

# MATH 1560 CALCULUS I

University of Lethbridge  
A version of the A<sup>P</sup>EX Calculus textbook edited by Sean Fitzpatrick

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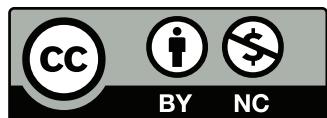
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This version of the text assembled and edited by Sean Fitzpatrick, University  
of Lethbridge, November, 2015.

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# PREFACE

This document is an attempt to provide a textbook that covers the entire curriculum (as of September 2015) for the course Math 1560 (Calculus I) at the University of Lethbridge at minimal cost to the student.

Most of this textbook is adapted from the *APEX Calculus* textbook project, which originated in the Department of Applied Mathematics at the Virginia Military Institute. (See [apexcalculus.com](http://apexcalculus.com).) On the following page you'll find the original preface from their text, which explains their project in more detail. They have produced calculus textbook that is **free** in two regards: it's free to you, the student, in the sense that you can download the PDF from their website at no cost, and do with it as you wish (share it, print it, etc.) It's also free in the sense of being an *open source* textbook: the authors have made all the files needed to produce the textbook freely available, and allow others (such as myself) to edit the text to suit the needs of various courses (such as Math 1560).

What's even better is that the textbook is of remarkably high production quality: unlike many free texts, it is polished and professionally produced, with graphics on almost every page, and a large collection of exercises (with selected answers!).

I hope that you find this textbook useful. If you find any errors, or if you have any suggestions as to how the material could be better arranged or presented, please let me know. (The beauty of the open source textbook is that it can be edited at any time!) In particular, if you find a particular topic that you think needs further explanation, or more examples, or more exercises, please let us know. My hope is that this text will be improved every time it is used for this course.

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November, 2015

# PREFACE TO APEX CALCULUS

## *A Note on Using this Text*

Thank you for reading this short preface. Allow us to share a few key points about the text so that you may better understand what you will find beyond this page.

This text is Part I of a three-text series on Calculus. The first part covers material taught in many “Calc 1” courses: limits, derivatives, and the basics of integration, found in Chapters 1 through 6.1. The second text covers material often taught in “Calc 2:” integration and its applications, along with an introduction to sequences, series and Taylor Polynomials, found in Chapters 5 through 8. The third text covers topics common in “Calc 3” or “multivariable calc:” parametric equations, polar coordinates, vector-valued functions, and functions of more than one variable, found in Chapters 9 through 13. All three are available separately for free at [www.apexcalculus.com](http://www.apexcalculus.com). These three texts are intended to work together and make one cohesive text, *APEX Calculus*, which can also be downloaded from the website.

Printing the entire text as one volume makes for a large, heavy, cumbersome book. One can certainly only print the pages they currently need, but some prefer to have a nice, bound copy of the text. Therefore this text has been split into these three manageable parts, each of which can be purchased for under \$15 at Amazon.com.

A result of this splitting is that sometimes a concept is said to be explored in a “later section,” though that section does not actually appear in this particular text. Also, the index makes reference to topics and page numbers that do not appear in this text. This is done intentionally to show the reader what topics are available for study. Downloading the .pdf of *APEX Calculus* will ensure that you have all the content.

### For Students: How to Read this Text

Mathematics textbooks have a reputation for being hard to read. High-level mathematical writing often seeks to say much with few words, and this style often seeps into texts of lower-level topics. This book was written with the goal of being easier to read than many other calculus textbooks, without becoming too verbose.

Each chapter and section starts with an introduction of the coming material, hopefully setting the stage for “why you should care,” and ends with a look ahead to see how the just-learned material helps address future problems.

*Please read the text;* it is written to explain the concepts of Calculus. There are numerous examples to demonstrate the meaning of definitions, the truth of theorems, and the application of mathematical techniques. When you encounter a sentence you don’t understand, read it again. If it still doesn’t make sense, read on anyway, as sometimes confusing sentences are explained by later sentences.

*You don’t have to read every equation.* The examples generally show “all” the steps needed to solve a problem. Sometimes reading through each step is helpful; sometimes it is confusing. When the steps are illustrating a new technique, one probably should follow each step closely to learn the new technique. When the steps are showing the mathematics needed to find a number to be used later, one can usually skip ahead and see how that number is being used, instead of getting bogged down in reading how the number was found.

*Most proofs have been omitted.* In mathematics, *proving* something is always true is extremely important, and entails much more than testing to see if it works twice. However, students often are confused by the details of a proof, or become concerned that they should have been able to construct this proof on their own. To alleviate this potential problem, we do not include the proofs to most theorems in the text. The interested reader is highly encouraged to find proofs online or from their instructor. In most cases, one is very capable of understanding what a theorem *means* and *how to apply it* without knowing fully *why* it is true.

## Interactive, 3D Graphics

New to Version 3.0 is the addition of interactive, 3D graphics in the .pdf version. Nearly all graphs of objects in space can be rotated, shifted, and zoomed in/out so the reader can better understand the object illustrated.

As of this writing, the only pdf viewers that support these 3D graphics are Adobe Reader & Acrobat (and only the versions for PC/Mac/Unix/Linux computers, not tablets or smartphones). To activate the interactive mode, click on the image. Once activated, one can click/drag to rotate the object and use the scroll wheel on a mouse to zoom in/out. (A great way to investigate an image is to first zoom in on the page of the pdf viewer so the graphic itself takes up much of the screen, then zoom inside the graphic itself.) A CTRL-click/drag pans the object left/right or up/down. By right-clicking on the graph one can access a menu of other options, such as changing the lighting scheme or perspective. One can also revert the graph back to its default view. If you wish to deactivate the interactivity, one can right-click and choose the “Disable Content” option.

## Thanks

There are many people who deserve recognition for the important role they have played in the development of this text. First, I thank Michelle for her support and encouragement, even as this “project from work” occupied my time and attention at home. Many thanks to Troy Siemers, whose most important contributions extend far beyond the sections he wrote or the 227 figures he coded in Asymptote for 3D interaction. He provided incredible support, advice and encouragement for which I am very grateful. My thanks to Brian Heinold and Dimplekumar Chalishajar for their contributions and to Jennifer Bowen for reading through so much material and providing great feedback early on. Thanks to Troy, Lee Dewald, Dan Joseph, Meagan Herald, Bill Lowe, John David, Vonda Walsh, Geoff Cox, Jessica Libertini and other faculty of VMI who have given me numerous suggestions and corrections based on their experience with teaching from the text. (Special thanks to Troy, Lee & Dan for their patience in teaching Calc III while I was still writing the Calc III material.) Thanks to Randy Cone for encouraging his tutors of VMI’s Open Math Lab to read through the text and check the solutions, and thanks to the tutors for spending their time doing so. A very special thanks to Kristi Brown and Paul Janiczek who took this opportunity far above & beyond what I expected, meticulously checking every solution and carefully reading every example. Their comments have been extraordinarily helpful. I am also thankful for the support provided by Wane Schneiter, who as my Dean provided me with extra time to work on this project. I am blessed to have so many people give of their time to make this book better.

## **A<sub>P</sub>E<sub>X</sub> – Affordable Print and Electronic teXts**

A<sub>P</sub>E<sub>X</sub> is a consortium of authors who collaborate to produce high-quality, low-cost textbooks. The current textbook-writing paradigm is facing a potential revolution as desktop publishing and electronic formats increase in popularity. However, writing a good textbook is no easy task, as the time requirements alone are substantial. It takes countless hours of work to produce text, write examples and exercises, edit and publish. Through collaboration, however, the cost to any individual can be lessened, allowing us to create texts that we freely distribute electronically and sell in printed form for an incredibly low cost. Having said that, nothing is entirely free; someone always bears some cost. This text “cost” the authors of this book their time, and that was not enough. *APEX Calculus* would not exist had not the Virginia Military Institute, through a generous Jackson–Hope grant, given the lead author significant time away from teaching so he could focus on this text.

Each text is available as a free .pdf, protected by a Creative Commons Attribution - Noncommercial 4.0 copyright. That means you can give the .pdf to anyone you like, print it in any form you like, and even edit the original content and redistribute it. If you do the latter, you must clearly reference this work and you cannot sell your edited work for money.

We encourage others to adapt this work to fit their own needs. One might add sections that are “missing” or remove sections that your students won’t need. The source files can be found at [github.com/APEXCalculus](https://github.com/APEXCalculus).

You can learn more at [www.vmi.edu/APEX](http://www.vmi.edu/APEX).

# 1: THE REAL NUMBERS

## 1.1 Some Basic Set Theory Notions

While the authors would like nothing more than to delve quickly and deeply into the sheer excitement that is *Precalculus*, experience has taught us that a brief refresher on some basic notions is welcome, if not completely necessary, at this stage. To that end, we present a brief summary of ‘set theory’ and some of the associated vocabulary and notations we use in the text. Like all good Math books, we begin with a definition.

### Definition 1    Set

A **set** is a well-defined collection of objects which are called the ‘elements’ of the set. Here, ‘well-defined’ means that it is possible to determine if something belongs to the collection or not, without prejudice.

For example, the collection of letters that make up the word “pronghorns” is well-defined and is a set, but the collection of the worst math teachers in the world is **not** well-defined, and so is **not** a set. In general, there are three ways to describe sets. They are

### Key Idea 1    Ways to Describe Sets

1. **The Verbal Method:** Use a sentence to define a set.
2. **The Roster Method:** Begin with a left brace ‘{’, list each element of the set *only once* and then end with a right brace ‘}’.
3. **The Set-Builder Method:** A combination of the verbal and roster methods using a “dummy variable” such as  $x$ .

One thing that student evaluations teach us is that any given Mathematics instructor can be simultaneously the best and worst teacher ever, depending on who is completing the evaluation.

For example, let  $S$  be the set described *verbally* as the set of letters that make up the word “pronghorns”. A **roster** description of  $S$  would be  $\{p, r, o, n, g, h, s\}$ . Note that we listed ‘r’, ‘o’, and ‘n’ only once, even though they appear twice in “pronghorns.” Also, the *order* of the elements doesn’t matter, so  $\{o, n, p, r, g, s, h\}$  is also a roster description of  $S$ . A **set-builder** description of  $S$  is:

$$\{x \mid x \text{ is a letter in the word “pronghorns”}\}$$

The way to read this is: ‘The set of elements  $x$  such that  $x$  is a letter in the word “pronghorns.”’ In each of the above cases, we may use the familiar equals sign ‘=’ and write  $S = \{p, r, o, n, g, h, s\}$  or  $S = \{x \mid x \text{ is a letter in the word “pronghorns”}\}$ . Clearly  $r$  is in  $S$  and  $q$  is not in  $S$ . We express these sentiments mathematically by writing  $r \in S$  and  $q \notin S$ .

More precisely, we have the following.

**Definition 2 Notation for set inclusion**

Let  $A$  be a set.

- If  $x$  is an element of  $A$  then we write  $x \in A$  which is read ‘ $x$  is in  $A$ ’.
- If  $x$  is *not* an element of  $A$  then we write  $x \notin A$  which is read ‘ $x$  is not in  $A$ ’.

Now let’s consider the set  $C = \{x \mid x \text{ is a consonant in the word “pronghorns”}\}$ . A roster description of  $C$  is  $C = \{p, r, n, g, h, s\}$ . Note that by construction, every element of  $C$  is also in  $S$ . We express this relationship by stating that the set  $C$  is a **subset** of the set  $S$ , which is written in symbols as  $C \subseteq S$ . The more formal definition is given below.

**Definition 3 Subset**

Given sets  $A$  and  $B$ , we say that the set  $A$  is a **subset** of the set  $B$  and write ‘ $A \subseteq B$ ’ if every element in  $A$  is also an element of  $B$ .

Note that in our example above  $C \subseteq S$ , but not vice-versa, since  $o \in S$  but  $o \notin C$ . Additionally, the set of vowels  $V = \{a, e, i, o, u\}$ , while it does have an element in common with  $S$ , is not a subset of  $S$ . (As an added note,  $S$  is not a subset of  $V$ , either.) We could, however, *build* a set which contains both  $S$  and  $V$  as subsets by gathering all of the elements in both  $S$  and  $V$  together into a single set, say  $U = \{p, r, o, n, g, h, s, a, e, i, u\}$ . Then  $S \subseteq U$  and  $V \subseteq U$ . The set  $U$  we have built is called the **union** of the sets  $S$  and  $V$  and is denoted  $S \cup V$ . Furthermore,  $S$  and  $V$  aren’t completely *different*<sup>1</sup> sets since they both contain the letter ‘o.’ The **intersection** of two sets is the set of elements (if any) the two sets have in common. In this case, the intersection of  $S$  and  $V$  is  $\{o\}$ , written  $S \cap V = \{o\}$ . We formalize these ideas below.

**Definition 4 Intersection and Union**

Suppose  $A$  and  $B$  are sets.

- The **intersection** of  $A$  and  $B$  is  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- The **union** of  $A$  and  $B$  is  $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ (or both)}\}$

The key words in Definition 4 to focus on are the conjunctions: ‘intersection’ corresponds to ‘and’ meaning the elements have to be in *both* sets to be in the intersection, whereas ‘union’ corresponds to ‘or’ meaning the elements have to be in one set, or the other set (or both). In other words, to belong to the union of two sets an element must belong to *at least one* of them.

Returning to the sets  $C$  and  $V$  above,  $C \cup V = \{p, r, n, g, h, s, a, e, i, o, u\}$ . When it comes to their intersection, however, we run into a bit of notational

<sup>1</sup>Since the word ‘different’ could be ambiguous, mathematicians use the word *disjoint* to refer to two sets that have no elements in common.

awkwardness since  $C$  and  $V$  have no elements in common. While we could write  $C \cap V = \{\}$ , this sort of thing happens often enough that we give the set with no elements a name.

### Definition 5    Empty set

The **Empty Set**  $\emptyset$  is the set which contains no elements. That is,

$$\emptyset = \{\} = \{x \mid x \neq x\}.$$

As promised, the empty set is the set containing no elements since no matter what ‘ $x$ ’ is, ‘ $x = x$ ’. Like the number ‘0’, the empty set plays a vital role in mathematics. We introduce it here more as a symbol of convenience as opposed to a contrivance. Using this new bit of notation, we have for the sets  $C$  and  $V$  above that  $C \cap V = \emptyset$ . A nice way to visualize relationships between sets and set operations is to draw a **Venn Diagram**. A Venn Diagram for the sets  $S$ ,  $C$  and  $V$  is drawn in Figure 1.1.

In Figure 1.1 we have three circles - one for each of the sets  $C$ ,  $S$  and  $V$ . We visualize the area enclosed by each of these circles as the elements of each set. Here, we’ve spelled out the elements for definitiveness. Notice that the circle representing the set  $C$  is completely inside the circle representing  $S$ . This is a geometric way of showing that  $C \subseteq S$ . Also, notice that the circles representing  $S$  and  $V$  overlap on the letter ‘o’. This common region is how we visualize  $S \cap V$ . Notice that since  $C \cap V = \emptyset$ , the circles which represent  $C$  and  $V$  have no overlap whatsoever.

All of these circles lie in a rectangle labelled  $U$  (for ‘universal’ set). A universal set contains all of the elements under discussion, so it could always be taken as the union of all of the sets in question, or an even larger set. In this case, we could take  $U = S \cup V$  or  $U$  as the set of letters in the entire alphabet. The usual triptych of Venn Diagrams indicating generic sets  $A$  and  $B$  along with  $A \cap B$  and  $A \cup B$  is given below.

(The reader may well wonder if there is an ultimate universal set which contains *everything*. The short answer is ‘no’. Our definition of a set turns out to be overly simplistic, but correcting this takes us well beyond the confines of this course. If you want the longer answer, you can begin by reading about [Russell’s Paradox](#) on Wikipedia.)

### 1.1.1 Sets of Real Numbers

The playground for most of this text is the set of **Real Numbers**. Many quantities in the ‘real world’ can be quantified using real numbers: the temperature at a given time, the revenue generated by selling a certain number of products and the maximum population of Sasquatch which can inhabit a particular region are just three basic examples. A succinct, but nonetheless incomplete definition of a real number is given below.

### Definition 6    The real numbers

A **real number** is any number which possesses a decimal representation. The set of real numbers is denoted by the character  $\mathbb{R}$ .

The full extent of the empty set’s role will not be explored in this text, but it is of fundamental importance in Set Theory. In fact, the empty set can be used to generate numbers - mathematicians can create something from nothing! If you’re interested, read about the von Neumann construction of the natural numbers or consider signing up for Math 2000.

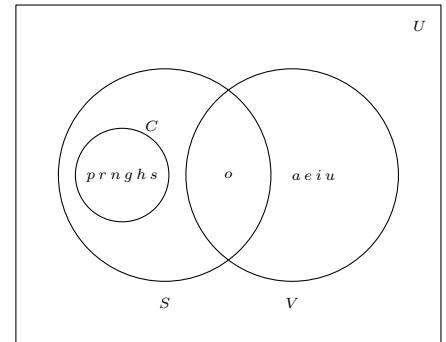
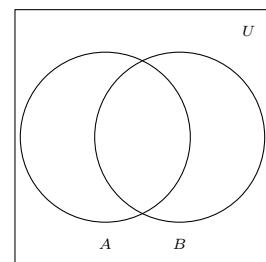
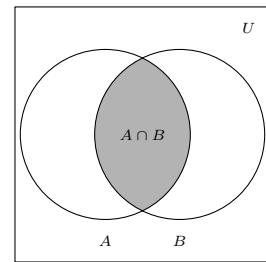


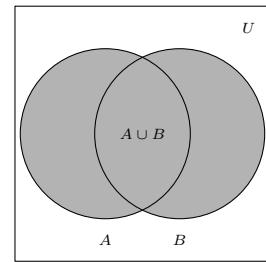
Figure 1.1: A Venn diagram for  $C$ ,  $S$ , and  $V$



Sets  $A$  and  $B$ .



$A \cup B$  is shaded.



$A \cap B$  is shaded.

Figure 1.2: Venn diagrams for intersection and union

Certain subsets of the real numbers are worthy of note and are listed below. In more advanced courses like Analysis, you learn that the real numbers can be *constructed* from the rational numbers, which in turn can be constructed from the integers (which themselves come from the natural numbers, which in turn can be defined as sets...).

### Definition 7 Sets of Numbers

An example of a number with a repeating decimal expansion is  $a = 2.13234234234\dots$ . This is rational since  $100a = 213.234234234\dots$ , and  $100000a = 213234.234234\dots$  so  $99900a = 100000a - 100a = 213021$ . This gives us the rational expression  $a = \frac{213021}{99900}$ .

The classic example of an irrational number is the number  $\pi$  (See Section ??), but numbers like  $\sqrt{2}$  and  $0.101001000100001\dots$  are other fine representatives.

1. The **Empty Set**:  $\emptyset = \{\} = \{x \mid x \neq x\}$ . This is the set with no elements. Like the number '0', it plays a vital role in mathematics.
2. The **Natural Numbers**:  $\mathbb{N} = \{1, 2, 3, \dots\}$  The periods of ellipsis here indicate that the natural numbers contain 1, 2, 3, 'and so forth'.
3. The **Integers**:  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
4. The **Rational Numbers**:  $\mathbb{Q} = \left\{ \frac{a}{b} \mid a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \right\}$ . Rational numbers are the ratios of integers (provided the denominator is not zero!) It turns out that another way to describe the rational numbers is:  

$$\mathbb{Q} = \{x \mid x \text{ possesses a repeating or terminating decimal representation.}\}$$
5. The **Real Numbers**:  $\mathbb{R} = \{x \mid x \text{ possesses a decimal representation.}\}$
6. The **Irrational Numbers**:  $\mathbb{P} = \{x \mid x \text{ is a non-rational real number.}\}$  Said another way, an irrational number is a decimal which neither repeats nor terminates.
7. The **Complex Numbers**:  $\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}$  (We will not deal with complex numbers in Math 1010, although they usually make an appearance in Math 1410.)

It is important to note that every natural number is a whole number is an integer. Each integer is a rational number (take  $b = 1$  in the above definition for  $\mathbb{Q}$ ) and the rational numbers are all real numbers, since they possess decimal representations (via long division!). If we take  $b = 0$  in the above definition of  $\mathbb{C}$ , we see that every real number is a complex number. In this sense, the sets  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$  are 'nested' like Matryoshka dolls. More formally, these sets form a subset chain:  $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$ . The reader is encouraged to sketch a Venn Diagram depicting  $\mathbb{R}$  and all of the subsets mentioned above. It is time for an example.

### Example 1 Sets of real numbers

1. Write a roster description for  $P = \{2^n \mid n \in \mathbb{N}\}$  and  $E = \{2n \mid n \in \mathbb{Z}\}$ .
2. Write a verbal description for  $S = \{x^2 \mid x \in \mathbb{R}\}$ .
3. Let  $A = \{-117, \frac{4}{5}, 0.\overline{202002}, 0.202002000200002\dots\}$ .  
 Which elements of  $A$  are natural numbers? Rational numbers? Real numbers?

### SOLUTION

1. To find a roster description for these sets, we need to list their elements. Starting with  $P = \{2^n \mid n \in \mathbb{N}\}$ , we substitute natural number values  $n$  into the formula  $2^n$ . For  $n = 1$  we get  $2^1 = 2$ , for  $n = 2$  we get  $2^2 = 4$ , for  $n = 3$  we get  $2^3 = 8$  and for  $n = 4$  we get  $2^4 = 16$ . Hence  $P$  describes the powers of 2, so a roster description for  $P$  is  $P = \{2, 4, 8, 16, \dots\}$  where the ‘ $\dots$ ’ indicates the pattern continues.

Proceeding in the same way, we generate elements in  $E = \{2n \mid n \in \mathbb{Z}\}$  by plugging in integer values of  $n$  into the formula  $2n$ . Starting with  $n = 0$  we obtain  $2(0) = 0$ . For  $n = 1$  we get  $2(1) = 2$ , for  $n = -1$  we get  $2(-1) = -2$  for  $n = 2$ , we get  $2(2) = 4$  and for  $n = -2$  we get  $2(-2) = -4$ . As  $n$  moves through the integers,  $2n$  produces all of the *even* integers. A roster description for  $E$  is  $E = \{0, \pm 2, \pm 4, \dots\}$ .

2. One way to verbally describe  $S$  is to say that  $S$  is the ‘set of all squares of real numbers’. While this isn’t incorrect, we’d like to take this opportunity to delve a little deeper. What makes the set  $S = \{x^2 \mid x \in \mathbb{R}\}$  a little trickier to wrangle than the sets  $P$  or  $E$  above is that the dummy variable here,  $x$ , runs through all *real* numbers. Unlike the natural numbers or the integers, the real numbers cannot be listed in any methodical way. Nevertheless, we can select some real numbers, square them and get a sense of what kind of numbers lie in  $S$ . For  $x = -2$ ,  $x^2 = (-2)^2 = 4$  so 4 is in  $S$ , as are  $(\frac{3}{2})^2 = \frac{9}{4}$  and  $(\sqrt{117})^2 = 117$ . Even things like  $(-\pi)^2$  and  $(0.101001000100001\dots)^2$  are in  $S$ .

So suppose  $s \in S$ . What can be said about  $s$ ? We know there is some real number  $x$  so that  $s = x^2$ . Since  $x^2 \geq 0$  for any real number  $x$ , we know  $s \geq 0$ . This tells us that everything in  $S$  is a non-negative real number. This begs the question: are all of the non-negative real numbers in  $S$ ? Suppose  $n$  is a non-negative real number, that is,  $n \geq 0$ . If  $n$  were in  $S$ , there would be a real number  $x$  so that  $x^2 = n$ . As you may recall, we can solve  $x^2 = n$  by ‘extracting square roots’:  $x = \pm\sqrt{n}$ . Since  $n \geq 0$ ,  $\sqrt{n}$  is a real number. Moreover,  $(\sqrt{n})^2 = n$  so  $n$  is the square of a real number which means  $n \in S$ . Hence,  $S$  is the set of non-negative real numbers.

3. The set  $A$  contains no natural numbers. Clearly,  $\frac{4}{5}$  is a rational number as is  $-117$  (which can be written as  $\frac{-117}{1}$ ). It’s the last two numbers listed in  $A$ ,  $0.\overline{202002}$  and  $0.2020020002\dots$ , that warrant some discussion. First, recall that the ‘line’ over the digits 2002 in  $0.20\overline{2002}$  (called the vinculum) indicates that these digits repeat, so it is a rational number. As for the number  $0.2020020002\dots$ , the ‘ $\dots$ ’ indicates the pattern of adding an extra ‘0’ followed by a ‘2’ is what defines this real number. Despite the fact there is a *pattern* to this decimal, this decimal is *not repeating*, so it is not a rational number - it is, in fact, an irrational number. All of the elements of  $A$  are real numbers, since all of them can be expressed as decimals (remember that  $\frac{4}{5} = 0.8$ ).  $\square$
4. The set  $\mathbb{N} \cup \mathbb{Q} = \{x \mid x \in \mathbb{N} \text{ or } x \in \mathbb{Q}\}$  is the union of the set of natural numbers with the set of rational numbers. Since every natural number is a rational number,  $\mathbb{N}$  doesn’t contribute any new elements to  $\mathbb{Q}$ , so  $\mathbb{N} \cup \mathbb{Q} = \mathbb{Q}$ . For the set  $\mathbb{Q} \cup \mathbb{P}$ , we note that every real number is either rational or not, hence  $\mathbb{Q} \cup \mathbb{P} = \mathbb{R}$ , pretty much by the definition of the set  $\mathbb{P}$ .

This isn’t the most *precise* way to describe this set - it’s always dangerous to use ‘ $\dots$ ’ since we assume that the pattern is clearly demonstrated and thus made evident to the reader. Formulas are more precise because the pattern is clear.

It shouldn’t be too surprising that  $E$  is the set of all even integers, since an even integer is *defined* to be an integer multiple of 2.

The fact that the real numbers cannot be listed is a nontrivial statement. Interested readers are directed to a discussion of [Cantor’s Diagonal Argument](#).

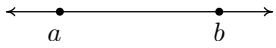


Figure 1.3: The real number line with two numbers  $a$  and  $b$ , where  $a < b$ .

As you may recall, we often visualize the set of real numbers  $\mathbb{R}$  as a line where each point on the line corresponds to one and only one real number. Given two different real numbers  $a$  and  $b$ , we write  $a < b$  if  $a$  is located to the left of  $b$  on the number line, as shown in Figure 1.3.

While this notion seems innocuous, it is worth pointing out that this convention is rooted in two deep properties of real numbers. The first property is that  $\mathbb{R}$  is complete. This means that there are no ‘holes’ or ‘gaps’ in the real number line.<sup>2</sup> Another way to think about this is that if you choose any two distinct (different) real numbers, and look between them, you’ll find a solid line segment (or interval) consisting of infinitely many real numbers.

The next result tells us what types of numbers we can expect to find.

### Theorem 1 Density Property of $\mathbb{Q}$ and $\mathbb{P}$ in $\mathbb{R}$

Between any two distinct real numbers, there is at least one rational number and irrational number. It then follows that between any two distinct real numbers there will be infinitely many rational and irrational numbers.

The Law of Trichotomy, strictly speaking, is an *axiom* of the real numbers: a basic requirement that we assume to be true. However, in any *construction* of the real, such as the method of Dedekind cuts, it is necessary to *prove* that the Law of Trichotomy is satisfied.

The root word ‘dense’ here communicates the idea that rationals and irrationals are ‘thoroughly mixed’ into  $\mathbb{R}$ . The reader is encouraged to think about how one would find both a rational and an irrational number between, say, 0.9999 and 1. Once you’ve done that, ask yourself whether there is any difference between the numbers  $0.\bar{9}$  and 1.

The second property  $\mathbb{R}$  possesses that lets us view it as a line is that the set is totally ordered. This means that given any two real numbers  $a$  and  $b$ , either  $a < b$ ,  $a > b$  or  $a = b$  which allows us to arrange the numbers from least (left) to greatest (right). You may have heard this property given as the ‘Law of Trichotomy’.

### Definition 8 Law of Trichotomy

If  $a$  and  $b$  are real numbers then **exactly one** of the following statements is true:

$$a < b$$

$$a > b$$

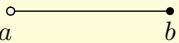
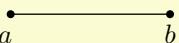
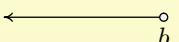
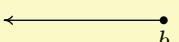
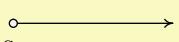
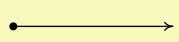
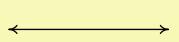
$$a = b$$

Segments of the real number line are called **intervals** of numbers. Below is a summary of the so-called **interval notation** associated with given sets of numbers. For intervals with finite endpoints, we list the left endpoint, then the right endpoint. We use square brackets, ‘[’ or ‘]’, if the endpoint is included in the interval and use a filled-in or ‘closed’ dot to indicate membership in the interval. Otherwise, we use parentheses, ‘(’ or ‘)’ and an ‘open’ circle to indicate that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbols  $-\infty$  to indicate that the interval extends indefinitely to the left and  $\infty$  to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use an appropriate arrow to indicate that the interval extends indefinitely in one (or both) directions.

<sup>2</sup>Alas, this intuitive feel for what it means to be ‘complete’ is as good as it gets at this level. Completeness does get a much more precise meaning later in courses like Analysis and Topology.

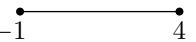
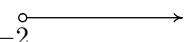
**Definition 9 Interval Notation**

Let  $a$  and  $b$  be real numbers with  $a < b$ .

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x   a < x < b\}$	$(a, b)$	
$\{x   a \leq x < b\}$	$[a, b)$	
$\{x   a < x \leq b\}$	$(a, b]$	
$\{x   a \leq x \leq b\}$	$[a, b]$	
$\{x   x < b\}$	$(-\infty, b)$	
$\{x   x \leq b\}$	$(-\infty, b]$	
$\{x   x > a\}$	$(a, \infty)$	
$\{x   x \geq a\}$	$[a, \infty)$	
$\mathbb{R}$	$(-\infty, \infty)$	

As you can glean from the table, for intervals with finite endpoints we start by writing ‘left endpoint, right endpoint’. We use square brackets, ‘[’ or ‘]’, if the endpoint is included in the interval. This corresponds to a ‘filled-in’ or ‘closed’ dot on the number line to indicate that the number is included in the set. Otherwise, we use parentheses, ‘(’ or ‘)’ that correspond to an ‘open’ circle which indicates that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbol  $-\infty$  to indicate that the interval extends indefinitely to the left and the symbol  $\infty$  to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use the appropriate arrow to indicate that the interval extends indefinitely in one or both directions.

Let’s do a few examples to make sure we have the hang of the notation:

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x   1 \leq x < 3\}$	$[1, 3)$	
$\{x   -1 \leq x \leq 4\}$	$[-1, 4]$	
$\{x   x \leq 5\}$	$(-\infty, 5]$	
$\{x   x > -2\}$	$(-2, \infty)$	

The importance of understanding interval notation in Calculus cannot be overstated. If you don’t find yourself getting the hang of it through repeated use, you may need to take the time to just memorize this chart.

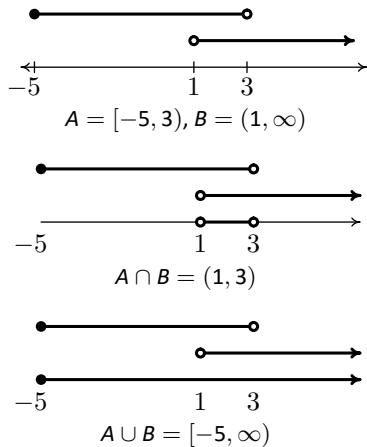


Figure 1.4: Union and intersection of intervals

We defined the intersection and union of arbitrary sets in Definition 4. Recall that the union of two sets consists of the totality of the elements in each of the sets, collected together. For example, if  $A = \{1, 2, 3\}$  and  $B = \{2, 4, 6\}$ , then  $A \cap B = \{2\}$  and  $A \cup B = \{1, 2, 3, 4, 6\}$ . If  $A = [-5, 3]$  and  $B = (1, \infty)$ , then we can find  $A \cap B$  and  $A \cup B$  graphically. To find  $A \cap B$ , we shade the overlap of the two and obtain  $A \cap B = (1, 3)$ . To find  $A \cup B$ , we shade each of  $A$  and  $B$  and describe the resulting shaded region to find  $A \cup B = [-5, \infty)$ .

While both intersection and union are important, we have more occasion to use union in this text than intersection, simply because most of the sets of real numbers we will be working with are either intervals or are unions of intervals, as the following example illustrates.

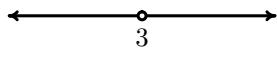
### Example 2 Expressing sets as unions of intervals

Express the following sets of numbers using interval notation.

1.  $\{x | x \leq -2 \text{ or } x \geq 2\}$
2.  $\{x | x \neq 3\}$
3.  $\{x | x \neq \pm 3\}$
4.  $\{x | -1 < x \leq 3 \text{ or } x = 5\}$

#### SOLUTION

1. The best way to proceed here is to graph the set of numbers on the number line and glean the answer from it. The inequality  $x \leq -2$  corresponds to the interval  $(-\infty, -2]$  and the inequality  $x \geq 2$  corresponds to the interval  $[2, \infty)$ . Since we are looking to describe the real numbers  $x$  in one of these *or* the other, we have  $\{x | x \leq -2 \text{ or } x \geq 2\} = (-\infty, -2] \cup [2, \infty)$ .
2. For the set  $\{x | x \neq 3\}$ , we shade the entire real number line except  $x = 3$ , where we leave an open circle. This divides the real number line into two intervals,  $(-\infty, 3)$  and  $(3, \infty)$ . Since the values of  $x$  could be in either one of these intervals *or* the other, we have that  $\{x | x \neq 3\} = (-\infty, 3) \cup (3, \infty)$ .
3. For the set  $\{x | x \neq \pm 3\}$ , we proceed as before and exclude both  $x = 3$  and  $x = -3$  from our set. This breaks the number line into *three* intervals,  $(-\infty, -3)$ ,  $(-3, 3)$  and  $(3, \infty)$ . Since the set describes real numbers which come from the first, second *or* third interval, we have  $\{x | x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ .
4. Graphing the set  $\{x | -1 < x \leq 3 \text{ or } x = 5\}$ , we get one interval,  $(-1, 3]$  along with a single number, or point,  $\{5\}$ . While we *could* express the latter as  $[5, 5]$  (Can you see why?), we choose to write our answer as  $\{x | -1 < x \leq 3 \text{ or } x = 5\} = (-1, 3] \cup \{5\}$ .

Figure 1.5: The set  $(-\infty, -2] \cup [2, \infty)$ Figure 1.6: The set  $(-\infty, 3) \cup (3, \infty)$ Figure 1.7: The set  $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ Figure 1.8: The set  $(-1, 3] \cup \{5\}$

# Exercises 1.1

## Problems

1. Fill in the chart below:

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$		
	$[0, 3)$	
		
$\{x \mid -5 < x \leq 0\}$		
	$(-3, 3)$	
		
$\{x \mid x \leq 3\}$		
	$(-\infty, 9)$	
		
$\{x \mid x \geq -3\}$		

In Exercises 2 – 7, find the indicated intersection or union and simplify if possible. Express your answers in interval notation.

2.  $(-1, 5] \cap [0, 8)$

3.  $(-1, 1) \cup [0, 6]$

4.  $(-\infty, 4] \cap (0, \infty)$

5.  $(-\infty, 0) \cap [1, 5]$

6.  $(-\infty, 0) \cup [1, 5]$

7.  $(-\infty, 5] \cap [5, 8)$

In Exercises 8 – 19, write the set using interval notation.

8.  $\{x \mid x \neq 5\}$

9.  $\{x \mid x \neq -1\}$

10.  $\{x \mid x \neq -3, 4\}$

11.  $\{x \mid x \neq 0, 2\}$

12.  $\{x \mid x \neq 2, -2\}$

13.  $\{x \mid x \neq 0, \pm 4\}$

14.  $\{x \mid x \leq -1 \text{ or } x \geq 1\}$

15.  $\{x \mid x < 3 \text{ or } x \geq 2\}$

16.  $\{x \mid x \leq -3 \text{ or } x > 0\}$

17.  $\{x \mid x \leq 5 \text{ or } x = 6\}$

18.  $\{x \mid x > 2 \text{ or } x = \pm 1\}$

19.  $\{x \mid -3 < x < 3 \text{ or } x = 4\}$

## 1.2 Real Number Arithmetic

In this section we list the properties of real number arithmetic. This is meant to be a succinct, targeted review so we'll resist the temptation to wax poetic about these axioms and their subtleties and refer the interested reader to a more formal course in Abstract Algebra. There are two (primary) operations one can perform with real numbers: addition and multiplication.

### Definition 10 Properties of Real Number Addition

- **Closure:** For all real numbers  $a$  and  $b$ ,  $a + b$  is also a real number.
- **Commutativity:** For all real numbers  $a$  and  $b$ ,  $a + b = b + a$ .
- **Associativity:** For all real numbers  $a$ ,  $b$  and  $c$ ,  $a + (b + c) = (a + b) + c$ .
- **Identity:** There is a real number '0' so that for all real numbers  $a$ ,  $a + 0 = a$ .
- **Inverse:** For all real numbers  $a$ , there is a real number  $-a$  such that  $a + (-a) = 0$ .
- **Definition of Subtraction:** For all real numbers  $a$  and  $b$ ,  $a - b = a + (-b)$ .

Next, we give real number multiplication a similar treatment. Recall that we may denote the product of two real numbers  $a$  and  $b$  a variety of ways:  $ab$ ,  $a \cdot b$ ,  $a(b)$ ,  $(a)b$  and so on. We'll refrain from using  $a \times b$  for real number multiplication in this text.

### Definition 11 Properties of Real Number Multiplication

- **Closure:** For all real numbers  $a$  and  $b$ ,  $ab$  is also a real number.
- **Commutativity:** For all real numbers  $a$  and  $b$ ,  $ab = ba$ .
- **Associativity:** For all real numbers  $a$ ,  $b$  and  $c$ ,  $a(bc) = (ab)c$ .
- **Identity:** There is a real number '1' so that for all real numbers  $a$ ,  $a \cdot 1 = a$ .
- **Inverse:** For all real numbers  $a \neq 0$ , there is a real number  $\frac{1}{a}$  such that  $a \left( \frac{1}{a} \right) = 1$ .
- **Definition of Division:** For all real numbers  $a$  and  $b \neq 0$ ,  $a \div b = \frac{a}{b} = a \left( \frac{1}{b} \right)$ .

While most students (and some faculty) tend to skip over these properties or give them a cursory glance at best, it is important to realize that the prop-

erties stated above are what drive the symbolic manipulation for all of Algebra. When listing a tally of more than two numbers,  $1 + 2 + 3$  for example, we don't need to specify the order in which those numbers are added. Notice though, try as we might, we can add only two numbers at a time and it is the associative property of addition which assures us that we could organize this sum as  $(1 + 2) + 3$  or  $1 + (2 + 3)$ . This brings up a note about 'grouping symbols'. Recall that parentheses and brackets are used in order to specify which operations are to be performed first. In the absence of such grouping symbols, multiplication (and hence division) is given priority over addition (and hence subtraction). For example,  $1 + 2 \cdot 3 = 1 + 6 = 7$ , but  $(1 + 2) \cdot 3 = 3 \cdot 3 = 9$ . As you may recall, we can 'distribute' the 3 across the addition if we really wanted to do the multiplication first:  $(1 + 2) \cdot 3 = 1 \cdot 3 + 2 \cdot 3 = 3 + 6 = 9$ . More generally, we have the following.

### Definition 12 The Distributive Property and Factoring

For all real numbers  $a$ ,  $b$  and  $c$ :

- **Distributive Property:**  $a(b + c) = ab + ac$  and  $(a + b)c = ac + bc$ .
- **Factoring:**  $ab + ac = a(b + c)$  and  $ac + bc = (a + b)c$ .

**Warning:** A common source of errors for beginning students is the misuse (that is, lack of use) of parentheses. When in doubt, more is better than less: redundant parentheses add clutter, but do not change meaning, whereas writing  $2x + 1$  when you meant to write  $2(x + 1)$  is almost guaranteed to cause you to make a mistake. (Even if you're able to proceed correctly in spite of your lack of proper notation, this is the sort of thing that will get you on your grader's bad side, so it's probably best to avoid the problem in the first place.)

It is worth pointing out that we didn't really need to list the Distributive Property both for  $a(b + c)$  (distributing from the left) and  $(a + b)c$  (distributing from the right), since the commutative property of multiplication gives us one from the other. Also, 'factoring' really is the same equation as the distributive property, just read from right to left. These are the first of many redundancies in this section, and they exist in this review section for one reason only - in our experience, many students see these things differently so we will list them as such.

It is hard to overstate the importance of the Distributive Property. For example, in the expression  $5(2 + x)$ , without knowing the value of  $x$ , we cannot perform the addition inside the parentheses first; we must rely on the distributive property here to get  $5(2 + x) = 5 \cdot 2 + 5 \cdot x = 10 + 5x$ . The Distributive Property is also responsible for combining 'like terms'. Why is  $3x + 2x = 5x$ ? Because  $3x + 2x = (3 + 2)x = 5x$ .

We continue our review with summaries of other properties of arithmetic, each of which can be derived from the properties listed above. First up are properties of the additive identity 0.

The Zero Product Property drives most of the equation solving algorithms in Algebra because it allows us to take complicated equations and reduce them to simpler ones. For example, you may recall that one way to solve  $x^2 + x - 6 = 0$  is by factoring the left hand side of this equation to get  $(x-2)(x+3) = 0$ . From here, we apply the Zero Product Property and set each factor equal to zero. This yields  $x-2 = 0$  or  $x+3 = 0$  so  $x = 2$  or  $x = -3$ . This application to solving equations leads, in turn, to some deep and profound structure theorems in Chapter ??.

The expression  $\frac{0}{0}$  is technically an ‘indeterminate form’ as opposed to being strictly ‘undefined’ meaning that with Calculus we can make some sense of it in certain situations. We’ll talk more about this in Chapter ??.

It’s always worth remembering that division is the same as multiplication by the reciprocal. You’d be surprised how often this comes in handy.

**Note:** A common denominator is **not** required to **multiply** or **divide** fractions!

**Note:** A common denominator is required to **add** or **subtract** fractions!

**Note:** The *only* way to change the denominator is to multiply both it and the numerator by the same nonzero value because we are, in essence, multiplying the fraction by 1.

We reduce fractions by ‘cancelling’ common factors - this is really just reading the previous property ‘from right to left’. **Caution:** We may only cancel common **factors** from both numerator and denominator.

### Theorem 2 Properties of Zero

Suppose  $a$  and  $b$  are real numbers.

- **Zero Product Property:**  $ab = 0$  if and only if  $a = 0$  or  $b = 0$  (or both)

**Note:** This not only says that  $0 \cdot a = 0$  for any real number  $a$ , it also says that the *only* way to get an answer of ‘0’ when multiplying two real numbers is to have one (or both) of the numbers be ‘0’ in the first place.

- **Zeros in Fractions:** If  $a \neq 0$ ,  $\frac{0}{a} = 0 \cdot \left(\frac{1}{a}\right) = 0$ .

**Note:** The quantity  $\frac{a}{0}$  is undefined.

We now continue with a review of arithmetic with fractions.

### Key Idea 2 Properties of Fractions

Suppose  $a, b, c$  and  $d$  are real numbers. Assume them to be nonzero whenever necessary; for example, when they appear in a denominator.

- **Identity Properties:**  $a = \frac{a}{1}$  and  $\frac{a}{a} = 1$ .

- **Fraction Equality:**  $\frac{a}{b} = \frac{c}{d}$  if and only if  $ad = bc$ .

- **Multiplication of Fractions:**  $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ . In particular:  $\frac{a}{b} \cdot c = \frac{a}{b} \cdot \frac{c}{1} = \frac{ac}{b}$

- **Division of Fractions:**  $\frac{a}{b} / \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$ .

In particular:  $1 / \frac{a}{b} = \frac{b}{a}$  and  $\frac{a}{b} / c = \frac{a}{b} / \frac{c}{1} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$

- **Addition and Subtraction of Fractions:**  $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$ .

- **Equivalent Fractions:**  $\frac{a}{b} = \frac{ad}{bd}$ , since  $\frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{d}{d} = \frac{ad}{bd}$

- **‘Reducing’ Fractions:**  $\frac{a\cancel{d}}{b\cancel{d}} = \frac{a}{b}$ , since  $\frac{ad}{bd} = \frac{a}{b} \cdot \frac{d}{d} = \frac{a}{b} \cdot 1 = \frac{a}{b}$ .

In particular,  $\frac{ab}{b} = a$  since  $\frac{ab}{b} = \frac{ab}{1 \cdot b} = \frac{a\cancel{b}}{1 \cdot \cancel{b}} = \frac{a}{1} = a$  and  $\frac{b-a}{a-b} = \frac{(-1)(a-b)}{(a-b)} = -1$ .

Next up is a review of the arithmetic of ‘negatives’. On page 10 we first introduced the dash which we all recognize as the ‘negative’ symbol in terms of the additive inverse. For example, the number  $-3$  (read ‘negative 3’) is defined

so that  $3 + (-3) = 0$ . We then defined subtraction using the concept of the additive inverse again so that, for example,  $5 - 3 = 5 + (-3)$ .

### Key Idea 3 Properties of Negatives

Given real numbers  $a$  and  $b$  we have the following.

- **Additive Inverse Properties:**  $-a = (-1)a$  and  $-(-a) = a$
- **Products of Negatives:**  $(-a)(-b) = ab$ .
- **Negatives and Products:**  $-ab = -(ab) = (-a)b = a(-b)$ .
- **Negatives and Fractions:** If  $b$  is nonzero,  $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$  and  $\frac{-a}{-b} = \frac{a}{b}$ .
- **'Distributing' Negatives:**  $-(a + b) = -a - b$  and  $-(a - b) = -a + b = b - a$ .
- **'Factoring' Negatives:**  $-a - b = -(a + b)$  and  $b - a = -(a - b)$ .

An important point here is that when we ‘distribute’ negatives, we do so across addition or subtraction only. This is because we are really distributing a factor of  $-1$  across each of these terms:  $-(a + b) = (-1)(a + b) = (-1)(a) + (-1)(b) = (-a) + (-b) = -a - b$ . Negatives do not ‘distribute’ across multiplication:  $-(2 \cdot 3) \neq (-2) \cdot (-3)$ . Instead,  $-(2 \cdot 3) = (-2) \cdot (3) = (2) \cdot (-3) = -6$ . The same sort of thing goes for fractions:  $-\frac{3}{5}$  can be written as  $\frac{-3}{5}$  or  $\frac{3}{-5}$ , but not  $\frac{-3}{-5}$ . It’s about time we did a few examples to see how these properties work in practice.

### Example 3 Arithmetic with fractions

Perform the indicated operations and simplify. By ‘simplify’ here, we mean to have the final answer written in the form  $\frac{a}{b}$  where  $a$  and  $b$  are integers which have no common factors. Said another way, we want  $\frac{a}{b}$  in ‘lowest terms’.

$$1. \frac{1}{4} + \frac{6}{7}$$

$$2. \frac{5}{12} - \left( \frac{47}{30} - \frac{7}{3} \right)$$

$$3. \frac{\frac{12}{5} - \frac{7}{24}}{1 + \left( \frac{12}{5} \right) \left( \frac{7}{24} \right)}$$

$$4. \frac{(2(2) + 1)(-3 - (-3)) - 5(4 - 7)}{4 - 2(3)} \quad 5. \left( \frac{3}{5} \right) \left( \frac{5}{13} \right) - \left( \frac{4}{5} \right) \left( -\frac{12}{13} \right)$$

### SOLUTION

1. It may seem silly to start with an example this basic but experience has taught us not to take much for granted. We start by finding the lowest common denominator and then we rewrite the fractions using that new denominator. Since 4 and 7 are **relatively prime**, meaning they have no

It might be junior high (elementary?) school material, but arithmetic with fractions is one of the most common sources of errors among university students. If you’re not comfortable working with fractions, we strongly recommend seeing your instructor (or a tutor) to go over this material until you’re completely confident that you understand it. Experience (and even formal educational studies) suggest that your success handling fractions corresponds pretty well with your overall success in passing your Mathematics courses.

In this text we do not distinguish typographically between the dashes in the expressions ‘ $5 - 3$ ’ and ‘ $-3$ ’ even though they are mathematically quite different. In the expression ‘ $5 - 3$ ’, the dash is a *binary* operation (that is, an operation requiring two numbers) whereas in ‘ $-3$ ’, the dash is a *unary* operation (that is, an operation requiring only one number). You might ask, ‘Who cares?’ Your calculator does – that’s who! In the text we can write  $-3 - 3 = -6$  but that will not work in your calculator. Instead you’d need to type  $-3 - 3$  to get  $-6$  where the first dash comes from the ‘ $+/-$ ’ key.

factors in common, the lowest common denominator is  $4 \cdot 7 = 28$ .

$$\begin{aligned}\frac{1}{4} + \frac{6}{7} &= \frac{1}{4} \cdot \frac{7}{7} + \frac{6}{7} \cdot \frac{4}{4} && \text{Equivalent Fractions} \\ &= \frac{7}{28} + \frac{24}{28} && \text{Multiplication of Fractions} \\ &= \frac{31}{28} && \text{Addition of Fractions}\end{aligned}$$

The result is in lowest terms because 31 and 28 are relatively prime so we're done.

We could have used  $12 \cdot 30 \cdot 3 = 1080$  as our common denominator but then the numerators would become unnecessarily large. It's best to use the *lowest* common denominator.

2. We could begin with the subtraction in parentheses, namely  $\frac{47}{30} - \frac{7}{3}$ , and then subtract that result from  $\frac{5}{12}$ . It's easier, however, to first distribute the negative across the quantity in parentheses and then use the Associative Property to perform all of the addition and subtraction in one step. The lowest common denominator for all three fractions is 60.

$$\begin{aligned}\frac{5}{12} - \left( \frac{47}{30} - \frac{7}{3} \right) &= \frac{5}{12} - \frac{47}{30} + \frac{7}{3} && \text{Distribute the Negative} \\ &= \frac{5}{12} \cdot \frac{5}{5} - \frac{47}{30} \cdot \frac{2}{2} + \frac{7}{3} \cdot \frac{20}{20} && \text{Equivalent Fractions} \\ &= \frac{25}{60} - \frac{94}{60} + \frac{140}{60} && \text{Multiplication of Fractions} \\ &= \frac{71}{60} && \text{Addition and Subtraction of Fractions}\end{aligned}$$

The numerator and denominator are relatively prime so the fraction is in lowest terms and we have our final answer.

3. What we are asked to simplify in this problem is known as a 'complex' or 'compound' fraction. Simply put, we have fractions within a fraction. The longest division line (also called a 'vinculum') acts as a grouping symbol, quite literally dividing the compound fraction into a numerator (containing fractions) and a denominator (which in this case does not contain fractions):

$$\frac{\frac{12}{5} - \frac{7}{24}}{1 + \left( \frac{12}{5} \right) \left( \frac{7}{24} \right)} = \frac{\left( \frac{12}{5} - \frac{7}{24} \right)}{\left( 1 + \left( \frac{12}{5} \right) \left( \frac{7}{24} \right) \right)}$$

The first step to simplifying a compound fraction like this one is to see if you can simplify the little fractions inside it. There are two ways to proceed. One is to simplify the numerator and denominator separately, and then use the fact that division is the same thing as multiplication by the reciprocal, as follows:

$$\begin{aligned}
 \frac{\left(\frac{12}{5} - \frac{7}{24}\right)}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} &= \frac{\left(\frac{12}{5} \cdot \frac{24}{24} - \frac{7}{24} \cdot \frac{5}{5}\right)}{\left(1 \cdot \frac{120}{120} + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} && \text{Equivalent Fractions} \\
 &= \frac{288/120 - 35/120}{120/120 + 84/120} && \text{Multiplication of fractions} \\
 &= \frac{253/120}{204/120} && \text{Addition and subtraction of fractions} \\
 &= \frac{253}{120} \cdot \frac{120}{204} && \text{Division of fractions and cancellation} \\
 &= \frac{253}{204}
 \end{aligned}$$

Since  $253 = 11 \cdot 23$  and  $204 = 2 \cdot 2 \cdot 3 \cdot 17$  have no common factors our result is in lowest terms which means we are done.

While there is nothing wrong with the above approach, we can also use our Equivalent Fractions property to rid ourselves of the ‘compound’ nature of this fraction straight away. The idea is to multiply both the numerator and denominator by the lowest common denominator of each of the ‘smaller’ fractions - in this case,  $24 \cdot 5 = 120$ .

$$\begin{aligned}
 \frac{\left(\frac{12}{5} - \frac{7}{24}\right)}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} &= \frac{\left(\frac{12}{5} - \frac{7}{24}\right) \cdot 120}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right) \cdot 120} && \text{Equivalent Fractions} \\
 &= \frac{\left(\frac{12}{5}\right)(120) - \left(\frac{7}{24}\right)(120)}{(1)(120) + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)(120)} && \text{Distributive Property} \\
 &= \frac{\frac{12 \cdot 120}{5} - \frac{7 \cdot 120}{24}}{120 + \frac{12 \cdot 7 \cdot 120}{5 \cdot 24}} && \text{Multiply fractions} \\
 &= \frac{\frac{12 \cdot 24 \cdot 5}{5} - \frac{7 \cdot 5 \cdot 24}{24}}{120 + \frac{12 \cdot 7 \cdot 5 \cdot 24}{5 \cdot 24}} && \text{Factor and cancel} \\
 &= \frac{(12 \cdot 24) - (7 \cdot 5)}{120 + (12 \cdot 7)} \\
 &= \frac{288 - 35}{120 + 84} = \frac{253}{204},
 \end{aligned}$$

which is the same as we obtained above.

4. This fraction may look simpler than the one before it, but the negative signs and parentheses mean that we shouldn’t get complacent. Again we note that the division line here acts as a grouping symbol. That is,

$$\frac{(2(2) + 1)(-3 - (-3)) - 5(4 - 7)}{4 - 2(3)} = \frac{((2(2) + 1)(-3 - (-3)) - 5(4 - 7))}{(4 - 2(3))}$$

This means that we should simplify the numerator and denominator first, then perform the division last. We tend to what's in parentheses first, giving multiplication priority over addition and subtraction.

$$\begin{aligned} \frac{(2(2) + 1)(-3 - (-3)) - 5(4 - 7)}{4 - 2(3)} &= \frac{(4 + 1)(-3 + 3) - 5(-3)}{4 - 6} \\ &= \frac{(5)(0) + 15}{-2} \\ &= \frac{15}{-2} \\ &= -\frac{15}{2} \quad \text{Properties of Negatives} \end{aligned}$$

Since  $15 = 3 \cdot 5$  and 2 have no common factors, we are done.

5. In this problem, we have multiplication and subtraction. Multiplication takes precedence so we perform it first. Recall that to multiply fractions, we do *not* need to obtain common denominators; rather, we multiply the corresponding numerators together along with the corresponding denominators. Like the previous example, we have parentheses and negative signs for added fun!

$$\begin{aligned} \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) &= \frac{3 \cdot 5}{5 \cdot 13} - \frac{4 \cdot (-12)}{5 \cdot 13} \quad \text{Multiply fractions} \\ &= \frac{15}{65} - \frac{-48}{65} \\ &= \frac{15}{65} + \frac{48}{65} \quad \text{Properties of Negatives} \\ &= \frac{15 + 48}{65} \quad \text{Add numerators} \\ &= \frac{63}{65} \end{aligned}$$

Since  $64 = 3 \cdot 3 \cdot 7$  and  $65 = 5 \cdot 13$  have no common factors, our answer  $\frac{63}{65}$  is in lowest terms and we are done.

Of the issues discussed in the previous set of examples none causes students more trouble than simplifying compound fractions. We presented two different methods for simplifying them: one in which we simplified the overall numerator and denominator and then performed the division and one in which we removed the compound nature of the fraction at the very beginning. We encourage the reader to go back and use both methods on each of the compound fractions presented. Keep in mind that when a compound fraction is encountered in the rest of the text it will usually be simplified using only one method and we may not choose your favourite method. Feel free to use the other one in your notes.

Next, we review exponents and their properties. Recall that  $2 \cdot 2 \cdot 2$  can be written as  $2^3$  because exponential notation expresses repeated multiplication. In the expression  $2^3$ , 2 is called the **base** and 3 is called the **exponent**. In order to generalize exponents from natural numbers to the integers, and eventually to rational and real numbers, it is helpful to think of the exponent as a count of the number of factors of the base we are multiplying by 1. For instance,

$$2^3 = 1 \cdot (\text{three factors of two}) = 1 \cdot (2 \cdot 2 \cdot 2) = 8.$$

From this, it makes sense that

$$2^0 = 1 \cdot (\text{zero factors of two}) = 1.$$

What about  $2^{-3}$ ? The ‘−’ in the exponent indicates that we are ‘taking away’ three factors of two, essentially dividing by three factors of two. So,

$$2^{-3} = 1 \div (\text{three factors of two}) = 1 \div (2 \cdot 2 \cdot 2) = \frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{8}.$$

We summarize the properties of integer exponents below.

### Definition 13 Properties of Integer Exponents

Suppose  $a$  and  $b$  are nonzero real numbers and  $n$  and  $m$  are integers.

- **Product Rules:**  $(ab)^n = a^n b^n$  and  $a^n a^m = a^{n+m}$ .
- **Quotient Rules:**  $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$  and  $\frac{a^n}{a^m} = a^{n-m}$ .
- **Power Rule:**  $(a^n)^m = a^{nm}$ .
- **Negatives in Exponents:**  $a^{-n} = \frac{1}{a^n}$ .  
In particular,  $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$  and  $\frac{1}{a^{-n}} = a^n$ .
- **Zero Powers:**  $a^0 = 1$ .  
**Note:** The expression  $0^0$  is an indeterminate form.<sup>a</sup>
- **Powers of Zero:** For any *natural* number  $n$ ,  $0^n = 0$ .  
**Note:** The expression  $0^n$  for integers  $n \leq 0$  is not defined.

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<sup>a</sup>See the comment regarding ' $\frac{0}{0}$ ' on page 12.

While it is important to state the Properties of Exponents, it is also equally important to take a moment to discuss one of the most common errors in Algebra. It is true that  $(ab)^2 = a^2 b^2$  (which some students refer to as ‘distributing’ the exponent to each factor) but you **cannot** do this sort of thing with addition. That is, in general,  $(a + b)^2 \neq a^2 + b^2$ . (For example, take  $a = 3$  and  $b = 4$ .) The same goes for any other powers.

With exponents now in the mix, we can now state the Order of Operations Agreement.

**Definition 14 Order of Operations Agreement**

When evaluating an expression involving real numbers:

1. Evaluate any expressions in parentheses (or other grouping symbols.)
2. Evaluate exponents.
3. Evaluate division and multiplication as you read from left to right.
4. Evaluate addition and subtraction as you read from left to right.

For example,  $2 + 3 \cdot 4^2 = 2 + 3 \cdot 16 = 2 + 48 = 50$ . Where students get into trouble is with things like  $-3^2$ . If we think of this as  $0 - 3^2$ , then it is clear that we evaluate the exponent first:  $-3^2 = 0 - 3^2 = 0 - 9 = -9$ . In general, we interpret  $-a^n = -(a^n)$ . If we want the ‘negative’ to also be raised to a power, we must write  $(-a)^n$  instead. To summarize,  $-3^2 = -9$  but  $(-3)^2 = 9$ .

Of course, many of the ‘properties’ we’ve stated in this section can be viewed as ways to circumvent the order of operations. We’ve already seen how the distributive property allows us to simplify  $5(2 + x)$  by performing the indicated multiplication **before** the addition that’s in parentheses. Similarly, consider trying to evaluate  $2^{30172} \cdot 2^{-30169}$ . The Order of Operations Agreement demands that the exponents be dealt with first, however, trying to compute  $2^{30172}$  is a challenge, even for a calculator. One of the Product Rules of Exponents, however, allow us to rewrite this product, essentially performing the multiplication first, to get:  $2^{30172-30169} = 2^3 = 8$ .

**Example 4 Operations with exponents**

Perform the indicated operations and simplify.

$$1. \frac{(4-2)(2 \cdot 4) - (4)^2}{(4-2)^2}$$

$$2. 12(-5)(-5 + 3)^{-4} + 6(-5)^2(-4)(-5+3)^{-5}$$

$$3. \frac{\left(\frac{5 \cdot 3^{51}}{4^{36}}\right)}{\left(\frac{5 \cdot 3^{49}}{4^{34}}\right)}$$

$$4. \frac{2 \left(\frac{5}{12}\right)^{-1}}{1 - \left(\frac{5}{12}\right)^{-2}}$$

Order of operations follows the “PEDMAS” rule some of you may have encountered.

**SOLUTION**

1. We begin working inside parentheses then deal with the exponents before working through the other operations. As we saw in Example 3, the division here acts as a grouping symbol, so we save the division to the end.

$$\begin{aligned} \frac{(4-2)(2 \cdot 4) - (4)^2}{(4-2)^2} &= \frac{(2)(8) - (4)^2}{(2)^2} = \frac{(2)(8) - 16}{4} \\ &= \frac{16 - 16}{4} = \frac{0}{4} = 0 \end{aligned}$$

2. As before, we simplify what’s in the parentheses first, then work our way

through the exponents, multiplication, and finally, the addition.

$$\begin{aligned}
 12(-5)(-5+3)^{-4} + 6(-5)^2(-4)(-5+3)^{-5} \\
 &= 12(-5)(-2)^{-4} + 6(-5)^2(-4)(-2)^{-5} \\
 &= 12(-5)\left(\frac{1}{(-2)^4}\right) + 6(-5)^2(-4)\left(\frac{1}{(-2)^5}\right) \\
 &= 12(-5)\left(\frac{1}{16}\right) + 6(25)(-4)\left(\frac{1}{-32}\right) \\
 &= (-60)\left(\frac{1}{16}\right) + (-600)\left(\frac{1}{-32}\right) \\
 &= \frac{-60}{16} + \left(\frac{-600}{-32}\right) \\
 &= \frac{-15 \cdot 4}{4 \cdot 4} + \frac{-75 \cdot 8}{-4 \cdot 8} \\
 &= \frac{-15}{4} + \frac{-75}{-4} \\
 &= \frac{-15}{4} + \frac{75}{4} \\
 &= \frac{-15 + 75}{4} \\
 &= \frac{60}{4} \\
 &= 15
 \end{aligned}$$

3. The Order of Operations Agreement mandates that we work within each set of parentheses first, giving precedence to the exponents, then the multiplication, and, finally the division. The trouble with this approach is that the exponents are so large that computation becomes a trifle unwieldy. What we observe, however, is that the bases of the exponential expressions, 3 and 4, occur in both the numerator and denominator of the compound fraction, giving us hope that we can use some of the Properties of Exponents (the Quotient Rule, in particular) to help us out. Our first step here is to invert and multiply. We see immediately that the 5's cancel after which we group the powers of 3 together and the powers of 4 together and apply the properties of exponents.

$$\begin{aligned}
 \frac{\left(\frac{5 \cdot 3^{51}}{4^{36}}\right)}{\left(\frac{5 \cdot 3^{49}}{4^{34}}\right)} &= \frac{5 \cdot 3^{51}}{4^{36}} \cdot \frac{4^{34}}{5 \cdot 3^{49}} = \frac{5 \cdot 3^{51} \cdot 4^{34}}{5 \cdot 3^{49} \cdot 4^{36}} = \frac{3^{51}}{3^{49}} \cdot \frac{4^{34}}{4^{36}} \\
 &= 3^{51-49} \cdot 4^{34-36} = 3^2 \cdot 4^{-2} = 3^2 \cdot \left(\frac{1}{4^2}\right) \\
 &= 9 \cdot \left(\frac{1}{16}\right) = \frac{9}{16}
 \end{aligned}$$

4. We have yet another instance of a compound fraction so our first order of business is to rid ourselves of the compound nature of the fraction like we did in Example 3. To do this, however, we need to tend to the exponents first so that we can determine what common denominator is needed to

It's important that you understand the difference between the statements  $y = \sqrt{x}$  and  $y^2 = x$ . As we'll discuss in Chapter 2, the equation  $y = \sqrt{x}$  defines  $y$  as a **function** of  $x$ , which means that for each value of  $x \geq 0$  there is only one value of  $y$  such that  $y = \sqrt{x}$ . For example,  $y = \sqrt{4}$  is equivalent to  $y = 2$ . On the other hand, there are **two** solutions to  $y^2 = x$ ; namely,  $y = \sqrt{x}$  and  $y = -\sqrt{x}$ . For example, the equation  $y^2 = 4$  is equivalent to the two equations  $y = 2$  and  $y = -2$  (or, more concisely,  $y = \pm 2$ ). Since these two equations are closely related, it's easy to mix them up. The main thing to remember is that  $\sqrt{x}$  always denotes the *positive* square root of  $x$ .

simplify the fraction.

$$\begin{aligned} \frac{2\left(\frac{5}{12}\right)^{-1}}{1-\left(\frac{5}{12}\right)^{-2}} &= \frac{2\left(\frac{12}{5}\right)}{1-\left(\frac{12}{5}\right)^2} = \frac{\left(\frac{24}{5}\right)}{1-\left(\frac{12^2}{5^2}\right)} \\ &= \frac{\left(\frac{24}{5}\right)}{1-\left(\frac{144}{25}\right)} = \frac{\left(\frac{24}{5}\right) \cdot 25}{\left(1-\frac{144}{25}\right) \cdot 25} \\ &= \frac{\left(\frac{24 \cdot 5 \cdot 5}{5}\right)}{\left(1 \cdot 25 - \frac{144 \cdot 25}{25}\right)} = \frac{120}{25 - 144} \\ &= \frac{120}{-119} = -\frac{120}{119} \end{aligned}$$

Since 120 and 119 have no common factors, we are done.

We close our review of real number arithmetic with a discussion of roots and radical notation. Just as subtraction and division were defined in terms of the inverse of addition and multiplication, respectively, we define roots by undoing natural number exponents.

### Definition 15 The principal $n^{\text{th}}$ root

Let  $a$  be a real number and let  $n$  be a natural number. If  $n$  is odd, then the **principal  $n^{\text{th}}$  root** of  $a$  (denoted  $\sqrt[n]{a}$ ) is the unique real number satisfying  $(\sqrt[n]{a})^n = a$ . If  $n$  is even,  $\sqrt[n]{a}$  is defined similarly provided  $a \geq 0$  and  $\sqrt[n]{a} \geq 0$ . The number  $n$  is called the **index** of the root and the number  $a$  is called the **radicand**. For  $n = 2$ , we write  $\sqrt{a}$  instead of  $\sqrt[2]{a}$ .

The reasons for the added stipulations for even-indexed roots in Definition 15 can be found in the Properties of Negatives. First, for all real numbers,  $x^{\text{even power}} \geq 0$ , which means it is never negative. Thus if  $a$  is a *negative* real number, there are no real numbers  $x$  with  $x^{\text{even power}} = a$ . This is why if  $n$  is even,  $\sqrt[n]{a}$  only exists if  $a \geq 0$ . The second restriction for even-indexed roots is that  $\sqrt[n]{a} \geq 0$ . This comes from the fact that  $x^{\text{even power}} = (-x)^{\text{even power}}$ , and we require  $\sqrt[n]{a}$  to have just one value. So even though  $2^4 = 16$  and  $(-2)^4 = 16$ , we require  $\sqrt[4]{16} = 2$  and ignore  $-2$ .

Dealing with odd powers is much easier. For example,  $x^3 = -8$  has one and only one real solution, namely  $x = -2$ , which means not only does  $\sqrt[3]{-8}$  exist, there is only one choice, namely  $\sqrt[3]{-8} = -2$ . Of course, when it comes to solving  $x^{5213} = -117$ , it's not so clear that there is one and only one real solution, let alone that the solution is  $\sqrt[5213]{-117}$ . Such pills are easier to swallow once we've thought a bit about such equations graphically, (see Chapter ??) and ultimately, these things come from the completeness property of the real numbers mentioned earlier.

We list properties of radicals below as a 'theorem' since they can be justified using the properties of exponents.

