0 is the object's initial velocity in meters per second and s_0 is its initial position in meters.
    (a) What is the applied domain of this function?
    (b) Discuss with your classmates what each of $v_0 > 0$, $v_0 = 0$ and $v_0 < 0$ would mean.
    (c) Come up with a scenario in which $s_0 < 0$.
    (d) Let's say a slingshot is used to shoot a marble straight up from the ground ($s_0 = 0$) with an initial velocity of 15 meters per second. What is the marble's maximum height above the ground? At what time will it hit the ground?
    (e) Now shoot the marble from the top of a tower which is 25 meters tall. When does it hit the ground?
    (f) What would the height function be if instead of shooting the marble up off of the tower, you were to shoot it straight DOWN from the top of the tower?

26. The two towers of a suspension bridge are 400 feet apart. The parabolic cable[2] attached to the tops of the towers is 10 feet above the point on the bridge deck that is midway between the towers. If the towers are 100 feet tall, find the height of the cable directly above a point of the bridge deck that is 50 feet to the right of the left-hand tower.

27. Graph $f(x) = |1 - x^2|$

28. Find all of the points on the line $y = 1 - x$ which are 2 units from $(1, -1)$.

29. Let L be the line $y = 2x + 1$. Find a function $D(x)$ which measures the distance *squared* from a point on L to $(0, 0)$. Use this to find the point on L closest to $(0, 0)$.

30. With the help of your classmates, show that if a quadratic function $f(x) = ax^2 + bx + c$ has two real zeros then the x-coordinate of the vertex is the midpoint of the zeros.

In Exercises 31 – 37, solve the quadratic equation for the indicated variable.

31. $x^2 - 10y^2 = 0$ for x

32. $y^2 - 4y = x^2 - 4$ for x

33. $x^2 - mx = 1$ for x

34. $y^2 - 3y = 4x$ for y

35. $y^2 - 4y = x^2 - 4$ for y

36. $-gt^2 + v_0t + s_0 = 0$ for t (Assume $g \neq 0$.)

[2] The weight of the bridge deck forces the bridge cable into a parabola and a free hanging cable such as a power line does not form a parabola. We shall see in Exercise **??** in Section **??** what shape a free hanging cable makes.

3.4 Inequalities with Absolute Value and Quadratic Functions

In this section, not only do we develop techniques for solving various classes of inequalities analytically, we also look at them graphically. The first example motivates the core ideas.

Example 57 **Inequalities with linear functions**

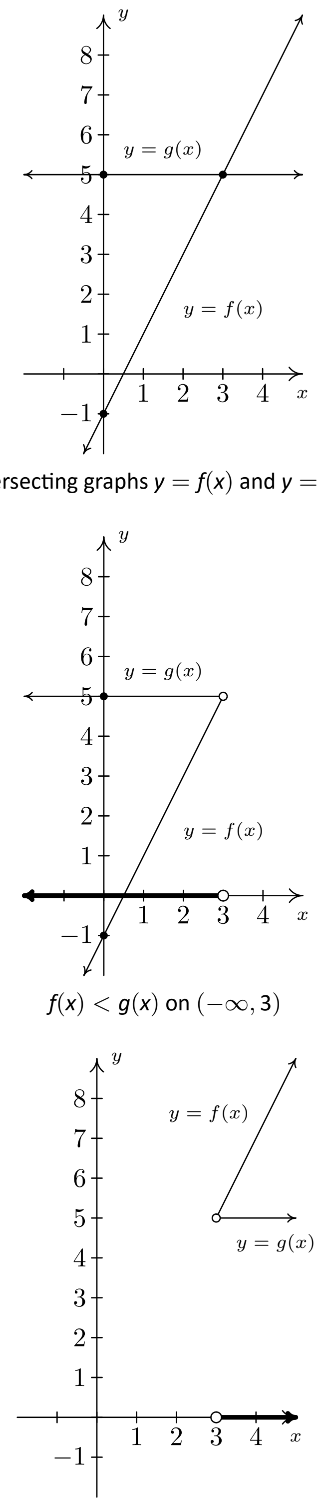
Let $f(x) = 2x - 1$ and $g(x) = 5$.

1. Solve $f(x) = g(x)$.
2. Solve $f(x) < g(x)$.
3. Solve $f(x) > g(x)$.
4. Graph $y = f(x)$ and $y = g(x)$ on the same set of axes and interpret your solutions to parts 1 through 3 above.

SOLUTION

1. To solve $f(x) = g(x)$, we replace $f(x)$ with $2x - 1$ and $g(x)$ with 5 to get $2x - 1 = 5$. Solving for x, we get $x = 3$.
2. The inequality $f(x) < g(x)$ is equivalent to $2x - 1 < 5$. Solving gives $x < 3$ or $(-\infty, 3)$.
3. To find where $f(x) > g(x)$, we solve $2x - 1 > 5$. We get $x > 3$, or $(3, \infty)$.
4. To graph $y = f(x)$, we graph $y = 2x - 1$, which is a line with a y-intercept of $(0, -1)$ and a slope of 2. The graph of $y = g(x)$ is $y = 5$ which is a horizontal line through $(0, 5)$.

   To see the connection between the graph and the Algebra, we recall the Fundamental Graphing Principle for Functions in Section 2.5: the point (a, b) is on the graph of f if and only if $f(a) = b$. In other words, a generic point on the graph of $y = f(x)$ is $(x, f(x))$, and a generic point on the graph of $y = g(x)$ is $(x, g(x))$. When we seek solutions to $f(x) = g(x)$, we are looking for x values whose y values on the graphs of f and g are the same. In part 1, we found $x = 3$ is the solution to $f(x) = g(x)$. Sure enough, $f(3) = 5$ and $g(3) = 5$ so that the point $(3, 5)$ is on both graphs. In other words, the graphs of f and g *intersect* at $(3, 5)$. In part 2, we set $f(x) < g(x)$ and solved to find $x < 3$. For $x < 3$, the point $(x, f(x))$ is *below* $(x, g(x))$ since the y values on the graph of f are less than the y values on the graph of g there. Analogously, in part 3, we solved $f(x) > g(x)$ and found $x > 3$. For $x > 3$, note that the graph of f is *above* the graph of g, since the y values on the graph of f are greater than the y values on the graph of g for those values of x: see Figure 3.29.



Intersecting graphs $y = f(x)$ and $y = g(x)$

$f(x) < g(x)$ on $(-\infty, 3)$

$f(x) > g(x)$ on $(3, \infty)$

Figure 3.29: Graphical interpretation of Example 57

The preceding example demonstrates the following, which is a consequence of the Fundamental Graphing Principle for Functions.

Key Idea 17 Graphical Interpretation of Equations and Inequalities

Suppose f and g are functions.

- The solutions to $f(x) = g(x)$ are the x values where the graphs of $y = f(x)$ and $y = g(x)$ intersect.
- The solution to $f(x) < g(x)$ is the set of x values where the graph of $y = f(x)$ is *below* the graph of $y = g(x)$.
- The solution to $f(x) > g(x)$ is the set of x values where the graph of $y = f(x)$ *above* the graph of $y = g(x)$.

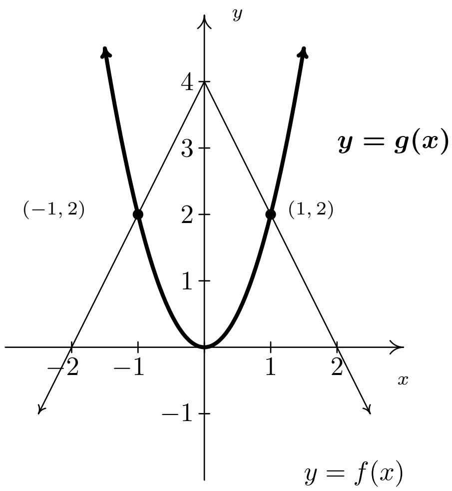


Figure 3.30: The graphs $y = f(x)$ and $y = g(x)$ for Example 58

The next example turns the tables and furnishes the graphs of two functions and asks for solutions to equations and inequalities.

Example 58 Using graphs to solve equations and inequalities

The graphs of f and g are shown in Figure 3.30. (The graph of $y = g(x)$ is in bold.) Use these graphs to answer the following questions.

1. Solve $f(x) = g(x)$.
2. Solve $f(x) < g(x)$.
3. Solve $f(x) \geq g(x)$.

Solution

1. To solve $f(x) = g(x)$, we look for where the graphs of f and g intersect. These appear to be at the points $(-1, 2)$ and $(1, 2)$, so our solutions to $f(x) = g(x)$ are $x = -1$ and $x = 1$.
2. To solve $f(x) < g(x)$, we look for where the graph of f is below the graph of g. This appears to happen for the x values less than -1 and greater than 1. Our solution is $(-\infty, -1) \cup (1, \infty)$.
3. To solve $f(x) \geq g(x)$, we look for solutions to $f(x) = g(x)$ as well as $f(x) > g(x)$. We solved the former equation and found $x = \pm 1$. To solve $f(x) > g(x)$, we look for where the graph of f is above the graph of g. This appears to happen between $x = -1$ and $x = 1$, on the interval $(-1, 1)$. Hence, our solution to $f(x) \geq g(x)$ is $[-1, 1]$.

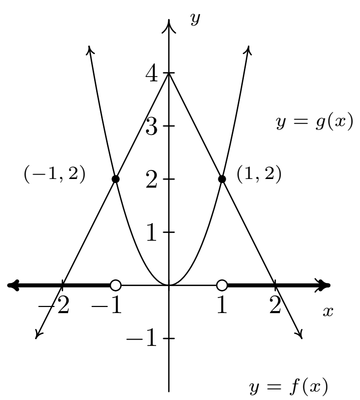


Figure 3.31: The solution to $f(x) < g(x)$

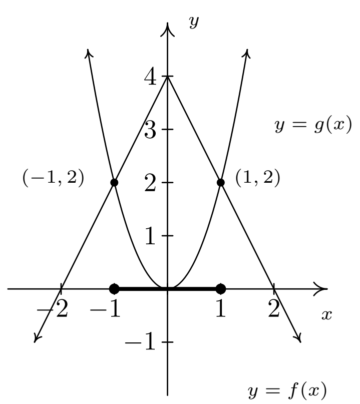


Figure 3.32: The solution to $f(x) \geq g(x)$

We now turn our attention to solving inequalities involving the absolute value. We have the following theorem to help us.

Theorem 18 Inequalities Involving the Absolute Value

Let c be a real number.

- For $c > 0$, $|x| < c$ is equivalent to $-c < x < c$.
- For $c > 0$, $|x| \leq c$ is equivalent to $-c \leq x \leq c$.
- For $c \leq 0$, $|x| < c$ has no solution, and for $c < 0$, $|x| \leq c$ has no solution.
- For $c \geq 0$, $|x| > c$ is equivalent to $x < -c$ or $x > c$.
- For $c \geq 0$, $|x| \geq c$ is equivalent to $x \leq -c$ or $x \geq c$.
- For $c < 0$, $|x| > c$ and $|x| \geq c$ are true for all real numbers.

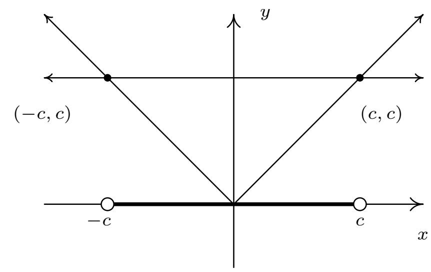


Figure 3.33: Solving $|x| < c$ graphically

As with Theorem 13 in Section 3.2, we could argue Theorem 18 using cases. However, in light of what we have developed in this section, we can understand these statements graphically. For instance, if $c > 0$, the graph of $y = c$ is a horizontal line which lies above the x-axis through $(0, c)$. To solve $|x| < c$, we are looking for the x values where the graph of $y = |x|$ is below the graph of $y = c$. We know that the graphs intersect when $|x| = c$, which, from Section 3.2, we know happens when $x = c$ or $x = -c$.

In Figure 3.33 we see that the graph of $y = |x|$ is below $y = c$ for x between $-c$ and c, and hence we get $|x| < c$ is equivalent to $-c < x < c$. The other properties in Theorem 18 can be shown similarly.

Example 59 Solving absolute value inequalities

Solve the following inequalities analytically; check your answers graphically.

1. $|x - 1| \geq 3$
2. $4 - 3|2x + 1| > -2$
3. $2 < |x - 1| \leq 5$
4. $|x + 1| \geq \dfrac{x + 4}{2}$

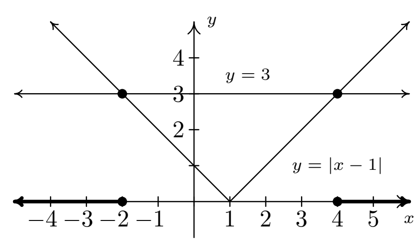


Figure 3.34: Solving $|x - 1| \geq 3$ in Example 59

SOLUTION

1. From Theorem 18, $|x - 1| \geq 3$ is equivalent to $x - 1 \leq -3$ or $x - 1 \geq 3$. Solving, we get $x \leq -2$ or $x \geq 4$, which, in interval notation is $(-\infty, -2] \cup [4, \infty)$. Graphically, we have Figure 3.34.

   We see that the graph of $y = |x - 1|$ is above the horizontal line $y = 3$ for $x < -2$ and $x > 4$ hence this is where $|x - 1| > 3$. The two graphs intersect when $x = -2$ and $x = 4$, so we have graphical confirmation of our analytic solution.

2. To solve $4 - 3|2x + 1| > -2$ analytically, we first isolate the absolute value before applying Theorem 18. To that end, we get $-3|2x + 1| > -6$ or $|2x + 1| < 2$. Rewriting, we now have $-2 < 2x + 1 < 2$ so that $-\frac{3}{2} < x < \frac{1}{2}$. In interval notation, we write $\left(-\frac{3}{2}, \frac{1}{2}\right)$. Graphically we see in Figure 3.35 that the graph of $y = 4 - 3|2x + 1|$ is above $y = -2$ for x values between $-\frac{3}{2}$ and $\frac{1}{2}$.

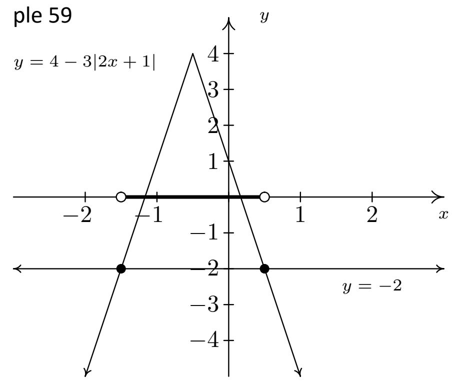


Figure 3.35: Solving $4 - 3|2x + 1| > -2$ in Example 59

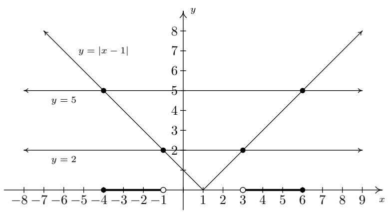


Figure 3.36: Solving $2 < |x-1| \leq 5$ in Example 59

3. Rewriting the compound inequality $2 < |x-1| \leq 5$ as '$2 < |x-1|$ and $|x-1| \leq 5$' allows us to solve each piece using Theorem 18. The first inequality, $2 < |x-1|$ can be re-written as $|x-1| > 2$ so $x-1 < -2$ or $x-1 > 2$. We get $x < -1$ or $x > 3$. Our solution to the first inequality is then $(-\infty,-1)\cup(3,\infty)$. For $|x-1| \leq 5$, we combine results in Theorems 13 and 18 to get $-5 \leq x-1 \leq 5$ so that $-4 \leq x \leq 6$, or $[-4,6]$. Our solution to $2 < |x-1| \leq 5$ is comprised of values of x which satisfy both parts of the inequality, so we take the intersection of $(-\infty,-1)\cup(3,\infty)$ and $[-4,6]$ to get $[-4,-1)\cup(3,6]$. (see Definition 4 in Section **??**.) Graphically, we see that the graph of $y = |x-1|$ is 'between' the horizontal lines $y = 2$ and $y = 5$ for x values between -4 and -1 as well as those between 3 and 6. Including the x values where $y = |x-1|$ and $y = 5$ intersect, we get Figure 3.36.

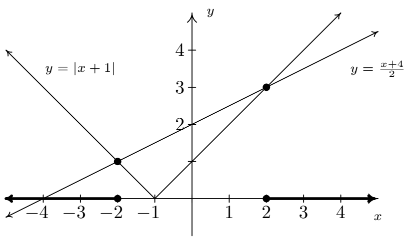


Figure 3.37: Solving $|x+1| \geq \frac{x+4}{2}$ in Example 59

4. We need to exercise some special caution when solving $|x+1| \geq \frac{x+4}{2}$. As we saw in Example 48 in Section 3.2, when variables are both inside and outside of the absolute value, it's usually best to refer to the definition of absolute value, Definition 33, to remove the absolute values and proceed from there. To that end, we have $|x+1| = -(x+1)$ if $x < -1$ and $|x+1| = x+1$ if $x \geq -1$. We break the inequality into cases, the first case being when $x < -1$. For these values of x, our inequality becomes $-(x+1) \geq \frac{x+4}{2}$. Solving, we get $-2x-2 \geq x+4$, so that $-3x \geq 6$, which means $x \leq -2$. Since all of these solutions fall into the category $x < -1$, we keep them all. For the second case, we assume $x \geq -1$. Our inequality becomes $x+1 \geq \frac{x+4}{2}$, which gives $2x+2 \geq x+4$ or $x \geq 2$. Since all of these values of x are greater than or equal to -1, we accept all of these solutions as well. Our final answer is $(-\infty,-2]\cup[2,\infty)$.

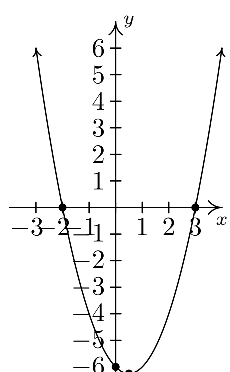
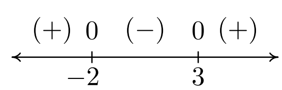


Figure 3.38: $y = x^2 - x - 6$

We now turn our attention to quadratic inequalities. In the last example of Section 3.3, we needed to determine the solution to $x^2 - x - 6 < 0$. We will now re-visit this problem using some of the techniques developed in this section not only to reinforce our solution in Section 3.3, but to also help formulate a general analytic procedure for solving all quadratic inequalities. If we consider $f(x) = x^2 - x - 6$ and $g(x) = 0$, then solving $x^2 - x - 6 < 0$ corresponds graphically to finding the values of x for which the graph of $y = f(x) = x^2 - x - 6$ (the parabola) is below the graph of $y = g(x) = 0$ (the x-axis). See Figure 3.38 for reference.

We can see that the graph of f does dip below the x-axis between its two x-intercepts. The zeros of f are $x = -2$ and $x = 3$ in this case and they divide the domain (the x-axis) into three intervals: $(-\infty,-2)$, $(-2,3)$ and $(3,\infty)$. For every number in $(-\infty,-2)$, the graph of f is above the x-axis; in other words, $f(x) > 0$ for all x in $(-\infty,-2)$. Similarly, $f(x) < 0$ for all x in $(-2,3)$, and $f(x) > 0$ for all x in $(3,\infty)$. We can schematically represent this with the **sign diagram** below.



Here, the $(+)$ above a portion of the number line indicates $f(x) > 0$ for those values of x; the $(-)$ indicates $f(x) < 0$ there. The numbers labeled on the number line are the zeros of f, so we place 0 above them. We see at once that the solution to $f(x) < 0$ is $(-2,3)$.

Our next goal is to establish a procedure by which we can generate the sign diagram without graphing the function. An important property of quadratic functions is that if the function is positive at one point and negative at another, the function must have at least one zero in between. Graphically, this means that a parabola can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis. This allows us to determine the sign of *all* of the function values on a given interval by testing the function at just *one* value in the interval. This gives us the following.

Key Idea 18 Steps for Solving a Quadratic Inequality

1. Rewrite the inequality, if necessary, as a quadratic function $f(x)$ on one side of the inequality and 0 on the other.
2. Find the zeros of f and place them on the number line with the number 0 above them.
3. Choose a real number, called a **test value**, in each of the intervals determined in step 2.
4. Determine the sign of $f(x)$ for each test value in step 3, and write that sign above the corresponding interval.
5. Choose the intervals which correspond to the correct sign to solve the inequality.

Example 60 Solving quadratic inequalities

Solve the following inequalities analytically using sign diagrams. Verify your answer graphically.

1. $2x^2 \leq 3 - x$
2. $x^2 - 2x > 1$
3. $x^2 + 1 \leq 2x$
4. $2x - x^2 \geq |x - 1| - 1$

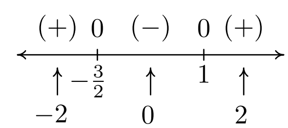


Figure 3.39: The sign diagram for $f(x) = 2x^2 + x - 3$

Solution

1. To solve $2x^2 \leq 3 - x$, we first get 0 on one side of the inequality which yields $2x^2 + x - 3 \leq 0$. We find the zeros of $f(x) = 2x^2 + x - 3$ by solving $2x^2 + x - 3 = 0$ for x. Factoring gives $(2x + 3)(x - 1) = 0$, so $x = -\frac{3}{2}$ or $x = 1$. We place these values on the number line with 0 above them and choose test values in the intervals $\left(-\infty, -\frac{3}{2}\right)$, $\left(-\frac{3}{2}, 1\right)$ and $(1, \infty)$. For the interval $\left(-\infty, -\frac{3}{2}\right)$, we choose $x = -2$; for $\left(-\frac{3}{2}, 1\right)$, we pick $x = 0$; and for $(1, \infty)$, $x = 2$. Evaluating the function at the three test values gives us $f(-2) = 3 > 0$, so we place $(+)$ above $\left(-\infty, -\frac{3}{2}\right)$; $f(0) = -3 < 0$, so $(-)$ goes above the interval $\left(-\frac{3}{2}, 1\right)$; and, $f(2) = 7$, which means $(+)$ is placed above $(1, \infty)$. Since we are solving $2x^2 + x - 3 \leq 0$, we look for solutions to $2x^2 + x - 3 < 0$ as well as solutions for $2x^2 + x - 3 = 0$. For $2x^2 + x - 3 < 0$, we need the intervals which we have a $(-)$. Checking the sign diagram, we see this is $\left(-\frac{3}{2}, 1\right)$. We know $2x^2 + x - 3 = 0$ when $x = -\frac{3}{2}$ and $x = 1$, so our final answer is $\left[-\frac{3}{2}, 1\right]$.

   We have to choose a test value in each interval to construct the sign diagram. You'll get the same sign chart if you choose different test values than the ones chosen here.

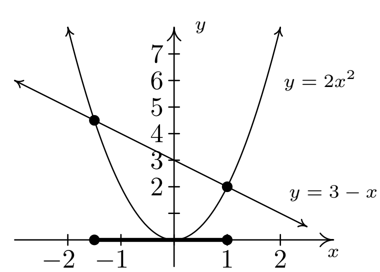
   

   Figure 3.40: Verifying the solution to $2x^2 \leq 3 - x$ graphically

   To verify our solution graphically, we refer to the original inequality, $2x^2 \leq 3 - x$. We let $g(x) = 2x^2$ and $h(x) = 3 - x$. We are looking for the x values where the graph of g is below that of h (the solution to $g(x) < h(x)$) as well as the points of intersection (the solutions to $g(x) = h(x)$). See Figure 3.40.

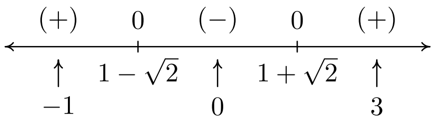


Figure 3.41: The sign diagram for $f(x) = x^2 - 2x - 1$

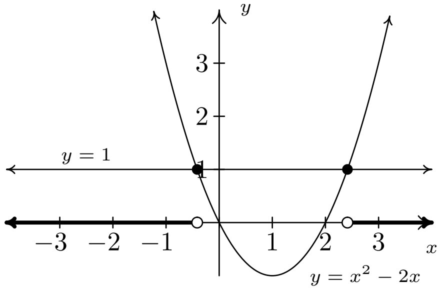


Figure 3.42: Verifying the solution to $x^2 - 2x > 1$ graphically

2. Once again, we re-write $x^2 - 2x > 1$ as $x^2 - 2x - 1 > 0$ and we identify $f(x) = x^2 - 2x - 1$. When we go to find the zeros of f, we find, to our chagrin, that the quadratic $x^2 - 2x - 1$ doesn't factor nicely. Hence, we resort to the quadratic formula to solve $x^2 - 2x - 1 = 0$, and arrive at $x = 1 \pm \sqrt{2}$. As before, these zeros divide the number line into three pieces. To help us decide on test values, we approximate $1 - \sqrt{2} \approx -0.4$ and $1 + \sqrt{2} \approx 2.4$. We choose $x = -1$, $x = 0$ and $x = 3$ as our test values and find $f(-1) = 2$, which is $(+)$; $f(0) = -1$ which is $(-)$; and $f(3) = 2$ which is $(+)$ again. Our solution to $x^2 - 2x - 1 > 0$ is where we have $(+)$, so, in interval notation $\left(-\infty, 1 - \sqrt{2}\right) \cup \left(1 + \sqrt{2}, \infty\right)$. To check the inequality $x^2 - 2x > 1$ graphically, we set $g(x) = x^2 - 2x$ and $h(x) = 1$. We are looking for the x values where the graph of g is above the graph of h: see Figure 3.42.

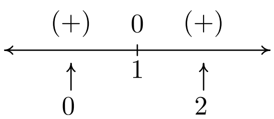


Figure 3.43: The sign diagram for $f(x) = x^2 - 2x + 1$

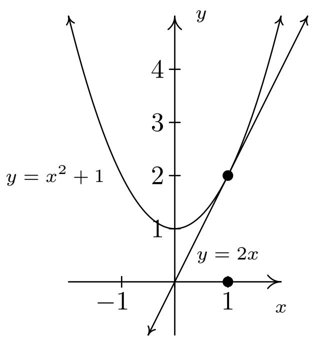


Figure 3.44: Verifying the solution to $x^2 + 1 \leq 2x$ graphically

3. To solve $x^2 + 1 \leq 2x$, as before, we solve $x^2 - 2x + 1 \leq 0$. Setting $f(x) = x^2 - 2x + 1 = 0$, we find the only one zero of f, $x = 1$. This one x value divides the number line into two intervals, from which we choose $x = 0$ and $x = 2$ as test values. We find $f(0) = 1 > 0$ and $f(2) = 1 > 0$. Since we are looking for solutions to $x^2 - 2x + 1 \leq 0$, we are looking for x values where $x^2 - 2x + 1 < 0$ as well as where $x^2 - 2x + 1 = 0$. Looking at our sign diagram, there are no places where $x^2 - 2x + 1 < 0$ (there are no $(-)$), so our solution is only $x = 1$ (where $x^2 - 2x + 1 = 0$). We write this as $\{1\}$. Graphically, we solve $x^2 + 1 \leq 2x$ by graphing $g(x) = x^2 + 1$ and $h(x) = 2x$. We are looking for the x values where the graph of g is below the graph of h (for $x^2 + 1 < 2x$) and where the two graphs intersect ($x^2 + 1 = 2x$); see Figure 3.44. Notice that the line and the parabola touch at $(1, 2)$, but the parabola is always above the line otherwise.

   In this case, we say the line $y = 2x$ is **tangent** to $y = x^2 + 1$ at $(1, 2)$. Finding tangent lines to arbitrary functions is a fundamental problem solved, in general, with Calculus.

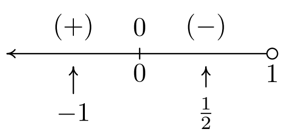


Figure 3.45: The sign diagram for $f(x) = x^2 - 3x$, where $x < 1$

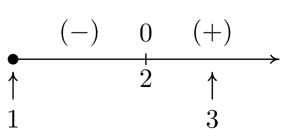


Figure 3.46: The sign diagram for $g(x) = x^2 - x - 2$, where $x \geq 1$

4. To solve our last inequality, $2x - x^2 \geq |x - 1| - 1$, we re-write the absolute value using cases. For $x < 1$, $|x - 1| = -(x - 1) = 1 - x$, so we get $2x - x^2 \geq 1 - x - 1$, or $x^2 - 3x \leq 0$. Finding the zeros of $f(x) = x^2 - 3x$, we get $x = 0$ and $x = 3$. However, we are only concerned with the portion of the number line where $x < 1$, so the only zero that we concern ourselves with is $x = 0$. This divides the interval $x < 1$ into two intervals: $(-\infty, 0)$ and $(0, 1)$. We choose $x = -1$ and $x = \frac{1}{2}$ as our test values. We find $f(-1) = 4$ and $f\left(\frac{1}{2}\right) = -\frac{5}{4}$, giving us the signs in Figure 3.45. Hence, our solution to $x^2 - 3x \leq 0$ for $x < 1$ is $[0, 1)$. Next, we turn our attention to the case $x \geq 1$. Here, $|x - 1| = x - 1$, so our original inequality becomes $2x - x^2 \geq x - 1 - 1$, or $x^2 - x - 2 \leq 0$. Setting $g(x) = x^2 - x - 2$, we find the zeros of g to be $x = -1$ and $x = 2$. Of these, only $x = 2$ lies in the region $x \geq 1$, so we ignore $x = -1$. Our test intervals are now $[1, 2)$ and $(2, \infty)$. We choose $x = 1$ and $x = 3$ as our test values and find $g(1) = -2$ and $g(3) = 4$, yielding the sign diagram in Figure **??**. Hence, our solution to $g(x) = x^2 - x - 2 \leq 0$, in this region is $[1, 2)$.

   Combining these into one sign diagram, we have that our solution is $[0, 2]$. Graphically, to check $2x - x^2 \geq |x - 1| - 1$, we set $h(x) = 2x - x^2$ and $i(x) = |x - 1| - 1$ and look for the x values where the graph of h is above the the graph of i (the solution of $h(x) > i(x)$) as well as the x-coordinates of the intersection points of both graphs (where $h(x) = i(x)$). The combined sign chart is given in Figure 3.47 and the graphs are plotted in Figure 3.48.

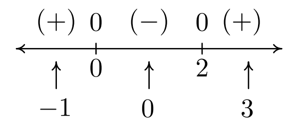


Figure 3.47: The overall sign diagram for Problem 4 in Example 60

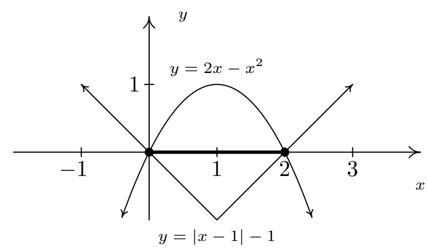


Figure 3.48: Verifying the inequality $2x - x^2 \geq |x-1| - 1$ graphically

One of the classic applications of inequalities is the notion of tolerances. Recall that for real numbers x and c, the quantity $|x-c|$ may be interpreted as the distance from x to c. Solving inequalities of the form $|x-c| \leq d$ for $d \geq 0$ can then be interpreted as finding all numbers x which lie within d units of c. We can think of the number d as a 'tolerance' and our solutions x as being within an accepted tolerance of c. We use this principle in the next example.