**Theorem 3 Properties of Radicals**

Let  $a$  and  $b$  be real numbers and let  $m$  and  $n$  be natural numbers. If  $\sqrt[n]{a}$  and  $\sqrt[n]{b}$  are real numbers, then

- **Product Rule:**  $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$
- **Quotient Rule:**  $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ , provided  $b \neq 0$ .
- **Power Rule:**  $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$

The proof of Theorem 3 is based on the definition of the principal  $n^{\text{th}}$  root and the Properties of Exponents. To establish the product rule, consider the following. If  $n$  is odd, then by definition  $\sqrt[n]{ab}$  is the unique real number such that  $(\sqrt[n]{ab})^n = ab$ . Given that  $(\sqrt[n]{a}\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab$  as well, it must be the case that  $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ . If  $n$  is even, then  $\sqrt[n]{ab}$  is the unique non-negative real number such that  $(\sqrt[n]{ab})^n = ab$ . Note that since  $n$  is even,  $\sqrt[n]{a}$  and  $\sqrt[n]{b}$  are also non-negative thus  $\sqrt[n]{a}\sqrt[n]{b} \geq 0$  as well. Proceeding as above, we find that  $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ . The quotient rule is proved similarly and is left as an exercise. The power rule results from repeated application of the product rule, so long as  $\sqrt[n]{a}$  is a real number to start with. We leave that as an exercise as well.

We pause here to point out one of the most common errors students make when working with radicals. Obviously  $\sqrt{9} = 3$ ,  $\sqrt{16} = 4$  and  $\sqrt{9+16} = \sqrt{25} = 5$ . Thus we can clearly see that  $5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3+4 = 7$  because we all know that  $5 \neq 7$ . The authors urge you to **never consider ‘distributing’ roots or exponents**. It’s wrong and no good will come of it because in general  $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ .

Since radicals have properties inherited from exponents, they are often written as such. We define rational exponents in terms of radicals in the box below.

**Definition 16 Rational exponents**

Let  $a$  be a real number, let  $m$  be an integer and let  $n$  be a natural number.

- $a^{\frac{1}{n}} = \sqrt[n]{a}$  whenever  $\sqrt[n]{a}$  is a real number. (If  $n$  is even we need  $a \geq 0$ .)
- $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$  whenever  $\sqrt[n]{a}$  is a real number.

Things get more complicated once complex numbers are involved. Fortunately (disappointingly?), that’s not a can of worms we’ll be opening in this course.

It would make life really nice if the rational exponents defined in Definition 16 had all of the same properties that integer exponents have as listed on page 17 - but they don’t. Why not? Let’s look at an example to see what goes wrong. Consider the Product Rule which says that  $(ab)^n = a^n b^n$  and let  $a = -16$ ,  $b = -81$  and  $n = \frac{1}{4}$ . Plugging the values into the Product Rule yields the equation  $((-16)(-81))^{1/4} = (-16)^{1/4}(-81)^{1/4}$ . The left side of this equation is  $1296^{1/4}$  which equals 6 but the right side is undefined because neither root is a real number. Would it help if, when it comes to even roots (as signified by even denominators in the fractional exponents), we ensure that everything they

apply to is non-negative? That works for some of the rules - we leave it as an exercise to see which ones - but does not work for the Power Rule.

Consider the expression  $(a^{2/3})^{3/2}$ . Applying the usual laws of exponents, we'd be tempted to simplify this as  $(a^{2/3})^{3/2} = a^{\frac{2}{3} \cdot \frac{3}{2}} = a^1 = a$ . However, if we substitute  $a = -1$  and apply Definition 16, we find  $(-1)^{2/3} = (\sqrt[3]{-1})^2 = (-1)^2 = 1$  so that  $((-1)^{2/3})^{3/2} = 1^{3/2} = (\sqrt{1})^3 = 1^3 = 1$ . Thus in this case we have  $(a^{2/3})^{3/2} \neq a$  even though all of the roots were defined. It is true, however, that  $(a^{3/2})^{2/3} = a$  and we leave this for the reader to show. The moral of the story is that when simplifying powers of rational exponents where the base is negative or worse, unknown, it's usually best to rewrite them as radicals.

### Example 5 Combining operations

Perform the indicated operations and simplify.

$$1. \frac{-(-4) - \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$$

$$2. \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1 - \left(\frac{\sqrt{3}}{3}\right)^2}$$

$$3. (\sqrt[3]{-2} - \sqrt[3]{-54})^2$$

$$4. 2\left(\frac{9}{4} - 3\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{9}{4} - 3\right)^{-2/3}$$

### SOLUTION

- We begin in the numerator and note that the radical here acts a grouping symbol,<sup>3</sup> so our first order of business is to simplify the radicand.

$$\begin{aligned} \frac{-(-4) - \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)} &= \frac{-(-4) - \sqrt{16 - 4(2)(-3)}}{2(2)} \\ &= \frac{-(-4) - \sqrt{16 - 4(-6)}}{2(2)} \\ &= \frac{-(-4) - \sqrt{16 - (-24)}}{2(2)} \\ &= \frac{-(-4) - \sqrt{16 + 24}}{2(2)} \\ &= \frac{-(-4) - \sqrt{40}}{2(2)} \end{aligned}$$

As you may recall, 40 can be factored using a perfect square as  $40 = 4 \cdot 10$  so we use the product rule of radicals to write  $\sqrt{40} = \sqrt{4 \cdot 10} =$

---

<sup>3</sup>The line extending horizontally from the square root symbol ' $\sqrt{}$ ' is, you guessed it, another vinculum.

$\sqrt{4}\sqrt{10} = 2\sqrt{10}$ . This lets us factor a ‘2’ out of both terms in the numerator, eventually allowing us to cancel it with a factor of 2 in the denominator.

$$\begin{aligned}\frac{-(-4) - \sqrt{40}}{2(2)} &= \frac{-(-4) - 2\sqrt{10}}{2(2)} = \frac{4 - 2\sqrt{10}}{2(2)} \\ &= \frac{2 \cdot 2 - 2\sqrt{10}}{2(2)} = \frac{2(2 - \sqrt{10})}{2(2)} \\ &= \frac{2(2 - \sqrt{10})}{2(2)} = \frac{2 - \sqrt{10}}{2}\end{aligned}$$

Since the numerator and denominator have no more common factors,<sup>4</sup> we are done.

- Once again we have a compound fraction, so we first simplify the exponent in the denominator to see which factor we’ll need to multiply by in order to clean up the fraction.

$$\begin{aligned}\frac{2\left(\frac{\sqrt{3}}{3}\right)}{1 - \left(\frac{\sqrt{3}}{3}\right)^2} &= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1 - \left(\frac{(\sqrt{3})^2}{3^2}\right)} = \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1 - \left(\frac{3}{9}\right)} \\ &= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1 - \left(\frac{1 \cdot 3}{3 \cdot 3}\right)} = \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1 - \left(\frac{1}{3}\right)} \\ &= \frac{2\left(\frac{\sqrt{3}}{3}\right) \cdot 3}{\left(1 - \left(\frac{1}{3}\right)\right) \cdot 3} = \frac{2 \cdot \sqrt{3} \cdot 3}{3} \\ &= \frac{2\sqrt{3}}{3 - 1} = \frac{2\sqrt{3}}{2} = \sqrt{3}\end{aligned}$$

- Working inside the parentheses, we first encounter  $\sqrt[3]{-2}$ . While the  $-2$  isn’t a perfect cube,<sup>5</sup> we may think of  $-2 = (-1)(2)$ . Since  $(-1)^3 = -1$ ,  $-1$  is a perfect cube, and we may write  $\sqrt[3]{-2} = \sqrt[3]{(-1)(2)} = \sqrt[3]{-1}\sqrt[3]{2} = -\sqrt[3]{2}$ . When it comes to  $\sqrt[3]{54}$ , we may write it as  $\sqrt[3]{(-27)(2)} = \sqrt[3]{-27}\sqrt[3]{2} = -3\sqrt[3]{2}$ . So,

$$\sqrt[3]{-2} - \sqrt[3]{-54} = -\sqrt[3]{2} - (-3\sqrt[3]{2}) = -\sqrt[3]{2} + 3\sqrt[3]{2}.$$

At this stage, we can simplify  $-\sqrt[3]{2} + 3\sqrt[3]{2} = 2\sqrt[3]{2}$ . You may remember this as being called ‘combining like radicals,’ but it is in fact just another application of the distributive property:

$$-\sqrt[3]{2} + 3\sqrt[3]{2} = (-1)\sqrt[3]{2} + 3\sqrt[3]{2} = (-1 + 3)\sqrt[3]{2} = 2\sqrt[3]{2}.$$

Putting all this together, we get:

$$\begin{aligned}(\sqrt[3]{-2} - \sqrt[3]{-54})^2 &= (-\sqrt[3]{2} + 3\sqrt[3]{2})^2 = (2\sqrt[3]{2})^2 \\ &= 2^2(\sqrt[3]{2})^2 = 4\sqrt[3]{2^2} = 4\sqrt[3]{4}\end{aligned}$$

Since there are no perfect integer cubes which are factors of 4 (apart from 1, of course), we are done.

---

<sup>4</sup>Do you see why we aren’t ‘cancelling’ the remaining 2’s?

<sup>5</sup>Of an integer, that is!

4. We start working in parentheses and get a common denominator to subtract the fractions:

$$\frac{9}{4} - 3 = \frac{9}{4} - \frac{3 \cdot 4}{1 \cdot 4} = \frac{9}{4} - \frac{12}{4} = \frac{-3}{4}$$

Since the denominators in the fractional exponents are odd, we can proceed using the properties of exponents:

$$\begin{aligned} 2 \left( \frac{9}{4} - 3 \right)^{1/3} + 2 \left( \frac{9}{4} \right) \left( \frac{1}{3} \right) \left( \frac{9}{4} - 3 \right)^{-2/3} \\ = 2 \left( \frac{-3}{4} \right)^{1/3} + 2 \left( \frac{9}{4} \right) \left( \frac{1}{3} \right) \left( \frac{-3}{4} \right)^{-2/3} \\ = 2 \left( \frac{(-3)^{1/3}}{(4)^{1/3}} \right) + 2 \left( \frac{9}{4} \right) \left( \frac{1}{3} \right) \left( \frac{4}{-3} \right)^{2/3} \\ = 2 \left( \frac{(-3)^{1/3}}{(4)^{1/3}} \right) + 2 \left( \frac{9}{4} \right) \left( \frac{1}{3} \right) \left( \frac{(4)^{2/3}}{(-3)^{2/3}} \right) \\ = \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{2 \cdot 9 \cdot 1 \cdot 4^{2/3}}{4 \cdot 3 \cdot (-3)^{2/3}} \\ = \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{2 \cdot 3 \cdot 4^{2/3}}{2 \cdot 3 \cdot (-3)^{2/3}} \\ = \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}} \end{aligned}$$

At this point, we could start looking for common denominators but it turns out that these fractions reduce even further. Since  $4 = 2^2$ ,  $4^{1/3} = (2^2)^{1/3} = 2^{2/3}$ . Similarly,  $4^{2/3} = (2^2)^{2/3} = 2^{4/3}$ . The expressions  $(-3)^{1/3}$  and  $(-3)^{2/3}$  contain negative bases so we proceed with caution and convert them back to radical notation to get:  $(-3)^{1/3} = \sqrt[3]{-3} = -\sqrt[3]{3} = -3^{1/3}$  and  $(-3)^{2/3} = (\sqrt[3]{-3})^2 = (-\sqrt[3]{3})^2 = (\sqrt[3]{3})^2 = 3^{2/3}$ . Hence:

$$\begin{aligned} \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}} &= \frac{2 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3 \cdot 2^{4/3}}{2 \cdot 3^{2/3}} \\ &= \frac{2^1 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3^1 \cdot 2^{4/3}}{2^1 \cdot 3^{2/3}} \\ &= 2^{1-2/3} \cdot (-3^{1/3}) + 3^{1-2/3} \cdot 2^{4/3-1} \\ &= 2^{1/3} \cdot (-3^{1/3}) + 3^{1/3} \cdot 2^{1/3} \\ &= -2^{1/3} \cdot 3^{1/3} + 3^{1/3} \cdot 2^{1/3} \\ &= 0 \end{aligned}$$

## Exercises 1.2

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### Problems

In Exercises 1–33, perform the indicated operations and simplify.

$$1. 5 - 2 + 3$$

$$2. 5 - (2 + 3)$$

$$3. \frac{2}{3} - \frac{4}{7}$$

$$4. \frac{3}{8} + \frac{5}{12}$$

$$5. \frac{5 - 3}{-2 - 4}$$

$$6. \frac{2(-3)}{3 - (-3)}$$

$$7. \frac{2(3) - (4 - 1)}{2^2 + 1}$$

$$8. \frac{4 - 5.8}{2 - 2.1}$$

$$9. \frac{1 - 2(-3)}{5(-3) + 7}$$

$$10. \frac{5(3) - 7}{2(3)^2 - 3(3) - 9}$$

$$11. \frac{2((-1)^2 - 1)}{((-1)^2 + 1)^2}$$

$$12. \frac{(-2)^2 - (-2) - 6}{(-2)^2 - 4}$$

$$13. \frac{3 - \frac{4}{9}}{-2 - (-3)}$$

$$14. \frac{\frac{2}{3} - \frac{4}{5}}{4 - \frac{7}{10}}$$

$$15. \frac{2\left(\frac{4}{3}\right)}{1 - \left(\frac{4}{3}\right)^2}$$

$$16. \frac{1 - \left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}{1 + \left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}$$

$$17. \left(\frac{2}{3}\right)^{-5}$$

$$18. 3^{-1} - 4^{-2}$$

$$19. \frac{1 + 2^{-3}}{3 - 4^{-1}}$$

$$20. \frac{3 \cdot 5^{100}}{12 \cdot 5^{98}}$$

$$21. \sqrt{3^2 + 4^2}$$

$$22. \sqrt{12} - \sqrt{75}$$

$$23. (-8)^{2/3} - 9^{-3/2}$$

$$24. \left(-\frac{32}{9}\right)^{-3/5}$$

$$25. \sqrt{(3 - 4)^2 + (5 - 2)^2}$$

$$26. \sqrt{(2 - (-1))^2 + \left(\frac{1}{2} - 3\right)^2}$$

$$27. \sqrt{(\sqrt{5} - 2\sqrt{5})^2 + (\sqrt{18} - \sqrt{8})^2}$$

$$28. \frac{-12 + \sqrt{18}}{21}$$

$$29. \frac{-2 - \sqrt{(2)^2 - 4(3)(-1)}}{2(3)}$$

$$30. \frac{-(-4) + \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)}$$

$$31. 2(-5)(-5 + 1)^{-1} + (-5)^2(-1)(-5 + 1)^{-2}$$

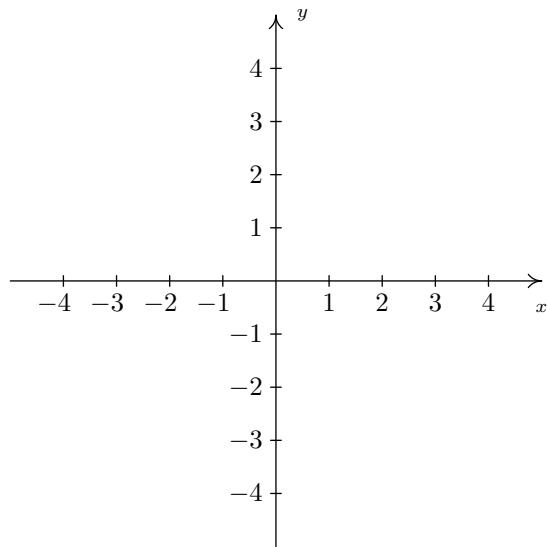
$$32. 3\sqrt{2(4) + 1} + 3(4)\left(\frac{1}{2}\right)(2(4) + 1)^{-1/2}(2)$$

$$33. 2(-7)\sqrt[3]{1 - (-7)} + (-7)^2\left(\frac{1}{3}\right)(1 - (-7))^{-2/3}(-1)$$

### 1.3 The Cartesian Coordinate Plane

The Cartesian Plane is named in honour of [René Descartes](#).

In order to visualize the pure excitement that is Precalculus, we need to unite Algebra and Geometry. Simply put, we must find a way to draw algebraic things. Let's start with possibly the greatest mathematical achievement of all time: the **Cartesian Coordinate Plane**. Imagine two real number lines crossing at a right angle at 0 as drawn below.

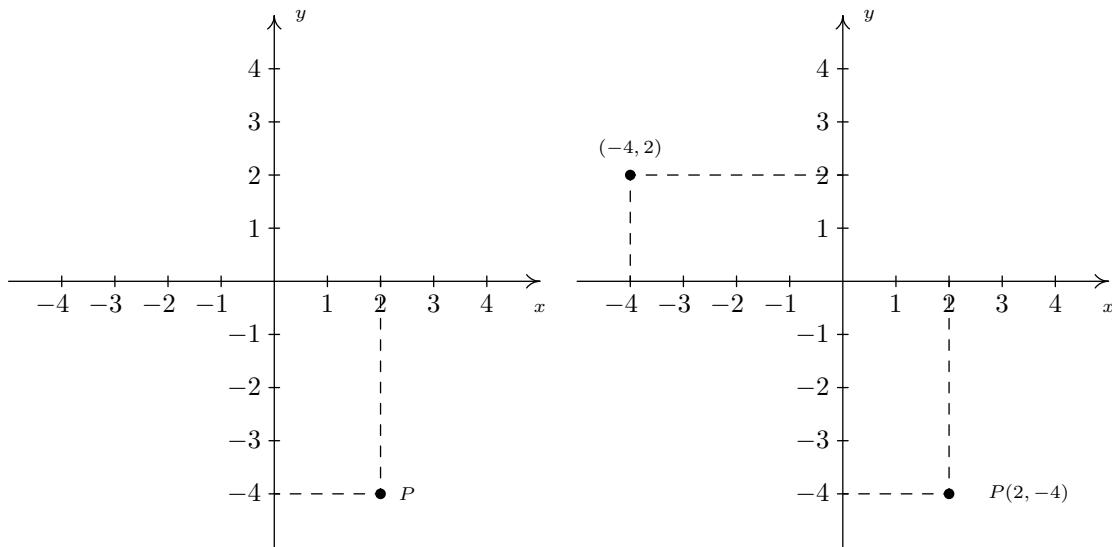


Usually extending off towards infinity is indicated by arrows, but here, the arrows are used to indicate the *direction* of increasing values of  $x$  and  $y$ .

The horizontal number line is usually called the  **$x$ -axis** while the vertical number line is usually called the  **$y$ -axis**. As with the usual number line, we imagine these axes extending off indefinitely in both directions. Having two number lines allows us to locate the positions of points off of the number lines as well as points on the lines themselves.

The names of the coordinates can vary depending on the context of the application. If, for example, the horizontal axis represented time we might choose to call it the  $t$ -axis. The first number in the ordered pair would then be the  $t$ -coordinate.

For example, consider the point  $P$  on the next page. To use the numbers on the axes to label this point, we imagine dropping a vertical line from the  $x$ -axis to  $P$  and extending a horizontal line from the  $y$ -axis to  $P$ . This process is sometimes called ‘projecting’ the point  $P$  to the  $x$ - (respectively  $y$ -) axis. We then describe the point  $P$  using the **ordered pair**  $(2, -4)$ . The first number in the ordered pair is called the **abscissa** or  **$x$ -coordinate** and the second is called the **ordinate** or  **$y$ -coordinate**. Taken together, the ordered pair  $(2, -4)$  comprise the **Cartesian coordinates** of the point  $P$ . In practice, the distinction between a point and its coordinates is blurred; for example, we often speak of ‘the point  $(2, -4)$ .’ We can think of  $(2, -4)$  as instructions on how to reach  $P$  from the **origin**  $(0, 0)$  by moving 2 units to the right and 4 units downwards. Notice that the order in the **ordered pair** is important – if we wish to plot the point  $(-4, 2)$ , we would move to the left 4 units from the origin and then move upwards 2 units, as below on the right.



When we speak of the Cartesian Coordinate Plane, we mean the set of all possible ordered pairs  $(x, y)$  as  $x$  and  $y$  take values from the real numbers. Below is a summary of important facts about Cartesian coordinates.

#### Key Idea 4      Important Facts about the Cartesian Coordinate Plane

- $(a, b)$  and  $(c, d)$  represent the same point in the plane if and only if  $a = c$  and  $b = d$ .
- $(x, y)$  lies on the  $x$ -axis if and only if  $y = 0$ .
- $(x, y)$  lies on the  $y$ -axis if and only if  $x = 0$ .
- The origin is the point  $(0, 0)$ . It is the only point common to both axes.

Cartesian coordinates are sometimes referred to as *rectangular coordinates*, to distinguish them from other coordinate systems such as *polar coordinates*. See Section ?? for more details.

#### Example 6      Plotting points in the Cartesian Plane

Plot the following points:  $A(5, 8)$ ,  $B\left(-\frac{5}{2}, 3\right)$ ,  $C(-5.8, -3)$ ,  $D(4.5, -1)$ ,  $E(5, 0)$ ,  $F(0, 5)$ ,  $G(-7, 0)$ ,  $H(0, -9)$ ,  $O(0, 0)$ .

The letter  $O$  is almost always reserved for the origin.

**SOLUTION** To plot these points, we start at the origin and move to the right if the  $x$ -coordinate is positive; to the left if it is negative. Next, we move up if the  $y$ -coordinate is positive or down if it is negative. If the  $x$ -coordinate is 0, we start at the origin and move along the  $y$ -axis only. If the  $y$ -coordinate is 0 we move along the  $x$ -axis only.

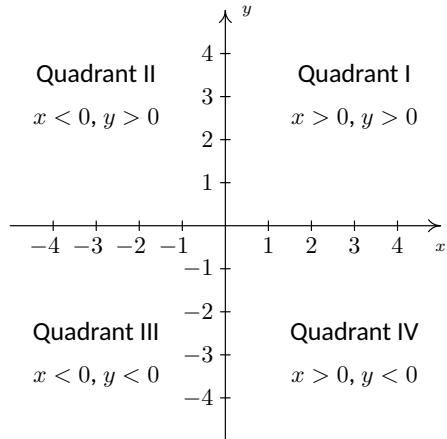
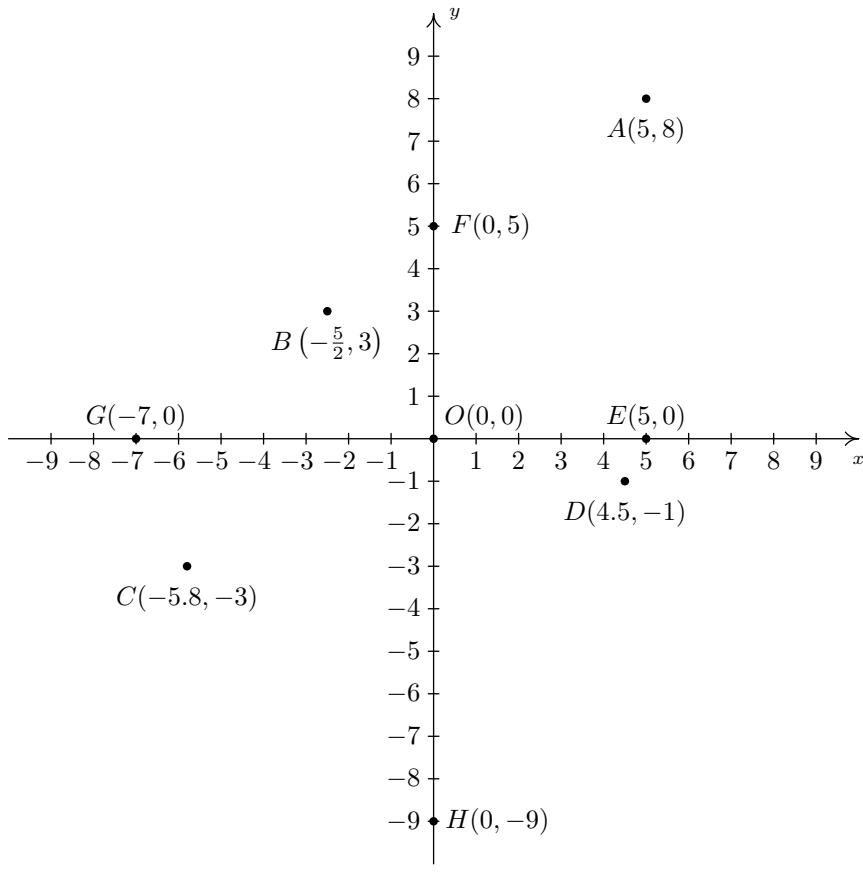


Figure 1.9: The four quadrants of the Cartesian plane



The axes divide the plane into four regions called **quadrants**. They are labelled with Roman numerals and proceed counterclockwise around the plane: see Figure 1.9.

For example,  $(1, 2)$  lies in Quadrant I,  $(-1, 2)$  in Quadrant II,  $(-1, -2)$  in Quadrant III and  $(1, -2)$  in Quadrant IV. If a point other than the origin happens to lie on the axes, we typically refer to that point as lying on the positive or negative  $x$ -axis (if  $y = 0$ ) or on the positive or negative  $y$ -axis (if  $x = 0$ ). For example,  $(0, 4)$  lies on the positive  $y$ -axis whereas  $(-117, 0)$  lies on the negative  $x$ -axis. Such points do not belong to any of the four quadrants.

One of the most important concepts in all of Mathematics is **symmetry**. There are many types of symmetry in Mathematics, but three of them can be discussed easily using Cartesian Coordinates.

#### Definition 17 Symmetry in the Cartesian Plane

Two points  $(a, b)$  and  $(c, d)$  in the plane are said to be

- **symmetric about the  $x$ -axis** if  $a = c$  and  $b = -d$
- **symmetric about the  $y$ -axis** if  $a = -c$  and  $b = d$
- **symmetric about the origin** if  $a = -c$  and  $b = -d$

In Figure 1.10,  $P$  and  $S$  are symmetric about the  $x$ -axis, as are  $Q$  and  $R$ ;  $P$  and  $Q$  are symmetric about the  $y$ -axis, as are  $R$  and  $S$ ; and  $P$  and  $R$  are symmetric about the origin, as are  $Q$  and  $S$ .

### Example 7 Finding points exhibiting symmetry

Let  $P$  be the point  $(-2, 3)$ . Find the points which are symmetric to  $P$  about the:

1.  $x$ -axis
2.  $y$ -axis
3. origin

Check your answer by plotting the points.

**SOLUTION** The figure after Definition 17 gives us a good way to think about finding symmetric points in terms of taking the opposites of the  $x$ - and/or  $y$ -coordinates of  $P(-2, 3)$ .

1. To find the point symmetric about the  $x$ -axis, we replace the  $y$ -coordinate with its opposite to get  $(-2, -3)$ .
2. To find the point symmetric about the  $y$ -axis, we replace the  $x$ -coordinate with its opposite to get  $(2, 3)$ .
3. To find the point symmetric about the origin, we replace the  $x$ - and  $y$ -coordinates with their opposites to get  $(2, -3)$ .

The points are plotted in Figure 1.11.

One way to visualize the processes in the previous example is with the concept of a **reflection**. If we start with our point  $(-2, 3)$  and pretend that the  $x$ -axis is a mirror, then the reflection of  $(-2, 3)$  across the  $x$ -axis would lie at  $(-2, -3)$ . If we pretend that the  $y$ -axis is a mirror, the reflection of  $(-2, 3)$  across that axis would be  $(2, 3)$ . If we reflect across the  $x$ -axis and then the  $y$ -axis, we would go from  $(-2, 3)$  to  $(-2, -3)$  then to  $(2, -3)$ , and so we would end up at the point symmetric to  $(-2, 3)$  about the origin. We summarize and generalize this process below.

### Key Idea 5 Reflections in the Cartesian Plane

To reflect a point  $(x, y)$  about the:

- $x$ -axis, replace  $y$  with  $-y$ .
- $y$ -axis, replace  $x$  with  $-x$ .
- origin, replace  $x$  with  $-x$  and  $y$  with  $-y$ .

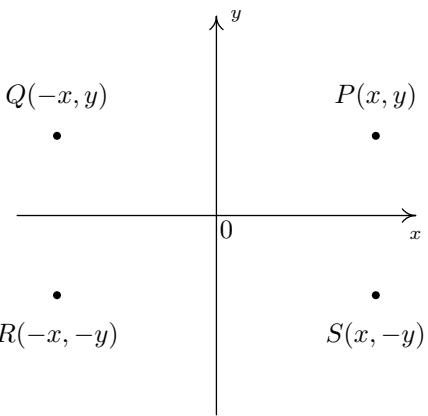


Figure 1.10: The three types of symmetry in the plane

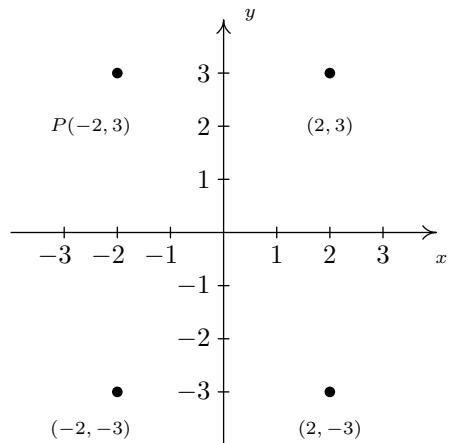


Figure 1.11: The point  $P(-2, 3)$  and its three reflections

## 1.3.1 Distance in the Plane

Another important concept in Geometry is the notion of length. If we are going to unite Algebra and Geometry using the Cartesian Plane, then we need to develop an algebraic understanding of what distance in the plane means. Suppose we have two points,  $P(x_0, y_0)$  and  $Q(x_1, y_1)$ , in the plane. By the **distance**  $d$  between  $P$  and  $Q$ , we mean the length of the line segment joining  $P$  with  $Q$ . (Remember, given any two distinct points in the plane, there is a unique line

containing both points.) Our goal now is to create an algebraic formula to compute the distance between these two points. Consider the generic situation in Figure 1.12.

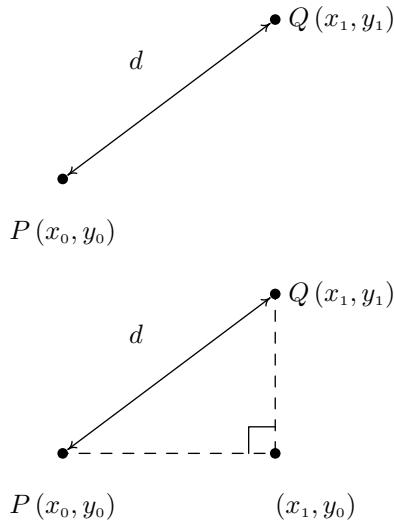


Figure 1.12: Distance between  $P$  and  $Q$

With a little more imagination, we can envision a right triangle whose hypotenuse has length  $d$  as drawn above on the right. From the latter figure, we see that the lengths of the legs of the triangle are  $|x_1 - x_0|$  and  $|y_1 - y_0|$  so the Pythagorean Theorem gives us

$$|x_1 - x_0|^2 + |y_1 - y_0|^2 = d^2$$

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = d^2$$

(Do you remember why we can replace the absolute value notation with parentheses?) By extracting the square root of both sides of the second equation and using the fact that distance is never negative, we get

#### Key Idea 6 The Distance Formula

The distance  $d$  between the points  $P(x_0, y_0)$  and  $Q(x_1, y_1)$  is:

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

It is not always the case that the points  $P$  and  $Q$  lend themselves to constructing such a triangle. If the points  $P$  and  $Q$  are arranged vertically or horizontally, or describe the exact same point, we cannot use the above geometric argument to derive the distance formula. It is left to the reader in Exercise 16 to verify Equation 6 for these cases.

#### Example 8 Distance between two points

Find and simplify the distance between  $P(-2, 3)$  and  $Q(1, -3)$ .

#### SOLUTION

$$\begin{aligned} d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\ &= \sqrt{(1 - (-2))^2 + (-3 - 3)^2} \\ &= \sqrt{9 + 36} \\ &= 3\sqrt{5} \end{aligned}$$

So the distance is  $3\sqrt{5}$ .

#### Example 9 Finding points at a given distance

Find all of the points with  $x$ -coordinate 1 which are 4 units from the point  $(3, 2)$ .

**SOLUTION** We shall soon see that the points we wish to find are on the line  $x = 1$ , but for now we'll just view them as points of the form  $(1, y)$ .

We require that the distance from  $(3, 2)$  to  $(1, y)$  be 4. The Distance Formula, Equation 6, yields

$$\begin{aligned}
 d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
 4 &= \sqrt{(1 - 3)^2 + (y - 2)^2} \\
 4 &= \sqrt{4 + (y - 2)^2} \\
 4^2 &= (\sqrt{4 + (y - 2)^2})^2 && \text{squaring both sides} \\
 16 &= 4 + (y - 2)^2 \\
 12 &= (y - 2)^2 \\
 (y - 2)^2 &= 12 \\
 y - 2 &= \pm\sqrt{12} && \text{extracting the square root} \\
 y - 2 &= \pm 2\sqrt{3} \\
 y &= 2 \pm 2\sqrt{3}
 \end{aligned}$$

We obtain two answers:  $(1, 2 + 2\sqrt{3})$  and  $(1, 2 - 2\sqrt{3})$ . The reader is encouraged to think about why there are two answers.

Related to finding the distance between two points is the problem of finding the **midpoint** of the line segment connecting two points. Given two points,  $P(x_0, y_0)$  and  $Q(x_1, y_1)$ , the **midpoint**  $M$  of  $P$  and  $Q$  is defined to be the point on the line segment connecting  $P$  and  $Q$  whose distance from  $P$  is equal to its distance from  $Q$ .

If we think of reaching  $M$  by going ‘halfway over’ and ‘halfway up’ we get the following formula.

### Key Idea 7 The Midpoint Formula

The midpoint  $M$  of the line segment connecting  $P(x_0, y_0)$  and  $Q(x_1, y_1)$  is:

$$M = \left( \frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2} \right)$$

If we let  $d$  denote the distance between  $P$  and  $Q$ , we leave it as Exercise 17 to show that the distance between  $P$  and  $M$  is  $d/2$  which is the same as the distance between  $M$  and  $Q$ . This suffices to show that Equation ?? gives the coordinates of the midpoint.

### Example 10 Finding the midpoint of a line segment

Find the midpoint of the line segment connecting  $P(-2, 3)$  and  $Q(1, -3)$ .

#### SOLUTION

$$\begin{aligned}
 M &= \left( \frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2} \right) \\
 &= \left( \frac{(-2) + 1}{2}, \frac{3 + (-3)}{2} \right) = \left( -\frac{1}{2}, 0 \right) \\
 &= \left( -\frac{1}{2}, 0 \right)
 \end{aligned}$$

The midpoint is  $(-\frac{1}{2}, 0)$ .

We close with a more abstract application of the Midpoint Formula. We will revisit the following example in Exercise 72 in Section 3.1.

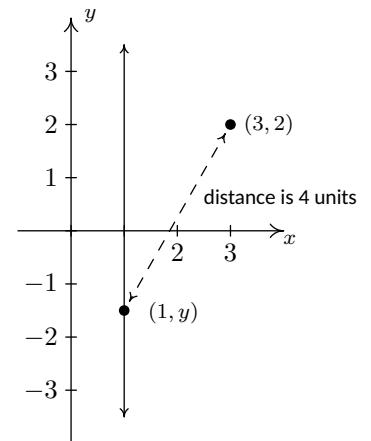


Figure 1.13: Diagram for Example 9

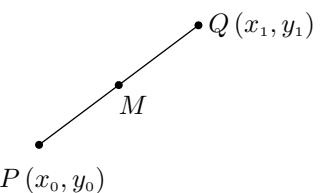


Figure 1.14: The midpoint of a line segment

**Example 11 An abstract midpoint problem**

If  $a \neq b$ , prove that the line  $y = x$  equally divides the line segment with endpoints  $(a, b)$  and  $(b, a)$ .

**SOLUTION**

To prove the claim, we use Equation ?? to find the midpoint

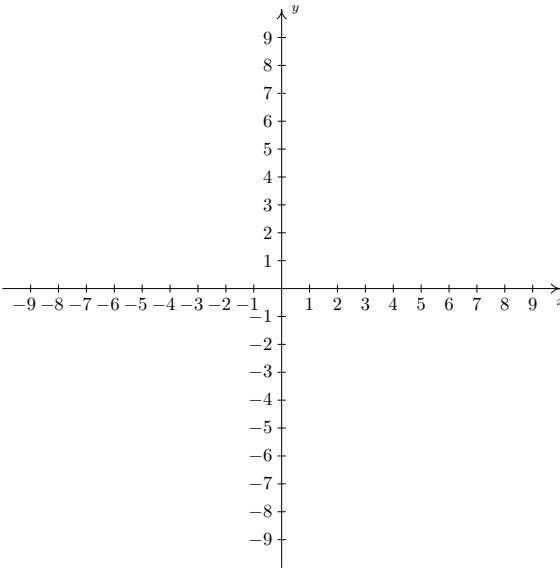
$$\begin{aligned} M &= \left( \frac{a+b}{2}, \frac{b+a}{2} \right) \\ &= \left( \frac{a+b}{2}, \frac{a+b}{2} \right) \end{aligned}$$

Since the  $x$  and  $y$  coordinates of this point are the same, we find that the midpoint lies on the line  $y = x$ , as required.

# Exercises 1.3

## Problems

1. Plot and label the points  $A(-3, -7)$ ,  $B(1.3, -2)$ ,  $C(\pi, \sqrt{10})$ ,  $D(0, 8)$ ,  $E(-5.5, 0)$ ,  $F(-8, 4)$ ,  $G(9.2, -7.8)$  and  $H(7, 5)$  in the Cartesian Coordinate Plane given below.



2. For each point given in Exercise 1 above

- Identify the quadrant or axis in/on which the point lies.
- Find the point symmetric to the given point about the  $x$ -axis.
- Find the point symmetric to the given point about the  $y$ -axis.
- Find the point symmetric to the given point about the origin.

In Exercises 3–10, find the distance  $d$  between the points and the midpoint  $M$  of the line segment which connects them.

3.  $(1, 2), (-3, 5)$

4.  $(3, -10), (-1, 2)$

5.  $\left(\frac{1}{2}, 4\right), \left(\frac{3}{2}, -1\right)$

6.  $\left(-\frac{2}{3}, \frac{3}{2}\right), \left(\frac{7}{3}, 2\right)$

7.  $\left(\frac{24}{5}, \frac{6}{5}\right), \left(-\frac{11}{5}, -\frac{19}{5}\right)$ .

8.  $(\sqrt{2}, \sqrt{3}), (-\sqrt{8}, -\sqrt{12})$

9.  $(2\sqrt{45}, \sqrt{12}), (\sqrt{20}, \sqrt{27})$ .

10.  $(0, 0), (x, y)$
11. Find all of the points of the form  $(x, -1)$  which are 4 units from the point  $(3, 2)$ .
12. Find all of the points on the  $y$ -axis which are 5 units from the point  $(-5, 3)$ .
13. Find all of the points on the  $x$ -axis which are 2 units from the point  $(-1, 1)$ .
14. Find all of the points of the form  $(x, -x)$  which are 1 unit from the origin.
15. Let's assume for a moment that we are standing at the origin and the positive  $y$ -axis points due North while the positive  $x$ -axis points due East. Our Sasquatch-o-meter tells us that Sasquatch is 3 miles West and 4 miles South of our current position. What are the coordinates of his position? How far away is he from us? If he runs 7 miles due East what would his new position be?
16. Verify the Distance Formula 6 for the cases when:
  - The points are arranged vertically. (Hint: Use  $P(a, y_0)$  and  $Q(a, y_1)$ .)
  - The points are arranged horizontally. (Hint: Use  $P(x_0, b)$  and  $Q(x_1, b)$ .)
  - The points are actually the same point. (You shouldn't need a hint for this one.)
17. Verify the Midpoint Formula by showing the distance between  $P(x_1, y_1)$  and  $M$  and the distance between  $M$  and  $Q(x_2, y_2)$  are both half of the distance between  $P$  and  $Q$ .
18. Show that the points  $A$ ,  $B$  and  $C$  below are the vertices of a right triangle.
  - $A(-3, 2)$ ,  $B(-6, 4)$ , and  $C(1, 8)$
  - $A(-3, 1)$ ,  $B(4, 0)$  and  $C(0, -3)$
19. Find a point  $D(x, y)$  such that the points  $A(-3, 1)$ ,  $B(4, 0)$ ,  $C(0, -3)$  and  $D$  are the corners of a square. Justify your answer.
20. Discuss with your classmates how many numbers are in the interval  $(0, 1)$ .
21. The world is not flat. (There are those who disagree with this statement. Look them up on the Internet some time when you're bored.) Thus the Cartesian Plane cannot possibly be the end of the story. Discuss with your classmates how you would extend Cartesian Coordinates to represent the three dimensional world. What would the Distance and Midpoint formulas look like, assuming those concepts make sense at all?

## 1.4 Complex Numbers

Historically, the lack of solutions to the equation  $x^2 = -1$  had nothing to do with the development of the complex numbers. Until the 19th century, equations such as  $x^2 = -1$  would have been considered in the context of the analytic geometry of Descartes. The lack of solutions simply indicated that the graph  $y = x^2$  did not intersect the line  $y = -1$ . The more remarkable case was that of *cubic* equations, of the form  $x^3 = ax + b$ . In this case a **real** solution is guaranteed, but there are cases where one needs **complex** numbers to find it! For details, see the excellent book *Visual Complex Analysis*, by Tristan Needham.

Note the use of the indefinite article ‘a’. Whatever beast is chosen to be  $i$ ,  $-i$  is the other square root of  $-1$ .

We conclude our first chapter with a review the set of **Complex Numbers**. As you may recall, the complex numbers fill an algebraic gap left by the real numbers. There is no real number  $x$  with  $x^2 = -1$ , since for any real number  $x^2 \geq 0$ . However, we could formally extract square roots and write  $x = \pm\sqrt{-1}$ . We build the complex numbers by relabelling the quantity  $\sqrt{-1}$  as  $i$ , the unfortunately mis-named **imaginary unit**.<sup>6</sup> The number  $i$ , while not a real number, is defined so that it plays along well with real numbers and acts very much like any other radical expression. For instance,  $3(2i) = 6i$ ,  $7i - 3i = 4i$ ,  $(2 - 7i) + (3 + 4i) = 5 - 3i$ , and so forth. The key properties which distinguish  $i$  from the real numbers are listed below.

### Definition 18 The imaginary unit

The imaginary unit  $i$  satisfies the two following properties:

1.  $i^2 = -1$
2. If  $c$  is a real number with  $c \geq 0$  then  $\sqrt{-c} = i\sqrt{c}$

Property 1 in Definition 18 establishes that  $i$  does act as a square root of  $-1$ , and property 2 establishes what we mean by the ‘principal square root’ of a negative real number. In property 2, it is important to remember the restriction on  $c$ . For example, it is perfectly acceptable to say  $\sqrt{-4} = i\sqrt{4} = i(2) = 2i$ . However,  $\sqrt{(-4)} \neq i\sqrt{-4}$ , otherwise, we’d get

$$2 = \sqrt{4} = \sqrt{-(-4)} = i\sqrt{-4} = i(2i) = 2i^2 = 2(-1) = -2,$$

which is unacceptable. The moral of this story is that the general properties of radicals do not apply for even roots of negative quantities. With Definition 18 in place, we are now in position to define the **complex numbers**.

### Definition 19 Complex number

A **complex number** is a number of the form  $a + bi$ , where  $a$  and  $b$  are real numbers and  $i$  is the imaginary unit. The set of complex numbers is denoted  $\mathbb{C}$ .

Complex numbers include things you’d normally expect, like  $3 + 2i$  and  $\frac{2}{5} - i\sqrt{3}$ . However, don’t forget that  $a$  or  $b$  could be zero, which means numbers like  $3i$  and  $6$  are also complex numbers. In other words, don’t forget that the complex numbers *include* the real numbers, so  $0$  and  $\pi - \sqrt{21}$  are both considered complex numbers. The arithmetic of complex numbers is as you would expect. The only things you need to remember are the two properties in Definition 18. The next example should help recall how these animals behave.

### Example 12 Arithmetic with complex numbers

Perform the indicated operations.

<sup>6</sup>Some Technical Mathematics textbooks label it ‘ $j$ ’. While it carries the adjective ‘imaginary’, these numbers have essential real-world implications. For example, every electronic device owes its existence to the study of ‘imaginary’ numbers.

1.  $(1 - 2i) - (3 + 4i)$
2.  $(1 - 2i)(3 + 4i)$
3.  $\frac{1 - 2i}{3 - 4i}$
  
4.  $\sqrt{-3}\sqrt{-12}$
5.  $\sqrt{(-3)(-12)}$
6.  $(x - [1 + 2i])(x - [1 - 2i])$

**SOLUTION**

1. As mentioned earlier, we treat expressions involving  $i$  as we would any other radical. We distribute and combine like terms:

$$\begin{aligned}(1 - 2i) - (3 + 4i) &= 1 - 2i - 3 - 4i && \text{Distribute} \\ &= -2 - 6i && \text{Gather like terms}\end{aligned}$$

Technically, we'd have to rewrite our answer  $-2 - 6i$  as  $(-2) + (-6)i$  to be (in the strictest sense) 'in the form  $a + bi$ '. That being said, even pedants have their limits, and we'll consider  $-2 - 6i$  good enough.

2. Using the Distributive Property (a.k.a. F.O.I.L.), we get

$$\begin{aligned}(1 - 2i)(3 + 4i) &= (1)(3) + (1)(4i) - (2i)(3) - (2i)(4i) && \text{F.O.I.L.} \\ &= 3 + 4i - 6i - 8i^2 \\ &= 3 - 2i - 8(-1) && i^2 = -1 \\ &= 3 - 2i + 8 \\ &= 11 - 2i\end{aligned}$$

3. How in the world are we supposed to simplify  $\frac{1-2i}{3-4i}$ ? Well, we deal with the denominator  $3 - 4i$  as we would any other denominator containing two terms, one of which is a square root: we and multiply both numerator and denominator by  $3 + 4i$ , the (complex) conjugate of  $3 - 4i$ . Doing so produces

$$\begin{aligned}\frac{1 - 2i}{3 - 4i} &= \frac{(1 - 2i)(3 + 4i)}{(3 - 4i)(3 + 4i)} && \text{Equivalent Fractions} \\ &= \frac{3 + 4i - 6i - 8i^2}{9 - 16i^2} && \text{F.O.I.L.} \\ &= \frac{3 - 2i - 8(-1)}{9 - 16(-1)} && i^2 = -1 \\ &= \frac{11 - 2i}{25} \\ &= \frac{11}{25} - \frac{2}{25}i\end{aligned}$$

4. We use property 2 of Definition 18 first, then apply the rules of radicals applicable to real numbers to get  $\sqrt{-3}\sqrt{-12} = (i\sqrt{3})(i\sqrt{12}) = i^2\sqrt{3 \cdot 12} = -\sqrt{36} = -6$ .

5. We adhere to the order of operations here and perform the multiplication before the radical to get  $\sqrt{(-3)(-12)} = \sqrt{36} = 6$ .

6. We can brute force multiply using the distributive property and see that

$$\begin{aligned}(x - [1 + 2i])(x - [1 - 2i]) &= x^2 - x[1 - 2i] - x[1 + 2i] + [1 - 2i][1 + 2i] \\ &\quad \text{F.O.I.L.} \\ &= x^2 - x + 2ix - x - 2ix + 1 - 2i + 2i - 4i^2 \\ &\quad \text{Distribute} \\ &= x^2 - 2x + 1 - 4(-1) \quad \text{Gather like terms} \\ &= x^2 - 2x + 5 \quad i^2 = -1\end{aligned}$$

This type of factoring will be revisited in Section ??.

In the previous example, we used the idea of a ‘conjugate’ to divide two complex numbers. (You may recall using conjugates to rationalize expressions involving square roots.) More generally, the **complex conjugate** of a complex number  $a + bi$  is the number  $a - bi$ . The notation commonly used for complex conjugation is a ‘bar’:  $\overline{a + bi} = a - bi$ . For example,  $\overline{3 + 2i} = 3 - 2i$  and  $\overline{3 - 2i} = 3 + 2i$ . To find  $\overline{6}$ , we note that  $\overline{6} = \overline{6 + 0i} = 6 - 0i = 6$ , so  $\overline{6} = 6$ . Similarly,  $\overline{4i} = -4i$ , since  $\overline{4i} = \overline{0 + 4i} = 0 - 4i = -4i$ . Note that  $3 + \sqrt{5} = 3 + \sqrt{5}$ , not  $3 - \sqrt{5}$ , since  $3 + \sqrt{5} = 3 + \sqrt{5} + 0i = 3 + \sqrt{5} - 0i = 3 + \sqrt{5}$ . Here, the conjugation specified by the ‘bar’ notation involves reversing the sign before  $i = \sqrt{-1}$ , not before  $\sqrt{5}$ . The properties of the conjugate are summarized in the following theorem.

#### Theorem 4 Properties of the Complex Conjugate

Let  $z$  and  $w$  be complex numbers.

- $\overline{\bar{z}} = z$
- $\overline{z + w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z}\bar{w}$
- $\overline{z^n} = (\bar{z})^n$ , for any natural number  $n$
- $z$  is a real number if and only if  $\bar{z} = z$ .

Essentially, Theorem 4 says that complex conjugation works well with addition, multiplication and powers. The proofs of these properties can best be achieved by writing out  $z = a + bi$  and  $w = c + di$  for real numbers  $a, b, c$  and  $d$ . Next, we compute the left and right sides of each equation and verify that they are the same.

The proof of the first property is a very quick exercise. To prove the second property, we compare  $\overline{z + w}$  with  $\bar{z} + \bar{w}$ . We have  $\overline{z + w} = \overline{a + bi + c + di} = a - bi + c - di$ . To find  $\bar{z} + \bar{w}$ , we first compute

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$$

so

$$\overline{z + w} = \overline{(a + c) + (b + d)i} = (a + c) - (b + d)i = a + c - bi - di = a - bi + c - di = \bar{z} + \bar{w}$$

As such, we have established  $\overline{z+w} = \bar{z} + \bar{w}$ . The proof for multiplication works similarly. The proof that the conjugate works well with powers can be viewed as a repeated application of the product rule, and is best proved using a technique called Mathematical Induction. The last property is a characterization of real numbers. If  $z$  is real, then  $z = a + 0i$ , so  $\bar{z} = a - 0i = a = z$ . On the other hand, if  $z = \bar{z}$ , then  $a + bi = a - bi$  which means  $b = -b$  so  $b = 0$ . Hence,  $z = a + 0i = a$  and is real.

We now consider the problem of solving quadratic equations. Consider  $x^2 - 2x + 5 = 0$ . The discriminant  $b^2 - 4ac = -16$  is negative, so we know by Theorem ?? there are no *real* solutions, since the Quadratic Formula would involve the term  $\sqrt{-16}$ . Complex numbers, however, are built just for such situations, so we can go ahead and apply the Quadratic Formula to get:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Proof by Mathematical Induction is usually taught in Math 2000.

### Example 13 Finding complex solutions

Find the complex solutions to the following equations.

$$1. \frac{2x}{x+1} = x+3$$

$$2. 2t^4 = 9t^2 + 5$$

$$3. z^3 + 1 = 0$$

#### SOLUTION

1. Clearing fractions yields a quadratic equation so we collect all terms on one side and apply the Quadratic Formula.

$$\frac{2x}{x+1} = x+3$$

$$2x = (x+3)(x+1) \quad \text{Multiply by } (x+1) \text{ to clear denominators}$$

$$2x = x^2 + x + 3x + 3 \quad \text{F.O.I.L.}$$

$$2x = x^2 + 4x + 3 \quad \text{Gather like terms}$$

$$0 = x^2 + 2x + 3 \quad \text{Subtract } 2x$$

From here, we apply the Quadratic Formula

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2(1)} \quad \text{Quadratic Formula}$$

$$= \frac{-2 \pm \sqrt{-8}}{2} \quad \text{Simplify}$$

$$= \frac{-2 \pm i\sqrt{8}}{2} \quad \text{Definition of } i$$

$$= \frac{-2 \pm i2\sqrt{2}}{2} \quad \text{Product Rule for Radicals}$$

$$= \frac{\cancel{i}(-1 \pm i\sqrt{2})}{\cancel{i}} \quad \text{Factor and reduce}$$

$$= -1 \pm i\sqrt{2}$$

Remember, all real numbers are complex numbers, so ‘complex solutions’ means both real and non-real answers.

We get two answers:  $x = -1 + i\sqrt{2}$  and its conjugate  $x = -1 - i\sqrt{2}$ . Checking both of these answers reviews all of the salient points about complex number arithmetic and is therefore strongly encouraged.

2. Since we have three terms, and the exponent on one term ('4' on  $t^4$ ) is exactly twice the exponent on the other ('2' on  $t^2$ ), we have a Quadratic in Disguise. We proceed accordingly.

$$\begin{aligned} 2t^4 &= 9t^2 + 5 \\ 2t^4 - 9t^2 - 5 &= 0 \quad \text{Subtract } 9t^2 \text{ and } 5 \\ (2t^2 + 1)(t^2 - 5) &= 0 \quad \text{Factor} \\ 2t^2 + 1 = 0 \quad \text{or} \quad t^2 &= 5 \quad \text{Zero Product Property} \end{aligned}$$

From  $2t^2 + 1 = 0$  we get  $2t^2 = -1$ , or  $t^2 = -\frac{1}{2}$ . We extract square roots as follows:

$$t = \pm \sqrt{-\frac{1}{2}} = \pm i\sqrt{\frac{1}{2}} = \pm i\frac{\sqrt{1}}{\sqrt{2}} = \pm i\frac{1}{\sqrt{2}} = \pm \frac{i\sqrt{2}}{2},$$

where we have rationalized the denominator per convention. From  $t^2 = 5$ , we get  $t = \pm\sqrt{5}$ . In total, we have four complex solutions - two real:  $t = \pm\sqrt{5}$  and two non-real:  $t = \pm\frac{i\sqrt{2}}{2}$ .

3. To find the *real* solutions to  $z^3 + 1 = 0$ , we can subtract the 1 from both sides and extract cube roots:  $z^3 = -1$ , so  $z = \sqrt[3]{-1} = -1$ . It turns out there are two more non-real complex number solutions to this equation. To get at these, we factor:

$$\begin{aligned} z^3 + 1 &= 0 \\ (z + 1)(z^2 - z + 1) &= 0 \quad \text{Factor (Sum of Two Cubes)} \\ z + 1 = 0 \quad \text{or} \quad z^2 - z + 1 &= 0 \end{aligned}$$

From  $z + 1 = 0$ , we get our real solution  $z = -1$ . From  $z^2 - z + 1 = 0$ , we apply the Quadratic Formula to get:

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

Thus we get *three* solutions to  $z^3 + 1 = 0$  - one real:  $z = -1$  and two non-real:  $z = \frac{1 \pm i\sqrt{3}}{2}$ . As always, the reader is encouraged to test their algebraic mettle and check these solutions.

It is no coincidence that the non-real solutions to the equations in Example 13 appear in complex conjugate pairs. Any time we use the Quadratic Formula to solve an equation with real coefficients, the answers will form a complex conjugate pair owing to the  $\pm$  in the Quadratic Formula. This leads us to a generalization of Theorem ?? which we state on the next page.

**Theorem 5 Discriminant Theorem**

Given a Quadratic Equation  $AX^2 + BX + C = 0$ , where  $A, B$  and  $C$  are real numbers, let  $D = B^2 - 4AC$  be the discriminant.

- If  $D > 0$ , there are two distinct real number solutions to the equation.
- If  $D = 0$ , there is one (repeated) real number solution.

**Note:** ‘Repeated’ here comes from the fact that ‘both’ solutions  $\frac{-B \pm 0}{2A}$  reduce to  $-\frac{B}{2A}$ .

- If  $D < 0$ , there are two non-real solutions which form a complex conjugate pair.

We will have much more to say about complex solutions to equations in Section ?? and we will revisit Theorem 5 then.

# Exercises 1.4

## Problems

In Exercises 1 – 10, use the given complex numbers  $z$  and  $w$  to find and simplify the following:

- $z + w$
- $zw$
- $z^2$
- $\frac{1}{z}$
- $\frac{z}{w}$
- $\frac{w}{z}$
- $\bar{z}$
- $z\bar{z}$
- $(\bar{z})^2$

1.  $z = 2 + 3i, w = 4i$

2.  $z = 1 + i, w = -i$

3.  $z = i, w = -1 + 2i$

4.  $z = 4i, w = 2 - 2i$

5.  $z = 3 - 5i, w = 2 + 7i$

6.  $z = -5 + i, w = 4 + 2i$

7.  $z = \sqrt{2} - i\sqrt{2}, w = \sqrt{2} + i\sqrt{2}$

8.  $z = 1 - i\sqrt{3}, w = -1 - i\sqrt{3}$

9.  $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i, w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

10.  $z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, w = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

In Exercises 11 – 20, simplify the quantity.

11.  $\sqrt{-49}$

15.  $\sqrt{-9}\sqrt{-16}$

12.  $\sqrt{-9}$

16.  $\sqrt{(-9)(-16)}$

13.  $\sqrt{-25}\sqrt{-4}$

17.  $\sqrt{-(-9)}$

14.  $\sqrt{(-25)(-4)}$

18.  $-\sqrt{(-9)}$

We know that  $i^2 = -1$  which means  $i^3 = i^2 \cdot i = (-1) \cdot i = -i$  and  $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$ . In Exercises 19 – 28, use this information to simplify the given power of  $i$ .

19.  $i^5$

23.  $i^{15}$

20.  $i^6$

24.  $i^{26}$

21.  $i^7$

25.  $i^{117}$

22.  $i^8$

26.  $i^{304}$

In Exercises 27 – 36, find all complex solutions.

27.  $3x^2 + 6 = 4x$

28.  $15t^2 + 2t + 5 = 3t(t^2 + 1)$

29.  $3y^2 + 4 = y^4$

30.  $\frac{2}{1-w} = w$

31.  $\frac{y}{3} - \frac{3}{y} = y$

32.  $\frac{x^3}{2x-1} = \frac{x}{3}$

33.  $x = \frac{2}{\sqrt{5}-x}$

34.  $\frac{5y^4+1}{y^2-1} = 3y^2$

35.  $z^4 = 16$

36. Multiply and simplify:  $(x - [3 - i\sqrt{23}]) (x - [3 + i\sqrt{23}])$

# 2: RELATIONS AND FUNCTIONS

## 2.1 Relations

From one point of view, all of Precalculus can be thought of as studying sets of points in the plane. With the Cartesian Plane now fresh in our memory we can discuss those sets in more detail and as usual, we begin with a definition.

### Definition 20      Relations in the Cartesian Plane

A relation is a set of points in the plane.

Since relations are sets, we can describe them using the techniques presented in Section ???. That is, we can describe a relation verbally, using the roster method, or using set-builder notation. Since the elements in a relation are points in the plane, we often try to describe the relation graphically or algebraically as well. Depending on the situation, one method may be easier or more convenient to use than another. As an example, consider the relation  $R = \{(-2, 1), (4, 3), (0, -3)\}$ . As written,  $R$  is described using the roster method. Since  $R$  consists of points in the plane, we follow our instinct and plot the points. Doing so produces the **graph** of  $R$ : see Figure 2.1.

In the following example, we graph a variety of relations.

### Example 14      Graphing relations

Graph the following relations.

1.  $A = \{(0, 0), (-3, 1), (4, 2), (-3, 2)\}$
2.  $HLS_1 = \{(x, 3) \mid -2 \leq x \leq 4\}$
3.  $HLS_2 = \{(x, 3) \mid -2 \leq x < 4\}$
4.  $V = \{(3, y) \mid y \text{ is a real number}\}$
5.  $H = \{(x, y) \mid y = -2\}$
6.  $R = \{(x, y) \mid 1 < y \leq 3\}$

#### SOLUTION

1. To graph  $A$ , we simply plot all of the points which belong to  $A$ , as shown below on the left.
2. Don't let the notation in this part fool you. The name of this relation is  $HLS_1$ , just like the name of the relation in number 1 was  $A$ . The letters and numbers are just part of its name, just like the numbers and letters of the phrase 'King George III' were part of George's name. In words,  $\{(x, 3) \mid -2 \leq x \leq 4\}$  reads 'the set of points  $(x, 3)$  such that  $-2 \leq x \leq 4$ '. All of these points have the same  $y$ -coordinate, 3, but the  $x$ -coordinate is allowed to vary between  $-2$  and  $4$ , inclusive. Some of the points which belong to  $HLS_1$  include some friendly points like:  $(-2, 3)$ ,  $(-1, 3)$ ,  $(0, 3)$ ,  $(1, 3)$ ,  $(2, 3)$ ,  $(3, 3)$ , and  $(4, 3)$ . However,  $HLS_1$  also contains the points  $(0.829, 3)$ ,  $(-\frac{5}{6}, 3)$ ,  $(\sqrt{\pi}, 3)$ , and so on. It is impossible to list all of these points, which is why the variable  $x$  is used. Plotting several friendly representative points should convince you that  $HLS_1$  describes the horizontal line segment from the point  $(-2, 3)$  up to and including the point  $(4, 3)$ .
3.  $HLS_2$  is hauntingly similar to  $HLS_1$ . In fact, the only difference between the two is that instead of ' $-2 \leq x \leq 4$ ' we have ' $-2 \leq x < 4$ '. This means that we still get a horizontal line segment which includes  $(-2, 3)$  and extends to  $(4, 3)$ , but we do *not* include  $(4, 3)$  because of the strict

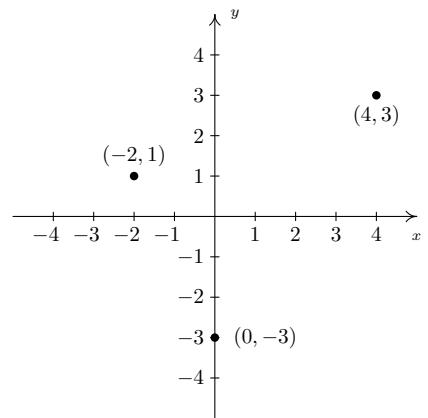


Figure 2.1: The graph of the relation  $R = \{(-2, 1), (4, 3), (0, -3)\}$

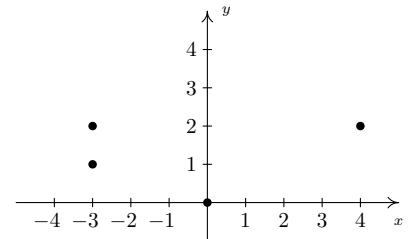


Figure 2.2: The graph of  $A$

Listing the points in a line segment is *really* impossible. The interested reader is encouraged to research countable versus uncountable sets.

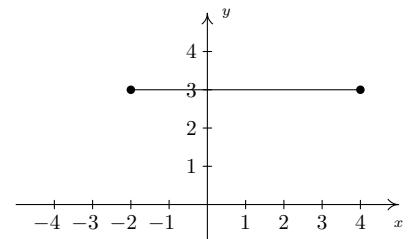


Figure 2.3: The graph of  $HLS_1$

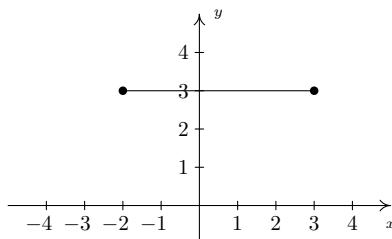
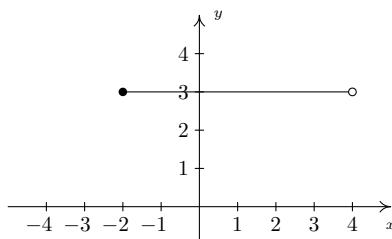
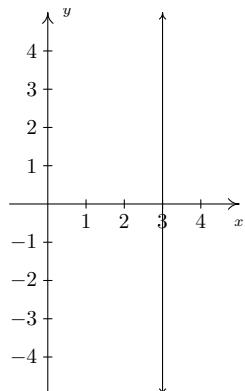
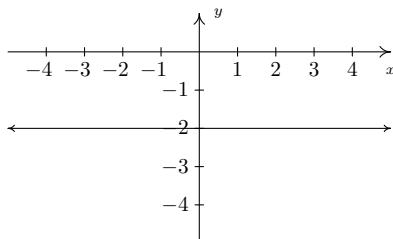

 This is NOT the correct graph of  $HLS_2$ 

 The graph of  $HLS_2$ 

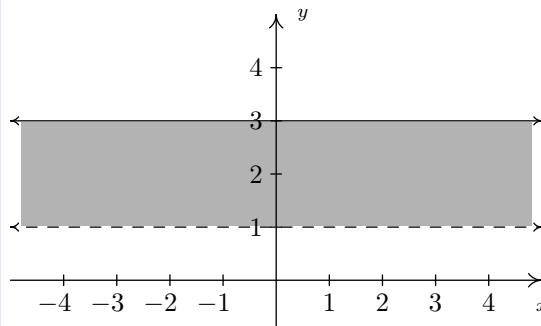
 Figure 2.5: Getting the right graph for  $HLS_2$ 

When we say you should plot some points in the relation  $H$ , the word ‘some’ is a relative term. It may take 5, 10, or 50 points until you see the pattern, depending on the relation.


 Figure 2.6: The graph of  $V$ 

 Figure 2.7: The graph of  $H$ 

inequality  $x < 4$ . How do we denote this on our graph? It is a common mistake to make the graph start at  $(-2, 3)$  end at  $(3, 3)$  as pictured below on the left. The problem with this graph is that we are forgetting about the points like  $(3.1, 3)$ ,  $(3.5, 3)$ ,  $(3.9, 3)$ ,  $(3.99, 3)$ , and so forth. There is no real number that comes ‘immediately before’ 4, so to describe the set of points we want, we draw the horizontal line segment starting at  $(-2, 3)$  and draw an open circle at  $(4, 3)$  as depicted below on the right.

4. Next, we come to the relation  $V$ , described as the set of points  $(3, y)$  such that  $y$  is a real number. All of these points have an  $x$ -coordinate of 3, but the  $y$ -coordinate is free to be whatever it wants to be, without restriction. Plotting a few ‘friendly’ points of  $V$  should convince you that all the points of  $V$  lie on the vertical line  $x = 3$ . Since there is no restriction on the  $y$ -coordinate, we put arrows on the end of the portion of the line we draw to indicate it extends indefinitely in both directions. The graph of  $V$  is below on the left.
5. Though written slightly differently, the relation  $H = \{(x, y) | y = -2\}$  is similar to the relation  $V$  above in that only one of the coordinates, in this case the  $y$ -coordinate, is specified, leaving  $x$  to be ‘free’. Plotting some representative points gives us the horizontal line  $y = -2$ .
6. For our last example, we turn to  $R = \{(x, y) | 1 < y \leq 3\}$ . As in the previous example,  $x$  is free to be whatever it likes. The value of  $y$ , on the other hand, while not completely free, is permitted to roam between 1 and 3 excluding 1, but including 3. After plotting some friendly elements of  $R$ , it should become clear that  $R$  consists of the region between the horizontal lines  $y = 1$  and  $y = 3$ . Since  $R$  requires that the  $y$ -coordinates be greater than 1, but not equal to 1, we dash the line  $y = 1$  to indicate that those points do not belong to  $R$ .


 Figure 2.4: The graph of  $R$ 

The relations  $V$  and  $H$  in the previous example lead us to our final way to describe relations: **algebraically**. We can more succinctly describe the points in  $V$  as those points which satisfy the equation ‘ $x = 3$ ’. Most likely, you have seen equations like this before. Depending on the context, ‘ $x = 3$ ’ could mean we have solved an equation for  $x$  and arrived at the solution  $x = 3$ . In this case, however, ‘ $x = 3$ ’ describes a set of points in the plane whose  $x$ -coordinate is 3. Similarly, the relation  $H$  above can be described by the equation ‘ $y = -2$ ’. At

some point in your mathematical upbringing, you probably learned the following.

#### Key Idea 8 Equations of Vertical and Horizontal Lines

- The graph of the equation  $x = a$  is a **vertical line** through  $(a, 0)$ .
- The graph of the equation  $y = b$  is a **horizontal line** through  $(0, b)$ .