Example 61 **Computing tolerance**

The area A (in square inches) of a square piece of particle board which measures x inches on each side is $A(x) = x^2$. Suppose a manufacturer needs to produce a 24 inch by 24 inch square piece of particle board as part of a home office desk kit. How close does the side of the piece of particle board need to be cut to 24 inches to guarantee that the area of the piece is within a tolerance of 0.25 square inches of the target area of 576 square inches?

Solution Mathematically, we express the desire for the area $A(x)$ to be within 0.25 square inches of 576 as $|A - 576| \leq 0.25$. Since $A(x) = x^2$, we get $|x^2 - 576| \leq 0.25$, which is equivalent to $-0.25 \leq x^2 - 576 \leq 0.25$. One way to proceed at this point is to solve the two inequalities $-0.25 \leq x^2 - 576$ and $x^2 - 576 \leq 0.25$ individually using sign diagrams and then taking the intersection of the solution sets. While this way will (eventually) lead to the correct answer, we take this opportunity to showcase the increasing property of the square root: if $0 \leq a \leq b$, then $\sqrt{a} \leq \sqrt{b}$. To use this property, we proceed as follows

$$\begin{array}{rcll}
-0.25 \leq & x^2 - 576 & \leq 0.25 & \\
575.75 \leq & x^2 & \leq 576.25 & \text{(add 576 across the inequalities.)} \\
\sqrt{575.75} \leq & \sqrt{x^2} & \leq \sqrt{576.25} & \text{(take square roots.)} \\
\sqrt{575.75} \leq & |x| & \leq \sqrt{576.25} & (\sqrt{x^2} = |x|)
\end{array}$$

By Theorem 18, we find the solution to $\sqrt{575.75} \leq |x|$ to be

$$\left(-\infty, -\sqrt{575.75}\right] \cup \left[\sqrt{575.75}, \infty\right)$$

and the solution to $|x| \leq \sqrt{576.25}$ to be $\left[-\sqrt{576.25}, \sqrt{576.25}\right]$. To solve $\sqrt{575.75} \leq |x| \leq \sqrt{576.25}$, we intersect these two sets to get

$$[-\sqrt{576.25}, -\sqrt{575.75}] \cup [\sqrt{575.75}, \sqrt{576.25}].$$

Since x represents a length, we discard the negative answers and get the interval $[\sqrt{575.75}, \sqrt{576.25}]$. This means that the side of the piece of particle board must be cut between $\sqrt{575.75} \approx 23.995$ and $\sqrt{576.25} \approx 24.005$ inches, a tolerance of (approximately) 0.005 inches of the target length of 24 inches.

Our last example in the section demonstrates how inequalities can be used to describe regions in the plane, as we saw earlier in Section 2.1.

Example 62 **Relations determined by inequalities**

Sketch the following relations.

1. $R = \{(x, y) : y > |x|\}$

2. $S = \{(x,y) : y \leq 2 - x^2\}$
3. $T = \{(x,y) : |x| < y \leq 2 - x^2\}$

Solution

1. The relation R consists of all points (x,y) whose y-coordinate is greater than $|x|$. If we graph $y = |x|$, then we want all of the points in the plane *above* the points on the graph. Dotting the graph of $y = |x|$ as we have done before to indicate that the points on the graph itself are not in the relation, we get the shaded region in Figure 3.49.

2. For a point to be in S, its y-coordinate must be less than or equal to the y-coordinate on the parabola $y = 2 - x^2$. This is the set of all points *below* or *on* the parabola $y = 2 - x^2$: see Figure 3.50.

3. Finally, the relation T takes the points whose y-coordinates satisfy both the conditions given in R and those of S. Thus we shade the region between $y = |x|$ and $y = 2 - x^2$, keeping those points on the parabola, but not the points on $y = |x|$. To get an accurate graph, we need to find where these two graphs intersect, so we set $|x| = 2 - x^2$. Proceeding as before, breaking this equation into cases, we get $x = -1, 1$. Graphing yields Figure 3.51.

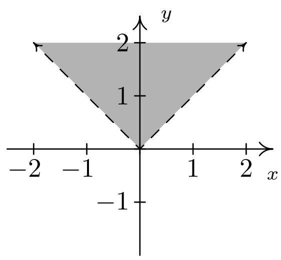


Figure 3.49: Graph of the relation R in Example 62

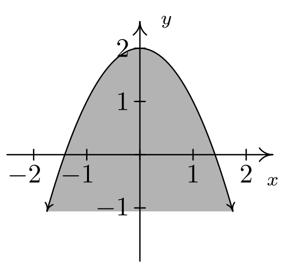


Figure 3.50: Graph of the relation S in Example 62

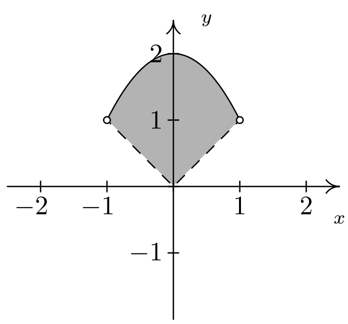


Figure 3.51: Graph of the relation T in Example 62

Exercises 3.4

Problems

In Exercises 1 – 33, solve the inequality. Write your answer using interval notation.

1. $|3x-5| \leq 4$

2. $|7x+2| > 10$

3. $|2x+1|-5 < 0$

4. $|2-x|-4 \geq -3$

5. $|3x+5|+2 < 1$

6. $2|7-x|+4 > 1$

7. $2 \leq |4-x| < 7$

8. $1 < |2x-9| \leq 3$

9. $|x+3| \geq |6x+9|$

10. $|x-3|-|2x+1| < 0$

11. $|1-2x| \geq x+5$

12. $x+5 < |x+5|$

13. $x \geq |x+1|$

14. $|2x+1| \leq 6-x$

15. $x+|2x-3| < 2$

16. $|3-x| \geq x-5$

17. $x^2+2x-3 \geq 0$

18. $16x^2+8x+1 > 0$

19. $x^2+9 < 6x$

20. $9x^2+16 \geq 24x$

21. $x^2+4 \leq 4x$

22. $x^2+1 < 0$

23. $3x^2 \leq 11x+4$

24. $x > x^2$

25. $2x^2-4x-1 > 0$

26. $5x+4 \leq 3x^2$

27. $2 \leq |x^2-9| < 9$

28. $x^2 \leq |4x-3|$

29. $x^2+x+1 \geq 0$

30. $x^2 \geq |x|$

31. $x|x+5| \geq -6$

32. $x|x-3| < 2$

33. The profit, in dollars, made by selling x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is given by $P(x) = -x^2+25x-100$, for $0 \leq x \leq 35$. How many bottles of tonic must be sold to make at least \$50 in profit?

34. Suppose $C(x) = x^2-10x+27$, $x \geq 0$ represents the costs, in *hundreds* of dollars, to produce x *thousand* pens. Find the number of pens which can be produced for no more than \$1100.

35. The temperature T, in degrees Fahrenheit, t hours after 6 AM is given by $T(t) = -\frac{1}{2}t^2+8t+32$, for $0 \leq t \leq 12$. When is it warmer than 42° Fahrenheit?

36. The height h in feet of a model rocket above the ground t seconds after lift-off is given by $h(t) = -5t^2+100t$, for $0 \leq t \leq 20$. When is the rocket at least 250 feet off the ground? Round your answer to two decimal places.

37. If a slingshot is used to shoot a marble straight up into the air from 2 meters above the ground with an initial velocity of 30 meters per second, for what values of time t will the marble be over 35 meters above the ground? (Refer to Exercise 25 in Section 3.3 for assistance if needed.) Round your answers to two decimal places.

38. What temperature values in degrees Celsius are equivalent to the temperature range $50°F$ to $95°F$? (Refer to Exercise 35 in Section 3.1 for assistance if needed.)

In Exercises 39 – 42, write and solve an inequality involving absolute values for the given statement.

39. Find all real numbers x so that x is within 4 units of 2.

40. Find all real numbers x so that $3x$ is within 2 units of -1.

41. Find all real numbers x so that x^2 is within 1 unit of 3.

42. Find all real numbers x so that x^2 is at least 7 units away from 4.

43. The surface area S of a cube with edge length x is given by $S(x) = 6x^2$ for $x > 0$. Suppose the cubes your company

manufactures are supposed to have a surface area of exactly 42 square centimetres, but the machines you own are old and cannot always make a cube with the precise surface area desired. Write an inequality using absolute value that says the surface area of a given cube is no more than 3 square centimetres away (high or low) from the target of 42 square centimetres. Solve the inequality and write your answer using interval notation.

44. Suppose f is a function, L is a real number and is a positive number. Discuss with your classmates what the inequality $|f(x) - L| <$ means algebraically and graphically.(Understanding this type of inequality is really important in Calculus.)

In Exercises 45 – 50, sketch the graph of the relation.

45. $R = \{(x, y) : y \leq x - 1\}$

46. $R = \left\{(x, y) : y > x^2 + 1\right\}$

47. $R = \{(x, y) : -1 < y \leq 2x + 1\}$

48. $R = \left\{(x, y) : x^2 \leq y < x + 2\right\}$

49. $R = \{(x, y) : |x| - 4 < y < 2 - x\}$

50. $R = \left\{(x, y) : x^2 < y \leq |4x - 3|\right\}$

4: Polynomial Functions

4.1 Graphs of Polynomial Functions

Three of the families of functions studied thus far – constant, linear and quadratic – belong to a much larger group of functions called **polynomials**. We begin our formal study of general polynomials with a definition and some examples.

Definition 37 Polynomial function

A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,$$

where a_0, a_1, ..., a_n are real numbers and $n \geq 1$ is a natural number. The domain of a polynomial function is $(-\infty, \infty)$.

There are several things about Definition 37 that may be off-putting or downright frightening. The best thing to do is look at an example. Consider $f(x) = 4x^5 - 3x^2 + 2x - 5$. Is this a polynomial function? We can re-write the formula for f as $f(x) = 4x^5 + 0x^4 + 0x^3 + (-3)x^2 + 2x + (-5)$. Comparing this with Definition 37, we identify $n = 5$, $a_5 = 4$, $a_4 = 0$, $a_3 = 0$, $a_2 = -3$, $a_1 = 2$ and $a_0 = -5$. In other words, a_5 is the coefficient of x^5, a_4 is the coefficient of x^4, and so forth; the subscript on the a's merely indicates to which power of x the coefficient belongs. The business of restricting n to be a natural number lets us focus on well-behaved algebraic animals. (Yes, there are examples of worse behaviour still to come!)

Example 63 Identifying polynomial functions

Determine if the following functions are polynomials. Explain your reasoning.

1. $g(x) = \dfrac{4 + x^3}{x}$
2. $p(x) = \dfrac{4x + x^3}{x}$
3. $q(x) = \dfrac{4x + x^3}{x^2 + 4}$
4. $f(x) = \sqrt[3]{x}$
5. $h(x) = |x|$
6. $z(x) = 0$

Solution

1. We note directly that the domain of $g(x) = \dfrac{x^3 + 4}{x}$ is $x \neq 0$. By definition, a polynomial has all real numbers as its domain. Hence, g can't be a polynomial.

2. Even though $p(x) = \dfrac{x^3 + 4x}{x}$ simplifies to $p(x) = x^2 + 4$, which certainly looks like the form given in Definition 37, the domain of p, which, as you may recall, we determine *before* we simplify, excludes 0. Alas, p is not a polynomial function for the same reason g isn't.

3. After what happened with p in the previous part, you may be a little shy about simplifying $q(x) = \dfrac{x^3 + 4x}{x^2 + 4}$ to $q(x) = x$, which certainly fits Definition 37. If we look at the domain of q before we simplified, we see that it is, indeed, all real numbers. A function which can be written in the form of Definition 37 whose domain is all real numbers is, in fact, a polynomial.

4. We can rewrite $f(x) = \sqrt[3]{x}$ as $f(x) = x^{\frac{1}{3}}$. Since $\frac{1}{3}$ is not a natural number, f is not a polynomial.

5. The function $h(x) = |x|$ isn't a polynomial, since it can't be written as a combination of powers of x even though it can be written as a piecewise function involving polynomials. As we shall see in this section, graphs of polynomials possess a quality that the graph of h does not.

6. There's nothing in Definition 37 which prevents all the coefficients a_n, etc., from being 0. Hence, $z(x) = 0$, is an honest-to-goodness polynomial.

Once we get to calculus, we'll see that the absolute value function is the classic example of a function which is continuous everywhere, but fails to have a derivative everywhere: the graph of $h(x) = |x|$ fails to be "smooth" at the origin.

Definition 38 Polynomial terminology

Suppose f is a polynomial function.

- Given $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with $a_n \neq 0$, we say
  - The natural number n is called the **degree** of the polynomial f.
  - The term $a_n x^n$ is called the **leading term** of the polynomial f.
  - The real number a_n is called the **leading coefficient** of the polynomial f.
  - The real number a_0 is called the **constant term** of the polynomial f.
- If $f(x) = a_0$, and $a_0 \neq 0$, we say f has degree 0.
- If $f(x) = 0$, we say f has no degree.

The reader may well wonder why we have chosen to separate off constant functions from the other polynomials in Definition 38. Why not just lump them all together and, instead of forcing n to be a natural number, $n = 1, 2, \ldots$, allow n to be a whole number, $n = 0, 1, 2, \ldots$. We could unify all of the cases, since, after all, isn't $a_0 x^0 = a_0$? The answer is 'yes, as long as $x \neq 0$.' The function $f(x) = 3$ and $g(x) = 3x^0$ are different, because their domains are different. The number $f(0) = 3$ is defined, whereas $g(0) = 3(0)^0$ is not. Indeed, much of the theory we will develop in this chapter doesn't include the constant functions, so we might as well treat them as outsiders from the start. One good thing that comes from Definition 38 is that we can now think of linear functions as degree 1 (or 'first degree') polynomial functions and quadratic functions as degree 2 (or 'second degree') polynomial functions.

In the context of limits, results such as 0^0 are known as *indeterminant forms*. These are cases where the function fails to be defined, but the methods of calculus might still be able to extract information.

Example 64 Using polynomial terminiology

Find the degree, leading term, leading coefficient and constant term of the following polynomial functions.

1. $f(x) = 4x^5 - 3x^2 + 2x - 5$
2. $g(x) = 12x + x^3$
3. $h(x) = \dfrac{4-x}{5}$
4. $p(x) = (2x-1)^3(x-2)(3x+2)$

Solution

1. There are no surprises with $f(x) = 4x^5 - 3x^2 + 2x - 5$. It is written in the form of Definition 38, and we see that the degree is 5, the leading term is $4x^5$, the leading coefficient is 4 and the constant term is -5.
2. The form given in Definition 38 has the highest power of x first. To that end, we re-write $g(x) = 12x + x^3 = x^3 + 12x$, and see that the degree of g is 3, the leading term is x^3, the leading coefficient is 1 and the constant term is 0.
3. We need to rewrite the formula for h so that it resembles the form given in Definition 38: $h(x) = \frac{4-x}{5} = \frac{4}{5} - \frac{x}{5} = -\frac{1}{5}x + \frac{4}{5}$. The degree of h is 1, the leading term is $-\frac{1}{5}x$, the leading coefficient is $-\frac{1}{5}$ and the constant term is $\frac{4}{5}$.
4. It may seem that we have some work ahead of us to get p in the form of Definition 38. However, it is possible to glean the information requested about p without multiplying out the entire expression $(2x-1)^3(x-2)(3x+2)$. The leading term of p will be the term which has the highest power of x. The way to get this term is to multiply the terms with the highest power of x from each factor together - in other words, the leading term of $p(x)$ is the product of the leading terms of the factors of $p(x)$. Hence, the leading term of p is $(2x)^3(x)(3x) = 24x^5$. This means that the degree of p is 5 and the leading coefficient is 24. As for the constant term, we can perform a similar trick. The constant term is obtained by multiplying the constant terms from each of the factors $(-1)^3(-2)(2) = 4$.

Our next example shows how polynomials of higher degree arise 'naturally' in even the most basic geometric applications.

Example 65 Optimizing a box construction

A box with no top is to be fashioned from a 10 inch $\times$ 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. Let x denote the length of the side of the square which is removed from each corner: see Figure 4.1.

1. Find the volume V of the box as a function of x. Include an appropriate applied domain.
2. Use software or a graphing calculator to graph $y = V(x)$ on the domain you found in part 1 and approximate the dimensions of the box with maximum volume to two decimal places. What is the maximum volume?

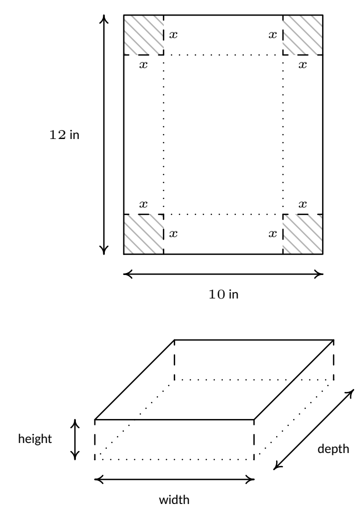


Figure 4.1: Constructing the box in Example 65

Solution

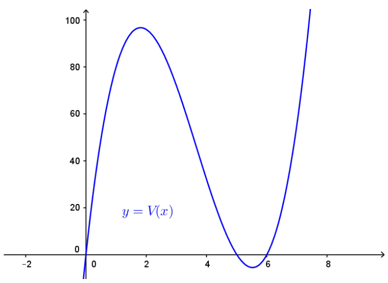
When we write $V(x)$, it is in the context of function notation, not the volume V times the quantity x. There's no harm in taking the time here to make sure that our definition of $V(x)$ makes sense. If we chopped out a 1 inch square from each side, then the width would be 8 inches, so chopping out x inches would leave $10 - 2x$ inches.

1. From Geometry, we know that Volume $=$ width $\times$ height $\times$ depth. The key is to find each of these quantities in terms of x. From the figure, we see that the height of the box is x itself. The cardboard piece is initially 10 inches wide. Removing squares with a side length of x inches from each corner leaves $10-2x$ inches for the width. As for the depth, the cardboard is initially 12 inches long, so after cutting out x inches from each side, we would have $12 - 2x$ inches remaining. As a function of x, the volume is

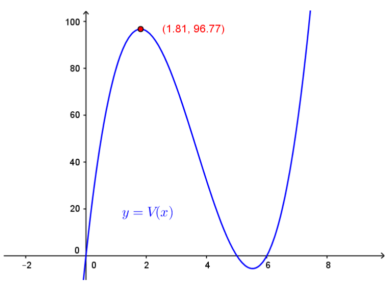
$$V(x) = x(10-2x)(12-2x) = 4x^3 - 44x^2 + 120x$$

To find a suitable applied domain, we note that to make a box at all we need $x > 0$. Also the shorter of the two dimensions of the cardboard is 10 inches, and since we are removing $2x$ inches from this dimension, we also require $10 - 2x > 0$ or $x < 5$. Hence, our applied domain is $0 < x < 5$.

2. Using GeoGebra to plot $V(x)$, we see that the graph of $y = V(x)$ has a relative maximum. The graph of V is shown in Figure 4.2; note that we had to rescale the y-axis significantly to get everything to fit on the screen. For $0 < x < 5$, this is also the absolute maximum. Using the 'Max' command, we get $x \approx 1.81$, $y \approx 96.77$. This yields a height of $x \approx 1.81$ inches, a width of $10 - 2x \approx 6.38$ inches, and a depth of $12 - 2x \approx 8.38$ inches. The y-coordinate is the maximum volume, which is approximately 96.77 cubic inches (also written in^3).



The graph $y = V(x)$



The graph $y = V(x)$ with maximum shown

Figure 4.2: Optimizing the volume of the box in Example 65

In order to solve Example 65, we made good use of the graph of the polynomial $y = V(x)$, so we ought to turn our attention to graphs of polynomials in general. In Figure 4.3 the graphs of $y = x^2$, $y = x^4$ and $y = x^6$, are shown. We have omitted the axes to allow you to see that as the exponent increases, the 'bottom' becomes 'flatter' and the 'sides' become 'steeper.' If you take the the time to graph these functions by hand, (make sure you choose some x-values between -1 and 1.) you will see why.

All of these functions are even, (Do you remember how to show this?) and it is exactly because the exponent is even. (Herein lies one of the possible origins of the term 'even' when applied to functions.) This symmetry is important, but we want to explore a different yet equally important feature of these functions which we can be seen graphically – their **end behaviour**.

When $x \to \infty$ we think of x as moving far to the right of zero and becoming a very large *positive* number. When $x \to -\infty$ we think of x as becoming a very large (in the sense of its absolute value) *negative* number far to the left of zero.

The end behaviour of a function is a way to describe what is happening to the function values (the y-values) as the x-values approach the 'ends' of the x-axis. (Of course, there are no ends to the x-axis.) That is, what happens to y as x becomes small without bound (written $x \to -\infty$) and, on the flip side, as x becomes large without bound (written $x \to \infty$).

See Theorems 9 and 10 in Section 2.6 if you need a reminder on the effect of scalings and reflections on the graph of a function.

For example, given $f(x) = x^2$, as $x \to -\infty$, we imagine substituting $x = -100$, $x = -1000$, etc., into f to get $f(-100) = 10000$, $f(-1000) = 1000000$, and so on. Thus the function values are becoming larger and larger positive numbers (without bound). To describe this behaviour, we write: as $x \to -\infty$, $f(x) \to \infty$. If we study the behaviour of f as $x \to \infty$, we see that in this case, too, $f(x) \to \infty$. (We told you that the symmetry was important!) The same can be said for any function of the form $f(x) = x^n$ where n is an even natural number. If we generalize just a bit to include vertical scalings and reflections across the x-axis, we have

Key Idea 19 End behaviour of functions $f(x) = ax^n$, n even.

Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and n is an even natural number. The end behaviour of the graph of $y = f(x)$ matches one of the following:

- for $a > 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to \infty$
- for $a < 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$

This is illustrated graphically below:

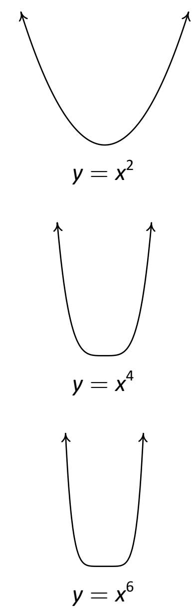


Figure 4.3: Graphing even powers of x

We now turn our attention to functions of the form $f(x) = x^n$ where $n \geq 3$ is an odd natural number. (We ignore the case when $n = 1$, since the graph of $f(x) = x$ is a line and doesn't fit the general pattern of higher-degree odd polynomials.) In Figure 4.4 we have graphed $y = x^3$, $y = x^5$, and $y = x^7$. The 'flattening' and 'steepening' that we saw with the even powers presents itself here as well, and, it should come as no surprise that all of these functions are odd. (And are, perhaps, the inspiration for the moniker 'odd function'.) The end behaviour of these functions is all the same, with $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.

As with the even degreed functions we studied earlier, we can generalize their end behaviour.

Key Idea 20 End behaviour of functions $f(x) = ax^n$, n odd.

Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and $n \geq 3$ is an odd natural number. The end behaviour of the graph of $y = f(x)$ matches one of the following:

- for $a > 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$
- for $a < 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to -\infty$

This is illustrated graphically as follows:

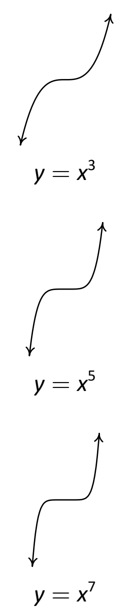


Figure 4.4: Graphing odd powers of x

Despite having different end behaviour, all functions of the form $f(x) = ax^n$ for natural numbers n share two properties which help distinguish them from other animals in the algebra zoo: they are **continuous** and **smooth**. While these concepts are formally defined using Calculus, informally, graphs of continuous functions have no 'breaks' or 'holes' in them, and the graphs of smooth functions

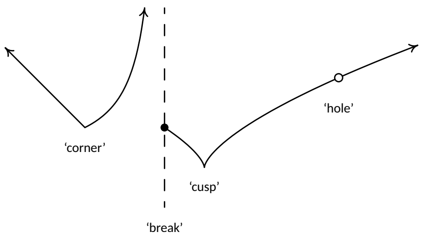


Figure 4.5: Pathologies not found on graphs of polynomials

have no 'sharp turns'. It turns out that these traits are preserved when functions are added together, so general polynomial functions inherit these qualities. In Figure 4.5, we find the graph of a function which is neither smooth nor continuous, and to its right we have a graph of a polynomial, for comparison. The function whose graph appears on the left fails to be continuous where it has a 'break' or 'hole' in the graph; everywhere else, the function is continuous. The function is continuous at the 'corner' and the 'cusp', but we consider these 'sharp turns', so these are places where the function fails to be smooth. Apart from these four places, the function is smooth and continuous. Polynomial functions are smooth and continuous everywhere, as exhibited in Figure 4.6.

In fact, when we get to Calculus, you'll find that smooth functions are automatically continuous, so that saying 'polynomials are continuous and smooth' is redundant.

The notion of smoothness is what tells us graphically that, for example, $f(x) = |x|$, whose graph is the characteristic '$\vee$' shape, cannot be a polynomial. The notion of continuity is what allowed us to construct the sign diagram for quadratic inequalities as we did in Section 3.4. This last result is formalized in the following theorem.

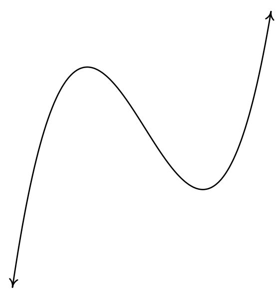


Figure 4.6: The graph of a polynomial

Theorem 19 The Intermediate Value Theorem (Zero Version)

Suppose f is a continuous function on an interval containing $x = a$ and $x = b$ with $a < b$. If $f(a)$ and $f(b)$ have different signs, then f has at least one zero between $x = a$ and $x = b$; that is, for at least one real number c such that $a < c < b$, we have $f(c) = 0$.

The Intermediate Value Theorem is extremely profound; it gets to the heart of what it means to be a real number, and is one of the most often used and under appreciated theorems in Mathematics. With that being said, most students see the result as common sense since it says, geometrically, that the graph of a polynomial function cannot be above the x-axis at one point and below the x-axis at another point without crossing the x-axis somewhere in between. We'll return to the Intermediate Value Theorem later in the Calculus portion of the course, when we study continuity in general. The following example uses the Intermediate Value Theorem to establish a fact that that most students take for granted. Many students, and sadly some instructors, will find it silly.

Example 66 Existence of $\sqrt{2}$

Use the Intermediate Value Theorem to establish that $\sqrt{2}$ is a real number.

The validity of the result in Example 66 of course relies on having a rigorous proof of Theorem 19. Although intuitive, its proof is one of the most difficult in single variable calculus. At most universities, you don't see a proof until a first course in Analysis, like Math 3500.

Solution Consider the polynomial function $f(x) = x^2 - 2$. Then $f(1) = -1$ and $f(3) = 7$. Since $f(1)$ and $f(3)$ have different signs, the Intermediate Value Theorem guarantees us a real number c between 1 and 3 with $f(c) = 0$. If $c^2 - 2 = 0$ then $c = \pm\sqrt{2}$. Since c is between 1 and 3, c is positive, so $c = \sqrt{2}$.

Our primary use of the Intermediate Value Theorem is in the construction of sign diagrams, as in Section 3.4, since it guarantees us that polynomial functions are always positive $(+)$ or always negative $(-)$ on intervals which do not contain any of its zeros. The general algorithm for polynomials is given below.

Key Idea 21 Steps for Constructing a Sign Diagram for a Polynomial Function

Suppose f is a polynomial function.

1. Find the zeros of f and place them on the number line with the number 0 above them.
2. Choose a real number, called a **test value**, in each of the intervals determined in step 1.
3. Determine the sign of $f(x)$ for each test value in step 2, and write that sign above the corresponding interval.

Example 67 Using a sign diagram to sketch a polynomial

Construct a sign diagram for $f(x) = x^3(x-3)^2(x+2)\left(x^2+1\right)$. Use it to give a rough sketch of the graph of $y = f(x)$.

SOLUTION First, we find the zeros of f by solving $x^3(x-3)^2(x+2)\left(x^2+1\right) = 0$. We get $x = 0$, $x = 3$ and $x = -2$. (The equation $x^2+1 = 0$ produces no real solutions.) These three points divide the real number line into four intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 3)$ and $(3, \infty)$. We select the test values $x = -3$, $x = -1$, $x = 1$ and $x = 4$. We find $f(-3)$ is $(+)$, $f(-1)$ is $(-)$ and $f(1)$ is $(+)$ as is $f(4)$. Wherever f is $(+)$, its graph is above the x-axis; wherever f is $(-)$, its graph is below the x-axis. The x-intercepts of the graph of f are $(-2, 0)$, $(0, 0)$ and $(3, 0)$. Knowing f is smooth and continuous allows us to sketch its graph in Figure 4.8.

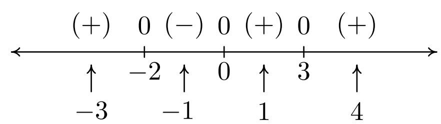


Figure 4.7: The sign diagram of f in Example 67

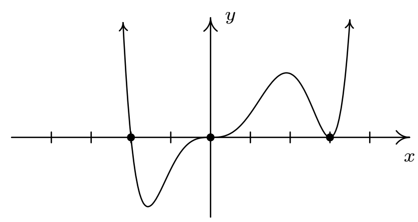


Figure 4.8: The graph $y = f(x)$ for Example 67

A couple of notes about the Example 67 are in order. First, note that we purposefully did not label the y-axis in the sketch of the graph of $y = f(x)$. This is because the sign diagram gives us the zeros and the relative position of the graph - it doesn't give us any information as to how high or low the graph strays from the x-axis. Furthermore, as we have mentioned earlier in the text, without Calculus, the values of the relative maximum and minimum can only be found approximately using a calculator. If we took the time to find the leading term of f, we would find it to be x^8. Looking at the end behaviour of f, we notice that it matches the end behaviour of $y = x^8$. This is no accident, as we find out in the next theorem.

Theorem 20 End behaviour for Polynomial Functions

The end behaviour of a polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ with $a_n \neq 0$ matches the end behaviour of $y = a_nx^n$.