Given that the very simple equations  $x = a$  and  $y = b$  produced lines, it's natural to wonder what shapes other equations might yield. Thus our next objective is to study the graphs of equations in a more general setting as we continue to unite Algebra and Geometry.

### 2.1.1 Graphs of Equations

In this section, we delve more deeply into the connection between Algebra and Geometry by focusing on graphing relations described by equations. The main idea of this section is the following.

#### Key Idea 9 The Fundamental Graphing Principle

The graph of an equation is the set of points which satisfy the equation. That is, a point  $(x, y)$  is on the graph of an equation if and only if  $x$  and  $y$  satisfy the equation.

Here, 'x and y satisfy the equation' means 'x and y make the equation true'. It is at this point that we gain some insight into the word 'relation'. If the equation to be graphed contains both  $x$  and  $y$ , then the equation itself is what is relating the two variables. More specifically, in the next two examples, we consider the graph of the equation  $x^2 + y^3 = 1$ . Even though it is not specifically spelled out, what we are doing is graphing the relation  $R = \{(x, y) | x^2 + y^3 = 1\}$ . The points  $(x, y)$  we graph belong to the *relation R* and are necessarily *related* by the equation  $x^2 + y^3 = 1$ , since it is those pairs of  $x$  and  $y$  which make the equation true.

#### Example 15 Checking to see if a point lies on a graph

Determine whether or not  $(2, -1)$  is on the graph of  $x^2 + y^3 = 1$ .

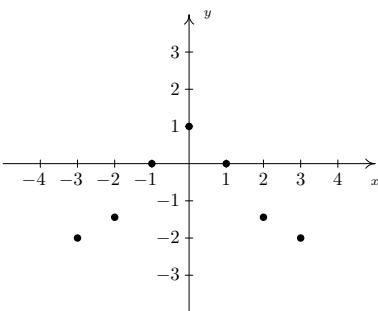
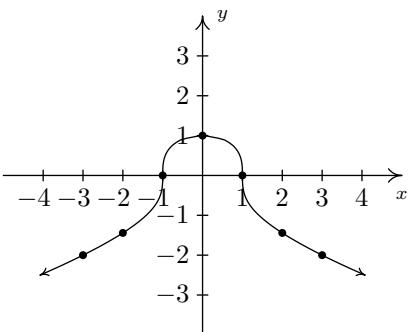
**SOLUTION** We substitute  $x = 2$  and  $y = -1$  into the equation to see if the equation is satisfied.

$$\begin{array}{rcl} (2)^2 + (-1)^3 & \stackrel{?}{=} & 1 \\ 4 + (-1) & \neq & 1 \end{array}$$

Hence,  $(2, -1)$  is **not** on the graph of  $x^2 + y^3 = 1$ . □

We could spend hours randomly guessing and checking to see if points are on the graph of the equation. A more systematic approach is outlined in the following example.

$x$	$y$	$(x, y)$
-3	-2	(-3, -2)
-2	$-\sqrt[3]{3}$	(-2, $-\sqrt[3]{3}$ )
-1	0	(-1, 0)
0	1	(0, 1)
1	0	(1, 0)
2	$-\sqrt[3]{3}$	(2, $-\sqrt[3]{3}$ )
3	-2	(3, -2)

Figure 2.8: Points on the curve  $x^2 + y^3 = 1$ Figure 2.9: The completed graph of  $x^2 + y^3 = 1$ **Example 16 Determining points on a graph systematically**Graph  $x^2 + y^3 = 1$ .

**SOLUTION** To efficiently generate points on the graph of this equation, we first solve for  $y$

$$\begin{aligned} x^2 + y^3 &= 1 \\ y^3 &= 1 - x^2 \\ \sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\ y &= \sqrt[3]{1 - x^2} \end{aligned}$$

We now substitute a value in for  $x$ , determine the corresponding value  $y$ , and plot the resulting point  $(x, y)$ . For example, substituting  $x = -3$  into the equation yields

$$y = \sqrt[3]{1 - x^2} = \sqrt[3]{1 - (-3)^2} = \sqrt[3]{-8} = -2,$$

so the point  $(-3, -2)$  is on the graph. Continuing in this manner, we generate a table of points which are on the graph of the equation. These points are then plotted in the plane as shown in Figure 2.8.

Remember, these points constitute only a small sampling of the points on the graph of this equation. To get a better idea of the shape of the graph, we could plot more points until we feel comfortable ‘connecting the dots’. Doing so would result in a curve similar to the one pictured in Figure 2.9.

Don’t worry if you don’t get all of the little bends and curves just right – Calculus is where the art of precise graphing takes center stage. For now, we will settle with our naive ‘plug and plot’ approach to graphing. If you feel like all of this tedious computation and plotting is beneath you, then you can try inputting the equation into a graphing calculator or an online tool such as Wolfram Alpha.

Of all of the points on the graph of an equation, the places where the graph crosses or touches the axes hold special significance. These are called the **intercepts** of the graph. Intercepts come in two distinct varieties:  $x$ -intercepts and  $y$ -intercepts. They are defined below.

**Definition 21  $x$ - and  $y$ -intercepts**

Suppose the graph of an equation is given.

- A point on a graph which is also on the  $x$ -axis is called an  **$x$ -intercept** of the graph.
- A point on a graph which is also on the  $y$ -axis is called an  **$y$ -intercept** of the graph.

In our previous example the graph had two  $x$ -intercepts,  $(-1, 0)$  and  $(1, 0)$ , and one  $y$ -intercept,  $(0, 1)$ . The graph of an equation can have any number of intercepts, including none at all! Since  $x$ -intercepts lie on the  $x$ -axis, we can find them by setting  $y = 0$  in the equation. Similarly, since  $y$ -intercepts lie on the  $y$ -axis, we can find them by setting  $x = 0$  in the equation. Keep in mind, intercepts are *points* and therefore must be written as ordered pairs. To summarize,

**Key Idea 10 Finding the Intercepts of the Graph of an Equation**

Given an equation involving  $x$  and  $y$ , we find the intercepts of the graph as follows:

- $x$ -intercepts have the form  $(x, 0)$ ; set  $y = 0$  in the equation and solve for  $x$ .
- $y$ -intercepts have the form  $(0, y)$ ; set  $x = 0$  in the equation and solve for  $y$ .

Another fact which you may have noticed about the graph in the previous example is that it seems to be symmetric about the  $y$ -axis. To actually prove this analytically, we assume  $(x, y)$  is a generic point on the graph of the equation. That is, we assume  $x^2 + y^3 = 1$  is true. As we learned in Section 1.3, the point symmetric to  $(x, y)$  about the  $y$ -axis is  $(-x, y)$ . To show that the graph is symmetric about the  $y$ -axis, we need to show that  $(-x, y)$  satisfies the equation  $x^2 + y^3 = 1$ , too. Substituting  $(-x, y)$  into the equation gives

$$\begin{array}{rcl} (-x)^2 + (y)^3 & \stackrel{?}{=} & 1 \\ x^2 + y^3 & \checkmark & 1 \end{array}$$

Since we are assuming the original equation  $x^2 + y^3 = 1$  is true, we have shown that  $(-x, y)$  satisfies the equation (since it leads to a true result) and hence is on the graph. In this way, we can check whether the graph of a given equation possesses any of the symmetries discussed in Section 1.3. We summarize the procedure in the following result.

**Key Idea 11 Testing the Graph of an Equation for Symmetry**

To test the graph of an equation for symmetry

- about the  $y$ -axis — substitute  $(-x, y)$  into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the  $y$ -axis.
- about the  $x$ -axis — substitute  $(x, -y)$  into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the  $x$ -axis.
- about the origin - substitute  $(-x, -y)$  into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the origin.

Intercepts and symmetry are two tools which can help us sketch the graph of an equation analytically, as demonstrated in the next example.

**Example 17 Finding intercepts and testing for symmetry**

Find the  $x$ - and  $y$ -intercepts (if any) of the graph of  $(x - 2)^2 + y^2 = 1$ . Test for symmetry. Plot additional points as needed to complete the graph.

**SOLUTION** To look for  $x$ -intercepts, we set  $y = 0$  and solve

$$\begin{aligned}
 (x - 2)^2 + y^2 &= 1 \\
 (x - 2)^2 + 0^2 &= 1 \\
 (x - 2)^2 &= 1 \\
 \sqrt{(x - 2)^2} &= \sqrt{1} \quad \text{extract square roots} \\
 x - 2 &= \pm 1 \\
 x &= 2 \pm 1 \\
 x &= 3, 1
 \end{aligned}$$

We get two answers for  $x$  which correspond to two  $x$ -intercepts:  $(1, 0)$  and  $(3, 0)$ . Turning our attention to  $y$ -intercepts, we set  $x = 0$  and solve

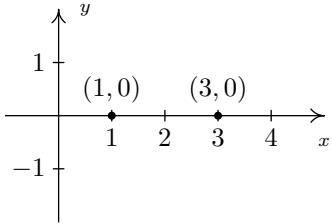


Figure 2.10: Plotting the data so far

$$\begin{aligned}
 (x - 2)^2 + y^2 &= 1 \\
 (0 - 2)^2 + y^2 &= 1 \\
 4 + y^2 &= 1 \\
 y^2 &= -3
 \end{aligned}$$

Since there is no real number which squares to a negative number (Do you remember why?), we are forced to conclude that the graph has no  $y$ -intercepts. We plot our results so far in Figure 2.10.

Moving along to symmetry, we can immediately dismiss the possibility that the graph is symmetric about the  $y$ -axis or the origin. If the graph possessed either of these symmetries, then the fact that  $(1, 0)$  is on the graph would mean  $(-1, 0)$  would have to be on the graph. (Why?) Since  $(-1, 0)$  would be another  $x$ -intercept (and we've found all of these), the graph can't have  $y$ -axis or origin symmetry. The only symmetry left to test is symmetry about the  $x$ -axis. To that end, we substitute  $(x, -y)$  into the equation and simplify

$$\begin{aligned}
 (x - 2)^2 + y^2 &= 1 \\
 (x - 2)^2 + (-y)^2 &\stackrel{?}{=} 1 \\
 (x - 2)^2 + y^2 &\stackrel{\checkmark}{=} 1
 \end{aligned}$$

Since we have obtained our original equation, we know the graph is symmetric about the  $x$ -axis. This means we can cut our 'plug and plot' time in half: whatever happens below the  $x$ -axis is reflected above the  $x$ -axis, and vice-versa. Proceeding as we did in the previous example, we obtain the plot shown in Figure 2.11.

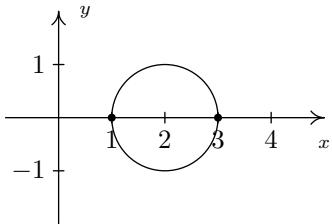


Figure 2.11: The final result

A couple of remarks are in order. First, it is entirely possible to choose a value for  $x$  which does not correspond to a point on the graph. For example, in the previous example, if we solve for  $y$  as is our custom, we get

$$y = \pm\sqrt{1 - (x - 2)^2}.$$

Upon substituting  $x = 0$  into the equation, we would obtain

$$y = \pm\sqrt{1 - (0 - 2)^2} = \pm\sqrt{1 - 4} = \pm\sqrt{-3},$$

which is not a real number. This means there are no points on the graph with an  $x$ -coordinate of 0. When this happens, we move on and try another point. This is another drawback of the 'plug-and-plot' approach to graphing equations. Luckily, we will devote much of the remainder of this book to developing techniques which allow us to graph entire families of equations quickly. Second, it is instructive to show what would have happened had we tested the equation

By the end of this course, you'll be able to accurately graph a wide variety of equations, without the use of a calculator, if you can believe it!

in the last example for symmetry about the  $y$ -axis. Substituting  $(-x, y)$  into the equation yields

$$\begin{aligned}(x - 2)^2 + y^2 &= 1 \\ (-x - 2)^2 + y^2 &\stackrel{?}{=} 1 \\ ((-1)(x + 2))^2 + y^2 &\stackrel{?}{=} 1 \\ (x + 2)^2 + y^2 &\stackrel{?}{=} 1.\end{aligned}$$

This last equation does not *appear* to be equivalent to our original equation. However, to actually prove that the graph is not symmetric about the  $y$ -axis, we need to find a point  $(x, y)$  on the graph whose reflection  $(-x, y)$  is not. Our  $x$ -intercept  $(1, 0)$  fits this bill nicely, since if we substitute  $(-1, 0)$  into the equation we get

$$\begin{aligned}(x - 2)^2 + y^2 &\stackrel{?}{=} 1 \\ (-1 - 2)^2 + 0^2 &\neq 1 \\ 9 &\neq 1.\end{aligned}$$

This proves that  $(-1, 0)$  is not on the graph.

# Exercises 2.1

## Problems

In Exercises 1 – 21, graph the given relation.

1.  $\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$

2.  $\{(-2, 0), (-1, 1), (-1, -1), (0, 2), (0, -2), (1, 3), (1, -3)\}$

3.  $\{(m, 2m) \mid m = 0, \pm 1, \pm 2\}$

4.  $\left\{\left(\frac{6}{k}, k\right) \mid k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\right\}$

5.  $\{(n, 4 - n^2) \mid n = 0, \pm 1, \pm 2\}$

6.  $\{(\sqrt{j}, j) \mid j = 0, 1, 4, 9\}$

7.  $\{(x, -2) \mid x > -4\}$

8.  $\{(x, 3) \mid x \leq 4\}$

9.  $\{(-1, y) \mid y > 1\}$

10.  $\{(2, y) \mid y \leq 5\}$

11.  $\{(-2, y) \mid -3 < y \leq 4\}$

12.  $\{(3, y) \mid -4 \leq y < 3\}$

13.  $\{(x, 2) \mid -2 \leq x < 3\}$

14.  $\{(x, -3) \mid -4 < x \leq 4\}$

15.  $\{(x, y) \mid x > -2\}$

16.  $\{(x, y) \mid x \leq 3\}$

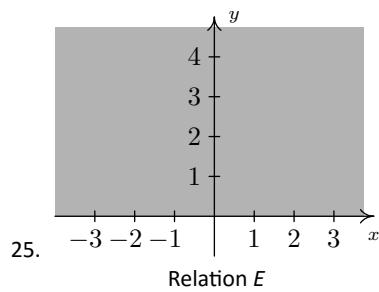
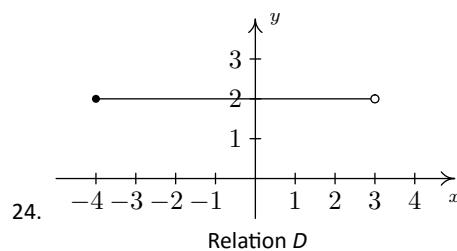
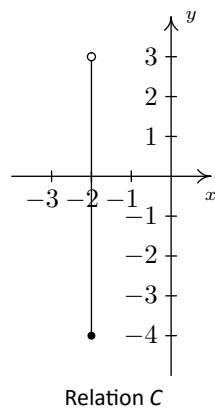
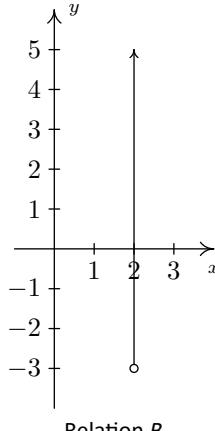
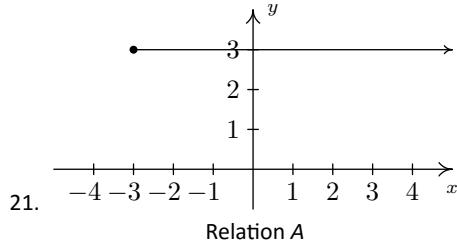
17.  $\{(x, y) \mid y < 4\}$

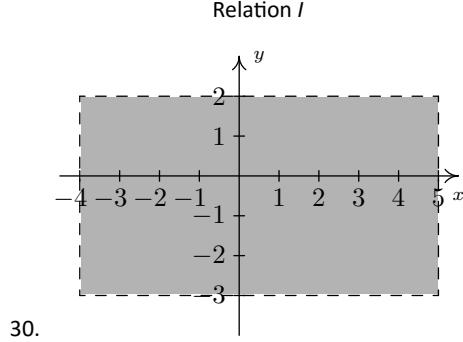
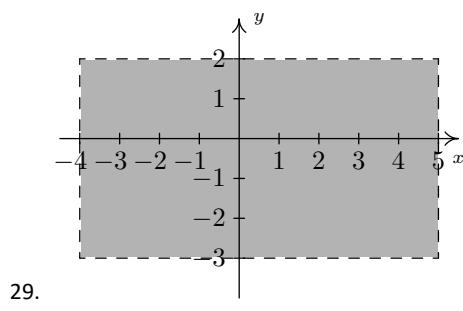
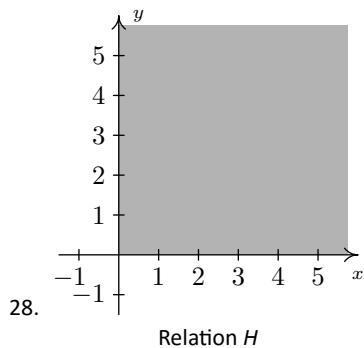
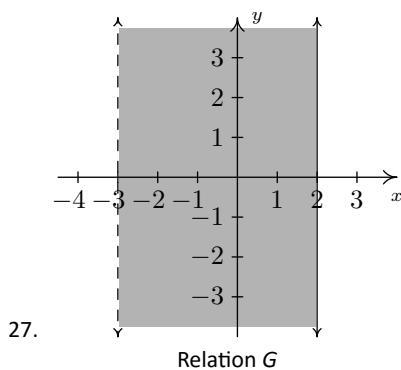
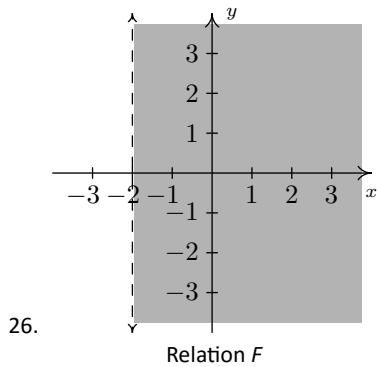
18.  $\{(x, y) \mid x \leq 3, y < 2\}$

19.  $\{(x, y) \mid x > 0, y < 4\}$

20.  $\{(x, y) \mid -\sqrt{2} \leq x \leq \frac{2}{3}, \pi < y \leq \frac{9}{2}\}$

In Exercises 21 – 30, describe the given relation using either the roster or set-builder method.





In Exercises 31 – 36, graph the given line.

31.  $x = -2$

32.  $x = 3$

33.  $y = 3$

34.  $y = -2$

35.  $x = 0$

36.  $y = 0$

Some relations are fairly easy to describe in words or with the roster method but are rather difficult, if not impossible, to graph. For Exercises 37 – 40, discuss with your classmates how you might graph the given relation.

37.  $\{(x, y) \mid x \text{ is an odd integer, and } y \text{ is an even integer}\}$

38.  $\{(x, 1) \mid x \text{ is an irrational number}\}$

39.  $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), \dots\}$

40.  $\{\dots, (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), \dots\}$

For each equation given in Exercises 41 – 52, (a) Find the  $x$  and  $y$  intercepts of the graph, if any exist; (b) Follow the procedure in Example 16 to create a table of sample points on the graph of the equation; (c) Plot the sample points and create a rough sketch of the graph of the equation; Test for symmetry. If the equation appears to fail any of the symmetry tests, find a point on the graph of the equation whose reflection fails to be on the graph as was done at the end of Example 17.

41.  $y = x^2 + 1$

42.  $y = x^2 - 2x - 8$

43.  $y = x^3 - x$

44.  $y = \frac{x^3}{4} - 3x$

45.  $y = \sqrt{x - 2}$

46.  $y = 2\sqrt{x + 4} - 2$

47.  $3x - y = 7$

48.  $3x - 2y = 10$

49.  $(x + 2)^2 + y^2 = 16$

50.  $x^2 - y^2 = 1$

51.  $4y^2 - 9x^2 = 36$

52.  $x^3y = -4$
53. With the help of your classmates, find examples of equations whose graphs possess
- symmetry about the  $x$ -axis only
  - symmetry about the  $y$ -axis only

- symmetry about the origin only
- symmetry about the  $x$ -axis,  $y$ -axis, and origin

Can you find an example of an equation whose graph possesses exactly two of the symmetries listed above? Why or why not?

## 2.2 Introduction to Functions

One of the core concepts in College Algebra is the **function**. There are many ways to describe a function and we begin by defining a function as a special kind of relation.

### Definition 22 Function

A relation in which each  $x$ -coordinate is matched with only one  $y$ -coordinate is said to describe  $y$  as a **function** of  $x$ .

### Example 18 Determining if a relation is a function

Which of the following relations describe  $y$  as a function of  $x$ ?

1.  $R_1 = \{(-2, 1), (1, 3), (1, 4), (3, -1)\}$
2.  $R_2 = \{(-2, 1), (1, 3), (2, 3), (3, -1)\}$

**SOLUTION** A quick scan of the points in  $R_1$  reveals that the  $x$ -coordinate 1 is matched with two *different*  $y$ -coordinates: namely 3 and 4. Hence in  $R_1$ ,  $y$  is not a function of  $x$ . On the other hand, every  $x$ -coordinate in  $R_2$  occurs only once which means each  $x$ -coordinate has only one corresponding  $y$ -coordinate. So,  $R_2$  does represent  $y$  as a function of  $x$ .

Note that in the previous example, the relation  $R_2$  contained two different points with the same  $y$ -coordinates, namely  $(1, 3)$  and  $(2, 3)$ . Remember, in order to say  $y$  is a function of  $x$ , we just need to ensure the same  $x$ -coordinate isn't used in more than one point.

To see what the function concept means geometrically, we graph  $R_1$  and  $R_2$  in the plane.

The fact that the  $x$ -coordinate 1 is matched with two different  $y$ -coordinates in  $R_1$  presents itself graphically as the points  $(1, 3)$  and  $(1, 4)$  lying on the same vertical line,  $x = 1$ . If we turn our attention to the graph of  $R_2$ , we see that no two points of the relation lie on the same vertical line. We can generalize this idea as follows

### Theorem 6 The Vertical Line Test

A set of points in the plane represents  $y$  as a function of  $x$  if and only if no two points lie on the same vertical line.

We will have occasion later in the text to concern ourselves with the concept of  $x$  being a function of  $y$ . In this case,  $R_1$  represents  $x$  as a function of  $y$ ;  $R_2$  does not.

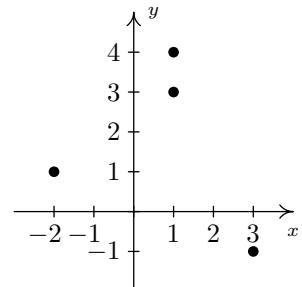


Figure 2.12: The graph of  $R_1$

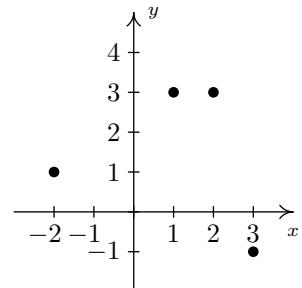
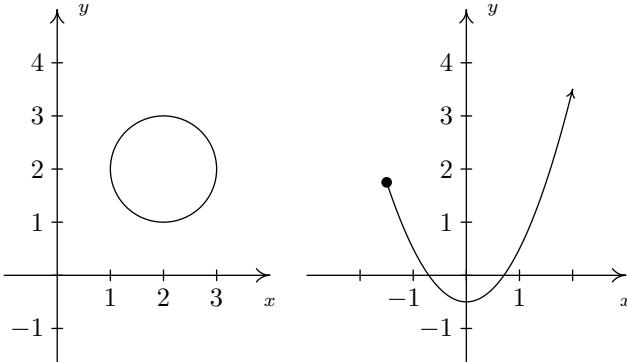


Figure 2.13: The graph of  $R_2$

It is worth taking some time to meditate on the Vertical Line Test; it will check to see how well you understand the concept of ‘function’ as well as the concept of ‘graph’.

### Example 19 Using the Vertical Line Test

Use the Vertical Line Test to determine which of the following relations describes  $y$  as a function of  $x$ .

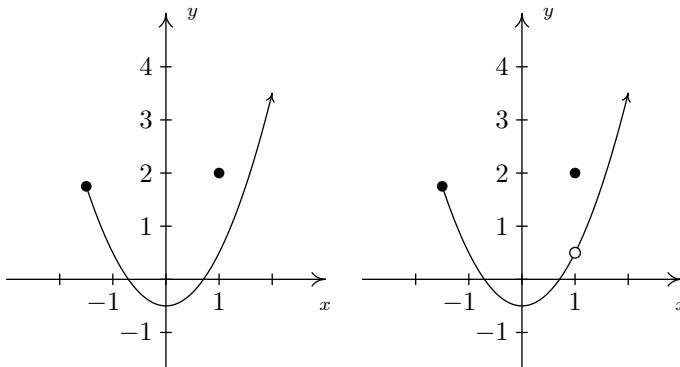
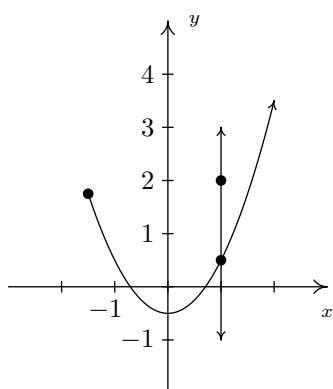
The graph of  $R$ The graph of  $S$ 

**SOLUTION** Looking at the graph of  $R$ , we can easily imagine a vertical line crossing the graph more than once. Hence,  $R$  does not represent  $y$  as a function of  $x$ . However, in the graph of  $S$ , every vertical line crosses the graph at most once, so  $S$  does represent  $y$  as a function of  $x$ .  $\square$

In the previous test, we say that the graph of the relation  $R$  **fails** the Vertical Line Test, whereas the graph of  $S$  **passes** the Vertical Line Test. Note that in the graph of  $R$  there are infinitely many vertical lines which cross the graph more than once. However, to fail the Vertical Line Test, all you need is one vertical line that fits the bill, as the next example illustrates.

#### Example 20 Using the Vertical Line Test

Use the Vertical Line Test to determine which of the following relations describes  $y$  as a function of  $x$ .

The graph of  $S_1$ The graph of  $S_2$ Figure 2.14:  $S_1$  and the line  $x = 1$ 

**SOLUTION** Both  $S_1$  and  $S_2$  are slight modifications to the relation  $S$  in the previous example whose graph we determined passed the Vertical Line Test. In both  $S_1$  and  $S_2$ , it is the addition of the point  $(1, 2)$  which threatens to cause trouble. In  $S_1$ , there is a point on the curve with  $x$ -coordinate 1 just below  $(1, 2)$ , which means that both  $(1, 2)$  and this point on the curve lie on the vertical line  $x = 1$ . (See the picture below and the left.) Hence, the graph of  $S_1$  fails the Vertical Line Test, so  $y$  is not a function of  $x$  here. However, in  $S_2$  notice that the point with  $x$ -coordinate 1 on the curve has been omitted, leaving an ‘open circle’ there. Hence, the vertical line  $x = 1$  crosses the graph of  $S_2$  only at the point  $(1, 2)$ . Indeed, any vertical line will cross the graph at most once, so we

have that the graph of  $S_2$  passes the Vertical Line Test. Thus it describes  $y$  as a function of  $x$ .

Suppose a relation  $F$  describes  $y$  as a function of  $x$ . The sets of  $x$ - and  $y$ -coordinates are given special names which we define below.

### Definition 23 Domain and range

Suppose  $F$  is a relation which describes  $y$  as a function of  $x$ .

- The set of the  $x$ -coordinates of the points in  $F$  is called the **domain** of  $F$ .
- The set of the  $y$ -coordinates of the points in  $F$  is called the **range** of  $F$ .

We demonstrate finding the domain and range of functions given to us either graphically or via the roster method in the following example.

### Example 21 Finding domain and range

Find the domain and range of the function  $F = \{(-3, 2), (0, 1), (4, 2), (5, 2)\}$  and of the function  $G$  whose graph is given in Figure 2.15.

**SOLUTION** The domain of  $F$  is the set of the  $x$ -coordinates of the points in  $F$ , namely  $\{-3, 0, 4, 5\}$  and the range of  $F$  is the set of the  $y$ -coordinates, namely  $\{1, 2\}$ .

To determine the domain and range of  $G$ , we need to determine which  $x$  and  $y$  values occur as coordinates of points on the given graph. To find the domain, it may be helpful to imagine collapsing the curve to the  $x$ -axis and determining the portion of the  $x$ -axis that gets covered. This is called **projecting** the curve to the  $x$ -axis. Before we start projecting, we need to pay attention to two subtle notations on the graph: the arrowhead on the lower left corner of the graph indicates that the graph continues to curve downwards to the left forever more; and the open circle at  $(1, 3)$  indicates that the point  $(1, 3)$  isn't on the graph, but all points on the curve leading up to that point are.

We see from Figures ?? and 2.17 that if we project the graph of  $G$  to the  $x$ -axis, we get all real numbers less than 1. Using interval notation, we write the domain of  $G$  as  $(-\infty, 1)$ . To determine the range of  $G$ , we project the curve to the  $y$ -axis as follows:

Note that even though there is an open circle at  $(1, 3)$ , we still include the  $y$  value of 3 in our range, since the point  $(-1, 3)$  is on the graph of  $G$ . Referring to Figures 2.18 and ??, we see that the range of  $G$  is all real numbers less than or equal to 4, or, in interval notation,  $(-\infty, 4]$ .

All functions are relations, but not all relations are functions. Thus the equations which described the relations in Section 2.1 may or may not describe  $y$  as a function of  $x$ . The algebraic representation of functions is possibly the most important way to view them so we need a process for determining whether or not an equation of a relation represents a function. (We delay the discussion of finding the domain of a function given algebraically until Section 2.3.)

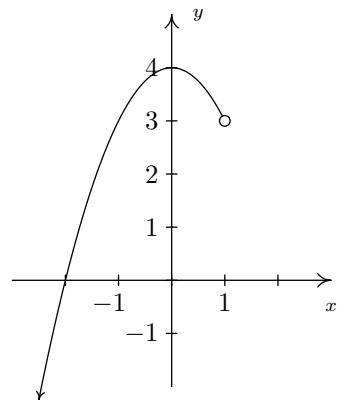


Figure 2.15: The graph of  $G$  for Example 21

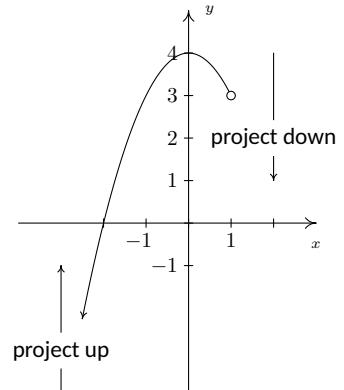


Figure 2.16: Projecting the graph onto the  $x$ -axis in Example 21

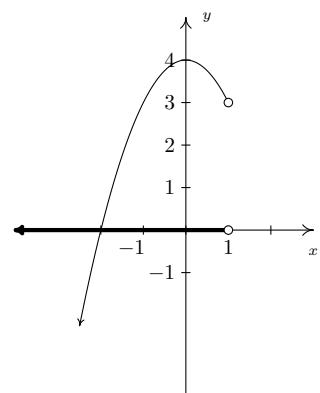


Figure 2.17: The domain of  $G$  in Example 21

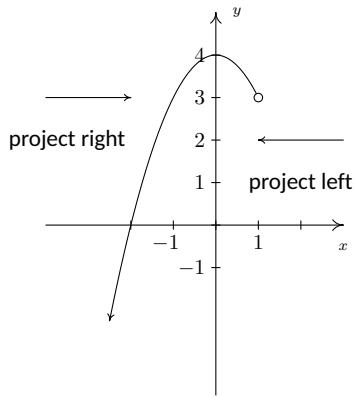
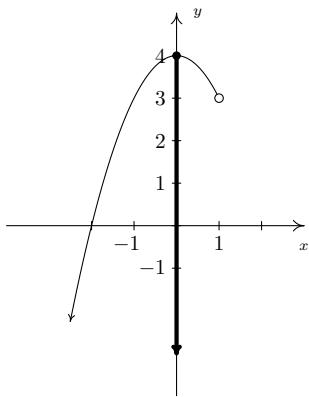


Figure 2.18: Projecting the graph onto the y-axis in Example 21

Figure 2.19: The range of  $G$  in Example 21

**Example 22 Functions defined by equations**  
Determine which equations represent  $y$  as a function of  $x$ .

1.  $x^3 + y^2 = 1$

2.  $x^2 + y^3 = 1$

3.  $x^2y = 1 - 3y$

**SOLUTION** For each of these equations, we solve for  $y$  and determine whether each choice of  $x$  will determine only one corresponding value of  $y$ .

1.

$$\begin{aligned}x^3 + y^2 &= 1 \\y^2 &= 1 - x^3 \\\sqrt{y^2} &= \sqrt{1 - x^3} \quad \text{extract square roots} \\y &= \pm\sqrt{1 - x^3}\end{aligned}$$

If we substitute  $x = 0$  into our equation for  $y$ , we get  $y = \pm\sqrt{1 - 0^3} = \pm 1$ , so that  $(0, 1)$  and  $(0, -1)$  are on the graph of this equation. Hence, this equation does not represent  $y$  as a function of  $x$ .

2.

$$\begin{aligned}x^2 + y^3 &= 1 \\y^3 &= 1 - x^2 \\\sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\y &= \sqrt[3]{1 - x^2}\end{aligned}$$

For every choice of  $x$ , the equation  $y = \sqrt[3]{1 - x^2}$  returns only **one** value of  $y$ . Hence, this equation describes  $y$  as a function of  $x$ .

3.

$$\begin{aligned}x^2y &= 1 - 3y \\x^2y + 3y &= 1 \\y(x^2 + 3) &= 1 \quad \text{factor} \\y &= \frac{1}{x^2 + 3}\end{aligned}$$

For each choice of  $x$ , there is only one value for  $y$ , so this equation describes  $y$  as a function of  $x$ .

## Exercises 2.2

### Problems

In Exercises 1 – 12, determine whether or not the relation represents  $y$  as a function of  $x$ . Find the domain and range of those relations which are functions.

1.  $\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$

2.  $\{(-3, 0), (1, 6), (2, -3), (4, 2), (-5, 6), (4, -9), (6, 2)\}$

3.  $\{(-3, 0), (-7, 6), (5, 5), (6, 4), (4, 9), (3, 0)\}$

4.  $\{(1, 2), (4, 4), (9, 6), (16, 8), (25, 10), (36, 12), \dots\}$

5.  $\{(x, y) \mid x \text{ is an odd integer, and } y \text{ is an even integer}\}$

6.  $\{(x, 1) \mid x \text{ is an irrational number}\}$

7.  $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), \dots\}$

8.  $\{\dots, (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), \dots\}^{17}$

9.  $\{(-2, y) \mid -3 < y < 4\}$

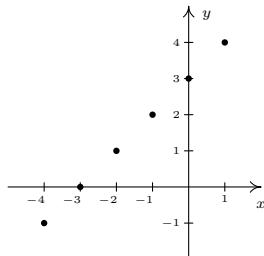
10.  $\{(x, 3) \mid -2 \leq x < 4\}$

11.  $\{(x, x^2) \mid x \text{ is a real number}\}$

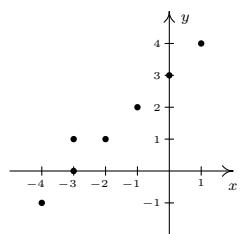
12.  $\{(x^2, x) \mid x \text{ is a real number}\}$

In Exercises 13 – 32, determine whether or not the relation represents  $y$  as a function of  $x$ . Find the domain and range of those relations which are functions.

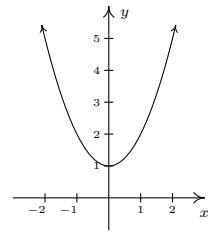
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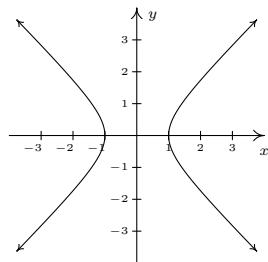
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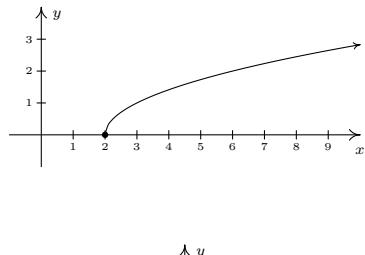
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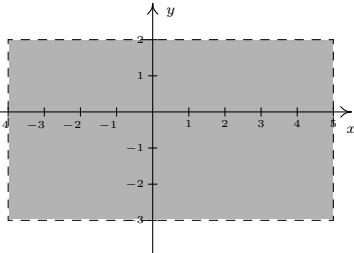
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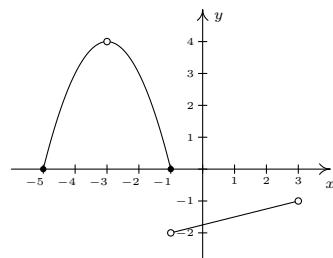
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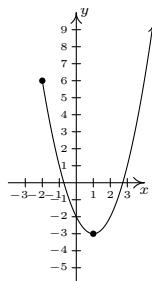
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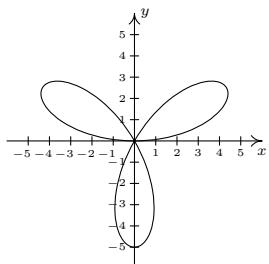
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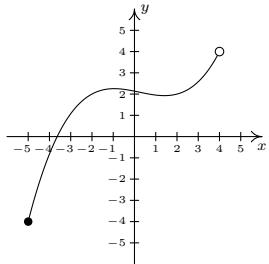
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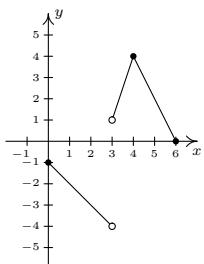
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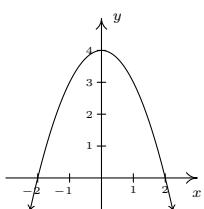
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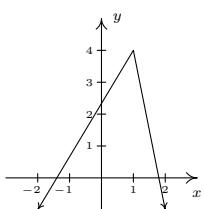
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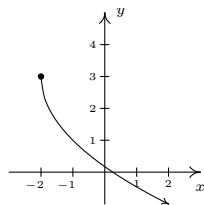
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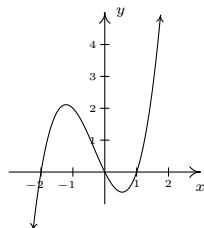
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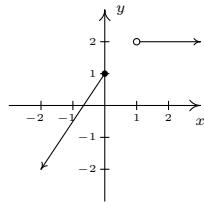
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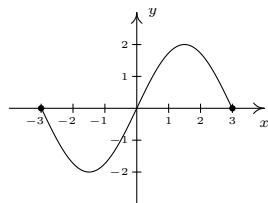
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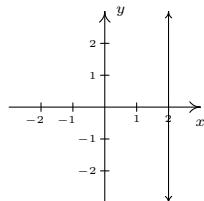
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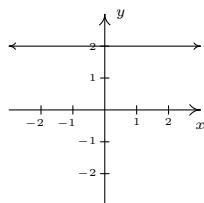
30.



31.



32.



In Exercises 33 – 47, determine whether or not the equation represents  $y$  as a function of  $x$ .

33.  $y = x^3 - x$

34.  $y = \sqrt{x - 2}$

35.  $x^3y = -4$

36.  $x^2 - y^2 = 1$

37.  $y = \frac{x}{x^2 - 9}$

38.  $x = -6$
39.  $x = y^2 + 4$
40.  $y = x^2 + 4$
41.  $x^2 + y^2 = 4$
42.  $y = \sqrt{4 - x^2}$
43.  $x^2 - y^2 = 4$
44.  $x^3 + y^3 = 4$
45.  $2x + 3y = 4$
46.  $2xy = 4$
47.  $x^2 = y^2$
48. Explain why the population  $P$  of Sasquatch in a given area is a function of time  $t$ . What would be the range of this function?
49. Explain why the relation between your classmates and their email addresses may not be a function. What about phone numbers and Social Security Numbers?
- Some relations are fairly easy to describe in words or with the roster method but are rather difficult, if not impossible, to graph. For Exercises 50 – 53, discuss with your classmates how you might graph the given relation.**
50.  $\{(x, y) \mid x \text{ is an odd integer, and } y \text{ is an even integer}\}$
51.  $\{(x, 1) \mid x \text{ is an irrational number}\}$
52.  $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), \dots\}$
53.  $\{\dots, (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), \dots\}$

## 2.3 Function Notation

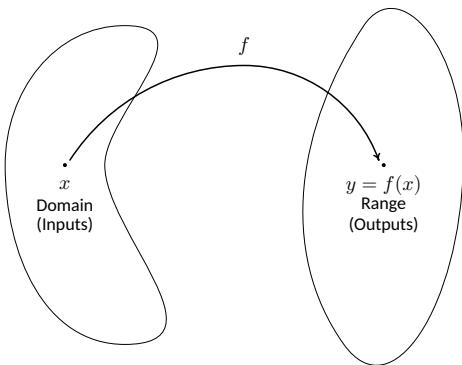


Figure 2.20: Graphical depiction of a function

In Definition 22, we described a function as a special kind of relation – one in which each  $x$ -coordinate is matched with only one  $y$ -coordinate. In this section, we focus more on the **process** by which the  $x$  is matched with the  $y$ . If we think of the domain of a function as a set of **inputs** and the range as a set of **outputs**, we can think of a function  $f$  as a process by which each input  $x$  is matched with only one output  $y$ . Since the output is completely determined by the input  $x$  and the process  $f$ , we symbolize the output with **function notation**: ' $f(x)$ ', read ' $f$  of  $x$ '. In other words,  $f(x)$  is the output which results by applying the process  $f$  to the input  $x$ . In this case, the parentheses here do not indicate multiplication, as they do elsewhere in Algebra. This can cause confusion if the context is not clear, so you must read carefully. This relationship is typically visualized using a diagram similar to the one in Figure 2.20.

The value of  $y$  is completely dependent on the choice of  $x$ . For this reason,  $x$  is often called the **independent variable**, or **argument** of  $f$ , whereas  $y$  is often called the **dependent variable**.

As we shall see, the process of a function  $f$  is usually described using an algebraic formula. For example, suppose a function  $f$  takes a real number and performs the following two steps, in sequence

1. Multiply by 3
2. Add 4

If we choose 5 as our input, in Step 1 we multiply by 3 to get  $(5)(3) = 15$ . In Step 2, we add 4 to our result from Step 1 which yields  $15 + 4 = 19$ . Using function notation, we would write  $f(5) = 19$  to indicate that the result of applying the process  $f$  to the input 5 gives the output 19. In general, if we use  $x$  for the input, applying Step 1 produces  $3x$ . Following with Step 2 produces  $3x + 4$  as our final output. Hence for an input  $x$ , we get the output  $f(x) = 3x + 4$ . Notice that to check our formula for the case  $x = 5$ , we replace the occurrence of  $x$  in the formula for  $f(x)$  with 5 to get  $f(5) = 3(5) + 4 = 15 + 4 = 19$ , as required.

### Example 23 Finding a formula for a function

Suppose a function  $g$  is described by applying the following steps, in sequence

1. add 4
2. multiply by 3

Determine  $g(5)$  and find an expression for  $g(x)$ .

**SOLUTION** Starting with 5, Step 1 gives  $5 + 4 = 9$ . Continuing with Step 2, we get  $(3)(9) = 27$ . To find a formula for  $g(x)$ , we start with our input  $x$ . Step 1 produces  $x + 4$ . We now wish to multiply this entire quantity by 3, so we use a parentheses:  $3(x + 4) = 3x + 12$ . Hence,  $g(x) = 3x + 12$ . We can check our formula by replacing  $x$  with 5 to get  $g(5) = 3(5) + 12 = 15 + 12 = 27 \checkmark$ .

Most of the functions we will encounter in Math 1010 will be described using formulas like the ones we developed for  $f(x)$  and  $g(x)$  above. Evaluating formulas using this function notation is a key skill for success in this and many other Math courses.

**Example 24** Using function notationLet  $f(x) = -x^2 + 3x + 4$ 

1. Find and simplify the following.

- (a)  $f(-1), f(0), f(2)$
- (b)  $f(2x), 2f(x)$
- (c)  $f(x+2), f(x)+2, f(x)+f(2)$

2. Solve  $f(x) = 4$ .**SOLUTION**1. (a) To find  $f(-1)$ , we replace every occurrence of  $x$  in the expression  $f(x)$  with  $-1$ 

$$\begin{aligned} f(-1) &= -(-1)^2 + 3(-1) + 4 \\ &= -(1) + (-3) + 4 \\ &= 0 \end{aligned}$$

Similarly,  $f(0) = -(0)^2 + 3(0) + 4 = 4$ , and  $f(2) = -(2)^2 + 3(2) + 4 = -4 + 6 + 4 = 6$ .(b) To find  $f(2x)$ , we replace every occurrence of  $x$  with the quantity  $2x$ 

$$\begin{aligned} f(2x) &= -(2x)^2 + 3(2x) + 4 \\ &= -(4x^2) + (6x) + 4 \\ &= -4x^2 + 6x + 4 \end{aligned}$$

The expression  $2f(x)$  means we multiply the expression  $f(x)$  by 2

$$\begin{aligned} 2f(x) &= 2(-x^2 + 3x + 4) \\ &= -2x^2 + 6x + 8 \end{aligned}$$

(c) To find  $f(x+2)$ , we replace every occurrence of  $x$  with the quantity  $x+2$ 

$$\begin{aligned} f(x+2) &= -(x+2)^2 + 3(x+2) + 4 \\ &= -(x^2 + 4x + 4) + (3x + 6) + 4 \\ &= -x^2 - 4x - 4 + 3x + 6 + 4 \\ &= -x^2 - x + 6 \end{aligned}$$

To find  $f(x) + 2$ , we add 2 to the expression for  $f(x)$ 

$$\begin{aligned} f(x) + 2 &= (-x^2 + 3x + 4) + 2 \\ &= -x^2 + 3x + 6 \end{aligned}$$

From our work above, we see  $f(2) = 6$  so that

$$\begin{aligned} f(x) + f(2) &= (-x^2 + 3x + 4) + 6 \\ &= -x^2 + 3x + 10 \end{aligned}$$

2. Since  $f(x) = -x^2 + 3x + 4$ , the equation  $f(x) = 4$  is equivalent to  $-x^2 + 3x + 4 = 4$ . Solving we get  $-x^2 + 3x = 0$ , or  $x(-x + 3) = 0$ . We get  $x = 0$  or  $x = 3$ , and we can verify these answers by checking that  $f(0) = 4$  and  $f(3) = 4$ .

A few notes about Example 24 are in order. First note the difference between the answers for  $f(2x)$  and  $2f(x)$ . For  $f(2x)$ , we are multiplying the *input* by 2; for  $2f(x)$ , we are multiplying the *output* by 2. As we see, we get entirely different results. Along these lines, note that  $f(x+2)$ ,  $f(x)+2$  and  $f(x)+f(2)$  are three *different* expressions as well. Even though function notation uses parentheses, as does multiplication, there is no general ‘distributive property’ of function notation. Finally, note the practice of using parentheses when substituting one algebraic expression into another; we highly recommend this practice as it will reduce careless errors.

Suppose now we wish to find  $r(3)$  for  $r(x) = \frac{2x}{x^2 - 9}$ . Substitution gives

$$r(3) = \frac{2(3)}{(3)^2 - 9} = \frac{6}{0},$$

which is undefined. (Why is this, again?) The number 3 is not an allowable input to the function  $r$ ; in other words, 3 is not in the domain of  $r$ . Which other real numbers are forbidden in this formula? We think back to arithmetic. The reason  $r(3)$  is undefined is because substitution results in a division by 0. To determine which other numbers result in such a transgression, we set the denominator equal to 0 and solve

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ \sqrt{x^2} &= \sqrt{9} \quad \text{extract square roots} \\ x &= \pm 3 \end{aligned}$$

As long as we substitute numbers other than 3 and  $-3$ , the expression  $r(x)$  is a real number. Hence, we write our domain in interval notation (see the Exercises for Section 1.3) as  $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ . When a formula for a function is given, we assume that the function is valid for all real numbers which make arithmetic sense when substituted into the formula. This set of numbers is often called the **implied domain** (or ‘implicit domain’) of the function. At this stage, there are only two mathematical sins we need to avoid: division by 0 and extracting even roots of negative numbers. The following example illustrates these concepts.

### Example 25 Determining an implied domain

Find the domain of the following functions.

- |   |  |
|---|--|
| 1. $g(x) = \sqrt{4 - 3x}$<br>3. $f(x) = \frac{2}{1 - \frac{4x}{x - 3}}$<br>5. $r(t) = \frac{4}{6 - \sqrt{t + 3}}$ | 2. $h(x) = \sqrt[5]{4 - 3x}$<br>4. $F(x) = \frac{\sqrt[4]{2x + 1}}{x^2 - 1}$<br>6. $I(x) = \frac{3x^2}{x}$ |
|---|--|

The ‘radicand’ is the expression ‘inside’ the radical.

### SOLUTION

- The potential disaster for  $g$  is if the radicand is negative. To avoid this, we set  $4 - 3x \geq 0$ . From this, we get  $3x \leq 4$  or  $x \leq \frac{4}{3}$ . What this shows is that as long as  $x \leq \frac{4}{3}$ , the expression  $4 - 3x \geq 0$ , and the formula  $g(x)$  returns a real number. Our domain is  $(-\infty, \frac{4}{3}]$ .
- The formula for  $h(x)$  is hauntingly close to that of  $g(x)$  with one key difference — whereas the expression for  $g(x)$  includes an even indexed root (namely a square root), the formula for  $h(x)$  involves an odd indexed root (the fifth root). Since odd roots of real numbers (even negative real numbers) are real numbers, there is no restriction on the inputs to  $h$ . Hence, the domain is  $(-\infty, \infty)$ .
- In the expression for  $f$ , there are two denominators. We need to make sure neither of them is 0. To that end, we set each denominator equal to 0 and solve. For the ‘small’ denominator, we get  $x - 3 = 0$  or  $x = 3$ . For the ‘large’ denominator

$$\begin{aligned} 1 - \frac{4x}{x-3} &= 0 \\ 1 &= \frac{4x}{x-3} \\ (1)(x-3) &= \left(\frac{4x}{x-3}\right)(x-3) \quad \text{clear denominators} \\ x-3 &= 4x \\ -3 &= 3x \\ -1 &= x \end{aligned}$$

So we get two real numbers which make denominators 0, namely  $x = -1$  and  $x = 3$ . Our domain is all real numbers except  $-1$  and  $3$ :  $(-\infty, -1) \cup (-1, 3) \cup (3, \infty)$ .

- In finding the domain of  $F$ , we notice that we have two potentially hazardous issues: not only do we have a denominator, we have a fourth (even-indexed) root. Our strategy is to determine the restrictions imposed by each part and select the real numbers which satisfy both conditions. To satisfy the fourth root, we require  $2x + 1 \geq 0$ . From this we get  $2x \geq -1$  or  $x \geq -\frac{1}{2}$ . Next, we round up the values of  $x$  which could cause trouble in the denominator by setting the denominator equal to 0. We get  $x^2 - 1 = 0$ , or  $x = \pm 1$ . Hence, in order for a real number  $x$  to be in the domain of  $F$ ,  $x \geq -\frac{1}{2}$  but  $x \neq \pm 1$ . In interval notation, this set is  $[-\frac{1}{2}, 1) \cup (1, \infty)$ .
- Don’t be put off by the ‘ $t$ ’ here. It is an independent variable representing a real number, just like  $x$  does, and is subject to the same restrictions. As in the previous problem, we have double danger here: we have a square root and a denominator. To satisfy the square root, we need a non-negative radicand so we set  $t + 3 \geq 0$  to get  $t \geq -3$ . Setting the denominator equal to zero gives  $6 - \sqrt{t+3} = 0$ , or  $\sqrt{t+3} = 6$ . Squaring both sides gives  $t + 3 = 36$ , or  $t = 33$ . Since we squared both sides in the course of solving this equation, we need to check our answer. Sure enough, when  $t = 33$ ,  $6 - \sqrt{t+3} = 6 - \sqrt{36} = 0$ , so  $t = 33$  will cause problems in the denominator. At last we can find the domain of  $r$ : we need  $t \geq -3$ , but  $t \neq 33$ . Our final answer is  $[-3, 33) \cup (33, \infty)$ .

Squaring both sides of an equation can introduce *extraneous solutions*. Do you remember why? Consider squaring both sides to ‘solve’  $\sqrt{t+1} = -2$ .

6. It's tempting to simplify  $I(x) = \frac{3x^2}{x} = 3x$ , and, since there are no longer any denominators, claim that there are no longer any restrictions. However, in simplifying  $I(x)$ , we are assuming  $x \neq 0$ , since  $\frac{0}{0}$  is undefined. Proceeding as before, we find the domain of  $I$  to be all real numbers except 0:  $(-\infty, 0) \cup (0, \infty)$ .

It is worth reiterating the importance of finding the domain of a function *before* simplifying, as evidenced by the function  $I$  in the previous example. Even though the formula  $I(x)$  simplifies to  $3x$ , it would be inaccurate to write  $I(x) = 3x$  without adding the stipulation that  $x \neq 0$ . It would be analogous to not reporting taxable income or some other sin of omission.

### 2.3.1 Modelling with Functions

The importance of Mathematics to our society lies in its value to approximate, or **model** real-world phenomenon. Whether it be used to predict the high temperature on a given day, determine the hours of daylight on a given day, or predict population trends of various and sundry real and mythical beasts, Mathematics is second only to literacy in the importance humanity's development.

It is important to keep in mind that any time Mathematics is used to approximate reality, there are always limitations to the model. For example, suppose grapes are on sale at the local market for \$1.50 per pound. Then one pound of grapes costs \$1.50, two pounds of grapes cost \$3.00, and so forth. Suppose we want to develop a formula which relates the cost of buying grapes to the amount of grapes being purchased. Since these two quantities vary from situation to situation, we assign them variables. Let  $c$  denote the cost of the grapes and let  $g$  denote the amount of grapes purchased. To find the cost  $c$  of the grapes, we multiply the amount of grapes  $g$  by the price \$1.50 dollars per pound to get

$$c = 1.5g$$

In order for the units to be correct in the formula,  $g$  must be measured in *pounds* of grapes in which case the computed value of  $c$  is measured in *dollars*. Since we're interested in finding the cost  $c$  given an amount  $g$ , we think of  $g$  as the independent variable and  $c$  as the dependent variable. Using the language of function notation, we write

$$c(g) = 1.5g$$

where  $g$  is the amount of grapes purchased (in pounds) and  $c(g)$  is the cost (in dollars). For example,  $c(5)$  represents the cost, in dollars, to purchase 5 pounds of grapes. In this case,  $c(5) = 1.5(5) = 7.5$ , so it would cost \$7.50. If, on the other hand, we wanted to find the *amount* of grapes we can purchase for \$5, we would need to set  $c(g) = 5$  and solve for  $g$ . In this case,  $c(g) = 1.5g$ , so solving  $c(g) = 5$  is equivalent to solving  $1.5g = 5$ . Doing so gives  $g = \frac{5}{1.5} = 3\bar{3}$ . This means we can purchase exactly  $3\bar{3}$  pounds of grapes for \$5. Of course, you would be hard-pressed to buy exactly  $3\bar{3}$  pounds of grapes, (you could get close... within a certain specified margin of error, perhaps) and this leads us to our next topic of discussion, the **applied domain**, or 'explicit domain' of a function.

Even though, mathematically,  $c(g) = 1.5g$  has no domain restrictions (there are no denominators and no even-indexed radicals), there are certain values of  $g$  that don't make any physical sense. For example,  $g = -1$  corresponds to 'purchasing'  $-1$  pounds of grapes. (Maybe this means *returning* a pound of grapes?)

Also, unless the ‘local market’ mentioned is the State of California (or some other exporter of grapes), it also doesn’t make much sense for  $g = 500,000,000$ , either. So the reality of the situation limits what  $g$  can be, and these limits determine the applied domain of  $g$ . Typically, an applied domain is stated explicitly. In this case, it would be common to see something like  $c(g) = 1.5g$ ,  $0 \leq g \leq 100$ , meaning the number of pounds of grapes purchased is limited from 0 up to 100. The upper bound here, 100 may represent the inventory of the market, or some other limit as set by local policy or law. Even with this restriction, our model has its limitations. As we saw above, it is virtually impossible to buy exactly 3.3 pounds of grapes so that our cost is exactly \$5. In this case, being sensible shoppers, we would most likely ‘round down’ and purchase 3 pounds of grapes or however close the market scale can read to 3.3 without being over. It is time for a more sophisticated example.

**Example 26 Height of a model rocket**

The height  $h$  in feet of a model rocket above the ground  $t$  seconds after lift-off is given by

$$h(t) = \begin{cases} -5t^2 + 100t, & \text{if } 0 \leq t \leq 20 \\ 0, & \text{if } t > 20 \end{cases}$$

1. Find and interpret  $h(10)$  and  $h(60)$ .
2. Solve  $h(t) = 375$  and interpret your answers.

**SOLUTION**

1. We first note that the independent variable here is  $t$ , chosen because it represents time. Secondly, the function is broken up into two rules: one formula for values of  $t$  between 0 and 20 inclusive, and another for values of  $t$  greater than 20. Since  $t = 10$  satisfies the inequality  $0 \leq t \leq 20$ , we use the first formula listed,  $h(t) = -5t^2 + 100t$ , to find  $h(10)$ . We get  $h(10) = -5(10)^2 + 100(10) = 500$ . Since  $t$  represents the number of seconds since lift-off and  $h(t)$  is the height above the ground in feet, the equation  $h(10) = 500$  means that 10 seconds after lift-off, the model rocket is 500 feet above the ground. To find  $h(60)$ , we note that  $t = 60$  satisfies  $t > 20$ , so we use the rule  $h(t) = 0$ . This function returns a value of 0 regardless of what value is substituted in for  $t$ , so  $h(60) = 0$ . This means that 60 seconds after lift-off, the rocket is 0 feet above the ground; in other words, a minute after lift-off, the rocket has already returned to Earth.
2. Since the function  $h$  is defined in pieces, we need to solve  $h(t) = 375$  in pieces. For  $0 \leq t \leq 20$ ,  $h(t) = -5t^2 + 100t$ , so for these values of  $t$ , we solve  $-5t^2 + 100t = 375$ . Rearranging terms, we get  $5t^2 - 100t + 375 = 0$ , and factoring gives  $5(t-5)(t-15) = 0$ . Our answers are  $t = 5$  and  $t = 15$ , and since both of these values of  $t$  lie between 0 and 20, we keep both solutions. For  $t > 20$ ,  $h(t) = 0$ , and in this case, there are no solutions to  $0 = 375$ . In terms of the model rocket, solving  $h(t) = 375$  corresponds to finding when, if ever, the rocket reaches 375 feet above the ground. Our two answers,  $t = 5$  and  $t = 15$  correspond to the rocket reaching this altitude twice – once 5 seconds after launch, and again 15 seconds after launch.

The type of function in the previous example is called a **piecewise-defined** function, or ‘piecewise’ function for short. Many real-world phenomena, income tax formulas for example, are modelled by such functions.

By the way, if we wanted to avoid using a piecewise function in Example 26, we could have used  $h(t) = -5t^2 + 100t$  on the explicit domain  $0 \leq t \leq 20$  because after 20 seconds, the rocket is on the ground and stops moving. In many cases, though, piecewise functions are your only choice, so it’s best to understand them well.

Mathematical modelling is not a one-section topic. It’s not even a one-*course* topic as is evidenced by undergraduate and graduate courses in mathematical modelling being offered at many universities. Thus our goal in this section cannot possibly be to tell you the whole story. What we can do is get you started. As we study new classes of functions, we will see what phenomena they can be used to model. In that respect, mathematical modelling cannot be a topic in a book, but rather, must be a theme of the book. For now, we have you explore some very basic models in the Exercises because you need to crawl to walk to run. As we learn more about functions, we’ll help you build your own models and get you on your way to applying Mathematics to your world.

## Exercises 2.3

### Problems

In Exercises 1 – 10, find an expression for  $f(x)$  and state its domain.

1.  $f$  is a function that takes a real number  $x$  and performs the following three steps in the order given: (1) multiply by 2; (2) add 3; (3) divide by 4.
2.  $f$  is a function that takes a real number  $x$  and performs the following three steps in the order given: (1) add 3; (2) multiply by 2; (3) divide by 4.
3.  $f$  is a function that takes a real number  $x$  and performs the following three steps in the order given: (1) divide by 4; (2) add 3; (3) multiply by 2.
4.  $f$  is a function that takes a real number  $x$  and performs the following three steps in the order given: (1) multiply by 2; (2) add 3; (3) take the square root.
5.  $f$  is a function that takes a real number  $x$  and performs the following three steps in the order given: (1) add 3; (2) multiply by 2; (3) take the square root.
6.  $f$  is a function that takes a real number  $x$  and performs the following three steps in the order given: (1) add 3; (2) take the square root; (3) multiply by 2.
7.  $f$  is a function that takes a real number  $x$  and performs the following three steps in the order given: (1) take the square root; (2) subtract 13; (3) make the quantity the denominator of a fraction with numerator 4.
8.  $f$  is a function that takes a real number  $x$  and performs the following three steps in the order given: (1) subtract 13; (2) take the square root; (3) make the quantity the denominator of a fraction with numerator 4.
9.  $f$  is a function that takes a real number  $x$  and performs the following three steps in the order given: (1) take the square root; (2) make the quantity the denominator of a fraction with numerator 4; (3) subtract 13.
10.  $f$  is a function that takes a real number  $x$  and performs the following three steps in the order given: (1) make the quantity the denominator of a fraction with numerator 4; (2) take the square root; (3) subtract 13.

In Exercises 11 – 18, use the given function  $f$  to find and simplify the following:

- $f(3)$
- $f(-1)$
- $f\left(\frac{3}{2}\right)$
- $f(4x)$
- $4f(x)$
- $f(-x)$
- $f(x - 4)$
- $f(x) - 4$
- $f(x^2)$

11.  $f(x) = 2x + 1$
12.  $f(x) = 3 - 4x$
13.  $f(x) = 2 - x^2$
14.  $f(x) = x^2 - 3x + 2$
15.  $f(x) = \frac{x}{x - 1}$
16.  $f(x) = \frac{2}{x^3}$
17.  $f(x) = 6$
18.  $f(x) = 0$

In Exercises 19 – 26, use the given function  $f$  to find and simplify the following:

- $f(2)$
- $f(-2)$
- $f(2a)$
- $2f(a)$
- $f(a + 2)$
- $f(a) + f(2)$
- $f\left(\frac{2}{a}\right)$
- $\frac{f(a)}{2}$
- $f(a + h)$

19.  $f(x) = 2x - 5$
20.  $f(x) = 5 - 2x$
21.  $f(x) = 2x^2 - 1$
22.  $f(x) = 3x^2 + 3x - 2$
23.  $f(x) = \sqrt{2x + 1}$
24.  $f(x) = 117$
25.  $f(x) = \frac{x}{2}$
26.  $f(x) = \frac{2}{x}$

In Exercises 27 – 34, use the given function  $f$  to find  $f(0)$  and solve  $f(x) = 0$ .

27.  $f(x) = 2x - 1$
28.  $f(x) = 3 - \frac{2}{5}x$
29.  $f(x) = 2x^2 - 6$
30.  $f(x) = x^2 - x - 12$
31.  $f(x) = \sqrt{x + 4}$

32.  $f(x) = \sqrt{1 - 2x}$

33.  $f(x) = \frac{3}{4 - x}$

34.  $f(x) = \frac{3x^2 - 12x}{4 - x^2}$

35. Let  $f(x) = \begin{cases} x + 5 & \text{if } x \leq -3 \\ \sqrt{9 - x^2} & \text{if } -3 < x \leq 3 \\ -x + 5 & \text{if } x > 3 \end{cases}$  Compute the following function values.

(a)  $f(-4)$

(d)  $f(3.001)$

(b)  $f(-3)$

(e)  $f(-3.001)$

(c)  $f(3)$

(f)  $f(2)$

36. Let  $f(x) = \begin{cases} x^2 & \text{if } x \leq -1 \\ \sqrt{1 - x^2} & \text{if } -1 < x \leq 1 \\ x & \text{if } x > 1 \end{cases}$  Compute the following function values.

(a)  $f(4)$

(d)  $f(0)$

(b)  $f(-3)$

(e)  $f(-1)$

(c)  $f(1)$

(f)  $f(-0.999)$

In Exercises 37 – 62, find the (implied) domain of the function.

37.  $f(x) = x^4 - 13x^3 + 56x^2 - 19$

38.  $f(x) = x^2 + 4$

39.  $f(x) = \frac{x - 2}{x + 1}$

40.  $f(x) = \frac{3x}{x^2 + x - 2}$

41.  $f(x) = \frac{2x}{x^2 + 3}$

42.  $f(x) = \frac{2x}{x^2 - 3}$

43.  $f(x) = \frac{x + 4}{x^2 - 36}$

44.  $f(x) = \frac{x - 2}{x - 2}$

45.  $f(x) = \sqrt{3 - x}$

46.  $f(x) = \sqrt{2x + 5}$

47.  $f(x) = 9x\sqrt{x + 3}$

48.  $f(x) = \frac{\sqrt{7 - x}}{x^2 + 1}$

49.  $f(x) = \sqrt{6x - 2}$

50.  $f(x) = \frac{6}{\sqrt{6x - 2}}$

51.  $f(x) = \sqrt[3]{6x - 2}$

52.  $f(x) = \frac{6}{4 - \sqrt{6x - 2}}$

53.  $f(x) = \frac{\sqrt{6x - 2}}{x^2 - 36}$

54.  $f(x) = \frac{\sqrt[3]{6x - 2}}{x^2 + 36}$

55.  $s(t) = \frac{t}{t - 8}$

56.  $Q(r) = \frac{\sqrt{r}}{r - 8}$

57.  $b(\theta) = \frac{\theta}{\sqrt{\theta - 8}}$

58.  $A(x) = \sqrt{x - 7} + \sqrt{9 - x}$

59.  $\alpha(y) = \sqrt[3]{\frac{y}{y - 8}}$

60.  $g(v) = \frac{1}{4 - \frac{1}{v^2}}$

61.  $T(t) = \frac{\sqrt{t} - 8}{5 - t}$

62.  $u(w) = \frac{w - 8}{5 - \sqrt{w}}$

63. The area  $A$  enclosed by a square, in square inches, is a function of the length of one of its sides  $x$ , when measured in inches. This relation is expressed by the formula  $A(x) = x^2$  for  $x > 0$ . Find  $A(3)$  and solve  $A(x) = 36$ . Interpret your answers to each. Why is  $x$  restricted to  $x > 0$ ?

64. The area  $A$  enclosed by a circle, in square meters, is a function of its radius  $r$ , when measured in meters. This relation is expressed by the formula  $A(r) = \pi r^2$  for  $r > 0$ . Find  $A(2)$  and solve  $A(r) = 16\pi$ . Interpret your answers to each. Why is  $r$  restricted to  $r > 0$ ?

65. The volume  $V$  enclosed by a cube, in cubic centimeters, is a function of the length of one of its sides  $x$ , when measured in centimeters. This relation is expressed by the formula  $V(x) = x^3$  for  $x > 0$ . Find  $V(5)$  and solve  $V(x) = 27$ . Interpret your answers to each. Why is  $x$  restricted to  $x > 0$ ?

66. The volume  $V$  enclosed by a sphere, in cubic feet, is a function of the radius of the sphere  $r$ , when measured in feet.

- This relation is expressed by the formula  $V(r) = \frac{4\pi}{3}r^3$  for  $r > 0$ . Find  $V(3)$  and solve  $V(r) = \frac{32\pi}{3}$ . Interpret your answers to each. Why is  $r$  restricted to  $r > 0$ ?
67. The volume  $V$  enclosed by a sphere, in cubic feet, is a function of the radius of the sphere  $r$ , when measured in feet. This relation is expressed by the formula  $V(r) = \frac{4\pi}{3}r^3$  for  $r > 0$ . Find  $V(3)$  and solve  $V(r) = \frac{32\pi}{3}$ . Interpret your answers to each. Why is  $r$  restricted to  $r > 0$ ?
68. The height of an object dropped from the roof of an eight story building is modeled by:  $h(t) = -16t^2 + 64$ ,  $0 \leq t \leq 2$ . Here,  $h$  is the height of the object off the ground, in feet,  $t$  seconds after the object is dropped. Find  $h(0)$  and solve  $h(t) = 0$ . Interpret your answers to each. Why is  $t$  restricted to  $0 \leq t \leq 2$ ?
69. The temperature  $T$  in degrees Fahrenheit  $t$  hours after 6 AM is given by  $T(t) = -\frac{1}{2}t^2 + 8t + 3$  for  $0 \leq t \leq 12$ . Find and interpret  $T(0)$ ,  $T(6)$  and  $T(12)$ .
70. The function  $C(x) = x^2 - 10x + 27$  models the cost, in *hundreds* of dollars, to produce  $x$  *thousand* pens. Find and interpret  $C(0)$ ,  $C(2)$  and  $C(5)$ .  
(The value  $C(0)$  is called the ‘fixed’ or ‘start-up’ cost. We’ll revisit this concept on page 73.)
71. Using data from the Bureau of Transportation Statistics, the average fuel economy  $F$  in miles per gallon for passenger cars in the US can be modelled by  $F(t) = -0.0076t^2 + 0.45t + 16$ ,  $0 \leq t \leq 28$ , where  $t$  is the number of years since 1980. Use your calculator to find  $F(0)$ ,  $F(14)$  and  $F(28)$ . Round your answers to two decimal places and interpret your answers to each.
72. The population of Sasquatch in Portage County can be modeled by the function  $P(t) = \frac{150t}{t+15}$ , where  $t$  represents the number of years since 1803. Find and interpret  $P(0)$  and  $P(205)$ . Discuss with your classmates what the applied domain and range of  $P$  should be.
73. For  $n$  copies of the book *Me and my Sasquatch*, a print on-demand company charges  $C(n)$  dollars, where  $C(n)$  is determined by the formula
- $$C(n) = \begin{cases} 15n & \text{if } 1 \leq n \leq 25 \\ 13.50n & \text{if } 25 < n \leq 50 \\ 12n & \text{if } n > 50 \end{cases}$$
- (a) Find and interpret  $C(20)$ .  
(b) How much does it cost to order 50 copies of the book? What about 51 copies?  
(c) Your answer to 73b should get you thinking. Suppose a bookstore estimates it will sell 50 copies of the book. How many books can, in fact, be ordered for the same price as those 50 copies? (Round your answer to a whole number of books.)
74. An on-line comic book retailer charges shipping costs according to the following formula
- $$S(n) = \begin{cases} 1.5n + 2.5 & \text{if } 1 \leq n \leq 14 \\ 0 & \text{if } n \geq 15 \end{cases}$$
- where  $n$  is the number of comic books purchased and  $S(n)$  is the shipping cost in dollars.
- (a) What is the cost to ship 10 comic books?  
(b) What is the significance of the formula  $S(n) = 0$  for  $n \geq 15$ ?
75. The cost  $C$  (in dollars) to talk  $m$  minutes a month on a mobile phone plan is modeled by
- $$C(m) = \begin{cases} 25 & \text{if } 0 \leq m \leq 1000 \\ 25 + 0.1(m - 1000) & \text{if } m > 1000 \end{cases}$$
- (a) How much does it cost to talk 750 minutes per month with this plan?  
(b) How much does it cost to talk 20 hours a month with this plan?  
(c) Explain the terms of the plan verbally.
76. In Section ?? we defined the set of **integers** as  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ . The **greatest integer of**  $x$ , denoted by  $[x]$ , is defined to be the largest integer  $k$  with  $k \leq x$ .  
**Note:** The use of the letter  $\mathbb{Z}$  for the integers is ostensibly because the German word *zahlen* means ‘to count’.
- (a) Find  $[0.785]$ ,  $[117]$ ,  $[-2.001]$ , and  $[\pi + 6]$   
(b) Discuss with your classmates how  $[x]$  may be described as a piecewise defined function.  
**HINT:** There are infinitely many pieces!  
(c) Is  $[a + b] = [a] + [b]$  always true? What if  $a$  or  $b$  is an integer? Test some values, make a conjecture, and explain your result.
- 77.
78. We have through our examples tried to convince you that, in general,  $f(a+b) \neq f(a)+f(b)$ . It has been our experience that students refuse to believe us so we’ll try again with a different approach. With the help of your classmates, find a function  $f$  for which the following properties are always true.
- (a)  $f(0) = f(-1+1) = f(-1) + f(1)$   
(b)  $f(5) = f(2+3) = f(2) + f(3)$   
(c)  $f(-6) = f(0-6) = f(0) - f(6)$   
(d)  $f(a+b) = f(a) + f(b)$  regardless of what two numbers we give you for  $a$  and  $b$ .
- How many functions did you find that failed to satisfy the conditions above? Did  $f(x) = x^2$  work? What about  $f(x) = \sqrt{x}$  or  $f(x) = 3x + 7$  or  $f(x) = \frac{1}{x}$ ? Did you find an attribute common to those functions that did succeed? You should have, because there is only one extremely special family of functions that actually works here. Thus we return to our previous statement, **in general**,  $f(a+b) \neq f(a) + f(b)$ .

## 2.4 Function Arithmetic

In the previous section we used the newly defined function notation to make sense of expressions such as ' $f(x) + 2$ ' and ' $2f(x)$ ' for a given function  $f$ . It would seem natural, then, that functions should have their own arithmetic which is consistent with the arithmetic of real numbers. The following definitions allow us to add, subtract, multiply and divide functions using the arithmetic we already know for real numbers.

### Definition 24 Function Arithmetic

Recall that if  $x$  is in the domains of both  $f$  and  $g$ , then we can say that  $x$  is an element of the intersection of the two domains.

- The **sum** of  $f$  and  $g$ , denoted  $f + g$ , is the function defined by the formula

$$(f + g)(x) = f(x) + g(x)$$

- The **difference** of  $f$  and  $g$ , denoted  $f - g$ , is the function defined by the formula

$$(f - g)(x) = f(x) - g(x)$$

- The **product** of  $f$  and  $g$ , denoted  $fg$ , is the function defined by the formula

$$(fg)(x) = f(x)g(x)$$

- The **quotient** of  $f$  and  $g$ , denoted  $\frac{f}{g}$ , is the function defined by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

provided  $g(x) \neq 0$ .

In other words, to add two functions, we add their outputs; to subtract two functions, we subtract their outputs, and so on. Note that while the formula  $(f+g)(x) = f(x)+g(x)$  looks suspiciously like some kind of distributive property, it is nothing of the sort; the addition on the left hand side of the equation is *function* addition, and we are using this equation to *define* the output of the new function  $f + g$  as the sum of the real number outputs from  $f$  and  $g$ .

### Example 27 Arithmetic with functions

Let  $f(x) = 6x^2 - 2x$  and  $g(x) = 3 - \frac{1}{x}$ .

1. Find  $(f + g)(-1)$
2. Find  $(fg)(2)$
3. Find the domain of  $g - f$  then find and simplify a formula for  $(g - f)(x)$ .
4. Find the domain of  $\left(\frac{g}{f}\right)$  then find and simplify a formula for  $\left(\frac{g}{f}\right)(x)$ .

### SOLUTION

1. To find  $(f + g)(-1)$  we first find  $f(-1) = 8$  and  $g(-1) = 4$ . By definition, we have that  $(f + g)(-1) = f(-1) + g(-1) = 8 + 4 = 12$ .

2. To find  $(fg)(2)$ , we first need  $f(2)$  and  $g(2)$ . Since  $f(2) = 20$  and  $g(2) = \frac{5}{2}$ , our formula yields  $(fg)(2) = f(2)g(2) = (20)\left(\frac{5}{2}\right) = 50$ .
3. One method to find the domain of  $g - f$  is to find the domain of  $g$  and of  $f$  separately, then find the intersection of these two sets. Owing to the denominator in the expression  $g(x) = 3 - \frac{1}{x}$ , we get that the domain of  $g$  is  $(-\infty, 0) \cup (0, \infty)$ . Since  $f(x) = 6x^2 - 2x$  is valid for all real numbers, we have no further restrictions. Thus the domain of  $g - f$  matches the domain of  $g$ , namely,  $(-\infty, 0) \cup (0, \infty)$ .

A second method is to analyze the formula for  $(g-f)(x)$  before simplifying and look for the usual domain issues. In this case,

$$(g - f)(x) = g(x) - f(x) = \left(3 - \frac{1}{x}\right) - (6x^2 - 2x),$$

so we find, as before, the domain is  $(-\infty, 0) \cup (0, \infty)$ .

Moving along, we need to simplify a formula for  $(g-f)(x)$ . In this case, we get common denominators and attempt to reduce the resulting fraction. Doing so, we get

$$\begin{aligned} (g - f)(x) &= g(x) - f(x) \\ &= \left(3 - \frac{1}{x}\right) - (6x^2 - 2x) \\ &= 3 - \frac{1}{x} - 6x^2 + 2x \\ &= \frac{3x}{x} - \frac{1}{x} - \frac{6x^3}{x} + \frac{2x^2}{x} \quad \text{get common denominators} \\ &= \frac{3x - 1 - 6x^3 - 2x^2}{x} \\ &= \frac{-6x^3 - 2x^2 + 3x - 1}{x} \end{aligned}$$

4. As in the previous example, we have two ways to approach finding the domain of  $\frac{g}{f}$ . First, we can find the domain of  $g$  and  $f$  separately, and find the intersection of these two sets. In addition, since  $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$ , we are introducing a new denominator, namely  $f(x)$ , so we need to guard against this being 0 as well. Our previous work tells us that the domain of  $g$  is  $(-\infty, 0) \cup (0, \infty)$  and the domain of  $f$  is  $(-\infty, \infty)$ . Setting  $f(x) = 0$  gives  $6x^2 - 2x = 0$  or  $x = 0, \frac{1}{3}$ . As a result, the domain of  $\frac{g}{f}$  is all real numbers except  $x = 0$  and  $x = \frac{1}{3}$ , or  $(-\infty, 0) \cup (0, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$ .

Alternatively, we may proceed as above and analyze the expression  $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$  before simplifying. In this case,

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{3 - \frac{1}{x}}{6x^2 - 2x}$$

We see immediately from the ‘little’ denominator that  $x \neq 0$ . To keep the ‘big’ denominator away from 0, we solve  $6x^2 - 2x = 0$  and get  $x = 0$  or

$x = \frac{1}{3}$ . Hence, as before, we find the domain of  $\frac{g}{f}$  to be

$$(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right).$$

Next, we find and simplify a formula for  $\left(\frac{g}{f}\right)(x)$ .

$$\begin{aligned} \left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} \\ &= \frac{3 - \frac{1}{x}}{6x^2 - 2x} \\ &= \frac{3 - \frac{1}{x}}{6x^2 - 2x} \cdot \frac{x}{x} \quad \text{simplify compound fractions} \\ &= \frac{\left(3 - \frac{1}{x}\right)x}{(6x^2 - 2x)x} \\ &= \frac{3x - 1}{(6x^2 - 2x)x} \\ &= \frac{3x - 1}{2x^2(3x - 1)} \quad \text{factor} \\ &= \frac{(3x - 1)^1}{2x^2(3x - 1)} \quad \text{cancel} \\ &= \frac{1}{2x^2} \end{aligned}$$

Please note the importance of finding the domain of a function *before* simplifying its expression. In number 4 in Example 27 above, had we waited to find the domain of  $\frac{g}{f}$  until after simplifying, we'd just have the formula  $\frac{1}{2x^2}$  to go by, and we would (incorrectly!) state the domain as  $(-\infty, 0) \cup (0, \infty)$ , since the other troublesome number,  $x = \frac{1}{3}$ , was cancelled away.

Next, we turn our attention to the **difference quotient** of a function.

We'll see what cancelling factors means geometrically in Chapter ??.

#### Definition 25 Difference quotient of a function

Given a function  $f$ , the **difference quotient** of  $f$  is the expression

$$\frac{f(x + h) - f(x)}{h}$$

We will revisit this concept in Section 3.1, but for now, we use it as a way to practice function notation and function arithmetic. For reasons which will become clear in Calculus, 'simplifying' a difference quotient means rewriting it in a form where the ' $h$ ' in the definition of the difference quotient cancels from the denominator. Once that happens, we consider our work to be done.

**Example 28 Computing difference quotients**

Find and simplify the difference quotients for the following functions

$$1. f(x) = x^2 - x - 2$$

$$2. g(x) = \frac{3}{2x+1}$$

$$3. r(x) = \sqrt{x}$$

**SOLUTION**

1. To find  $f(x + h)$ , we replace every occurrence of  $x$  in the formula  $f(x) = x^2 - x - 2$  with the quantity  $(x + h)$  to get

$$\begin{aligned} f(x + h) &= (x + h)^2 - (x + h) - 2 \\ &= x^2 + 2xh + h^2 - x - h - 2. \end{aligned}$$

So the difference quotient is

$$\begin{aligned} \frac{f(x + h) - f(x)}{h} &= \frac{(x^2 + 2xh + h^2 - x - h - 2) - (x^2 - x - 2)}{h} \\ &= \frac{x^2 + 2xh + h^2 - x - h - 2 - x^2 + x + 2}{h} \\ &= \frac{2xh + h^2 - h}{h} \\ &= \frac{h(2x + h - 1)}{h} && \text{factor} \\ &= \frac{h(2x + h - 1)}{h} && \text{cancel} \\ &= 2x + h - 1. \end{aligned}$$

2. To find  $g(x + h)$ , we replace every occurrence of  $x$  in the formula  $g(x) = \frac{3}{2x+1}$  with the quantity  $(x + h)$  to get

$$\begin{aligned} g(x + h) &= \frac{3}{2(x + h) + 1} \\ &= \frac{3}{2x + 2h + 1}, \end{aligned}$$

which yields

$$\begin{aligned} \frac{g(x + h) - g(x)}{h} &= \frac{\frac{3}{2x + 2h + 1} - \frac{3}{2x + 1}}{h} \\ &= \frac{\frac{3}{2x + 2h + 1} - \frac{3}{2x + 1}}{h} \cdot \frac{(2x + 2h + 1)(2x + 1)}{(2x + 2h + 1)(2x + 1)} \end{aligned}$$

$$\begin{aligned}
&= \frac{3(2x+1) - 3(2x+2h+1)}{h(2x+2h+1)(2x+1)} \\
&= \frac{6x+3 - 6x - 6h - 3}{h(2x+2h+1)(2x+1)} \\
&= \frac{-6h}{h(2x+2h+1)(2x+1)} \\
&= \frac{-6h}{h(2x+2h+1)(2x+1)} \\
&= \frac{-6}{(2x+2h+1)(2x+1)}.
\end{aligned}$$

Since we have managed to cancel the original ' $h$ ' from the denominator, we are done.

3. For  $r(x) = \sqrt{x}$ , we get  $r(x+h) = \sqrt{x+h}$  so the difference quotient is

$$\frac{r(x+h) - r(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

In order to cancel the ' $h$ ' from the denominator, we rationalize the *numerator* by multiplying by its conjugate.

$$\begin{aligned}
\frac{r(x+h) - r(x)}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} \\
&\quad \text{Multiply by the conjugate.} \\
&= \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} \quad \text{Difference of Squares.} \\
&= \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{\cancel{h}^1}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{1}{\sqrt{x+h} + \sqrt{x}}
\end{aligned}$$

Since we have removed the original ' $h$ ' from the denominator, we are done.

As mentioned before, we will revisit difference quotients in Section 3.1 where we will explain them geometrically. For now, we want to move on to some classic applications of function arithmetic from Economics and for that, we need to think like an entrepreneur.

Suppose you are a manufacturer making a certain product. Let  $x$  be the **production level**, that is, the number of items produced in a given time period. It is customary to let  $C(x)$  denote the function which calculates the total **cost** of producing  $x$  items. The quantity  $C(0)$ , which represents the cost of producing no items, is called the **fixed cost**, and represents the amount of money required to begin production. Associated with the total cost  $C(x)$  is cost per item, or **average cost**, denoted  $\bar{C}(x)$  and read ‘ $C$ -bar’ of  $x$ . To compute  $\bar{C}(x)$ , we take the total cost  $C(x)$  and divide by the number of items produced  $x$  to get

$$\bar{C}(x) = \frac{C(x)}{x}$$

On the retail end, we have the **price**  $p$  charged per item. To simplify the dialogue and computations in this text, we assume that *the number of items sold equals the number of items produced*. From a retail perspective, it seems natural to think of the number of items sold,  $x$ , as a function of the price charged,  $p$ . After all, the retailer can easily adjust the price to sell more product. In the language of functions,  $x$  would be the *dependent* variable and  $p$  would be the *independent* variable or, using function notation, we have a function  $x(p)$ . While we will adopt this convention later in the text, (see Example ?? in Section ??) we will hold with tradition at this point and consider the price  $p$  as a function of the number of items sold,  $x$ . That is, we regard  $x$  as the independent variable and  $p$  as the dependent variable and speak of the **price-demand** function,  $p(x)$ . Hence,  $p(x)$  returns the price charged per item when  $x$  items are produced and sold. Our next function to consider is the **revenue** function,  $R(x)$ . The function  $R(x)$  computes the amount of money collected as a result of selling  $x$  items. Since  $p(x)$  is the price charged per item, we have  $R(x) = xp(x)$ . Finally, the **profit** function,  $P(x)$  calculates how much money is earned after the costs are paid. That is,  $P(x) = (R - C)(x) = R(x) - C(x)$ . We summarize all of these functions below.

### Key Idea 12 Summary of Common Economic Functions

Suppose  $x$  represents the quantity of items produced and sold.

- The price-demand function  $p(x)$  calculates the price per item.
- The revenue function  $R(x)$  calculates the total money collected by selling  $x$  items at a price  $p(x)$ ,  $R(x) = xp(x)$ .
- The cost function  $C(x)$  calculates the cost to produce  $x$  items. The value  $C(0)$  is called the fixed cost or start-up cost.
- The average cost function  $\bar{C}(x) = \frac{C(x)}{x}$  calculates the cost per item when making  $x$  items. Here, we necessarily assume  $x > 0$ .
- The profit function  $P(x)$  calculates the money earned after costs are paid when  $x$  items are produced and sold,  $P(x) = (R - C)(x) = R(x) - C(x)$ .

### Example 29 Computing (and interpreting) cost and profit functions

Let  $x$  represent the number of dOpi media players (‘dOpis’—pronounced ‘dopeys’...) produced and sold in a typical week. Suppose the cost, in dollars, to produce  $x$  dOpis is given by  $C(x) = 100x + 2000$ , for  $x \geq 0$ , and the price, in dollars per dOpi, is given by  $p(x) = 450 - 15x$  for  $0 \leq x \leq 30$ .

1. Find and interpret  $C(0)$ .
2. Find and interpret  $\bar{C}(10)$ .
3. Find and interpret  $p(0)$  and  $p(20)$ .
4. Solve  $p(x) = 0$  and interpret the result.
5. Find and simplify expressions for the revenue function  $R(x)$  and the profit function  $P(x)$ .
6. Find and interpret  $R(0)$  and  $P(0)$ .
7. Solve  $P(x) = 0$  and interpret the result.

**SOLUTION**

1. We substitute  $x = 0$  into the formula for  $C(x)$  and get  $C(0) = 100(0) + 2000 = 2000$ . This means to produce 0 dOpis, it costs \$2000. In other words, the fixed (or start-up) costs are \$2000. The reader is encouraged to contemplate what sorts of expenses these might be.
2. Since  $\bar{C}(x) = \frac{C(x)}{x}$ ,  $\bar{C}(10) = \frac{C(10)}{10} = \frac{3000}{10} = 300$ . This means when 10 dOpis are produced, the cost to manufacture them amounts to \$300 per dOpi.
3. Plugging  $x = 0$  into the expression for  $p(x)$  gives  $p(0) = 450 - 15(0) = 450$ . This means no dOpis are sold if the price is \$450 per dOpi. On the other hand,  $p(20) = 450 - 15(20) = 150$  which means to sell 20 dOpis in a typical week, the price should be set at \$150 per dOpi.
4. Setting  $p(x) = 0$  gives  $450 - 15x = 0$ . Solving gives  $x = 30$ . This means in order to sell 30 dOpis in a typical week, the price needs to be set to \$0. What's more, this means that even if dOpis were given away for free, the retailer would only be able to move 30 of them.
5. To find the revenue, we compute  $R(x) = xp(x) = x(450 - 15x) = 450x - 15x^2$ . Since the formula for  $p(x)$  is valid only for  $0 \leq x \leq 30$ , our formula  $R(x)$  is also restricted to  $0 \leq x \leq 30$ . For the profit,  $P(x) = (R - C)(x) = R(x) - C(x)$ . Using the given formula for  $C(x)$  and the derived formula for  $R(x)$ , we get  $P(x) = (450x - 15x^2) - (100x + 2000) = -15x^2 + 350x - 2000$ . As before, the validity of this formula is for  $0 \leq x \leq 30$  only.
6. We find  $R(0) = 0$  which means if no dOpis are sold, we have no revenue, which makes sense. Turning to profit,  $P(0) = -2000$  since  $P(x) = R(x) - C(x)$  and  $P(0) = R(0) - C(0) = -2000$ . This means that if no dOpis are sold, more money (\$2000 to be exact!) was put into producing the dOpis than was recouped in sales. In number 1, we found the fixed costs to be \$2000, so it makes sense that if we sell no dOpis, we are out those start-up costs.
7. Setting  $P(x) = 0$  gives  $-15x^2 + 350x - 2000 = 0$ . Factoring gives  $-5(x - 10)(3x - 40) = 0$  so  $x = 10$  or  $x = \frac{40}{3}$ . What do these values mean in the context of the problem? Since  $P(x) = R(x) - C(x)$ , solving  $P(x) = 0$  is the same as solving  $R(x) = C(x)$ . This means that the solutions to  $P(x) = 0$  are the production (and sales) figures for which the sales revenue exactly balances the total production costs. These are the so-called '**break even**' points. The solution  $x = 10$  means 10 dOpis should be produced (and

sold) during the week to recoup the cost of production. For  $x = \frac{40}{3} = 13.\overline{3}$ , things are a bit more complicated. Even though  $x = 13.\overline{3}$  satisfies  $0 \leq x \leq 30$ , and hence is in the domain of  $P$ , it doesn't make sense in the context of this problem to produce a fractional part of a dOpi. Evaluating  $P(13) = 15$  and  $P(14) = -40$ , we see that producing and selling 13 dOpis per week makes a (slight) profit, whereas producing just one more puts us back into the red. While breaking even is nice, we ultimately would like to find what production level (and price) will result in the largest profit, and we'll do just that ...in Section 3.3.

Recall from Section 2.3.1 that in problems such as this, it is necessary to take the **applied domain** of the function into account.

## Exercises 2.4

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### Problems

In Exercises 1 – 10, use the pair of functions  $f$  and  $g$  to find the following values if they exist:

- $(f + g)(2)$
- $(f - g)(-1)$
- $(g - f)(1)$
- $(fg) \left(\frac{1}{2}\right)$

- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

1.  $f(x) = 3x + 1$  and  $g(x) = 4 - x$

2.  $f(x) = x^2$  and  $g(x) = -2x + 1$

3.  $f(x) = x^2 - x$  and  $g(x) = 12 - x^2$

4.  $f(x) = 2x^3$  and  $g(x) = -x^2 - 2x - 3$

5.  $f(x) = \sqrt{x+3}$  and  $g(x) = 2x - 1$

6.  $f(x) = \sqrt{4-x}$  and  $g(x) = \sqrt{x+2}$

7.  $f(x) = 2x$  and  $g(x) = \frac{1}{2x+1}$

8.  $f(x) = x^2$  and  $g(x) = \frac{3}{2x-3}$

9.  $f(x) = x^2$  and  $g(x) = \frac{1}{x^2}$

10.  $f(x) = x^2 + 1$  and  $g(x) = \frac{1}{x^2 + 1}$

In Exercises 11 – 20, use the pair of functions  $f$  and  $g$  to find the domain of the indicated function then find and simplify an expression for it.

- $(f + g)(x)$
- $(f - g)(x)$

- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

11.  $f(x) = 2x + 1$  and  $g(x) = x - 2$

12.  $f(x) = 1 - 4x$  and  $g(x) = 2x - 1$

13.  $f(x) = x^2$  and  $g(x) = 3x - 1$

14.  $f(x) = x^2 - x$  and  $g(x) = 7x$

15.  $f(x) = x^2 - 4$  and  $g(x) = 3x + 6$

16.  $f(x) = -x^2 + x + 6$  and  $g(x) = x^2 - 9$

17.  $f(x) = \frac{x}{2}$  and  $g(x) = \frac{2}{x}$

18.  $f(x) = x - 1$  and  $g(x) = \frac{1}{x - 1}$

19.  $f(x) = x$  and  $g(x) = \sqrt{x+1}$

20.  $f(x) = \sqrt{x-5}$  and  $g(x) = f(x) = \sqrt{x-5}$

**In Exercises 21 – 45, find and simplify the difference quotient  $\frac{f(x+h) - f(x)}{h}$  for the given function.**

21.  $f(x) = 2x - 5$

22.  $f(x) = -3x + 5$

23.  $f(x) = 6$

24.  $f(x) = 3x^2 - x$

25.  $f(x) = -x^2 + 2x - 1$

26.  $f(x) = 4x^2$

27.  $f(x) = x - x^2$

28.  $f(x) = x^3 + 1$

29.  $f(x) = mx + b$  where  $m \neq 0$

30.  $f(x) = ax^2 + bx + c$  where  $a \neq 0$

31.  $f(x) = \frac{2}{x}$

32.  $f(x) = \frac{3}{1-x}$

33.  $f(x) = \frac{1}{x^2}$

34.  $f(x) = \frac{2}{x+5}$

35.  $f(x) = \frac{1}{4x-3}$

36.  $f(x) = \frac{3x}{x+1}$

37.  $f(x) = \frac{x}{x-9}$

38.  $f(x) = \frac{x^2}{2x+1}$

39.  $f(x) = \sqrt{x-9}$

40.  $f(x) = \sqrt{2x+1}$

41.  $f(x) = \sqrt{-4x+5}$

42.  $f(x) = \sqrt{4-x}$

43.  $f(x) = \sqrt{ax + b}$ , where  $a \neq 0$ .

44.  $f(x) = x\sqrt{x}$

45.  $f(x) = \sqrt[3]{x}$ . HINT:  $(a - b)(a^2 + ab + b^2) = a^3 - b^3$

In Exercises 46 – 50,  $C(x)$  denotes the cost to produce  $x$  items and  $p(x)$  denotes the price-demand function in the given economic scenario. In each Exercise, do the following:

- Find and interpret  $C(0)$ .
  - Find and interpret  $\bar{C}(10)$ .
  - Find and interpret  $p(5)$
  - Find and simplify  $R(x)$ .
  - Find and simplify  $P(x)$ .
  - Solve  $P(x) = 0$  and interpret.
46. The cost, in dollars, to produce  $x$  “I’d rather be a Sasquatch” T-Shirts is  $C(x) = 2x + 26$ ,  $x \geq 0$  and the price-demand function, in dollars per shirt, is  $p(x) = 30 - 2x$ ,  $0 \leq x \leq 15$ .
47. The cost, in dollars, to produce  $x$  bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is  $C(x) = 10x + 100$ ,  $x \geq 0$  and the price-demand function, in dollars per bottle, is  $p(x) = 35 - x$ ,  $0 \leq x \leq 35$ .
48. The cost, in cents, to produce  $x$  cups of Mountain Thunder Lemonade at Junior’s Lemonade Stand is  $C(x) = 18x + 240$ ,  $x \geq 0$  and the price-demand function, in cents per cup, is  $p(x) = 90 - 3x$ ,  $0 \leq x \leq 30$ .
49. The daily cost, in dollars, to produce  $x$  Sasquatch Berry Pies  $C(x) = 3x + 36$ ,  $x \geq 0$  and the price-demand function, in dollars per pie, is  $p(x) = 12 - 0.5x$ ,  $0 \leq x \leq 24$ .
50. The monthly cost, in hundreds of dollars, to produce  $x$  custom built electric scooters is  $C(x) = 20x + 1000$ ,  $x \geq 0$  and the price-demand function, in hundreds of dollars per scooter, is  $p(x) = 140 - 2x$ ,  $0 \leq x \leq 70$ .

In Exercises 51 – 64, let  $f$  be the function defined by

$$f = \{(-3, 4), (-2, 2), (-1, 0), (0, 1), (1, 3), (2, 4), (3, -1)\}$$

and let  $g$  be the function defined

$$g = \{(-3, -2), (-2, 0), (-1, -4), (0, 0), (1, -3), (2, 1), (3, 2)\}.$$

Compute the indicated value if it exists.

51.  $(f + g)(-3)$

58.  $\left(\frac{f}{g}\right)(-1)$

52.  $(f - g)(2)$

59.  $\left(\frac{f}{g}\right)(2)$

53.  $(fg)(-1)$

54.  $(g + f)(1)$

60.  $\left(\frac{g}{f}\right)(-1)$

55.  $(g - f)(3)$

61.  $\left(\frac{g}{f}\right)(3)$

56.  $(gf)(-3)$

57.  $\left(\frac{f}{g}\right)(-2)$

62.  $\left(\frac{g}{f}\right)(-3)$

## 2.5 Graphs of Functions

In Section 2.2 we defined a function as a special type of relation; one in which each  $x$ -coordinate was matched with only one  $y$ -coordinate. We spent most of our time in that section looking at functions graphically because they were, after all, just sets of points in the plane. Then in Section 2.3 we described a function as a process and defined the notation necessary to work with functions algebraically. So now it's time to look at functions graphically again, only this time we'll do so with the notation defined in Section 2.3. We start with what should not be a surprising connection.

### Key Idea 13 The Fundamental Graphing Principle for Functions

The graph of a function  $f$  is the set of points which satisfy the equation  $y = f(x)$ . That is, the point  $(x, y)$  is on the graph of  $f$  if and only if  $y = f(x)$ .

### Example 30 Graphing a function

Graph  $f(x) = x^2 - x - 6$ .

$x$	$f(x)$	$(x, f(x))$
-3	6	(-3, 6)
-2	0	(-2, 0)
-1	-4	(-1, -4)
0	-6	(0, -6)
1	-6	(1, -6)
2	-4	(2, -4)
3	0	(3, 0)
4	6	(4, 6)

**SOLUTION** To graph  $f$ , we graph the equation  $y = f(x)$ . To this end, we use the techniques outlined in Section 2.1.1. Specifically, we check for intercepts, test for symmetry, and plot additional points as needed. To find the  $x$ -intercepts, we set  $y = 0$ . Since  $y = f(x)$ , this means  $f(x) = 0$ .

$$\begin{aligned} f(x) &= x^2 - x - 6 \\ 0 &= x^2 - x - 6 \\ 0 &= (x - 3)(x + 2) \quad \text{factor} \\ x - 3 = 0 &\quad \text{or} \quad x + 2 = 0 \\ x &= 3 \quad \text{or} \quad x = -2 \end{aligned}$$

So we get  $(-2, 0)$  and  $(3, 0)$  as  $x$ -intercepts. To find the  $y$ -intercept, we set  $x = 0$ . Using function notation, this is the same as finding  $f(0)$  and  $f(0) = 0^2 - 0 - 6 = -6$ . Thus the  $y$ -intercept is  $(0, -6)$ . As far as symmetry is concerned, we can tell from the intercepts that the graph possesses none of the three symmetries discussed thus far. (You should verify this.) We can make a table analogous to the ones we made in Section 2.1.1, plot the points and connect the dots in a somewhat pleasing fashion to get the graph shown in Figure 2.21.

Graphing piecewise-defined functions is a bit more of a challenge.

### Example 31 Graphing a piecewise-defined function

Graph:  $f(x) = \begin{cases} 4 - x^2 & \text{if } x < 1 \\ x - 3, & \text{if } x \geq 1 \end{cases}$

**SOLUTION** We proceed as before – finding intercepts, testing for symmetry and then plotting additional points as needed. To find the  $x$ -intercepts, as before, we set  $f(x) = 0$ . The twist is that we have two formulas for  $f(x)$ . For  $x < 1$ , we use the formula  $f(x) = 4 - x^2$ . Setting  $f(x) = 0$  gives  $0 = 4 - x^2$ , so that  $x = \pm 2$ . However, of these two answers, only  $x = -2$  fits in the domain  $x < 1$  for this piece. This means the only  $x$ -intercept for the  $x < 1$  region of the  $x$ -axis is  $(-2, 0)$ . For  $x \geq 1$ ,  $f(x) = x - 3$ . Setting  $f(x) = 0$  gives  $0 = x - 3$ , or  $x = 3$ . Since  $x = 3$  satisfies the inequality  $x \geq 1$ , we get  $(3, 0)$  as another

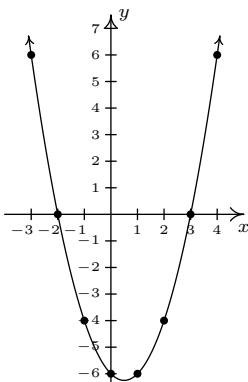


Figure 2.21: Graphing the function  $f(x) = x^2 - x - 6$

$x$ -intercept. Next, we seek the  $y$ -intercept. Notice that  $x = 0$  falls in the domain  $x < 1$ . Thus  $f(0) = 4 - 0^2 = 4$  yields the  $y$ -intercept  $(0, 4)$ . As far as symmetry is concerned, you can check that the equation  $y = 4 - x^2$  is symmetric about the  $y$ -axis; unfortunately, this equation (and its symmetry) is valid only for  $x < 1$ . You can also verify  $y = x - 3$  possesses none of the symmetries discussed in the Section 2.1.1. When plotting additional points, it is important to keep in mind the restrictions on  $x$  for each piece of the function. The sticking point for this function is  $x = 1$ , since this is where the equations change. When  $x = 1$ , we use the formula  $f(x) = x - 3$ , so the point on the graph  $(1, f(1))$  is  $(1, -2)$ . However, for all values less than 1, we use the formula  $f(x) = 4 - x^2$ . As we have discussed earlier in Section 2.1, there is no real number which immediately precedes  $x = 1$  on the number line. Thus for the values  $x = 0.9, x = 0.99, x = 0.999$ , and so on, we find the corresponding  $y$  values using the formula  $f(x) = 4 - x^2$ . Making a table as before, we see that as the  $x$  values sneak up to  $x = 1$  in this fashion, the  $f(x)$  values inch closer and closer to  $4 - 1^2 = 3$ . To indicate this graphically, we use an open circle at the point  $(1, 3)$ . Putting all of this information together and plotting additional points, we get the result in Figure 2.22.

In the previous two examples, the  $x$ -coordinates of the  $x$ -intercepts of the graph of  $y = f(x)$  were found by solving  $f(x) = 0$ . For this reason, they are called the **zeros** of  $f$ .

#### Definition 26 Zeros of a function

The **zeros** of a function  $f$  are the solutions to the equation  $f(x) = 0$ . In other words,  $x$  is a zero of  $f$  if and only if  $(x, 0)$  is an  $x$ -intercept of the graph of  $y = f(x)$ .

Of the three symmetries discussed in Section 2.1.1, only two are of significance to functions: symmetry about the  $y$ -axis and symmetry about the origin. Recall that we can test whether the graph of an equation is symmetric about the  $y$ -axis by replacing  $x$  with  $-x$  and checking to see if an equivalent equation results. If we are graphing the equation  $y = f(x)$ , substituting  $-x$  for  $x$  results in the equation  $y = f(-x)$ . In order for this equation to be equivalent to the original equation  $y = f(x)$  we need  $f(-x) = f(x)$ . In a similar fashion, we recall that to test an equation's graph for symmetry about the origin, we replace  $x$  and  $y$  with  $-x$  and  $-y$ , respectively. Doing this substitution in the equation  $y = f(x)$  results in  $-y = f(-x)$ . Solving the latter equation for  $y$  gives  $y = -f(-x)$ . In order for this equation to be equivalent to the original equation  $y = f(x)$  we need  $-f(-x) = f(x)$ , or, equivalently,  $f(-x) = -f(x)$ . These results are summarized below.

#### Key Idea 14 Testing the Graph of a Function for Symmetry

The graph of a function  $f$  is symmetric

- about the  $y$ -axis if and only if  $f(-x) = f(x)$  for all  $x$  in the domain of  $f$ .
- about the origin if and only if  $f(-x) = -f(x)$  for all  $x$  in the domain of  $f$ .

$x$	$f(x)$	$(x, f(x))$
0.9	3.19	$(0.9, 3.19)$
0.99	$\approx 3.02$	$(0.99, 3.02)$
0.999	$\approx 3.002$	$(0.999, 3.002)$

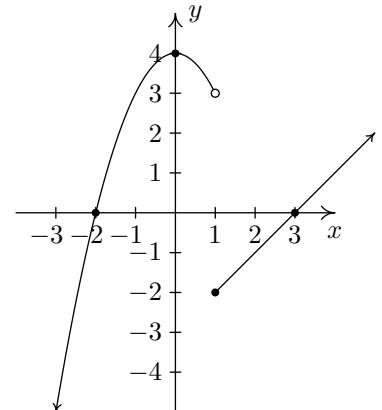


Figure 2.22: The graph of  $f(x)$  from Example 31

Note that for graphs of functions, we don't bother to discuss symmetry about the  $x$ -axis. Why do you suppose this is?

A good resource when you need to quickly check something like the graph of a function is Wolfram Alpha.

If you want a good (and free!) program you can run locally on a computer or tablet, we recommend trying Geogebra. It's free to download, works on all major operating systems, and it's pretty easy to figure out the basics.

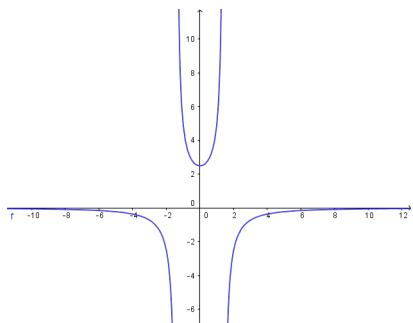


Figure 2.23: The graph of  $f(x)$  in Example 32

While the plot provided by the software can provide us with visual evidence that a function is even or odd, this evidence is never conclusive. The only way to know for sure is to check analytically using the definitions of even and odd functions.

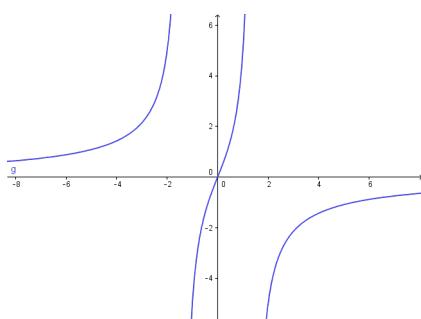


Figure 2.24: The graph of  $g(x)$  in Example 32

For reasons which won't become clear until we study polynomials, we call a function **even** if its graph is symmetric about the  $y$ -axis or **odd** if its graph is symmetric about the origin. Apart from a very specialized family of functions which are both even and odd, (any ideas?) functions fall into one of three distinct categories: even, odd, or neither even nor odd.

### Example 32 Even and odd functions

Determine analytically if the following functions are even, odd, or neither even nor odd. Verify your result with a graphing calculator or computer software.

1.  $f(x) = \frac{5}{2-x^2}$
2.  $g(x) = \frac{5x}{2-x^2}$
3.  $h(x) = \frac{5x}{2-x^3}$
4.  $i(x) = \frac{5x}{2x-x^3}$
5.  $j(x) = x^2 - \frac{x}{100} - 1$
6.  $p(x) = \begin{cases} rx + 3 & \text{if } x < 0 \\ -x + 3, & \text{if } x \geq 0 \end{cases}$

#### SOLUTION

The first step in all of these problems is to replace  $x$  with  $-x$  and simplify.

1.

$$\begin{aligned} f(x) &= \frac{5}{2-x^2} \\ f(-x) &= \frac{5}{2-(-x)^2} \\ f(-x) &= \frac{5}{2-x^2} \\ f(-x) &= f(x) \end{aligned}$$

Hence,  $f$  is **even**. A plot of  $f(x)$  using GeoGebra is given in Figure 2.23.

This suggests that the graph of  $f$  is symmetric about the  $y$ -axis, as expected.

2.

$$\begin{aligned} g(x) &= \frac{5x}{2-x^2} \\ g(-x) &= \frac{5(-x)}{2-(-x)^2} \\ g(-x) &= \frac{-5x}{2-x^2} \end{aligned}$$

It doesn't appear that  $g(-x)$  is equivalent to  $g(x)$ . To prove this, we check with an  $x$  value. After some trial and error, we see that  $g(1) = 5$  whereas  $g(-1) = -5$ . This proves that  $g$  is not even, but it doesn't rule out the possibility that  $g$  is odd. (Why not?) To check if  $g$  is odd, we compare  $g(-x)$  with  $-g(x)$

$$\begin{aligned} -g(x) &= -\frac{5x}{2-x^2} \\ &= \frac{-5x}{2-x^2} \\ -g(x) &= g(-x) \end{aligned}$$

Hence,  $g$  is odd: see Figure 2.24.

3.

$$\begin{aligned} h(x) &= \frac{5x}{2-x^3} \\ h(-x) &= \frac{5(-x)}{2-(-x)^3} \\ h(-x) &= \frac{-5x}{2+x^3} \end{aligned}$$

Once again,  $h(-x)$  doesn't appear to be equivalent to  $h(x)$ . We check with an  $x$  value, for example,  $h(1) = 5$  but  $h(-1) = -\frac{5}{3}$ . This proves that  $h$  is not even and it also shows  $h$  is not odd. (Why?)

In Figure 2.25, the graph of  $h$  appears to be neither symmetric about the  $y$ -axis nor the origin.

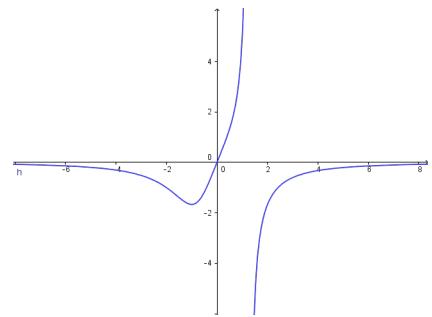


Figure 2.25: The graph of  $h(x)$  in Example 32

4.

$$\begin{aligned} i(x) &= \frac{5x}{2x-x^3} \\ i(-x) &= \frac{5(-x)}{2(-x)-(-x)^3} \\ i(-x) &= \frac{-5x}{-2x+x^3} \end{aligned}$$

The expression  $i(-x)$  doesn't appear to be equivalent to  $i(x)$ . However, after checking some  $x$  values, for example  $x = 1$  yields  $i(1) = 5$  and  $i(-1) = 5$ , it appears that  $i(-x)$  does, in fact, equal  $i(x)$ . However, while this suggests  $i$  is even, it doesn't prove it. (It does, however, prove  $i$  is not odd.) To prove  $i(-x) = i(x)$ , we need to manipulate our expressions for  $i(x)$  and  $i(-x)$  and show that they are equivalent. A clue as to how to proceed is in the numerators: in the formula for  $i(x)$ , the numerator is  $5x$  and in  $i(-x)$  the numerator is  $-5x$ . To re-write  $i(x)$  with a numerator of  $-5x$ , we need to multiply its numerator by  $-1$ . To keep the value of the fraction the same, we need to multiply the denominator by  $-1$  as well. Thus

$$\begin{aligned} i(x) &= \frac{5x}{2x-x^3} \\ &= \frac{(-1)5x}{(-1)(2x-x^3)} \\ &= \frac{-5x}{-2x+x^3} \end{aligned}$$

Hence,  $i(x) = i(-x)$ , so  $i$  is even. See Figure 2.26 for the graph.

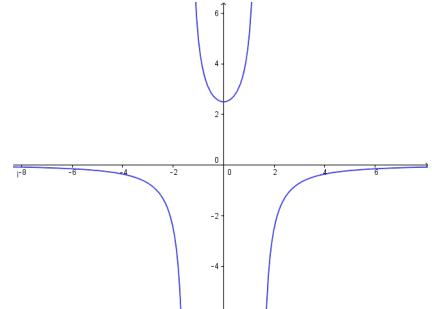


Figure 2.26: The graph of  $i(x)$  in Example 32

5.

$$\begin{aligned} j(x) &= x^2 - \frac{x}{100} - 1 \\ j(-x) &= (-x)^2 - \frac{-x}{100} - 1 \\ j(-x) &= x^2 + \frac{x}{100} - 1 \end{aligned}$$

The expression for  $j(-x)$  doesn't seem to be equivalent to  $j(x)$ , so we check using  $x = 1$  to get  $j(1) = -\frac{1}{100}$  and  $j(-1) = \frac{1}{100}$ . This rules out  $j$  being even. However, it doesn't rule out  $j$  being odd. Examining  $-j(x)$  gives

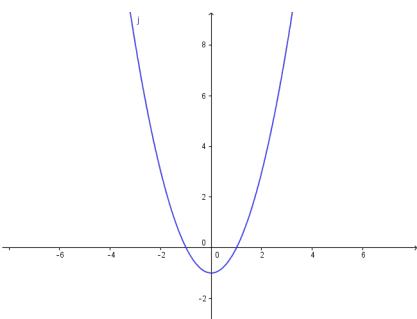


Figure 2.27: The graph of  $j(x)$  in Example 32

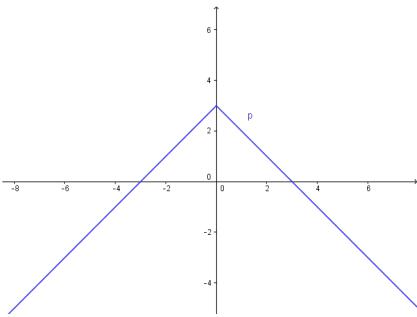


Figure 2.28: The graph of  $p(x)$  in Example 32

$$\begin{aligned}j(x) &= x^2 - \frac{x}{100} - 1 \\-j(x) &= -\left(x^2 - \frac{x}{100} - 1\right) \\-j(x) &= -x^2 + \frac{x}{100} + 1\end{aligned}$$

The expression  $-j(x)$  doesn't seem to match  $j(-x)$  either. Testing  $x = 2$  gives  $j(2) = \frac{149}{50}$  and  $j(-2) = \frac{151}{50}$ , so  $j$  is not odd, either.

Notice in Figure 2.27 that the computer plot seems to suggest that the graph of  $j$  is symmetric about the  $y$ -axis which would imply that  $j$  is even. However, we have proven that is not the case. The problem is that the effect of the  $x/100$  term is so small, our eyes don't detect it in the graph.

6. Testing the graph of  $y = p(x)$  for symmetry is complicated by the fact  $p(x)$  is a piecewise-defined function. As always, we handle this by checking the condition for symmetry by checking it on each piece of the domain. We first consider the case when  $x < 0$  and set about finding the correct expression for  $p(-x)$ . Even though  $p(x) = x+3$  for  $x < 0$ ,  $p(-x) \neq -x+3$  here. The reason for this is that since  $x < 0$ ,  $-x > 0$  which means to find  $p(-x)$ , we need to use the *other* formula for  $p(x)$ , namely  $p(x) = -x+3$ . Hence, for  $x < 0$ ,  $p(-x) = -(-x)+3 = x+3 = p(x)$ . For  $x \geq 0$ ,  $p(x) = -x+3$  and we have two cases. If  $x > 0$ , then  $-x < 0$  so  $p(-x) = (-x)+3 = -x+3 = p(x)$ . If  $x = 0$ , then  $p(0) = 3 = p(-0)$ . Hence, in all cases,  $p(-x) = p(x)$ , so  $p$  is even. Since  $p(0) = 3$  but  $p(-0) = p(0) = 3 \neq -3$ , we also have  $p$  is not odd.

In Figure 2.28, we see that the graph appears to be symmetric about the  $y$ -axis.

There are two lessons to be learned from the last example. The first is that sampling function values at particular  $x$  values is not enough to prove that a function is even or odd – despite the fact that  $j(-1) = -j(1)$ ,  $j$  turned out not to be odd. Secondly, while the calculator may *suggest* mathematical truths, it is the Algebra which *proves* mathematical truths. (Or, in other words, don't rely too heavily on the machine!)

### 2.5.1 General Function Behaviour

The last topic we wish to address in this section is general function behaviour. As you shall see in the next several chapters, each family of functions has its own unique attributes and we will study them all in great detail. The purpose of this section's discussion, then, is to lay the foundation for that further study by investigating aspects of function behaviour which apply to all functions. To start, we will examine the concepts of **increasing**, **decreasing** and **constant**. Before defining the concepts algebraically, it is instructive to first look at them graphically. Consider the graph of the function  $f$  in Figure 2.29.

Reading from left to right, the graph 'starts' at the point  $(-4, -3)$  and 'ends' at the point  $(6, 5.5)$ . If we imagine walking from left to right on the graph, between  $(-4, -3)$  and  $(-2, 4.5)$ , we are walking 'uphill'; then between  $(-2, 4.5)$  and  $(3, -8)$ , we are walking 'downhill'; and between  $(3, -8)$  and  $(4, -6)$ , we are walking 'uphill' once more. From  $(4, -6)$  to  $(5, -6)$ , we 'level off', and then

resume walking ‘uphill’ from  $(5, -6)$  to  $(6, 5.5)$ . In other words, for the  $x$  values between  $-4$  and  $-2$  (inclusive), the  $y$ -coordinates on the graph are getting larger, or **increasing**, as we move from left to right. Since  $y = f(x)$ , the  $y$  values on the graph are the function values, and we say that the function  $f$  is **increasing** on the interval  $[-4, -2]$ . Analogously, we say that  $f$  is **decreasing** on the interval  $[-2, 3]$  increasing once more on the interval  $[3, 4]$ , **constant** on  $[4, 5]$ , and finally increasing once again on  $[5, 6]$ . It is extremely important to notice that the behaviour (increasing, decreasing or constant) occurs on an interval on the  $x$ -axis. When we say that the function  $f$  is increasing on  $[-4, -2]$  we do not mention the actual  $y$  values that  $f$  attains along the way. Thus, we report *where* the behaviour occurs, not to what extent the behaviour occurs. Also notice that we do not say that a function is increasing, decreasing or constant at a single  $x$  value. In fact, we would run into serious trouble in our previous example if we tried to do so because  $x = -2$  is contained in an interval on which  $f$  was increasing and one on which it is decreasing. (There’s more on this issue – and many others – in the Exercises.)

We’re now ready for the more formal algebraic definitions of what it means for a function to be increasing, decreasing or constant.

### Definition 27 Increasing, decreasing, and constant functions

Suppose  $f$  is a function defined on an interval  $I$ . We say  $f$  is:

- **increasing** on  $I$  if and only if  $f(a) < f(b)$  for all real numbers  $a, b$  in  $I$  with  $a < b$ .
- **decreasing** on  $I$  if and only if  $f(a) > f(b)$  for all real numbers  $a, b$  in  $I$  with  $a < b$ .
- **constant** on  $I$  if and only if  $f(a) = f(b)$  for all real numbers  $a, b$  in  $I$ .

It is worth taking some time to see that the algebraic descriptions of increasing, decreasing and constant as stated in Definition 27 agree with our graphical descriptions given earlier. You should look back through the examples and exercise sets in previous sections where graphs were given to see if you can determine the intervals on which the functions are increasing, decreasing or constant. Can you find an example of a function for which none of the concepts in Definition 27 apply?

Now let’s turn our attention to a few of the points on the graph. Clearly the point  $(-2, 4.5)$  does not have the largest  $y$  value of all of the points on the graph of  $f$  – indeed that honour goes to  $(6, 5.5)$  – but  $(-2, 4.5)$  should get some sort of consolation prize for being ‘the top of the hill’ between  $x = -4$  and  $x = 3$ . We say that the function  $f$  has a **local maximum** (or **relative maximum**) at the point  $(-2, 4.5)$ , because the  $y$ -coordinate  $4.5$  is the largest  $y$ -value (hence, function value) on the curve ‘near’  $x = -2$ . Similarly, we say that the function  $f$  has a **local minimum** (or **relative minimum**) at the point  $(3, -8)$ , since the  $y$ -coordinate  $-8$  is the smallest function value near  $x = 3$ . Although it is tempting to say that local extrema occur when the function changes from increasing to decreasing or vice versa, it is not a precise enough way to define the concepts for the needs of Calculus. At the risk of being pedantic, we will present the traditional definitions and thoroughly vet the pathologies they induce in the Exercises. We have one

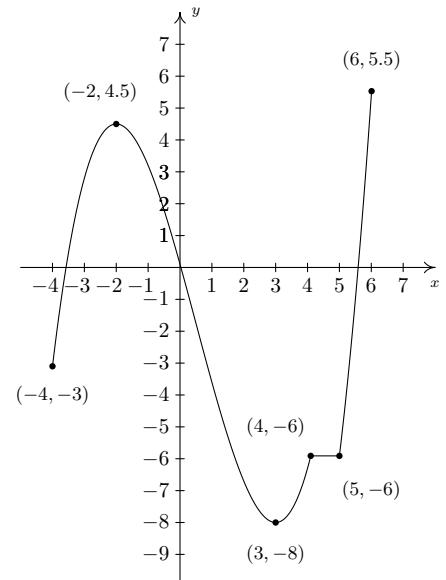


Figure 2.29: The graph  $y = f(x)$

The notions of how quickly or how slowly a function increases or decreases are explored in Calculus.

Typically, in (pre)calculus, whenever you’re told that something occurs ‘near’ a given point, you should read this as ‘on some open interval  $I$  containing that point’.

'Maxima' is the plural of 'maximum' and 'minima' is the plural of 'minimum'. 'Extrema' is the plural of 'extremum' which combines maximum and minimum.

last observation to make before we proceed to the algebraic definitions and look at a fairly tame, yet helpful, example.

If we look at the entire graph, we see that the largest  $y$  value (the largest function value) is 5.5 at  $x = 6$ . In this case, we say the **maximum** (often called the 'absolute' or 'global' maximum) of  $f$  is 5.5; similarly, the **minimum** (again, 'absolute' or 'global' minimum can be used.) of  $f$  is  $-8$ .

We formalize these concepts in the following definitions.

### Definition 28 Local maximum and minimum

Suppose  $f$  is a function with  $f(a) = b$ .

- We say  $f$  has a **local maximum** at the point  $(a, b)$  if and only if there is an open interval  $I$  containing  $a$  for which  $f(a) \geq f(x)$  for all  $x$  in  $I$ . The value  $f(a) = b$  is called 'a local maximum value of  $f$ ' in this case.
- We say  $f$  has a **local minimum** at the point  $(a, b)$  if and only if there is an open interval  $I$  containing  $a$  for which  $f(a) \leq f(x)$  for all  $x$  in  $I$ . The value  $f(a) = b$  is called 'a local minimum value of  $f$ ' in this case.
- The value  $b$  is called the **maximum** of  $f$  if  $b \geq f(x)$  for all  $x$  in the domain of  $f$ .
- The value  $b$  is called the **minimum** of  $f$  if  $b \leq f(x)$  for all  $x$  in the domain of  $f$ .

It's important to note that not every function will have all of these features. Indeed, it is possible to have a function with no local or absolute extrema at all! (Any ideas of what such a function's graph would have to look like?) We shall see examples of functions in the Exercises which have one or two, but not all, of these features, some that have instances of each type of extremum and some functions that seem to defy common sense. In all cases, though, we shall adhere to the algebraic definitions above as we explore the wonderful diversity of graphs that functions provide us.

Here is the 'tame' example which was promised earlier. It summarizes all of the concepts presented in this section as well as some from previous sections so you should spend some time thinking deeply about it before proceeding to the Exercises.

### Example 33 A 'tame' example

Given the graph of  $y = f(x)$  in Figure ??, answer all of the following questions.

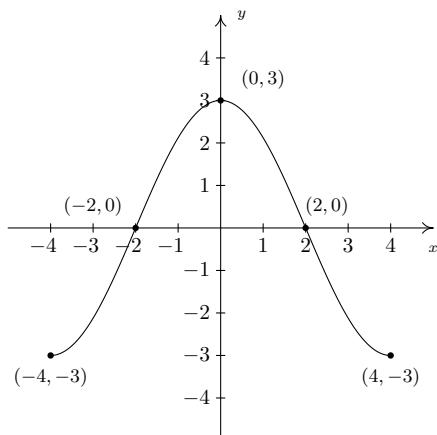


Figure 2.30: The graph for Example 33

1. Find the domain of  $f$ .
2. Find the range of  $f$ .
3. List the  $x$ -intercepts, if any exist.
4. List the  $y$ -intercepts, if any exist.
5. Find the zeros of  $f$ .
6. Solve  $f(x) < 0$ .

7. Determine  $f(2)$ .
8. Solve  $f(x) = -3$ .
9. Find the number of solutions to  $f(x) = 1$ .
10. Does  $f$  appear to be even, odd, or neither?
11. List the intervals on which  $f$  is increasing.
12. List the intervals on which  $f$  is decreasing.
13. List the local maximums, if any exist.
14. List the local minimums, if any exist.
15. Find the maximum, if it exists.
16. Find the minimum, if it exists.

**SOLUTION**

1. To find the domain of  $f$ , we proceed as in Section 2.2. By projecting the graph to the  $x$ -axis, we see that the portion of the  $x$ -axis which corresponds to a point on the graph is everything from  $-4$  to  $4$ , inclusive. Hence, the domain is  $[-4, 4]$ .
2. To find the range, we project the graph to the  $y$ -axis. We see that the  $y$  values from  $-3$  to  $3$ , inclusive, constitute the range of  $f$ . Hence, our answer is  $[-3, 3]$ .
3. The  $x$ -intercepts are the points on the graph with  $y$ -coordinate  $0$ , namely  $(-2, 0)$  and  $(2, 0)$ .
4. The  $y$ -intercept is the point on the graph with  $x$ -coordinate  $0$ , namely  $(0, 3)$ .
5. The zeros of  $f$  are the  $x$ -coordinates of the  $x$ -intercepts of the graph of  $y = f(x)$  which are  $x = -2, 2$ .
6. To solve  $f(x) < 0$ , we look for the  $x$  values of the points on the graph where the  $y$ -coordinate is less than  $0$ . Graphically, we are looking for where the graph is below the  $x$ -axis. This happens for the  $x$  values from  $-4$  to  $-2$  and again from  $2$  to  $4$ . So our answer is  $[-4, -2) \cup (2, 4]$ .
7. Since the graph of  $f$  is the graph of the equation  $y = f(x)$ ,  $f(2)$  is the  $y$ -coordinate of the point which corresponds to  $x = 2$ . Since the point  $(2, 0)$  is on the graph, we have  $f(2) = 0$ .
8. To solve  $f(x) = -3$ , we look where  $y = f(x) = -3$ . We find two points with a  $y$ -coordinate of  $-3$ , namely  $(-4, -3)$  and  $(4, -3)$ . Hence, the solutions to  $f(x) = -3$  are  $x = \pm 4$ .
9. As in the previous problem, to solve  $f(x) = 1$ , we look for points on the graph where the  $y$ -coordinate is  $1$ . Even though these points aren't specified, we see that the curve has two points with a  $y$  value of  $1$ , as seen in the graph below. That means there are two solutions to  $f(x) = 1$ : see Figure 2.31.
10. The graph appears to be symmetric about the  $y$ -axis. This suggests (but does not prove) that  $f$  is even.
11. As we move from left to right, the graph rises from  $(-4, -3)$  to  $(0, 3)$ . This means  $f$  is increasing on the interval  $[-4, 0]$ . (Remember, the answer here is an interval on the  $x$ -axis.)

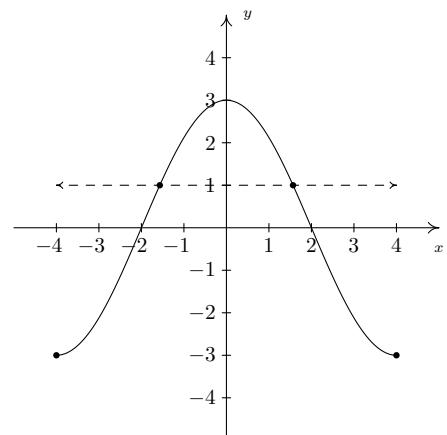


Figure 2.31: Solving  $f(x) = 1$  in Example 33

12. As we move from left to right, the graph falls from  $(0, 3)$  to  $(4, -3)$ . This means  $f$  is decreasing on the interval  $[0, 4]$ . (Remember, the answer here is an interval on the  $x$ -axis.)
13. The function has its only local maximum at  $(0, 3)$  so  $f(0) = 3$  is the local minimum value.
14. There are no local minimums. Why don't  $(-4, -3)$  and  $(4, -3)$  count? Let's consider the point  $(-4, -3)$  for a moment. Recall that, in the definition of local minimum, there needs to be an open interval  $I$  which contains  $x = -4$  such that  $f(-4) < f(x)$  for all  $x$  in  $I$  different from  $-4$ . But if we put an open interval around  $x = -4$  a portion of that interval will lie outside of the domain of  $f$ . Because we are unable to satisfy the requirements of the definition for a local minimum, we cannot claim that  $f$  has one at  $(-4, -3)$ . The point  $(4, -3)$  fails for the same reason — no open interval around  $x = 4$  stays within the domain of  $f$ .
15. The maximum value of  $f$  is the largest  $y$ -coordinate which is 3.
16. The minimum value of  $f$  is the smallest  $y$ -coordinate which is  $-3$ .

In general, the problem of finding maximum and minimum values, requires the techniques of Calculus. We will explore this in Chapter ???. In the meantime, we'll have to rely on technology to assist us. Most graphing calculators and many mathematics software programs have 'Minimum' and 'Maximum' features which can be used to approximate these values, as we now demonstrate.

**Example 34** Using the computer to find maxima and minima  
 Let  $f(x) = \frac{15x}{x^2 + 3}$ . Use the computer or a graphing calculator to approximate the intervals on which  $f$  is increasing and those on which it is decreasing. Approximate all extrema.

**SOLUTION** Using GeoGebra, we enter  $f(x) = 15x/(x^2+3)$  to plot the graph of  $f$ . The command  $\text{Max}[f, -3, 3]$  then calculates the maximum value of  $f$  on the interval  $[-3, 3]$ . Similarly,  $\text{Min}[f, -3, 3]$  gives the minimum value of  $f$  on the interval  $[-3, 3]$ . The graph of  $f$ , together with the local maximum and local minimum, are plotted in Figure 2.32.

To two decimal places,  $f$  appears to have its only local minimum at  $(-1.73, -4.33)$  and its only local maximum at  $(1.73, 4.33)$ . Given the symmetry about the origin suggested by the graph, the relation between these points shouldn't be too surprising. The function appears to be increasing on  $[-1.73, 1.73]$  and decreasing on  $(-\infty, -1.73] \cup [1.73, \infty)$ . This makes  $-4.33$  the (absolute) minimum and  $4.33$  the (absolute) maximum.

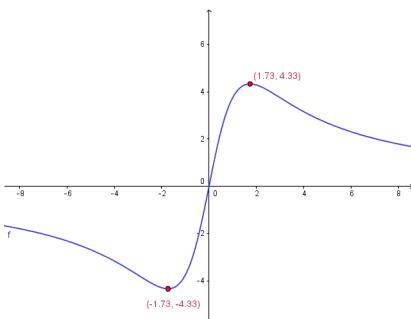


Figure 2.32: The local maximum and minimum of  $f(x) = \frac{15x}{x^2 + 3}$  in Example 34

**Example 35** Minimizing distance from a graph to the origin  
 Find the points on the graph of  $y = (x - 3)^2$  which are closest to the origin. Round your answers to two decimal places.

**SOLUTION** Suppose a point  $(x, y)$  is on the graph of  $y = (x - 3)^2$ . Its distance to the origin  $(0, 0)$  is given by

$$\begin{aligned}
 d &= \sqrt{(x-0)^2 + (y-0)^2} \\
 &= \sqrt{x^2 + y^2} \\
 &= \sqrt{x^2 + [(x-3)^2]^2} \quad \text{Since } y = (x-3)^2 \\
 &= \sqrt{x^2 + (x-3)^4}
 \end{aligned}$$

Given a value for  $x$ , the formula  $d = \sqrt{x^2 + (x-3)^4}$  is the distance from  $(0,0)$  to the point  $(x,y)$  on the curve  $y = (x-3)^2$ . What we have defined, then, is a function  $d(x)$  which we wish to minimize over all values of  $x$ . To accomplish this task analytically would require Calculus so as we've mentioned before, we can use a graphing calculator to find an approximate solution. Using Geogebra, we enter the function  $d(x)$  as shown below and graph.

Using the Minimum feature, we see above on the right that the (absolute) minimum occurs near  $x = 2$ . Rounding to two decimal places, we get that the minimum distance occurs when  $x = 2.00$ . To find the  $y$  value on the parabola associated with  $x = 2.00$ , we substitute 2.00 into the equation to get  $y = (x-3)^2 = (2.00-3)^2 = 1.00$ . So, our final answer is  $(2.00, 1.00)$ .

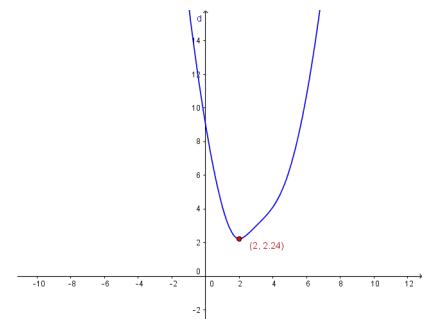


Figure 2.33: Minimizing  $d(x)$  in Example 35

It seems silly to list a final answer as  $(2.00, 1.00)$ . Indeed, Calculus confirms that the *exact* answer to this problem is, in fact,  $(2, 1)$ . As you are well aware by now, the authors are overly pedantic, and as such, use the decimal places to remind the reader that *any* result garnered from a calculator in this fashion is an approximation, and should be treated as such. (What does the  $y$  value calculated by Geogebra in Figure 2.33 mean in this problem?)

## Exercises 2.5

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### Problems

In Exercises 1 – 12, sketch the graph of the given function. State the domain of the function, identify any intercepts and test for symmetry.

1.  $f(x) = 2 - x$

2.  $f(x) = \frac{x-2}{3}$

3.  $f(x) = x^2 + 1$

4.  $f(x) = 4 - x^2$

5.  $f(x) = 2$

6.  $f(x) = x^3$

7.  $f(x) = x(x-1)(x+2)$

8.  $f(x) = \sqrt{x-2}$

9.  $f(x) = \sqrt{5-x}$

10.  $f(x) = 3 - 2\sqrt{x+2}$

11.  $f(x) = \sqrt[3]{x}$

12.  $f(x) = \frac{1}{x^2 + 1}$

In Exercises 13 – 20, sketch the graph of the given piecewise-defined function.

13. 
$$f(x) = \begin{cases} 4-x & \text{if } x \leq 3 \\ 2 & \text{if } x > 3 \end{cases}$$

14. 
$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ 2x & \text{if } x > 0 \end{cases}$$

15. 
$$f(x) = \begin{cases} -3 & \text{if } x < 0 \\ 2x-3 & \text{if } 0 \leq x \leq 3 \\ 3 & \text{if } x > 3 \end{cases}$$

16. 
$$f(x) = \begin{cases} x^2 - 4 & \text{if } x \leq -2 \\ 4 - x^2 & \text{if } -2 < x < 2 \\ x^2 - 4 & \text{if } x \geq 2 \end{cases}$$

17. 
$$f(x) = \begin{cases} -2x-4 & \text{if } x < 0 \\ 3x & \text{if } x \geq 0 \end{cases}$$

18. 
$$f(x) = \begin{cases} \sqrt{x+4} & \text{if } -4 \leq x < 5 \\ \sqrt{x-1} & \text{if } x \geq 5 \end{cases}$$

19. 
$$f(x) = \begin{cases} x^2 & \text{if } x \leq -2 \\ 3-x & \text{if } -2 < x < 2 \\ 4 & \text{if } x \geq 2 \end{cases}$$

20. 
$$f(x) = \begin{cases} \frac{1}{x} & \text{if } -6 < x < -1 \\ x & \text{if } -1 < x < 1 \\ \sqrt{x} & \text{if } 1 < x < 9 \end{cases}$$

In Exercises 21 – 41, determine analytically if the following functions are even, odd or neither.