To see why Theorem 20 is true, let's first look at a specific example. Consider $f(x) = 4x^3 - x + 5$. If we wish to examine end behaviour, we look to see the behaviour of f as $x \to \pm\infty$. Since we're concerned with x's far down the x-axis, we are far away from $x = 0$ so can rewrite $f(x)$ for these values of x as

$$f(x) = 4x^3\left(1 - \frac{1}{4x^2} + \frac{5}{4x^3}\right)$$

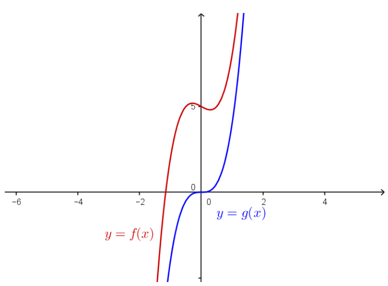
As x becomes unbounded (in either direction), the terms $\frac{1}{4x^2}$ and $\frac{5}{4x^3}$ become closer and closer to 0, as the table below indicates.

x	$\frac{1}{4x^2}$	$\frac{5}{4x^3}$
-1000	0.00000025	-0.00000000125
-100	0.000025	-0.00000125
-10	0.0025	-0.00125
10	0.0025	0.00125
100	0.000025	0.00000125
1000	0.00000025	0.00000000125

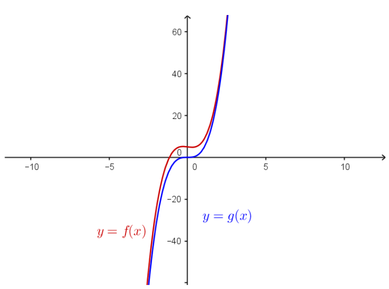
In other words, as $x \to \pm\infty$, $f(x) \approx 4x^3(1 - 0 + 0) = 4x^3$, which is the leading term of f. The formal proof of Theorem 20 works in much the same way. Factoring out the leading term leaves

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \ldots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right)$$

As $x \to \pm\infty$, any term with an x in the denominator becomes closer and closer to 0, and we have $f(x) \approx a_n x^n$. Geometrically, Theorem 20 says that if we graph $y = f(x)$ using a graphing calculator, and continue to 'zoom out', the graph of it and its leading term become indistinguishable. In Figure 4.9 the graphs of $y = 4x^3 - x + 5$ and $y = 4x^3$) in two different windows.



A view close to the origin



A 'zoomed out' view

Figure 4.9: Two views of the polynomials $f(x)$ and $g(x)$

Let's return to the function in Example 67, $f(x) = x^3(x-3)^2(x+2)\left(x^2+1\right)$, whose sign diagram and graph are given in Figures 4.7 and 4.8. Theorem 20 tells us that the end behaviour is the same as that of its leading term x^8. This tells us that the graph of $y = f(x)$ starts and ends above the x-axis. In other words, $f(x)$ is $(+)$ as $x \to \pm\infty$, and as a result, we no longer need to evaluate f at the test values $x = -3$ and $x = 4$. Is there a way to eliminate the need to evaluate f at the other test values? What we would really need to know is how the function behaves near its zeros - does it cross through the x-axis at these points, as it does at $x = -2$ and $x = 0$, or does it simply touch and rebound like it does at $x = 3$. From the sign diagram, the graph of f will cross the x-axis whenever the signs on either side of the zero switch (like they do at $x = -2$ and $x = 0$); it will touch when the signs are the same on either side of the zero (as is the case with $x = 3$). What we need to determine is the reason behind whether or not the sign change occurs.

Fortunately, f was given to us in factored form: $f(x) = x^3(x-3)^2(x+2)$. When we attempt to determine the sign of $f(-4)$, we are attempting to find the sign of the number $(-4)^3(-7)^2(-2)$, which works out to be $(-)(+)(-)$ which is $(+)$. If we move to the other side of $x = -2$, and find the sign of $f(-1)$, we are determining the sign of $(-1)^3(-4)^2(+1)$, which is $(-)(+)(+)$ which gives us the $(-)$. Notice that signs of the first two factors in both expressions are the same in $f(-4)$ and $f(-1)$. The only factor which switches sign is the third factor, $(x+2)$, precisely the factor which gave us the zero $x = -2$. If we move to the other side of 0 and look closely at $f(1)$, we get the sign pattern $(+1)^3(-2)^2(+3)$ or $(+)(+)(+)$ and we note that, once again, going from $f(-1)$ to $f(1)$, the only factor which changed sign was the first factor, x^3, which corresponds to the zero $x = 0$. Finally, to find $f(4)$, we substitute to get $(+4)^3(+2)^2(+5)$ which is $(+)(+)(+)$ or $(+)$. The sign didn't change for the middle factor $(x-3)^2$. Even though this is the factor which corresponds to the zero $x = 3$, the fact that the quantity is *squared* kept the sign of the middle factor the same on either side of 3. If we look back at the exponents on the factors $(x+2)$ and x^3, we see that they are both odd, so as we substitute values to the left and right of the corresponding zeros, the signs of the corresponding factors change which results in the sign of the function value changing. This is the key to the behaviour of the function near the zeros. We need a definition and then a theorem.

Definition 39 Multiplicity of a zero

Suppose f is a polynomial function and m is a natural number. If $(x-c)^m$ is a factor of $f(x)$ but $(x-c)^{m+1}$ is not, then we say $x=c$ is a zero of **multiplicity** m.

Hence, rewriting $f(x)=x^3(x-3)^2(x+2)$ as $f(x)=(x-0)^3(x-3)^2(x-(-2))^1$, we see that $x=0$ is a zero of multiplicity 3, $x=3$ is a zero of multiplicity 2 and $x=-2$ is a zero of multiplicity 1.

Theorem 21 The Role of Multiplicity

Suppose f is a polynomial function and $x=c$ is a zero of multiplicity m.

- If m is even, the graph of $y=f(x)$ touches and rebounds from the x-axis at $(c,0)$.
- If m is odd, the graph of $y=f(x)$ crosses through the x-axis at $(c,0)$.

Our last example shows how end behaviour and multiplicity allow us to sketch a decent graph without appealing to a sign diagram.

Example 68 Using end behaviour and multiplicity

Sketch the graph of $f(x)=-3(2x-1)(x+1)^2$ using end behaviour and the multiplicity of its zeros.

Solution The end behaviour of the graph of f will match that of its leading term. To find the leading term, we multiply by the leading terms of each factor to get $(-3)(2x)(x)^2=-6x^3$. This tells us that the graph will start above the x-axis, in Quadrant II, and finish below the x-axis, in Quadrant IV. Next, we find the zeros of f. Fortunately for us, f is factored. (Obtaining the factored form of a polynomial is the main focus of the next few sections.) Setting each factor equal to zero gives is $x=\frac{1}{2}$ and $x=-1$ as zeros. To find the multiplicity of $x=\frac{1}{2}$ we note that it corresponds to the factor $(2x-1)$. This isn't strictly in the form required in Definition 39. If we factor out the 2, however, we get $(2x-1)=2\left(x-\frac{1}{2}\right)$, and we see that the multiplicity of $x=\frac{1}{2}$ is 1. Since 1 is an odd number, we know from Theorem 21 that the graph of f will cross through the x-axis at $\left(\frac{1}{2},0\right)$. Since the zero $x=-1$ corresponds to the factor $(x+1)^2=(x-(-1))^2$, we find its multiplicity to be 2 which is an even number. As such, the graph of f will touch and rebound from the x-axis at $(-1,0)$. Though we're not asked to, we can find the y-intercept by finding $f(0)=-3(2(0)-1)(0+1)^2=3$. Thus $(0,3)$ is an additional point on the graph. Putting this together gives us the graph in Figure 4.10.

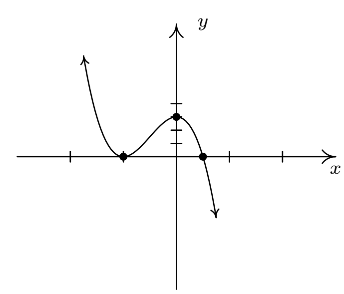


Figure 4.10: The graph $y=f(x)$ for Example 68

Exercises 4.1

Problems

In Exercises 1 – 11, solve the inequality. Write your answer using interval notation.

1. $f(x) = 4 - x - 3x^2$

2. $g(x) = 3x^5 - 2x^2 + x + 1$

3. $q(r) = 1 - 16r^4$

4. $Z(b) = 42b - b^3$

5. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$

6. $s(t) = -4.9t^2 + v_0 t + s_0$

7. $P(x) = (x-1)(x-2)(x-3)(x-4)$

8. $p(t) = -t^2(3-5t)(t^2+t+4)$

9. $f(x) = -2x^3(x+1)(x+2)^2$

10. $G(t) = 4(t-2)^2\left(t + \frac{1}{2}\right)$

In Exercises 11 – 21, find the real zeros of the given polynomial and their corresponding multiplicities. Use this information along with a sign chart to provide a rough sketch of the graph of the polynomial. Compare your answer with the result from a graphing utility.

11. $a(x) = x(x+2)^2$

12. $g(x) = x(x+2)^3$

13. $f(x) = -2(x-2)^2(x+1)$

14. $g(x) = (2x+1)^2(x-3)$

15. $F(x) = x^3(x+2)^2$

16. $P(x) = (x-1)(x-2)(x-3)(x-4)$

17. $Q(x) = (x+5)^2(x-3)^4$

18. $h(x) = x^2(x-2)^2(x+2)^2$

19. $H(t) = (3-t)(t^2+1)$

20. $Z(b) = b(42-b^2)$

In Exercises 21 – 27, given the pair of functions f and g, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice through the transformations. State the domain and range of g.

21. $f(x) = x^3$, $g(x) = (x+2)^3 + 1$

22. $f(x) = x^4$, $g(x) = (x+2)^4 + 1$

23. $f(x) = x^4$, $g(x) = 2 - 3(x-1)^4$

24. $f(x) = x^5$, $g(x) = -x^5 - 3$

25. $f(x) = x^5$, $g(x) = (x+1)^5 + 10$

26. $f(x) = x^6$, $g(x) = 8 - x^6$

27. Use the Intermediate Value Theorem to prove that $f(x) = x^3 - 9x + 5$ has a real zero in each of the following intervals: $[-4, -3]$, $[0, 1]$ and $[2, 3]$.

28. Rework Example **??** assuming the box is to be made from an 8.5 inch by 11 inch sheet of paper. Using scissors and tape, construct the box. Are you surprised?[1]

In Exercises 29 – 32, suppose the revenue R, in *thousands* of dollars, from producing and selling x *hundred* LCD TVs is given by $R(x) = -5x^3 + 35x^2 + 155x$ for $0 \leq x \leq 10.07$.

29. Use a graphing utility to graph $y = R(x)$ and determine the number of TVs which should be sold to maximize revenue. What is the maximum revenue?

30. Assume that the cost, in *thousands* of dollars, to produce x *hundred* LCD TVs is given by $C(x) = 200x + 25$ for $x \geq 0$. Find and simplify an expression for the profit function $P(x)$. (Remember: Profit = Revenue - Cost.)

31. Use a graphing utility to graph $y = P(x)$ and determine the number of TVs which should be sold to maximize profit. What is the maximum profit?

32. While developing their newest game, Sasquatch Attack!, the makers of the PortaBoy (from Example 45) revised their cost function and now use $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \geq 0$. As before, $C(x)$ is the cost to make x PortaBoy Game Systems. Market research indicates that the demand function $p(x) = -1.5x + 250$ remains unchanged. Use a graphing utility to find the production level x that maximizes the *profit* made by producing and selling x PortaBoy game systems.

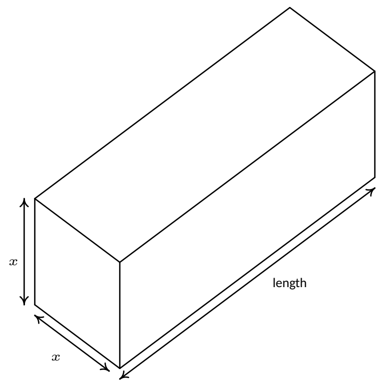
33. According to US Postal regulations, a rectangular shipping box must satisfy the inequality "Length + Girth $\leq$ 130 inches" for Parcel Post and "Length + Girth $\leq$ 108 inches" for other services. Let's assume we have a closed rectangular box with a square face of side length x as drawn below. The length is the longest side and is clearly labeled. The girth is the distance around the box in the other two dimensions so in our case it is the sum of the four sides of the square, $4x$.

[1] Consider decorating the box and presenting it to your instructor. If done well enough, maybe your instructor will issue you some bonus points. Or maybe not.

(a) Assuming that we'll be mailing a box via Parcel Post where Length + Girth $= 130$ inches, express the length of the box in terms of x and then express the volume V of the box in terms of x.

(b) Find the dimensions of the box of maximum volume that can be shipped via Parcel Post.

(c) Repeat parts 33a and 33b if the box is shipped using "other services".



34. Show that the end behaviour of a linear function $f(x) = mx + b$ is as it should be according to the results we've established in the section for polynomials of odd degree.[2] (That is, show that the graph of a linear function is "up on one side and down on the other" just like the graph of $y = a_n x^n$ for odd numbers n.)

35. There is one subtlety about the role of multiplicity that we need to discuss further; specifically we need to see 'how' the graph crosses the x-axis at a zero of odd multiplicity. In the section, we deliberately excluded the function $f(x) = x$ from the discussion of the end behaviour of $f(x) = x^n$ for odd numbers n and we said at the time that it was due to the fact that $f(x) = x$ didn't fit the pattern we were trying to establish. You just showed in the previous exercise that the end behaviour of a linear function behaves like every other polynomial of odd degree, so what doesn't $f(x) = x$ do that $g(x) = x^3$ does? It's the 'flattening' for values of x near zero. It is this local behaviour that will distinguish between a zero of multiplicity 1 and one of higher odd multiplicity. Look again closely at the graphs of $a(x) = x(x+2)^2$ and $F(x) = x^3(x+2)^2$ from Exercise 21. Discuss with your classmates how the graphs are fundamentally different at the origin. It might help to use a graphing calculator to zoom in on the origin to see the different crossing behaviour. Also compare the behaviour of $a(x) = x(x+2)^2$ to that of $g(x) = x(x+2)^3$ near the point $(-2, 0)$. What do you predict will happen at the zeros of $f(x) = (x-1)(x-2)^2(x-3)^3(x-4)^4(x-5)^5$?

36. Here are a few other questions for you to discuss with your classmates.

(a) How many local extrema could a polynomial of degree n have? How few local extrema can it have?

(b) Could a polynomial have two local maxima but no local minima?

(c) If a polynomial has two local maxima and two local minima, can it be of odd degree? Can it be of even degree?

(d) Can a polynomial have local extrema without having any real zeros?

(e) Why must every polynomial of odd degree have at least one real zero?

(f) Can a polynomial have two distinct real zeros and no local extrema?

(g) Can an x-intercept yield a local extrema? Can it yield an absolute extrema?

(h) If the y-intercept yields an absolute minimum, what can we say about the degree of the polynomial and the sign of the leading coefficient?

[2]Remember, to be a linear function, $m \neq 0$.

4.2 The Factor Theorem and the Remainder Theorem

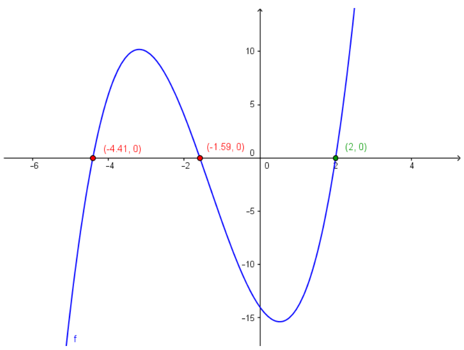


Figure 4.11: The graph $y = x^3 + 4x^2 - 5x - 14$

Suppose we wish to find the zeros of $f(x) = x^3 + 4x^2 - 5x - 14$. Setting $f(x) = 0$ results in the polynomial equation $x^3 + 4x^2 - 5x - 14 = 0$. Despite all of the factoring techniques we learned (and probably forgot) in high school, this equation foils us at every turn. If we graph f using GeoGebra, we get the result in Figure 4.11.

The graph suggests that the function has three zeros, one of which is $x = 2$. It's easy to show that $f(2) = 0$, but the other two zeros seem to be less friendly. Asking GeoGebra to intersect the graph with the x-axis gives us the decimal approximations shown in the figure, but we seek a method to find the remaining zeros *exactly*. Based on our experience, if $x = 2$ is a zero, it seems that there should be a factor of $(x - 2)$ lurking around in the factorization of $f(x)$. In other words, we should expect that $x^3 + 4x^2 - 5x - 14 = (x - 2)\,q(x)$, where $q(x)$ is some other polynomial. How could we find such a $q(x)$, if it even exists? The answer comes from our old friend, polynomial division. Dividing $x^3 + 4x^2 - 5x - 14$ by $x - 2$ gives

$$
\begin{array}{rrrrr}
 & x^2 & +\ 6x & +\ 7 & \\ \cline{2-5}
x-2 \,\big) & x^3 & +\ 4x^2 & -\ 5x & -\ 14 \\
 & -\left(x^3\right. & \left.-\ 2x^2\right) & & \\ \cline{2-3}
 & & 6x^2 & -\ 5x & \\
 & & -\left(6x^2\right. & \left.-\ 12x\right) & \\ \cline{3-4}
 & & & 7x & -\ 14 \\
 & & & -\left(7x\right. & \left.-\ 14\right) \\ \cline{4-5}
 & & & & 0
\end{array}
$$

As you may recall, this means $x^3 + 4x^2 - 5x - 14 = (x - 2)\left(x^2 + 6x + 7\right)$, so to find the zeros of f, we now solve $(x - 2)\left(x^2 + 6x + 7\right) = 0$. We get $x - 2 = 0$ (which gives us our known zero, $x = 2$) as well as $x^2 + 6x + 7 = 0$. The latter doesn't factor nicely, so we apply the Quadratic Formula to get $x = -3 \pm \sqrt{2}$. The point of this section is to generalize the technique applied here. First up is a friendly reminder of what we can expect when we divide polynomials.

Theorem 22 Polynomial Division

Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d. There exist two unique polynomials, $q(x)$ and $r(x)$, such that $p(x) = d(x)\,q(x) + r(x)$, where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

As you may recall, all of the polynomials in Theorem 22 have special names. The polynomial p is called the **dividend**; d is the **divisor**; q is the **quotient**; r is the **remainder**. If $r(x) = 0$ then d is called a **factor** of p. The proof of Theorem 22 is usually relegated to a course in Abstract Algebra, but we can still use the result to establish two important facts which are the basis of the rest of the chapter.

Theorem 23 The Remainder Theorem

Suppose p is a polynomial of degree at least 1 and c is a real number. When $p(x)$ is divided by $x-c$ the remainder is $p(c)$.

The proof of Theorem 23 is a direct consequence of Theorem 22. When a polynomial is divided by $x-c$, the remainder is either 0 or has degree less than the degree of $x-c$. Since $x-c$ is degree 1, the degree of the remainder must be 0, which means the remainder is a constant. Hence, in either case, $p(x) = (x-c)\,q(x) + r$, where r, the remainder, is a real number, possibly 0. It follows that $p(c) = (c-c)\,q(c) + r = 0 \cdot q(c) + r = r$, so we get $r = p(c)$ as required. There is one more piece of 'low hanging fruit' to collect, which we present below.

Theorem 24 The Factor Theorem

Suppose p is a nonzero polynomial. The real number c is a zero of p if and only if $(x-c)$ is a factor of $p(x)$.

The proof of The Factor Theorem is a consequence of what we already know. If $(x-c)$ is a factor of $p(x)$, this means $p(x) = (x-c)\,q(x)$ for some polynomial q. Hence, $p(c) = (c-c)\,q(c) = 0$, so c is a zero of p. Conversely, if c is a zero of p, then $p(c) = 0$. In this case, The Remainder Theorem tells us the remainder when $p(x)$ is divided by $(x-c)$, namely $p(c)$, is 0, which means $(x-c)$ is a factor of p. What we have established is the fundamental connection between zeros of polynomials and factors of polynomials.

The next example pulls together all of the concepts discussed in this section.

Example 69 Factoring a cubic polynomial

Let $p(x) = 2x^3 - 5x + 3$.

1. Find $p(-2)$ using The Remainder Theorem. Check your answer by substitution.

2. Use the fact that $x = 1$ is a zero of p to factor $p(x)$ and then find all of the real zeros of p.

Solution

1. The Remainder Theorem states $p(-2)$ is the remainder when $p(x)$ is divided by by $x-(-2) = x+2$. Using long division, we get:

$$
\begin{array}{r|rrrr}
 & 2x^2 & -\,4x & +\,3 & \\
\cline{2-5}
x+2 & 2x^3 & & -\,5x & +\,3 \\
 & -(2x^3 & +\,4x^2) & & \\
\cline{2-3}
 & & -\,4x^2 & -\,5x & \\
 & & -(4x^2 & +\,8x) & \\
\cline{3-4}
 & & & 3x & +\,3 \\
 & & & -(3x & +\,6) \\
\cline{4-5}
 & & & & -3
\end{array}
$$

According to the Remainder Theorem, $p(-2) = -3$. We can check this by direct substitution into the formula for $p(x)$: $p(-2) = 2(-2)^3 - 5(-2) + 3 = -16 + 10 + 3 = -3$.

2. The Factor Theorem tells us that since $x = 1$ is a zero of p, $x - 1$ is a factor of $p(x)$. To factor $p(x)$, we divide by $x - 1$, giving us

$$\begin{array}{r|l} & \quad 2x^2 + 2x - 3 \\ \hline x-1 & 2x^3 \qquad\quad + x - 3 \\ & -\underline{(2x^3 - 2x^2)} \\ & \qquad - 4x^2 - 5x \\ & \qquad -\underline{(2x^2 - 2x)} \\ & \qquad\qquad - 3x + 3 \\ & \qquad\qquad -\underline{(3x+3)} \\ & \qquad\qquad\qquad 0 \end{array}$$

We get a remainder of 0 which verifies that, indeed, $p(1) = 0$. Our quotient polynomial is a second degree polynomial with coefficients 2, 2, and -3. So $q(x) = 2x^2 + 2x - 3$. Theorem 22 tells us $p(x) = (x - 1)\left(2x^2 + 2x - 3\right)$. To find the remaining real zeros of p, we need to solve $2x^2 + 2x - 3 = 0$ for x. Since this doesn't factor nicely, we use the quadratic formula to find that the remaining zeros are $x = \frac{-1\pm\sqrt{7}}{2}$.

In Section 4.1, we discussed the notion of the multiplicity of a zero. Roughly speaking, a zero with multiplicity 2 can be divided twice into a polynomial; multiplicity 3, three times and so on. This is illustrated in the next example.

Example 70 **Factoring out a zero of multiplicity two**

Let $p(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3$. Given that $x = \frac{1}{2}$ is a zero of multiplicity 2, find all of the real zeros of p.

Solution Since $x = \frac{1}{2}$ is a zero of multiplicity 2, the Factor Theorem guarantees that $\left(x - \frac{1}{2}\right)$ is a factor of $p(x)$, *twice*; that is, that $\left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{4}$ is a factor of $p(x)$. Using long division, we have

$$\begin{array}{r|l} & \qquad 4x^2 \qquad\quad -12 \\ \hline x^2-x+\frac{1}{4} & 4x^4 - 4x^3 - 11x^2 + 12x - 3 \\ & -\underline{(4x^4 - 4x^3 + x^2)} \\ & \qquad\qquad - 12x^2 + 12x - 3 \\ & \qquad\qquad -\underline{(12x^2 + 12x - 3)} \\ & \qquad\qquad\qquad\qquad 0 \end{array}$$

We obtain a remainder of 0, confirming that $\left(x - \frac{1}{2}\right)$ is a factor, and our quotient of $4x^2 - 12$ tells us that $p(x) = \left(x - \frac{1}{2}\right)^2(4x^2 - 12)$. Setting $4x^2 - 12 = 0$ gives us $x = \pm\sqrt{3}$, so the remaining zeros of p are $x = \sqrt{3}$ and $x = -\sqrt{3}$.

Since we found that $x = \pm\sqrt{3}$ are zeros of p, the Factor Theorem guarantees that $(x - \sqrt{3})$ and $(x + \sqrt{3})$ are factors of p. Of course, you were probably able to spot this yourself: since $4x^2 - 12$ has a common factor of 4, we have

$$4x^2 - 12 = 4(x^2 - 3) = 4(x - \sqrt{3})(x + \sqrt{3}),$$

by recognizing $x^2 - 3$ as a difference of squares. (If you're rusty on your factoring, see Review Worksheet #5.) Thus, we obtain

$$p(x) = 4\left(x - \frac{1}{2}\right)^2 (x - \sqrt{3})(x + \sqrt{3})$$

in completely factored form.

We have shown that p is a product of its leading coefficient times linear factors of the form $(x - c)$ where c are zeros of p. It may surprise and delight the reader that, in theory, all polynomials can be reduced to this kind of factorization. We leave that discussion to Section 4.4, because the zeros may not be real numbers. Our final theorem in the section gives us an upper bound on the number of real zeros.

Theorem 25 Number of zeros is bounded above by degree

Suppose f is a polynomial of degree $n \geq 1$. Then f has at most n real zeros, counting multiplicities.

Theorem 25 is a consequence of the Factor Theorem and polynomial multiplication. Every zero c of f gives us a factor of the form $(x - c)$ for $f(x)$. Since f has degree n, there can be at most n of these factors. The next section provides us some tools which not only help us determine where the real zeros are to be found, but which real numbers they may be.

We close this section with a summary of several concepts previously presented. You should take the time to look back through the text to see where each concept was first introduced and where each connection to the other concepts was made.

Key Idea 22 Connections Between Zeros, Factors and Graphs of Polynomial Functions

Suppose p is a polynomial function of degree $n \geq 1$. The following statements are equivalent:

- The real number c is a zero of p
- $p(c) = 0$
- $x = c$ is a solution to the polynomial equation $p(x) = 0$
- $(x - c)$ is a factor of $p(x)$
- The point $(c, 0)$ is an x-intercept of the graph of $y = p(x)$

Exercises 4.2

Problems

In Exercises 1 – 7, use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.

1. $\left(4x^2 + 3x - 1\right) \div (x - 3)$
2. $\left(2x^3 - x + 1\right) \div \left(x^2 + x + 1\right)$
3. $\left(5x^4 - 3x^3 + 2x^2 - 1\right) \div \left(x^2 + 4\right)$
4. $\left(-x^5 + 7x^3 - x\right) \div \left(x^3 - x^2 + 1\right)$
5. $\left(9x^3 + 5\right) \div (2x - 3)$
6. $\left(4x^2 - x - 23\right) \div \left(x^2 - 1\right)$

In Exercises 7 – 17, determine $p(c)$ using the Remainder Theorem for the given polynomial functions and value of c. If $p(c) = 0$, factor $p(x) = (x - c)q(x)$.

7. $p(x) = 2x^2 - x + 1$, $c = 4$
8. $p(x) = 4x^2 - 33x - 180$, $c = 12$
9. $p(x) = 2x^3 - x + 6$, $c = -3$
10. $p(x) = x^3 + 2x^2 + 3x + 4$, $c = -1$
11. $p(x) = 3x^3 - 6x^2 + 4x - 8$, $c = 2$
12. $p(x) = 8x^3 + 12x^2 + 6x + 1$, $c = -\frac{1}{2}$
13. $p(x) = x^4 - 2x^2 + 4$, $c = \frac{3}{2}$
14. $p(x) = 6x^4 - x^2 + 2$, $c = -\frac{2}{3}$
15. $p(x) = x^4 + x^3 - 6x^2 - 7x - 7$, $c = -\sqrt{7}$
16. $p(x) = x^2 - 4x + 1$, $c = 2 - \sqrt{3}$

In Exercises 17 – 27, you are given a polynomial and one of its zeros. Use the techniques in this section to find the rest of the real zeros and factor the polynomial.

17. $x^3 - 6x^2 + 11x - 6, \quad c = 1$
18. $x^3 - 24x^2 + 192x - 512, \quad c = 8$
19. $3x^3 + 4x^2 - x - 2, \quad c = \frac{2}{3}$
20. $2x^3 - 3x^2 - 11x + 6, \quad c = \frac{1}{2}$
21. $x^3 + 2x^2 - 3x - 6, \quad c = -2$
22. $2x^3 - x^2 - 10x + 5, \quad c = \frac{1}{2}$
23. $4x^4 - 28x^3 + 61x^2 - 42x + 9$, $c = \frac{1}{2}$ is a zero of multiplicity 2
24. $x^5 + 2x^4 - 12x^3 - 38x^2 - 37x - 12$, $c = -1$ is a zero of multiplicity 3
25. $125x^5 - 275x^4 - 2265x^3 - 3213x^2 - 1728x - 324$, $c = -\frac{3}{5}$ is a zero of multiplicity 3
26. $x^2 - 2x - 2, \quad c = 1 - \sqrt{3}$

In Exercises 27 – 32, create a polynomial p which has the desired characteristics. You may leave the polynomial in factored form.

27.
    - The zeros of p are $c = \pm 2$ and $c = \pm 1$
    - The leading term of $p(x)$ is $117x^4$.
28.
    - The zeros of p are $c = 1$ and $c = 3$
    - $c = 3$ is a zero of multiplicity 2.
    - The leading term of $p(x)$ is $-5x^3$
29.
    - The solutions to $p(x) = 0$ are $x = \pm 3$ and $x = 6$
    - The leading term of $p(x)$ is $7x^4$
    - The point $(-3, 0)$ is a local minimum on the graph of $y = p(x)$.
30.
    - The solutions to $p(x) = 0$ are $x = \pm 3$, $x = -2$, and $x = 4$.
    - The leading term of $p(x)$ is $-x^5$.
    - The point $(-2, 0)$ is a local maximum on the graph of $y = p(x)$.
31.
    - p is degree 4.
    - as $x \to \infty$, $p(x) \to -\infty$
    - p has exactly three x-intercepts: $(-6, 0)$, $(1, 0)$ and $(117, 0)$
    - The graph of $y = p(x)$ crosses through the x-axis at $(1, 0)$.
32. Find a quadratic polynomial with integer coefficients which has $x = \dfrac{3}{5} \pm \dfrac{\sqrt{29}}{5}$ as its real zeros.

4.3 Real Zeros of Polynomials

In Section 4.2, we found that we can use long division to determine if a given real number is a zero of a polynomial function. This section presents results which will help us determine good candidates to test using long division. There are two approaches to the topic of finding the real zeros of a polynomial. The first approach (which is gaining popularity) is to use a little bit of Mathematics followed by a good use of technology like graphing calculators. The second approach (for purists) makes good use of mathematical machinery (theorems) only. For completeness, we include the two approaches but in separate subsections. Both approaches benefit from the following two theorems, the first of which is due to the famous mathematician Augustin Cauchy. It gives us an interval on which all of the real zeros of a polynomial can be found.

Carl is the purist and is responsible for all of the theorems in this section. Jeff, on the other hand, has spent too much time in school politics and has been polluted with notions of 'compromise.' You can blame the slow decline of civilization on him and those like him who mingle Mathematics with technology.

Theorem 26 Cauchy's Bound

Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$. Let M be the largest of the numbers: $\frac{|a_0|}{|a_n|}$, $\frac{|a_1|}{|a_n|}$, ..., $\frac{|a_{n-1}|}{|a_n|}$. Then all the real zeros of f lie in in the interval $[-(M+1), M+1]$.

The proof of Theorem 26 is not easily explained within the confines of this text. This paper contains the result and gives references to its proof.

Like many of the results in this section, Cauchy's Bound is best understood with an example.

Example 71 Using Cauchy's Bound

Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Determine an interval which contains all of the real zeros of f.

SOLUTION To find the M stated in Cauchy's Bound, we take the absolute value of the leading coefficient, in this case $|2| = 2$ and divide it into the largest (in absolute value) of the remaining coefficients, in this case $|-6| = 6$. This yields $M = 3$ so it is guaranteed that all of the real zeros of f lie in the interval $[-4, 4]$.

Whereas the previous result tells us *where* we can find the real zeros of a polynomial, the next theorem gives us a list of *possible* real zeros.

Theorem 27 Rational Zeros Theorem

Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$, and a_0, a_1, ...a_n are integers. If r is a rational zero of f, then r is of the form $\pm\frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

The Rational Zeros Theorem gives us a list of numbers to try in our long division and that is a lot nicer than simply guessing. If none of the numbers in the list are zeros, then either the polynomial has no real zeros at all, or all of the real zeros are irrational numbers. To see why the Rational Zeros Theorem works, suppose c is a zero of f and $c = \frac{p}{q}$ in lowest terms. This means p and q have no common factors. Since $f(c) = 0$, we have

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + a_1\left(\frac{p}{q}\right) + a_0 = 0.$$

Multiplying both sides of this equation by q^n, we clear the denominators to get

$$a_np^n + a_{n-1}p^{n-1}q + \ldots + a_1pq^{n-1} + a_0q^n = 0$$

Rearranging this equation, we get

$$a_np^n = -a_{n-1}p^{n-1}q - \ldots - a_1pq^{n-1} - a_0q^n$$

Now, the left hand side is an integer multiple of p, and the right hand side is an integer multiple of q. (Can you see why?) This means a_np^n is both a multiple of p and a multiple of q. Since p and q have no common factors, a_n must be a multiple of q. If we rearrange the equation

$$a_np^n + a_{n-1}p^{n-1}q + \ldots + a_1pq^{n-1} + a_0q^n = 0$$

as

$$a_0q^n = -a_np^n - a_{n-1}p^{n-1}q - \ldots - a_1pq^{n-1}$$

we can play the same game and conclude a_0 is a multiple of p, and we have the result.

Example 72 **Finding rational zeros**
Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use the Rational Zeros Theorem to list all of the possible rational zeros of f.

Solution To generate a complete list of rational zeros, we need to take each of the factors of constant term, $a_0 = -3$, and divide them by each of the factors of the leading coefficient $a_4 = 2$. The factors of -3 are ± 1 and ± 3. Since the Rational Zeros Theorem tacks on a $\pm$ anyway, for the moment, we consider only the positive factors 1 and 3. The factors of 2 are 1 and 2, so the Rational Zeros Theorem gives the list $\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}\right\}$ or $\left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3\right\}$.

Our discussion now diverges between those who wish to use technology and those who do not.

4.3.1 For Those Wishing to use Technology

At this stage, we know not only the interval in which all of the zeros of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ are located, but we also know some potential candidates. We can now use software or a graphing calculator to help us determine all of the real zeros of f, as illustrated in the next example.

Example 73 **Using technology to find the zeros of a polynomial**
Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

1. Graph $y = f(x)$ on the calculator using the interval obtained in Example 71 as a guide.

2. Use the graph to shorten the list of possible rational zeros obtained in Example 72.

3. Use long division to find the real zeros of f, and state their multiplicities.

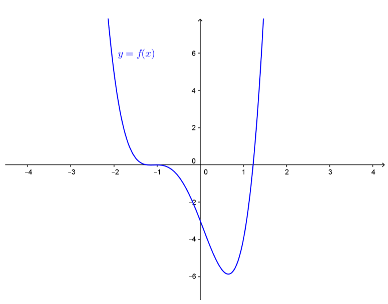


Figure 4.12: The graph $y = f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$

Solution

1. In Example 71, we determined all of the real zeros of f lie in the interval $[-4, 4]$. We plot $f(x)$ using GeoGebra, and zoom in to show the portion of the graph where $-4 \leq x \leq 4$: see Figure 4.12.

2. In Example 72, we learned that any rational zero of f must be in the list $\left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3\right\}$. From the graph, it looks as if we can rule out any of the positive rational zeros, since the graph seems to cross the x-axis at a value just a little greater than 1. On the negative side, -1 looks good, so we try that for our long division.

$$
\begin{array}{r|rrrrr}
 & 2x^3 & +\,2x^2 & -\,3x & -\,3 & \\ \hline
x+1 & 2x^4 & +\,4x^3 & -\,x^2 & -\,6x & -\,3 \\
 & -\left(2x^4\right. & \left.+\,2x^3\right) & & & \\ \hline
 & & 2x^3 & -\,x^2 & & \\
 & & -\left(2x^3\right. & \left.+\,2x^2\right) & & \\ \hline
 & & & -\,3x^2 & -\,6x & \\
 & & & -\left(-\,3x^2\right. & \left.-\,3x\right) & \\ \hline
 & & & & -\,3x & -\,3 \\
 & & & & -\left(-\,3x\right. & \left.-\,3\right) \\ \hline
 & & & & & 0
\end{array}
$$

We have a winner! Remembering that f was a fourth degree polynomial, we know that our quotient is a third degree polynomial. If we can do one more successful division, we will have knocked the quotient down to a quadratic, and, if all else fails, we can use the quadratic formula to find the last two zeros. Since there seems to be no other rational zeros to try, we continue with -1. Also, the shape of the crossing at $x = -1$ leads us to wonder if the zero $x = -1$ has multiplicity 3.

$$
\begin{array}{r|rrrr}
 & 2x^2 & & -\,3 & \\ \hline
x+1 & 2x^3 & +\,2x^2 & -\,3x & -\,3 \\
 & -\left(2x^3\right. & \left.+\,2x^2\right) & & \\ \hline
 & & 0 & -\,3x & -\,3 \\
 & & & -\left(-\,3x\right. & \left.-\,3\right) \\ \hline
 & & & & 0
\end{array}
$$

Success! Our quotient polynomial is now $2x^2 - 3$. Setting this to zero gives $2x^2 - 3 = 0$, or $x^2 = \frac{3}{2}$, which gives us $x = \pm\frac{\sqrt{6}}{2}$. Concerning multiplicities, based on our division, we have that -1 has a multiplicity of at least 2. The Factor Theorem tells us our remaining zeros, $\pm\frac{\sqrt{6}}{2}$, each have multiplicity at least 1. However, Theorem 25 tells us f can have at most 4 real zeros, counting multiplicity, and so we conclude that -1 is of multiplicity exactly 2 and $\pm\frac{\sqrt{6}}{2}$ each has multiplicity 1. (Thus, we were wrong to think that -1 had multiplicity 3.)

It is interesting to note that we could greatly improve on the graph of $y = f(x)$ in the previous example given to us by GeoGebra. For instance, from our determination of the zeros of f and their multiplicities, we know the graph crosses at $x = -\frac{\sqrt{6}}{2} \approx -1.22$ then turns back upwards to touch the $x-$axis at $x = -1$. This tells us that, despite what the software showed us the first time, there is a relative maximum occurring at $x = -1$ and not a 'flattened crossing' as we originally believed. After zooming in and rescaling the coordinate axes, we see not only the relative maximum but also a relative minimum (this is an example of what is called 'hidden behaviour.') just to the left of $x = -1$ which shows us,

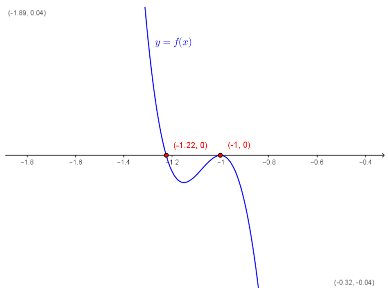


Figure 4.13: Zooming in on the repeated zero in Example 73

The y-axis isn't visible in Figure 4.13, so it's worth pointing out that in order to get a good view of the two local extrema, we had to shrink the y scale significantly: the y-value of the local minimum at $x = -\sqrt{6}/2$ is just shy of -0.01.

once again, that Mathematics enhances the technology, instead of vice-versa: see Figure 4.13.

Our next example shows how even a mild-mannered polynomial can cause problems.

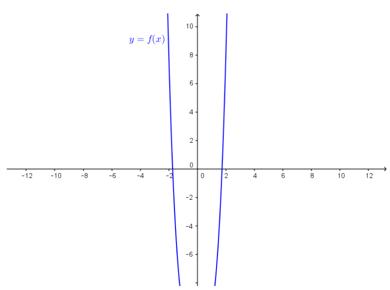
Example 74 **Factoring using a *u*-substitution**

Let $f(x) = x^4 + x^2 - 12$.

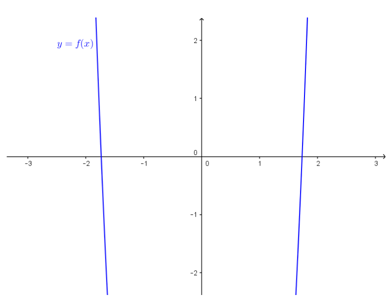
1. Use Cauchy's Bound to determine an interval in which all of the real zeros of f lie.
2. Use the Rational Zeros Theorem to determine a list of possible rational zeros of f.
3. Graph $y = f(x)$ using your graphing calculator.
4. Find all of the real zeros of f and their multiplicities.

Solution

1. Applying Cauchy's Bound, we find $M = 12$, so all of the real zeros lie in the interval $[-13, 13]$.
2. Applying the Rational Zeros Theorem with constant term $a_0 = -12$ and leading coefficient $a_4 = 1$, we get the list $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$.
3. Graphing $y = f(x)$ on the interval $[-13, 13]$ produces the graph in Figure 4.14 (a). Zooming in a bit gives the graph (b). Based on the graph, none of our rational zeros will work. (Do you see why not?)
4. From the graph, we know f has two real zeros, one positive, and one negative. Our only hope at this point is to try and find the zeros of f by setting $f(x) = x^4 + x^2 - 12 = 0$ and solving. If we stare at this equation long enough, we may recognize it as a 'quadratic in disguise' or 'quadratic in form'. In other words, we have three terms: x^4, x^2 and 12, and the exponent on the first term, x^4, is exactly twice that of the second term, x^2. We may rewrite this as $\left(x^2\right)^2 + \left(x^2\right) - 12 = 0$. To better see the forest for the trees, we momentarily replace x^2 with the variable u. In terms of u, our equation becomes $u^2 + u - 12 = 0$, which we can readily factor as $(u+4)(u-3) = 0$. In terms of x, this means $x^4 + x^2 - 12 = \left(x^2 - 3\right)\left(x^2 + 4\right) = 0$. We get $x^2 = 3$, which gives us $x = \pm\sqrt{3}$, or $x^2 = -4$, which admits no real solutions. Since $\sqrt{3} \approx 1.73$, the two zeros match what we expected from the graph. In terms of multiplicity, the Factor Theorem guarantees $\left(x - \sqrt{3}\right)$ and $\left(x + \sqrt{3}\right)$ are factors of $f(x)$. Since $f(x)$ can be factored as $f(x) = \left(x^2 - 3\right)\left(x^2 + 4\right)$, and $x^2 + 4$ has no real zeros, the quantities $\left(x - \sqrt{3}\right)$ and $\left(x + \sqrt{3}\right)$ must both be factors of $x^2 - 3$. According to Theorem 25, $x^2 - 3$ can have at most 2 zeros, counting multiplicity, hence each of $\pm\sqrt{3}$ is a zero of f of multiplicity 1.



(a)



(b)

Figure 4.14: Two views of the graph $y = f(x) = x^4 + x^2 - 12$ in Example 74

The technique used to factor $f(x)$ in Example 74 is called ***u*-substitution**. We shall see more of this technique in Section **??**. In general, substitution can help us identify a 'quadratic in disguise' provided that there are exactly three terms and the exponent of the first term is exactly twice that of the second. It is entirely possible that a polynomial has no real roots at all, or worse, it has real roots but none of the techniques discussed in this section can help us find them exactly. In the latter case, we are forced to approximate, which in this subsection means we use the 'Zero' command on the graphing calculator.

4.3.2 For Those Wishing NOT to use Technology

Suppose we wish to find the zeros of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ *without* using the calculator. In this subsection, we present some more advanced mathematical tools (theorems) to help us. Our first result is due to René Descartes.

Theorem 28 Descartes' Rule of Signs

Suppose $f(x)$ is the formula for a polynomial function written with descending powers of x.

- If P denotes the number of variations of sign in the formula for $f(x)$, then the number of positive real zeros (counting multiplicity) is one of the numbers $\{P, P-2, P-4, \dots\}$.
- If N denotes the number of variations of sign in the formula for $f(-x)$, then the number of negative real zeros (counting multiplicity) is one of the numbers $\{N, N-2, N-4, \dots\}$.

A few remarks are in order. First, to use Descartes' Rule of Signs, we need to understand what is meant by a '**variation in sign**' of a polynomial function. Consider $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. If we focus on only the *signs* of the coefficients, we start with a $(+)$, followed by another $(+)$, then switch to $(-)$, and stay $(-)$ for the remaining two coefficients. Since the signs of the coefficients switched *once* as we read from left to right, we say that $f(x)$ has *one* variation in sign. When we speak of the variations in sign of a polynomial function f we assume the formula for $f(x)$ is written with descending powers of x, as in Definition 37, and concern ourselves only with the nonzero coefficients. Second, unlike the Rational Zeros Theorem, Descartes' Rule of Signs gives us an estimate to the *number* of positive and negative real zeros, not the actual *value* of the zeros. Lastly, Descartes' Rule of Signs counts multiplicities. This means that, for example, if one of the zeros has multiplicity 2, Descsartes' Rule of Signs would count this as *two* zeros. Lastly, note that the number of positive or negative real zeros always starts with the number of sign changes and decreases by an even number. For example, if $f(x)$ has 7 sign changes, then, counting multplicities, f has either 7, 5, 3 or 1 positive real zero. This implies that the graph of $y = f(x)$ crosses the positive x-axis at least once. If $f(-x)$ results in 4 sign changes, then, counting multiplicities, f has 4, 2 or 0 negative real zeros; hence, the graph of $y = f(x)$ may not cross the negative x-axis at all. The proof of Descartes' Rule of Signs is a bit technical, and can be found here.

Example 75 Using Descartes' Rule of Signs

Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use Descartes' Rule of Signs to determine the possible number and location of the real zeros of f.

Solution As noted above, the variations of sign of $f(x)$ is 1. This means, counting multiplicities, f has exactly 1 positive real zero. Since $f(-x) = 2(-x)^4 + 4(-x)^3 - (-x)^2 - 6(-x) - 3 = 2x^4 - 4x^3 - x^2 + 6x - 3$ has 3 variations in sign, f has either 3 negative real zeros or 1 negative real zero, counting multiplicities.

Example 76 Finding real zeros by hand

Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

1. Find all of the real zeros of f and their multiplicities.

2. Sketch the graph of $y = f(x)$.

Solution

1. We know from Cauchy's Bound that all of the real zeros lie in the interval $[-4, 4]$ and that our possible rational zeros are $\pm\frac{1}{2}$, ± 1, $\pm\frac{3}{2}$ and ± 3. Descartes' Rule of Signs guarantees us at least one negative real zero and exactly one positive real zero, counting multiplicity. We try our positive rational zeros, starting with the smallest, $\frac{1}{2}$. Plugging in $x = \frac{1}{2}$, we get $f\left(\frac{1}{2}\right) = -\frac{45}{8} \neq 0$, so we know that $x = \frac{1}{2}$ is not a zero of f. Trying the next possible rational zero, we find $f(1) = -4 \neq 0$, so $x = 1$ is not a zero of f either. We can similarly check that $f\left(\frac{3}{2}\right) = \frac{75}{8} \neq 0$ and $f(3) = 240$, so that none of the possible rational zeros are, in fact zeros. Descartes' Rule of Signs guaranteed us a positive real zero, and at this point we have shown this zero is irrational. Furthermore, the Intermediate Value Theorem, Theorem 19, tells us the zero lies between 1 and $\frac{3}{2}$, since $f(1) < 0$ and $f\left(\frac{3}{2}\right) > 0$.

   We now turn our attention to negative real zeros. We try the largest possible zero, $-\frac{1}{2}$. We calculate $f\left(-\frac{1}{2}\right) = -\frac{5}{8}$, so $x = -\frac{1}{2}$ is not a zero of f. Next, we try $f(-1)$, and we find

   $$f(-1) = 2(-1)^4 + 4(-1)^3 - (-1)^2 - 6(-1) - 3 = 2 - 4 - 1 + 6 - 3 = 0,$$

   so $x = -1$ is a zero of f! We can now proceed to find the quotient using long division:

   $$\begin{array}{r|l}
   & 2x^3 + 2x^2 - 3x - 3 \\ \hline
   x+1 & 2x^4 + 4x^3 - x^2 - 6x - 3 \\
   & -\underline{\left(2x^4 + 2x^3\right)} \\
   & \qquad 2x^3 - x^2 \\
   & \qquad -\underline{\left(2x^3 + 2x^2\right)} \\
   & \qquad\qquad -3x^2 - 6x \\
   & \qquad\qquad -\underline{\left(-3x^2 - 3x\right)} \\
   & \qquad\qquad\qquad -3x - 3 \\
   & \qquad\qquad\qquad -\underline{\left(-3x - 3\right)} \\
   & \qquad\qquad\qquad\qquad 0
   \end{array}$$

   Descartes' Rule of Signs told us that we may have up to three negative real zeros, counting multiplicity, so we try -1 again, and it works once more: with $q(x) = 2x^3 + 2x^2 - 3x - 3$ we get $q(-1) = -2 + 2 + 2 - 3 = 0$, so the Factor Theorem tells us that the quotient is still divisible by $x + 1$. Indeed, long division gives us

   $$\begin{array}{r|l}
   & \qquad\qquad 2x^2 - 3 \\ \hline
   x+1 & 2x^3 + 2x^2 - 3x - 3 \\
   & -\underline{\left(2x^3 + 2x^2\right)} \\
   & \qquad 0 - 3x - 3 \\
   & \qquad -\underline{\left(-3x - 3\right)} \\
   & \qquad\qquad 0,
   \end{array}$$

   so the quotient after dividing twice by $x + 1$ is $2x^2 - 3$.

Setting the quotient polynomial equal to zero yields $2x^2 - 3 = 0$, so that $x^2 = \frac{3}{2}$, or $x = \pm\frac{\sqrt{6}}{2}$. Descartes' Rule of Signs tells us that the positive real zero we found, $\frac{\sqrt{6}}{2}$, has multiplicity 1. Descartes also tells us the total multiplicity of negative real zeros is 3, which forces -1 to be a zero of multiplicity 2 and $-\frac{\sqrt{6}}{2}$ to have multiplicity 1.

2. We know the end behaviour of $y = f(x)$ resembles that of its leading term $y = 2x^4$. This means that the graph enters the scene in Quadrant II and exits in Quadrant I. Since $\pm\frac{\sqrt{6}}{2}$ are zeros of odd multiplicity, we have that the graph crosses through the x-axis at the points $\left(-\frac{\sqrt{6}}{2}, 0\right)$ and $\left(\frac{\sqrt{6}}{2}, 0\right)$. Since -1 is a zero of multiplicity 2, the graph of $y = f(x)$ touches and rebounds off the x-axis at $(-1, 0)$. Putting this together, we get the graph in Figure 4.15.

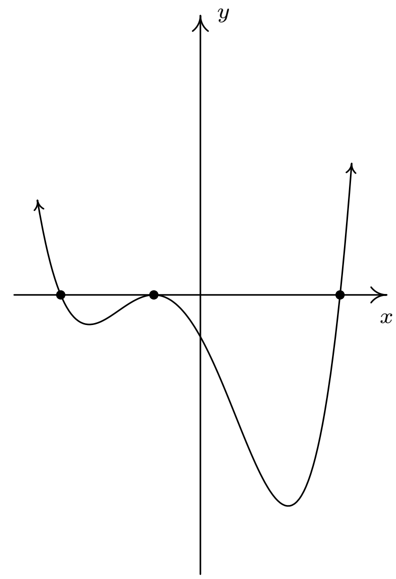


Figure 4.15: The graph $y = 2x^4 + 4x^3 - x^2 - 6x - 3$

You can see why the 'no calculator' approach is not very popular these days. It requires more computation and more theorems than the alternative. (This is apparently a bad thing.) In general, no matter how many theorems you throw at a polynomial, it may well be impossible to find their zeros exactly. The polynomial $f(x) = x^5 - x - 1$ is one such beast. According to Descartes' Rule of Signs, f has exactly one positive real zero, and it could have two negative real zeros, or none at all. The Rational Zeros Test gives us ± 1 as rational zeros to try but neither of these work since $f(1) = f(-1) = -1$. If we try the substitution technique we used in Example 74, we find $f(x)$ has three terms, but the exponent on the x^5 isn't exactly twice the exponent on x. How could we go about approximating the positive zero without resorting to the 'Zero' command of a graphing calculator? We use the **Bisection Method**. The first step in the Bisection Method is to find an interval on which f changes sign. We know $f(1) = -1$ and we find $f(2) = 29$. By the Intermediate Value Theorem, we know that the zero of f lies in the interval $[1, 2]$. Next, we 'bisect' this interval and find the midpoint is 1.5. We have that $f(1.5) \approx 5.09$. This means that our zero is between 1 and 1.5, since f changes sign on this interval. Now, we 'bisect' the interval $[1, 1.5]$ and find $f(1.25) \approx 0.80$, so now we have the zero between 1 and 1.25. Bisecting $[1, 1.25]$, we find $f(1.125) \approx -0.32$, which means the zero of f is between 1.125 and 1.25. We continue in this fashion until we have 'sandwiched' the zero between two numbers which differ by no more than a desired accuracy. You can think of the Bisection Method as reversing the sign diagram process: instead of finding the zeros and checking the sign of f using test values, we are using test values to determine where the signs switch to find the zeros. It is a slow and tedious, yet fool-proof, method for approximating a real zero.

We don't use the word "impossible" lightly; it can be proven that the zeros of some polynomials cannot be expressed using the usual algebraic symbols. See this page, for example.

Our next example reminds us of the role finding zeros plays in solving equations and inequalities.

Example 77 **Solving a polynomial equation and inequality**

1. Find all of the real solutions to the equation $2x^5 + 6x^3 + 3 = 3x^4 + 8x^2$.
2. Solve the inequality $2x^5 + 6x^3 + 3 \leq 3x^4 + 8x^2$.
3. Interpret your answer to part 2 graphically, and verify using a graphing calculator.

SOLUTION

1. Finding the real solutions to $2x^5+6x^3+3=3x^4+8x^2$ is the same as finding the real solutions to $2x^5-3x^4+6x^3-8x^2+3=0$. In other words, we are looking for the real zeros of $p(x)=2x^5-3x^4+6x^3-8x^2+3$. The Rational Roots Theorem tells us that the possible zeros of p include $\pm1,\pm3,\pm\frac{1}{2}$, and $\pm\frac{3}{2}$. Neither Cauchy's Bound nor Descartes' Rule of Signs narrow things down much for us, so we simply test values, starting with the easiest ones to plug in. Our first attempt is $p(1)=2-3+6-8+3=0$, so we get lucky: the Factor Theorem tells us that $x-1$ is a factor. Using long division, we find (exercise) that

$$p(x)=2x^5-3x^4+6x^3-8x^2+3=(x-1)(2x^4-x^3+5x^2-3x-3).$$

The possible rational zeros of the quotient $q_1(x)=2x^4-x^3+5x^2-3x-3$ are the same as before, and again we get lucky testing $x=1$ first: we have $q_1(1)=2-1+5-3-3=0$, so $x=1$ is a zero of multiplicity at least 2. Dividing by $x-1$ again, we have

$$p(x)=2x^5-3x^4+6x^3-8x^2+3=(x-1)^2(2x^3+x^2+6x+3).$$

This time, trying $x=\pm1$ in the quotient $q_2(x)=2x^3+x^2+6x+3$ gets us $q_2(1)=12\neq0$ and $q_2(-1)=-4\neq0$, so neither value is a zero; however, since $q_2(-1)<0$ and $q_2(1)>0$, the Intermediate Value Theorem guarantees us a zero between -1 and 1. Trying possible rational roots first, we find $q_2\left(\frac{1}{2}\right)=\frac{13}{2}\neq0$, but

$$q_2\left(-\frac{1}{2}\right)=2\left(-\frac{1}{8}\right)+\frac{1}{4}+6\left(-\frac{1}{2}\right)+3=-\frac{1}{4}+\frac{1}{4}-3+3=0,$$

so $x=-\frac{1}{2}$ is a root. Dividing once more by $x+\frac{1}{2}$, we find

$$p(x)=(x-1)^2\left(x+\frac{1}{2}\right)(2x^2+6).$$

The quotient polynomial is $2x^2+6$ which has no real zeros so we get $x=-\frac{1}{2}$ and $x=1$ as the zeros of p.

2. To solve this nonlinear inequality, we follow the same guidelines set forth in Section 3.4: we get 0 on one side of the inequality and construct a sign diagram. Our original inequality can be rewritten as $2x^5-3x^4+6x^3-8x^2+3\leq0$. We found the zeros of $p(x)=2x^5-3x^4+6x^3-8x^2+3$ in part 1 to be $x=-\frac{1}{2}$ and $x=1$. We construct our sign diagram as before, giving us Figure 4.16.

   The solution to $p(x)<0$ is $\left(-\infty,-\frac{1}{2}\right)$, and we know $p(x)=0$ at $x=-\frac{1}{2}$ and $x=1$. Hence, the solution to $p(x)\leq0$ is $\left(-\infty,-\frac{1}{2}\right]\cup\{1\}$.

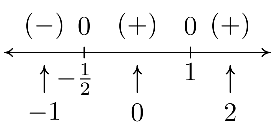
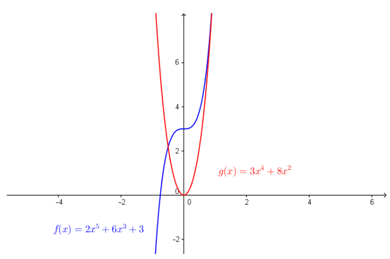
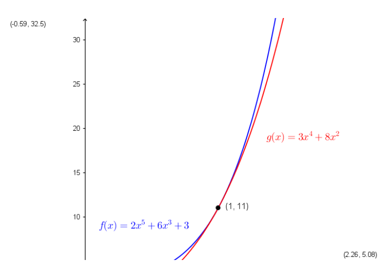


Figure 4.16: The sign diagram for $p(x)$ in Example 77

3. To interpret this solution graphically, we set $f(x)=2x^5+6x^3+3$ and $g(x)=3x^4+8x^2$. We recall that the solution to $f(x)\leq g(x)$ is the set of x values for which the graph of f is below the graph of g (where $f(x)<g(x)$) along with the x values where the two graphs intersect ($f(x)=g(x)$). Graphing f and g using GeoGebra produces Figure 4.17(a). (The end behaviour should tell you which is which.) We see that the graph of f is below the graph of g on $\left(-\infty,-\frac{1}{2}\right)$. However, it is difficult to see what is happening near $x=1$. Zooming in (and making the graph of g thicker), we see in Figure 4.17(b) that the graphs of f and g do intersect at $x=1$, but the graph of g remains below the graph of f on either side of $x=1$.



(a)



(b)

Figure 4.17: The polynomials $f(x)$ and $g(x)$ from Example 77, part 3

Our last example revisits an application from page 162 in the Exercises of Section 4.1.

Example 78 **Calculating sales profits**

Suppose the profit P, in *thousands* of dollars, from producing and selling x *hundred* LCD TVs is given by $P(x) = -5x^3 + 35x^2 - 45x - 25$, $0 \leq x \leq 10.07$. How many TVs should be produced to make a profit? Check your answer using a graphing utility.

SOLUTION To 'make a profit' means to solve $P(x) = -5x^3+35x^2-45x-25 > 0$, which we do analytically using a sign diagram. To simplify things, we first factor out the -5 common to all the coefficients to get $-5\left(x^3 - 7x^2 + 9x - 5\right) > 0$, so we can just focus on finding the zeros of $f(x) = x^3 - 7x^2 + 9x + 5$. The possible rational zeros of f are ± 1 and ± 5, and going through the usual computations, we find $x = 5$ is the only rational zero. Using this, we factor $f(x) = x^3 - 7x^2 + 9x + 5 = (x-5)\left(x^2 - 2x - 1\right)$, and we find the remaining zeros by applying the Quadratic Formula to $x^2 - 2x - 1 = 0$. We find three real zeros, $x = 1 - \sqrt{2} = -0.414\ldots$, $x = 1 + \sqrt{2} = 2.414\ldots$, and $x = 5$, of which only the last two fall in the applied domain of $[0, 10.07]$. We choose $x = 0$, $x = 3$ and $x = 10.07$ as our test values and plug them into the function $P(x) = -5x^3 + 35x^2 - 45x - 25$ (not $f(x) = x^3 - 7x^2 + 9x - 5$) to get the sign diagram in Figure 4.18.

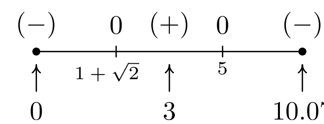


Figure 4.18: The sign diagram for $P(x)$ in Example 78

We see immediately that $P(x) > 0$ on $(1+\sqrt{2}, 5)$. Since x measures the number of TVs in *hundreds*, $x = 1 + \sqrt{2}$ corresponds to $241.4\ldots$ TVs. Since we can't produce a fractional part of a TV, we need to choose between producing 241 and 242 TVs. From the sign diagram, we see that $P(2.41) < 0$ but $P(2.42) > 0$ so, in this case we take the next *larger* integer value and set the minimum production to 242 TVs. At the other end of the interval, we have $x = 5$ which corresponds to 500 TVs. Here, we take the next *smaller* integer value, 499 TVs to ensure that we make a profit. Hence, in order to make a profit, at least 242, but no more than 499 TVs need to be produced. To check our answer using GeoGebra, we graph $y = P(x)$ and use the Intersect tool to see where $y = P(x)$ intersects the x-axis. We see in Figure 4.19 that the software approximations bear out our analysis.

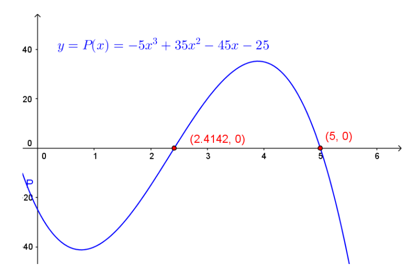


Figure 4.19: Plotting the profit function $P(x)$ in Example 78

Exercises 4.3

Problems

In Exercises 1 – 11, for the given polynomial:

- **Use Cauchy's Bound to find an interval containing all of the real zeros.**
- **Use the Rational Zeros Theorem to make a list of possible rational zeros.**
- **Use Descartes' Rule of Signs to list the possible number of positive and negative real zeros, counting multiplicities.**

1. $f(x) = x^3 - 2x^2 - 5x + 6$
2. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
3. $f(x) = x^4 - 9x^2 - 4x + 12$
4. $f(x) = x^3 + 4x^2 - 11x + 6$
5. $f(x) = x^3 - 7x^2 + x - 7$
6. $f(x) = -2x^3 + 19x^2 - 49x + 20$
7. $f(x) = -17x^3 + 5x^2 + 34x - 10$
8. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
9. $f(x) = 3x^3 + 3x^2 - 11x - 10$
10. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

In Exercises 11 – 31, find the real zeros of the polynomial using the techniques specified by your instructor. State the multiplicity of each real zero.