21.  $f(x) = 7x$

22.  $f(x) = 7x + 2$

23.  $f(x) = 7$

24.  $f(x) = 3x^2 - 4$

25.  $f(x) = 4 - x^2$

26.  $f(x) = x^2 - x - 6$

27.  $f(x) = 2x^3 - x$

28.  $f(x) = -x^5 + 2x^3 - x$

29.  $f(x) = x^6 - x^4 + x^2 + 9$

30.  $f(x) = x^3 + x^2 + x + 1$

31.  $f(x) = \sqrt{1-x}$

32.  $f(x) = \sqrt{1-x^2}$

33.  $f(x) = 0$

34.  $f(x) = \sqrt[3]{x^2}$

35.  $f(x) = \sqrt[3]{x^2}$

36.  $f(x) = \frac{3}{x^2}$

37.  $f(x) = \frac{2x-1}{x+1}$

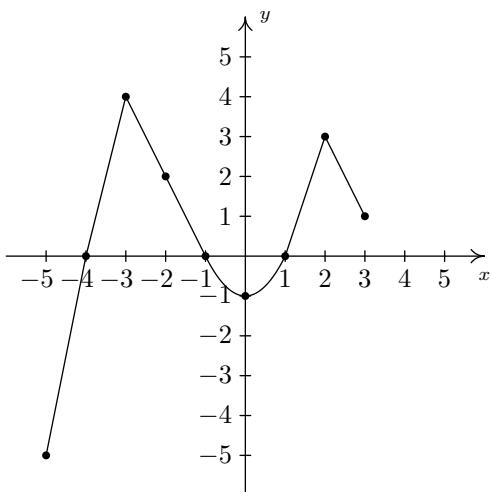
38.  $f(x) = \frac{3x}{x^2 + 1}$

39.  $f(x) = \frac{x^2 - 3}{x - 4x^3}$

40.  $f(x) = \frac{9}{\sqrt{4-x^2}}$

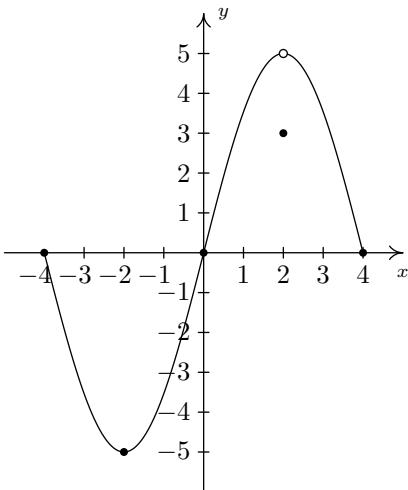
41.  $f(x) = \frac{\sqrt[3]{x^3+x}}{5x}$

In Exercises 42 – 57, use the graph of  $y = f(x)$  given below to answer the question.



42. Find the domain of  $f$ .
43. Find the range of  $f$ .
44. Determine  $f(-2)$ .
45. Solve  $f(x) = 4$ .
46. List the  $x$ -intercepts, if any exist.
47. List the  $y$ -intercepts, if any exist.
48. Find the zeros of  $f$ .
49. Solve  $f(x) \geq 0$ .
50. Find the number of solutions to  $f(x) = 1$ .
51. Does  $f$  appear to be even, odd, or neither?
52. List the intervals where  $f$  is increasing.
53. List the intervals where  $f$  is decreasing.
54. List the local maximums, if any exist.
55. List the local minimums, if any exist.
56. Find the maximum, if it exists.
57. Find the minimum, if it exists.

In Exercises 58 – 73, use the graph of  $y = f(x)$  given below to answer the question.



58. Find the domain of  $f$ .
59. Find the range of  $f$ .
60. Determine  $f(2)$ .
61. Solve  $f(x) = -5$ .
62. List the  $x$ -intercepts, if any exist.
63. List the  $y$ -intercepts, if any exist.
64. Find the zeros of  $f$ .
65. Solve  $f(x) \leq 0$ .
66. Find the number of solutions to  $f(x) = 3$ .
67. Does  $f$  appear to be even, odd, or neither?
68. List the intervals where  $f$  is increasing.
69. List the intervals where  $f$  is decreasing.
70. List the local maximums, if any exist.
71. List the local minimums, if any exist.
72. Find the maximum, if it exists.
73. Find the minimum, if it exists.

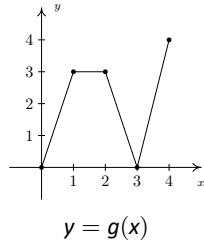
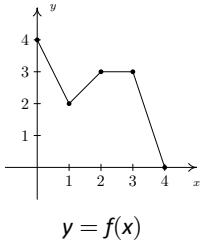
**In Exercises 74 – 77, use a graphing calculator or software (such as GeoGebra) to approximate the local and absolute extrema of the given function. Approximate the intervals on which the function is increasing and those on which it is decreasing. Round your answers to two decimal places.**

74.  $f(x) = x^4 - 3x^3 - 24x^2 + 28x + 48$
75.  $f(x) = x^{2/3}(x - 4)$

76.  $f(x) = \sqrt{9 - x^2}$

77.  $f(x) = x\sqrt{9 - x^2}$

In Exercises 78 – 87, use the graphs of  $y = f(x)$  and  $y = g(x)$  below to find the function value.



78.  $(f + g)(0)$

83.  $(fg)(1)$

79.  $(f + g)(1)$

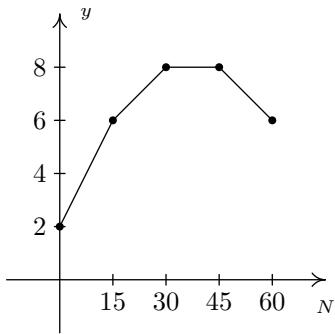
84.  $\left(\frac{f}{g}\right)(4)$

80.  $(f - g)(1)$

85.  $\left(\frac{g}{f}\right)(2)$

82.  $(fg)(2)$

The graph below represents the height  $h$  of a Sasquatch (in feet) as a function of its age  $N$  in years. Use it to answer the questions in Exercises 86 – 90.



86. Find and interpret  $h(0)$ .

87. How tall is the Sasquatch when she is 15 years old?

88. Solve  $h(N) = 6$  and interpret.

89. List the interval over which  $h$  is constant and interpret your answer.

90. List the interval over which  $h$  is decreasing and interpret your answer.

For Exercises 91 – 93, let  $f(x) = \lfloor x \rfloor$  be the greatest integer function as defined in Exercise 76 in Section 2.3.

91. Graph  $y = f(x)$ . Be careful to correctly describe the behaviour of the graph near the integers.

92. Is  $f$  even, odd, or neither? Explain.

93. Discuss with your classmates which points on the graph are local minimums, local maximums or both. Is  $f$  ever increasing? Decreasing? Constant?

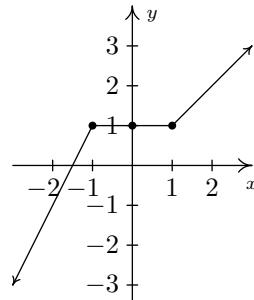
94. In Exercise 72 in Section 2.3, we saw that the population of Sasquatch in Portage County could be modeled by the function  $P(t) = \frac{150t}{t+15}$ , where  $t = 0$  represents the year 1803. Use your graphing calculator to analyze the general function behaviour of  $P$ . Will there ever be a time when 200 Sasquatch roam Portage County?

95. Suppose  $f$  and  $g$  are both even functions. What can be said about the functions  $f + g$ ,  $f - g$ ,  $fg$  and  $\frac{f}{g}$ ? What if  $f$  and  $g$  are both odd? What if  $f$  is even but  $g$  is odd?

96. One of the most important aspects of the Cartesian Coordinate Plane is its ability to put Algebra into geometric terms and Geometry into algebraic terms. We've spent most of this chapter looking at this very phenomenon and now you should spend some time with your classmates reviewing what we've done. What major results do we have that tie Algebra and Geometry together? What concepts from Geometry have we not yet described algebraically? What topics from Intermediate Algebra have we not yet discussed geometrically?

It's now time to "thoroughly vet the pathologies induced" by the precise definitions of local maximum and local minimum. You and your classmates should carefully discuss Exercises 97 – 99. You will need to refer back to Definition 27 (Increasing, Decreasing and Constant) and Definition 28 (Maximum and Minimum) during the discussion.

97. Consider the graph of the function  $f$  given below.



(a) Show that  $f$  has a local maximum but not a local minimum at the point  $(-1, 1)$ .

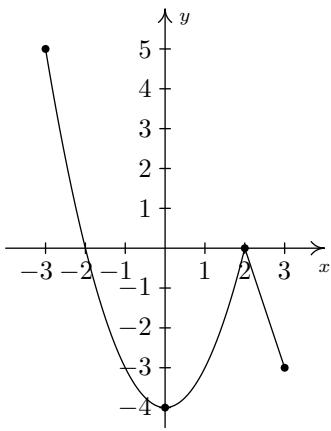
(b) Show that  $f$  has a local minimum but not a local maximum at the point  $(1, 1)$ .

(c) Show that  $f$  has a local maximum AND a local minimum at the point  $(0, 1)$ .

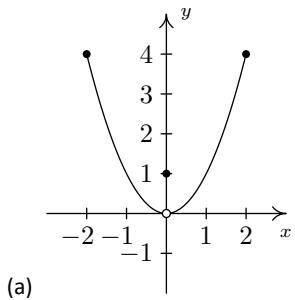
(d) Show that  $f$  is constant on the interval  $[-1, 1]$  and thus has both a local maximum AND a local minimum at every point  $(x, f(x))$  where  $-1 < x < 1$ .

98. Using Example 33 as a guide, show that the function  $g$  whose graph is given below does not have a local maximum at  $(-3, 5)$  nor does it have a local minimum at  $(3, -3)$ . Find

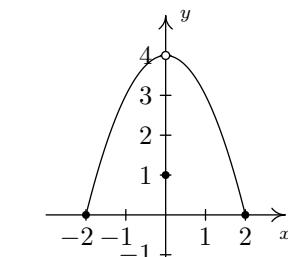
its extrema, both local and absolute. What's unique about the point  $(0, -4)$  on this graph? Also find the intervals on which  $g$  is increasing and those on which  $g$  is decreasing.



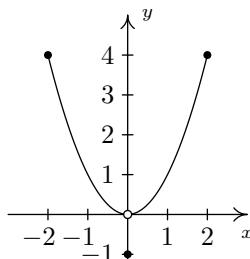
99. We said earlier in the section that it is not good enough to say local extrema exist where a function changes from increasing to decreasing or vice versa. As a previous exercise showed, we could have local extrema when a function is constant so now we need to examine some functions whose graphs do indeed change direction. Consider the functions graphed below. Notice that all four of them change direction at an open circle on the graph. Examine each for local extrema. What is the effect of placing the "dot" on the  $y$ -axis above or below the open circle? What could you say if no function value were assigned to  $x = 0$ ?



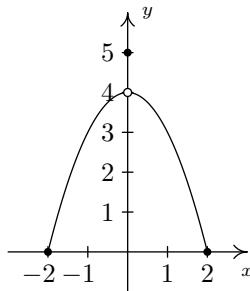
(a)



(b)



(c)



(d)

## 2.6 Transformations

In this section, we study how the graphs of functions change, or **transform**, when certain specialized modifications are made to their formulas. The transformations we will study fall into three broad categories: shifts, reflections and scalings, and we will present them in that order. Suppose that Figure 2.35 the complete graph of a function  $f$ .

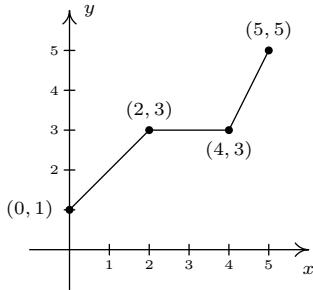


Figure 2.35: The graph of a function  $f$

The Fundamental Graphing Principle for Functions says that for a point  $(a, b)$  to be on the graph,  $f(a) = b$ . In particular, we know  $f(0) = 1, f(2) = 3, f(4) = 3$  and  $f(5) = 5$ . Suppose we wanted to graph the function defined by the formula  $g(x) = f(x) + 2$ . Let's take a minute to remind ourselves of what  $g$  is doing. We start with an input  $x$  to the function  $f$  and we obtain the output  $f(x)$ . The function  $g$  takes the output  $f(x)$  and adds 2 to it. In order to graph  $g$ , we need to graph the points  $(x, g(x))$ . How are we to find the values for  $g(x)$  without a formula for  $f(x)$ ? The answer is that we don't need a *formula* for  $f(x)$ , we just need the *values* of  $f(x)$ . The values of  $f(x)$  are the  $y$  values on the graph of  $y = f(x)$ . For example, using the points indicated on the graph of  $f$ , we can make the following table.

$x$	$(x, f(x))$	$f(x)$	$g(x) = f(x) + 2$	$(x, g(x))$
0	$(0, 1)$	1	3	$(0, 3)$
2	$(2, 3)$	3	5	$(2, 5)$
4	$(4, 3)$	3	5	$(4, 5)$
5	$(5, 5)$	5	7	$(5, 7)$

In general, if  $(a, b)$  is on the graph of  $y = f(x)$ , then  $f(a) = b$ , so  $g(a) = f(a) + 2 = b + 2$ . Hence,  $(a, b + 2)$  is on the graph of  $g$ . In other words, to obtain the graph of  $g$ , we add 2 to the  $y$ -coordinate of each point on the graph of  $f$ . Geometrically, adding 2 to the  $y$ -coordinate of a point moves the point 2 units above its previous location. Adding 2 to every  $y$ -coordinate on a graph *en masse* is usually described as 'shifting the graph up 2 units'. Notice that the graph retains the same basic shape as before, it is just 2 units above its original location. In other words, we connect the four points we moved in the same manner in which they were connected before: see Figure 2.34.

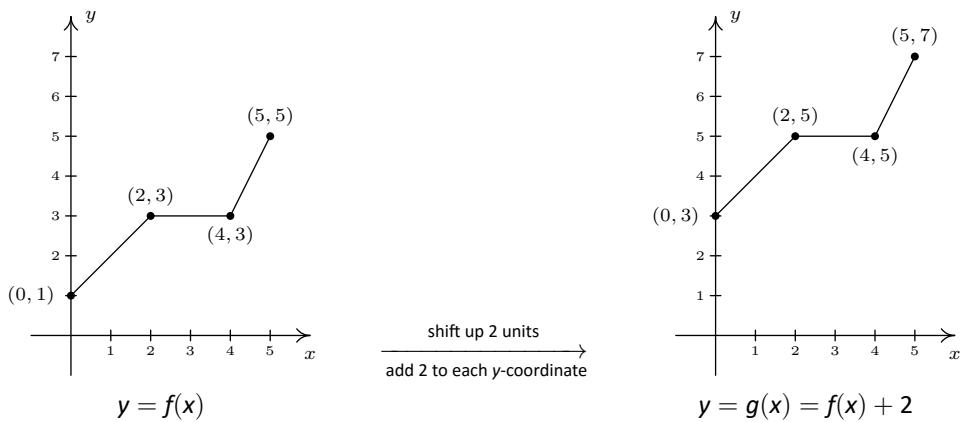


Figure 2.34: Shifting the graph of  $f$  up by 2 units

You'll note that the domain of  $f$  and the domain of  $g$  are the same, namely  $[0, 5]$ , but that the range of  $f$  is  $[1, 5]$  while the range of  $g$  is  $[3, 7]$ . In general, shifting a function vertically like this will leave the domain unchanged, but could very well affect the range. You can easily imagine what would happen if we

wanted to graph the function  $j(x) = f(x) - 2$ . Instead of adding 2 to each of the  $y$ -coordinates on the graph of  $f$ , we'd be subtracting 2. Geometrically, we would be moving the graph down 2 units. We leave it to the reader to verify that the domain of  $j$  is the same as  $f$ , but the range of  $j$  is  $[-1, 3]$ . What we have discussed is generalized in the following theorem.

**Theorem 7    Vertical Shifts**

Suppose  $f$  is a function and  $k$  is a positive number.

- To graph  $y = f(x) + k$ , shift the graph of  $y = f(x)$  up  $k$  units by adding  $k$  to the  $y$ -coordinates of the points on the graph of  $f$ .
- To graph  $y = f(x) - k$ , shift the graph of  $y = f(x)$  down  $k$  units by subtracting  $k$  from the  $y$ -coordinates of the points on the graph of  $f$ .

The key to understanding Theorem 7 and, indeed, all of the theorems in this section comes from an understanding of the Fundamental Graphing Principle for Functions. If  $(a, b)$  is on the graph of  $f$ , then  $f(a) = b$ . Substituting  $x = a$  into the equation  $y = f(x) + k$  gives  $y = f(a) + k = b + k$ . Hence,  $(a, b + k)$  is on the graph of  $y = f(x) + k$ , and we have the result. In the language of ‘inputs’ and ‘outputs’, Theorem 7 can be paraphrased as “Adding to, or subtracting from, the *output* of a function causes the graph to shift up or down, respectively.” So what happens if we add to or subtract from the *input* of the function?

Keeping with the graph of  $y = f(x)$  above, suppose we wanted to graph  $g(x) = f(x + 2)$ . In other words, we are looking to see what happens when we add 2 to the input of the function.<sup>1</sup> Let’s try to generate a table of values of  $g$  based on those we know for  $f$ . We quickly find that we run into some difficulties.

$x$	$(x, f(x))$	$f(x)$	$g(x) = f(x + 2)$	$(x, g(x))$
0	$(0, 1)$	1	$f(0 + 2) = f(2) = 3$	$(0, 3)$
2	$(2, 3)$	3	$f(2 + 2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$f(4 + 2) = f(6) = ?$	
5	$(5, 5)$	5	$f(5 + 2) = f(7) = ?$	

When we substitute  $x = 4$  into the formula  $g(x) = f(x + 2)$ , we are asked to find  $f(4 + 2) = f(6)$  which doesn’t exist because the domain of  $f$  is only  $[0, 5]$ . The same thing happens when we attempt to find  $g(5)$ . What we need here is a new strategy. We know, for instance,  $f(0) = 1$ . To determine the corresponding point on the graph of  $g$ , we need to figure out what value of  $x$  we must substitute into  $g(x) = f(x + 2)$  so that the quantity  $x + 2$ , works out to be 0. Solving  $x + 2 = 0$  gives  $x = -2$ , and  $g(-2) = f((-2) + 2) = f(0) = 1$  so  $(-2, 1)$  is on the graph of  $g$ . To use the fact  $f(2) = 3$ , we set  $x + 2 = 2$  to get  $x = 0$ . Substituting gives

<sup>1</sup>We have spent a lot of time in this text showing you that  $f(x + 2)$  and  $f(x) + 2$  are, in general, wildly different algebraic animals. We will see momentarily that their geometry is also dramatically different.

$g(0) = f(0 + 2) = f(2) = 3$ . Continuing in this fashion, we get

$x$	$x + 2$	$g(x) = f(x + 2)$	$(x, g(x))$
-2	0	$g(-2) = f(0) = 1$	$(-2, 1)$
0	2	$g(0) = f(2) = 3$	$(0, 3)$
2	4	$g(2) = f(4) = 3$	$(2, 3)$
3	5	$g(3) = f(5) = 5$	$(3, 5)$

In summary, the points  $(0, 1)$ ,  $(2, 3)$ ,  $(4, 3)$  and  $(5, 5)$  on the graph of  $y = f(x)$  give rise to the points  $(-2, 1)$ ,  $(0, 3)$ ,  $(2, 3)$  and  $(3, 5)$  on the graph of  $y = g(x)$ , respectively. In general, if  $(a, b)$  is on the graph of  $y = f(x)$ , then  $f(a) = b$ . Solving  $x + 2 = a$  gives  $x = a - 2$  so that  $g(a - 2) = f((a - 2) + 2) = f(a) = b$ . As such,  $(a - 2, b)$  is on the graph of  $y = g(x)$ . The point  $(a - 2, b)$  is exactly 2 units to the *left* of the point  $(a, b)$  so the graph of  $y = g(x)$  is obtained by shifting the graph  $y = f(x)$  to the left 2 units, as pictured below.

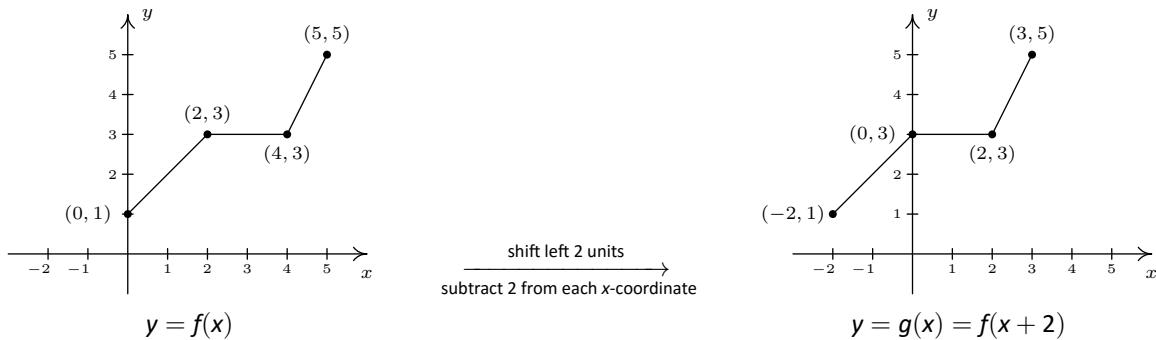


Figure 2.36: Shifting the graph of  $f$  left by 2 units

Note that while the ranges of  $f$  and  $g$  are the same, the domain of  $g$  is  $[-2, 3]$  whereas the domain of  $f$  is  $[0, 5]$ . In general, when we shift the graph horizontally, the range will remain the same, but the domain could change. If we set out to graph  $j(x) = f(x - 2)$ , we would find ourselves *adding* 2 to all of the  $x$  values of the points on the graph of  $y = f(x)$  to effect a shift to the *right* 2 units. Generalizing these notions produces the following result.

### Theorem 8     Horizontal Shifts

Suppose  $f$  is a function and  $h$  is a positive number.

- To graph  $y = f(x + h)$ , shift the graph of  $y = f(x)$  left  $h$  units by subtracting  $h$  from the  $x$ -coordinates of the points on the graph of  $f$ .
- To graph  $y = f(x - h)$ , shift the graph of  $y = f(x)$  right  $h$  units by adding  $h$  to the  $x$ -coordinates of the points on the graph of  $f$ .

In other words, Theorem 8 says that adding to or subtracting from the *input* to a function amounts to shifting the graph left or right, respectively. Theorems 7 and 8 present a theme which will run common throughout the section: changes to the outputs from a function affect the  $y$ -coordinates of the graph, resulting in some kind of vertical change; changes to the inputs to a function affect the  $x$ -coordinates of the graph, resulting in some kind of horizontal change.

**Example 36** Transforming with vertical and horizontal shifts

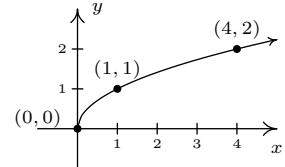
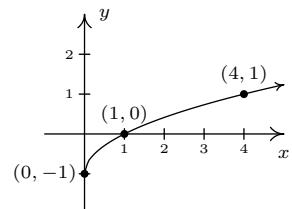
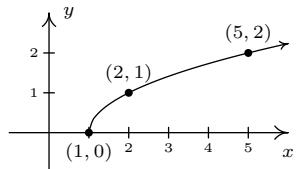
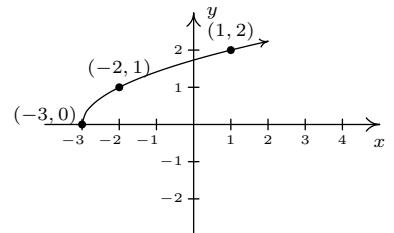
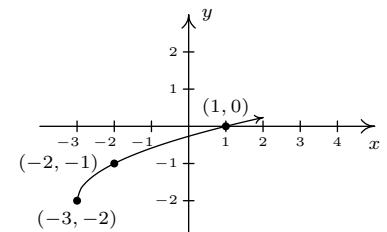
1. Graph  $f(x) = \sqrt{x}$ . Plot at least three points.
2. Use your graph in 1 to graph  $g(x) = \sqrt{x} - 1$ .
3. Use your graph in 1 to graph  $j(x) = \sqrt{x - 1}$ .
4. Use your graph in 1 to graph  $m(x) = \sqrt{x + 3} - 2$ .

**SOLUTION**

1. Owing to the square root, the domain of  $f$  is  $x \geq 0$ , or  $[0, \infty)$ . We choose perfect squares to build our table and graph below. From the graph we verify the domain of  $f$  is  $[0, \infty)$  and the range of  $f$  is also  $[0, \infty)$ . The original function is plotted in Figure 2.37
2. The domain of  $g$  is the same as the domain of  $f$ , since the only condition on both functions is that  $x \geq 0$ . If we compare the formula for  $g(x)$  with  $f(x)$ , we see that  $g(x) = f(x) - 1$ . In other words, we have subtracted 1 from the output of the function  $f$ . By Theorem 7, we know that in order to graph  $g$ , we shift the graph of  $f$  down one unit by subtracting 1 from each of the  $y$ -coordinates of the points on the graph of  $f$ . Applying this to the three points we have specified on the graph, we move  $(0, 0)$  to  $(0, -1)$ ,  $(1, 1)$  to  $(1, 0)$ , and  $(4, 2)$  to  $(4, 1)$ . The rest of the points follow suit, and we connect them with the same basic shape as before. We confirm the domain of  $g$  is  $[0, \infty)$  and find the range of  $g$  to be  $[-1, \infty)$ . The graph of  $g$  is given in Figure 2.38.
3. Solving  $x - 1 \geq 0$  gives  $x \geq 1$ , so the domain of  $j$  is  $[1, \infty)$ . To graph  $j$ , we note that  $j(x) = f(x - 1)$ . In other words, we are subtracting 1 from the *input* of  $f$ . According to Theorem 8, this induces a shift to the right of the graph of  $f$ . We add 1 to the  $x$ -coordinates of the points on the graph of  $f$  and get the result below. The graph reaffirms that the domain of  $j$  is  $[1, \infty)$  and tells us that the range of  $j$  is  $[0, \infty)$ .
4. To find the domain of  $m$ , we solve  $x + 3 \geq 0$  and get  $[-3, \infty)$ . Comparing the formulas of  $f(x)$  and  $m(x)$ , we have  $m(x) = f(x + 3) - 2$ . We have 3 being added to an input, indicating a horizontal shift, and 2 being subtracted from an output, indicating a vertical shift. We leave it to the reader to verify that, in this particular case, the order in which we perform these transformations is immaterial; we will arrive at the same graph regardless as to which transformation we apply first. (We shall see in the next example that order is generally important when applying more than one transformation to a graph.) We follow the convention ‘inputs first’, and to that end we first tackle the horizontal shift. Letting  $m_1(x) = f(x + 3)$  denote this intermediate step, Theorem 8 tells us that the graph of  $y = m_1(x)$  is the graph of  $f$  shifted to the left 3 units. Hence, we subtract 3 from each of the  $x$ -coordinates of the points on the graph of  $f$ .

Since  $m(x) = f(x+3)-2$  and  $f(x+3) = m_1(x)$ , we have  $m(x) = m_1(x)-2$ . We can apply Theorem 7 and obtain the graph of  $m$  by subtracting 2 from the  $y$ -coordinates of each of the points on the graph of  $m_1(x)$ . The graph verifies that the domain of  $m$  is  $[-3, \infty)$  and we find the range of  $m$  to be  $[-2, \infty)$ .

$x$	$f(x)$	$(x, f(x))$
0	0	$(0, 0)$
1	1	$(1, 1)$
4	2	$(4, 2)$

Figure 2.37: The graph  $y = f(x) = \sqrt{x}$ Figure 2.38: Graphing  $g(x) = \sqrt{x} - 1$ Figure 2.39: Graphing  $j(x) = \sqrt{x - 1}$ Figure 2.40: Graphing  $m_1(x) = \sqrt{x + 3}$ Figure 2.41: Graphing  $m(x) = \sqrt{x + 3} - 2$

Keep in mind that we can check our answer to any of these kinds of problems by showing that any of the points we've moved lie on the graph of our final answer. For example, we can check that  $(-3, -2)$  is on the graph of  $m$  by computing  $m(-3) = \sqrt{(-3) + 3} - 2 = \sqrt{0} - 2 = -2 \checkmark$

We now turn our attention to reflections. We know from Section 1.3 that to reflect a point  $(x, y)$  across the  $x$ -axis, we replace  $y$  with  $-y$ . If  $(x, y)$  is on the graph of  $f$ , then  $y = f(x)$ , so replacing  $y$  with  $-y$  is the same as replacing  $f(x)$  with  $-f(x)$ . Hence, the graph of  $y = -f(x)$  is the graph of  $f$  reflected across the  $x$ -axis. Similarly, the graph of  $y = f(-x)$  is the graph of  $f$  reflected across the  $y$ -axis. Returning to the language of inputs and outputs, multiplying the output from a function by  $-1$  reflects its graph across the  $x$ -axis, while multiplying the input to a function by  $-1$  reflects the graph across the  $y$ -axis.<sup>2</sup>

### Theorem 9 Reflections

Suppose  $f$  is a function.

- To graph  $y = -f(x)$ , reflect the graph of  $y = f(x)$  across the  $x$ -axis by multiplying the  $y$ -coordinates of the points on the graph of  $f$  by  $-1$ .
- To graph  $y = f(-x)$ , reflect the graph of  $y = f(x)$  across the  $y$ -axis by multiplying the  $x$ -coordinates of the points on the graph of  $f$  by  $-1$ .

Applying Theorem 9 to the graph of  $y = f(x)$  given at the beginning of the section, we can graph  $y = -f(x)$  by reflecting the graph of  $f$  about the  $x$ -axis

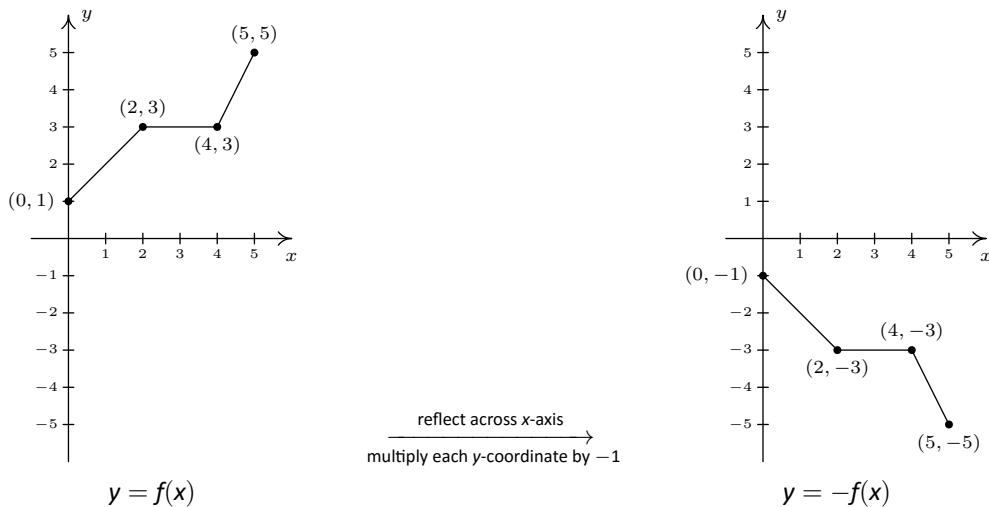
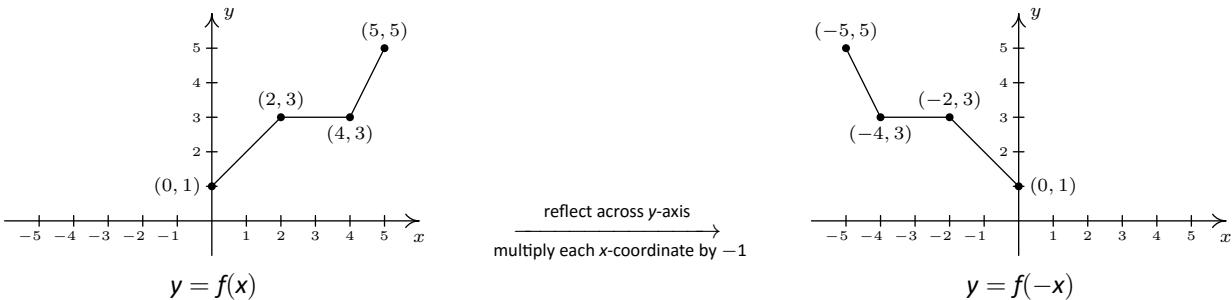


Figure 2.42: Reflecting the graph of  $f$  across the  $x$ -axis

By reflecting the graph of  $f$  across the  $y$ -axis, we obtain the graph of  $y = f(-x)$ .

<sup>2</sup>The expressions  $-f(x)$  and  $f(-x)$  should look familiar - they are the quantities we used in Section 2.5 to test if a function was even, odd or neither. The interested reader is invited to explore the role of reflections and symmetry of functions. What happens if you reflect an even function across the  $y$ -axis? What happens if you reflect an odd function across the  $y$ -axis? What about the  $x$ -axis?

Figure 2.43: Reflecting the graph of  $f$  across the  $y$ -axis

With the addition of reflections, it is now more important than ever to consider the order of transformations, as the next example illustrates.

### Example 37 Graphing reflections

Let  $f(x) = \sqrt{x}$ . Use the graph of  $f$  from Example 36 to graph the following functions. Also, state their domains and ranges.

$$1. g(x) = \sqrt{-x} \quad 2. j(x) = \sqrt{3-x} \quad 3. m(x) = 3 - \sqrt{x}$$

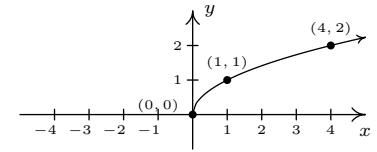
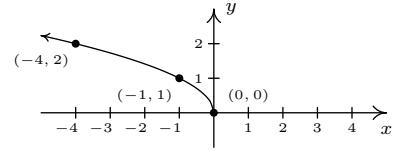
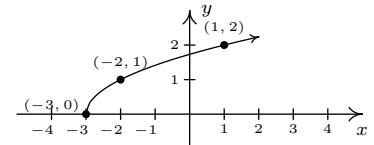
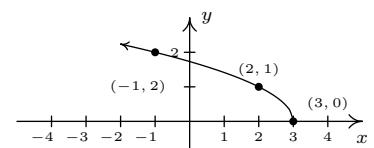
#### SOLUTION

1. The mere sight of  $\sqrt{-x}$  usually causes alarm, if not panic. When we discussed domains in Section 2.3, we clearly banished negatives from the radicands of even roots. However, we must remember that  $x$  is a variable, and as such, the quantity  $-x$  isn't always negative. For example, if  $x = -4$ ,  $-x = 4$ , thus  $\sqrt{-x} = \sqrt{-(-4)} = 2$  is perfectly well-defined. To find the domain analytically, we set  $-x \geq 0$  which gives  $x \leq 0$ , so that the domain of  $g$  is  $(-\infty, 0]$ . Since  $g(x) = f(-x)$ , Theorem 9 tells us that the graph of  $g$  is the reflection of the graph of  $f$  across the  $y$ -axis. We accomplish this by multiplying each  $x$ -coordinate on the graph of  $f$  by  $-1$ , so that the points  $(0, 0)$ ,  $(1, 1)$ , and  $(4, 2)$  move to  $(0, 0)$ ,  $(-1, 1)$ , and  $(-4, 2)$ , respectively. Graphically, we see that the domain of  $g$  is  $(-\infty, 0]$  and the range of  $g$  is the same as the range of  $f$ , namely  $[0, \infty)$ .

If we had done the reflection first, then  $j_1(x) = f(-x)$ . Following this by a shift left would give us  $j(x) = j_1(x + 3) = f(-(x + 3)) = f(-x - 3) = \sqrt{-x - 3}$  which isn't what we want. However, if we did the reflection first and followed it by a shift to the right 3 units, we would have arrived at the function  $j(x)$ . We leave it to the reader to verify the details.

2. To determine the domain of  $j(x) = \sqrt{3-x}$ , we solve  $3-x \geq 0$  and get  $x \leq 3$ , or  $(-\infty, 3]$ . To determine which transformations we need to apply to the graph of  $f$  to obtain the graph of  $j$ , we rewrite  $j(x) = \sqrt{-x+3} = f(-x+3)$ . Comparing this formula with  $f(x) = \sqrt{x}$ , we see that not only are we multiplying the input  $x$  by  $-1$ , which results in a reflection across the  $y$ -axis, but also we are adding 3, which indicates a horizontal shift to the left. Does it matter in which order we do the transformations? If so, which order is the correct order? Let's consider the point  $(4, 2)$  on the graph of  $f$ . We refer to the discussion leading up to Theorem 8. We know  $f(4) = 2$  and wish to find the point on  $y = j(x) = f(-x+3)$  which corresponds to  $(4, 2)$ . We set  $-x+3 = 4$  and solve. Our first step is to subtract 3 from both sides to get  $-x = 1$ . Subtracting 3 from the  $x$ -coordinate 4 is shifting the point  $(4, 2)$  to the left. From  $-x = 1$ , we then multiply<sup>3</sup>

<sup>3</sup>Or divide - it amounts to the same thing.

Figure 2.44: The graph  $y = f(x)$  from Example 36Figure 2.45: Reflecting  $y = f(x)$  across the  $y$ -axis to obtain the graph of  $g(x) = \sqrt{-x}$ Figure 2.46: The intermediate function  $j_1(x) = f(x + 3)$ Figure 2.47: Reflecting  $y = j_1(x)$  across the  $y$ -axis to obtain the graph of  $j(x) = \sqrt{3 - x}$

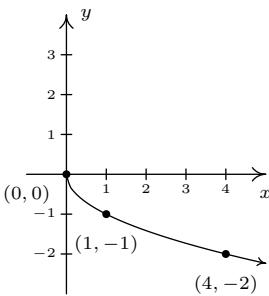


Figure 2.48: Reflecting  $y = f(x)$  across the  $x$ -axis to obtain the graph of  $m_1(x) = -\sqrt{x}$

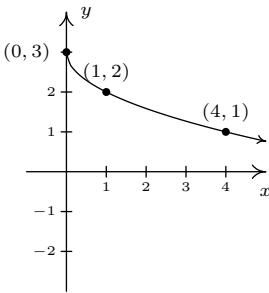
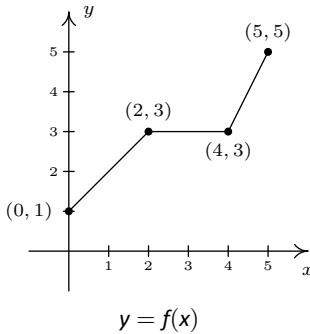
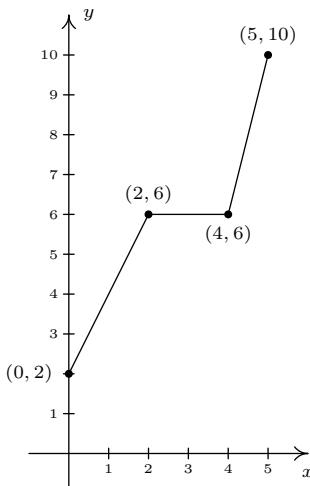


Figure 2.49: Shifting  $y = m_1(x)$  up by three units to obtain the graph of  $m(x) = 3 - \sqrt{x}$



$y = f(x)$



$y = 2f(x) = g(x)$

Figure 2.50: Graphing  $g(x) = 2f(x)$

both sides by  $-1$  to get  $x = -1$ . Multiplying the  $x$ -coordinate by  $-1$  corresponds to reflecting the point about the  $y$ -axis. Hence, we perform the horizontal shift first, then follow it with the reflection about the  $y$ -axis. Starting with  $f(x) = \sqrt{x}$ , we let  $j_1(x)$  be the intermediate function which shifts the graph of  $f$  3 units to the left,  $j_1(x) = f(x + 3)$ .

To obtain the function  $j$ , we reflect the graph of  $j_1$  about  $y$ -axis. Theorem 9 tells us we have  $j(x) = j_1(-x)$ . Putting it all together, we have  $j(x) = j_1(-x) = f(-x + 3) = \sqrt{-x + 3}$ , which is what we want. From the graph, we confirm the domain of  $j$  is  $(-\infty, 3]$  and we get that the range is  $[0, \infty)$ .

3. The domain of  $m$  works out to be the domain of  $f$ ,  $[0, \infty)$ . Rewriting  $m(x) = -\sqrt{x} + 3$ , we see  $m(x) = -f(x) + 3$ . Since we are multiplying the output of  $f$  by  $-1$  and then adding 3, we once again have two transformations to deal with: a reflection across the  $x$ -axis and a vertical shift. To determine the correct order in which to apply the transformations, we imagine trying to determine the point on the graph of  $m$  which corresponds to  $(4, 2)$  on the graph of  $f$ . Since in the formula for  $m(x)$ , the input to  $f$  is just  $x$ , we substitute to find  $m(4) = -f(4) + 3 = -2 + 3 = 1$ . Hence,  $(4, 1)$  is the corresponding point on the graph of  $m$ . If we closely examine the arithmetic, we see that we first multiply  $f(4)$  by  $-1$ , which corresponds to the reflection across the  $x$ -axis, and then we add 3, which corresponds to the vertical shift. If we define an intermediate function  $m_1(x) = -f(x)$  to take care of the reflection, we get the graph in Figure 2.48.

To shift the graph of  $m_1$  up 3 units, we set  $m(x) = m_1(x) + 3$ . Since  $m_1(x) = -f(x)$ , when we put it all together, we get  $m(x) = m_1(x) + 3 = -f(x) + 3 = -\sqrt{x} + 3$ . We see from the graph that the range of  $m$  is  $(-\infty, 3]$ .

We now turn our attention to our last class of transformations known as **scalings**. A thorough discussion of scalings can get complicated because they are not as straight-forward as the previous transformations. A quick review of what we've covered so far, namely vertical shifts, horizontal shifts and reflections, will show you why those transformations are known as **rigid transformations**. Simply put, they do not change the *shape* of the graph, only its position and orientation in the plane. If, however, we wanted to make a new graph twice as tall as a given graph, or one-third as wide, we would be changing the shape of the graph. This type of transformation is called **non-rigid** for obvious reasons. Not only will it be important for us to differentiate between modifying inputs versus outputs, we must also pay close attention to the magnitude of the changes we make. As you will see shortly, the Mathematics turns out to be easier than the associated grammar.

Suppose we wish to graph the function  $g(x) = 2f(x)$  where  $f(x)$  is the function whose graph is given in Figure 2.35 the beginning of the section. From its graph, we can build a table of values for  $g$  as before:

$x$	$(x, f(x))$	$f(x)$	$g(x) = 2f(x)$	$(x, g(x))$
0	$(0, 1)$	1	2	$(0, 2)$
2	$(2, 3)$	3	6	$(2, 6)$
4	$(4, 3)$	3	6	$(4, 6)$
5	$(5, 5)$	5	10	$(5, 10)$

In general, if  $(a, b)$  is on the graph of  $f$ , then  $f(a) = b$  so that  $g(a) = 2f(a) = 2b$  puts  $(a, 2b)$  on the graph of  $g$ . In other words, to obtain the graph of  $g$ , we

multiply all of the  $y$ -coordinates of the points on the graph of  $f$  by 2. Multiplying all of the  $y$ -coordinates of all of the points on the graph of  $f$  by 2 causes what is known as a ‘vertical scaling (or ‘vertical stretching’, or ‘vertical expansion’ or ‘vertical dilation’) by a factor of 2’, and the results are given in Figure 2.50.

If we wish to graph  $y = \frac{1}{2}f(x)$ , we multiply the all of the  $y$ -coordinates of the points on the graph of  $f$  by  $\frac{1}{2}$ . This creates a ‘vertical scaling by a factor of  $\frac{1}{2}$ ’ (also called ‘vertical shrinking’, ‘vertical compression’ or ‘vertical contraction’ by a factor of 2) as seen in Figure 2.51

These results are generalized in the following theorem.

### Theorem 10 Vertical Scalings

Suppose  $f$  is a function and  $a > 0$ . To graph  $y = af(x)$ , multiply all of the  $y$ -coordinates of the points on the graph of  $f$  by  $a$ . We say the graph of  $f$  has been vertically scaled by a factor of  $a$ .

- If  $a > 1$ , we say the graph of  $f$  has undergone a vertical stretching (expansion, dilation) by a factor of  $a$ .
- If  $0 < a < 1$ , we say the graph of  $f$  has undergone a vertical shrinking (compression, contraction) by a factor of  $\frac{1}{a}$ .

A few remarks about Theorem 10 are in order. First, a note about the verbiage. To the authors, the words ‘stretching’, ‘expansion’, and ‘dilation’ all indicate something getting bigger. Hence, ‘stretched by a factor of 2’ makes sense if we are scaling something by multiplying it by 2. Similarly, we believe words like ‘shrinking’, ‘compression’ and ‘contraction’ all indicate something getting smaller, so if we scale something by a factor of  $\frac{1}{2}$ , we would say it ‘shrinks by a factor of 2’ - not ‘shrinks by a factor of  $\frac{1}{2}$ ’. This is why we have written the descriptions ‘stretching by a factor of  $a$ ’ and ‘shrinking by a factor of  $\frac{1}{a}$ ’ in the statement of the theorem. Second, in terms of inputs and outputs, Theorem 10 says multiplying the *outputs* from a function by positive number  $a$  causes the graph to be vertically scaled by a factor of  $a$ . It is natural to ask what would happen if we multiply the *inputs* of a function by a positive number. This leads us to our last transformation of the section.

Referring to the graph of  $f$  given at the beginning of this section, suppose we want to graph  $g(x) = f(2x)$ . In other words, we are looking to see what effect multiplying the inputs to  $f$  by 2 has on its graph. If we attempt to build a table directly, we quickly run into the same problem we had in our discussion leading up to Theorem 8, as seen in the table below.

$x$	$(x, f(x))$	$f(x)$	$g(x) = f(2x)$	$(x, g(x))$
0	$(0, 1)$	1	$f(2 \cdot 0) = f(0) = 1$	$(0, 1)$
2	$(2, 3)$	3	$f(2 \cdot 2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$f(2 \cdot 4) = f(8) = ?$	
5	$(5, 5)$	5	$f(2 \cdot 5) = f(10) = ?$	

We solve this problem in the same way we solved this problem before. For example, if we want to determine the point on  $g$  which corresponds to the point  $(2, 3)$  on the graph of  $f$ , we set  $2x = 2$  so that  $x = 1$ . Substituting  $x = 1$  into  $g(x)$ , we obtain  $g(1) = f(2 \cdot 1) = f(2) = 3$ , so that  $(1, 3)$  is on the graph of  $g$ . Continuing in this fashion, we can complete our table as follows:

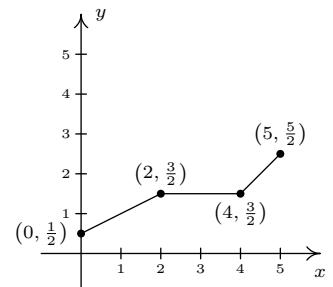
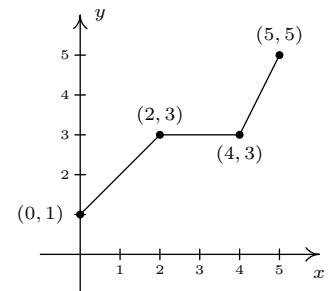
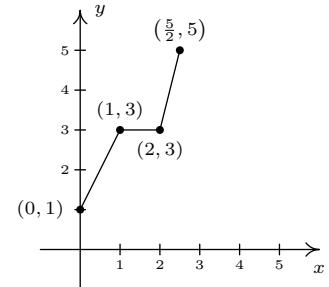


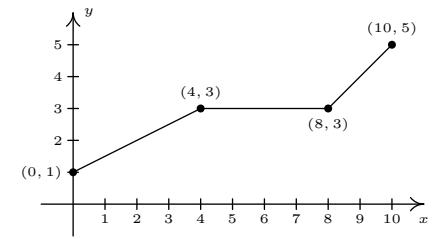
Figure 2.51: Vertical scaling by  $\frac{1}{2}$



The graph  $y = f(x)$  from Figure 2.35



The graph  $y = g(x) = f(2x)$



The graph  $y = h(x) = f(\frac{1}{2}x)$

Figure 2.52: The effect of horizontal scaling on a graph

$x$	$2x$	$g(x) = f(2x)$	$(x, g(x))$
0	0	$g(0) = f(0) = 1$	$(0, 0)$
1	2	$g(1) = f(2) = 3$	$(1, 3)$
2	4	$g(2) = f(4) = 3$	$(2, 3)$
$\frac{5}{2}$	5	$g\left(\frac{5}{2}\right) = f(5) = 5$	$\left(\frac{5}{2}, 5\right)$

In general, if  $(a, b)$  is on the graph of  $f$ , then  $f(a) = b$ . Hence  $g\left(\frac{a}{2}\right) = f(2 \cdot \frac{a}{2}) = f(a) = b$  so that  $\left(\frac{a}{2}, b\right)$  is on the graph of  $g$ . In other words, to graph  $g$  we divide the  $x$ -coordinates of the points on the graph of  $f$  by 2. This results in a horizontal scaling by a factor of  $\frac{1}{2}$  (also called ‘horizontal shrinking’, ‘horizontal compression’ or ‘horizontal contraction’ by a factor of 2).

If, on the other hand, we wish to graph  $y = f(\frac{1}{2}x)$ , we end up multiplying the  $x$ -coordinates of the points on the graph of  $f$  by 2 which results in a horizontal scaling<sup>4</sup> by a factor of 2. The effect of both horizontal scalings is shown in Figure 2.52.

We have the following theorem.

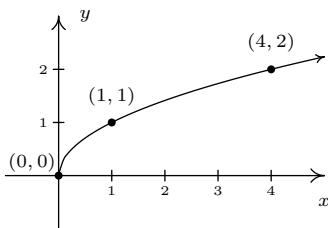


Figure 2.53: The graph  $y = \sqrt{x}$

### Theorem 11 Horizontal Scalings.

Suppose  $f$  is a function and  $b > 0$ . To graph  $y = f(bx)$ , divide all of the  $x$ -coordinates of the points on the graph of  $f$  by  $b$ . We say the graph of  $f$  has been horizontally scaled by a factor of  $\frac{1}{b}$ .

- If  $0 < b < 1$ , we say the graph of  $f$  has undergone a horizontal stretching (expansion, dilation) by a factor of  $\frac{1}{b}$ .
- If  $b > 1$ , we say the graph of  $f$  has undergone a horizontal shrinking (compression, contraction) by a factor of  $b$ .

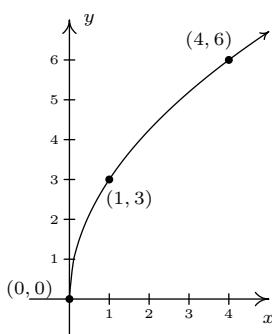


Figure 2.54: The graph  $y = g(x) = 3\sqrt{x}$

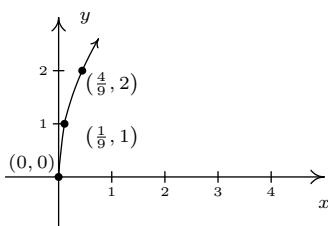


Figure 2.55: The graph  $y = j(x) = \sqrt{9x}$

Theorem 11 tells us that if we multiply the input to a function by  $b$ , the resulting graph is scaled horizontally by a factor of  $\frac{1}{b}$  since the  $x$ -values are divided by  $b$  to produce corresponding points on the graph of  $y = f(bx)$ . The next example explores how vertical and horizontal scalings sometimes interact with each other and with the other transformations introduced in this section.

### Example 38 Applying vertical and horizontal scalings

Let  $f(x) = \sqrt{x}$ . Use the graph of  $f$  from Example 36 (see Figure 2.53) to graph the following functions. Also, state their domains and ranges.

$$\begin{array}{lll} 1. \quad g(x) = 3\sqrt{x} & 2. \quad j(x) = \sqrt{9x} & 3. \quad m(x) = 1 - \sqrt{\frac{x+3}{2}} \end{array}$$

### SOLUTION

- First we note that the domain of  $g$  is  $[0, \infty)$  for the usual reason. Next, we have  $g(x) = 3f(x)$  so by Theorem 10, we obtain the graph of  $g$  by multiplying all of the  $y$ -coordinates of the points on the graph of  $f$  by 3. The result is a vertical scaling of the graph of  $f$  by a factor of 3. We find the range of  $g$  is also  $[0, \infty)$ . The graph of  $g$  is given in Figure 2.54.

<sup>4</sup>Also called ‘horizontal stretching’, ‘horizontal expansion’ or ‘horizontal dilation’ by a factor of 2.

2. To determine the domain of  $j$ , we solve  $9x \geq 0$  to find  $x \geq 0$ . Our domain is once again  $[0, \infty)$ . We recognize  $j(x) = f(9x)$  and by Theorem 11, we obtain the graph of  $j$  by dividing the  $x$ -coordinates of the points on the graph of  $f$  by 9. From the graph in Figure 2.55, we see the range of  $j$  is also  $[0, \infty)$ .

3. Solving  $\frac{x+3}{2} \geq 0$  gives  $x \geq -3$ , so the domain of  $m$  is  $[-3, \infty)$ . To take advantage of what we know of transformations, we rewrite  $m(x) = -\sqrt{\frac{1}{2}x + \frac{3}{2}} + 1$ , or  $m(x) = -f(\frac{1}{2}x + \frac{3}{2}) + 1$ . Focusing on the inputs first, we note that the input to  $f$  in the formula for  $m(x)$  is  $\frac{1}{2}x + \frac{3}{2}$ . Multiplying the  $x$  by  $\frac{1}{2}$  corresponds to a horizontal stretching by a factor of 2, and adding the  $\frac{3}{2}$  corresponds to a shift to the left by  $\frac{3}{2}$ . As before, we resolve which to perform first by thinking about how we would find the point on  $m$  corresponding to a point on  $f$ , in this case,  $(4, 2)$ . To use  $f(4) = 2$ , we solve  $\frac{1}{2}x + \frac{3}{2} = 4$ . Our first step is to subtract the  $\frac{3}{2}$  (the horizontal shift) to obtain  $\frac{1}{2}x = \frac{5}{2}$ . Next, we multiply by 2 (the horizontal stretching) and obtain  $x = 5$ . We define two intermediate functions to handle first the shift, then the stretching. In accordance with Theorem 8,  $m_1(x) = f(x + \frac{3}{2}) = \sqrt{x + \frac{3}{2}}$  will shift the graph of  $f$  to the left  $\frac{3}{2}$  units: see Figure 2.56

Next,  $m_2(x) = m_1(\frac{1}{2}x) = \sqrt{\frac{1}{2}x + \frac{3}{2}}$  will, according to Theorem 11, horizontally stretch the graph of  $m_1$  by a factor of 2: see Figure 2.57

We now examine what's happening to the outputs. From  $m(x) = -f(\frac{1}{2}x + \frac{3}{2}) + 1$ , we see that the output from  $f$  is being multiplied by  $-1$  (a reflection about the  $x$ -axis) and then a  $1$  is added (a vertical shift up  $1$ ). As before, we can determine the correct order by looking at how the point  $(4, 2)$  is moved. We already know that to make use of the equation  $f(4) = 2$ , we need to substitute  $x = 5$ . We get  $m(5) = -f(\frac{1}{2}(5) + \frac{3}{2}) + 1 = -f(4) + 1 = -2 + 1 = -1$ . We see that  $f(4)$  (the output from  $f$ ) is first multiplied by  $-1$  then the  $1$  is added meaning we first reflect the graph about the  $x$ -axis then shift up  $1$ . Theorem 9 tells us  $m_3(x) = -m_2(x)$  will handle the reflection.

Finally, to handle the vertical shift, Theorem 7 gives  $m(x) = m_3(x) + 1$ , and we see that the range of  $m$  is  $(-\infty, 1]$ . The graph of  $m$  is given in Figure 2.59.

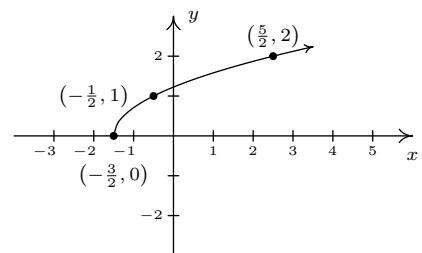


Figure 2.56: The graph  $y = m_1(x) = f(x + \frac{3}{2}) = \sqrt{x + \frac{3}{2}}$

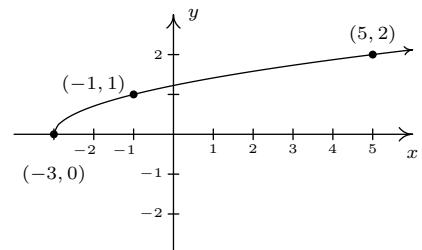


Figure 2.57: The graph  $y = m_2(x) = m_1(\frac{1}{2}x) = \sqrt{\frac{1}{2}x + \frac{3}{2}}$

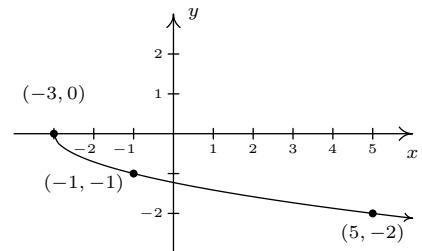


Figure 2.58: The graph  $y = m_3(x) = -m_2(x) = -\sqrt{\frac{1}{2}x + \frac{3}{2}}$

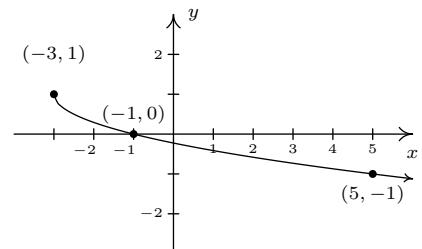


Figure 2.59: The graph  $y = m(x) = m_3(x) + 1 = -\sqrt{\frac{1}{2}x + \frac{3}{2}} + 1$

Some comments about Example 38 are in order. First, recalling the properties of radicals from Intermediate Algebra, we know that the functions  $g$  and  $j$  are the same, since  $j$  and  $g$  have the same domains and  $j(x) = \sqrt{9x} = \sqrt{9}\sqrt{x} = 3\sqrt{x} = g(x)$ . (We invite the reader to verify that all of the points we plotted on the graph of  $g$  lie on the graph of  $j$  and vice-versa.) Hence, for  $f(x) = \sqrt{x}$ , a vertical stretch by a factor of 3 and a horizontal shrinking by a factor of 9 result in the same transformation. While this kind of phenomenon is not universal, it happens commonly enough with some of the families of functions studied in College Algebra that it is worthy of note. Secondly, to graph the function  $m$ , we applied a series of four transformations. While it would have been easier on the authors to simply inform the reader of which steps to take, we have strived to explain why the order in which the transformations were applied made sense. We generalize the procedure in the theorem below.

**Theorem 12      Transformations**

Suppose  $f$  is a function. If  $A \neq 0$  and  $B \neq 0$ , then to graph

$$g(x) = Af(Bx + H) + K$$

1. Subtract  $H$  from each of the  $x$ -coordinates of the points on the graph of  $f$ . This results in a horizontal shift to the left if  $H > 0$  or right if  $H < 0$ .
2. Divide the  $x$ -coordinates of the points on the graph obtained in Step 1 by  $B$ . This results in a horizontal scaling, but may also include a reflection about the  $y$ -axis if  $B < 0$ .
3. Multiply the  $y$ -coordinates of the points on the graph obtained in Step 2 by  $A$ . This results in a vertical scaling, but may also include a reflection about the  $x$ -axis if  $A < 0$ .
4. Add  $K$  to each of the  $y$ -coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if  $K > 0$  or down if  $K < 0$ .

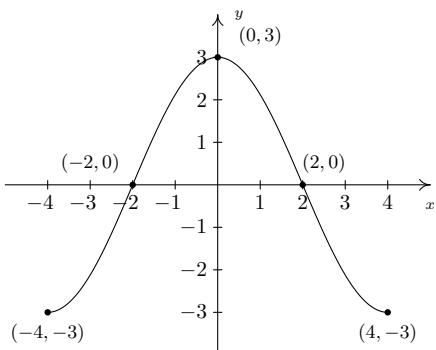


Figure 2.60: The graph  $y = f(x)$  for Example 39

Theorem 12 can be established by generalizing the techniques developed in this section. Suppose  $(a, b)$  is on the graph of  $f$ . Then  $f(a) = b$ , and to make good use of this fact, we set  $Bx + H = a$  and solve. We first subtract the  $H$  (causing the horizontal shift) and then divide by  $B$ . If  $B$  is a positive number, this induces only a horizontal scaling by a factor of  $\frac{1}{B}$ . If  $B < 0$ , then we have a factor of  $-1$  in play, and dividing by it induces a reflection about the  $y$ -axis. So we have  $x = \frac{a-H}{B}$  as the input to  $g$  which corresponds to the input  $x = a$  to  $f$ . We now evaluate  $g\left(\frac{a-H}{B}\right) = Af\left(B \cdot \frac{a-H}{B} + H\right) + K = Af(a) + K = Ab + K$ . We notice that the output from  $f$  is first multiplied by  $A$ . As with the constant  $B$ , if  $A > 0$ , this induces only a vertical scaling. If  $A < 0$ , then the  $-1$  induces a reflection across the  $x$ -axis. Finally, we add  $K$  to the result, which is our vertical shift. A less precise, but more intuitive way to paraphrase Theorem 12 is to think of the quantity  $Bx + H$  is the ‘inside’ of the function  $f$ . What’s happening inside  $f$  affects the inputs or  $x$ -coordinates of the points on the graph of  $f$ . To find the  $x$ -coordinates of the corresponding points on  $g$ , we undo what has been done to  $x$  in the same way we would solve an equation. What’s happening to the output can be thought of as things happening ‘outside’ the function,  $f$ . Things happening outside affect the outputs or  $y$ -coordinates of the points on the graph of  $f$ . Here, we follow the usual order of operations agreement: we first multiply by  $A$  then add  $K$  to find the corresponding  $y$ -coordinates on the graph of  $g$ .

**Example 39      Graphing a general transformation**

The complete graph of  $y = f(x)$  is shown in Figure 2.60. Use it to graph  $g(x) = \frac{4-3f(1-2x)}{2}$ .

**SOLUTION**      We use Theorem 12 to track the five ‘key points’  $(-4, -3)$ ,  $(-2, 0)$ ,  $(0, 3)$ ,  $(2, 0)$  and  $(4, -3)$  indicated on the graph of  $f$  to their new locations. We first rewrite  $g(x)$  in the form presented in Theorem 12,  $g(x) = -\frac{3}{2}f(-2x + 1) + 2$ . We set  $-2x + 1$  equal to the  $x$ -coordinates of the key points and solve. For example, solving  $-2x + 1 = -4$ , we first subtract 1 to get  $-2x = -5$  then divide by  $-2$  to get  $x = \frac{5}{2}$ . Subtracting the 1 is a horizontal shift to the left 1 unit. Dividing by  $-2$  can be thought of as a two step process:

dividing by 2 which compresses the graph horizontally by a factor of 2 followed by dividing (multiplying) by  $-1$  which causes a reflection across the  $y$ -axis. We summarize the results in the table in Figure 2.62

Next, we take each of the  $x$  values and substitute them into  $g(x) = -\frac{3}{2}f(-2x+1) + 2$  to get the corresponding  $y$ -values. Substituting  $x = \frac{5}{2}$ , and using the fact that  $f(-4) = -3$ , we get

$$g\left(\frac{5}{2}\right) = -\frac{3}{2}f\left(-2\left(\frac{5}{2}\right) + 1\right) + 2 = -\frac{3}{2}f(-4) + 2 = -\frac{3}{2}(-3) + 2 = \frac{9}{2} + 2 = \frac{13}{2}$$

We see that the output from  $f$  is first multiplied by  $-\frac{3}{2}$ . Thinking of this as a two step process, multiplying by  $\frac{3}{2}$  then by  $-1$ , we have a vertical stretching by a factor of  $\frac{3}{2}$  followed by a reflection across the  $x$ -axis. Adding 2 results in a vertical shift up 2 units. Continuing in this manner, we get the table in Figure 2.63.

To graph  $g$ , we plot each of the points in the table above and connect them in the same order and fashion as the points to which they correspond. Plotting  $f$  and  $g$  side-by-side gives

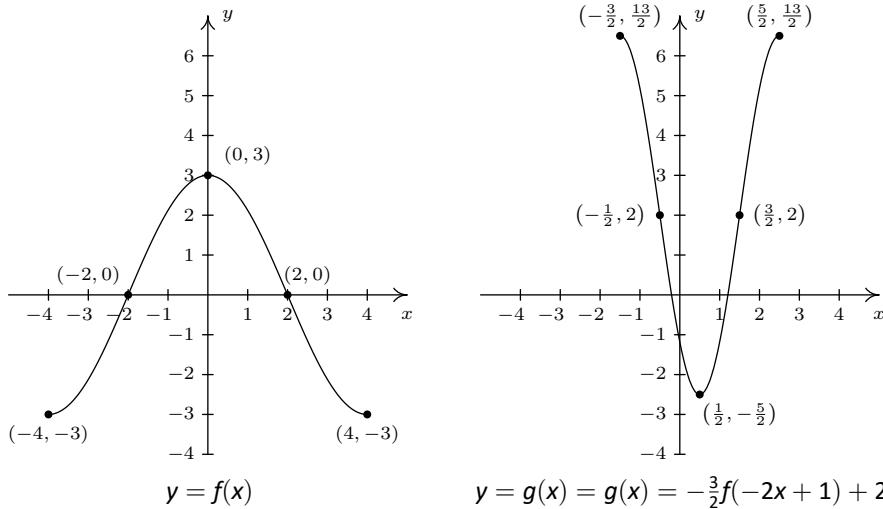


Figure 2.61: Determining the graph of  $g(x) = -\frac{3}{2}f(-2x+1)+2$

The reader is strongly encouraged to graph the series of functions which shows the gradual transformation of the graph of  $f$  into the graph of  $g$ . (You really should do this once in your life.) We have outlined the sequence of transformations in the above exposition; all that remains is to plot the five intermediate stages.

Our last example turns the tables and asks for the formula of a function given a desired sequence of transformations. If nothing else, it is a good review of function notation.

**Example 40 Determining the formula for a transformed function**  
Let  $f(x) = x^2$ . Find and simplify the formula of the function  $g(x)$  whose graph is the result of  $f$  undergoing the following sequence of transformations. Check your answer using a graphing calculator.