11. $f(x) = x^3 - 2x^2 - 5x + 6$
12. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
13. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
14. $f(x) = x^3 + 4x^2 - 11x + 6$
15. $f(x) = x^3 - 7x^2 + x - 7$
16. $f(x) = -2x^3 + 19x^2 - 49x + 20$
17. $f(x) = -17x^3 + 5x^2 + 34x - 10$
18. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
19. $f(x) = 3x^3 + 3x^2 - 11x - 10$
20. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
21. $f(x) = 9x^3 - 5x^2 - x$
22. $f(x) = 6x^4 - 5x^3 - 9x^2$
23. $f(x) = x^4 + 2x^2 - 15$
24. $f(x) = x^4 - 9x^2 + 14$
25. $f(x) = 3x^4 - 14x^2 - 5$
26. $f(x) = 2x^4 - 7x^2 + 6$
27. $f(x) = x^6 - 3x^3 - 10$
28. $f(x) = 2x^6 - 9x^3 + 10$
29. $f(x) = x^5 - 2x^4 - 4x + 8$
30. $f(x) = 2x^5 + 3x^4 - 18x - 27$

In Exercises 31 – 34, use software or a graphing calculator[3] to help you find the real zeros of the polynomial. State the multiplicity of each real zero.

31. $f(x) = x^5 - 60x^3 - 80x^2 + 960x + 2304$
32. $f(x) = 25x^5 - 105x^4 + 174x^3 - 142x^2 + 57x - 9$
33. $f(x) = 90x^4 - 399x^3 + 622x^2 - 399x + 90$
34. Find the real zeros of $f(x) = x^3 - \frac{1}{12}x^2 - \frac{7}{72}x + \frac{1}{72}$ by first finding a polynomial $q(x)$ with integer coefficients such that $q(x) = N \cdot f(x)$ for some integer N. (Recall that the Rational Zeros Theorem required the polynomial in question to have integer coefficients.) Show that f and q have the same real zeros.

In Exercises 35 – 45, find the real solutions of the polynomial equation. (See Example 77.)

35. $9x^3 = 5x^2 + x$
36. $9x^2 + 5x^3 = 6x^4$
37. $x^3 + 6 = 2x^2 + 5x$
38. $x^4 + 2x^3 = 12x^2 + 40x + 32$
39. $x^3 - 7x^2 = 7 - x$
40. $2x^3 = 19x^2 - 49x + 20$
41. $x^3 + x^2 = \dfrac{11x + 10}{3}$
42. $x^4 + 2x^2 = 15$

[3] You *can* do theseby hand, but it may test your mettle!

43. $14x^2 + 5 = 3x^4$

44. $2x^5 + 3x^4 = 18x + 27$

In Exercises 45 – 55, solve the polynomial inequality and state your answer using interval notation.

45. $-2x^3 + 19x^2 - 49x + 20 > 0$

46. $x^4 - 9x^2 \leq 4x - 12$

47. $(x-1)^2 \geq 4$

48. $4x^3 \geq 3x + 1$

49. $x^4 \leq 16 + 4x - x^3$

50. $3x^2 + 2x < x^4$

51. $\dfrac{x^3 + 2x^2}{2} < x + 2$

52. $\dfrac{x^3 + 20x}{8} \geq x^2 + 2$

53. $2x^4 > 5x^2 + 3$

54. $2x^4 > 5x^2 + 3$

55. In Example **??** in Section 4.1, a box with no top is constructed from a 10 inch $\times$ 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. We determined the volume of that box (in cubic inches) is given by $V(x) = 4x^3 - 44x^2 + 120x$, where x denotes the length of the side of the square which is removed from each corner (in inches), $0 < x < 5$. Solve the inequality $V(x) \geq 80$ analytically and interpret your answer in the context of that example.

56. From Exercise 32 in Section 4.1, $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \geq 0$ models the cost, in dollars, to produce x PortaBoy game systems. If the production budget is $\$5000$, find the number of game systems which can be produced and still remain under budget.

57. Let $f(x) = 5x^7 - 33x^6 + 3x^5 - 71x^4 - 597x^3 + 2097x^2 - 1971x + 567$. With the help of your classmates, find the x- and y- intercepts of the graph of f. Find the intervals on which the function is increasing, the intervals on which it is decreasing and the local extrema. Sketch the graph of f, using more than one picture if necessary to show all of the important features of the graph.

58. With the help of your classmates, create a list of five polynomials with different degrees whose real zeros cannot be found using any of the techniques in this section.

4.4 Complex Zeros of Polynomials

In Section 4.3, we were focused on finding the real zeros of a polynomial function. In this section, we expand our horizons and look for the non-real zeros as well. Consider the polynomial $p(x) = x^2 + 1$. The zeros of p are the solutions to $x^2 + 1 = 0$, or $x^2 = -1$. This equation has no real solutions, but you may recall Section 1.4 that we can formally extract the square roots of both sides to get $x = \pm\sqrt{-1}$. You may want to review the basics of complex numbers in Section 1.4 before proceeding.

Suppose we wish to find the zeros of $f(x) = x^2 - 2x + 5$. To solve the equation $x^2 - 2x + 5 = 0$, we note that the quadratic doesn't factor nicely, so we resort to the Quadratic Formula, Equation 16 and obtain

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Two things are important to note. First, the zeros $1 + 2i$ and $1 - 2i$ are complex conjugates. If ever we obtain non-real zeros to a quadratic function with real coefficients, the zeros will be a complex conjugate pair. (Do you see why?) Next, we note that in Example 12, part 6, we found $(x - [1 + 2i])(x - [1 - 2i]) = x^2 - 2x + 5$. This demonstrates that the factor theorem holds even for non-real zeros, i.e, $x = 1 + 2i$ is a zero of f, and, sure enough, $(x - [1 + 2i])$ is a factor of $f(x)$. It turns out that polynomial division works the same way for all complex numbers, real and non-real alike, so the Factor and Remainder Theorems hold as well. But how do we know if a general polynomial has any complex zeros at all? We have many examples of polynomials with no real zeros. Can there be polynomials with no zeros whatsoever? The answer to that last question is "No." and the theorem which provides that answer is The Fundamental Theorem of Algebra.

Theorem 29 The Fundamental Theorem of Algebra

Suppose f is a polynomial function with complex number coefficients of degree $n \geq 1$, then f has at least one complex zero.

The Fundamental Theorem of Algebra is an example of an 'existence' theorem in Mathematics. Like the Intermediate Value Theorem, Theorem 19, the Fundamental Theorem of Algebra guarantees the existence of at least one zero, but gives us no algorithm to use in finding it. In fact, as we mentioned in Section 4.3, there are polynomials whose real zeros, though they exist, cannot be expressed using the 'usual' combinations of arithmetic symbols, and must be approximated. The authors are fully aware that the full impact and profound nature of the Fundamental Theorem of Algebra is lost on most students studying College Algebra, and that's fine. It took mathematicians literally hundreds of years to prove the theorem in its full generality, and some of that history is recorded in this Wikipedia article. Note that the Fundamental Theorem of Algebra applies to not only polynomial functions with real coefficients, but to those with complex number coefficients as well.

The Fundamental Theorem of Algebra has since been proved many times, using many different methods, by many mathematicians. There are probably very few, if any, results in mathematics with the variety of proofs this result has. Unfortunately, none of the proofs can be understood within the realm of this text, but if the reader is sufficiently interested, a collection of proofs can be found at this website.

Suppose f is a polynomial of degree $n \geq 1$. The Fundamental Theorem of Algebra guarantees us at least one complex zero, z_1, and as such, the Factor Theorem guarantees that $f(x)$ factors as $f(x) = (x - z_1)\, q_1(x)$ for a polynomial function q_1, of degree exactly $n - 1$. If $n - 1 \geq 1$, then the Fundamental Theorem of Algebra guarantees a complex zero of q_1 as well, say z_2, so then the Factor Theorem gives us $q_1(x) = (x - z_2)\, q_2(x)$, and hence $f(x) = (x - z_1)(x - z_2)\, q_2(x)$. We can continue this process exactly n times, at which point our quotient polynomial q_n has degree 0 so it's a constant. This argument gives us the following factorization theorem.

Theorem 30 Complex Factorization Theorem

Suppose f is a polynomial function with complex number coefficients. If the degree of f is n and $n \geq 1$, then f has exactly n complex zeros, counting multiplicity. If z_1, z_2, ..., z_k are the distinct zeros of f, with multiplicities m_1, m_2, ..., m_k, respectively, then $f(x) = a\,(x - z_1)^{m_1}\,(x - z_2)^{m_2} \cdots (x - z_k)^{m_k}$.

Note that the value a in Theorem 30 is the leading coefficient of $f(x)$ (Can you see why?) and as such, we see that a polynomial is completely determined by its zeros, their multiplicities, and its leading coefficient. We put this theorem to good use in the next example.

Example 79 Factoring using complex numbers

Let $f(x) = 12x^5 - 20x^4 + 19x^3 - 6x^2 - 2x + 1$.

1. Find all of the complex zeros of f and state their multiplicities.
2. Factor $f(x)$ using Theorem 30

Solution

1. Since f is a fifth degree polynomial, we know that we need to perform at least three successful divisions to get the quotient down to a quadratic function. At that point, we can find the remaining zeros using the Quadratic Formula, if necessary. We first look for (rational) real zeros. The Rational Zeros Theorem tells us that our candidates in this case are ± 1, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$, $\pm\frac{1}{4}$, $\pm\frac{1}{6}$, and $\pm\frac{1}{12}$. We find $f(1) = 4$ and $f(-1) = -54$, but

   $$f\left(\frac{1}{2}\right) = \frac{12}{32} - \frac{20}{16} + \frac{19}{8} - \frac{6}{4} - \frac{2}{2} + 1 = \frac{3}{8} - \frac{10}{8} + \frac{19}{8} - \frac{12}{8} - 1 + 1 = 0,$$

   so $x = \frac{1}{2}$ is a zero. In fact, $x = \frac{1}{2}$ is a zero of multiplicity 2: dividing $f(x)$ by $\left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{4}$, we have:

   $$\begin{array}{r|l}
   & \quad\quad\quad\quad\quad\quad 12x^3 - 4x^2 + 8x + 4 \\
   \hline
   x^2 - x + \frac{1}{4} & 12x^5 - 20x^4 + 19x^3 - 6x^2 - 2x + 1 \\
   & -\left(12x^5 - 12x^4 + 3x^3\right) \\
   & \quad\quad\quad 8x^3 - 4x^2 - 2x \\
   & \quad\quad -\left(8x^3 - 8x^2 + 2x\right) \\
   & \quad\quad\quad\quad 4x^2 - 4x + 1 \\
   & \quad\quad\quad -\left(4x^2 - 4x + 1\right) \\
   & \quad\quad\quad\quad\quad\quad\quad 0
   \end{array}$$

   Thus, we can conclude that $f(x) = \left(x - \frac{1}{2}\right)^2 \left(12x^3 - 8x^2 + 8x + 4\right)$. Checking the remaining candidates for rational zeros, we find that $x = -\frac{1}{3}$ works as

We don't really recommend jumping straight to division by $\left(x - \frac{1}{2}\right)^2$ to see if $x = \frac{1}{2}$ has multiplicity 2. In the back rooms of the Mathematics Department, we secretly divided first by $x - \frac{1}{2}$ and then checked that $x = \frac{1}{2}$ was still a divisor of the resulting quotient. We're presenting things this way since it lets us write out one long division instead of two (and since it's always good to get practice dividing by higher-degree polynomials).

well: plugging this value into the quotient gives us

$$-\frac{12}{27}-\frac{8}{9}-\frac{8}{3}+4=-\frac{4}{9}-\frac{8}{9}-\frac{24}{9}+\frac{36}{9}=0.$$

Dividing by $12x^3-8x^2+8x+4$ by $x+\frac{1}{3}$ leaves us with the quotient $12x^2-12x+12$ (exercise), so we've factored $f(x)$ into the form

$$f(x)=\left(x-\frac{1}{2}\right)\left(x+\frac{1}{3}\right)(12x^2-12x+12).$$

Our quotient is $12x^2-12x+12$, whose zeros we find to be $\frac{1\pm i\sqrt{3}}{2}$. From Theorem 30, we know f has exactly 5 zeros, counting multiplicities, and as such we have the zero $\frac{1}{2}$ with multiplicity 2, and the zeros $-\frac{1}{3}$, $\frac{1+i\sqrt{3}}{2}$ and $\frac{1-i\sqrt{3}}{2}$, each of multiplicity 1.

2. Applying Theorem 30, we are guaranteed that f factors as

$$f(x)=12\left(x-\frac{1}{2}\right)^2\left(x+\frac{1}{3}\right)\left(x-\left[\frac{1+i\sqrt{3}}{2}\right]\right)\left(x-\left[\frac{1-i\sqrt{3}}{2}\right]\right)$$

A true test of Theorem 30 (and a student's mettle!) would be to take the factored form of $f(x)$ in the previous example and multiply it out to see that it really does reduce to the original formula $f(x)=12x^5-20x^4+19x^3-6x^2-2x+1$. (You really should do this once in your life to convince yourself that all of the theory actually does work!) When factoring a polynomial using Theorem 30, we say that it is **factored completely over the complex numbers**, meaning that it is impossible to factor the polynomial any further using complex numbers. If we wanted to completely factor $f(x)$ over the **real numbers** then we would have stopped short of finding the nonreal zeros of f and factored f using our work from the synthetic division to write $f(x)=\left(x-\frac{1}{2}\right)^2\left(x+\frac{1}{3}\right)\left(12x^2-12x+12\right)$, or $f(x)=12\left(x-\frac{1}{2}\right)^2\left(x+\frac{1}{3}\right)\left(x^2-x+1\right)$. Since the zeros of x^2-x+1 are nonreal, we call x^2-x+1 an **irreducible quadratic** meaning it is impossible to break it down any further using *real* numbers.

The last two results of the section show us that, at least in theory, if we have a polynomial function with real coefficients, we can always factor it down enough so that any nonreal zeros come from irreducible quadratics.

Theorem 31 Conjugate Pairs Theorem

If f is a polynomial function with real number coefficients and z is a zero of f, then so is $\bar{z}$.

To prove the theorem, suppose f is a polynomial with real number coefficients. Specifically, let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$. If z is a zero of f, then $f(z)=0$, which means $a_nz^n+a_{n-1}z^{n-1}+\ldots+a_2z^2+a_1z+a_0=0$. Next, we consider $f(\bar{z})$ and apply Theorem 4 below.

$$\begin{aligned} f(\bar{z}) &= a_n(\bar{z})^n + a_{n-1}(\bar{z})^{n-1} + \ldots + a_2(\bar{z})^2 + a_1\bar{z} + a_0 \\ &= a_n\overline{z^n} + a_{n-1}\overline{z^{n-1}} + \ldots + a_2\overline{z^2} + a_1\bar{z} + a_0 && \text{since } (\bar{z})^n = \overline{z^n} \\ &= \overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \ldots + \overline{a_2}\,\overline{z^2} + \overline{a_1}\,\bar{z} + \overline{a_0} && \text{since the coefficients are real} \\ &= \overline{a_nz^n} + \overline{a_{n-1}z^{n-1}} + \ldots + \overline{a_2z^2} + \overline{a_1z} + \overline{a_0} && \text{since } \bar{z}\,\bar{w} = \overline{zw} \\ &= \overline{a_nz^n + a_{n-1}z^{n-1} + \ldots + a_2z^2 + a_1z + a_0} && \text{since } \bar{z} + \bar{w} = \overline{z+w} \\ &= \overline{f(z)} \\ &= \overline{0} \\ &= 0 \end{aligned}$$

This shows that $\bar{z}$ is a zero of f. So, if f is a polynomial function with real number coefficients, Theorem 31 tells us that if $a + bi$ is a nonreal zero of f, then so is $a - bi$. In other words, nonreal zeros of f come in conjugate pairs. The Factor Theorem kicks in to give us both $(x - [a + bi])$ and $(x - [a - bi])$ as factors of $f(x)$ which means $(x - [a + bi])(x - [a - bi]) = x^2 + 2ax + \left(a^2 + b^2\right)$ is an irreducible quadratic factor of f. As a result, we have our last theorem of the section.

Theorem 32 Real Factorization Theorem

Suppose f is a polynomial function with real number coefficients. Then $f(x)$ can be factored into a product of linear factors corresponding to the real zeros of f and irreducible quadratic factors which give the nonreal zeros of f.

We now present an example which pulls together all of the major ideas of this section.

Example 80 Factoring over the complex numbers

Let $f(x) = x^4 + 64$.

1. Verify that $x = 2 + 2i$ is a zero of f (a) directly, and (b) using long division.
2. Find the remaining complex zeros of f.
3. Completely factor $f(x)$ over the complex numbers.
4. Completely factor $f(x)$ over the real numbers.

Solution

1. We divide $f(x)$ by $x - (2 + 2i)$ using long division as follows:
2. Since f is a fourth degree polynomial, we need to make two successful divisions to get a quadratic quotient. Since $2 + 2i$ is a zero, we know from Theorem 31 that $2 - 2i$ is also a zero. Using long division, we divide the above quotient $x^3 + (2 + i)x^2 + 8ix + (-16 + 16i)$ by $x - (2 - i)$ as follows:

   Our quotient polynomial is $x^2 + 4x + 8$. Using the quadratic formula, we obtain the remaining zeros $-2 + 2i$ and $-2 - 2i$.
3. Using Theorem 30, we get $f(x) = (x - [2 - 2i])(x - [2 + 2i])(x - [-2 + 2i])(x - [-2 - 2i])$.

4. We multiply the linear factors of $f(x)$ which correspond to complex conjugate pairs. We find $(x - [2 - 2i])(x - [2 + 2i]) = x^2 - 4x + 8$, and $(x - [-2 + 2i])(x - [-2 - 2i]) = x^2 + 4x + 8$. Our final answer is $f(x) = \left(x^2 - 4x + 8\right)\left(x^2 + 4x + 8\right)$.

Our last example turns the tables and asks us to manufacture a polynomial with certain properties of its graph and zeros.

Example 81 Constructing a polynomial

Find a polynomial p of lowest degree that has integer coefficients and satisfies all of the following criteria:

- the graph of $y = p(x)$ touches (but doesn't cross) the x-axis at $\left(\frac{1}{3}, 0\right)$
- $x = 3i$ is a zero of p.
- as $x \to -\infty$, $p(x) \to -\infty$
- as $x \to \infty$, $p(x) \to -\infty$

Solution To solve this problem, we will need a good understanding of the relationship between the x-intercepts of the graph of a function and the zeros of a function, the Factor Theorem, the role of multiplicity, complex conjugates, the Complex Factorization Theorem, and end behaviour of polynomial functions. (In short, you'll need most of the major concepts of this chapter.) Since the graph of p touches the x-axis at $\left(\frac{1}{3}, 0\right)$, we know $x = \frac{1}{3}$ is a zero of even multiplicity. Since we are after a polynomial of lowest degree, we need $x = \frac{1}{3}$ to have multiplicity exactly 2. The Factor Theorem now tells us $\left(x - \frac{1}{3}\right)^2$ is a factor of $p(x)$. Since $x = 3i$ is a zero and our final answer is to have integer (real) coefficients, $x = -3i$ is also a zero. The Factor Theorem kicks in again to give us $(x - 3i)$ and $(x + 3i)$ as factors of $p(x)$. We are given no further information about zeros or intercepts so we conclude, by the Complex Factorization Theorem that $p(x) = a\left(x - \frac{1}{3}\right)^2 (x-3i)(x+3i)$ for some real number a. Expanding this, we get $p(x) = ax^4 - \frac{2a}{3}x^3 + \frac{82a}{9}x^2 - 6ax + a$. In order to obtain integer coefficients, we know a must be an integer multiple of 9. Our last concern is end behavior. Since the leading term of $p(x)$ is ax^4, we need $a < 0$ to get $p(x) \to -\infty$ as $x \to \pm\infty$. Hence, if we choose $x = -9$, we get $p(x) = -9x^4 + 6x^3 - 82x^2 + 54x - 9$. We can verify our handiwork using the techniques developed in this chapter.

This example concludes our study of polynomial functions. (With the exception of the Exercises on the next page, of course.) The last few sections have contained what is considered by many to be 'heavy' Mathematics. Like a heavy meal, heavy Mathematics takes time to digest. Don't be overly concerned if it doesn't seem to sink in all at once, and pace yourself in the Exercises or you're liable to get mental cramps. But before we get to the Exercises, we'd like to offer a bit of an epilogue.

Our main goal in presenting the material on the complex zeros of a polynomial was to give the chapter a sense of completeness. Given that it can be shown that some polynomials have real zeros which cannot be expressed using the usual algebraic operations, and still others have no real zeros at all, it was nice to discover that every polynomial of degree $n \geq 1$ has n complex zeros. So like we said, it gives us a sense of closure. But the observant reader will note that we did not give any examples of applications which involve complex numbers. Students often wonder when complex numbers will be used in 'real-world' applications. After all, didn't we call i the imaginary unit? How can imaginary things be used in reality? It turns out that complex numbers are very useful in many applied fields such as fluid dynamics, electromagnetism and quantum mechanics, but most of the applications require Mathematics well beyond College Algebra to fully understand them. That does not mean you'll never be be able to understand them; in fact, it is the authors' sincere hope that all of you will reach a point in your studies when the glory, awe and splendour of complex numbers are revealed to you. For now, however, the really good stuff is beyond the scope of this text. We invite you and your classmates to find a few examples of complex number applications and see what you can make of them. A simple Internet search with the phrase 'complex numbers in real life' should get you started. Basic electronics classes are another place to look, but remember, they might use the letter j where we have used i.

For the remainder of the text, with the exception of Section **??** and a few exploratory exercises scattered about, we will restrict our attention to real numbers. We do this primarily because the first Calculus sequence you will take, ostensibly the one that this text is preparing you for, studies only functions of real variables. Also, lots of really cool scientific things don't require any deep understanding of complex numbers to study them, but they do need more Mathematics like exponential, logarithmic and trigonometric functions. We believe it makes more sense pedagogically for you to learn about those functions now then take a course in Complex Function Theory in your junior or senior year once you've completed the Calculus sequence. It is in that course that the true power of the complex numbers is released. But for now, in order to fully prepare you for life immediately after College Algebra, we will say that functions like $f(x) = \frac{1}{x^2+1}$ have a domain of all real numbers, even though we know $x^2 + 1 = 0$ has two complex solutions, namely $x = \pm i$. Because $x^2 + 1 > 0$ for all *real* numbers x, the fraction $\frac{1}{x^2+1}$ is never undefined in the real variable setting.

Exercises 4.4

Problems

In Exercises 1 – 23, find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.

1. $f(x) = x^2 - 4x + 13$
2. $f(x) = x^2 - 2x + 5$
3. $f(x) = 3x^2 + 2x + 10$
4. $f(x) = x^3 - 2x^2 + 9x - 18$
5. $f(x) = x^3 + 6x^2 + 6x + 5$
6. $f(x) = 3x^3 - 13x^2 + 43x - 13$
7. $f(x) = x^3 + 3x^2 + 4x + 12$
8. $f(x) = 4x^3 - 6x^2 - 8x + 15$
9. $f(x) = x^3 + 7x^2 + 9x - 2$
10. $f(x) = 9x^3 + 2x + 1$
11. $f(x) = 4x^4 - 4x^3 + 13x^2 - 12x + 3$
12. $f(x) = 2x^4 - 7x^3 + 14x^2 - 15x + 6$
13. $f(x) = x^4 + x^3 + 7x^2 + 9x - 18$
14. $f(x) = 6x^4 + 17x^3 - 55x^2 + 16x + 12$
15. $f(x) = -3x^4 - 8x^3 - 12x^2 - 12x - 5$
16. $f(x) = 8x^4 + 50x^3 + 43x^2 + 2x - 4$
17. $f(x) = x^4 + 9x^2 + 20$
18. $f(x) = x^4 + 5x^2 - 24$
19. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12$
20. $f(x) = x^6 - 64$
21. $f(x) = x^4 - 2x^3 + 27x^2 - 2x + 26$ (Hint: $x = i$ is one of the zeros.)
22. $f(x) = 2x^4 + 5x^3 + 13x^2 + 7x + 5$ (Hint: $x = -1 + 2i$ is a zero.)

In Exercises 23 – 28, create a polynomial *f* with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.

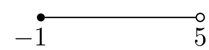
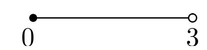
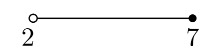
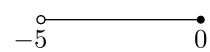
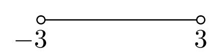
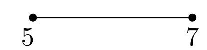
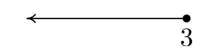
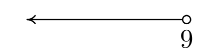
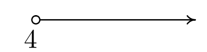
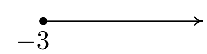
23.
   - The zeros of f are $c = \pm 1$ and $c = \pm i$
   - The leading term of $f(x)$ is $42x^4$
24.
   - $c = 2i$ is a zero.
   - the point $(-1, 0)$ is a local minimum on the graph of $y = f(x)$
   - the leading term of $f(x)$ is $117x^4$
25.
   - The solutions to $f(x) = 0$ are $x = \pm 2$ and $x = \pm 7i$
   - The leading term of $f(x)$ is $-3x^5$
   - The point $(2, 0)$ is a local maximum on the graph of $y = f(x)$.
26.
   - f is degree 5.
   - $x = 6$, $x = i$ and $x = 1 - 3i$ are zeros of f
   - as $x \to -\infty$, $f(x) \to \infty$
27.
   - The leading term of $f(x)$ is $-2x^3$
   - $c = 2i$ is a zero
   - $f(0) = -16$
28. Let z and w be arbitrary complex numbers. Show that $\overline{z}\,\overline{w} = \overline{zw}$ and $\overline{\overline{z}} = z$.

A: Answers To Selected Problems

Chapter 1

Section 1.1

1.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \le x < 5\}$	$[-1, 5)$	
$\{x \mid 0 \le x < 3\}$	$[0, 3)$	
$\{x \mid 2 < x \le 7\}$	$(2, 7]$	
$\{x \mid -5 < x \le 0\}$	$(-5, 0]$	
$\{x \mid -3 < x < 3\}$	$(-3, 3)$	
$\{x \mid 5 \le x \le 7\}$	$[5, 7]$	
$\{x \mid x \le 3\}$	$(-\infty, 3]$	
$\{x \mid x < 9\}$	$(-\infty, 9)$	
$\{x \mid x > 4\}$	$(4, \infty)$	
$\{x \mid x \ge -3\}$	$[-3, \infty)$	

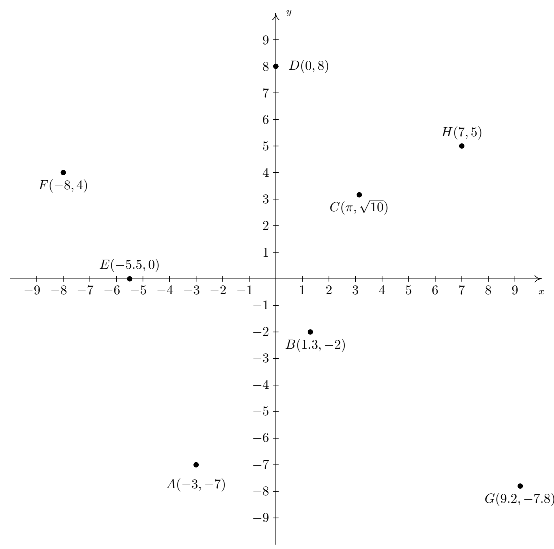
3. $(-1, 1) \cup [0, 6] = (-1, 6]$
5. $(-\infty, 0) \cap [1, 5] = \emptyset$
7. $(-\infty, 5] \cap [5, 8) = \{5\}$
9. $(-\infty, -1) \cup (-1, \infty)$
11. $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$
13. $(-\infty, -4) \cup (-4, 0) \cup (0, 4) \cup (4, \infty)$
15. $(-\infty, \infty)$
17. $(-\infty, 5] \cup \{6\}$
19. $(-3, 3) \cup \{4\}$

Section 1.2

1. 6
3. $\dfrac{2}{21}$
5. $-\dfrac{1}{3}$
7. $\dfrac{3}{5}$
9. $-\dfrac{7}{8}$
11. 0
13. $\dfrac{23}{9}$
15. $-\dfrac{24}{7}$
17. $\dfrac{243}{32}$
19. $\dfrac{9}{22}$
21. 5
23. $\dfrac{107}{27}$
25. $\sqrt{10}$
27. $\sqrt{7}$
29. -1
31. $\dfrac{15}{16}$
33. $-\dfrac{385}{12}$

Section 1.3

1. The required points $A(-3, -7)$, $B(1.3, -2)$, $C(\pi, \sqrt{10})$, $D(0, 8)$, $E(-5.5, 0)$, $F(-8, 4)$, $G(9.2, -7.8)$, and $H(7, 5)$ are plotted in the Cartesian Coordinate Plane below.



3. $d = 5$, $M = \left(-1, \frac{7}{2}\right)$
5. $d = \sqrt{26}$, $M = \left(1, \frac{3}{2}\right)$
7. $d = \sqrt{74}$, $M = \left(\frac{13}{10}, -\frac{13}{10}\right)$
9. $d = \sqrt{83}$, $M = \left(4\sqrt{5}, \frac{5\sqrt{3}}{2}\right)$
11. $(3 + \sqrt{7}, -1)$, $(3 - \sqrt{7}, -1)$
13. $(-1 + \sqrt{3}, 0)$, $(-1 - \sqrt{3}, 0)$
15. $(-3, -4)$, 5 miles, $(4, -4)$
17.
19.
21.

Section 1.4

1. For $z = 2 + 3i$ and $w = 4i$
   - $z + w = 2 + 7i$
   - $zw = -12 + 8i$
   - $z^2 = -5 + 12i$

- $\frac{1}{z} = \frac{2}{13} - \frac{3}{13}\,i$
- $\frac{z}{w} = \frac{3}{4} - \frac{1}{2}\,i$
- $\frac{w}{z} = \frac{12}{13} + \frac{8}{13}\,i$
- $\bar{z} = 2 - 3i$
- $z\bar{z} = 13$
- $(\bar{z})^2 = -5 - 12i$

3. For $z = i$ and $w = -1 + 2i$
   - $z + w = -1 + 3i$
   - $zw = -2 - i$
   - $z^2 = -1$
   - $\frac{1}{z} = -i$
   - $\frac{z}{w} = \frac{2}{5} - \frac{1}{5}\,i$
   - $\frac{w}{z} = 2 + i$
   - $\bar{z} = -i$
   - $z\bar{z} = 1$
   - $(\bar{z})^2 = -1$

5. For $z = 3 - 5i$ and $w = 2 + 7i$
   - $z + w = 5 + 2i$
   - $zw = 41 + 11i$
   - $z^2 = -16 - 30i$
   - $\frac{1}{z} = \frac{3}{34} + \frac{5}{34}\,i$
   - $\frac{z}{w} = -\frac{29}{53} - \frac{31}{53}\,i$
   - $\frac{w}{z} = -\frac{29}{34} + \frac{31}{34}\,i$
   - $\bar{z} = 3 + 5i$
   - $z\bar{z} = 34$
   - $(\bar{z})^2 = -16 + 30i$

7. For $z = \sqrt{2} - i\sqrt{2}$ and $w = \sqrt{2} + i\sqrt{2}$
   - $z + w = 2\sqrt{2}$
   - $zw = 4$
   - $z^2 = -4i$
   - $\frac{1}{z} = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}\,i$
   - $\frac{z}{w} = -i$
   - $\frac{w}{z} = i$
   - $\bar{z} = \sqrt{2} + i\sqrt{2}$
   - $z\bar{z} = 4$
   - $(\bar{z})^2 = 4i$

9. For $z = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i$ and $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$
   - $z + w = i\sqrt{3}$
   - $zw = -1$
   - $z^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$
   - $\frac{1}{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i$
   - $\frac{z}{w} = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i$
   - $\frac{w}{z} = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i$
   - $\bar{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}\,i$
   - $z\bar{z} = 1$
   - $(\bar{z})^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i$

11. $7i$

13. -10

15. -12

17. 3

19. $i^5 = i^4 \cdot i = 1 \cdot i = i$

21. $i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$

23. $i^{15} = \left(i^4\right)^3 \cdot i^3 = 1 \cdot (-i) = -i$

25. $i^{117} = \left(i^4\right)^{29} \cdot i = 1 \cdot i = i$

27. $x = \dfrac{2 \pm i\sqrt{14}}{3}$

29. $y = \pm 2, \pm i$

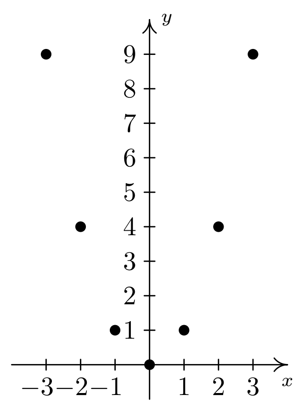
31. $y = \pm\dfrac{3i\sqrt{2}}{2}$

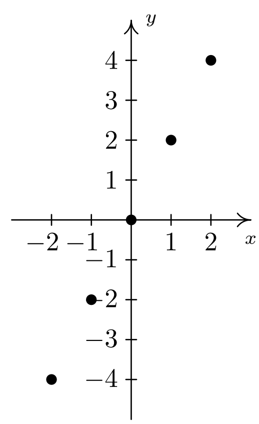
33. $x = \dfrac{\sqrt{5} \pm i\sqrt{3}}{2}$

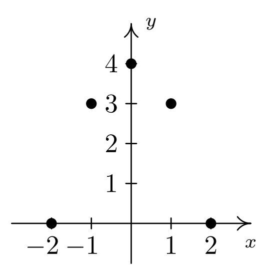
35. $z = \pm 2, \pm 2i$

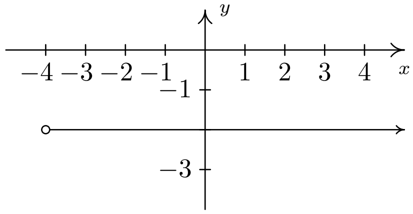
Chapter 2

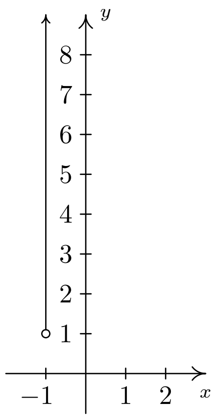
Section 2.1

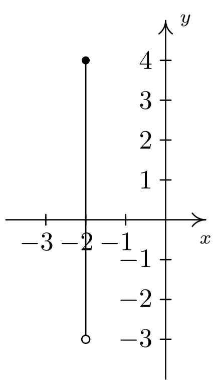
1. 

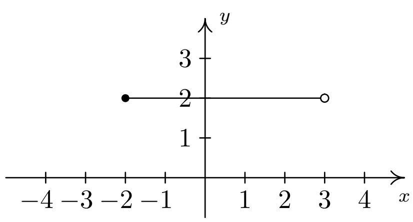
3. 

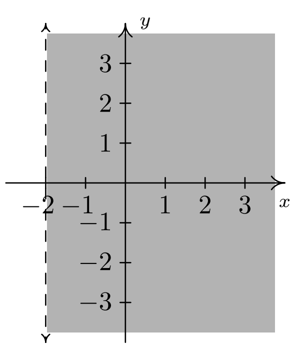
5. 

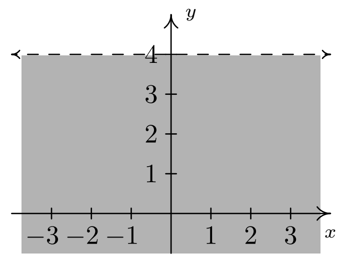
7. 

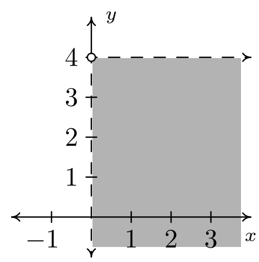
9. 

11. 

13. 

15. 

17. 

19. 

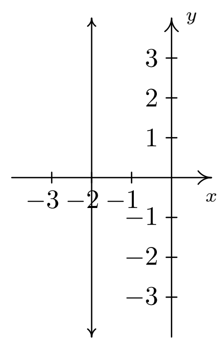
21. $A = \{(-4,-1),(-2,1),(0,3),(1,4)\}$

23. $C = \{(2,y)\,|\,y > -3\}$

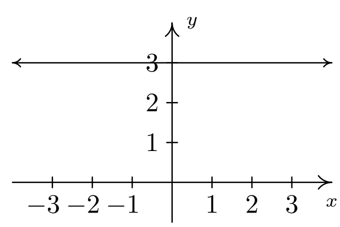
25. $E = \{(x,2)\mid -4 \leq x < 3\}$

27. $G = \{(x,y)\,|\,x > -2\}$

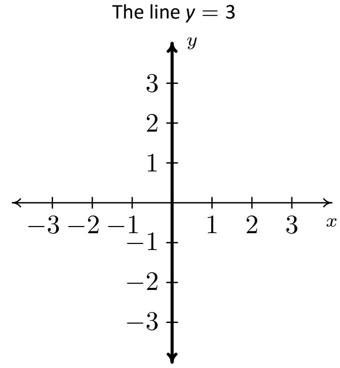
29. $I = \{(x,y)\,|\,x \geq 0, y \geq 0\}$

31. 

The line $x = -2$

33. 

The line $y = 3$

35. 

The line $x = 0$ is the y-axis

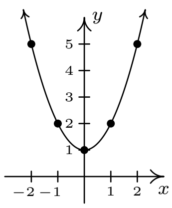
37.

39.

41. The graph has no x-intercepts

y-intercept: $(0,1)$

x	y	(x,y)
-2	5	$(-2,5)$
-1	2	$(-1,2)$
0	1	$(0,1)$
1	2	$(1,2)$
2	5	$(2,5)$



The graph is not symmetric about the x-axis (e.g. $(2,5)$ is on the graph but $(2,-5)$ is not)

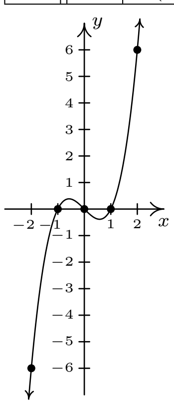
The graph is symmetric about the y-axis

The graph is not symmetric about the origin (e.g. $(2,5)$ is on the graph but $(-2,-5)$ is not)

43. x-intercepts: $(-1,0),(0,0),(1,0)$

y-intercept: $(0,0)$

x	y	(x,y)
-2	-6	$(-2,-6)$
-1	0	$(-1,0)$
0	0	$(0,0)$
1	0	$(1,0)$
2	6	$(2,6)$



The graph is not symmetric about the x-axis. (e.g. $(2,6)$ is on the graph but $(2,-6)$ is not)

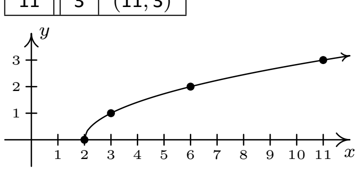
The graph is not symmetric about the y-axis. (e.g. $(2,6)$ is on the graph but $(-2,6)$ is not)

The graph is symmetric about the origin.

45. x-intercept: $(2,0)$

The graph has no y-intercepts

x	y	(x,y)
2	0	$(2,0)$
3	1	$(3,1)$
6	2	$(6,2)$
11	3	$(11,3)$



The graph is not symmetric about the x-axis (e.g. $(3,1)$ is on the graph but $(3,-1)$ is not)

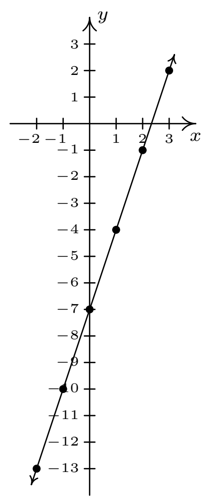
The graph is not symmetric about the y-axis (e.g. $(3,1)$ is on the graph but $(-3,1)$ is not)

The graph is not symmetric about the origin (e.g. $(3,1)$ is on the graph but $(-3,-1)$ is not)

47. x-intercept: $(\frac{7}{3},0)$

y-intercept: $(0,-7)$

x	y	(x,y)
-2	-13	$(-2,-13)$
-1	-10	$(-1,-10)$
0	-7	$(0,-7)$
1	-4	$(1,-4)$
2	-1	$(2,-1)$
3	2	$(3,2)$



The graph is not symmetric about the x-axis (e.g. $(3,2)$ is on the graph but $(3,-2)$ is not)

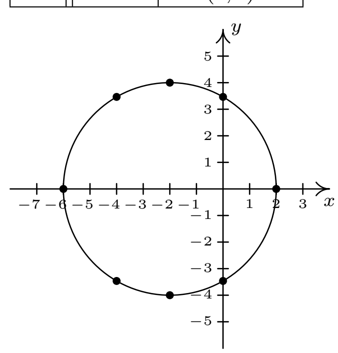
The graph is not symmetric about the y-axis (e.g. $(3,2)$ is on the graph but $(-3,2)$ is not)

The graph is not symmetric about the origin (e.g. $(3,2)$ is on the graph but $(-3,-2)$ is not)

49. x-intercepts: $(-6,0)$, $(2,0)$

y-intercepts: $\left(0,\pm 2\sqrt{3}\right)$

x	y	(x,y)
-6	0	$(-6,0)$
-4	$\pm 2\sqrt{3}$	$\left(-4,\pm 2\sqrt{3}\right)$
-2	± 4	$(-2,\pm 4)$
0	$\pm 2\sqrt{3}$	$\left(0,\pm 2\sqrt{3}\right)$
2	0	$(2,0)$



The graph is symmetric about the x-axis

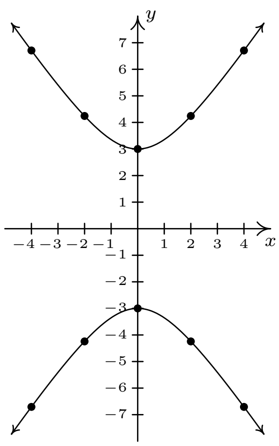
The graph is not symmetric about the y-axis (e.g. $(-6,0)$ is on the graph but $(6,0)$ is not)

The graph is not symmetric about the origin (e.g. $(-6,0)$ is on the graph but $(6,0)$ is not)

51. $4y^2 - 9x^2 = 36$
Re-write as: $y = \pm\frac{\sqrt{9x^2+36}}{2}$.
The graph has no x-intercepts

y-intercepts: $(0, \pm 3)$

x	y	(x,y)
-4	$\pm 3\sqrt{5}$	$(-4, \pm 3\sqrt{5})$
-2	$\pm 3\sqrt{2}$	$(-2, \pm 3\sqrt{2})$
0	± 3	$(0, \pm 3)$
2	$\pm 3\sqrt{2}$	$(2, \pm 3\sqrt{2})$
4	$\pm 3\sqrt{5}$	$(4, \pm 3\sqrt{5})$



The graph is symmetric about the x-axis

The graph is symmetric about the y-axis

The graph is symmetric about the origin

53.

Section 2.2

1. Function
domain = $\{-3,-2,-1,0,1,2,3\}$
range = $\{0,1,4,9\}$

3. Function
domain = $\{-7,-3,3,4,5,6\}$
range = $\{0,4,5,6,9\}$

5. Not a function

7. Function
domain = $\{x | x = 2^n$ for some whole number $n\}$
range = $\{y \,|\, y \geq 0$ is an integer$\}$

9. Not a function

11. Function
domain = $(-\infty, \infty)$
range = $[0, \infty)$

13. Function
domain = $\{-4, -3, -2, -1, 0, 1\}$
range = $\{-1, 0, 1, 2, 3, 4\}$

15. Function
domain = $(-\infty, \infty)$
range = $[1, \infty)$

17. Function
domain = $[2, \infty)$
range = $[0, \infty)$

19. Not a function

21. Function
domain = $[-2, \infty)$
range = $[-3, \infty)$

23. Function
domain = $[-5, 4)$
range = $[-4, 4)$

25. Function
domain = $(-\infty, \infty)$
range = $(-\infty, 4]$

27. Function
domain = $[-2, \infty)$
range = $(-\infty, 3]$

29. Function
domain = $(-\infty, 0] \cup (1, \infty)$
range = $(-\infty, 1] \cup \{2\}$

31. Not a function

33. Function

35. Function

37. Function

39. Not a function

41. Function

43. Not a function

45. Function

47. Not a function

49.

51.

53.

Section 2.3

1. $f(x) = \frac{2x+3}{4}$
Domain: $(-\infty, \infty)$

3. $f(x) = 2\left(\frac{x}{4}+3\right) = \frac{1}{2}x + 6$
Domain: $(-\infty, \infty)$

5. $f(x) = \sqrt{2(x+3)} = \sqrt{2x+6}$
Domain: $[-3, \infty)$

7. $f(x) = \frac{4}{\sqrt{x}-13}$
Domain: $[0, 169) \cup (169, \infty)$

9. $f(x) = \frac{4}{\sqrt{x}} - 13$
Domain: $(0, \infty)$

11. For $f(x) = 2x + 1$

- $f(3) = 7$
- $f(-1) = -1$
- $f\left(\frac{3}{2}\right) = 4$
- $f(4x) = 8x + 1$
- $4f(x) = 8x + 4$
- $f(-x) = -2x + 1$
- $f(x-4) = 2x - 7$
- $f(x) - 4 = 2x - 3$
- $f\left(x^2\right) = 2x^2 + 1$

13. For $f(x) = 2 - x^2$

- $f(3) = -7$
- $f(-1) = 1$
- $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
- $f(4x) = 2 - 16x^2$
- $4f(x) = 8 - 4x^2$
- $f(-x) = 2 - x^2$
- $f(x-4) =$
$-x^2 + 8x - 14$
- $f(x) - 4 = -x^2 - 2$
- $f\left(x^2\right) = 2 - x^4$

15. For $f(x) = \dfrac{x}{x-1}$

- $f(3) = \dfrac{3}{2}$
- $f(-1) = \dfrac{1}{2}$
- $f\left(\dfrac{3}{2}\right) = 3$
- $f(4x) = \dfrac{4x}{4x-1}$
- $4f(x) = \dfrac{4x}{x-1}$
- $f(-x) = \dfrac{x}{x+1}$
- $f(x-4) = \dfrac{x-4}{x-5}$
- $f(x) - 4 = \dfrac{x}{x-1} - 4 = \frac{4-3x}{x-1}$
- $f\left(x^2\right) = \dfrac{x^2}{x^2-1}$

17. For $f(x) = 6$

- $f(3) = 6$
- $f(-1) = 6$
- $f\left(\frac{3}{2}\right) = 6$
- $f(4x) = 6$
- $4f(x) = 24$
- $f(-x) = 6$
- $f(x-4) = 6$
- $f(x) - 4 = 2$
- $f\left(x^2\right) = 6$

19. For $f(x) = 2x - 5$

- $f(2) = -1$
- $f(-2) = -9$
- $f(2a) = 4a - 5$
- $2f(a) = 4a - 10$
- $f(a+2) = 2a - 1$
- $f(a) + f(2) = 2a - 6$
- $f\left(\frac{2}{a}\right) = \frac{4}{a} - 5 = \frac{4-5a}{a}$
- $\frac{f(a)}{2} = \frac{2a-5}{2}$
- $f(a+h) = 2a + 2h - 5$

21. For $f(x) = 2x^2 - 1$

- $f(2) = 7$
- $f(-2) = 7$
- $f(2a) = 8a^2 - 1$
- $2f(a) = 4a^2 - 2$
- $f(a+2) = 2a^2 + 8a + 7$
- $f(a) + f(2) = 2a^2 + 6$
- $f\left(\frac{2}{a}\right) = \frac{8}{a^2} - 1 = \frac{8-a^2}{a^2}$
- $\frac{f(a)}{2} = \frac{2a^2-1}{2}$
- $f(a+h) = 2a^2 + 4ah + 2h^2 - 1$

23. For $f(x) = \sqrt{2x+1}$

- $f(2) = \sqrt{5}$
- $f(-2)$ is not real
- $f(2a) = \sqrt{4a+1}$
- $2f(a) = 2\sqrt{2a+1}$
- $f(a+2) = \sqrt{2a+5}$
- $f(a) + f(2) = \sqrt{2a+1} + \sqrt{5}$
- $f\left(\frac{2}{a}\right) = \sqrt{\frac{4}{a} + 1} = \sqrt{\frac{a+4}{a}}$
- $\frac{f(a)}{2} = \frac{\sqrt{2a+1}}{2}$
- $f(a+h) = \sqrt{2a+2h+1}$

25. For $f(x) = \frac{x}{2}$

- $f(2) = 1$
- $f(-2) = -1$
- $f(2a) = a$
- $2f(a) = a$
- $f(a+2) = \frac{a+2}{2}$
- $f(a) + f(2) = \frac{a}{2} + 1 = \frac{a+2}{2}$
- $f\left(\frac{2}{a}\right) = \frac{1}{a}$
- $\frac{f(a)}{2} = \frac{a}{4}$
- $f(a+h) = \frac{a+h}{2}$

27. For $f(x) = 2x - 1$, $f(0) = -1$ and $f(x) = 0$ when $x = \frac{1}{2}$

29. For $f(x) = 2x^2 - 6$, $f(0) = -6$ and $f(x) = 0$ when $x = \pm\sqrt{3}$

31. For $f(x) = \sqrt{x+4}$, $f(0) = 2$ and $f(x) = 0$ when $x = -4$

33. For $f(x) = \frac{3}{4-x}$, $f(0) = \frac{3}{4}$ and $f(x)$ is never equal to 0

35. (a) $f(-4) = 1$
    (b) $f(-3) = 2$
    (c) $f(3) = 0$
    (d) $f(3.001) = 1.999$
    (e) $f(-3.001) = 1.999$
    (f) $f(2) = \sqrt{5}$

37. $(-\infty, \infty)$

39. $(-\infty, -1) \cup (-1, \infty)$

41. $(-\infty, \infty)$

43. $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$

45. $(-\infty, 3]$

47. $[-3, \infty)$

49. $\left[\frac{1}{3}, \infty\right)$

51. $(-\infty, \infty)$

53. $\left[\frac{1}{3}, 6\right) \cup (6, \infty)$

55. $(-\infty, 8) \cup (8, \infty)$

57. $(8, \infty)$

59. $(-\infty, 8) \cup (8, \infty)$

61. $[0, 5) \cup (5, \infty)$

63. $A(3) = 9$, so the area enclosed by a square with a side of length 3 inches is 9 square inches. The solutions to $A(x) = 36$ are $x = \pm 6$. Since x is restricted to $x > 0$, we only keep $x = 6$. This means for the area enclosed by the square to be 36 square inches, the length of the side needs to be 6 inches. Since x represents a length, $x > 0$.

65. $V(5) = 125$, so the volume enclosed by a cube with a side of length 5 centimeters is 125 cubic centimeters. The solution to $V(x) = 27$ is $x = 3$. This means for the volume enclosed by the cube to be 27 cubic centimeters, the length of the side needs to 3 centimeters. Since x represents a length, $x > 0$.

67. $V(3) = 36\pi$, so the volume enclosed by a sphere with radius 3 feet is 36π cubic feet. The solution to $V(r) = \frac{32\pi}{3}$ is $r = 2$. This means for the volume enclosed by the sphere to be $\frac{32\pi}{3}$ cubic feet, the radius needs to 2 feet. Since r represents a radius (length), $r > 0$.

69. $T(0) = 3$, so at 6 AM (0 hours after 6 AM), it is 3° Fahrenheit. $T(6) = 33$, so at noon (6 hours after 6 AM), the temperature is 33° Fahrenheit. $T(12) = 27$, so at 6 PM (12 hours after 6 AM), it is 27° Fahrenheit.

71. $F(0) = 16.00$, so in 1980 (0 years after 1980), the average fuel economy of passenger cars in the US was 16.00 miles per gallon. $F(14) = 20.81$, so in 1994 (14 years after 1980), the average fuel economy of passenger cars in the US was 20.81 miles per gallon. $F(28) = 22.64$, so in 2008 (28 years after 1980), the average fuel economy of passenger cars in the US was 22.64 miles per gallon.

73. (a) $C(20) = 300$. It costs \$300 for 20 copies of the book.
    (b) $C(50) = 675$, so it costs \$675 for 50 copies of the book. $C(51) = 612$, so it costs \$612 for 51 copies of the book.
    (c) 56 books.

75. (a) $C(750) = 25$, so it costs \$25 to talk 750 minutes per month with this plan.
    (b) Since 20 hours $= 1200$ minutes, we substitute $m = 1200$ and get $C(1200) = 45$. It costs \$45 to talk 20 hours per month with this plan.
    (c) It costs \$25 for up to 1000 minutes and 10 cents per minute for each minute over 1000 minutes.

77.

Section 2.4

1. For $f(x) = 3x + 1$ and $g(x) = 4 - x$
   - $(f+g)(2) = 9$
   - $(f-g)(-1) = -7$
   - $(g-f)(1) = -1$
   - $(fg)\left(\frac{1}{2}\right) = \frac{35}{4}$
   - $\left(\frac{f}{g}\right)(0) = \frac{1}{4}$
   - $\left(\frac{g}{f}\right)(-2) = -\frac{6}{5}$

3. For $f(x) = x^2 - x$ and $g(x) = 12 - x^2$
   - $(f+g)(2) = 10$
   - $(f-g)(-1) = -9$
   - $(g-f)(1) = 11$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{47}{16}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{4}{3}$

5. For $f(x) = \sqrt{x+3}$ and $g(x) = 2x - 1$
   - $(f+g)(2) = 3 + \sqrt{5}$
   - $(f-g)(-1) = 3 + \sqrt{2}$
   - $(g-f)(1) = -1$
   - $(fg)\left(\frac{1}{2}\right) = 0$
   - $\left(\frac{f}{g}\right)(0) = -\sqrt{3}$
   - $\left(\frac{g}{f}\right)(-2) = -5$

7. For $f(x) = 2x$ and $g(x) = \dfrac{1}{2x+1}$
   - $(f+g)(2) = \frac{21}{5}$
   - $(f-g)(-1) = -1$
   - $(g-f)(1) = -\frac{5}{3}$
   - $(fg)\left(\frac{1}{2}\right) = \frac{1}{2}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{1}{12}$

9. For $f(x) = x^2$ and $g(x) = \dfrac{1}{x^2}$
   - $(f+g)(2) = \frac{17}{4}$
   - $(f-g)(-1) = 0$
   - $(g-f)(1) = 0$
   - $(fg)\left(\frac{1}{2}\right) = 1$
   - $\left(\frac{f}{g}\right)(0)$ is undefined.
   - $\left(\frac{g}{f}\right)(-2) = \frac{1}{16}$

11. For $f(x) = 2x + 1$ and $g(x) = x - 2$
   - $(f+g)(x) = 3x - 1$ Domain: $(-\infty, \infty)$
   - $(f-g)(x) = x + 3$ Domain: $(-\infty, \infty)$
   - $(fg)(x) = 2x^2 - 3x - 2$ Domain: $(-\infty, \infty)$
   - $\left(\frac{f}{g}\right)(x) = \frac{2x+1}{x-2}$ Domain: $(-\infty, 2) \cup (2, \infty)$

13. For $f(x) = x^2$ and $g(x) = 3x - 1$
   - $(f+g)(x) = x^2 + 3x - 1$ Domain: $(-\infty, \infty)$
   - $(f-g)(x) = x^2 - 3x + 1$ Domain: $(-\infty, \infty)$
   - $(fg)(x) = 3x^3 - x^2$ Domain: $(-\infty, \infty)$
   - $\left(\frac{f}{g}\right)(x) = \frac{x^2}{3x-1}$ Domain: $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

15. For $f(x) = x^2 - 4$ and $g(x) = 3x + 6$
   - $(f+g)(x) = x^2 + 3x + 2$ Domain: $(-\infty, \infty)$
   - $(f-g)(x) = x^2 - 3x - 10$ Domain: $(-\infty, \infty)$
   - $(fg)(x) = 3x^3 + 6x^2 - 12x - 24$ Domain: $(-\infty, \infty)$
   - $\left(\frac{f}{g}\right)(x) = \frac{x-2}{3}$ Domain: $(-\infty, -2) \cup (-2, \infty)$

17. For $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$
   - $(f+g)(x) = \frac{x^2+4}{2x}$ Domain: $(-\infty, 0) \cup (0, \infty)$
   - $(f-g)(x) = \frac{x^2-4}{2x}$ Domain: $(-\infty, 0) \cup (0, \infty)$
   - $(fg)(x) = 1$ Domain: $(-\infty, 0) \cup (0, \infty)$
   - $\left(\frac{f}{g}\right)(x) = \frac{x^2}{4}$ Domain: $(-\infty, 0) \cup (0, \infty)$

19. For $f(x) = x$ and $g(x) = \sqrt{x+1}$
   - $(f+g)(x) = x + \sqrt{x+1}$ Domain: $[-1, \infty)$
   - $(f-g)(x) = x - \sqrt{x+1}$ Domain: $[-1, \infty)$
   - $(fg)(x) = x\sqrt{x+1}$ Domain: $[-1, \infty)$
   - $\left(\frac{f}{g}\right)(x) = \frac{x}{\sqrt{x+1}}$ Domain: $(-1, \infty)$

21. 2

23. 0

25. $-2x - h + 2$

27. $-2x - h + 1$

29. m

31. $\dfrac{-2}{x(x+h)}$

33. $\dfrac{-(2x+h)}{x^2(x+h)^2}$

35. $\dfrac{-4}{(4x-3)(4x+4h-3)}$

37. $\dfrac{-9}{(x-9)(x+h-9)}$

39. $\dfrac{1}{\sqrt{x+h-9}+\sqrt{x-9}}$

41. $\dfrac{-4}{\sqrt{-4x-4h+5}+\sqrt{-4x+5}}$

43. $\dfrac{a}{\sqrt{ax+ah+b}+\sqrt{ax+b}}$

45. $\dfrac{1}{(x+h)^{2/3}+(x+h)^{1/3}x^{1/3}+x^{2/3}}$

47.
   - $C(0) = 100$, so the fixed costs are \$100.
   - $\overline{C}(10) = 20$, so when 10 bottles of tonic are produced, the cost per bottle is \$20.
   - $p(5) = 30$, so to sell 5 bottles of tonic, set the price at \$30 per bottle.

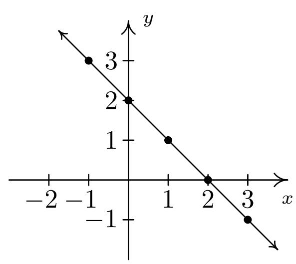
- $R(x) = -x^2 + 35x$, $0 \leq x \leq 35$
- $P(x) = -x^2 + 25x - 100$, $0 \leq x \leq 35$
- $P(x) = 0$ when $x = 5$ and $x = 20$. These are the 'break even' points, so selling 5 bottles of tonic or 20 bottles of tonic will guarantee the revenue earned exactly recoups the cost of production.

49.
- $C(0) = 36$, so the daily fixed costs are \$36.
- $\overline{C}(10) = 6.6$, so when 10 pies are made, the cost per pie is \$6.60.
- $p(5) = 9.5$, so to sell 5 pies a day, set the price at \$9.50 per pie.
- $R(x) = -0.5x^2 + 12x$, $0 \leq x \leq 24$
- $P(x) = -0.5x^2 + 9x - 36$, $0 \leq x \leq 24$
- $P(x) = 0$ when $x = 6$ and $x = 12$. These are the 'break even' points, so selling 6 pies or 12 pies a day will guarantee the revenue earned exactly recoups the cost of production.

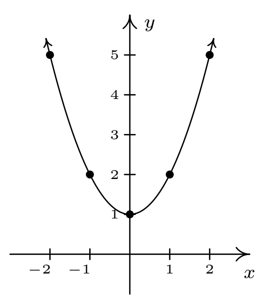
51. $(f+g)(-3) = 2$
53. $(fg)(-1) = 0$
55. $(g-f)(3) = 3$
57. $\left(\frac{f}{g}\right)(-2)$ does not exist
59. $\left(\frac{f}{g}\right)(2) = 4$
61. $\left(\frac{g}{f}\right)(3) = -2$

Section 2.5

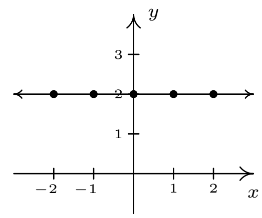
1. $f(x) = 2 - x$
Domain: $(-\infty, \infty)$
x-intercept: $(2, 0)$
y-intercept: $(0, 2)$
No symmetry



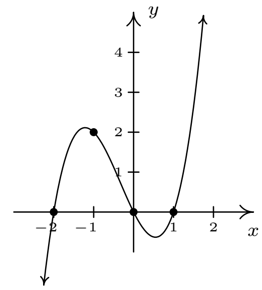
3. $f(x) = x^2 + 1$
Domain: $(-\infty, \infty)$
x-intercept: None
y-intercept: $(0, 1)$
Even



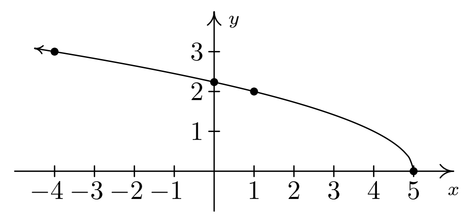
5. $f(x) = 2$
Domain: $(-\infty, \infty)$
x-intercept: None
y-intercept: $(0, 2)$
Even



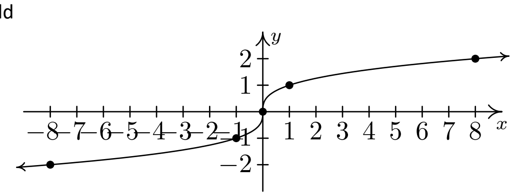
7. $f(x) = x(x-1)(x+2)$
Domain: $(-\infty, \infty)$
x-intercepts: $(-2, 0)$, $(0, 0)$, $(1, 0)$
y-intercept: $(0, 0)$
No symmetry

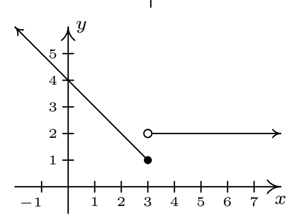


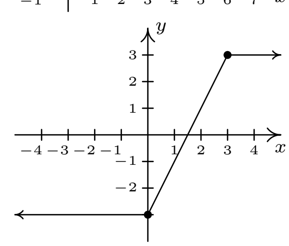
9. $f(x) = \sqrt{5 - x}$
Domain: $(-\infty, 5]$
x-intercept: $(5, 0)$
y-intercept: $(0, \sqrt{5})$
No symmetry

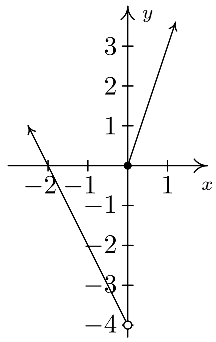


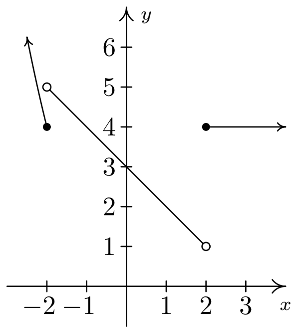
11. $f(x) = \sqrt[3]{x}$
Domain: $(-\infty, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
Odd



13. 