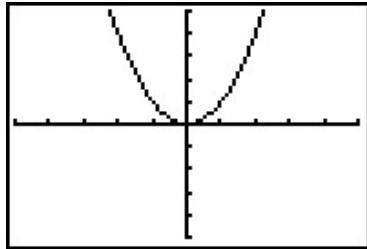
1. Vertical shift up 2 units
2. Reflection across the  $x$ -axis
3. Horizontal shift right 1 unit
4. Horizontal stretching by a factor of 2

$(a, f(a))$	$-2x + 1 = a$	$x$
$(-4, -3)$	$-2x + 1 = -4$	$x = \frac{5}{2}$
$(-2, 0)$	$-2x + 1 = -2$	$x = \frac{3}{2}$
$(0, 3)$	$-2x + 1 = 0$	$x = \frac{1}{2}$
$(2, 0)$	$-2x + 1 = 2$	$x = -\frac{1}{2}$
$(4, -3)$	$-2x + 1 = 4$	$x = -\frac{3}{2}$

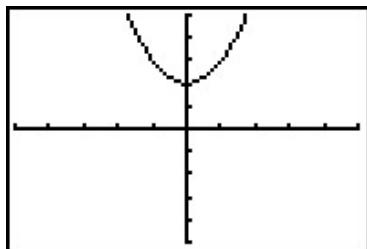
Figure 2.62: Tracking the  $x$  coordinates of transformed points

$x$	$g(x)$	$(x, g(x))$
$\frac{5}{2}$	$\frac{13}{2}$	$(\frac{5}{2}, \frac{13}{2})$
$\frac{3}{2}$	2	$(\frac{3}{2}, 2)$
$\frac{1}{2}$	$-\frac{5}{2}$	$(\frac{1}{2}, -\frac{5}{2})$
$-\frac{1}{2}$	2	$(-\frac{1}{2}, 2)$
$-\frac{3}{2}$	$\frac{13}{2}$	$(-\frac{3}{2}, \frac{13}{2})$

Figure 2.63: Getting the corresponding  $y$  coordinates

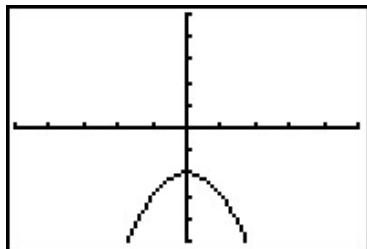


$$y = f(x) = x^2$$



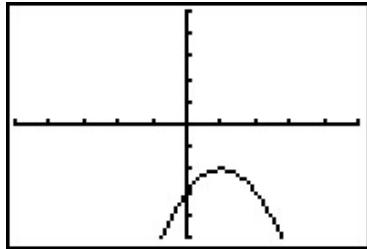
$$y = g_1(x) = f(x) + 2 = x^2 + 2$$

(Shift up by 2)



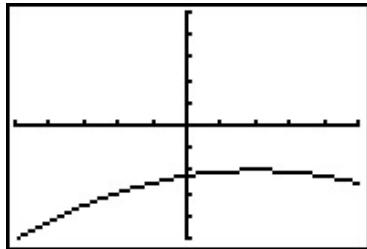
$$y = g_2(x) = -g_1(x) = -x^2 - 2$$

(Reflect across x-axis)



$$y = g_3(x) = g_2(x - 1) = -x^2 + 2x - 3$$

(Shift right one unit)

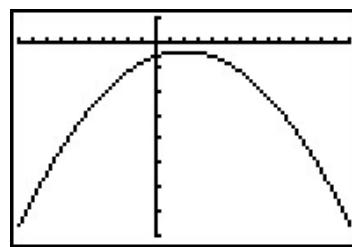


$$y = g(x) = g_3((\frac{1}{2}x)) = -\frac{1}{4}x^2 + x - 3$$

(Horizontal stretch by a factor of 2)

**SOLUTION** We build up to a formula for  $g(x)$  using intermediate functions as we've seen in previous examples. We let  $g_1$  take care of our first step. Theorem 7 tells us  $g_1(x) = f(x) + 2 = x^2 + 2$ . Next, we reflect the graph of  $g_1$  about the  $x$ -axis using Theorem 9:  $g_2(x) = -g_1(x) = -(x^2 + 2) = -x^2 - 2$ . We shift the graph to the right 1 unit, according to Theorem 8, by setting  $g_3(x) = g_2(x - 1) = -(x - 1)^2 - 2 = -x^2 + 2x - 3$ . Finally, we induce a horizontal stretch by a factor of 2 using Theorem 11 to get  $g(x) = g_3(\frac{1}{2}x) = -(\frac{1}{2}x)^2 + 2(\frac{1}{2}x) - 3$  which yields  $g(x) = -\frac{1}{4}x^2 + x - 3$ . We use the calculator to graph the stages below to confirm our result.

We have kept the viewing window the same in all of the graphs above. This had the undesirable consequence of making the last graph look 'incomplete' in that we cannot see the original shape of  $f(x) = x^2$ . Altering the viewing window results in a more complete graph of the transformed function shown below:



This example brings our first chapter to a close. In the chapters which lie ahead, be on the lookout for the concepts developed here to resurface as we study different families of functions.

Figure 2.64: The sequence of transformations in Example 40

## Exercises 2.6

### Problems

Suppose  $(2, -3)$  is on the graph of  $y = f(x)$ . In Exercises 1 – 20, use Theorem 12 to find a point on the graph of the given transformed function.

1.  $y = f(x) + 3$

11.  $y = 10 - f(x)$

2.  $y = f(x + 3)$

12.  $y = 3f(2x) - 1$

3.  $y = f(x) - 1$

13.  $y = \frac{1}{2}f(4 - x)$

4.  $y = f(x - 1)$

14.  $y = 5f(2x + 1) + 3$

5.  $y = 3f(x)$

15.  $y = 2f(1 - x) - 1$

6.  $y = f(3x)$

16.  $y = f\left(\frac{7 - 2x}{4}\right)$

8.  $y = f(-x)$

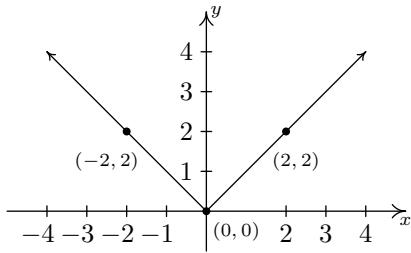
17.  $y = \frac{f(3x) - 1}{2}$

9.  $y = f(x - 3) + 1$

10.  $y = 2f(x + 1)$

18.  $y = \frac{4 - f(3x - 1)}{7}$

The complete graph of  $y = f(x)$  is given below. In Exercises 19 – 27, use it and Theorem 12 to graph the given transformed function.



19.  $y = f(x) + 1$

20.  $y = f(x) - 2$

21.  $y = f(x + 1)$

22.  $y = f(x - 2)$

23.  $y = 2f(x)$

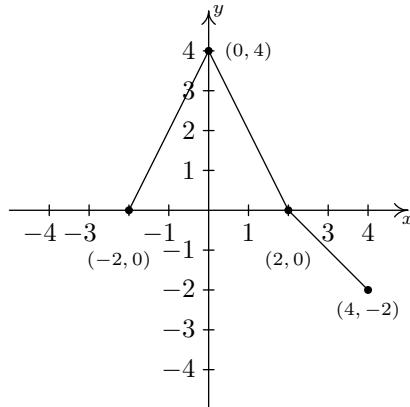
24.  $y = f(2x)$

25.  $y = 2 - f(x)$

26.  $y = f(2 - x)$

27.  $y = 2 - f(2 - x)$

The complete graph of  $y = f(x)$  is given below. In Exercises 28 – 36, use it and Theorem 12 to graph the given transformed function.



28.  $y = f(x) - 1$

29.  $y = f(x + 1)$

30.  $y = \frac{1}{2}f(x)$

31.  $y = f(2x)$

32.  $y = -f(x)$

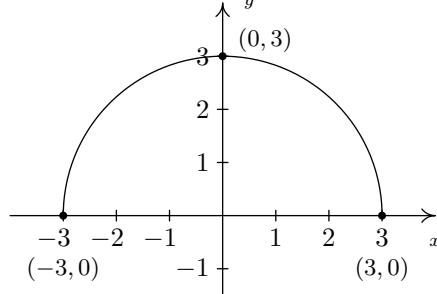
33.  $y = f(-x)$

34.  $y = f(x + 1) - 1$

35.  $y = 1 - f(x)$

36.  $y = \frac{1}{2}f(x + 1) - 1$

The complete graph of  $y = f(x)$  is given below. In Exercises 37 – 48, use it and Theorem 12 to graph the given transformed function.



37.  $g(x) = f(x) + 3$

38.  $h(x) = f(x) - \frac{1}{2}$

39.  $j(x) = f\left(x - \frac{2}{3}\right)$

40.  $a(x) = f(x + 4)$

41.  $b(x) = f(x + 1) - 1$

42.  $c(x) = \frac{3}{5}f(x)$

43.  $d(x) = -2f(x)$

44.  $k(x) = f\left(\frac{2}{3}x\right)$

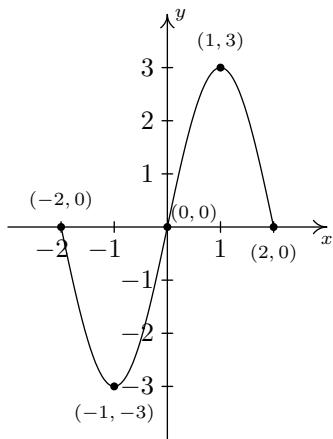
45.  $m(x) = -\frac{1}{4}f(3x)$

46.  $n(x) = 4f(x - 3) - 6$

47.  $p(x) = 4 + f(1 - 2x)$

48.  $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3$

**The complete graph of  $y = S(x)$  is given below. The purpose of Exercises 49–52 is to graph  $y = \frac{1}{2}S(-x+1)+1$  by graphing each transformation, one step at a time.**



49.  $y = S_1(x) = S(x + 1)$

50.  $y = S_2(x) = S_1(-x) = S(-x + 1)$

51.  $y = S_3(x) = \frac{1}{2}S_2(x) = \frac{1}{2}S(-x + 1)$

52.  $y = S_4(x) = S_3(x) + 1 = \frac{1}{2}S(-x + 1) + 1$

Let  $f(x) = \sqrt{x}$ . In Exercises 53–62, find a formula for a function  $g$  whose graph is obtained from  $f$  from the given sequence of transformations.

53. (1) shift right 2 units; (2) shift down 3 units

54. (1) shift down 3 units; (2) shift right 2 units

55. (1) reflect across the  $x$ -axis; (2) shift up 1 unit

56. (1) shift up 1 unit; (2) reflect across the  $x$ -axis

57. (1) shift left 1 unit; (2) reflect across the  $y$ -axis; (3) shift up 2 units

58. (1) reflect across the  $y$ -axis; (2) shift left 1 unit; (3) shift up 2 units

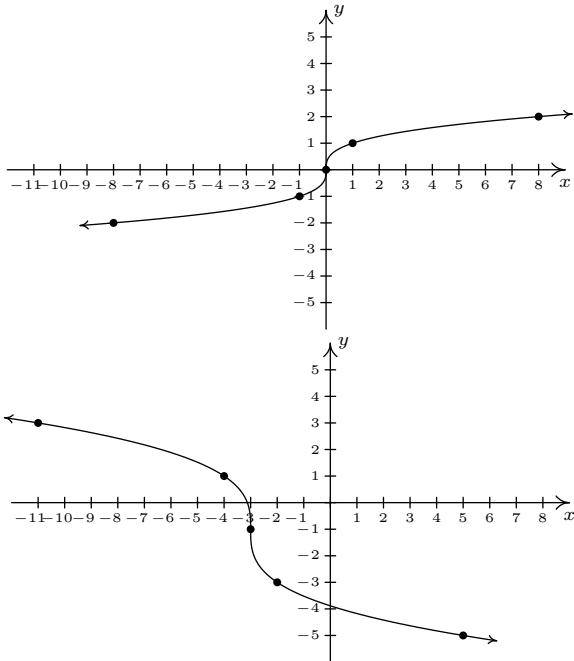
59. (1) shift left 3 units; (2) vertical stretch by a factor of 2; (3) shift down 4 units

60. (1) shift left 3 units; (2) shift down 4 units; (3) vertical stretch by a factor of 2

61. (1) shift right 3 units; (2) horizontal shrink by a factor of 2; (3) shift up 1 unit

62. (1) horizontal shrink by a factor of 2; (2) shift right 3 units; (3) shift up 1 unit

63. The graph of  $y = f(x) = \sqrt[3]{x}$  is given immediately below, and the graph of  $y = g(x)$  is given below that of  $y = f(x)$ . Find a formula for  $g$  based on transformations of the graph of  $f$ . Check your answer by confirming that the points shown on the graph of  $g$  satisfy the equation  $y = g(x)$ .



64. For many common functions, the properties of Algebra make a horizontal scaling the same as a vertical scaling by (possibly) a different factor. For example, we stated earlier that  $\sqrt{9x} = 3\sqrt{x}$ . With the help of your classmates, find the equivalent vertical scaling produced by the horizontal scalings  $y = (2x)^3$ ,  $y = |5x|$ ,  $y = \sqrt[3]{27x}$  and  $y = \left(\frac{1}{2}x\right)^2$ . What about  $y = (-2x)^3$ ,  $y = |-5x|$ ,  $y = \sqrt[3]{-27x}$  and  $y = \left(-\frac{1}{2}x\right)^2$ ?

65. We mentioned earlier in the section that, in general, the order in which transformations are applied matters, yet in our first example with two transformations the order did not matter. (You could perform the shift to the left followed by the shift down or you could shift down and then left to achieve the same result.) With the help of your classmates,

- determine the situations in which order does matter and those in which it does not.
66. What happens if you reflect an even function across the  $y$ -axis?
  67. What happens if you reflect an odd function across the  $y$ -axis?
  68. What happens if you reflect an even function across the  $x$ -axis?
  69. What happens if you reflect an odd function across the  $x$ -axis?
  70. How would you describe symmetry about the origin in terms of reflections?
  71. As we saw in Example 40, the viewing window on the graphing calculator affects how we see the transformations done to a graph. Using two different calculators, find viewing windows so that  $f(x) = x^2$  on the one calculator looks like  $g(x) = 3x^2$  on the other.



# 3: LINEAR AND QUADRATIC FUNCTIONS

## 3.1 Linear Functions

We now begin the study of families of functions. Our first family, linear functions, are old friends as we shall soon see. Recall from Geometry that two distinct points in the plane determine a unique line containing those points, as indicated in Figure 3.1.

To give a sense of the ‘steepness’ of the line, we recall that we can compute the **slope** of the line using the formula below.

### Definition 29 Slope

The **slope**  $m$  of the line containing the points  $P(x_0, y_0)$  and  $Q(x_1, y_1)$  is:

$$m = \frac{y_1 - y_0}{x_1 - x_0},$$

provided  $x_1 \neq x_0$ .

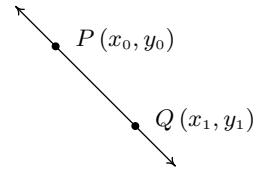


Figure 3.1: The line between two points  $P$  and  $Q$

A couple of notes about Definition 29 are in order. First, don’t ask why we use the letter ‘ $m$ ’ to represent slope. There are many explanations out there, but apparently no one really knows for sure. Secondly, the stipulation  $x_1 \neq x_0$  ensures that we aren’t trying to divide by zero. The reader is invited to pause to think about what is happening geometrically; the anxious reader can skip along to the next example.

### Example 41 Finding the slope of a line

Find the slope of the line containing the following pairs of points, if it exists. Plot each pair of points and the line containing them.

See [www.mathforum.org](http://www.mathforum.org) or [www.mathworld.wolfram.com](http://www.mathworld.wolfram.com) for discussions on the use of the letter  $m$  to indicate slope.

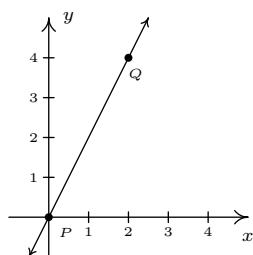
1.  $P(0, 0), Q(2, 4)$       2.  $P(-1, 2), Q(3, 4)$

3.  $P(-2, 3), Q(2, -3)$       4.  $P(-3, 2), Q(4, 2)$

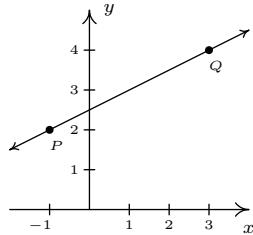
5.  $P(2, 3), Q(2, -1)$       6.  $P(2, 3), Q(2.1, -1)$

**SOLUTION** In each of these examples, we apply the slope formula, from Definition 29.

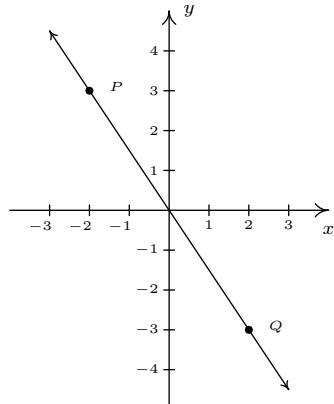
1.  $m = \frac{4 - 0}{2 - 0} = \frac{4}{2} = 2$



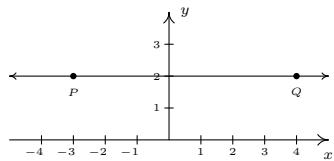
2.  $m = \frac{4 - 2}{3 - (-1)} = \frac{2}{4} = \frac{1}{2}$



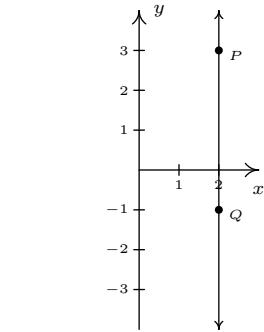
3.  $m = \frac{-3 - 3}{2 - (-2)} = \frac{-6}{4} = -\frac{3}{2}$



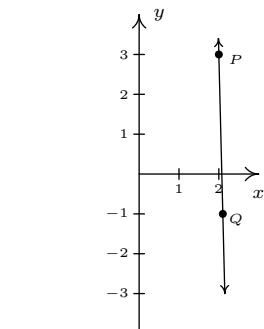
4.  $m = \frac{2 - 2}{4 - (-3)} = \frac{0}{7} = 0$



5.  $m = \frac{-1 - 3}{2 - 2} = \frac{-4}{0}$ , which is undefined



6.  $m = \frac{-1 - 3}{2.1 - 2} = \frac{-4}{0.1} = -40$



A few comments about Example 41 are in order. First, for reasons which will be made clear soon, if the slope is positive then the resulting line is said to be increasing. If it is negative, we say the line is decreasing. A slope of 0 results in a horizontal line which we say is constant, and an undefined slope results in a

vertical line. Second, the larger the slope is in absolute value, the steeper the line. You may recall from high school that slope can be described as the ratio  $\frac{\text{rise}}{\text{run}}$ . For example, in the second part of Example 41, we found the slope to be  $\frac{1}{2}$ . We can interpret this as a rise of 1 unit upward for every 2 units to the right we travel along the line, as shown in Figure 3.2.

Using more formal notation, given points  $(x_0, y_0)$  and  $(x_1, y_1)$ , we use the Greek letter delta ' $\Delta$ ' to write  $\Delta y = y_1 - y_0$  and  $\Delta x = x_1 - x_0$ . In most scientific circles, the symbol  $\Delta$  means 'change in'.

Hence, we may write

$$m = \frac{\Delta y}{\Delta x},$$

which describes the slope as the **rate of change** of  $y$  with respect to  $x$ . Rates of change abound in the 'real world', as the next example illustrates.

#### Example 42 Temperature rate of change

Suppose that two separate temperature readings were taken at the ranger station on the top of Mt. Sasquatch: at 6 AM the temperature was  $2^\circ\text{C}$  and at 10 AM it was  $8^\circ\text{C}$ .

1. Find the slope of the line containing the points  $(6, 2)$  and  $(10, 8)$ .
2. Interpret your answer to the first part in terms of temperature and time.
3. Predict the temperature at noon.

#### SOLUTION

1. For the slope, we have  $m = \frac{8-2}{10-6} = \frac{6}{4} = \frac{3}{2}$ .
2. Since the values in the numerator correspond to the temperatures in  $^\circ\text{C}$ , and the values in the denominator correspond to time in hours, we can interpret the slope as  $\frac{3}{2} = \frac{3^\circ\text{C}}{2 \text{ hour}}$ , or  $1.5^\circ\text{C}$  per hour. Since the slope is positive, we know this corresponds to an increasing line. Hence, the temperature is increasing at a rate of  $1.5^\circ\text{C}$  per hour.
3. Noon is two hours after 10 AM. Assuming a temperature increase of  $1.5^\circ\text{C}$  per hour, in two hours the temperature should rise  $3^\circ\text{C}$ . Since the temperature at 10 AM is  $82^\circ\text{C}$ , we would expect the temperature at noon to be  $8 + 3 = 11^\circ\text{C}$ .

Now it may well happen that in the previous scenario, at noon the temperature is only  $10^\circ\text{C}$ . This doesn't mean our calculations are incorrect, rather, it means that the temperature change throughout the day isn't a constant  $1.5^\circ\text{C}$  per hour. As discussed in Section 2.3.1, mathematical models are just that: models. The predictions we get out of the models may be mathematically accurate, but may not resemble what happens in the real world.

In Section 2.1, we discussed the equations of vertical and horizontal lines. Using the concept of slope, we can develop equations for the other varieties of lines. Suppose a line has a slope of  $m$  and contains the point  $(x_0, y_0)$ . Suppose  $(x, y)$  is another point on the line, as indicated in Figure 3.3.

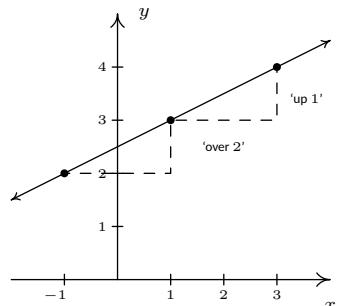


Figure 3.2: Slope as "rise over run"

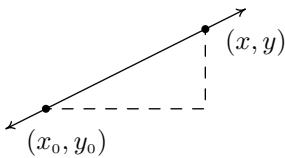


Figure 3.3: Deriving the point-slope formula

Definition 29 yields

$$\begin{aligned} m &= \frac{y - y_0}{x - x_0} \\ m(x - x_0) &= y - y_0 \\ y - y_0 &= m(x - x_0) \end{aligned}$$

We have just derived the **point-slope form** of a line.

#### Key Idea 15 The point-slope form of a line

The **point-slope form** of the equation of a line with slope  $m$  containing the point  $(x_0, y_0)$  is the equation  $y - y_0 = m(x - x_0)$ .

#### Example 43 Using the point-slope form

Write the equation of the line containing the points  $(-1, 3)$  and  $(2, 1)$ .

**SOLUTION** In order to use Key Idea 15 we need to find the slope of the line in question so we use Definition 29 to get  $m = \frac{\Delta y}{\Delta x} = \frac{1-3}{2-(-1)} = -\frac{2}{3}$ . We are spoiled for choice for a point  $(x_0, y_0)$ . We'll use  $(-1, 3)$  and leave it to the reader to check that using  $(2, 1)$  results in the same equation. Substituting into the point-slope form of the line, we get

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 3 &= -\frac{2}{3}(x - (-1)) \\ y - 3 &= -\frac{2}{3}(x + 1) \\ y - 3 &= -\frac{2}{3}x - \frac{2}{3} \\ y &= -\frac{2}{3}x + \frac{7}{3}. \end{aligned}$$

We can check our answer by showing that both  $(-1, 3)$  and  $(2, 1)$  are on the graph of  $y = -\frac{2}{3}x + \frac{7}{3}$  algebraically, as we did in Section 2.1.1.

In simplifying the equation of the line in the previous example, we produced another form of a line, the **slope-intercept form**. This is the familiar  $y = mx + b$  form you have probably seen in high school. The ‘intercept’ in ‘slope-intercept’ comes from the fact that if we set  $x = 0$ , we get  $y = b$ . In other words, the  $y$ -intercept of the line  $y = mx + b$  is  $(0, b)$ .

#### Key Idea 16 Slope intercept form of a line

The **slope-intercept form** of the line with slope  $m$  and  $y$ -intercept  $(0, b)$  is the equation  $y = mx + b$ .

Note that if we have slope  $m = 0$ , we get the equation  $y = b$  which matches our formula for a horizontal line given in Section 2.1. The formula given in Key Idea 16 can be used to describe all lines except vertical lines. All lines except vertical lines are functions (Why is this?) so we have finally reached a good point to introduce **linear functions**.

**Definition 30 Linear function**

A **linear function** is a function of the form

$$f(x) = mx + b,$$

where  $m$  and  $b$  are real numbers with  $m \neq 0$ . The domain of a linear function is  $(-\infty, \infty)$ .

For the case  $m = 0$ , we get  $f(x) = b$ . These are given their own classification.

**Definition 31 Constant function**

A **constant function** is a function of the form

$$f(x) = b,$$

where  $b$  is real number. The domain of a constant function is  $(-\infty, \infty)$ .

Recall that to graph a function,  $f$ , we graph the equation  $y = f(x)$ . Hence, the graph of a linear function is a line with slope  $m$  and  $y$ -intercept  $(0, b)$ ; the graph of a constant function is a horizontal line (a line with slope  $m = 0$ ) and a  $y$ -intercept of  $(0, b)$ . Now think back to Section ??, specifically Definition 27 concerning increasing, decreasing and constant functions. A line with positive slope was called an increasing line because a linear function with  $m > 0$  is an increasing function. Similarly, a line with a negative slope was called a decreasing line because a linear function with  $m < 0$  is a decreasing function. And horizontal lines were called constant because, well, we hope you've already made the connection.

**Example 44 Graphing linear functions**

Graph the following functions. Identify the slope and  $y$ -intercept.

1.  $f(x) = 3$

3.  $f(x) = \frac{3-2x}{4}$

2.  $f(x) = 3x - 1$

4.  $f(x) = \frac{x^2 - 4}{x - 2}$

**SOLUTION**

1. To graph  $f(x) = 3$ , we graph  $y = 3$ . This is a horizontal line ( $m = 0$ ) through  $(0, 3)$ : see Figure 3.4.

2. The graph of  $f(x) = 3x - 1$  is the graph of the line  $y = 3x - 1$ . Comparison of this equation with Equation 16 yields  $m = 3$  and  $b = -1$ . Hence, our slope is 3 and our  $y$ -intercept is  $(0, -1)$ . To get another point on the line, we can plot  $(1, f(1)) = (1, 2)$ . Constructing the line through these points gives us Figure 3.5.

3. At first glance, the function  $f(x) = \frac{3-2x}{4}$  does not fit the form in Definition 30 but after some rearranging we get  $f(x) = \frac{3-2x}{4} = \frac{3}{4} - \frac{2x}{4} =$

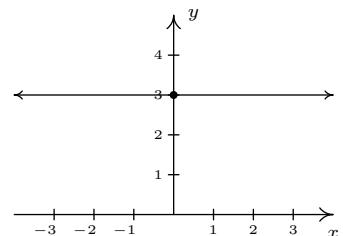


Figure 3.4: The graph of  $f(x) = 3$

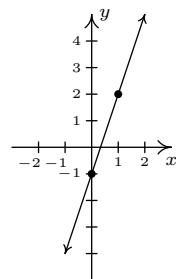
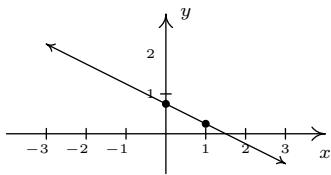


Figure 3.5: The graph of  $f(x) = 3x - 1$

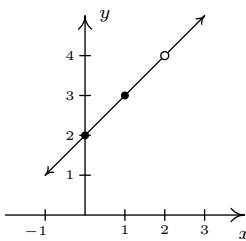
Figure 3.6: The graph of  $f(x) = \frac{3 - 2x}{4}$ 

$-\frac{1}{2}x + \frac{3}{4}$ . We identify  $m = -\frac{1}{2}$  and  $b = \frac{3}{4}$ . Hence, our graph is a line with a slope of  $-\frac{1}{2}$  and a  $y$ -intercept of  $(0, \frac{3}{4})$ . Plotting an additional point, we can choose  $(1, f(1))$  to get  $(1, \frac{1}{4})$ : see Figure 3.6.

- If we simplify the expression for  $f$ , we get

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{(x - 2)} = x + 2.$$

If we were to state  $f(x) = x + 2$ , we would be committing a sin of omission. Remember, to find the domain of a function, we do so **before** we simplify! In this case,  $f$  has big problems when  $x = 2$ , and as such, the domain of  $f$  is  $(-\infty, 2) \cup (2, \infty)$ . To indicate this, we write  $f(x) = x + 2, x \neq 2$ . So, except at  $x = 2$ , we graph the line  $y = x + 2$ . The slope  $m = 1$  and the  $y$ -intercept is  $(0, 2)$ . A second point on the graph is  $(1, f(1)) = (1, 3)$ . Since our function  $f$  is not defined at  $x = 2$ , we put an open circle at the point that would be on the line  $y = x + 2$  when  $x = 2$ , namely  $(2, 4)$ , as shown in Figure 3.7.

Figure 3.7: The graph of  $f(x) = \frac{x^2 - 4}{x - 2}$ 

The last two functions in the previous example showcase some of the difficulty in defining a linear function using the phrase ‘of the form’ as in Definition 30, since some algebraic manipulations may be needed to rewrite a given function to match ‘the form’. Keep in mind that the domains of linear and constant functions are all real numbers  $(-\infty, \infty)$ , so while  $f(x) = \frac{x^2 - 4}{x - 2}$  simplified to a formula  $f(x) = x + 2$ ,  $f$  is not considered a linear function since its domain excludes  $x = 2$ . However, we would consider

$$f(x) = \frac{2x^2 + 2}{x^2 + 1}$$

to be a constant function since its domain is all real numbers (Can you tell us why?) and

$$f(x) = \frac{2x^2 + 2}{x^2 + 1} = \frac{2(x^2 + 1)}{(x^2 + 1)} = 2$$

The following example uses linear functions to model some basic economic relationships.

#### Example 45 Pricing for a game system

The cost  $C$ , in dollars, to produce  $x$  PortaBoy game systems for a local retailer is given by  $C(x) = 80x + 150$  for  $x \geq 0$ .

- Find and interpret  $C(10)$ .
- How many PortaBoys can be produced for \$15,000?
- Explain the significance of the restriction on the domain,  $x \geq 0$ .
- Find and interpret  $C(0)$ .
- Find and interpret the slope of the graph of  $y = C(x)$ .

#### SOLUTION

- To find  $C(10)$ , we replace every occurrence of  $x$  with 10 in the formula for  $C(x)$  to get  $C(10) = 80(10) + 150 = 950$ . Since  $x$  represents the number of PortaBoys produced, and  $C(x)$  represents the cost, in dollars,  $C(10) = 950$  means it costs \$950 to produce 10 PortaBoys for the local retailer.
- To find how many PortaBoys can be produced for \$15,000, we solve  $C(x) = 15000$ , or  $80x + 150 = 15000$ . Solving, we get  $x = \frac{14850}{80} = 185.625$ . Since we can only produce a whole number amount of PortaBoys, we can produce 185 PortaBoys for \$15,000.
- The restriction  $x \geq 0$  is the applied domain, as discussed in Section 2.3.1. In this context,  $x$  represents the number of PortaBoys produced. It makes no sense to produce a negative quantity of game systems.
- We find  $C(0) = 80(0) + 150 = 150$ . This means it costs \$150 to produce 0 PortaBoys. As mentioned on page 73, this is the fixed, or start-up cost of this venture.
- If we were to graph  $y = C(x)$ , we would be graphing the portion of the line  $y = 80x + 150$  for  $x \geq 0$ . We recognize the slope,  $m = 80$ . Like any slope, we can interpret this as a rate of change. Here,  $C(x)$  is the cost in dollars, while  $x$  measures the number of PortaBoys so

$$m = \frac{\Delta y}{\Delta x} = \frac{\Delta C}{\Delta x} = 80 = \frac{80}{1} = \frac{\$80}{1 \text{ PortaBoy}}.$$

In other words, the cost is increasing at a rate of \$80 per PortaBoy produced. This is often called the **variable cost** for this venture.

Actually, it makes no sense to produce a fractional part of a game system, either, as we saw in the previous part of this example. This absurdity, however, seems quite forgiveable in some textbooks but not to us.

The next example asks us to find a linear function to model a related economic problem.

#### Example 46 Modelling demand

The local retailer in Example 45 has determined that the number  $x$  of PortaBoy game systems sold in a week is related to the price  $p$  in dollars of each system. When the price was \$220, 20 game systems were sold in a week. When the systems went on sale the following week, 40 systems were sold at \$190 a piece.

- Find a linear function which fits this data. Use the weekly sales  $x$  as the independent variable and the price  $p$  as the dependent variable.
- Find a suitable applied domain.
- Interpret the slope.
- If the retailer wants to sell 150 PortaBoys next week, what should the price be?
- What would the weekly sales be if the price were set at \$150 per system?

#### SOLUTION

- We recall from Section 2.3 the meaning of ‘independent’ and ‘dependent’ variable. Since  $x$  is to be the independent variable, and  $p$  the dependent variable, we treat  $x$  as the input variable and  $p$  as the output variable. Hence, we are looking for a function of the form  $p(x) = mx + b$ . To

determine  $m$  and  $b$ , we use the fact that 20 PortaBoys were sold during the week when the price was 220 dollars and 40 units were sold when the price was 190 dollars. Using function notation, these two facts can be translated as  $p(20) = 220$  and  $p(40) = 190$ . Since  $m$  represents the rate of change of  $p$  with respect to  $x$ , we have

$$m = \frac{\Delta p}{\Delta x} = \frac{190 - 220}{40 - 20} = \frac{-30}{20} = -1.5.$$

We now have determined  $p(x) = -1.5x + b$ . To determine  $b$ , we can use our given data again. Using  $p(20) = 220$ , we substitute  $x = 20$  into  $p(x) = 1.5x + b$  and set the result equal to 220:  $-1.5(20) + b = 220$ . Solving, we get  $b = 250$ . Hence, we get  $p(x) = -1.5x + 250$ . We can check our formula by computing  $p(20)$  and  $p(40)$  to see if we get 220 and 190, respectively. You may recall from page 73 that the function  $p(x)$  is called the price-demand (or simply demand) function for this venture.

2. To determine the applied domain, we look at the physical constraints of the problem. Certainly, we can't sell a negative number of PortaBoys, so  $x \geq 0$ . However, we also note that the slope of this linear function is negative, and as such, the price is decreasing as more units are sold. Thus another constraint on the price is  $p(x) \geq 0$ . Solving  $-1.5x + 250 \geq 0$  results in  $-1.5x \geq -250$  or  $x \leq \frac{500}{3} = 166.\bar{6}$ . Since  $x$  represents the number of PortaBoys sold in a week, we round down to 166. As a result, a reasonable applied domain for  $p$  is  $[0, 166]$ .
3. The slope  $m = -1.5$ , once again, represents the rate of change of the price of a system with respect to weekly sales of PortaBoys. Since the slope is negative, we have that the price is decreasing at a rate of \$1.50 per PortaBoy sold. (Said differently, you can sell one more PortaBoy for every \$1.50 drop in price.)
4. To determine the price which will move 150 PortaBoys, we find  $p(150) = -1.5(150) + 250 = 25$ . That is, the price would have to be \$25.
5. If the price of a PortaBoy were set at \$150, we have  $p(x) = 150$ , or  $-1.5x + 250 = 150$ . Solving, we get  $-1.5x = -100$  or  $x = 66.\bar{6}$ . This means you would be able to sell 66 PortaBoys a week if the price were \$150 per system.

Not all real-world phenomena can be modelled using linear functions. Nevertheless, it is possible to use the concept of slope to help analyze non-linear functions using the following.

#### Definition 32 Average rate of change

Let  $f$  be a function defined on the interval  $[a, b]$ . The **average rate of change** of  $f$  over  $[a, b]$  is defined as:

$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Geometrically, if we have the graph of  $y = f(x)$ , the average rate of change over  $[a, b]$  is the slope of the line which connects  $(a, f(a))$  and  $(b, f(b))$ . This is

called the **secant line** through these points. For that reason, some textbooks use the notation  $m_{\text{sec}}$  for the average rate of change of a function. Note that for a linear function  $m = m_{\text{sec}}$ , or in other words, its rate of change over an interval is the same as its average rate of change.

The interested reader may question the adjective ‘average’ in the phrase ‘average rate of change’. In the figure above, we can see that the function changes wildly on  $[a, b]$ , yet the slope of the secant line only captures a snapshot of the action at  $a$  and  $b$ . This situation is entirely analogous to the average speed on a trip. Suppose it takes you 2 hours to travel 100 kilometres. Your average speed is  $\frac{100 \text{ km}}{2 \text{ h}} = 50 \text{ km/h}$ . However, it is entirely possible that at the start of your journey, you travelled 25 kilometres per hour, then sped up to 65 kilometres per hour, and so forth. The average rate of change is akin to your average speed on the trip. Your speedometer measures your speed at any one instant along the trip, your **instantaneous rate of change**, and this is one of the central themes of Calculus.

When interpreting rates of change, we interpret them the same way we did slopes. In the context of functions, it may be helpful to think of the average rate of change as:

$$\frac{\text{change in outputs}}{\text{change in inputs}}$$

#### Example 47 A non-linear revenue model

Recall from page 73, the revenue from selling  $x$  units at a price  $p$  per unit is given by the formula  $R = xp$ . Suppose we are in the scenario of Examples 45 and 46.

1. Find and simplify an expression for the weekly revenue  $R(x)$  as a function of weekly sales  $x$ .
2. Find and interpret the average rate of change of  $R(x)$  over the interval  $[0, 50]$ .
3. Find and interpret the average rate of change of  $R(x)$  as  $x$  changes from 50 to 100 and compare that to your result in part 2.
4. Find and interpret the average rate of change of weekly revenue as weekly sales increase from 100 PortaBoys to 150 PortaBoys.

#### SOLUTION

1. Since  $R = xp$ , we substitute  $p(x) = -1.5x + 250$  from Example 46 to get  $R(x) = x(-1.5x + 250) = -1.5x^2 + 250x$ . Since we determined the price-demand function  $p(x)$  is restricted to  $0 \leq x \leq 166$ ,  $R(x)$  is restricted to these values of  $x$  as well.
2. Using Definition 32, we get that the average rate of change is

$$\frac{\Delta R}{\Delta x} = \frac{R(50) - R(0)}{50 - 0} = \frac{8750 - 0}{50 - 0} = 175.$$

Interpreting this slope as we have in similar situations, we conclude that for every additional PortaBoy sold during a given week, the weekly revenue increases \$175.

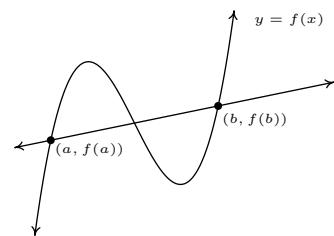


Figure 3.8: The graph of  $y = f(x)$  and its secant line through  $(a, f(a))$  and  $(b, f(b))$

3. The wording of this part is slightly different than that in Definition 32, but its meaning is to find the average rate of change of  $R$  over the interval  $[50, 100]$ . To find this rate of change, we compute

$$\frac{\Delta R}{\Delta x} = \frac{R(100) - R(50)}{100 - 50} = \frac{10000 - 8750}{50} = 25.$$

In other words, for each additional PortaBoy sold, the revenue increases by \$25. Note that while the revenue is still increasing by selling more game systems, we aren't getting as much of an increase as we did in part 2 of this example. (Can you think of why this would happen?)

4. Translating the English to the mathematics, we are being asked to find the average rate of change of  $R$  over the interval  $[100, 150]$ . We find

$$\frac{\Delta R}{\Delta x} = \frac{R(150) - R(100)}{150 - 100} = \frac{3750 - 10000}{50} = -125.$$

This means that we are losing \$125 dollars of weekly revenue for each additional PortaBoy sold. (Can you think why this is possible?)

We close this section with a new look at difference quotients which were first introduced in Section 2.3. If we wish to compute the average rate of change of a function  $f$  over the interval  $[x, x + h]$ , then we would have

$$\frac{\Delta f}{\Delta x} = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}$$

As we have indicated, the rate of change of a function (average or otherwise) is of great importance in Calculus. (So we are not torturing you with these for nothing.) Also, we have the geometric interpretation of difference quotients which was promised to you back on page 72 – a difference quotient yields the slope of a secant line.

# Exercises 3.1

## Problems

In Exercises 1 – 11, find both the point-slope form and the slope-intercept form of the line with the given slope which passes through the given point.

1.  $m = 3$ ,  $P(3, -1)$

2.  $m = -2$ ,  $P(-5, 8)$

3.  $m = -1$ ,  $P(-7, -1)$

4.  $m = \frac{2}{3}$ ,  $P(-2, 1)$

5.  $m = \frac{2}{3}$ ,  $P(-2, 1)$

6.  $m = \frac{1}{7}$ ,  $P(-1, 4)$

7.  $m = 0$ ,  $P(3, 117)$

8.  $m = -\sqrt{2}$ ,  $P(0, -3)$

9.  $m = -5$ ,  $P(\sqrt{3}, 2\sqrt{3})$

10.  $m = 678$ ,  $P(-1, -12)$

In Exercises 11 – 21, find the slope-intercept form of the line which passes through the given points.

11.  $P(0, 0)$ ,  $Q(-3, 5)$

12.  $P(-1, -2)$ ,  $Q(3, -2)$

13.  $P(5, 0)$ ,  $Q(0, -8)$

14.  $P(3, -5)$ ,  $Q(7, 4)$

15.  $P(-1, 5)$ ,  $Q(7, 5)$

16.  $P(4, -8)$ ,  $Q(5, -8)$

17.  $P\left(\frac{1}{2}, \frac{3}{4}\right)$ ,  $Q\left(\frac{5}{2}, -\frac{7}{4}\right)$

18.  $P\left(\frac{2}{3}, \frac{7}{2}\right)$ ,  $Q\left(-\frac{1}{3}, \frac{3}{2}\right)$

19.  $P(\sqrt{2}, -\sqrt{2})$ ,  $Q(-\sqrt{2}, \sqrt{2})$

20.  $P(-\sqrt{3}, -1)$ ,  $Q(\sqrt{3}, 1)$

In Exercises 21 – 27, graph the function. Find the slope,  $y$ -intercept and  $x$ -intercept, if any exist.

21.  $f(x) = 2x - 1$

22.  $f(x) = 3 - x$

23.  $f(x) = 3$

24.  $f(x) = 0$

25.  $f(x) = \frac{2}{3}x + \frac{1}{3}$

26.  $f(x) = \frac{1-x}{2}$

27. Find all of the points on the line  $y = 2x + 1$  which are 4 units from the point  $(-1, 3)$ .

28. Jeff can walk comfortably at 3 miles per hour. Find a linear function  $d$  that represents the total distance Jeff can walk in  $t$  hours, assuming he doesn't take any breaks.

29. Carl can stuff 6 envelopes per minute. Find a linear function  $E$  that represents the total number of envelopes Carl can stuff after  $t$  hours, assuming he doesn't take any breaks.

30. A landscaping company charges \$45 per cubic yard of mulch plus a delivery charge of \$20. Find a linear function which computes the total cost  $C$  (in dollars) to deliver  $x$  cubic yards of mulch.

31. A plumber charges \$50 for a service call plus \$80 per hour. If she spends no longer than 8 hours a day at any one site, find a linear function that represents her total daily charges  $C$  (in dollars) as a function of time  $t$  (in hours) spent at any one given location.

32. A salesperson is paid \$200 per week plus 5% commission on her weekly sales of  $x$  dollars. Find a linear function that represents her total weekly pay,  $W$  (in dollars) in terms of  $x$ . What must her weekly sales be in order for her to earn \$475.00 for the week?

33. An on-demand publisher charges \$22.50 to print a 600 page book and \$15.50 to print a 400 page book. Find a linear function which models the cost of a book  $C$  as a function of the number of pages  $p$ . Interpret the slope of the linear function and find and interpret  $C(0)$ .

34. The Topology Taxi Company charges \$2.50 for the first fifth of a mile and \$0.45 for each additional fifth of a mile. Find a linear function which models the taxi fare  $F$  as a function of the number of miles driven,  $m$ . Interpret the slope of the linear function and find and interpret  $F(0)$ .

35. Water freezes at  $0^\circ$  Celsius and  $32^\circ$  Fahrenheit and it boils at  $100^\circ\text{C}$  and  $212^\circ\text{F}$ .

(a) Find a linear function  $F$  that expresses temperature in the Fahrenheit scale in terms of degrees Celsius. Use this function to convert  $20^\circ\text{C}$  into Fahrenheit.

(b) Find a linear function  $C$  that expresses temperature in the Celsius scale in terms of degrees Fahrenheit. Use this function to convert  $110^\circ\text{F}$  into Celsius.

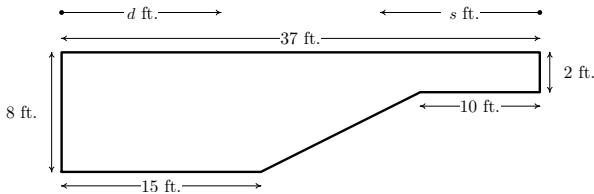
(c) Is there a temperature  $n$  such that  $F(n) = C(n)$ ?

36. Legend has it that a bull Sasquatch in rut will howl approximately 9 times per hour when it is  $40^{\circ}\text{F}$  outside and only 5 times per hour if it's  $70^{\circ}\text{F}$ . Assuming that the number of howls per hour,  $N$ , can be represented by a linear function of temperature Fahrenheit, find the number of howls per hour he'll make when it's only  $20^{\circ}\text{F}$  outside. What is the applied domain of this function? Why?
37. Economic forces beyond anyone's control have changed the cost function for PortaBoys to  $C(x) = 105x + 175$ . Rework Example 45 with this new cost function.
38. In response to the economic forces in Exercise 37 above, the local retailer sets the selling price of a PortaBoy at \$250. Remarkably, 30 units were sold each week. When the systems went on sale for \$220, 40 units per week were sold. Rework Examples 46 and 47 with this new data. What difficulties do you encounter?
39. A local pizza store offers medium two-topping pizzas delivered for \$6.00 per pizza plus a \$1.50 delivery charge per order. On weekends, the store runs a 'game day' special: if six or more medium two-topping pizzas are ordered, they are \$5.50 each with no delivery charge. Write a piecewise-defined linear function which calculates the cost  $C$  (in dollars) of  $p$  medium two-topping pizzas delivered during a weekend.
40. A restaurant offers a buffet which costs \$15 per person. For parties of 10 or more people, a group discount applies, and the cost is \$12.50 per person. Write a piecewise-defined linear function which calculates the total bill  $T$  of a party of  $n$  people who all choose the buffet.
41. A mobile plan charges a base monthly rate of \$10 for the first 500 minutes of air time plus a charge of 15¢ for each additional minute. Write a piecewise-defined linear function which calculates the monthly cost  $C$  (in dollars) for using  $m$  minutes of air time.

**HINT:** You may want to revisit Exercise 75 in Section 2.3

42. The local pet shop charges 12¢ per cricket up to 100 crickets, and 10¢ per cricket thereafter. Write a piecewise-defined linear function which calculates the price  $P$ , in dollars, of purchasing  $c$  crickets.
43. The cross-section of a swimming pool is below. Write a piecewise-defined linear function which describes the depth of the pool,  $D$  (in feet) as a function of:

- the distance (in feet) from the edge of the shallow end of the pool,  $d$ .
- the distance (in feet) from the edge of the deep end of the pool,  $s$ .
- Graph each of the functions in (a) and (b). Discuss with your classmates how to transform one into the other and how they relate to the diagram of the pool.



**In Exercises 44 – 50, compute the average rate of change of the function over the specified interval.**

44.  $f(x) = x^3$ ,  $[-1, 2]$

45.  $f(x) = \frac{1}{x}$ ,  $[1, 5]$

46.  $f(x) = \sqrt{x}$ ,  $[0, 16]$

47.  $f(x) = x^2$ ,  $[-3, 3]$

48.  $f(x) = \frac{x+4}{x-3}$ ,  $[5, 7]$

49.  $f(x) = 3x^2 + 2x - 7$ ,  $[-4, 2]$

**In Exercises 50 – 54, compute the average rate of change of the given function over the interval  $[x, x+h]$ . Here we assume  $[x, x+h]$  is in the domain of the function.**

50.  $f(x) = x^3$

51.  $f(x) = \frac{1}{x}$

52.  $f(x) = \frac{x+4}{x-3}$

53.  $f(x) = 3x^2 + 2x - 7$

54. Using data from [Bureau of Transportation Statistics](#), the average fuel economy  $F$  in miles per gallon for passenger cars in the US can be modeled by  $F(t) = -0.0076t^2 + 0.45t + 16$ ,  $0 \leq t \leq 28$ , where  $t$  is the number of years since 1980. Find and interpret the average rate of change of  $F$  over the interval  $[0, 28]$ .

55. The temperature  $T$  in degrees Fahrenheit  $t$  hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

- Find and interpret  $T(4)$ ,  $T(8)$  and  $T(12)$ .
- Find and interpret the average rate of change of  $T$  over the interval  $[4, 8]$ .
- Find and interpret the average rate of change of  $T$  from  $t = 8$  to  $t = 12$ .
- Find and interpret the average rate of temperature change between 10 AM and 6 PM.

56. Suppose  $C(x) = x^2 - 10x + 27$  represents the costs, in hundreds, to produce  $x$  thousand pens. Find and interpret the average rate of change as production is increased from making 3000 to 5000 pens.
57. With the help of your classmates find several other “real-world” examples of rates of change that are used to describe non-linear phenomena.
58. With the help of your classmates find several other “real-world” examples of rates of change that are used to describe non-linear phenomena.

**(Parallel Lines)** Recall from high school that parallel lines have the same slope. (Please note that two vertical lines are also parallel to one another even though they have an undefined slope.) In Exercises 59 – 65, you are given a line and a point which is not on that line. Find the line parallel to the given line which passes through the given point.

59.  $y = 3x + 2$ ,  $P(0, 0)$

60.  $y = -6x + 5$ ,  $P(3, 2)$

61.  $y = \frac{2}{3}x - 7$ ,  $P(6, 0)$

62.  $y = \frac{4-x}{3}$ ,  $P(1, -1)$

63.  $y = 6$ ,  $P(3, -2)$

64.  $x = 1$ ,  $P(-5, 0)$

**(Perpendicular Lines)** Recall from high school that two non-vertical lines are perpendicular if and only if they have negative reciprocal slopes. That is to say, if one line has slope  $m_1$  and the other has slope  $m_2$  then  $m_1 \cdot m_2 = -1$ . (You will be guided through a proof of this result in Exercise 71.) Please note that a horizontal line is perpendicular to a vertical line and vice versa, so we assume  $m_1 \neq 0$  and  $m_2 \neq 0$ . In Exercises 65 – 71, you are given a line and a point which is not on that line. Find the line perpendicular to the given line which passes through the given point.

65.  $y = \frac{1}{3}x + 2$ ,  $P(0, 0)$

66.  $y = -6x + 5$ ,  $P(3, 2)$

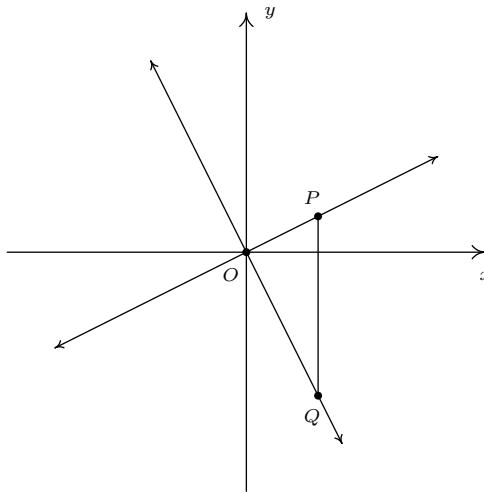
67.  $y = \frac{2}{3}x - 7$ ,  $P(6, 0)$

68.  $y = \frac{4-x}{3}$ ,  $P(1, -1)$

69.  $y = 6$ ,  $P(3, -2)$

70.  $x = 1$ ,  $P(-5, 0)$

71. We shall now prove that  $y = m_1x + b_1$  is perpendicular to  $y = m_2x + b_2$  if and only if  $m_1 \cdot m_2 = -1$ . To make our lives easier we shall assume that  $m_1 > 0$  and  $m_2 < 0$ . We can also “move” the lines so that their point of intersection is the origin without messing things up, so we’ll assume  $b_1 = b_2 = 0$ . (Take a moment with your classmates to discuss why this is okay.) Graphing the lines and plotting the points  $O(0, 0)$ ,  $P(1, m_1)$  and  $Q(1, m_2)$  gives us the following set up.

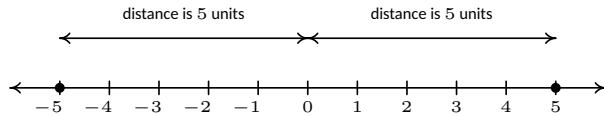


The line  $y = m_1x$  will be perpendicular to the line  $y = m_2x$  if and only if  $\triangle OPQ$  is a right triangle. Let  $d_1$  be the distance from  $O$  to  $P$ , let  $d_2$  be the distance from  $O$  to  $Q$  and let  $d_3$  be the distance from  $P$  to  $Q$ . Use the Pythagorean Theorem to show that  $\triangle OPQ$  is a right triangle if and only if  $m_1 \cdot m_2 = -1$  by showing  $d_1^2 + d_2^2 = d_3^2$  if and only if  $m_1 \cdot m_2 = -1$ .

72. Show that if  $a \neq b$ , the line containing the points  $(a, b)$  and  $(b, a)$  is perpendicular to the line  $y = x$ . (Coupled with the result from Example 11 on page 31, we have now shown that the line  $y = x$  is a *perpendicular bisector* of the line segment connecting  $(a, b)$  and  $(b, a)$ . This means the points  $(a, b)$  and  $(b, a)$  are symmetric about the line  $y = x$ . We will revisit this symmetry in section ??.)
73. The function defined by  $I(x) = x$  is called the Identity Function.
- Discuss with your classmates why this name makes sense.
  - Show that the point-slope form of a line (Equation 15) can be obtained from  $I$  using a sequence of the transformations defined in Section 2.6.

## 3.2 Absolute Value Functions

There are a few ways to describe what is meant by the absolute value  $|x|$  of a real number  $x$ . You may have been taught that  $|x|$  is the distance from the real number  $x$  to 0 on the number line. So, for example,  $|5| = 5$  and  $|-5| = 5$ , since each is 5 units from 0 on the number line.



Another way to define absolute value is by the equation  $|x| = \sqrt{x^2}$ . Using this definition, we have  $|5| = \sqrt{(5)^2} = \sqrt{25} = 5$  and  $|-5| = \sqrt{(-5)^2} = \sqrt{25} = 5$ . The long and short of both of these procedures is that  $|x|$  takes negative real numbers and assigns them to their positive counterparts while it leaves positive numbers alone. This last description is the one we shall adopt, and is summarized in the following definition.

### Definition 33      Absolute value function

The **absolute value** of a real number  $x$ , denoted  $|x|$ , is given by

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

In Definition 33, we define  $|x|$  using a piecewise-defined function. (See page 64 in Section 2.3.) To check that this definition agrees with what we previously understood as absolute value, note that since  $5 \geq 0$ , to find  $|5|$  we use the rule  $|x| = x$ , so  $|5| = 5$ . Similarly, since  $-5 < 0$ , we use the rule  $|x| = -x$ , so that  $|-5| = -(-5) = 5$ . This is one of the times when it's best to interpret the expression ' $-x$ ' as 'the opposite of  $x$ ' as opposed to 'negative  $x$ '. Before we begin studying absolute value functions, we remind ourselves of the properties of absolute value.

### Theorem 13      Properties of Absolute Value

Let  $a$ ,  $b$  and  $x$  be real numbers and let  $n$  be an integer. Then

- **Product Rule:**  $|ab| = |a||b|$
- **Power Rule:**  $|a^n| = |a|^n$  whenever  $a^n$  is defined
- **Quotient Rule:**  $\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$ , provided  $b \neq 0$

#### Equality Properties:

- $|x| = 0$  if and only if  $x = 0$ .
- For  $c > 0$ ,  $|x| = c$  if and only if  $x = c$  or  $-x = c$ .
- For  $c < 0$ ,  $|x| = c$  has no solution.

The proofs of the Product and Quotient Rules in Theorem 13 boil down to checking four cases: when both  $a$  and  $b$  are positive; when they are both negative; when one is positive and the other is negative; and when one or both are zero.

For example, suppose we wish to show that  $|ab| = |a||b|$ . We need to show that this equation is true for all real numbers  $a$  and  $b$ . If  $a$  and  $b$  are both positive, then so is  $ab$ . Hence,  $|a| = a$ ,  $|b| = b$  and  $|ab| = ab$ . Hence, the equation  $|ab| = |a||b|$  is the same as  $ab = ab$  which is true. If both  $a$  and  $b$  are negative, then  $ab$  is positive. Hence,  $|a| = -a$ ,  $|b| = -b$  and  $|ab| = ab$ . The equation  $|ab| = |a||b|$  becomes  $ab = (-a)(-b)$ , which is true. Suppose  $a$  is positive and  $b$  is negative. Then  $ab$  is negative, and we have  $|ab| = -ab$ ,  $|a| = a$  and  $|b| = -b$ . The equation  $|ab| = |a||b|$  reduces to  $-ab = a(-b)$  which is true. A symmetric argument shows the equation  $|ab| = |a||b|$  holds when  $a$  is negative and  $b$  is positive. Finally, if either  $a$  or  $b$  (or both) are zero, then both sides of  $|ab| = |a||b|$  are zero, so the equation holds in this case, too. All of this rhetoric has shown that the equation  $|ab| = |a||b|$  holds true in all cases.

The proof of the Quotient Rule is very similar, with the exception that  $b \neq 0$ . The Power Rule can be shown by repeated application of the Product Rule. The ‘Equality Properties’ can be proved using Definition 33 and by looking at the cases when  $x \geq 0$ , in which case  $|x| = x$ , or when  $x < 0$ , in which case  $|x| = -x$ . For example, if  $c > 0$ , and  $|x| = c$ , then if  $x \geq 0$ , we have  $x = |x| = c$ . If, on the other hand,  $x < 0$ , then  $-x = |x| = c$ , so  $x = -c$ . The remaining properties are proved similarly and are left for the Exercises. Our first example reviews how to solve basic equations involving absolute value using the properties listed in Theorem 13.

#### Example 48 Solving equations with absolute values

Solve each of the following equations.

$$1. |3x - 1| = 6 \quad 2. 3 - |x + 5| = 1 \quad 3. 3|2x + 1| - 5 = 0$$

$$4. 4 - |5x + 3| = 5 \quad 5. |x| = x^2 - 6 \quad 6. |x - 2| + 1 = x$$

#### SOLUTION

- The equation  $|3x - 1| = 6$  is of the form  $|x| = c$  for  $c > 0$ , so by the Equality Properties,  $|3x - 1| = 6$  is equivalent to  $3x - 1 = 6$  or  $3x - 1 = -6$ . Solving the former, we arrive at  $x = \frac{7}{3}$ , and solving the latter, we get  $x = -\frac{5}{3}$ . We may check both of these solutions by substituting them into the original equation and showing that the arithmetic works out.
- To use the Equality Properties to solve  $3 - |x + 5| = 1$ , we first isolate the absolute value.

$$\begin{aligned} 3 - |x + 5| &= 1 \\ -|x + 5| &= -2 \quad \text{subtract 3} \\ |x + 5| &= 2 \quad \text{divide by } -1 \end{aligned}$$

From the Equality Properties, we have  $x + 5 = 2$  or  $x + 5 = -2$ , and get our solutions to be  $x = -3$  or  $x = -7$ . We leave it to the reader to check both answers in the original equation.

3. As in the previous example, we first isolate the absolute value in the equation  $3|2x + 1| - 5 = 0$  and get  $|2x + 1| = \frac{5}{3}$ . Using the Equality Properties, we have  $2x + 1 = \frac{5}{3}$  or  $2x + 1 = -\frac{5}{3}$ . Solving the former gives  $x = \frac{1}{3}$  and solving the latter gives  $x = -\frac{4}{3}$ . As usual, we may substitute both answers in the original equation to check.
4. Upon isolating the absolute value in the equation  $4 - |5x + 3| = 5$ , we get  $|5x + 3| = -1$ . At this point, we know there cannot be any real solution, since, by definition, the absolute value of *anything* is never negative. We are done.
5. The equation  $|x| = x^2 - 6$  presents us with some difficulty, since  $x$  appears both inside and outside of the absolute value. Moreover, there are values of  $x$  for which  $x^2 - 6$  is positive, negative and zero, so we cannot use the Equality Properties without the risk of introducing extraneous solutions, or worse, losing solutions. For this reason, we break equations like this into cases by rewriting the term in absolute values,  $|x|$ , using Definition 33. For  $x < 0$ ,  $|x| = -x$ , so for  $x < 0$ , the equation  $|x| = x^2 - 6$  is equivalent to  $-x = x^2 - 6$ . Rearranging this gives us  $x^2 + x - 6 = 0$ , or  $(x + 3)(x - 2) = 0$ . We get  $x = -3$  or  $x = 2$ . Since only  $x = -3$  satisfies  $x < 0$ , this is the answer we keep. For  $x \geq 0$ ,  $|x| = x$ , so the equation  $|x| = x^2 - 6$  becomes  $x = x^2 - 6$ . From this, we get  $x^2 - x - 6 = 0$  or  $(x - 3)(x + 2) = 0$ . Our solutions are  $x = 3$  or  $x = -2$ , and since only  $x = 3$  satisfies  $x \geq 0$ , this is the one we keep. Hence, our two solutions to  $|x| = x^2 - 6$  are  $x = -3$  and  $x = 3$ .
6. To solve  $|x - 2| + 1 = x$ , we first isolate the absolute value and get  $|x - 2| = x - 1$ . Since we see  $x$  both inside and outside of the absolute value, we break the equation into cases. The term with absolute values here is  $|x - 2|$ , so we replace ' $x$ ' with the quantity '( $x - 2$ )' in Definition 33 to get

$$|x - 2| = \begin{cases} -(x - 2), & \text{if } (x - 2) < 0 \\ (x - 2), & \text{if } (x - 2) \geq 0 \end{cases}$$

Simplifying yields

$$|x - 2| = \begin{cases} -x + 2, & \text{if } x < 2 \\ x - 2, & \text{if } x \geq 2 \end{cases}$$

So, for  $x < 2$ ,  $|x - 2| = -x + 2$  and our equation  $|x - 2| = x - 1$  becomes  $-x + 2 = x - 1$ , which gives  $x = \frac{3}{2}$ . Since this solution satisfies  $x < 2$ , we keep it. Next, for  $x \geq 2$ ,  $|x - 2| = x - 2$ , so the equation  $|x - 2| = x - 1$  becomes  $x - 2 = x - 1$ . Here, the equation reduces to  $-2 = -1$ , which signifies we have no solutions here. Hence, our only solution is  $x = \frac{3}{2}$ .

Next, we turn our attention to graphing absolute value functions. Our strategy in the next example is to make liberal use of Definition 33 along with what we know about graphing linear functions (from Section 3.1) and piecewise-defined functions (from Section 2.3).

**Example 49 Graphing absolute value functions**

Graph each of the following functions.

$$1. f(x) = |x|$$

$$3. h(x) = |x| - 3$$

$$2. g(x) = |x - 3|$$

$$4. i(x) = 4 - 2|3x + 1|$$

Find the zeros of each function and the  $x$ - and  $y$ -intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, list the intervals on which the function is increasing, decreasing or constant, and find the relative and absolute extrema, if they exist.

**SOLUTION**

1. To find the zeros of  $f$ , we set  $f(x) = 0$ . We get  $|x| = 0$ , which, by Theorem 13 gives us  $x = 0$ . Since the zeros of  $f$  are the  $x$ -coordinates of the  $x$ -intercepts of the graph of  $y = f(x)$ , we get  $(0, 0)$  as our only  $x$ -intercept. To find the  $y$ -intercept, we set  $x = 0$ , and find  $y = f(0) = 0$ , so that  $(0, 0)$  is our  $y$ -intercept as well. Using Definition 33, we get

$$f(x) = |x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Hence, for  $x < 0$ , we are graphing the line  $y = -x$ ; for  $x \geq 0$ , we have the line  $y = x$ . Proceeding as we did in Section 2.5, we get the first two graphs in Figure 3.9.

Notice that we have an ‘open circle’ at  $(0, 0)$  in the graph when  $x < 0$ . As we have seen before, this is due to the fact that the points on  $y = -x$  approach  $(0, 0)$  as the  $x$ -values approach 0. Since  $x$  is required to be strictly less than zero on this stretch, the open circle is drawn at the origin. However, notice that when  $x \geq 0$ , we get to fill in the point at  $(0, 0)$ , which effectively ‘plugs’ the hole indicated by the open circle. Thus our final result is the graph at the bottom of Figure 3.9.

By projecting the graph to the  $x$ -axis, we see that the domain is  $(-\infty, \infty)$ . Projecting to the  $y$ -axis gives us the range  $[0, \infty)$ . The function is increasing on  $[0, \infty)$  and decreasing on  $(-\infty, 0]$ . The relative minimum value of  $f$  is the same as the absolute minimum, namely 0 which occurs at  $(0, 0)$ . There is no relative maximum value of  $f$ . There is also no absolute maximum value of  $f$ , since the  $y$  values on the graph extend infinitely upwards.

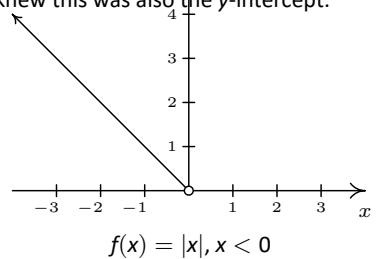
2. To find the zeros of  $g$ , we set  $g(x) = |x - 3| = 0$ . By Theorem 13, we get  $x - 3 = 0$  so that  $x = 3$ . Hence, the  $x$ -intercept is  $(3, 0)$ . To find our  $y$ -intercept, we set  $x = 0$  so that  $y = g(0) = |0 - 3| = 3$ , which yields  $(0, 3)$  as our  $y$ -intercept. To graph  $g(x) = |x - 3|$ , we use Definition 33 to rewrite  $g$  as

$$g(x) = |x - 3| = \begin{cases} -(x - 3), & \text{if } (x - 3) < 0 \\ (x - 3), & \text{if } (x - 3) \geq 0 \end{cases}$$

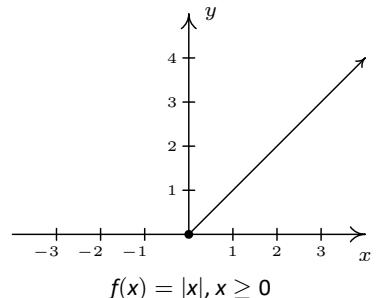
Simplifying, we get

$$g(x) = \begin{cases} -x + 3, & \text{if } x < 3 \\ x - 3, & \text{if } x \geq 3 \end{cases}$$

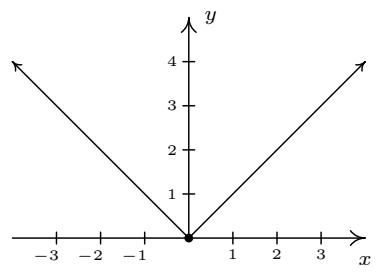
Since functions can have at most one  $y$ -intercept (Do you know why?), as soon as we found  $(0, 0)$  as the  $x$ -intercept, we knew this was also the  $y$ -intercept.



$$f(x) = |x|, x < 0$$

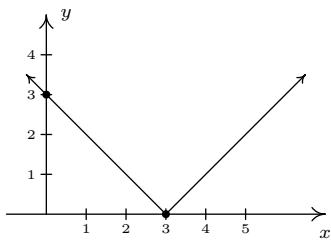
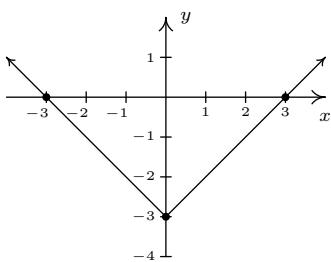


$$f(x) = |x|, x \geq 0$$



$$f(x) = |x|$$

Figure 3.9: Constructing the graph of  $f(x) = |x|$

Figure 3.10:  $g(x) = |x - 3|$ Figure 3.11:  $h(x) = |x| - 3$ 

As before, the open circle we introduce at  $(3, 0)$  from the graph of  $y = -x + 3$  is filled by the point  $(3, 0)$  from the line  $y = x - 3$ . We determine the domain as  $(-\infty, \infty)$  and the range as  $[0, \infty)$ . The function  $g$  is increasing on  $[3, \infty)$  and decreasing on  $(-\infty, 3]$ . The relative and absolute minimum value of  $g$  is 0 which occurs at  $(3, 0)$ . As before, there is no relative or absolute maximum value of  $g$ .

3. Setting  $h(x) = 0$  to look for zeros gives  $|x| - 3 = 0$ . As in Example 48, we isolate the absolute value to get  $|x| = 3$  so that  $x = 3$  or  $x = -3$ . As a result, we have a pair of  $x$ -intercepts:  $(-3, 0)$  and  $(3, 0)$ . Setting  $x = 0$  gives  $y = h(0) = |0| - 3 = -3$ , so our  $y$ -intercept is  $(0, -3)$ . As before, we rewrite the absolute value in  $h$  to get

$$h(x) = \begin{cases} -x - 3, & \text{if } x < 0 \\ x - 3, & \text{if } x \geq 0 \end{cases}$$

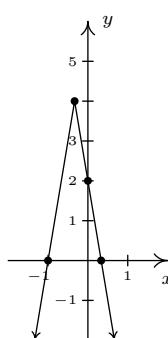
Once again, the open circle at  $(0, -3)$  from one piece of the graph of  $h$  is filled by the point  $(0, -3)$  from the other piece of  $h$ . From the graph, we determine the domain of  $h$  is  $(-\infty, \infty)$  and the range is  $[-3, \infty)$ . On  $[0, \infty)$ ,  $h$  is increasing; on  $(-\infty, 0]$  it is decreasing. The relative minimum occurs at the point  $(0, -3)$  on the graph, and we see  $-3$  is both the relative and absolute minimum value of  $h$ . Also,  $h$  has no relative or absolute maximum value.

4. As before, we set  $i(x) = 0$  to find the zeros of  $i$  and get  $4 - 2|3x + 1| = 0$ . Isolating the absolute value term gives  $|3x + 1| = 2$ , so either  $3x + 1 = 2$  or  $3x + 1 = -2$ . We get  $x = \frac{1}{3}$  or  $x = -1$ , so our  $x$ -intercepts are  $(\frac{1}{3}, 0)$  and  $(-1, 0)$ . Substituting  $x = 0$  gives  $y = i(0) = 4 - 2|3(0) + 1| = 2$ , for a  $y$ -intercept of  $(0, 2)$ . Rewriting the formula for  $i(x)$  without absolute values gives

$$\begin{aligned} i(x) &= \begin{cases} 4 - 2(-(3x + 1)), & \text{if } (3x + 1) < 0 \\ 4 - 2(3x + 1), & \text{if } (3x + 1) \geq 0 \end{cases} \\ &= \begin{cases} 6x + 6, & \text{if } x < -\frac{1}{3} \\ -6x + 2, & \text{if } x \geq -\frac{1}{3} \end{cases} \end{aligned}$$

The usual analysis near the trouble spot  $x = -\frac{1}{3}$  gives that the ‘corner’ of this graph is  $(-\frac{1}{3}, 4)$ , and we get the distinctive ‘V’ shape: see Figure 3.12.

The domain of  $i$  is  $(-\infty, \infty)$  while the range is  $(-\infty, 4]$ . The function  $i$  is increasing on  $(-\infty, -\frac{1}{3}]$  and decreasing on  $[-\frac{1}{3}, \infty)$ . The relative maximum occurs at the point  $(-\frac{1}{3}, 4)$  and the relative and absolute maximum value of  $i$  is 4. Since the graph of  $i$  extends downwards forever more, there is no absolute minimum value. As we can see from the graph, there is no relative minimum, either.

Figure 3.12:  $i(x) = 4 - 2|3x + 1|$ 

Note that all of the functions in the previous example bear the characteristic ‘V’ shape of the graph of  $y = |x|$ . We could have graphed the functions  $g$ ,  $h$  and  $i$  in Example 49 starting with the graph of  $f(x) = |x|$  and applying transformations as in Section 2.6 as our next example illustrates.

**Example 50 Graphing using transformations**

Graph the following functions starting with the graph of  $f(x) = |x|$  and using transformations.

1.  $g(x) = |x - 3|$
2.  $h(x) = |x| - 3$
3.  $i(x) = 4 - 2|3x + 1|$

**SOLUTION** We begin by graphing  $f(x) = |x|$  and labelling three points,  $(-1, 1)$ ,  $(0, 0)$  and  $(1, 1)$ , as in Figure 3.13

1. Since  $g(x) = |x - 3| = f(x - 3)$ , Theorem 12 tells us to *add* 3 to each of the  $x$ -values of the points on the graph of  $y = f(x)$  to obtain the graph of  $y = g(x)$ . This shifts the graph of  $y = f(x)$  to the *right* 3 units and moves the point  $(-1, 1)$  to  $(2, 1)$ ,  $(0, 0)$  to  $(3, 0)$  and  $(1, 1)$  to  $(4, 1)$ . Connecting these points in the classic ‘ $\vee$ ’ fashion produces the graph of  $y = g(x)$  in Figure 3.14.
2. For  $h(x) = |x| - 3 = f(x) - 3$ , Theorem 12 tells us to *subtract* 3 from each of the  $y$ -values of the points on the graph of  $y = f(x)$  to obtain the graph of  $y = h(x)$ . This shifts the graph of  $y = f(x)$  *down* 3 units and moves  $(-1, 1)$  to  $(-1, -2)$ ,  $(0, 0)$  to  $(0, -3)$  and  $(1, 1)$  to  $(1, -2)$ . Connecting these points with the ‘ $\vee$ ’ shape produces our graph of  $y = h(x)$ : see Figure 3.15.
3. We re-write  $i(x) = 4 - 2|3x + 1| = 4 - 2f(3x + 1) = -2f(3x + 1) + 4$  and apply Theorem 12. First, we take care of the changes on the ‘inside’ of the absolute value. Instead of  $|x|$ , we have  $|3x + 1|$ , so, in accordance with Theorem 12, we first *subtract* 1 from each of the  $x$ -values of points on the graph of  $y = f(x)$ , then *divide* each of those new values by 3. This effects a horizontal shift *left* 1 unit followed by a horizontal *shrink* by a factor of 3. These transformations move  $(-1, 1)$  to  $(-\frac{2}{3}, 1)$ ,  $(0, 0)$  to  $(-\frac{1}{3}, 0)$  and  $(1, 1)$  to  $(0, 1)$ . Next, we take care of what’s happening ‘outside of’ the absolute value. Theorem 12 instructs us to first *multiply* each  $y$ -value of these new points by  $-2$  then *add* 4. Geometrically, this corresponds to a vertical *stretch* by a factor of 2, a reflection across the  $x$ -axis and finally, a vertical shift *up* 4 units. These transformations move  $(-\frac{2}{3}, 1)$  to  $(-\frac{2}{3}, 2)$ ,  $(-\frac{1}{3}, 0)$  to  $(-\frac{1}{3}, 4)$ , and  $(0, 1)$  to  $(0, 2)$ . Connecting these points with the usual ‘ $\vee$ ’ shape produces our graph of  $y = i(x)$ .

While the methods in Section 2.6 can be used to graph an entire family of absolute value functions, not all functions involving absolute values possess the characteristic ‘ $\vee$ ’ shape. As the next example illustrates, often there is no substitute for appealing directly to the definition.

**Example 51 A more complicated example**

Graph each of the following functions. Find the zeros of each function and the  $x$ - and  $y$ -intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, list the intervals on which the function is increasing, decreasing or constant, and find the relative and absolute extrema, if they exist.

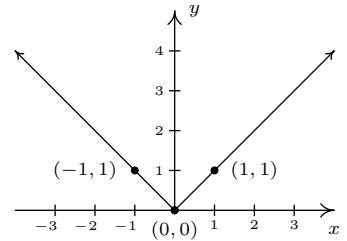


Figure 3.13:  $f(x) = |x|$  with three labelled points

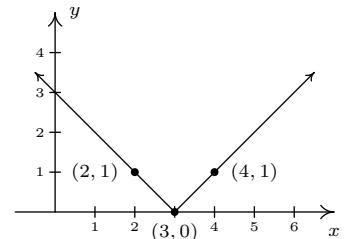


Figure 3.14:  $g(x) = |x - 3| = f(x - 3)$

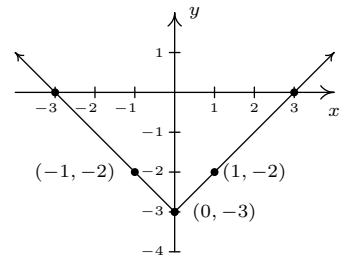


Figure 3.15:  $h(x) = |x| - 3 = f(x) - 3$

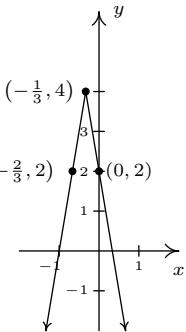


Figure 3.16:  $i(x) = 4 - 2|3x + 1| = -2f(3x + 1) + 4$

$$1. f(x) = \frac{|x|}{x}$$

$$2. g(x) = |x+2| - |x-3| + 1$$

**SOLUTION**

1. We first note that, due to the fraction in the formula of  $f(x)$ ,  $x \neq 0$ . Thus the domain is  $(-\infty, 0) \cup (0, \infty)$ . To find the zeros of  $f$ , we set  $f(x) = \frac{|x|}{x} = 0$ . This last equation implies  $|x| = 0$ , which, from Theorem 13, implies  $x = 0$ . However,  $x = 0$  is not in the domain of  $f$ , which means we have, in fact, no  $x$ -intercepts. We have no  $y$ -intercepts either, since  $f(0)$  is undefined. Re-writing the absolute value in the function gives

$$f(x) = \begin{cases} \frac{-x}{x}, & \text{if } x < 0 \\ \frac{x}{x}, & \text{if } x > 0 \end{cases} = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$$

To graph this function, we graph two horizontal lines:  $y = -1$  for  $x < 0$  and  $y = 1$  for  $x > 0$ . We have open circles at  $(0, -1)$  and  $(0, 1)$  (Can you explain why?) so we get the graph in figure 3.17.

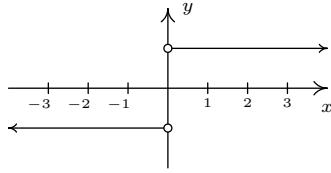


Figure 3.17:  $f(x) = \frac{|x|}{x}$

As we found earlier, the domain is  $(-\infty, 0) \cup (0, \infty)$ . The range consists of just two  $y$ -values:  $\{-1, 1\}$ . The function  $f$  is constant on  $(-\infty, 0)$  and  $(0, \infty)$ . The local minimum value of  $f$  is the absolute minimum value of  $f$ , namely  $-1$ ; the local maximum and absolute maximum values for  $f$  also coincide – they both are  $1$ . Every point on the graph of  $f$  is simultaneously a relative maximum and a relative minimum. (Can you remember why in light of Definition 28? This was explored in the Exercises in Section ??.)

2. To find the zeros of  $g$ , we set  $g(x) = 0$ . The result is  $|x+2| - |x-3| + 1 = 0$ . Attempting to isolate the absolute value term is complicated by the fact that there are **two** terms with absolute values. In this case, it easier to proceed using cases by re-writing the function  $g$  with two separate applications of Definition 33 to remove each instance of the absolute values, one at a time. In the first round we get

$$\begin{aligned} g(x) &= \begin{cases} -(x+2) - |x-3| + 1, & \text{if } (x+2) < 0 \\ (x+2) - |x-3| + 1, & \text{if } (x+2) \geq 0 \end{cases} \\ &= \begin{cases} -x-1-|x-3|, & \text{if } x < -2 \\ x+3-|x-3|, & \text{if } x \geq -2 \end{cases} \end{aligned}$$

Given that

$$|x-3| = \begin{cases} -(x-3), & \text{if } (x-3) < 0 \\ x-3, & \text{if } (x-3) \geq 0 \end{cases} = \begin{cases} -x+3, & \text{if } x < 3 \\ x-3, & \text{if } x \geq 3 \end{cases}$$

we need to break up the domain again at  $x = 3$ . Note that if  $x < -2$ , then  $x < 3$ , so we replace  $|x-3|$  with  $-x+3$  for that part of the domain, too. Our completed revision of the form of  $g$  yields

$$\begin{aligned}
 g(x) &= \begin{cases} -x - 1 - (-x + 3), & \text{if } x < -2 \\ x + 3 - (-x + 3), & \text{if } x \geq -2 \text{ and } x < 3 \\ x + 3 - (x - 3), & \text{if } x \geq 3 \end{cases} \\
 &= \begin{cases} -4, & \text{if } x < -2 \\ 2x, & \text{if } -2 \leq x < 3 \\ 6, & \text{if } x \geq 3 \end{cases}
 \end{aligned}$$

To solve  $g(x) = 0$ , we see that the only piece which contains a variable is  $g(x) = 2x$  for  $-2 \leq x < 3$ . Solving  $2x = 0$  gives  $x = 0$ . Since  $x = 0$  is in the interval  $[-2, 3]$ , we keep this solution and have  $(0, 0)$  as our only  $x$ -intercept. Accordingly, the  $y$ -intercept is also  $(0, 0)$ . To graph  $g$ , we start with  $x < -2$  and graph the horizontal line  $y = -4$  with an open circle at  $(-2, -4)$ . For  $-2 \leq x < 3$ , we graph the line  $y = 2x$  and the point  $(-2, -4)$  patches the hole left by the previous piece. An open circle at  $(3, 6)$  completes the graph of this part. Finally, we graph the horizontal line  $y = 6$  for  $x \geq 3$ , and the point  $(3, 6)$  fills in the open circle left by the previous part of the graph. The finished graph is given in Figure 3.18

The domain of  $g$  is all real numbers,  $(-\infty, \infty)$ , and the range of  $g$  is all real numbers between  $-4$  and  $6$  inclusive,  $[-4, 6]$ . The function is increasing on  $[-2, 3]$  and constant on  $(-\infty, -2]$  and  $[3, \infty)$ . The relative minimum value of  $-4$  which matches the absolute minimum. The relative and absolute maximum values also coincide at  $6$ . Every point on the graph of  $y = g(x)$  for  $x < -2$  and  $x > 3$  yields both a relative minimum and relative maximum. The point  $(-2, -4)$ , however, gives only a relative minimum and the point  $(3, 6)$  yields only a relative maximum. (Recall the Exercises in Section ?? which dealt with constant functions.)

Many of the applications that the authors are aware of involving absolute values also involve absolute value inequalities. For that reason, we save our discussion of applications for Section ??.

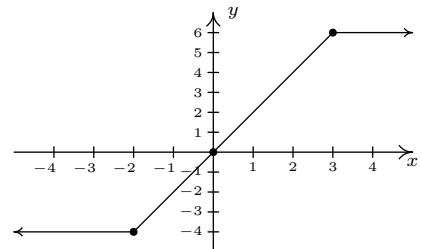


Figure 3.18:  $g(x) = |x+2| - |x-3| + 1$

## Exercises 3.2

### Problems

In Exercises 1 – 16, solve the equation.

1.  $|x| = 6$

2.  $|3x - 1| = 10$

3.  $|4 - x| = 7$

4.  $4 - |x| = 3$

5.  $2|5x + 1| - 3 = 0$

6.  $|7x - 1| + 2 = 0$

7.  $\frac{5 - |x|}{2} = 1$

8.  $\frac{2}{3}|5 - 2x| - \frac{1}{2} = 5$

9.  $|x| = x + 3$

10.  $|2x - 1| = x + 1$

11.  $4 - |x| = 2x + 1$

12.  $|x - 4| = x - 5$

13.  $|x| = x^2$

14.  $|x| = 12 - x^2$

15.  $|x^2 - 1| = 3$

Prove that if  $|f(x)| = |g(x)|$  then either  $f(x) = g(x)$  or  $f(x) = -g(x)$ . Use that result to solve the equations in Exercises 16 – 22.

16.  $|3x - 2| = |2x + 7|$

17.  $|3x + 1| = |4x|$

18.  $|1 - 2x| = |x + 1|$

19.  $|4 - x| - |x + 2| = 0$

20.  $|2 - 5x| = 5|x + 1|$

21.  $3|x - 1| = 2|x + 1|$

In Exercises 22 – 34, graph the function. Find the zeros of each function and the  $x$ - and  $y$ -intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, list the intervals on which the function is increasing, decreasing or constant, and find the relative and absolute extrema, if they exist.

22.  $f(x) = |x + 4|$

23.  $f(x) = |x| + 4$

24.  $f(x) = |4x|$

25.  $f(x) = -3|x|$

26.  $f(x) = 3|x + 4| - 4$

27.  $f(x) = \frac{1}{3}|2x - 1|$

28.  $f(x) = \frac{|x + 4|}{x + 4}$

29.  $f(x) = \frac{|2 - x|}{2 - x}$

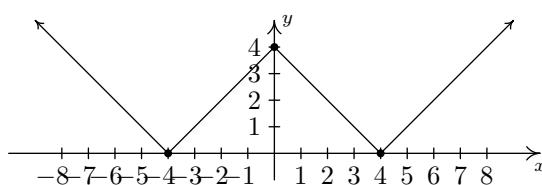
30.  $f(x) = x + |x| - 3$

31.  $f(x) = |x + 2| - x$

32.  $f(x) = |x + 2| - |x|$

33.  $f(x) = |x + 4| + |x - 2|$

34. With the help of your classmates, find an absolute value function whose graph is given below.



35. With help from your classmates, prove the second, third and fifth parts of Theorem 13.

36. Prove **The Triangle Inequality**: For all real numbers  $a$  and  $b$ ,  $|a + b| \leq |a| + |b|$ .

### 3.3 Quadratic Functions

You may recall studying quadratic equations in high school. In this section, we review those equations in the context of our next family of functions: the quadratic functions.

#### Definition 34 Quadratic function

A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

where  $a, b$  and  $c$  are real numbers with  $a \neq 0$ . The domain of a quadratic function is  $(-\infty, \infty)$ .

The most basic quadratic function is  $f(x) = x^2$ , whose graph appears below. Its shape should look familiar from high school – it is called a **parabola**. The point  $(0, 0)$  is called the **vertex** of the parabola. In this case, the vertex is a relative minimum and is also the where the absolute minimum value of  $f$  can be found.

Much like many of the absolute value functions in Section 3.2, knowing the graph of  $f(x) = x^2$  enables us to graph an entire family of quadratic functions using transformations.

#### Example 52 Graphics quadratic functions

Graph the following functions starting with the graph of  $f(x) = x^2$  and using transformations. Find the vertex, state the range and find the  $x$ - and  $y$ -intercepts, if any exist.

1.  $g(x) = (x + 2)^2 - 3$
2.  $h(x) = -2(x - 3)^2 + 1$

#### SOLUTION

1. Since  $g(x) = (x + 2)^2 - 3 = f(x + 2) - 3$ , Theorem 12 instructs us to first *subtract 2* from each of the  $x$ -values of the points on  $y = f(x)$ . This shifts the graph of  $y = f(x)$  to the *left* 2 units and moves  $(-2, 4)$  to  $(-4, 4)$ ,  $(-1, 1)$  to  $(-3, 1)$ ,  $(0, 0)$  to  $(-2, 0)$ ,  $(1, 1)$  to  $(-1, 1)$  and  $(2, 4)$  to  $(0, 4)$ . Next, we *subtract 3* from each of the  $y$ -values of these new points. This moves the graph *down* 3 units and moves  $(-4, 4)$  to  $(-4, 1)$ ,  $(-3, 1)$  to  $(-3, -2)$ ,  $(-2, 0)$  to  $(-2, -3)$ ,  $(-1, 1)$  to  $(-1, -2)$  and  $(0, 4)$  to  $(0, 1)$ . We connect the dots in parabolic fashion to get the graph in Figure 3.21.

From the graph, we see that the vertex has moved from  $(0, 0)$  on the graph of  $y = f(x)$  to  $(-2, -3)$  on the graph of  $y = g(x)$ . This sets  $[-3, \infty)$  as the range of  $g$ . We see that the graph of  $y = g(x)$  crosses the  $x$ -axis twice, so we expect two  $x$ -intercepts. To find these, we set  $y = g(x) = 0$  and solve. Doing so yields the equation  $(x + 2)^2 - 3 = 0$ , or  $(x + 2)^2 = 3$ . Extracting square roots gives  $x + 2 = \pm\sqrt{3}$ , or  $x = -2 \pm \sqrt{3}$ . Our  $x$ -intercepts are  $(-2 - \sqrt{3}, 0) \approx (-3.73, 0)$  and  $(-2 + \sqrt{3}, 0) \approx (-0.27, 0)$ . The  $y$ -intercept of the graph,  $(0, 1)$  was one of the points we originally plotted, so we are done.

2. Following Theorem 12 once more, to graph  $h(x) = -2(x - 3)^2 + 1 = -2f(x - 3) + 1$ , we first start by *adding 3* to each of the  $x$ -values of the

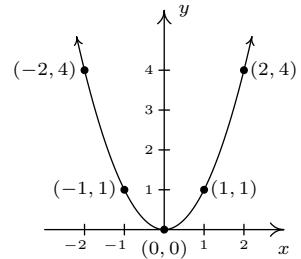


Figure 3.19: The graph of the basic quadratic function  $f(x) = x^2$

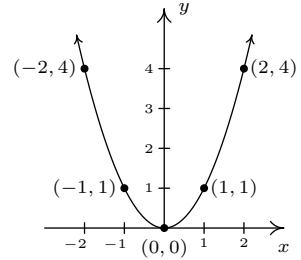


Figure 3.20: The graph  $y = x^2$  with points labelled

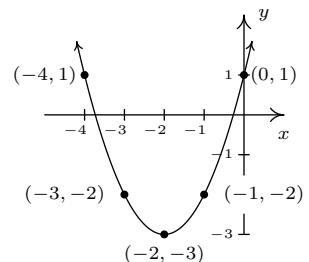


Figure 3.21:  $g(x) = f(x + 2) - 3 = (x + 2)^2 - 3$

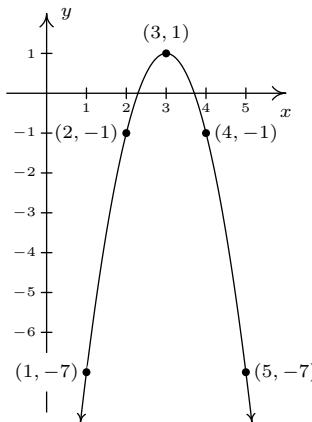


Figure 3.22:  $h(x) = -2f(x - 3) + 1 = -2(x - 3)^2 + 1$

points on the graph of  $y = f(x)$ . This effects a horizontal shift *right* 3 units and moves  $(-2, 4)$  to  $(1, 4)$ ,  $(-1, 1)$  to  $(2, 1)$ ,  $(0, 0)$  to  $(3, 0)$ ,  $(1, 1)$  to  $(4, 1)$  and  $(2, 4)$  to  $(5, 4)$ . Next, we *multiply* each of our  $y$ -values first by  $-2$  and then *add* 1 to that result. Geometrically, this is a vertical *stretch* by a factor of 2, followed by a reflection about the  $x$ -axis, followed by a vertical shift *up* 1 unit. This moves  $(1, 4)$  to  $(1, -7)$ ,  $(2, 1)$  to  $(2, -1)$ ,  $(3, 0)$  to  $(3, 1)$ ,  $(4, 1)$  to  $(4, -1)$  and  $(5, 4)$  to  $(5, -7)$ , giving us the graph in Figure 3.22.

The vertex is  $(3, 1)$  which makes the range of  $h$   $(-\infty, 1]$ . From our graph, we know that there are two  $x$ -intercepts, so we set  $y = h(x) = 0$  and solve. We get  $-2(x - 3)^2 + 1 = 0$  which gives  $(x - 3)^2 = \frac{1}{2}$ . Extracting square roots (and rationalizing denominators!) gives  $x - 3 = \pm\frac{\sqrt{2}}{2}$ , so that when we add 3 to each side, (and get common denominators!) we get  $x = \frac{6 \pm \sqrt{2}}{2}$ . Hence, our  $x$ -intercepts are  $\left(\frac{6 - \sqrt{2}}{2}, 0\right) \approx (2.29, 0)$  and  $\left(\frac{6 + \sqrt{2}}{2}, 0\right) \approx (3.71, 0)$ . Although our graph doesn't show it, there is a  $y$ -intercept which can be found by setting  $x = 0$ . With  $h(0) = -2(0 - 3)^2 + 1 = -17$ , we have that our  $y$ -intercept is  $(0, -17)$ .

A few remarks about Example 52 are in order. First note that neither the formula given for  $g(x)$  nor the one given for  $h(x)$  match the form given in Definition 34. We could, of course, convert both  $g(x)$  and  $h(x)$  into that form by expanding and collecting like terms. Doing so, we find  $g(x) = (x+2)^2 - 3 = x^2 + 4x + 1$  and  $h(x) = -2(x - 3)^2 + 1 = -2x^2 + 12x - 17$ . While these ‘simplified’ formulas for  $g(x)$  and  $h(x)$  satisfy Definition 34, they do not lend themselves to graphing easily. For that reason, the form of  $g$  and  $h$  presented in Example 53 is given a special name, which we list below, along with the form presented in Definition 34.

### Definition 35 Standard and General Form of Quadratic Functions

Suppose  $f$  is a quadratic function.

- The **general form** of the quadratic function  $f$  is  $f(x) = ax^2 + bx + c$ , where  $a, b$  and  $c$  are real numbers with  $a \neq 0$ .
- The **standard form** of the quadratic function  $f$  is  $f(x) = a(x - h)^2 + k$ , where  $a, h$  and  $k$  are real numbers with  $a \neq 0$ .

It is important to note at this stage that we have no guarantees that *every* quadratic function can be written in standard form. This is actually true, and we prove this later in the exposition, but for now we celebrate the advantages of the standard form, starting with the following theorem.

### Theorem 14 Vertex Formula for Quadratics in Standard Form

For the quadratic function  $f(x) = a(x - h)^2 + k$ , where  $a, h$  and  $k$  are real numbers with  $a \neq 0$ , the vertex of the graph of  $y = f(x)$  is  $(h, k)$ .

We can readily verify the formula given Theorem 14 with the two functions given in Example 52. After a (slight) rewrite,  $g(x) = (x+2)^2 - 3 = (x-(-2))^2 + (-3)$ , and we identify  $h = -2$  and  $k = -3$ . Sure enough, we found the vertex of the graph of  $y = g(x)$  to be  $(-2, -3)$ . For  $h(x) = -2(x-3)^2 + 1$ , no rewrite is needed. We can directly identify  $h = 3$  and  $k = 1$  and, sure enough, we found the vertex of the graph of  $y = h(x)$  to be  $(3, 1)$ .

To see why the formula in Theorem 14 produces the vertex, consider the graph of the equation  $y = a(x-h)^2 + k$ . When we substitute  $x = h$ , we get  $y = k$ , so  $(h, k)$  is on the graph. If  $x \neq h$ , then  $x-h \neq 0$  so  $(x-h)^2$  is a positive number. If  $a > 0$ , then  $a(x-h)^2$  is positive, thus  $y = a(x-h)^2 + k$  is always a number larger than  $k$ . This means that when  $a > 0$ ,  $(h, k)$  is the lowest point on the graph and thus the parabola must open upwards, making  $(h, k)$  the vertex. A similar argument shows that if  $a < 0$ ,  $(h, k)$  is the highest point on the graph, so the parabola opens downwards, and  $(h, k)$  is also the vertex in this case.

Alternatively, we can apply the machinery in Section 2.6. Since the vertex of  $y = x^2$  is  $(0, 0)$ , we can determine the vertex of  $y = a(x-h)^2 + k$  by determining the final destination of  $(0, 0)$  as it is moved through each transformation. To obtain the formula  $f(x) = a(x-h)^2 + k$ , we start with  $g(x) = x^2$  and first define  $g_1(x) = ag(x) = ax^2$ . This results in a vertical scaling and/or reflection. (Just a scaling if  $a > 0$ . If  $a < 0$ , there is a reflection involved.) Since we multiply the output by  $a$ , we multiply the  $y$ -coordinates on the graph of  $g$  by  $a$ , so the point  $(0, 0)$  remains  $(0, 0)$  and remains the vertex. Next, we define  $g_2(x) = g_1(x-h) = a(x-h)^2$ . This induces a horizontal shift right or left  $h$  units (right if  $h > 0$ , left if  $h < 0$ ) moves the vertex, in either case, to  $(h, 0)$ . Finally,  $f(x) = g_2(x) + k = a(x-h)^2 + k$  which effects a vertical shift up or down  $k$  units (up if  $k > 0$ , down if  $k < 0$ ) resulting in the vertex moving from  $(h, 0)$  to  $(h, k)$ .

In addition to verifying Theorem 14, the arguments in the two preceding paragraphs have also shown us the role of the number  $a$  in the graphs of quadratic functions. The graph of  $y = a(x-h)^2 + k$  is a parabola ‘opening upwards’ if  $a > 0$ , and ‘opening downwards’ if  $a < 0$ . Moreover, the symmetry enjoyed by the graph of  $y = x^2$  about the  $y$ -axis is translated to a symmetry about the vertical line  $x = h$  which is the vertical line through the vertex. (You should use transformations to verify this!) This line is called the **axis of symmetry** of the parabola and is shown as the dashed line in Figure 3.23

Without a doubt, the standard form of a quadratic function, coupled with the machinery in Section 2.6, allows us to list the attributes of the graphs of such functions quickly and elegantly. What remains to be shown, however, is the fact that every quadratic function *can be written* in standard form. To convert a quadratic function given in general form into standard form, we employ the ancient rite of ‘Completing the Square’. We remind the reader how this is done in our next example.

### Example 53 Converting from general to standard form

Convert the functions below from general form to standard form. Find the vertex, axis of symmetry and any  $x$ - or  $y$ -intercepts. Graph each function and determine its range.

1.  $f(x) = x^2 - 4x + 3$ .
2.  $g(x) = 6 - x - x^2$

### SOLUTION

1. To convert from general form to standard form, we complete the square. First, we verify that the coefficient of  $x^2$  is 1. Next, we find the coefficient

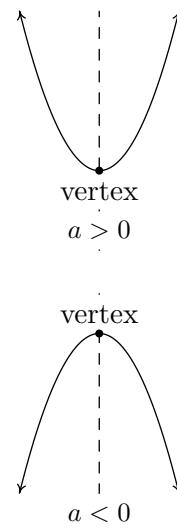
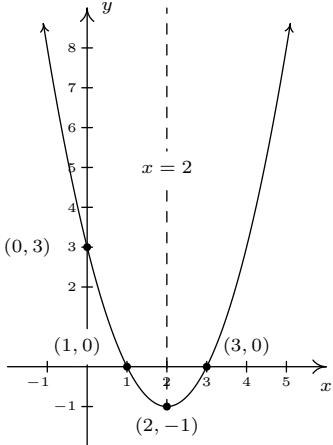


Figure 3.23: The axis of symmetry of a parabola

If you forget why we do what we do to complete the square, start with  $a(x - h)^2 + k$ , multiply it out, step by step, and then reverse the process.

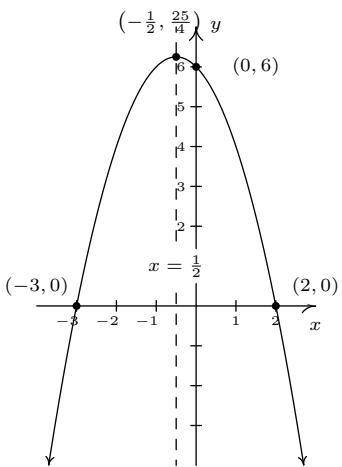
Figure 3.24:  $f(x) = x^2 - 4x + 3$ 

of  $x$ , in this case  $-4$ , and take half of it to get  $\frac{1}{2}(-4) = -2$ . This tells us that our target perfect square quantity is  $(x - 2)^2$ . To get an expression equivalent to  $(x - 2)^2$ , we need to add  $(-2)^2 = 4$  to the  $x^2 - 4x$  to create a perfect square trinomial, but to keep the balance, we must also subtract it. We collect the terms which create the perfect square and gather the remaining constant terms. Putting it all together, we get

$$\begin{aligned} f(x) &= x^2 - 4x + 3 && \text{(Compute } \frac{1}{2}(-4) = -2\text{.)} \\ &= (x^2 - 4x + 4 - 4) + 3 && \text{(Add and subtract } (-2)^2 = 4 \text{ to } (x^2 + 4x)\text{.)} \\ &= (x^2 - 4x + 4) - 4 + 3 && \text{(Group the perfect square trinomial.)} \\ &= (x - 2)^2 - 1 && \text{(Factor the perfect square trinomial.)} \end{aligned}$$

Of course, we can always check our answer by multiplying out  $f(x) = (x - 2)^2 - 1$  to see that it simplifies to  $f(x) = x^2 - 4x + 3$ . In the form  $f(x) = (x - 2)^2 - 1$ , we readily find the vertex to be  $(2, -1)$  which makes the axis of symmetry  $x = 2$ . To find the  $x$ -intercepts, we set  $y = f(x) = 0$ . We are spoiled for choice, since we have two formulas for  $f(x)$ . Since we recognize  $f(x) = x^2 - 4x + 3$  to be easily factorable, (experience pays off, here!) we proceed to solve  $x^2 - 4x + 3 = 0$ . Factoring gives  $(x - 3)(x - 1) = 0$  so that  $x = 3$  or  $x = 1$ . The  $x$ -intercepts are then  $(1, 0)$  and  $(3, 0)$ . To find the  $y$ -intercept, we set  $x = 0$ . Once again, the general form  $f(x) = x^2 - 4x + 3$  is easiest to work with here, and we find  $y = f(0) = 3$ . Hence, the  $y$ -intercept is  $(0, 3)$ . With the vertex, axis of symmetry and the intercepts, we get a pretty good graph without the need to plot additional points. We see that the range of  $f$  is  $[-1, \infty)$  and we are done. The graph of  $f$  is given in Figure 3.24.

2. To get started, we rewrite  $g(x) = 6 - x - x^2 = -x^2 - x + 6$  and note that the coefficient of  $x^2$  is  $-1$ , not  $1$ . This means our first step is to factor out the  $(-1)$  from both the  $x^2$  and  $x$  terms. We then follow the completing the square recipe as above.

Figure 3.25:  $g(x) = 6 - x - x^2$ 

$$\begin{aligned} g(x) &= -x^2 - x + 6 \\ &= (-1)(x^2 + x) + 6 && \text{(Factor the coefficient of } x^2 \text{ from } x^2 \text{ and } x\text{.)} \\ &= (-1)\left(x^2 + x + \frac{1}{4} - \frac{1}{4}\right) + 6 \\ &= (-1)\left(x^2 + x + \frac{1}{4}\right) + (-1)\left(-\frac{1}{4}\right) + 6 && \text{(Group the perfect square trinomial.)} \\ &= -\left(x + \frac{1}{2}\right)^2 + \frac{25}{4} \end{aligned}$$

From  $g(x) = -\left(x + \frac{1}{2}\right)^2 + \frac{25}{4}$ , we get the vertex to be  $(-\frac{1}{2}, \frac{25}{4})$  and the axis of symmetry to be  $x = -\frac{1}{2}$ . To get the  $x$ -intercepts, we opt to set the given formula  $g(x) = 6 - x - x^2 = 0$ . Solving, we get  $x = -3$  and  $x = 2$ , so the  $x$ -intercepts are  $(-3, 0)$  and  $(2, 0)$ . Setting  $x = 0$ , we find  $g(0) = 6$ , so the  $y$ -intercept is  $(0, 6)$ . Plotting these points gives us the graph in Figure 3.25. We see that the range of  $g$  is  $(-\infty, \frac{25}{4}]$ .



With Example 53 fresh in our minds, we are now in a position to show that every quadratic function can be written in standard form. We begin with  $f(x) = ax^2 + bx + c$ , assume  $a \neq 0$ , and complete the square in *complete* generality.

$$\begin{aligned}
 f(x) &= ax^2 + bx + c \\
 &= a\left(x^2 + \frac{b}{a}x\right) + c && \text{(Factor out coefficient of } x^2 \text{ from } x^2 \text{ and } x.) \\
 &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c \\
 &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - a\left(\frac{b^2}{4a^2}\right) + c && \text{(Group the perfect square trinomial.)} \\
 &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} && \text{(Factor and get a common denominator.)}
 \end{aligned}$$

Comparing this last expression with the standard form, we identify  $(x - h)$  with  $(x + \frac{b}{2a})$  so that  $h = -\frac{b}{2a}$ . Instead of memorizing the value  $k = \frac{4ac - b^2}{4a}$ , we see that  $f(-\frac{b}{2a}) = \frac{4ac - b^2}{4a}$ . As such, we have derived a vertex formula for the general form. We summarize both vertex formulas in the box at the top of the next page.

### Theorem 15 Vertex Formulas for Quadratic Functions

Suppose  $a, b, c, h$  and  $k$  are real numbers with  $a \neq 0$ .

- If  $f(x) = a(x - h)^2 + k$ , the vertex of the graph of  $y = f(x)$  is the point  $(h, k)$ .
- If  $f(x) = ax^2 + bx + c$ , the vertex of the graph of  $y = f(x)$  is the point  $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ .

There are two more results which can be gleaned from the completed-square form of the general form of a quadratic function,

$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

We have seen that the number  $a$  in the standard form of a quadratic function determines whether the parabola opens upwards (if  $a > 0$ ) or downwards (if  $a < 0$ ). We see here that this number  $a$  is none other than the coefficient of  $x^2$  in the general form of the quadratic function. In other words, it is the coefficient of  $x^2$  alone which determines this behavior – a result that is generalized in Section ???. The second treasure is a re-discovery of the **quadratic formula**.

**Theorem 16      The Quadratic Formula**

If  $a$ ,  $b$  and  $c$  are real numbers with  $a \neq 0$ , then the solutions to  $ax^2 + bx + c = 0$  are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Assuming the conditions of Equation 16, the solutions to  $ax^2 + bx + c = 0$  are precisely the zeros of  $f(x) = ax^2 + bx + c$ . Since

$$f(x) = ax^2 + bx + c = a \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}$$

the equation  $ax^2 + bx + c = 0$  is equivalent to

$$a \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a} = 0.$$

Solving gives

$$\begin{aligned} a \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a} &= 0 \\ a \left( x + \frac{b}{2a} \right)^2 &= -\frac{4ac - b^2}{4a} \\ \frac{1}{a} \left[ a \left( x + \frac{b}{2a} \right)^2 \right] &= \frac{1}{a} \left( \frac{b^2 - 4ac}{4a} \right) \\ \left( x + \frac{b}{2a} \right)^2 &= \frac{b^2 - 4ac}{4a^2} \\ x + \frac{b}{2a} &= \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{extract square roots} \\ x + \frac{b}{2a} &= \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

In our discussions of domain, we were warned against having negative numbers underneath the square root. Given that  $\sqrt{b^2 - 4ac}$  is part of the Quadratic Formula, we will need to pay special attention to the radicand  $b^2 - 4ac$ . It turns out that the quantity  $b^2 - 4ac$  plays a critical role in determining the nature of the solutions to a quadratic equation. It is given a special name.

**Definition 36      Discriminant**

If  $a$ ,  $b$  and  $c$  are real numbers with  $a \neq 0$ , then the **discriminant** of the quadratic equation  $ax^2 + bx + c = 0$  is the quantity  $b^2 - 4ac$ .

The discriminant ‘discriminates’ between the kinds of solutions we get from a quadratic equation. These cases, and their relation to the discriminant, are summarized below.

**Theorem 17 Discriminant Trichotomy**

Let  $a$ ,  $b$  and  $c$  be real numbers with  $a \neq 0$ .

- If  $b^2 - 4ac < 0$ , the equation  $ax^2 + bx + c = 0$  has no real solutions.
- If  $b^2 - 4ac = 0$ , the equation  $ax^2 + bx + c = 0$  has exactly one real solution.
- If  $b^2 - 4ac > 0$ , the equation  $ax^2 + bx + c = 0$  has exactly two real solutions.

The proof of Theorem 17 stems from the position of the discriminant in the quadratic equation, and is left as a good mental exercise for the reader. The next example exploits the fruits of all of our labor in this section thus far.

**Example 54 Computing and maximizing profit**

Recall that the profit (defined on page 73) for a product is defined by the equation Profit = Revenue – Cost, or  $P(x) = R(x) - C(x)$ . In Example 47 the weekly revenue, in dollars, made by selling  $x$  PortaBoy Game Systems was found to be  $R(x) = -1.5x^2 + 250x$  with the restriction (carried over from the price-demand function) that  $0 \leq x \leq 166$ . The cost, in dollars, to produce  $x$  PortaBoy Game Systems is given in Example 45 as  $C(x) = 80x + 150$  for  $x \geq 0$ .

1. Determine the weekly profit function  $P(x)$ .
2. Graph  $y = P(x)$ . Include the  $x$ - and  $y$ -intercepts as well as the vertex and axis of symmetry.
3. Interpret the zeros of  $P$ .
4. Interpret the vertex of the graph of  $y = P(x)$ .
5. Recall that the weekly price-demand equation for PortaBoys is  $p(x) = -1.5x + 250$ , where  $p(x)$  is the price per PortaBoy, in dollars, and  $x$  is the weekly sales. What should the price per system be in order to maximize profit?

**SOLUTION**

1. To find the profit function  $P(x)$ , we subtract

$$P(x) = R(x) - C(x) = (-1.5x^2 + 250x) - (80x + 150) = -1.5x^2 + 170x - 150.$$

Since the revenue function is valid when  $0 \leq x \leq 166$ ,  $P$  is also restricted to these values.

2. To find the  $x$ -intercepts, we set  $P(x) = 0$  and solve  $-1.5x^2 + 170x - 150 = 0$ . The mere thought of trying to factor the left hand side of this equation could do serious psychological damage, so we resort to the quadratic formula, Equation 16. Identifying  $a = -1.5$ ,  $b = 170$ , and  $c = -150$ , we obtain

$$\begin{aligned}
 x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 &= \frac{-170 \pm \sqrt{170^2 - 4(-1.5)(-150)}}{2(-1.5)} \\
 &= \frac{-170 \pm \sqrt{28000}}{-3} \\
 &= \frac{170 \pm 20\sqrt{70}}{3}
 \end{aligned}$$

We get two  $x$ -intercepts:  $\left(\frac{170-20\sqrt{70}}{3}, 0\right)$  and  $\left(\frac{170+20\sqrt{70}}{3}, 0\right)$ . To find the  $y$ -intercept, we set  $x = 0$  and find  $y = P(0) = -150$  for a  $y$ -intercept of  $(0, -150)$ . To find the vertex, we use the fact that  $P(x) = -1.5x^2 + 170x - 150$  is in the general form of a quadratic function and appeal to Equation 15. Substituting  $a = -1.5$  and  $b = 170$ , we get  $x = -\frac{170}{2(-1.5)} = \frac{170}{3}$ . To find the  $y$ -coordinate of the vertex, we compute  $P\left(\frac{170}{3}\right) = \frac{14000}{3}$  and find that our vertex is  $\left(\frac{170}{3}, \frac{14000}{3}\right)$ . The axis of symmetry is the vertical line passing through the vertex so it is the line  $x = \frac{170}{3}$ . To sketch a reasonable graph, we approximate the  $x$ -intercepts,  $(0.89, 0)$  and  $(112.44, 0)$ , and the vertex,  $(56.67, 4666.67)$ . (Note that in order to get the  $x$ -intercepts and the vertex to show up in the same picture, we had to scale the  $x$ -axis differently than the  $y$ -axis in Figure 3.26. This results in the left-hand  $x$ -intercept and the  $y$ -intercept being uncomfortably close to each other and to the origin in the picture.)

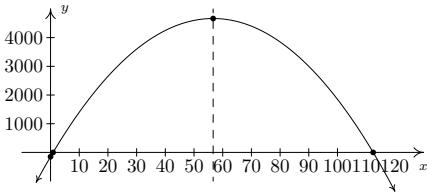


Figure 3.26: The graph of the profit function  $P(x)$

3. The zeros of  $P$  are the solutions to  $P(x) = 0$ , which we have found to be approximately 0.89 and 112.44. As we saw in Example 29, these are the ‘break-even’ points of the profit function, where enough product is sold to recover the cost spent to make the product. More importantly, we see from the graph that as long as  $x$  is between 0.89 and 112.44, the graph  $y = P(x)$  is above the  $x$ -axis, meaning  $y = P(x) > 0$  there. This means that for these values of  $x$ , a profit is being made. Since  $x$  represents the weekly sales of PortaBoy Game Systems, we round the zeros to positive integers and have that as long as 1, but no more than 112 game systems are sold weekly, the retailer will make a profit.
4. From the graph, we see that the maximum value of  $P$  occurs at the vertex, which is approximately  $(56.67, 4666.67)$ . As above,  $x$  represents the weekly sales of PortaBoy systems, so we can’t sell 56.67 game systems. Comparing  $P(56) = 4666$  and  $P(57) = 4666.5$ , we conclude that we will make a maximum profit of \$4666.50 if we sell 57 game systems.
5. In the previous part, we found that we need to sell 57 PortaBoys per week to maximize profit. To find the price per PortaBoy, we substitute  $x = 57$  into the price-demand function to get  $p(57) = -1.5(57) + 250 = 164.5$ . The price should be set at \$164.50.

Our next example is another classic application of quadratic functions.

**Example 55 Optimizing pasture dimensions**

Much to Donnie's surprise and delight, he inherits a large parcel of land in Ashtabula County from one of his (e)strange(d) relatives. The time is finally right for him to pursue his dream of farming alpaca. He wishes to build a rectangular pasture, and estimates that he has enough money for 200 linear feet of fencing material. If he makes the pasture adjacent to a stream (so no fencing is required on that side), what are the dimensions of the pasture which maximize the area? What is the maximum area? If an average alpaca needs 25 square feet of grazing area, how many alpaca can Donnie keep in his pasture?

**SOLUTION** It is always helpful to sketch the problem situation, so we do so in Figure 3.27.

We are tasked to find the dimensions of the pasture which would give a maximum area. We let  $w$  denote the width of the pasture and we let  $l$  denote the length of the pasture. Since the units given to us in the statement of the problem are feet, we assume  $w$  and  $l$  are measured in feet. The area of the pasture, which we'll call  $A$ , is related to  $w$  and  $l$  by the equation  $A = wl$ . Since  $w$  and  $l$  are both measured in feet,  $A$  has units of feet<sup>2</sup>, or square feet. We are given the total amount of fencing available is 200 feet, which means  $w + l + w = 200$ , or,  $l + 2w = 200$ . We now have two equations,  $A = wl$  and  $l + 2w = 200$ . In order to use the tools given to us in this section to *maximize*  $A$ , we need to use the information given to write  $A$  as a function of just *one* variable, either  $w$  or  $l$ . This is where we use the equation  $l + 2w = 200$ . Solving for  $l$ , we find  $l = 200 - 2w$ , and we substitute this into our equation for  $A$ . We get  $A = wl = w(200 - 2w) = 200w - 2w^2$ . We now have  $A$  as a function of  $w$ ,  $A(w) = 200w - 2w^2 = -2w^2 + 200w$ .

Before we go any further, we need to find the applied domain of  $A$  so that we know what values of  $w$  make sense in this problem situation. (Donnie would be very upset if, for example, we told him the width of the pasture needs to be  $-50$  feet.) Since  $w$  represents the width of the pasture,  $w > 0$ . Likewise,  $l$  represents the length of the pasture, so  $l = 200 - 2w > 0$ . Solving this latter inequality, we find  $w < 100$ . Hence, the function we wish to maximize is  $A(w) = -2w^2 + 200w$  for  $0 < w < 100$ . Since  $A$  is a quadratic function (of  $w$ ), we know that the graph of  $y = A(w)$  is a parabola. Since the coefficient of  $w^2$  is  $-2$ , we know that this parabola opens downwards. This means that there is a maximum value to be found, and we know it occurs at the vertex. Using the vertex formula, we find  $w = -\frac{200}{2(-2)} = 50$ , and  $A(50) = -2(50)^2 + 200(50) = 5000$ . Since  $w = 50$  lies in the applied domain,  $0 < w < 100$ , we have that the area of the pasture is maximized when the width is 50 feet. To find the length, we use  $l = 200 - 2w$  and find  $l = 200 - 2(50) = 100$ , so the length of the pasture is 100 feet. The maximum area is  $A(50) = 5000$ , or 5000 square feet. If an average alpaca requires 25 square feet of pasture, Donnie can raise  $\frac{5000}{25} = 200$  average alpaca.

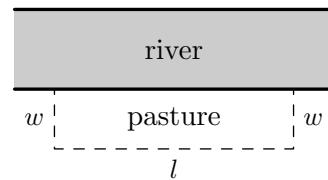


Figure 3.27: A diagram of pasture dimensions

We conclude this section with the graph of a more complicated absolute value function.

**Example 56 Graphing the absolute value of a quadratic function**

Graph  $f(x) = |x^2 - x - 6|$ .

**SOLUTION** Using the definition of absolute value, Definition 33, we have

$$f(x) = \begin{cases} -(x^2 - x - 6), & \text{if } x^2 - x - 6 < 0 \\ x^2 - x - 6, & \text{if } x^2 - x - 6 \geq 0 \end{cases}$$

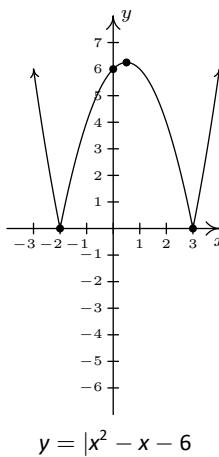
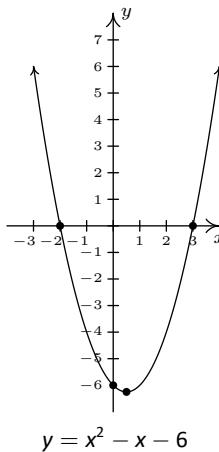


Figure 3.28: Obtaining the graph of  $f(x) = |x^2 - x - 6|$

The trouble is that we have yet to develop any analytic techniques to solve nonlinear inequalities such as  $x^2 - x - 6 < 0$ . You won't have to wait long; this is one of the main topics of Section ???. Nevertheless, we can attack this problem graphically. To that end, we graph  $y = g(x) = x^2 - x - 6$  using the intercepts and the vertex. To find the  $x$ -intercepts, we solve  $x^2 - x - 6 = 0$ . Factoring gives  $(x-3)(x+2) = 0$  so  $x = -2$  or  $x = 3$ . Hence,  $(-2, 0)$  and  $(3, 0)$  are  $x$ -intercepts. The  $y$ -intercept  $(0, -6)$  is found by setting  $x = 0$ . To plot the vertex, we find  $x = -\frac{b}{2a} = -\frac{-1}{2(1)} = \frac{1}{2}$ , and  $y = (\frac{1}{2})^2 - (\frac{1}{2}) - 6 = -\frac{25}{4} = -6.25$ . Plotting, we get the parabola seen below on the left. To obtain points on the graph of  $y = f(x) = |x^2 - x - 6|$ , we can take points on the graph of  $g(x) = x^2 - x - 6$  and apply the absolute value to each of the  $y$  values on the parabola. We see from the graph of  $g$  that for  $x \leq -2$  or  $x \geq 3$ , the  $y$  values on the parabola are greater than or equal to zero (since the graph is on or above the  $x$ -axis), so the absolute value leaves these portions of the graph alone. For  $x$  between  $-2$  and  $3$ , however, the  $y$  values on the parabola are negative. For example, the point  $(0, -6)$  on  $y = x^2 - x - 6$  would result in the point  $(0, |-6|) = (0, -(-6)) = (0, 6)$  on the graph of  $f(x) = |x^2 - x - 6|$ . Proceeding in this manner for all points with  $x$ -coordinates between  $-2$  and  $3$  results in the graph seen at the bottom of Figure 3.28.

If we take a step back and look at the graphs of  $g$  and  $f$  in the last example, we notice that to obtain the graph of  $f$  from the graph of  $g$ , we reflect a *portion* of the graph of  $g$  about the  $x$ -axis. We can see this analytically by substituting  $g(x) = x^2 - x - 6$  into the formula for  $f(x)$  and calling to mind Theorem 9 from Section 2.6.

$$f(x) = \begin{cases} -g(x), & \text{if } g(x) < 0 \\ g(x), & \text{if } g(x) \geq 0 \end{cases}$$

The function  $f$  is defined so that when  $g(x)$  is negative (i.e., when its graph is below the  $x$ -axis), the graph of  $f$  is its reflection across the  $x$ -axis. This is a general template to graph functions of the form  $f(x) = |g(x)|$ . From this perspective, the graph of  $f(x) = |x|$  can be obtained by reflecting the portion of the line  $g(x) = x$  which is below the  $x$ -axis back above the  $x$ -axis creating the characteristic 'V' shape.

# Exercises 3.3

## Problems

In Exercises 1 – 16, solve the equation.

1.  $|x| = 6$

2.  $|3x - 1| = 10$

3.  $|4 - x| = 7$

4.  $4 - |x| = 3$

5.  $2|5x + 1| - 3 = 0$

6.  $|7x - 1| + 2 = 0$

7.  $\frac{5 - |x|}{2} = 1$

8.  $\frac{2}{3}|5 - 2x| - \frac{1}{2} = 5$

9.  $|x| = x + 3$

10.  $|2x - 1| = x + 1$

11.  $4 - |x| = 2x + 1$

12.  $|x - 4| = x - 5$

13.  $|x| = x^2$

14.  $|x| = 12 - x^2$

15.  $|x^2 - 1| = 3$

Prove that if  $|f(x)| = |g(x)|$  then either  $f(x) = g(x)$  or  $f(x) = -g(x)$ . Use that result to solve the equations in Exercises 16 – 22.

16.  $|3x - 2| = |2x + 7|$

17.  $|3x + 1| = |4x|$

18.  $|1 - 2x| = |x + 1|$

19.  $|4 - x| - |x + 2| = 0$

20.  $|2 - 5x| = 5|x + 1|$

21.  $3|x - 1| = 2|x + 1|$

22. The International Silver Strings Submarine Band holds a bake sale each year to fund their trip to the National Sasquatch Convention. It has been determined that the cost in dollars of baking  $x$  cookies is  $C(x) = 0.1x + 25$  and that the demand function for their cookies is  $p = 10 - .01x$ . How many cookies should they bake in order to maximize their profit?

23. Using data from Bureau of Transportation Statistics, the average fuel economy  $F$  in miles per gallon for passenger cars in the US can be modelled by  $F(t) = -0.0076t^2 + 0.45t + 16$ ,  $0 \leq t \leq 28$ , where  $t$  is the number of years since 1980. Find and interpret the coordinates of the vertex of the graph of  $y = F(t)$ .

24. The temperature  $T$ , in degrees Fahrenheit,  $t$  hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

What is the warmest temperature of the day? When does this happen?

25. Suppose  $C(x) = x^2 - 10x + 27$  represents the costs, in hundreds, to produce  $x$  thousand pens. How many pens should be produced to minimize the cost? What is this minimum cost?

26. Skippy wishes to plant a vegetable garden along one side of his house. In his garage, he found 32 linear feet of fencing. Since one side of the garden will border the house, Skippy doesn't need fencing along that side. What are the dimensions of the garden which will maximize the area of the garden? What is the maximum area of the garden?

27. In the situation of Example 55, Donnie has a nightmare that one of his alpaca herd fell into the river and drowned. To avoid this, he wants to move his rectangular pasture away from the river. This means that all four sides of the pasture require fencing. If the total amount of fencing available is still 200 linear feet, what dimensions maximize the area of the pasture now? What is the maximum area? Assuming an average alpaca requires 25 square feet of pasture, how many alpaca can he raise now?

28. What is the largest rectangular area one can enclose with 14 inches of string?

29. The height of an object dropped from the roof of an eight story building is modelled by  $h(t) = -16t^2 + 64$ ,  $0 \leq t \leq 2$ . Here,  $h$  is the height of the object off the ground, in feet,  $t$  seconds after the object is dropped. How long before the object hits the ground?

30. The height  $h$  in feet of a model rocket above the ground  $t$  seconds after lift-off is given by  $h(t) = -5t^2 + 100t$ , for  $0 \leq t \leq 20$ . When does the rocket reach its maximum height above the ground? What is its maximum height?

31. Carl's friend Jason participates in the Highland Games. In one event, the hammer throw, the height  $h$  in feet of the hammer above the ground  $t$  seconds after Jason lets it go is modeled by  $h(t) = -16t^2 + 22.08t + 6$ . What is the hammer's maximum height? What is the hammer's total time in the air? Round your answers to two decimal places.

32. Assuming no air resistance or forces other than the Earth's gravity, the height above the ground at time  $t$  of a falling object is given by  $s(t) = -4.9t^2 + v_0t + s_0$  where  $s$  is in meters,  $t$  is in seconds,  $v_0$  is the object's initial velocity in meters per second and  $s_0$  is its initial position in meters.
- What is the applied domain of this function?
  - Discuss with your classmates what each of  $v_0 > 0$ ,  $v_0 = 0$  and  $v_0 < 0$  would mean.
  - Come up with a scenario in which  $s_0 < 0$ .
  - Let's say a slingshot is used to shoot a marble straight up from the ground ( $s_0 = 0$ ) with an initial velocity of 15 meters per second. What is the marble's maximum height above the ground? At what time will it hit the ground?
  - Now shoot the marble from the top of a tower which is 25 meters tall. When does it hit the ground?
  - What would the height function be if instead of shooting the marble up off of the tower, you were to shoot it straight DOWN from the top of the tower?
33. The two towers of a suspension bridge are 400 feet apart. The parabolic cable<sup>1</sup> attached to the tops of the towers is 10 feet above the point on the bridge deck that is midway between the towers. If the towers are 100 feet tall, find the height of the cable directly above a point of the bridge deck that is 50 feet to the right of the left-hand tower.
34. Graph  $f(x) = |1 - x^2|$
35. Find all of the points on the line  $y = 1 - x$  which are 2 units from  $(1, -1)$ .
36. Let  $L$  be the line  $y = 2x + 1$ . Find a function  $D(x)$  which measures the distance *squared* from a point on  $L$  to  $(0, 0)$ . Use this to find the point on  $L$  closest to  $(0, 0)$ .
37. With the help of your classmates, show that if a quadratic function  $f(x) = ax^2 + bx + c$  has two real zeros then the  $x$ -coordinate of the vertex is the midpoint of the zeros.
- In Exercises 38 – 44, solve the quadratic equation for the indicated variable.**
- $x^2 - 10y^2 = 0$  for  $x$
  - $y^2 - 4y = x^2 - 4$  for  $x$
  - $x^2 - mx = 1$  for  $x$
  - $y^2 - 3y = 4x$  for  $y$
  - $y^2 - 4y = x^2 - 4$  for  $y$
  - $-gt^2 + v_0t + s_0 = 0$  for  $t$  (Assume  $g \neq 0$ .)

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<sup>1</sup>The weight of the bridge deck forces the bridge cable into a parabola and a free hanging cable such as a power line does not form a parabola. We shall see in Exercise ?? in Section ?? what shape a free hanging cable makes.

# A: ANSWERS TO SELECTED PROBLEMS

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## Chapter 1

### Section 1.1

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$	$[-1, 5)$	
$\{x \mid 0 \leq x < 3\}$	$[0, 3)$	
$\{x \mid 2 < x \leq 7\}$	$(2, 7]$	
$\{x \mid -5 < x \leq 0\}$	$(-5, 0]$	
1. $\{x \mid -3 < x < 3\}$	$(-3, 3)$	
$\{x \mid 5 \leq x \leq 7\}$	$[5, 7]$	
$\{x \mid x \leq 3\}$	$(-\infty, 3]$	
$\{x \mid x < 9\}$	$(-\infty, 9)$	
$\{x \mid x > 4\}$	$(4, \infty)$	
$\{x \mid x \geq -3\}$	$[-3, \infty)$	

3.  $(-1, 1) \cup [0, 6] = (-1, 6]$   
 5.  $(-\infty, 0) \cap [1, 5] = \emptyset$   
 7.  $(-\infty, 5] \cap [5, 8) = \{5\}$   
 9.  $(-\infty, -1) \cup (-1, \infty)$   
 11.  $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$   
 13.  $(-\infty, -4) \cup (-4, 0) \cup (0, 4) \cup (4, \infty)$   
 15.  $(-\infty, \infty)$   
 17.  $(-\infty, 5] \cup \{6\}$   
 19.  $(-3, 3) \cup \{4\}$

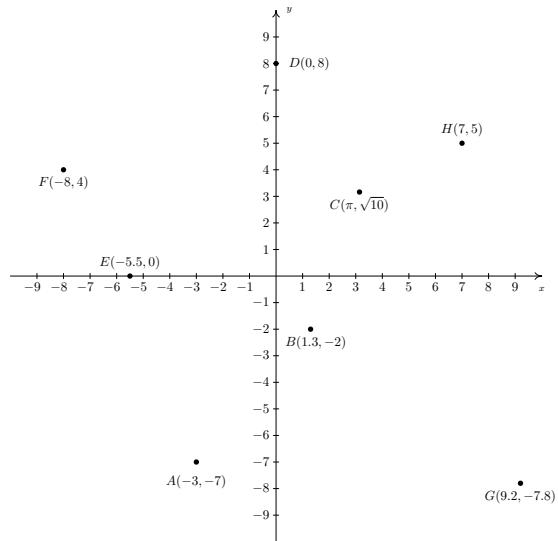
### Section 1.2

1. 6  
 3.  $\frac{2}{21}$   
 5.  $-\frac{1}{3}$   
 7.  $\frac{3}{5}$   
 9.  $-\frac{7}{8}$   
 11. 0  
 13.  $\frac{23}{9}$

15.  $-\frac{24}{7}$   
 17.  $\frac{243}{32}$   
 19.  $\frac{9}{22}$   
 21. 5  
 23.  $\frac{107}{27}$   
 25.  $\sqrt{10}$   
 27.  $\sqrt{7}$   
 29.  $-1$   
 31.  $\frac{15}{16}$   
 33.  $-\frac{385}{12}$

### Section 1.3

1. The required points  $A(-3, -7)$ ,  $B(1.3, -2)$ ,  $C(\pi, \sqrt{10})$ ,  $D(0, 8)$ ,  $E(-5.5, 0)$ ,  $F(-8, 4)$ ,  $G(9.2, -7.8)$ , and  $H(7, 5)$  are plotted in the Cartesian Coordinate Plane below.



3.  $d = 5, M = (-1, \frac{7}{2})$   
 5.  $d = \sqrt{26}, M = (1, \frac{3}{2})$   
 7.  $d = \sqrt{74}, M = (\frac{13}{10}, -\frac{13}{10})$   
 9.  $d = \sqrt{83}, M = (4\sqrt{5}, \frac{5\sqrt{3}}{2})$

11.  $(3 + \sqrt{7}, -1), (3 - \sqrt{7}, -1)$   
 13.  $(-1 + \sqrt{3}, 0), (-1 - \sqrt{3}, 0)$   
 15.  $(-3, -4)$ , 5 miles,  $(4, -4)$   
 17.  
 19.  
 21.

### Section 1.4

1. For  $z = 2 + 3i$  and  $w = 4i$   
 •  $z + w = 2 + 7i$

- $zw = -12 + 8i$
  - $z^2 = -5 + 12i$
  - $\frac{1}{z} = \frac{2}{13} - \frac{3}{13}i$
  - $\frac{z}{w} = \frac{3}{4} - \frac{1}{2}i$
  - $\frac{w}{z} = \frac{12}{13} + \frac{8}{13}i$
  - $\bar{z} = 2 - 3i$
  - $z\bar{z} = 13$
  - $(\bar{z})^2 = -5 - 12i$
11.  $7i$       15.  $-12$   
 13.  $-10$       17.  $3$   
 19.  $i^5 = i^4 \cdot i = 1 \cdot i = i$       23.  $i^{15} = (i^4)^3 \cdot i^3 = 1 \cdot (-i) = -i$   
 21.  $i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$       25.  $i^{117} = (i^4)^{29} \cdot i = 1 \cdot i = i$

3. For  $z = i$  and  $w = -1 + 2i$

- $z + w = -1 + 3i$
- $zw = -2 - i$
- $z^2 = -1$
- $\frac{1}{z} = -i$
- $\frac{z}{w} = \frac{2}{5} - \frac{1}{5}i$
- $\frac{w}{z} = 2 + i$
- $\bar{z} = -i$
- $z\bar{z} = 1$
- $(\bar{z})^2 = -1$

5. For  $z = 3 - 5i$  and  $w = 2 + 7i$

- $z + w = 5 + 2i$
- $zw = 41 + 11i$
- $z^2 = -16 - 30i$
- $\frac{1}{z} = \frac{3}{34} + \frac{5}{34}i$
- $\frac{z}{w} = -\frac{29}{53} - \frac{31}{53}i$
- $\frac{w}{z} = -\frac{29}{34} + \frac{31}{34}i$
- $\bar{z} = 3 + 5i$
- $z\bar{z} = 34$
- $(\bar{z})^2 = -16 + 30i$

7. For  $z = \sqrt{2} - i\sqrt{2}$  and  $w = \sqrt{2} + i\sqrt{2}$

- $z + w = 2\sqrt{2}$
- $zw = 4$
- $z^2 = -4i$
- $\frac{1}{z} = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}i$
- $\frac{z}{w} = -i$
- $\frac{w}{z} = i$
- $\bar{z} = \sqrt{2} + i\sqrt{2}$
- $z\bar{z} = 4$
- $(\bar{z})^2 = 4i$

9. For  $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$  and  $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

- $z + w = i\sqrt{3}$
- $zw = -1$
- $z^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
- $\frac{1}{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$
- $\frac{z}{w} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$
- $\frac{w}{z} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$
- $\bar{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

27.  $x = \frac{2 \pm i\sqrt{14}}{3}$

29.  $y = \pm 2, \pm i$

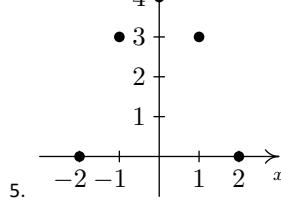
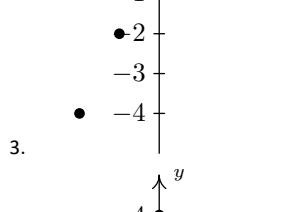
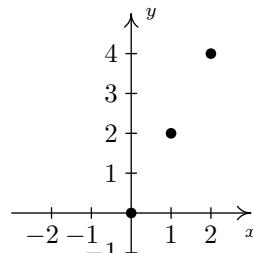
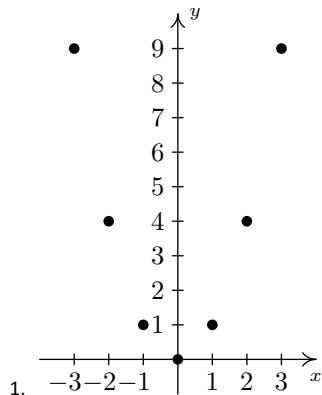
31.  $y = \pm \frac{3i\sqrt{2}}{2}$

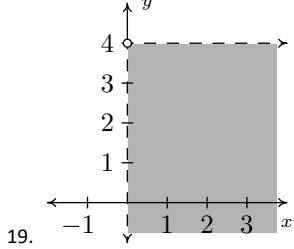
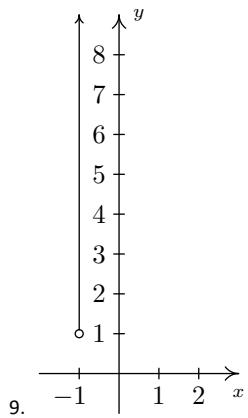
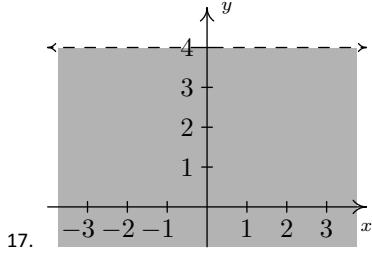
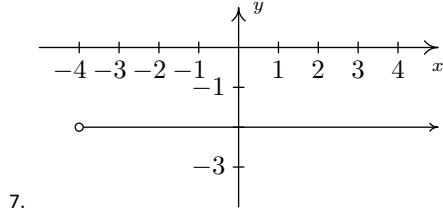
33.  $x = \frac{\sqrt{5} \pm i\sqrt{3}}{2}$

35.  $z = \pm 2, \pm 2i$

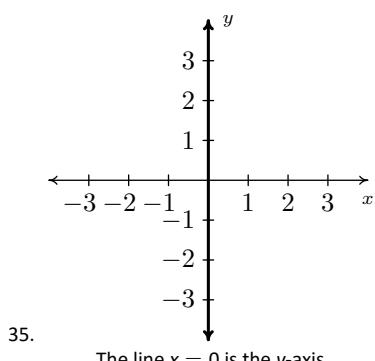
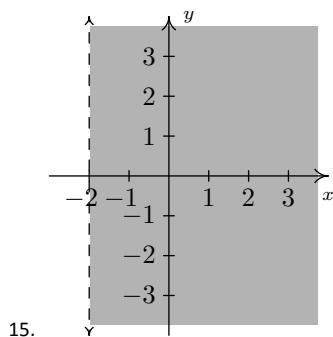
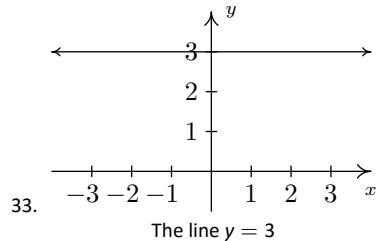
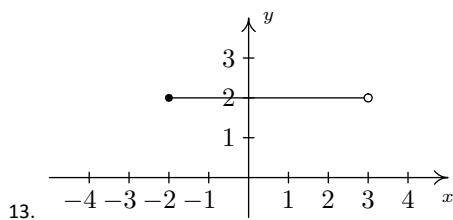
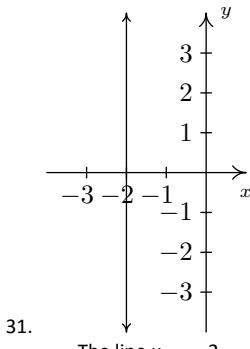
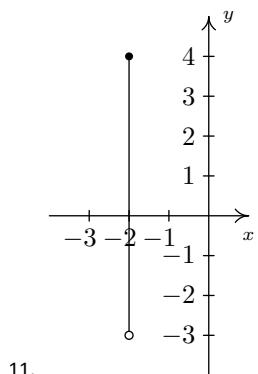
## Chapter 2

### Section 2.1





21.  $A = \{(-4, -1), (-2, 1), (0, 3), (1, 4)\}$   
 23.  $C = \{(2, y) \mid y > -3\}$   
 25.  $E = \{(x, 2) \mid -4 \leq x < 3\}$   
 27.  $G = \{(x, y) \mid x > -2\}$   
 29.  $I = \{(x, y) \mid x \geq 0, y \geq 0\}$



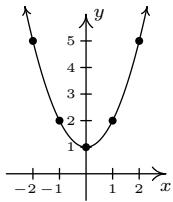
37.