15. 

17. 

19. 

21. odd

23. even

25. even

27. odd

29. even

31. neither

33. even **and** odd

35. even

37. neither

39. odd

41. even

43. $[-5, 4]$

45. $x = -3$

47. $(0, -1)$

49. $[-4, -1] \cup [1, 3]$

51. neither

53. $[-3, 0]$, $[2, 3]$

55. $f(0) = -1$

57. $f(-5) = -5$

59. $[-5, 5)$

61. $x = -2$

63. $(0, 0)$

65. $[-4, 0] \cup \{4\}$

67. neither

69. $[-4, -2]$, $(2, 4]$

71. $f(-2) = -5, f(2) = 3$

73. $f(-2) = -5$

75. No absolute maximum
No absolute minimum
Local maximum at $(0, 0)$
Local minimum at $(1.60, -3.28)$
Increasing on $(-\infty, 0]$, $[1.60, \infty)$
Decreasing on $[0, 1.60]$

77. Absolute maximum $f(2.12) \approx 4.50$
Absolute minimum $f(-2.12) \approx -4.50$
Local maximum $(2.12, 4.50)$
Local minimum $(-2.12, -4.50)$
Increasing on $[-2.12, 2.12]$
Decreasing on $[-3, -2.12]$, $[2.12, 3]$

79. $(f + g)(1) = 5$

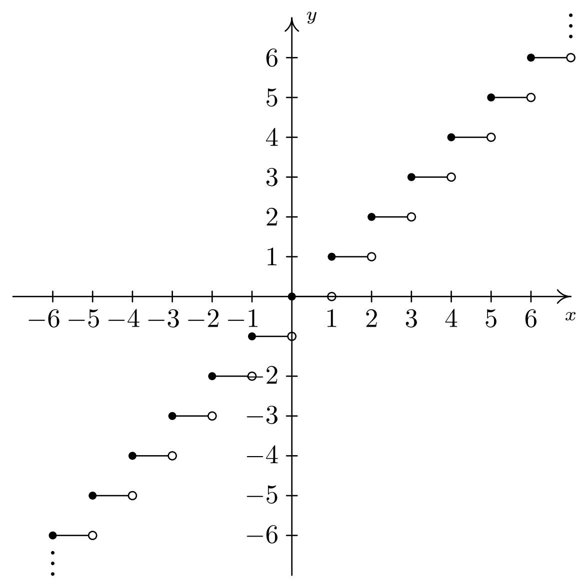
81. $(g - f)(2) = 0$

83. $(fg)(1) = 6$

85. $\left(\frac{g}{f}\right)(2) = 1$

87. $h(15) = 6$, so the Saquatch is 6 feet tall when she is 15 years old.

89. h is constant on $[30, 45]$. This means the Sasquatch's height is constant (at 8 feet) for these years.

91. 

The graph of $f(x) = \lfloor x \rfloor$.

93.

95.

97.

99.

Section 2.6

1. $(2, 0)$

3. $(2, -4)$

5. $(2, -9)$

7. $(2, 3)$

9. $(5, -2)$

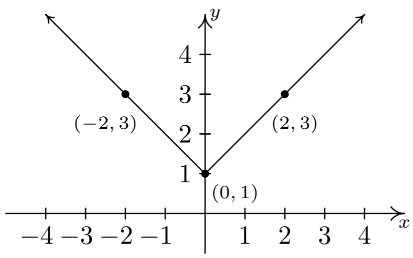
11. $(2, 13)$

13. $(2, -\frac{3}{2})$

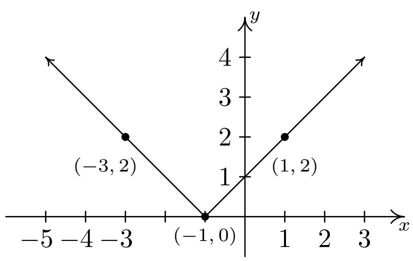
15. $(-1, -7)$

17. $(\frac{2}{3}, -2)$

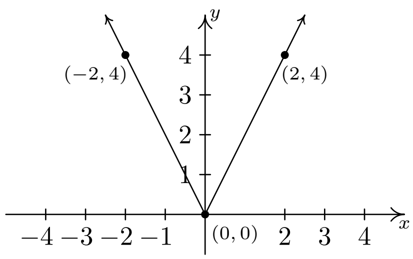
19. $y = f(x) + 1$



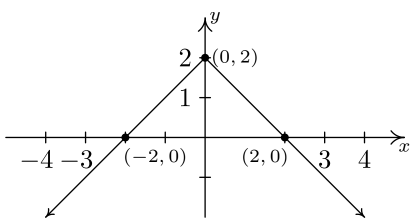
21. $y = f(x+1)$



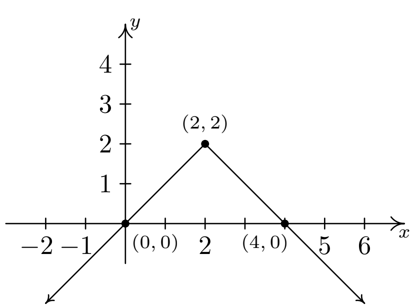
23. $y = 2f(x)$



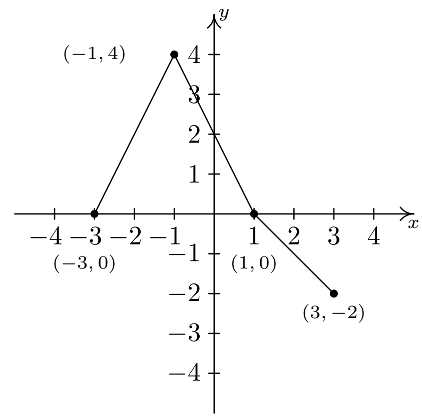
25. $y = 2 - f(x)$



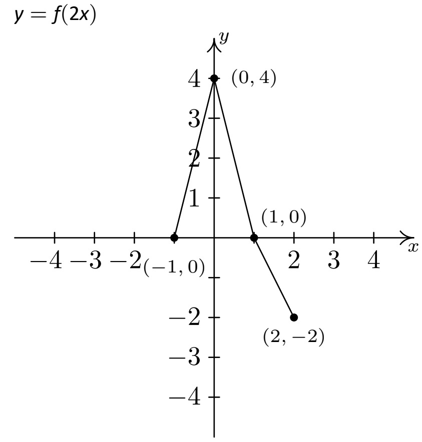
27. $y = 2 - f(2-x)$



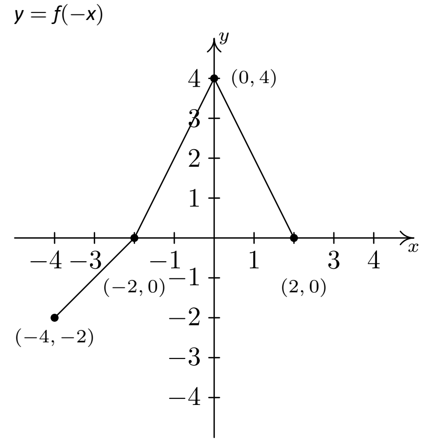
29. $y = f(x+1)$



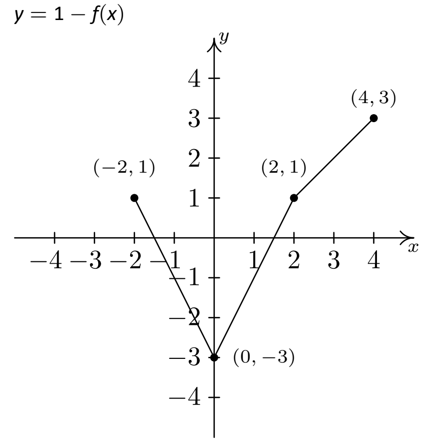
31. $y = f(2x)$



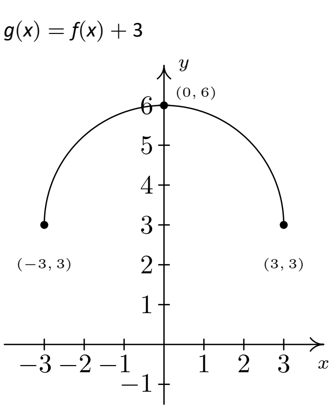
33. $y = f(-x)$



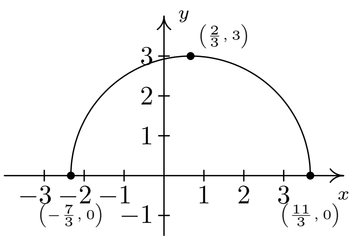
35. $y = 1 - f(x)$



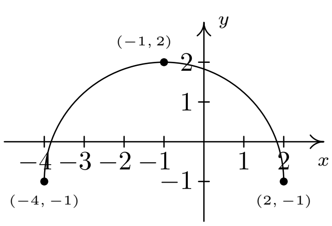
37. $g(x) = f(x) + 3$



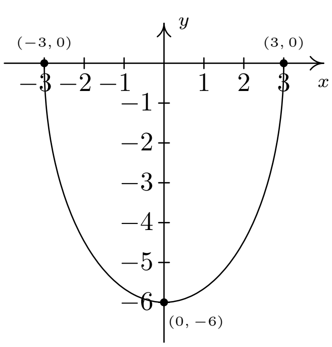
39. $j(x) = f\left(x - \frac{2}{3}\right)$



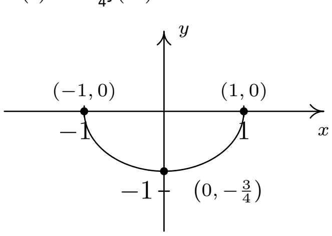
41. $b(x) = f(x + 1) - 1$



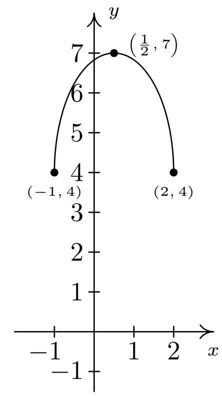
43. $d(x) = -2f(x)$



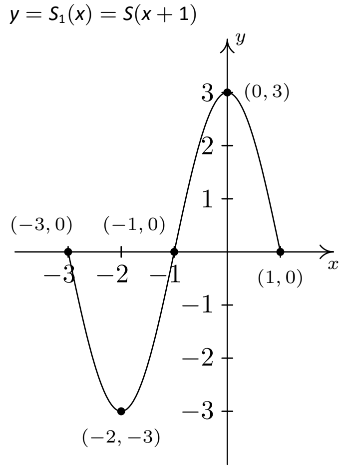
45. $m(x) = -\frac{1}{4}f(3x)$



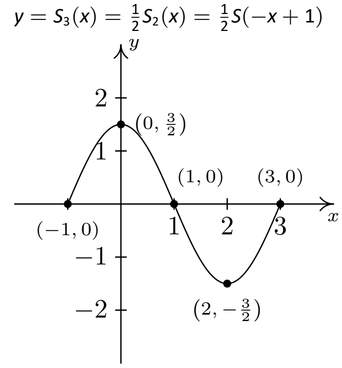
47. $p(x) = 4 + f(1 - 2x)$



49. $y = S_1(x) = S(x + 1)$



51. $y = S_3(x) = \frac{1}{2}S_2(x) = \frac{1}{2}S(-x + 1)$



53. $g(x) = \sqrt{x - 2} - 3$

55. $g(x) = -\sqrt{x} + 1$

57. $g(x) = \sqrt{-(x + 1)} + 2 = \sqrt{-x - 1} + 2$

59. $g(x) = 2\left(\sqrt{x + 3} - 4\right) = 2\sqrt{x + 3} - 8$

61. $g(x) = \sqrt{2(x - 3)} + 1 = \sqrt{2x - 6} + 1$

63.

65.

67. The same thing as reflecting it across the *x*-axis.

69. The same thing as reflecting it across the *y*-axis.

71.

Chapter 3

Section 3.1

1. $y + 1 = 3(x - 3)$
   $y = 3x - 10$

3. $y + 1 = -(x + 7)$
   $y = -x - 8$

5. $y - 4 = -\frac{1}{5}(x - 10)$
   $y = -\frac{1}{5}x + 6$

7. $y - 117 = 0$
   $y = 117$

9. $y - 2\sqrt{3} = -5(x - \sqrt{3})$
   $y = -5x + 7\sqrt{3}$

11. $y = -\frac{5}{3}x$

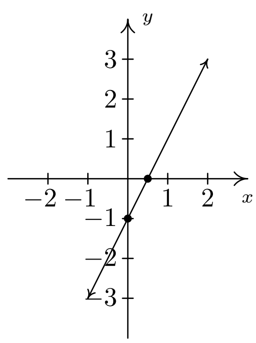
13. $y = \frac{8}{5}x - 8$

15. $y = 5$

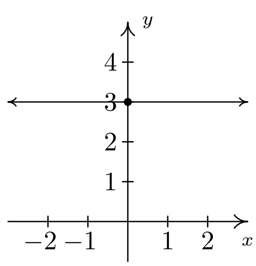
17. $y = -\frac{5}{4}x + \frac{11}{8}$

19. $y = -x$

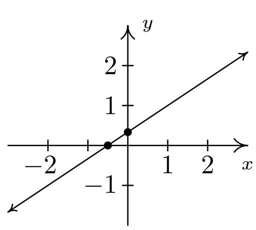
21. $f(x) = 2x - 1$
    slope: $m = 2$
    y-intercept: $(0, -1)$
    x-intercept: $\left(\frac{1}{2}, 0\right)$



23. $f(x) = 3$
    slope: $m = 0$
    y-intercept: $(0, 3)$
    x-intercept: none



25. $f(x) = \frac{2}{3}x + \frac{1}{3}$
    slope: $m = \frac{2}{3}$
    y-intercept: $\left(0, \frac{1}{3}\right)$
    x-intercept: $\left(-\frac{1}{2}, 0\right)$



27. $(-1, -1)$ and $\left(\frac{11}{5}, \frac{27}{5}\right)$

29. $E(t) = 360t$, $t \geq 0$.

31. $C(t) = 80t + 50$, $0 \leq t \leq 8$

33. $C(p) = 0.035p + 1.5$ The slope 0.035 means it costs 3.5¢ per page. $C(0) = 1.5$ means there is a fixed, or start-up, cost of \$1.50 to make each book.

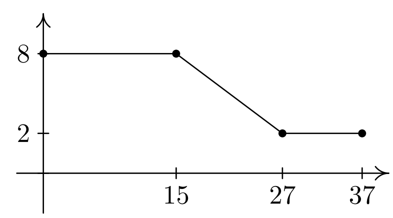
35. (a) $F(C) = \frac{9}{5}C + 32$
    (b) $C(F) = \frac{5}{9}(F - 32) = \frac{5}{9}F - \frac{160}{9}$
    (c) $F(-40) = -40 = C(-40)$.

37.

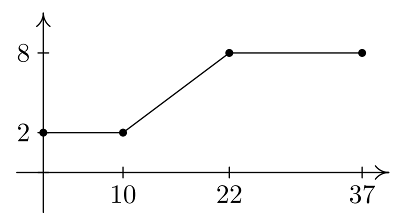
39. $C(p) = \begin{cases} 6p + 1.5 & \text{if} \quad 1 \leq p \leq 5 \\ 5.5p & \text{if} \quad p \geq 6 \end{cases}$

41. $C(m) = \begin{cases} 10 & \text{if} \quad 0 \leq m \leq 500 \\ 10 + 0.15(m - 500) & \text{if} \quad m > 500 \end{cases}$

43. (a)
$$D(d) = \begin{cases} 8 & \text{if} \quad 0 \leq d \leq 15 \\ -\frac{1}{2}d + \frac{31}{2} & \text{if} \quad 15 \leq d \leq 27 \\ 2 & \text{if} \quad 27 \leq d \leq 37 \end{cases}$$
    (b)
$$D(s) = \begin{cases} 2 & \text{if} \quad 0 \leq s \leq 10 \\ \frac{1}{2}s - 3 & \text{if} \quad 10 \leq s \leq 22 \\ 8 & \text{if} \quad 22 \leq s \leq 37 \end{cases}$$
    (c)



$y = D(d)$



$y = D(s)$

45. $\dfrac{\frac{1}{5} - \frac{1}{1}}{5 - 1} = -\dfrac{1}{5}$

47. $\dfrac{3^2 - (-3)^2}{3 - (-3)} = 0$

49. $\dfrac{(3(2)^2 + 2(2) - 7) - (3(-4)^2 + 2(-4) - 7)}{2 - (-4)} = -4$

51. $\dfrac{-1}{x(x + h)}$

53. $6x + 3h + 2$

55. (a) $T(4) = 56$, so at 10 AM (4 hours after 6 AM), it is 56°F.
    $T(8) = 64$, so at 2 PM (8 hours after 6 AM), it is 64°F.
    $T(12) = 56$, so at 6 PM (12 hours after 6 AM), it is 56°F.
    (b) The average rate of change is $\frac{T(8) - T(4)}{8 - 4} = 2$. Between 10 AM and 2 PM, the temperature increases, on average, at a rate of 2°F per hour.
    (c) The average rate of change is $\frac{T(12) - T(8)}{12 - 8} = -2$. Between 2 PM and 6 PM, the temperature decreases, on average, at a rate of 2°F per hour.

(d) The average rate of change is $\frac{T(12)-T(4)}{12-4} = 0$. Between 10 AM and 6 PM, the temperature, on average, remains constant.

57.

59. $y = 3x$

61. $y = \frac{2}{3}x - 4$

63. $y = -2$

65. $y = -3x$

67. $y = -\frac{3}{2}x + 9$

69. $x = 3$

71.

73.

Section 3.2

1. $x = -6$ or $x = 6$

3. $x = -3$ or $x = 11$

5. $x = -\frac{1}{2}$ or $x = \frac{1}{10}$

7. $x = -3$ or $x = 3$

9. $x = -\frac{3}{2}$

11. $x = 1$

13. $x = -1, x = 0$ or $x = 1$

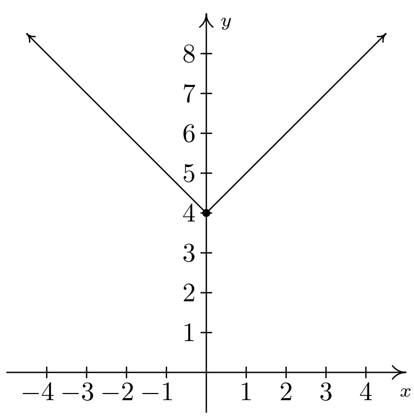
15. $x = -2$ or $x = 2$

17. $x = -\frac{1}{7}$ or $x = 1$

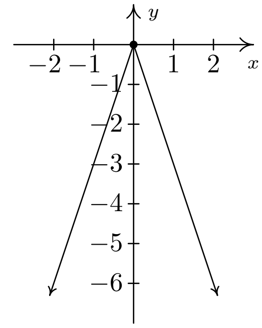
19. $x = 1$

21. $x = \frac{1}{5}$ or $x = 5$

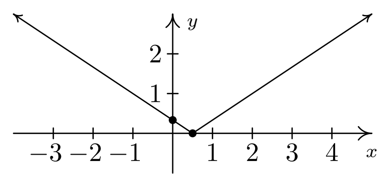
23. $f(x) = |x| + 4$
No zeros
No x-intercepts
y-intercept $(0, 4)$
Domain $(-\infty, \infty)$
Range $[4, \infty)$
Decreasing on $(-\infty, 0]$
Increasing on $[0, \infty)$
Relative and absolute minimum at $(0, 4)$
No relative or absolute maximum



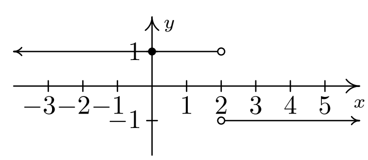
25. $f(x) = -3|x|$
$f(0) = 0$
x-intercept $(0, 0)$
y-intercept $(0, 0)$
Domain $(-\infty, \infty)$
Range $(-\infty, 0]$
Increasing on $(-\infty, 0]$
Decreasing on $[0, \infty)$
Relative and absolute maximum at $(0, 0)$
No relative or absolute minimum



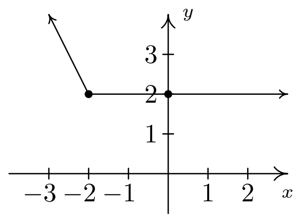
27. $f(x) = \frac{1}{3}|2x - 1|$
$f\left(\frac{1}{2}\right) = 0$
x-intercepts $\left(\frac{1}{2}, 0\right)$
y-intercept $\left(0, \frac{1}{3}\right)$
Domain $(-\infty, \infty)$
Range $[0, \infty)$
Decreasing on $\left(-\infty, \frac{1}{2}\right]$
Increasing on $\left[\frac{1}{2}, \infty\right)$
Relative and absolute min. at $\left(\frac{1}{2}, 0\right)$
No relative or absolute maximum



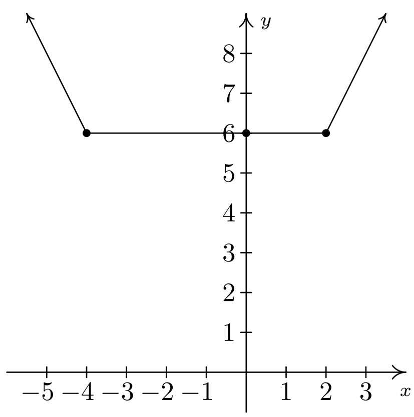
29. $f(x) = \dfrac{|2 - x|}{2 - x}$
No zeros
No x-intercept
y-intercept $(0, 1)$
Domain $(-\infty, 2) \cup (2, \infty)$
Range $\{-1, 1\}$
Constant on $(-\infty, 2)$
Constant on $(2, \infty)$
Absolute minimum at every point $(x, -1)$ where $x > 2$
Absolute maximum at every point $(x, 1)$ where $x < 2$
Relative maximum AND minimum at every point on the graph



31. Re-write $f(x) = |x + 2| - x$ as
$f(x) = \begin{cases} -2x - 2 & \text{if} \quad x < -2 \\ 2 & \text{if} \quad x \geq -2 \end{cases}$
No zeros
No x-intercepts
y-intercept $(0, 2)$
Domain $(-\infty, \infty)$
Range $[2, \infty)$
Decreasing on $(-\infty, -2]$
Constant on $[-2, \infty)$
Absolute minimum at every point $(x, 2)$ where $x \geq -2$
No absolute maximum
Relative minimum at every point $(x, 2)$ where $x \geq -2$
Relative maximum at every point $(x, 2)$ where $x > -2$



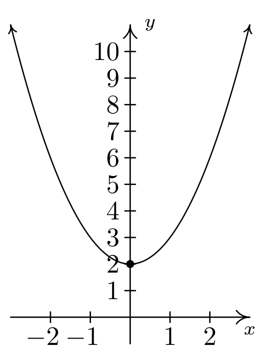
33. Re-write $f(x) = |x+4| + |x-2|$ as
$$f(x) = \begin{cases} -2x-2 & \text{if} \quad x < -4 \\ 6 & \text{if} \quad -4 \le x < 2 \\ 2x+2 & \text{if} \quad x \ge 2 \end{cases}$$
No zeros
No x-intercept
y-intercept $(0, 6)$
Domain $(-\infty, \infty)$
Range $[6, \infty)$
Decreasing on $(-\infty, -4]$
Constant on $[-4, 2]$
Increasing on $[2, \infty)$
Absolute minimum at every point $(x, 6)$ where $-4 \le x \le 2$
No absolute maximum
Relative minimum at every point $(x, 6)$ where $-4 \le x \le 2$
Relative maximum at every point $(x, 6)$ where $-4 < x < 2$



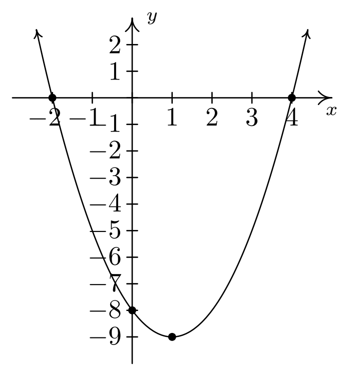
35.

Section 3.3

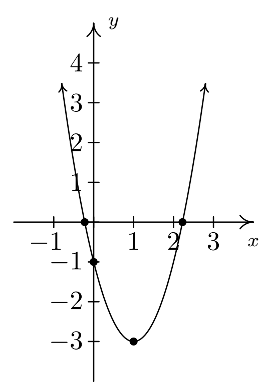
1. $f(x) = x^2 + 2$ (this is both forms!)
No x-intercepts
y-intercept $(0, 2)$
Domain: $(-\infty, \infty)$
Range: $[2, \infty)$
Decreasing on $(-\infty, 0]$
Increasing on $[0, \infty)$
Vertex $(0, 2)$ is a minimum
Axis of symmetry $x = 0$



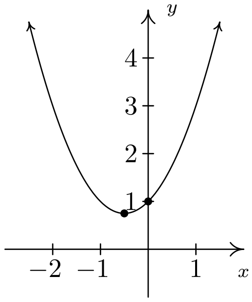
3. $f(x) = x^2 - 2x - 8 = (x-1)^2 - 9$
x-intercepts $(-2, 0)$ and $(4, 0)$
y-intercept $(0, -8)$
Domain: $(-\infty, \infty)$
Range: $[-9, \infty)$
Decreasing on $(-\infty, 1]$
Increasing on $[1, \infty)$
Vertex $(1, -9)$ is a minimum
Axis of symmetry $x = 1$



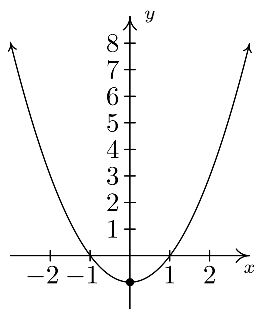
5. $f(x) = 2x^2 - 4x - 1 = 2(x-1)^2 - 3$
x-intercepts $\left(\frac{2-\sqrt{6}}{2}, 0\right)$ and $\left(\frac{2+\sqrt{6}}{2}, 0\right)$
y-intercept $(0, -1)$
Domain: $(-\infty, \infty)$
Range: $[-3, \infty)$
Increasing on $[1, \infty)$
Decreasing on $(-\infty, 1]$
Vertex $(1, -3)$ is a minimum
Axis of symmetry $x = 1$



7. $f(x) = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$
No x-intercepts
y-intercept $(0, 1)$
Domain: $(-\infty, \infty)$
Range: $\left[\frac{3}{4}, \infty\right)$
Increasing on $\left[-\frac{1}{2}, \infty\right)$
Decreasing on $\left(-\infty, -\frac{1}{2}\right]$
Vertex $\left(-\frac{1}{2}, \frac{3}{4}\right)$ is a minimum
Axis of symmetry $x = -\frac{1}{2}$



9. $f(x) = x^2 - \frac{1}{100}x - 1 = \left(x - \frac{1}{200}\right)^2 - \frac{40001}{40000}$
x-intercepts $\left(\frac{1+\sqrt{40001}}{200}\right)$ and $\left(\frac{1-\sqrt{40001}}{200}\right)$
y-intercept $(0, -1)$
Domain: $(-\infty, \infty)$
Range: $\left[-\frac{40001}{40000}, \infty\right)$
Decreasing on $\left(-\infty, \frac{1}{200}\right]$
Increasing on $\left[\frac{1}{200}, \infty\right)$
Vertex $\left(\frac{1}{200}, -\frac{40001}{40000}\right)$ is a minimum
Axis of symmetry $x = \frac{1}{200}$



Note: You'll need to plot this on a computer to zoom in far enough to see that the vertex is not the y-intercept.

11.
- $P(x) = -x^2 + 25x - 100$, for $0 \le x \le 35$
- Since the vertex occurs at $x = 12.5$, and it is impossible to make or sell 12.5 bottles of tonic, maximum profit occurs when either 12 or 13 bottles of tonic are made and sold.
- The maximum profit is $\$56$.
- The price per bottle can be either $\$23$ (to sell 12 bottles) or $\$22$ (to sell 13 bottles.) Both will result in the maximum profit.
- The break even points are $x = 5$ and $x = 20$, so to make a profit, between 5 and 20 bottles of tonic need to be made and sold.

13.
- $P(x) = -0.5x^2 + 9x - 36$, for $0 \le x \le 24$
- 9 pies should be made and sold to maximize the daily profit.
- The maximum daily profit is $\$4.50$.
- The price per pie should be set at $\$7.50$ to maximize profit.
- The break even points are $x = 6$ and $x = 12$, so to make a profit, between 6 and 12 pies need to be made and sold daily.

15. 495 cookies

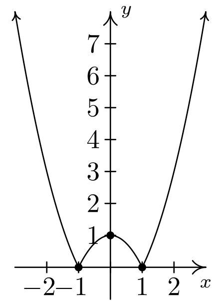
17. 64° at 2 PM (8 hours after 6 AM.)

19. 8 feet by 16 feet; maximum area is 128 square feet.

21. The largest rectangle has area 12.25 square inches.

23. The rocket reaches its maximum height of 500 feet 10 seconds after lift-off.

25. (a) The applied domain is $[0, \infty)$.
(d) The height function is this case is $s(t) = -4.9t^2 + 15t$. The vertex of this parabola is approximately $(1.53, 11.48)$ so the maximum height reached by the marble is 11.48 meters. It hits the ground again when $t \approx 3.06$ seconds.
(e) The revised height function is $s(t) = -4.9t^2 + 15t + 25$ which has zeros at $t \approx -1.20$ and $t \approx 4.26$. We ignore the negative value and claim that the marble will hit the ground after 4.26 seconds.
(f) Shooting down means the initial velocity is negative so the height functions becomes $s(t) = -4.9t^2 - 15t + 25$.

27. 

29. $D(x) = x^2 + (2x+1)^2 = 5x^2 + 4x + 1$, D is minimized when $x = -\frac{2}{5}$, so the point on $y = 2x + 1$ closest to $(0, 0)$ is $\left(-\frac{2}{5}, \frac{1}{5}\right)$

31. $x = \pm y\sqrt{10}$

33. $x = \dfrac{m \pm \sqrt{m^2 + 4}}{2}$

35. $y = 2 \pm x$

Section 3.4

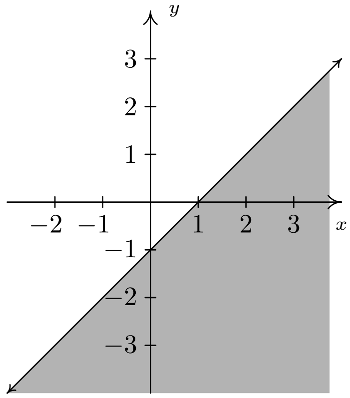
1. $\left[\frac{1}{3}, 3\right]$
3. $(-3, 2)$
5. No solution
7. $(-3, 2] \cup [6, 11)$
9. $\left[-\frac{12}{7}, -\frac{6}{5}\right]$
11. $\left(-\infty, -\frac{4}{3}\right] \cup [6, \infty)$
13. No Solution
15. $\left(1, \frac{5}{3}\right)$
17. $(-\infty, -3] \cup [1, \infty)$
19. No solution
21. $\{2\}$
23. $\left[-\frac{1}{3}, 4\right]$
25. $\left(-\infty, 1 - \frac{\sqrt{6}}{2}\right) \cup \left(1 + \frac{\sqrt{6}}{2}, \infty\right)$
27. $(-3\sqrt{2}, -\sqrt{11}] \cup [-\sqrt{7}, 0) \cup (0, \sqrt{7}] \cup [\sqrt{11}, 3\sqrt{2})$
29. $(-\infty, \infty)$
31. $[-6, -3] \cup [-2, \infty)$
33. $P(x) \ge 50$ on $[10, 15]$. This means anywhere between 10 and 15 bottles of tonic need to be sold to earn at least $\$50$ in profit.
35. $T(t) > 42$ on $(8 - 2\sqrt{11}, 8 + 2\sqrt{11}) \approx (1.37, 14.63)$, which corresponds to between 7:22 AM (1.37 hours after 6 AM) to 8:38 PM (14.63 hours after 6 AM.) However, since the model is valid only for t, $0 \le t \le 12$, we restrict our answer and find it is warmer than 42° Fahrenheit from 7:22 AM to 6 PM.
37. $s(t) = -4.9t^2 + 30t + 2$. $s(t) > 35$ on (approximately) $(1.44, 4.68)$. This means between 1.44 and 4.68 seconds after it is launched into the air, the marble is more than 35 feet off the ground.

39. $|x-2| \leq 4$, $[-2, 6]$

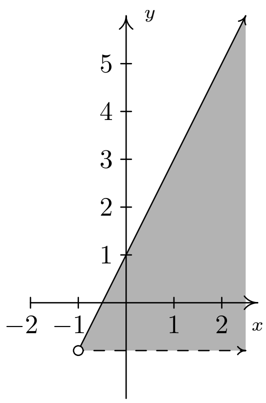
41. $|x^2-3| \leq 1$, $[-2, -\sqrt{2}] \cup [\sqrt{2}, 2]$

43. Solving $|S(x)-42| \leq 3$, and disregarding the negative solutions yields $\left[\sqrt{\frac{13}{2}}, \sqrt{\frac{15}{2}}\right] \approx [2.550, 2.739]$. The edge length must be within 2.550 and 2.739 centimetres.

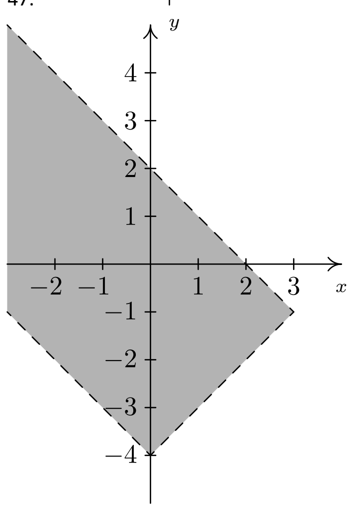
45.



47.



49.



Chapter 4

Section 4.1

1. $f(x) = 4 - x - 3x^2$
   Degree 2
   Leading term $-3x^2$
   Leading coefficient -3
   Constant term 4
   As $x \to -\infty,\ f(x) \to -\infty$
   As $x \to \infty,\ f(x) \to -\infty$

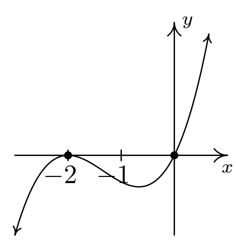
3. $q(r) = 1 - 16r^4$
   Degree 4
   Leading term $-16r^4$
   Leading coefficient -16
   Constant term 1
   As $r \to -\infty,\ q(r) \to -\infty$
   As $r \to \infty,\ q(r) \to -\infty$

5. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
   Degree 17
   Leading term $\sqrt{3}x^{17}$
   Leading coefficient $\sqrt{3}$
   Constant term $\frac{1}{3}$
   As $x \to -\infty,\ f(x) \to -\infty$
   As $x \to \infty,\ f(x) \to \infty$

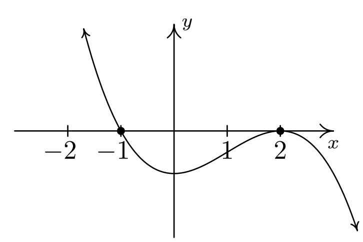
7. $P(x) = (x-1)(x-2)(x-3)(x-4)$
   Degree 4
   Leading term x^4
   Leading coefficient 1
   Constant term 24
   As $x \to -\infty,\ P(x) \to \infty$
   As $x \to \infty,\ P(x) \to \infty$

9. $f(x) = -2x^3(x+1)(x+2)^2$
   Degree 6
   Leading term $-2x^6$
   Leading coefficient -2
   Constant term 0
   As $x \to -\infty,\ f(x) \to -\infty$
   As $x \to \infty,\ f(x) \to -\infty$

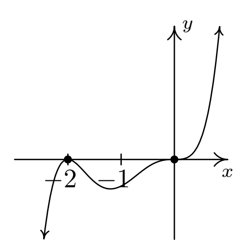
11. $a(x) = x(x+2)^2$
    $x = 0$ multiplicity 1
    $x = -2$ multiplicity 2



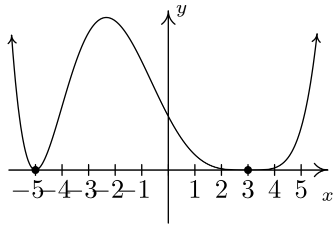
13. $f(x) = -2(x-2)^2(x+1)$
    $x = 2$ multiplicity 2
    $x = -1$ multiplicity 1



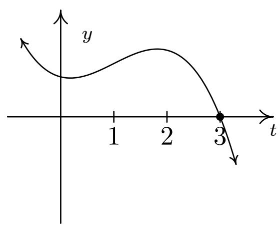
15. $F(x) = x^3(x+2)^2$
    $x = 0$ multiplicity 3
    $x = -2$ multiplicity 2



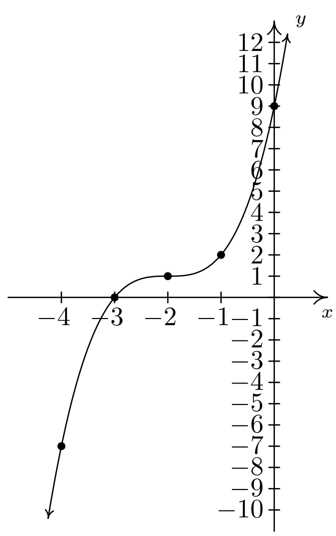
17. $Q(x) = (x+5)^2(x-3)^4$
$x = -5$ multiplicity 2
$x = 3$ multiplicity 4



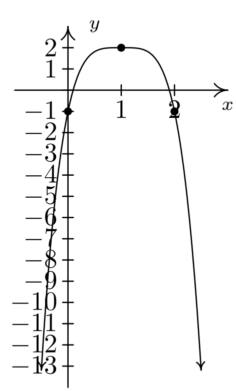
19. $H(t) = (3-t)\left(t^2+1\right)$
$x = 3$ multiplicity 1



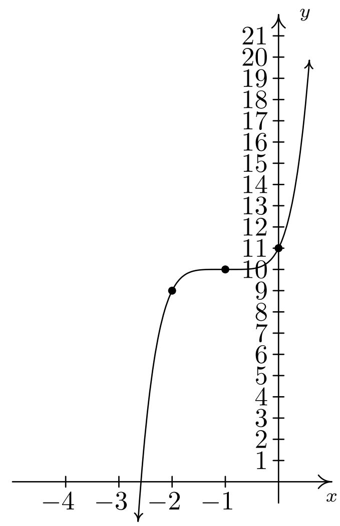
21. $g(x) = (x+2)^3 + 1$
domain: $(-\infty, \infty)$
range: $(-\infty, \infty)$



23. $g(x) = 2 - 3(x-1)^4$
domain: $(-\infty, \infty)$
range: $(-\infty, 2]$



25. $g(x) = (x+1)^5 + 10$
domain: $(-\infty, \infty)$
range: $(-\infty, \infty)$



27. We have
$f(-4) = -23,\ f(-3) = 5,\ f(0) = 5,\ f(1) = -3,\ f(2) = -5$ and $f(3) = 5$ so the Intermediate Value Theorem tells us that $f(x) = x^3 - 9x + 5$ has real zeros in the intervals $[-4, -3]$, $[0, 1]$ and $[2, 3]$.

29. The calculator gives the location of the absolute maximum (rounded to three decimal places) as $x \approx 6.305$ and $y \approx 1115.417$. Since x represents the number of TVs sold in hundreds, $x = 6.305$ corresponds to 630.5 TVs. Since we can't sell half of a TV, we compare $R(6.30) \approx 1115.415$ and $R(6.31) \approx 1115.416$, so selling 631 TVs results in a (slightly) higher revenue. Since y represents the revenue in *thousands* of dollars, the maximum revenue is $\$1{,}115{,}416$.

31. The calculator gives the location of the absolute maximum (rounded to three decimal places) as $x \approx 3.897$ and $y \approx 35.255$. Since x represents the number of TVs sold in hundreds, $x = 3.897$ corresponds to 389.7 TVs. Since we can't sell 0.7 of a TV, we compare $P(3.89) \approx 35.254$ and $P(3.90) \approx 35.255$, so selling 390 TVs results in a (slightly) higher revenue. Since y represents the revenue in *thousands* of dollars, the maximum revenue is $\$35{,}255$.

33. (a) Our ultimate goal is to maximize the volume, so we'll start with the maximum Length + Girth of 130. This means the length is $130 - 4x$. The volume of a rectangular box is always length $\times$ width $\times$ height so we get $V(x) = x^2(130 - 4x) = -4x^3 + 130x^2$.

(b) Graphing $y = V(x)$ on $[0, 33] \times [0, 21000]$ shows a maximum at $(21.67, 20342.59)$ so the dimensions of the box with maximum volume are 21.67in. $\times$ 21.67in. $\times$ 43.32in. for a volume of 20342.59in.^{3}.

(c) If we start with Length + Girth $= 108$ then the length is $108 - 4x$ and the volume is $V(x) = -4x^3 + 108x^2$. Graphing $y = V(x)$ on $[0, 27] \times [0, 11700]$ shows a maximum at $(18.00, 11664.00)$ so the dimensions of the box with maximum volume are 18.00in. $\times$ 18.00in. $\times$ 36in. for a volume of 11664.00in.^{3}. (Calculus will confirm that the measurements which maximize the volume are exactly 18in. by 18in. by 36in., however, as I'm sure you are aware by now, we treat all calculator results as approximations and list them as such.)

35.

Section 4.2

1. $4x^2 + 3x - 1 = (x-3)(4x+15) + 44$
3. $5x^4 - 3x^3 + 2x^2 - 1 = \left(x^2+4\right)\left(5x^2 - 3x - 18\right) + (12x + 71)$
5. $9x^3 + 5 = (2x-3)\left(\frac{9}{2}x^2 + \frac{27}{4}x + \frac{81}{8}\right) + \frac{283}{8}$
7. $p(4) = 29$
9. $p(-3) = -45$
11. $p(2) = 0$, $p(x) = (x-2)\left(3x^2+4\right)$
13. $p\left(\frac{3}{2}\right) = \frac{73}{16}$
15. $p(-\sqrt{7}) = 0$,
$p(x) = (x+\sqrt{7})\left(x^3 + (1-\sqrt{7})x^2 + (1-\sqrt{7})x - \sqrt{7}\right)$
17. $x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$
19. $3x^3 + 4x^2 - x - 2 = 3\left(x - \frac{2}{3}\right)(x+1)^2$
21. $x^3 + 2x^2 - 3x - 6 = (x+2)(x+\sqrt{3})(x-\sqrt{3})$
23. $4x^4 - 28x^3 + 61x^2 - 42x + 9 = 4\left(x - \frac{1}{2}\right)^2(x-3)^2$
25. $125x^5 - 275x^4 - 2265x^3 - 3213x^2 - 1728x - 324 =$
$125\left(x + \frac{3}{5}\right)^3(x+2)(x-6)$
27. $p(x) = 117(x+2)(x-2)(x+1)(x-1)$
29. $p(x) = 7(x+3)^2(x-3)(x-6)$
31. $p(x) = a(x+6)^2(x-1)(x-117)$ or
$p(x) = a(x+6)(x-1)(x-117)^2$ where a can be any negative real number

Section 4.3

1. For $f(x) = x^3 - 2x^2 - 5x + 6$
    - All of the real zeros lie in the interval $[-7, 7]$
    - Possible rational zeros are $\pm1, \pm2, \pm3, \pm6$
    - There are 2 or 0 positive real zeros; there is 1 negative real zero
3. For $f(x) = x^4 - 9x^2 - 4x + 12$
    - All of the real zeros lie in the interval $[-13, 13]$
    - Possible rational zeros are $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$
    - There are 2 or 0 positive real zeros; there are 2 or 0 negative real zeros
5. For $f(x) = x^3 - 7x^2 + x - 7$
    - All of the real zeros lie in the interval $[-8, 8]$
    - Possible rational zeros are $\pm1, \pm7$
    - There are 3 or 1 positive real zeros; there are no negative real zeros
7. For $f(x) = -17x^3 + 5x^2 + 34x - 10$
    - All of the real zeros lie in the interval $[-3, 3]$
    - Possible rational zeros are $\pm\frac{1}{17}, \pm\frac{2}{17}, \pm\frac{5}{17}, \pm\frac{10}{17}, \pm1, \pm2, \pm5, \pm10$
    - There are 2 or 0 positive real zeros; there is 1 negative real zero
9. For $f(x) = 3x^3 + 3x^2 - 11x - 10$
    - All of the real zeros lie in the interval $\left[-\frac{14}{3}, \frac{14}{3}\right]$
    - Possible rational zeros are $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}, \pm1, \pm2, \pm5, \pm10$
    - There is 1 positive real zero; there are 2 or 0 negative real zeros
11. $f(x) = x^3 - 2x^2 - 5x + 6$
$x = -2, x = 1, x = 3$ (each has mult. 1)
13. $f(x) = x^4 - 9x^2 - 4x + 12$
$x = -2$ (mult. 2), $x = 1$ (mult. 1), $x = 3$ (mult. 1)
15. $f(x) = x^3 - 7x^2 + x - 7$
$x = 7$ (mult. 1)
17. $f(x) = -17x^3 + 5x^2 + 34x - 10$
$x = \frac{5}{17}, x = \pm\sqrt{2}$ (each has mult. 1)
19. $f(x) = 3x^3 + 3x^2 - 11x - 10$
$x = -2, x = \frac{3\pm\sqrt{69}}{6}$ (each has mult. 1)
21. $f(x) = 9x^3 - 5x^2 - x$
$x = 0, x = \frac{5\pm\sqrt{61}}{18}$ (each has mult. 1)
23. $f(x) = x^4 + 2x^2 - 15$
$x = \pm\sqrt{3}$ (each has mult. 1)
25. $f(x) = 3x^4 - 14x^2 - 5$
$x = \pm\sqrt{5}$ (each has mult. 1)
27. $f(x) = x^6 - 3x^3 - 10$
$x = \sqrt[3]{-2} = -\sqrt[3]{2}, x = \sqrt[3]{5}$ (each has mult. 1)
29. $f(x) = x^5 - 2x^4 - 4x + 8$
$x = 2, x = \pm\sqrt{2}$ (each has mult. 1)
31. $f(x) = x^5 - 60x^3 - 80x^2 + 960x + 2304$
$x = -4$ (mult. 3), $x = 6$ (mult. 2)
33. $f(x) = 90x^4 - 399x^3 + 622x^2 - 399x + 90$
$x = \frac{2}{3}, x = \frac{3}{2}, x = \frac{5}{3}, x = \frac{3}{5}$ (each has mult. 1)
35. $x = 0, \frac{5\pm\sqrt{61}}{18}$
37. $x = -2, 1, 3$
39. $x = 7$
41. $x = -2, \frac{3\pm\sqrt{69}}{6}$
43. $x = \pm\sqrt{5}$
45. $\left(-\infty, \frac{1}{2}\right) \cup (4, 5)$
47. $(-\infty, -1] \cup [3, \infty)$
49. $[-2, 2]$
51. $(-\infty, -2) \cup \left(-\sqrt{2}, \sqrt{2}\right)$
53. $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$
55. $V(x) \geq 80$ on $[1, 5-\sqrt{5}] \cup [5+\sqrt{5}, \infty)$. Only the portion $[1, 5-\sqrt{5}]$ lies in the applied domain, however. In the context of the problem, this says for the volume of the box to be at least 80 cubic inches, the square removed from each corner needs to have a side length of at least 1 inch, but no more than $5 - \sqrt{5} \approx 2.76$ inches.
57.

Section 4.4

1. $f(x) = x^2 - 4x + 13 = (x - (2+3i))(x - (2-3i))$
Zeros: $x = 2 \pm 3i$
3. $f(x) = 3x^2 + 2x + 10 =$
$3\left(x - \left(-\frac{1}{3} + \frac{\sqrt{29}}{3}i\right)\right)\left(x - \left(-\frac{1}{3} - \frac{\sqrt{29}}{3}i\right)\right)$
Zeros: $x = -\frac{1}{3} \pm \frac{\sqrt{29}}{3}i$
5. $f(x) = x^3 + 6x^2 + 6x + 5 = (x+5)(x^2 + x + 1) =$
$(x+5)\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)$
Zeros: $x = -5,\ x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$
7. $f(x) = x^3 + 3x^2 + 4x + 12 = (x+3)\left(x^2+4\right) =$
$(x+3)(x+2i)(x-2i)$
Zeros: $x = -3,\ \pm 2i$

9. $f(x) = x^3 + 7x^2 + 9x - 2 =$
$(x+2)\left(x - \left(-\frac{5}{2} + \frac{\sqrt{29}}{2}\right)\right)\left(x - \left(-\frac{5}{2} - \frac{\sqrt{29}}{2}\right)\right)$
Zeros: $x = -2,\ x = -\frac{5}{2} \pm \frac{\sqrt{29}}{2}$

11. $f(x) = 4x^4 - 4x^3 + 13x^2 - 12x + 3 = \left(x - \frac{1}{2}\right)^2\left(4x^2 + 12\right) =$
$4\left(x - \frac{1}{2}\right)^2(x + i\sqrt{3})(x - i\sqrt{3})$
Zeros: $x = \frac{1}{2},\ x = \pm\sqrt{3}i$

13. $f(x) = x^4 + x^3 + 7x^2 + 9x - 18 = (x+2)(x-1)\left(x^2 + 9\right) =$
$(x+2)(x-1)(x+3i)(x-3i)$
Zeros: $x = -2,\ 1,\ \pm 3i$

15. $f(x) = -3x^4 - 8x^3 - 12x^2 - 12x - 5 = (x+1)^2\left(-3x^2 - 2x - 5\right)$
$= -3(x+1)^2\left(x - \left(-\frac{1}{3} + \frac{\sqrt{14}}{3}i\right)\right)\left(x - \left(-\frac{1}{3} - \frac{\sqrt{14}}{3}i\right)\right)$
Zeros: $x = -1,\ x = -\frac{1}{3} \pm \frac{\sqrt{14}}{3}i$

17. $f(x) = x^4 + 9x^2 + 20 = \left(x^2 + 4\right)\left(x^2 + 5\right) =$
$(x - 2i)(x + 2i)\left(x - i\sqrt{5}\right)\left(x + i\sqrt{5}\right)$
Zeros: $x = \pm 2i, \pm i\sqrt{5}$

19. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12 = (x-1)\left(x^2 + 3\right)\left(x^2 + 4\right)$
$= (x-1)(x - i\sqrt{3})(x + i\sqrt{3})(x - 2i)(x + 2i)$
Zeros: $x = 1,\ \pm\sqrt{3}i,\ \pm 2i$

21. $f(x) = x^4 - 2x^3 + 27x^2 - 2x + 26 = (x^2 - 2x + 26)(x^2 + 1) =$
$(x - (1 + 5i))(x - (1 - 5i))(x + i)(x - i)$
Zeros: $x = 1 \pm 5i,\ x = \pm i$

23. $f(x) = 42(x-1)(x+1)(x-i)(x+i)$

25. $f(x) = -3(x-2)^2(x+2)(x-7i)(x+7i)$

27. $f(x) = -2(x-2i)(x+2i)(x+2)$

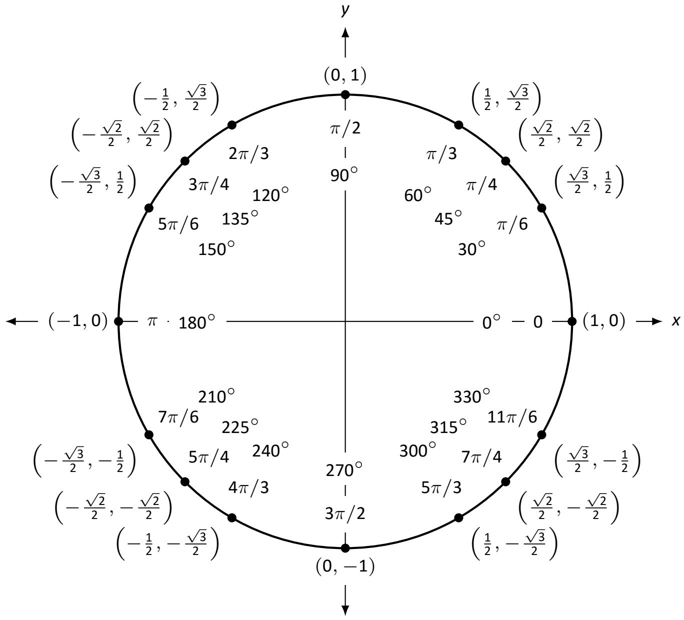
Differentiation Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$
8. $\frac{d}{dx}(x^n) = nx^{n-1}$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}(\sin x) = \cos x$
14. $\frac{d}{dx}(\cos x) = -\sin x$
15. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
16. $\frac{d}{dx}(\sec x) = \sec x \tan x$
17. $\frac{d}{dx}(\tan x) = \sec^2 x$
18. $\frac{d}{dx}(\cot x) = -\csc^2 x$
19. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$
22. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$
23. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$
25. $\frac{d}{dx}(\cosh x) = \sinh x$
26. $\frac{d}{dx}(\sinh x) = \cosh x$
27. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
29. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
30. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
31. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$
32. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$
34. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$
35. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$
36. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$

Integration Rules

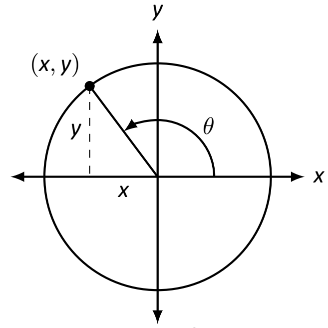
1. $\int c \cdot f(x)\,dx = c\int f(x)\,dx$
2. $\int f(x) \pm g(x)\,dx =$
   $\int f(x)\,dx \pm \int g(x)\,dx$
3. $\int 0\,dx = C$
4. $\int 1\,dx = x + C$
5. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C,\ n \neq -1$
   $n \neq -1$
6. $\int e^x\,dx = e^x + C$
7. $\int a^x\,dx = \frac{1}{\ln a} \cdot a^x + C$
8. $\int \frac{1}{x}\,dx = \ln|x| + C$
9. $\int \cos x\,dx = \sin x + C$
10. $\int \sin x\,dx = -\cos x + C$
11. $\int \tan x\,dx = -\ln|\cos x| + C$
12. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
13. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
14. $\int \cot x\,dx = \ln|\sin x| + C$
15. $\int \sec^2 x\,dx = \tan x + C$
16. $\int \csc^2 x\,dx = -\cot x + C$
17. $\int \sec x \tan x\,dx = \sec x + C$
18. $\int \csc x \cot x\,dx = -\csc x + C$
19. $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
20. $\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
21. $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
22. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
23. $\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$
24. $\int \cosh x\,dx = \sinh x + C$
25. $\int \sinh x\,dx = \cosh x + C$
26. $\int \tanh x\,dx = \ln(\cosh x) + C$
27. $\int \coth x\,dx = \ln|\sinh x| + C$
28. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$
29. $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$
30. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2}\ln\left|\frac{a+x}{a-x}\right| + C$
31. $\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$
32. $\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

The Unit Circle



Definitions of the Trigonometric Functions

Unit Circle Definition

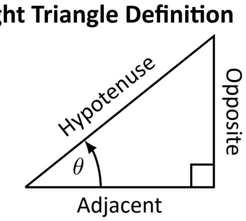


$\sin\theta = y \qquad \cos\theta = x$

$\csc\theta = \frac{1}{y} \qquad \sec\theta = \frac{1}{x}$

$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$

Right Triangle Definition



$\sin\theta = \frac{\mathrm{O}}{\mathrm{H}} \qquad \csc\theta = \frac{\mathrm{H}}{\mathrm{O}}$

$\cos\theta = \frac{\mathrm{A}}{\mathrm{H}} \qquad \sec\theta = \frac{\mathrm{H}}{\mathrm{A}}$

$\tan\theta = \frac{\mathrm{O}}{\mathrm{A}} \qquad \cot\theta = \frac{\mathrm{A}}{\mathrm{O}}$

Common Trigonometric Identities

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$$
$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$$
$$\tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$$

Double Angle Formulas

$$\sin 2x = 2\sin x\cos x$$
$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x\end{aligned}$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Sum to Product Formulas

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Power–Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$
$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Even/Odd Identities

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$
$$\csc(-x) = -\csc x$$
$$\sec(-x) = \sec x$$
$$\cot(-x) = -\cot x$$

Product to Sum Formulas

$$\sin x\sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$
$$\cos x\cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$
$$\sin x\cos y = \frac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$$

Angle Sum/Difference Formulas

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$
$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

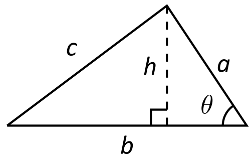
Areas and Volumes

Triangles

$h = a \sin\theta$

Area = $\frac{1}{2}bh$

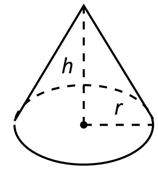
Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos\theta$



Right Circular Cone

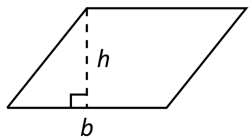
Volume = $\frac{1}{3}\pi r^2 h$

Surface Area =
$\pi r\sqrt{r^2 + h^2} + \pi r^2$



Parallelograms

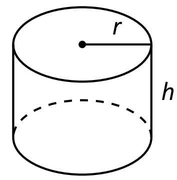
Area = bh



Right Circular Cylinder

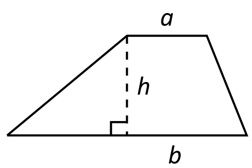
Volume = $\pi r^2 h$

Surface Area =
$2\pi rh + 2\pi r^2$



Trapezoids

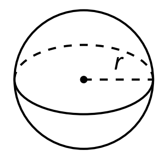
Area = $\frac{1}{2}(a + b)h$



Sphere

Volume = $\frac{4}{3}\pi r^3$

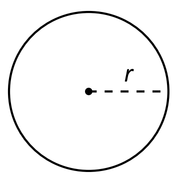
Surface Area =$4\pi r^2$



Circles

Area = πr^2

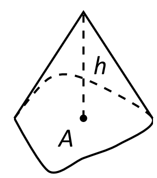
Circumference = $2\pi r$



General Cone

Area of Base = A

Volume = $\frac{1}{3}Ah$

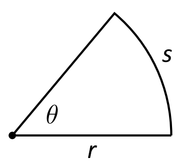


Sectors of Circles

θ in radians

Area = $\frac{1}{2}\theta r^2$

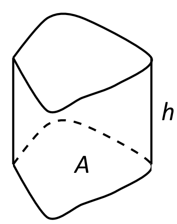
$s = r\theta$



General Right Cylinder

Area of Base = A

Volume = Ah



Algebra

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, and $0 \le b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors

$x^2 - a^2 = (x-a)(x+a)$ $\qquad$ $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x+a)(x^2 - ax + a^2)$ $\qquad$ $x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$

$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$

$(x-y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$

Binomial Theorem

$(x+y)^2 = x^2 + 2xy + y^2$ $\qquad$ $(x-y)^2 = x^2 - 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ $\qquad$ $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ $\qquad$ $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$acx^3 + adx^2 + bcx + bd = ax^2(cs + d) + b(cx + d) = (ax^2 + b)(cx + d)$

Arithmetic Operations

$$ab + ac = a(b+c) \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc} \qquad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{a-b}{c-d} = \frac{b-a}{d-c} \qquad \frac{ab + ac}{a} = b + c$$

Exponents and Radicals

$$a^0 = 1,\ \ a \ne 0 \qquad (ab)^x = a^x b^x \qquad a^x a^y = a^{x+y} \qquad \sqrt{a} = a^{1/2} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \sqrt[n]{a} = a^{1/n}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \sqrt[n]{a^m} = a^{m/n} \qquad a^{-x} = \frac{1}{a^x} \qquad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \qquad (a^x)^y = a^{xy} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Additional Formulas

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \ldots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right]$

Arc Length:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\;\,dx$$

Surface of Revolution:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\;\,dx$$

(where $f(x) \geq 0$)

$$S = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\;\,dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:

$$W = \int_a^b F(x)\;\,dx$$

Force Exerted by a Fluid:

$$F = \int_a^b w\,d(y)\,\ell(y)\;\,dy$$

Taylor Series Expansion for $f(x)$:

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	$\text{Sum} = \frac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		$\text{Sum} = \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\, dn$ is convergent	$\int_1^{\infty} a(n)\, dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} a_n/b_n > 0$	Also diverges if $\lim_{n\to\infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} (a_n)^{1/n} < 1$	$\lim_{n\to\infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} (a_n)^{1/n} = \infty$