39.

41. The graph has no  $x$ -intercepts

$y$ -intercept:  $(0, 1)$

$x$	$y$	$(x, y)$
-2	5	$(-2, 5)$
-1	2	$(-1, 2)$
0	1	$(0, 1)$
1	2	$(1, 2)$
2	5	$(2, 5)$



The graph is not symmetric about the  $x$ -axis (e.g.  $(2, 5)$  is on the graph but  $(2, -5)$  is not)

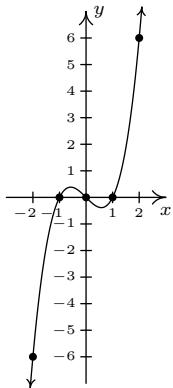
The graph is symmetric about the  $y$ -axis

The graph is not symmetric about the origin (e.g.  $(2, 5)$  is on the graph but  $(-2, -5)$  is not)

43.  $x$ -intercepts:  $(-1, 0), (0, 0), (1, 0)$

$y$ -intercept:  $(0, 0)$

$x$	$y$	$(x, y)$
-2	-6	$(-2, -6)$
-1	0	$(-1, 0)$
0	0	$(0, 0)$
1	0	$(1, 0)$
2	6	$(2, 6)$



The graph is not symmetric about the  $x$ -axis. (e.g.  $(2, 6)$  is on the graph but  $(2, -6)$  is not)

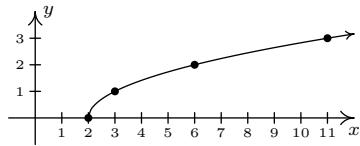
The graph is not symmetric about the  $y$ -axis. (e.g.  $(2, 6)$  is on the graph but  $(-2, 6)$  is not)

The graph is symmetric about the origin.

45.  $x$ -intercept:  $(2, 0)$

The graph has no  $y$ -intercepts

$x$	$y$	$(x, y)$
2	0	$(2, 0)$
3	1	$(3, 1)$
6	2	$(6, 2)$
11	3	$(11, 3)$



The graph is not symmetric about the  $x$ -axis (e.g.  $(3, 1)$  is on the graph but  $(3, -1)$  is not)

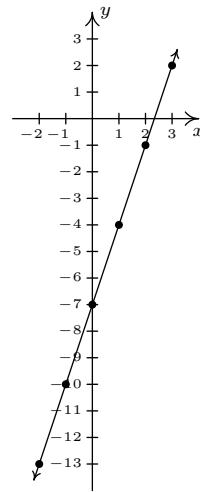
The graph is not symmetric about the  $y$ -axis (e.g.  $(3, 1)$  is on the graph but  $(-3, 1)$  is not)

The graph is not symmetric about the origin (e.g.  $(3, 1)$  is on the graph but  $(-3, -1)$  is not)

47.  $x$ -intercept:  $(\frac{7}{3}, 0)$

$y$ -intercept:  $(0, -7)$

$x$	$y$	$(x, y)$
-2	-13	$(-2, -13)$
-1	-10	$(-1, -10)$
0	-7	$(0, -7)$
1	-4	$(1, -4)$
2	-1	$(2, -1)$
3	2	$(3, 2)$



The graph is not symmetric about the  $x$ -axis (e.g.  $(3, 2)$  is on the graph but  $(3, -2)$  is not)

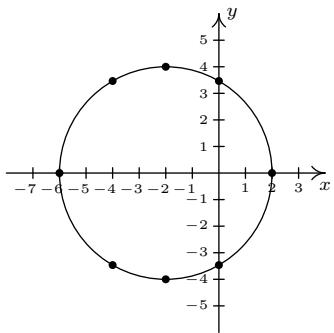
The graph is not symmetric about the  $y$ -axis (e.g.  $(3, 2)$  is on the graph but  $(-3, 2)$  is not)

The graph is not symmetric about the origin (e.g.  $(3, 2)$  is on the graph but  $(-3, -2)$  is not)

49.  $x$ -intercepts:  $(-6, 0), (2, 0)$

$y$ -intercepts:  $(0, \pm 2\sqrt{3})$

$x$	$y$	$(x, y)$
-6	0	$(-6, 0)$
-4	$\pm 2\sqrt{3}$	$(-4, \pm 2\sqrt{3})$
-2	$\pm 4$	$(-2, \pm 4)$
0	$\pm 2\sqrt{3}$	$(0, \pm 2\sqrt{3})$
2	0	$(2, 0)$



The graph is symmetric about the  $x$ -axis

The graph is not symmetric about the  $y$ -axis (e.g.  $(-6, 0)$  is on the graph but  $(6, 0)$  is not)

The graph is not symmetric about the origin (e.g.  $(-6, 0)$  is on the graph but  $(6, 0)$  is not)

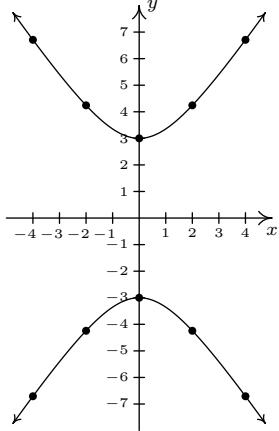
51.  $4y^2 - 9x^2 = 36$

Re-write as:  $y = \pm \frac{\sqrt{9x^2 + 36}}{2}$ .

The graph has no  $x$ -intercepts

$y$ -intercepts:  $(0, \pm 3)$

$x$	$y$	$(x, y)$
-4	$\pm 3\sqrt{5}$	$(-4, \pm 3\sqrt{5})$
-2	$\pm 3\sqrt{2}$	$(-2, \pm 3\sqrt{2})$
0	$\pm 3$	$(0, \pm 3)$
2	$\pm 3\sqrt{2}$	$(2, \pm 3\sqrt{2})$
4	$\pm 3\sqrt{5}$	$(4, \pm 3\sqrt{5})$



The graph is symmetric about the  $x$ -axis

The graph is symmetric about the  $y$ -axis

The graph is symmetric about the origin

53.

## Section 2.2

1. Function

domain =  $\{-3, -2, -1, 0, 1, 2, 3\}$   
range =  $\{0, 1, 4, 9\}$

3. Function

domain =  $\{-7, -3, 3, 4, 5, 6\}$   
range =  $\{0, 4, 5, 6, 9\}$

5. Not a function

7. Function

domain =  $\{x | x = 2^n \text{ for some whole number } n\}$   
range =  $\{y | y \geq 0 \text{ is an integer}\}$

9. Not a function

11. Function

domain =  $(-\infty, \infty)$   
range =  $[0, \infty)$

13. Function

domain =  $\{-4, -3, -2, -1, 0, 1\}$   
range =  $\{-1, 0, 1, 2, 3, 4\}$

15. Function

domain =  $(-\infty, \infty)$   
range =  $[1, \infty)$

17. Function

domain =  $[2, \infty)$   
range =  $[0, \infty)$

19. Not a function

21. Function

domain =  $[-2, \infty)$   
range =  $[-3, \infty)$

23. Function

domain =  $[-5, 4)$   
range =  $[-4, 4)$

25. Function

domain =  $(-\infty, \infty)$   
range =  $(-\infty, 4]$

27. Function

domain =  $[-2, \infty)$   
range =  $(-\infty, 3]$

29. Function

domain =  $(-\infty, 0] \cup (1, \infty)$   
range =  $(-\infty, 1] \cup \{2\}$

31. Not a function

33. Function

35. Function

37. Function

39. Not a function

41. Function

43. Not a function

45. Function

47. Not a function

49.

51.

53.

## Section 2.3

1.  $f(x) = \frac{2x+3}{4}$   
Domain:  $(-\infty, \infty)$

3.  $f(x) = 2\left(\frac{x}{4} + 3\right) = \frac{1}{2}x + 6$   
Domain:  $(-\infty, \infty)$

5.  $f(x) = \sqrt{2(x+3)} = \sqrt{2x+6}$   
Domain:  $[-3, \infty)$

7.  $f(x) = \frac{4}{\sqrt{x-13}}$   
Domain:  $[0, 169) \cup (169, \infty)$

9.  $f(x) = \frac{4}{\sqrt{x}} - 13$   
Domain:  $(0, \infty)$

11. For  $f(x) = 2x + 1$

- $f(3) = 7$
- $f(-1) = -1$
- $f\left(\frac{3}{2}\right) = 4$
- $f(4x) = 8x + 1$
- $4f(x) = 8x + 4$
- $f(-x) = -2x + 1$
- $f(x - 4) = 2x - 7$
- $f(x) - 4 = 2x - 3$
- $f(x^2) = 2x^2 + 1$

13. For  $f(x) = 2 - x^2$

- $f(3) = -7$
- $f(-1) = 1$
- $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
- $f(4x) = 2 - 16x^2$
- $4f(x) = 8 - 4x^2$
- $f(-x) = 2 - x^2$
- $f(x - 4) = -x^2 + 8x - 14$
- $f(x) - 4 = -x^2 - 2$
- $f(x^2) = 2 - x^4$

15. For  $f(x) = \frac{x}{x - 1}$

- $f(3) = \frac{3}{2}$
- $f(-1) = \frac{1}{2}$
- $f\left(\frac{3}{2}\right) = 3$
- $f(4x) = \frac{4x}{4x - 1}$
- $4f(x) = \frac{4x}{x - 1}$
- $f(-x) = \frac{x}{x + 1}$
- $f(x - 4) = \frac{x - 4}{x - 5}$
- $f(x) - 4 = \frac{x}{x - 1} - 4 = \frac{4 - 3x}{x - 1}$
- $f(x^2) = \frac{x^2}{x^2 - 1}$

17. For  $f(x) = 6$

- $f(3) = 6$
- $f(-1) = 6$
- $f\left(\frac{3}{2}\right) = 6$
- $f(4x) = 6$
- $4f(x) = 24$
- $f(-x) = 6$
- $f(x - 4) = 6$
- $f(x) - 4 = 2$
- $f(x^2) = 6$

19. For  $f(x) = 2x - 5$

- $f(2) = -1$
- $f(-2) = -9$
- $f(2a) = 4a - 5$
- $2f(a) = 4a - 10$
- $f(a + 2) = 2a - 1$
- $f(a) + f(2) = 2a - 6$
- $f\left(\frac{2}{a}\right) = \frac{4}{a} - 5 = \frac{4-5a}{a}$
- $\frac{f(a)}{2} = \frac{2a-5}{2}$
- $f(a + h) = 2a + 2h - 5$
- $f(-x) = -2x - 5$
- $f(x - 4) = 2x - 1$
- $f(x) - 4 = 2x - 9$
- $f(x^2) = 2x^2 - 5$

21. For  $f(x) = 2x^2 - 1$

- $f(2) = 7$
- $f(-2) = 7$
- $f(2a) = 8a^2 - 1$
- $2f(a) = 4a^2 - 2$
- $f(a + 2) = 2a^2 + 8a + 7$
- $f(a) + f(2) = 2a^2 + 6$
- $f\left(\frac{2}{a}\right) = \frac{8}{a^2} - 1 = \frac{8-a^2}{a^2}$
- $f(-x) = 2x^2 - 1$
- $f(x - 4) = 2x^2 - 17$
- $f(x) - 4 = 2x^2 - 5$
- $f(x^2) = 2x^4 - 1$

$$\bullet \frac{f(a)}{2} = \frac{2a^2 - 1}{2}$$

$$\bullet f(a + h) = 2a^2 + 4ah + 2h^2 - 1$$

23. For  $f(x) = \sqrt{2x + 1}$

- $f(2) = \sqrt{5}$
- $f(-2)$  is not real
- $f(2a) = \sqrt{4a + 1}$
- $2f(a) = 2\sqrt{2a + 1}$
- $f(a + 2) = \sqrt{2a + 5}$
- $f(a) + f(2) = \sqrt{2a + 1} + \sqrt{5}$
- $f\left(\frac{2}{a}\right) = \sqrt{\frac{4}{a} + 1} = \sqrt{\frac{a+4}{a}}$
- $\frac{f(a)}{2} = \frac{\sqrt{2a+1}}{2}$
- $f(a + h) = \sqrt{2a + 2h + 1}$

25. For  $f(x) = \frac{x}{2}$

- $f(2) = 1$
- $f(-2) = -1$
- $f(2a) = a$
- $2f(a) = a$
- $f(a + 2) = \frac{a+2}{2}$
- $f(a) + f(2) = \frac{a}{2} + 1 = \frac{a+2}{2}$
- $f\left(\frac{2}{a}\right) = \frac{1}{a}$
- $\frac{f(a)}{2} = \frac{a}{4}$
- $f(a + h) = \frac{a+h}{2}$

27. For  $f(x) = 2x - 1$ ,  $f(0) = -1$  and  $f(x) = 0$  when  $x = \frac{1}{2}$

29. For  $f(x) = 2x^2 - 6$ ,  $f(0) = -6$  and  $f(x) = 0$  when  $x = \pm\sqrt{3}$

31. For  $f(x) = \sqrt{x + 4}$ ,  $f(0) = 2$  and  $f(x) = 0$  when  $x = -4$

33. For  $f(x) = \frac{3}{4-x}$ ,  $f(0) = \frac{3}{4}$  and  $f(x)$  is never equal to 0

35. (a)  $f(-4) = 1$

(b)  $f(-3) = 2$

(c)  $f(3) = 0$

(d)  $f(3.001) = 1.999$

(e)  $f(-3.001) = 1.999$

(f)  $f(2) = \sqrt{5}$

37.  $(-\infty, \infty)$

39.  $(-\infty, -1) \cup (-1, \infty)$

41.  $(-\infty, \infty)$

43.  $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$

45.  $(-\infty, 3]$

47.  $[-3, \infty)$

49.  $\left[\frac{1}{3}, \infty\right)$

51.  $(-\infty, \infty)$

53.  $\left[\frac{1}{3}, 6\right) \cup (6, \infty)$

55.  $(-\infty, 8) \cup (8, \infty)$

57.  $(8, \infty)$

59.  $(-\infty, 8) \cup (8, \infty)$

61.  $[0, 5) \cup (5, \infty)$

63.  $A(3) = 9$ , so the area enclosed by a square with a side of length 3 inches is 9 square inches. The solutions to  $A(x) = 36$  are  $x = \pm 6$ . Since  $x$  is restricted to  $x > 0$ , we only keep  $x = 6$ . This means for the area enclosed by the square to be 36 square inches, the length of the side needs to be 6 inches. Since  $x$  represents a length,  $x > 0$ .
65.  $V(5) = 125$ , so the volume enclosed by a cube with a side of length 5 centimeters is 125 cubic centimeters. The solution to  $V(x) = 27$  is  $x = 3$ . This means for the volume enclosed by the cube to be 27 cubic centimeters, the length of the side needs to 3 centimeters. Since  $x$  represents a length,  $x > 0$ .
67.  $V(3) = 36\pi$ , so the volume enclosed by a sphere with radius 3 feet is  $36\pi$  cubic feet. The solution to  $V(r) = \frac{32\pi}{3}$  is  $r = 2$ . This means for the volume enclosed by the sphere to be  $\frac{32\pi}{3}$  cubic feet, the radius needs to 2 feet. Since  $r$  represents a radius (length),  $r > 0$ .
69.  $T(0) = 3$ , so at 6 AM (0 hours after 6 AM), it is  $3^\circ$  Fahrenheit.  $T(6) = 33$ , so at noon (6 hours after 6 AM), the temperature is  $33^\circ$  Fahrenheit.  $T(12) = 27$ , so at 6 PM (12 hours after 6 AM), it is  $27^\circ$  Fahrenheit.
71.  $F(0) = 16.00$ , so in 1980 (0 years after 1980), the average fuel economy of passenger cars in the US was 16.00 miles per gallon.  $F(14) = 20.81$ , so in 1994 (14 years after 1980), the average fuel economy of passenger cars in the US was 20.81 miles per gallon.  $F(28) = 22.64$ , so in 2008 (28 years after 1980), the average fuel economy of passenger cars in the US was 22.64 miles per gallon.
73. (a)  $C(20) = 300$ . It costs \$300 for 20 copies of the book.  
(b)  $C(50) = 675$ , so it costs \$675 for 50 copies of the book.  
 $C(51) = 612$ , so it costs \$612 for 51 copies of the book.  
(c) 56 books.
75. (a)  $C(750) = 25$ , so it costs \$25 to talk 750 minutes per month with this plan.  
(b) Since  $20 \text{ hours} = 1200 \text{ minutes}$ , we substitute  $m = 1200$  and get  $C(1200) = 45$ . It costs \$45 to talk 20 hours per month with this plan.  
(c) It costs \$25 for up to 1000 minutes and 10 cents per minute for each minute over 1000 minutes.
- 77.
- ### Section 2.4
1. For  $f(x) = 3x + 1$  and  $g(x) = 4 - x$
- $(f+g)(2) = 9$  •  $\left(\frac{f}{g}\right)(0) = \frac{1}{4}$
  - $(f-g)(-1) = -7$
  - $(g-f)(1) = -1$
  - $(fg)\left(\frac{1}{2}\right) = \frac{35}{4}$  •  $\left(\frac{g}{f}\right)(-2) = -\frac{6}{5}$
3. For  $f(x) = x^2 - x$  and  $g(x) = 12 - x^2$
- $(f+g)(2) = 10$  •  $\left(\frac{f}{g}\right)(0) = 0$
  - $(f-g)(-1) = -9$
  - $(g-f)(1) = 11$
  - $(fg)\left(\frac{1}{2}\right) = -\frac{47}{16}$  •  $\left(\frac{g}{f}\right)(-2) = \frac{4}{3}$
5. For  $f(x) = \sqrt{x+3}$  and  $g(x) = 2x - 1$
- $(f+g)(2) = 3 + \sqrt{5}$  •  $\left(\frac{f}{g}\right)(0) = -\sqrt{3}$
  - $(f-g)(-1) = 3 + \sqrt{2}$
  - $(g-f)(1) = -1$
  - $(fg)\left(\frac{1}{2}\right) = 0$  •  $\left(\frac{g}{f}\right)(-2) = -5$
7. For  $f(x) = 2x$  and  $g(x) = \frac{1}{2x+1}$
- $(f+g)(2) = \frac{21}{5}$
  - $(f-g)(-1) = -1$
  - $(g-f)(1) = -\frac{5}{3}$
  - $(fg)\left(\frac{1}{2}\right) = \frac{1}{2}$
  - $\left(\frac{f}{g}\right)(0) = 0$
  - $\left(\frac{g}{f}\right)(-2) = \frac{1}{12}$
9. For  $f(x) = x^2$  and  $g(x) = \frac{1}{x^2}$
- $(f+g)(2) = \frac{17}{4}$
  - $(f-g)(-1) = 0$
  - $(g-f)(1) = 0$
  - $(fg)\left(\frac{1}{2}\right) = 1$
  - $\left(\frac{f}{g}\right)(0)$  is undefined.
  - $\left(\frac{g}{f}\right)(-2) = \frac{1}{16}$
11. For  $f(x) = 2x + 1$  and  $g(x) = x - 2$
- $(f+g)(x) = 3x - 1$  Domain:  $(-\infty, \infty)$
  - $(f-g)(x) = x + 3$  Domain:  $(-\infty, \infty)$
  - $(fg)(x) = 2x^2 - 3x - 2$  Domain:  $(-\infty, \infty)$
  - $\left(\frac{f}{g}\right)(x) = \frac{2x+1}{x-2}$  Domain:  $(-\infty, 2) \cup (2, \infty)$
13. For  $f(x) = x^2$  and  $g(x) = 3x - 1$
- $(f+g)(x) = x^2 + 3x - 1$  Domain:  $(-\infty, \infty)$
  - $(f-g)(x) = x^2 - 3x + 1$  Domain:  $(-\infty, \infty)$
  - $(fg)(x) = 3x^3 - x^2$  Domain:  $(-\infty, \infty)$
  - $\left(\frac{f}{g}\right)(x) = \frac{x^2}{3x-1}$  Domain:  $(-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty)$
15. For  $f(x) = x^2 - 4$  and  $g(x) = 3x + 6$
- $(f+g)(x) = x^2 + 3x + 2$  Domain:  $(-\infty, \infty)$
  - $(f-g)(x) = x^2 - 3x - 10$  Domain:  $(-\infty, \infty)$
  - $(fg)(x) = 3x^3 + 6x^2 - 12x - 24$  Domain:  $(-\infty, \infty)$
  - $\left(\frac{f}{g}\right)(x) = \frac{x-2}{3}$  Domain:  $(-\infty, -2) \cup (-2, \infty)$
17. For  $f(x) = \frac{x}{2}$  and  $g(x) = \frac{2}{x}$
- $(f+g)(x) = \frac{x^2+4}{2x}$  Domain:  $(-\infty, 0) \cup (0, \infty)$
  - $(f-g)(x) = \frac{x^2-4}{2x}$  Domain:  $(-\infty, 0) \cup (0, \infty)$
  - $(fg)(x) = 1$  Domain:  $(-\infty, 0) \cup (0, \infty)$
  - $\left(\frac{f}{g}\right)(x) = \frac{x^2}{4}$  Domain:  $(-\infty, 0) \cup (0, \infty)$
19. For  $f(x) = x$  and  $g(x) = \sqrt{x+1}$
- $(f+g)(x) = x + \sqrt{x+1}$  Domain:  $[-1, \infty)$
  - $(f-g)(x) = x - \sqrt{x+1}$  Domain:  $[-1, \infty)$
  - $(fg)(x) = x\sqrt{x+1}$  Domain:  $[-1, \infty)$
  - $\left(\frac{f}{g}\right)(x) = \frac{x}{\sqrt{x+1}}$  Domain:  $(-1, \infty)$
21. 2
23. 0
25.  $-2x - h + 2$
27.  $-2x - h + 1$
29.  $m$
31.  $\frac{-2}{x(x+h)}$
33.  $\frac{-(2x+h)}{x^2(x+h)^2}$

35. 
$$\frac{-4}{(4x-3)(4x+4h-3)}$$

37. 
$$\frac{-9}{(x-9)(x+h-9)}$$

39. 
$$\frac{1}{\sqrt{x+h-9} + \sqrt{x-9}}$$

41. 
$$\frac{-4}{\sqrt{-4x-4h+5} + \sqrt{-4x+5}}$$

43. 
$$\frac{a}{\sqrt{ax+ah+b} + \sqrt{ax+b}}$$

45. 
$$\frac{1}{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}}$$

47. •  $C(0) = 100$ , so the fixed costs are \$100.  
 •  $\bar{C}(10) = 20$ , so when 10 bottles of tonic are produced, the cost per bottle is \$20.  
 •  $p(5) = 30$ , so to sell 5 bottles of tonic, set the price at \$30 per bottle.  
 •  $R(x) = -x^2 + 35x, 0 \leq x \leq 35$   
 •  $P(x) = -x^2 + 25x - 100, 0 \leq x \leq 35$   
 •  $P(x) = 0$  when  $x = 5$  and  $x = 20$ . These are the 'break even' points, so selling 5 bottles of tonic or 20 bottles of tonic will guarantee the revenue earned exactly recoups the cost of production.  
 •  $C(0) = 36$ , so the daily fixed costs are \$36.  
 •  $\bar{C}(10) = 6.6$ , so when 10 pies are made, the cost per pie is \$6.60.  
 •  $p(5) = 9.5$ , so to sell 5 pies a day, set the price at \$9.50 per pie.  
 •  $R(x) = -0.5x^2 + 12x, 0 \leq x \leq 24$   
 •  $P(x) = -0.5x^2 + 9x - 36, 0 \leq x \leq 24$   
 •  $P(x) = 0$  when  $x = 6$  and  $x = 12$ . These are the 'break even' points, so selling 6 pies or 12 pies a day will guarantee the revenue earned exactly recoups the cost of production.

51.  $(f+g)(-3) = 2$

53.  $(fg)(-1) = 0$

55.  $(g-f)(3) = 3$

57.  $\left(\frac{f}{g}\right)(-2)$  does not exist

59.  $\left(\frac{f}{g}\right)(2) = 4$

61.  $\left(\frac{g}{f}\right)(3) = -2$

## Section 2.5

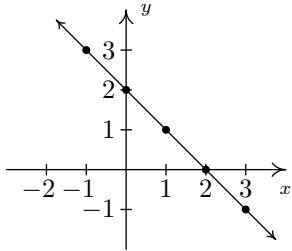
1.  $f(x) = 2 - x$

Domain:  $(-\infty, \infty)$

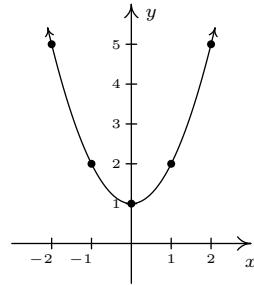
x-intercept: (2, 0)

y-intercept: (0, 2)

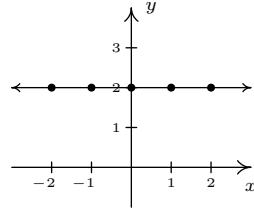
No symmetry



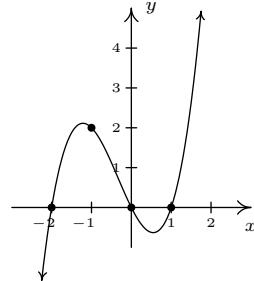
3.  $f(x) = x^2 + 1$   
 Domain:  $(-\infty, \infty)$   
 x-intercept: None  
 y-intercept: (0, 1)  
 Even



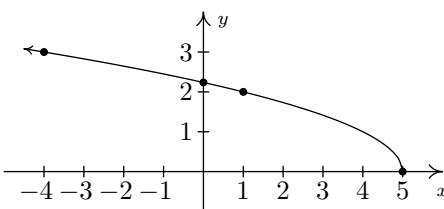
5.  $f(x) = 2$   
 Domain:  $(-\infty, \infty)$   
 x-intercept: None  
 y-intercept: (0, 2)  
 Even



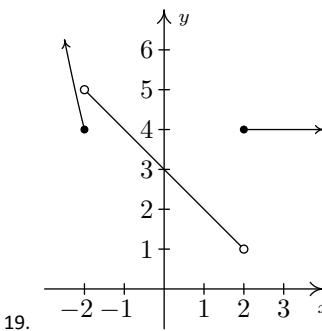
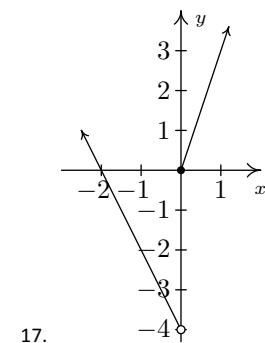
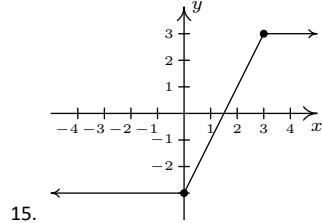
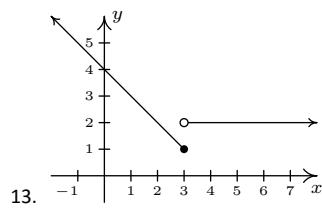
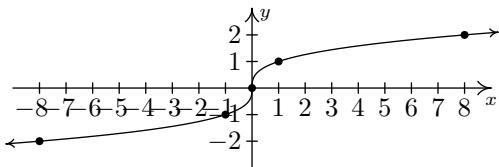
7.  $f(x) = x(x-1)(x+2)$   
 Domain:  $(-\infty, \infty)$   
 x-intercepts: (-2, 0), (0, 0), (1, 0)  
 y-intercept: (0, 0)  
 No symmetry



9.  $f(x) = \sqrt{5-x}$   
 Domain:  $(-\infty, 5]$   
 x-intercept: (5, 0)  
 y-intercept:  $(0, \sqrt{5})$   
 No symmetry



11.  $f(x) = \sqrt[3]{x}$   
 Domain:  $(-\infty, \infty)$   
 x-intercept: (0, 0)  
 y-intercept: (0, 0)  
 Odd



21. odd  
23. even  
25. even  
27. odd  
29. even  
31. neither  
33. even and odd  
35. even  
37. neither  
39. odd  
41. even  
43.  $[-5, 4]$   
45.  $x = -3$

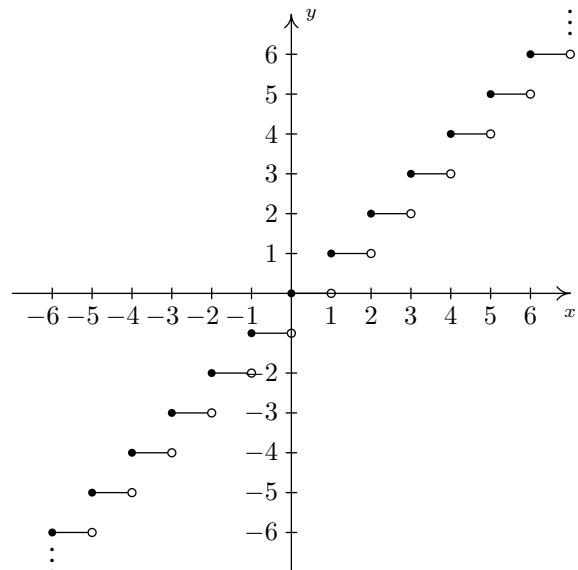
47.  $(0, -1)$   
49.  $[-4, -1] \cup [1, 3]$   
51. neither  
53.  $[-3, 0], [2, 3]$

55.  $f(0) = -1$   
57.  $f(-5) = -5$   
59.  $[-5, 5)$   
61.  $x = -2$   
63.  $(0, 0)$   
65.  $[-4, 0] \cup \{4\}$   
67. neither  
69.  $[-4, -2], (2, 4]$   
71.  $f(-2) = -5, f(2) = 3$   
73.  $f(-2) = -5$   
75. No absolute maximum  
No absolute minimum  
Local maximum at  $(0, 0)$   
Local minimum at  $(1.60, -3.28)$   
Increasing on  $(-\infty, 0], [1.60, \infty)$   
Decreasing on  $[0, 1.60]$

77. Absolute maximum  $f(2.12) \approx 4.50$   
Absolute minimum  $f(-2.12) \approx -4.50$   
Local maximum  $(2.12, 4.50)$   
Local minimum  $(-2.12, -4.50)$   
Increasing on  $[-2.12, 2.12]$   
Decreasing on  $[-3, -2.12], [2.12, 3]$

79.  $(f + g)(1) = 5$       83.  $(fg)(1) = 6$   
81.  $(g - f)(2) = 0$       85.  $\left(\frac{g}{f}\right)(2) = 1$

87.  $h(15) = 6$ , so the Saquatch is 6 feet tall when she is 15 years old.  
89.  $h$  is constant on  $[30, 45]$ . This means the Sasquatch's height is constant (at 8 feet) for these years.



The graph of  $f(x) = [x]$ .

93.  
95.  
97.

99.

**Section 2.6**

1.  $(2, 0)$

3.  $(2, -4)$

5.  $(2, -9)$

7.  $(2, 3)$

9.  $(5, -2)$

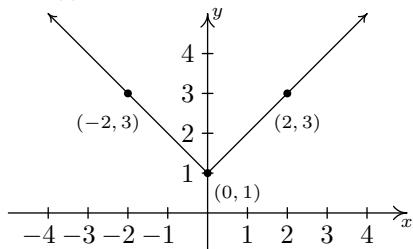
11.  $(2, 13)$

13.  $(2, -\frac{3}{2})$

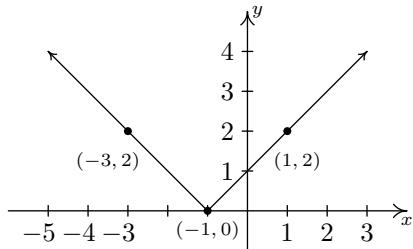
15.  $(-1, -7)$

17.  $(\frac{2}{3}, -2)$

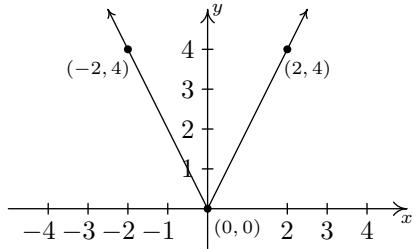
19.  $y = f(x) + 1$



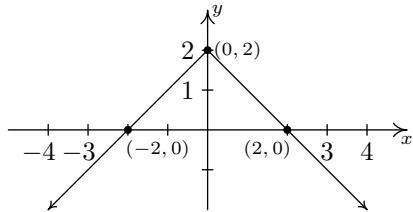
21.  $y = f(x + 1)$



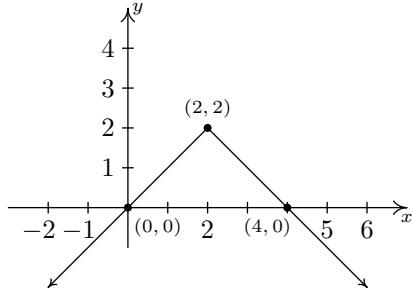
23.  $y = 2f(x)$



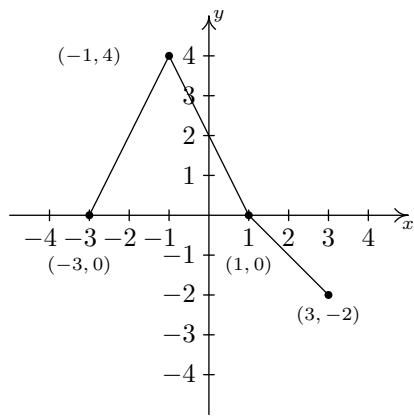
25.  $y = 2 - f(x)$



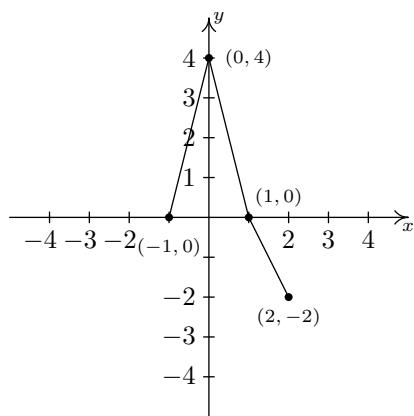
27.  $y = 2 - f(2 - x)$



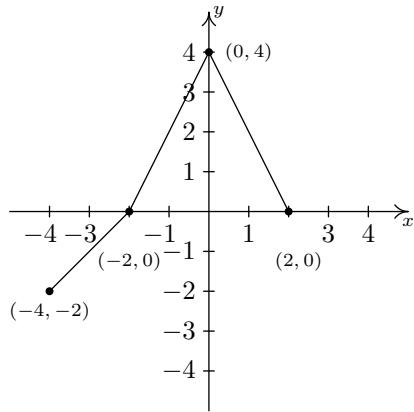
29.  $y = f(x + 1)$



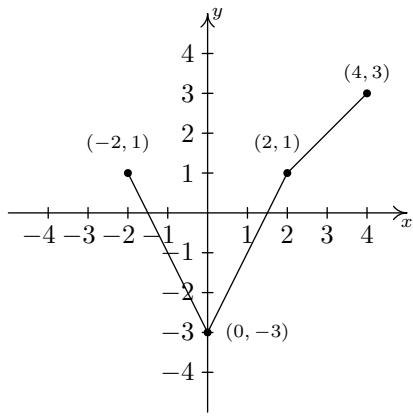
31.  $y = f(2x)$



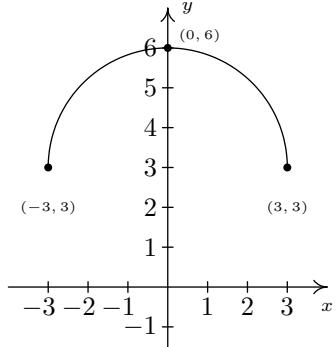
33.  $y = f(-x)$



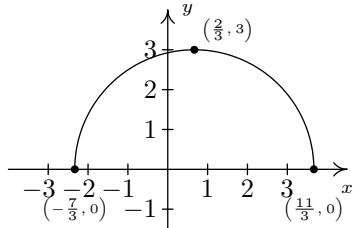
35.  $y = 1 - f(x)$



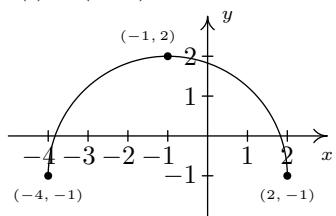
37.  $g(x) = f(x) + 3$



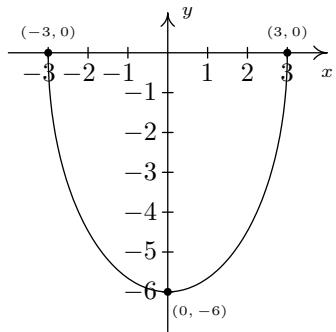
39.  $j(x) = f\left(x - \frac{2}{3}\right)$



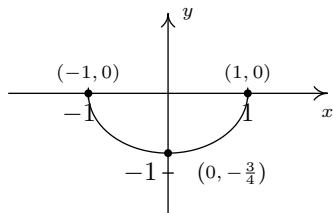
41.  $b(x) = f(x + 1) - 1$



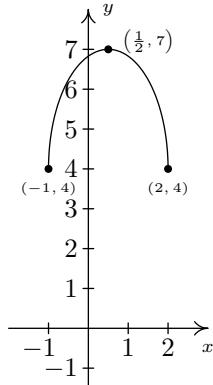
43.  $d(x) = -2f(x)$



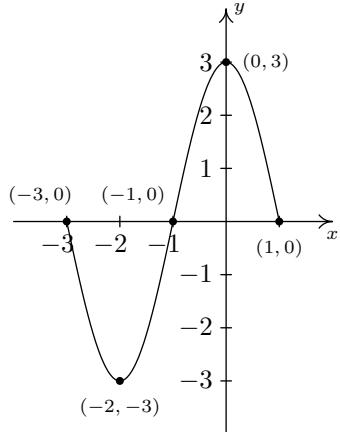
45.  $m(x) = -\frac{1}{4}f(3x)$



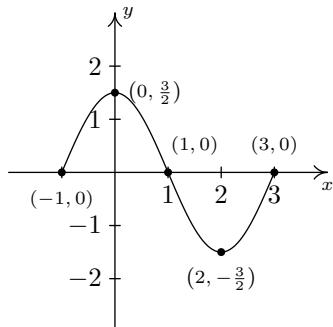
47.  $p(x) = 4 + f(1 - 2x)$



49.  $y = S_1(x) = S(x + 1)$



51.  $y = S_3(x) = \frac{1}{2}S_2(x) = \frac{1}{2}S(-x + 1)$



53.  $g(x) = \sqrt{x - 2} - 3$

55.  $g(x) = -\sqrt{x} + 1$

57.  $g(x) = \sqrt{-(x + 1)} + 2 = \sqrt{-x - 1} + 2$

59.  $g(x) = 2(\sqrt{x + 3} - 4) = 2\sqrt{x + 3} - 8$

61.  $g(x) = \sqrt{2(x - 3)} + 1 = \sqrt{2x - 6} + 1$

63.

65.

67. The same thing as reflecting it across the  $x$ -axis.

69. The same thing as reflecting it across the  $y$ -axis.

71.

## Chapter 3

### Section 3.1

1.  $y + 1 = 3(x - 3)$   
 $y = 3x - 10$

3.  $y + 1 = -(x + 7)$   
 $y = -x - 8$

5.  $y - 4 = -\frac{1}{5}(x - 10)$   
 $y = -\frac{1}{5}x + 6$

7.  $y - 117 = 0$   
 $y = 117$

9.  $y - 2\sqrt{3} = -5(x - \sqrt{3})$   
 $y = -5x + 7\sqrt{3}$

11.  $y = -\frac{5}{3}x$

13.  $y = \frac{8}{5}x - 8$

15.  $y = 5$

17.  $y = -\frac{5}{4}x + \frac{11}{8}$

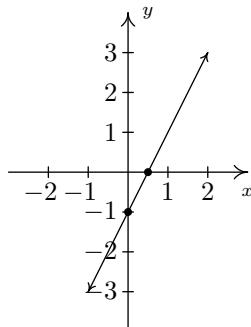
19.  $y = -x$

21.  $f(x) = 2x - 1$

slope:  $m = 2$

$y$ -intercept:  $(0, -1)$

$x$ -intercept:  $(\frac{1}{2}, 0)$

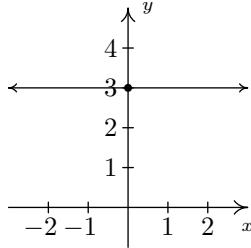


23.  $f(x) = 3$

slope:  $m = 0$

$y$ -intercept:  $(0, 3)$

$x$ -intercept: none

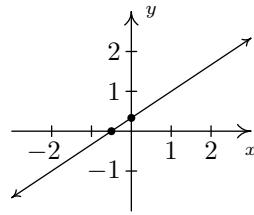


25.  $f(x) = \frac{2}{3}x + \frac{1}{3}$

slope:  $m = \frac{2}{3}$

$y$ -intercept:  $(0, \frac{1}{3})$

$x$ -intercept:  $(-\frac{1}{2}, 0)$



27.  $(-1, -1)$  and  $(\frac{11}{5}, \frac{27}{5})$

29.  $E(t) = 360t, t \geq 0$ .

31.  $C(t) = 80t + 50, 0 \leq t \leq 8$

33.  $C(p) = 0.035p + 1.5$  The slope 0.035 means it costs 3.5¢ per page.  $C(0) = 1.5$  means there is a fixed, or start-up, cost of \$1.50 to make each book.

35. (a)  $F(C) = \frac{9}{5}C + 32$

(b)  $C(F) = \frac{5}{9}(F - 32) = \frac{5}{9}F - \frac{160}{9}$

(c)  $F(-40) = -40 = C(-40)$ .

37.

39.  $C(p) = \begin{cases} 6p + 1.5 & \text{if } 1 \leq p \leq 5 \\ 5.5p & \text{if } p \geq 6 \end{cases}$

41.  $C(m) = \begin{cases} 10 & \text{if } 0 \leq m \leq 500 \\ 10 + 0.15(m - 500) & \text{if } m > 500 \end{cases}$

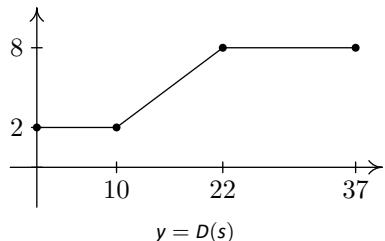
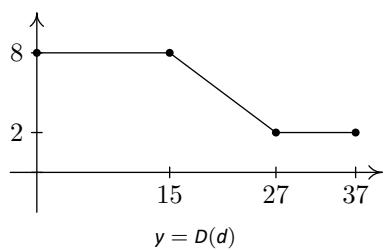
43. (a)

$$D(d) = \begin{cases} 8 & \text{if } 0 \leq d \leq 15 \\ -\frac{1}{2}d + \frac{31}{2} & \text{if } 15 \leq d \leq 27 \\ 2 & \text{if } 27 \leq d \leq 37 \end{cases}$$

(b)

$$D(s) = \begin{cases} 2 & \text{if } 0 \leq s \leq 10 \\ \frac{1}{2}s - 3 & \text{if } 10 \leq s \leq 22 \\ 8 & \text{if } 22 \leq s \leq 37 \end{cases}$$

(c)



45.  $\frac{\frac{1}{5} - \frac{1}{1}}{5 - 1} = -\frac{1}{5}$

47.  $\frac{3^2 - (-3)^2}{3 - (-3)} = 0$

49.  $\frac{(3(2)^2 + 2(2) - 7) - (3(-4)^2 + 2(-4) - 7)}{2 - (-4)} = -4$

51.  $\frac{-1}{x(x+h)}$

53.  $6x + 3h + 2$

55. (a)  $T(4) = 56$ , so at 10 AM (4 hours after 6 AM), it is  $56^{\circ}\text{F}$ .  
 $T(8) = 64$ , so at 2 PM (8 hours after 6 AM), it is  $64^{\circ}\text{F}$ .  
 $T(12) = 56$ , so at 6 PM (12 hours after 6 AM), it is  $56^{\circ}\text{F}$ .
- (b) The average rate of change is  $\frac{T(8)-T(4)}{8-4} = 2$ . Between 10 AM and 2 PM, the temperature increases, on average, at a rate of  $2^{\circ}\text{F}$  per hour.
- (c) The average rate of change is  $\frac{T(12)-T(8)}{12-8} = -2$ . Between 2 PM and 6 PM, the temperature decreases, on average, at a rate of  $2^{\circ}\text{F}$  per hour.
- (d) The average rate of change is  $\frac{T(12)-T(4)}{12-4} = 0$ . Between 10 AM and 6 PM, the temperature, on average, remains constant.

57.

59.  $y = 3x$

61.  $y = \frac{2}{3}x - 4$

63.  $y = -2$

65.  $y = -3x$

67.  $y = -\frac{3}{2}x + 9$

69.  $x = 3$

71.

73.

### Section 3.2

1.  $x = -6$  or  $x = 6$

3.  $x = -3$  or  $x = 11$

5.  $x = -\frac{1}{2}$  or  $x = \frac{1}{10}$

7.  $x = -3$  or  $x = 3$

9.  $x = -\frac{3}{2}$

11.  $x = 1$

13.  $x = -1$ ,  $x = 0$  or  $x = 1$

15.  $x = -2$  or  $x = 2$

17.  $x = -\frac{1}{7}$  or  $x = 1$

19.  $x = 1$

21.  $x = \frac{1}{5}$  or  $x = 5$

23.  $f(x) = |x| + 4$

No zeros

No x-intercepts

y-intercept  $(0, 4)$

Domain  $(-\infty, \infty)$

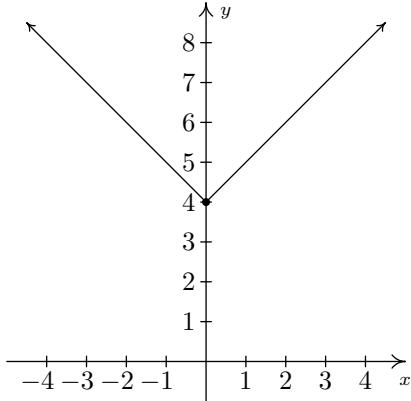
Range  $[4, \infty)$

Decreasing on  $(-\infty, 0]$

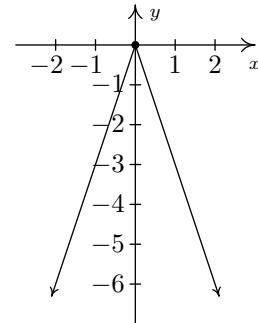
Increasing on  $[0, \infty)$

Relative and absolute minimum at  $(0, 4)$

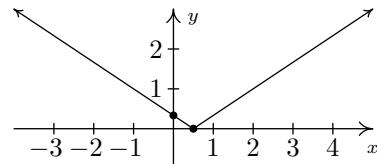
No relative or absolute maximum



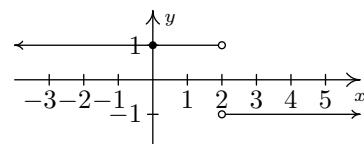
25.  $f(x) = -3|x|$   
 $f(0) = 0$   
x-intercept  $(0, 0)$   
y-intercept  $(0, 0)$   
Domain  $(-\infty, \infty)$   
Range  $(-\infty, 0]$   
Increasing on  $(-\infty, 0]$   
Decreasing on  $[0, \infty)$   
Relative and absolute maximum at  $(0, 0)$   
No relative or absolute minimum



27.  $f(x) = \frac{1}{3}|2x - 1|$   
 $f(\frac{1}{2}) = 0$   
x-intercepts  $(\frac{1}{2}, 0)$   
y-intercept  $(0, \frac{1}{3})$   
Domain  $(-\infty, \infty)$   
Range  $[0, \infty)$   
Decreasing on  $(-\infty, \frac{1}{2}]$   
Increasing on  $[\frac{1}{2}, \infty)$   
Relative and absolute min. at  $(\frac{1}{2}, 0)$   
No relative or absolute maximum



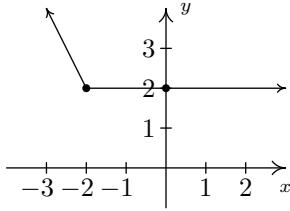
29.  $f(x) = \frac{|2-x|}{2-x}$   
No zeros  
No x-intercept  
y-intercept  $(0, 1)$   
Domain  $(-\infty, 2) \cup (2, \infty)$   
Range  $\{-1, 1\}$   
Constant on  $(-\infty, 2)$   
Constant on  $(2, \infty)$   
Absolute minimum at every point  $(x, -1)$  where  $x > 2$   
Absolute maximum at every point  $(x, 1)$  where  $x < 2$   
Relative maximum AND minimum at every point on the graph



31. Re-write  $f(x) = |x+2| - x$  as  

$$f(x) = \begin{cases} -2x-2 & \text{if } x < -2 \\ 2 & \text{if } x \geq -2 \end{cases}$$
  
No zeros  
No x-intercepts  
y-intercept  $(0, 2)$   
Domain  $(-\infty, \infty)$

Range  $[2, \infty)$   
 Decreasing on  $(-\infty, -2]$   
 Constant on  $[-2, \infty)$   
 Absolute minimum at every point  $(x, 2)$  where  $x \geq -2$   
 No absolute maximum  
 Relative minimum at every point  $(x, 2)$  where  $x \geq -2$   
 Relative maximum at every point  $(x, 2)$  where  $x > -2$



33. Re-write  $f(x) = |x + 4| + |x - 2|$  as

$$f(x) = \begin{cases} -2x - 2 & \text{if } x < -4 \\ 6 & \text{if } -4 \leq x < 2 \\ 2x + 2 & \text{if } x \geq 2 \end{cases}$$

No zeros

No x-intercept

y-intercept  $(0, 6)$

Domain  $(-\infty, \infty)$

Range  $[6, \infty)$

Decreasing on  $(-\infty, -4]$

Constant on  $[-4, 2]$

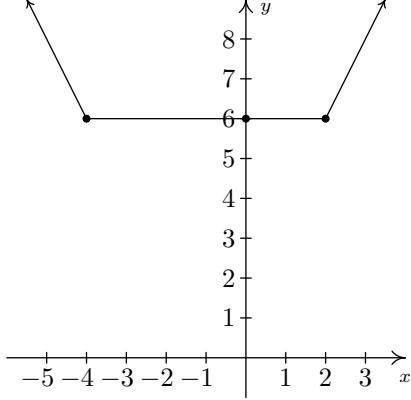
Increasing on  $[2, \infty)$

Absolute minimum at every point  $(x, 6)$  where  $-4 \leq x \leq 2$

No absolute maximum

Relative minimum at every point  $(x, 6)$  where  $-4 \leq x \leq 2$

Relative maximum at every point  $(x, 6)$  where  $-4 < x < 2$



- 35.

### Section 3.3

1.  $x = -6$  or  $x = 6$
3.  $x = -3$  or  $x = 11$
5.  $x = -\frac{1}{2}$  or  $x = \frac{1}{10}$
7.  $x = -3$  or  $x = 3$
9.  $x = -\frac{3}{2}$
11.  $x = 1$
13.  $x = -1$ ,  $x = 0$  or  $x = 1$
15.  $x = -2$  or  $x = 2$
17.  $x = -\frac{1}{7}$  or  $x = 1$
19.  $x = 1$
21.  $x = \frac{1}{5}$  or  $x = 5$
23. The vertex is (approximately)  $(29.60, 22.66)$ , which corresponds to a maximum fuel economy of 22.66 miles per gallon, reached sometime between 2009 and 2010 (29–30 years after 1980.) Unfortunately, the model is only valid up until 2008 (28 years after 1980.) So, at this point, we are using the model to predict the maximum fuel economy.
25. 5000 pens should be produced for a cost of \$200.
27. 50 feet by 50 feet; maximum area is 2500 feet; he can raise 100 average alpacas.
29. 2 seconds.
31. The hammer reaches a maximum height of approximately 13.62 feet. The hammer is in the air approximately 1.61 seconds.
33. Make the vertex of the parabola  $(0, 10)$  so that the point on the top of the left-hand tower where the cable connects is  $(-200, 100)$  and the point on the top of the right-hand tower is  $(200, 100)$ . Then the parabola is given by  $p(x) = \frac{9}{4000}x^2 + 10$ . Standing 50 feet to the right of the left-hand tower means you're standing at  $x = -150$  and  $p(-150) = 60.625$ . So the cable is 60.625 feet above the bridge deck there.
35.  $\left(\frac{3 - \sqrt{7}}{2}, \frac{-1 + \sqrt{7}}{2}\right), \left(\frac{3 + \sqrt{7}}{2}, \frac{-1 - \sqrt{7}}{2}\right)$
- 37.
39.  $x = \pm(y - 2)$
41.  $y = \frac{3 \pm \sqrt{16x + 9}}{2}$
43.  $t = \frac{v_0 \pm \sqrt{v_0^2 + 4gs_0}}{2g}$

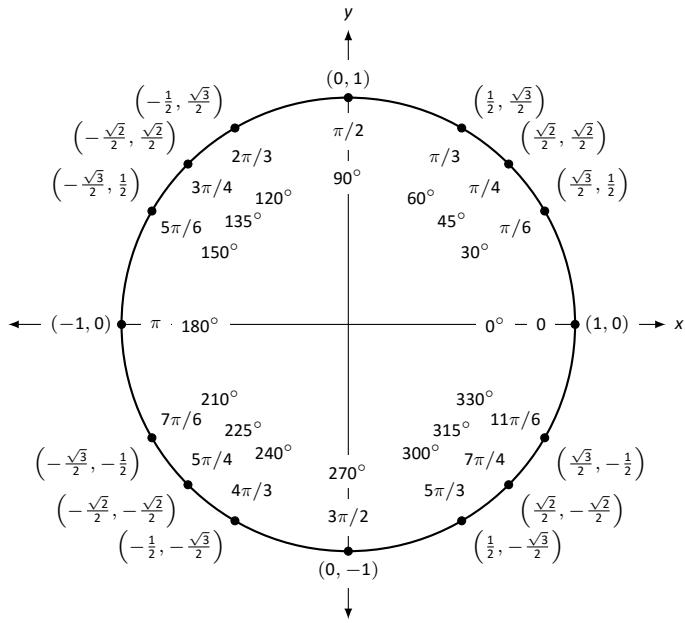
## Differentiation Rules

1.  $\frac{d}{dx}(cx) = c$
2.  $\frac{d}{dx}(u \pm v) = u' \pm v'$
3.  $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4.  $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5.  $\frac{d}{dx}(u(v)) = u'(v)v'$
6.  $\frac{d}{dx}(c) = 0$
7.  $\frac{d}{dx}(x) = 1$
8.  $\frac{d}{dx}(x^n) = nx^{n-1}$
9.  $\frac{d}{dx}(e^x) = e^x$
10.  $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11.  $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12.  $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13.  $\frac{d}{dx}(\sin x) = \cos x$
14.  $\frac{d}{dx}(\cos x) = -\sin x$
15.  $\frac{d}{dx}(\csc x) = -\csc x \cot x$
16.  $\frac{d}{dx}(\sec x) = \sec x \tan x$
17.  $\frac{d}{dx}(\tan x) = \sec^2 x$
18.  $\frac{d}{dx}(\cot x) = -\csc^2 x$
19.  $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
20.  $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
21.  $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$
22.  $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
23.  $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
24.  $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
25.  $\frac{d}{dx}(\cosh x) = \sinh x$
26.  $\frac{d}{dx}(\sinh x) = \cosh x$
27.  $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28.  $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
29.  $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
30.  $\frac{d}{dx}(\operatorname{coth} x) = -\operatorname{csch}^2 x$
31.  $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$
32.  $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$
33.  $\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$
34.  $\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}$
35.  $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$
36.  $\frac{d}{dx}(\operatorname{coth}^{-1} x) = \frac{1}{1-x^2}$

## Integration Rules

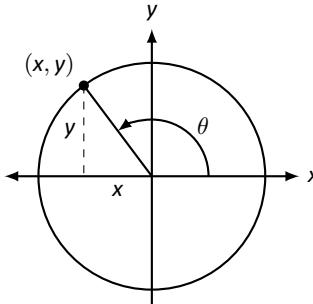
1.  $\int c \cdot f(x) dx = c \int f(x) dx$
2.  $\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$
3.  $\int 0 dx = C$
4.  $\int 1 dx = x + C$
5.  $\int x^n dx = \frac{1}{n+1} x^{n+1} + C, n \neq -1$
6.  $\int e^x dx = e^x + C$
7.  $\int a^x dx = \frac{1}{\ln a} \cdot a^x + C$
8.  $\int \frac{1}{x} dx = \ln|x| + C$
9.  $\int \cos x dx = \sin x + C$
10.  $\int \sin x dx = -\cos x + C$
11.  $\int \tan x dx = -\ln|\cos x| + C$
12.  $\int \sec x dx = \ln|\sec x + \tan x| + C$
13.  $\int \csc x dx = -\ln|\csc x + \cot x| + C$
14.  $\int \cot x dx = \ln|\sin x| + C$
15.  $\int \sec^2 x dx = \tan x + C$
16.  $\int \csc^2 x dx = -\cot x + C$
17.  $\int \sec x \tan x dx = \sec x + C$
18.  $\int \csc x \cot x dx = -\csc x + C$
19.  $\int \cos^2 x dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
20.  $\int \sin^2 x dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
21.  $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$
22.  $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
23.  $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{|x|}{a}\right) + C$
24.  $\int \cosh x dx = \sinh x + C$
25.  $\int \sinh x dx = \cosh x + C$
26.  $\int \tanh x dx = \ln(\cosh x) + C$
27.  $\int \coth x dx = \ln|\sinh x| + C$
28.  $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln|x+\sqrt{x^2-a^2}| + C$
29.  $\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln|x+\sqrt{x^2+a^2}| + C$
30.  $\int \frac{1}{a^2-x^2} dx = \frac{1}{2} \ln\left|\frac{a+x}{a-x}\right| + C$
31.  $\int \frac{1}{x\sqrt{a^2-x^2}} dx = \frac{1}{a} \ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$
32.  $\int \frac{1}{x\sqrt{x^2+a^2}} dx = \frac{1}{a} \ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

## The Unit Circle



## Definitions of the Trigonometric Functions

### Unit Circle Definition

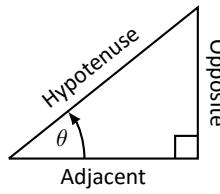


$$\sin \theta = y \quad \cos \theta = x$$

$$\csc \theta = \frac{1}{y} \quad \sec \theta = \frac{1}{x}$$

$$\tan \theta = \frac{y}{x} \quad \cot \theta = \frac{x}{y}$$

### Right Triangle Definition



$$\sin \theta = \frac{O}{H} \quad \csc \theta = \frac{H}{O}$$

$$\cos \theta = \frac{A}{H} \quad \sec \theta = \frac{H}{A}$$

$$\tan \theta = \frac{O}{A} \quad \cot \theta = \frac{A}{O}$$

## Common Trigonometric Identities

### Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

### Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

$$\csc\left(\frac{\pi}{2} - x\right) = \sec x$$

$$\sec\left(\frac{\pi}{2} - x\right) = \csc x$$

$$\cot\left(\frac{\pi}{2} - x\right) = \tan x$$

### Double Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 2 \cos^2 x - 1$$

$$= 1 - 2 \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

### Sum to Product Formulas

$$\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2 \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x+y}{2}\right)$$

$$\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

### Power-Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

### Even/Odd Identities

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

$$\csc(-x) = -\csc x$$

$$\sec(-x) = \sec x$$

$$\cot(-x) = -\cot x$$

### Product to Sum Formulas

$$\sin x \sin y = \frac{1}{2} (\cos(x-y) - \cos(x+y))$$

$$\cos x \cos y = \frac{1}{2} (\cos(x-y) + \cos(x+y))$$

$$\sin x \cos y = \frac{1}{2} (\sin(x+y) + \sin(x-y))$$

### Angle Sum/Difference Formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

## Areas and Volumes

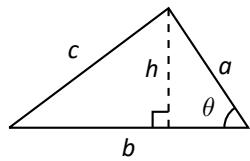
### Triangles

$$h = a \sin \theta$$

$$\text{Area} = \frac{1}{2}bh$$

Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

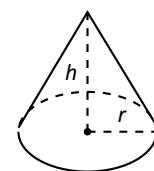


### Right Circular Cone

$$\text{Volume} = \frac{1}{3}\pi r^2 h$$

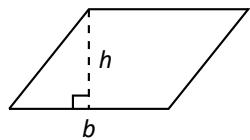
Surface Area =

$$\pi r \sqrt{r^2 + h^2} + \pi r^2$$



### Parallelograms

$$\text{Area} = bh$$

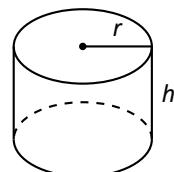


### Right Circular Cylinder

$$\text{Volume} = \pi r^2 h$$

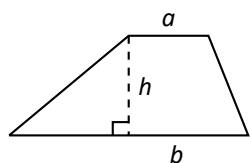
Surface Area =

$$2\pi rh + 2\pi r^2$$



### Trapezoids

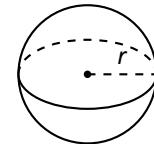
$$\text{Area} = \frac{1}{2}(a + b)h$$



### Sphere

$$\text{Volume} = \frac{4}{3}\pi r^3$$

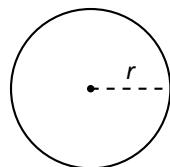
$$\text{Surface Area} = 4\pi r^2$$



### Circles

$$\text{Area} = \pi r^2$$

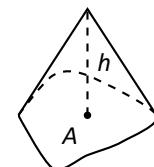
$$\text{Circumference} = 2\pi r$$



### General Cone

$$\text{Area of Base} = A$$

$$\text{Volume} = \frac{1}{3}Ah$$

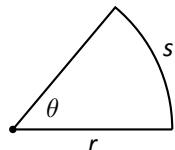


### Sectors of Circles

$\theta$  in radians

$$\text{Area} = \frac{1}{2}\theta r^2$$

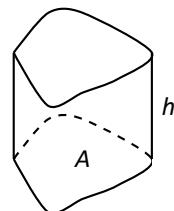
$$s = r\theta$$



### General Right Cylinder

$$\text{Area of Base} = A$$

$$\text{Volume} = Ah$$



# Algebra

## Factors and Zeros of Polynomials

Let  $p(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$  be a polynomial. If  $p(a) = 0$ , then  $a$  is a *zero* of the polynomial and a solution of the equation  $p(x) = 0$ . Furthermore,  $(x - a)$  is a *factor* of the polynomial.

## Fundamental Theorem of Algebra

An  $n$ th degree polynomial has  $n$  (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

## Quadratic Formula

If  $p(x) = ax^2 + bx + c$ , and  $0 \leq b^2 - 4ac$ , then the real zeros of  $p$  are  $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

## Special Factors

$$\begin{aligned}x^2 - a^2 &= (x - a)(x + a) & x^3 - a^3 &= (x - a)(x^2 + ax + a^2) \\x^3 + a^3 &= (x + a)(x^2 - ax + a^2) & x^4 - a^4 &= (x^2 - a^2)(x^2 + a^2) \\(x + y)^n &= x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \dots + nxy^{n-1} + y^n \\(x - y)^n &= x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \dots \mp nxy^{n-1} \mp y^n\end{aligned}$$

## Binomial Theorem

$$\begin{aligned}(x + y)^2 &= x^2 + 2xy + y^2 & (x - y)^2 &= x^2 - 2xy + y^2 \\(x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 & (x - y)^3 &= x^3 - 3x^2y + 3xy^2 - y^3 \\(x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 & (x - y)^4 &= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4\end{aligned}$$

## Rational Zero Theorem

If  $p(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$  has integer coefficients, then every *rational zero* of  $p$  is of the form  $x = r/s$ , where  $r$  is a factor of  $a_0$  and  $s$  is a factor of  $a_n$ .

## Factoring by Grouping

$$acx^3 + adx^2 + bcx + bd = ax^2(cs + d) + b(cx + d) = (ax^2 + b)(cx + d)$$

## Arithmetic Operations

$$ab + ac = a(b + c) \quad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc} \quad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \quad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \quad \frac{a-b}{c-d} = \frac{b-a}{d-c} \quad \frac{ab+ac}{a} = b+c$$

## Exponents and Radicals

$$a^0 = 1, \quad a \neq 0 \quad (ab)^x = a^x b^x \quad a^x a^y = a^{x+y} \quad \sqrt{a} = a^{1/2} \quad \frac{a^x}{a^y} = a^{x-y} \quad \sqrt[n]{a} = a^{1/n}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \quad \sqrt[n]{a^m} = a^{m/n} \quad a^{-x} = \frac{1}{a^x} \quad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \quad (a^x)^y = a^{xy} \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

## Additional Formulas

### Summation Formulas:

$$\sum_{i=1}^n c = cn$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^3 = \left( \frac{n(n+1)}{2} \right)^2$$

### Trapezoidal Rule:

$$\int_a^b f(x) dx \approx \frac{\Delta x}{2} [f(x_1) + 2f(x_2) + 2f(x_3) + \dots + 2f(x_n) + f(x_{n+1})]$$

$$\text{with Error} \leq \frac{(b-a)^3}{12n^2} [\max |f''(x)|]$$

### Simpson's Rule:

$$\int_a^b f(x) dx \approx \frac{\Delta x}{3} [f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \dots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})]$$

$$\text{with Error} \leq \frac{(b-a)^5}{180n^4} [\max |f^{(4)}(x)|]$$

### Arc Length:

$$L = \int_a^b \sqrt{1+f'(x)^2} dx$$

### Surface of Revolution:

$$S = 2\pi \int_a^b f(x) \sqrt{1+f'(x)^2} dx$$

(where  $f(x) \geq 0$ )

$$S = 2\pi \int_a^b x \sqrt{1+f'(x)^2} dx$$

(where  $a, b \geq 0$ )

### Work Done by a Variable Force:

$$W = \int_a^b F(x) dx$$

### Force Exerted by a Fluid:

$$F = \int_a^b w d(y) \ell(y) dy$$

### Taylor Series Expansion for $f(x)$ :

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

### Maclaurin Series Expansion for $f(x)$ , where $c = 0$ :

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

## Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
<i>n</i> th-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n \rightarrow \infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$ r  < 1$	$ r  \geq 1$	Sum = $\frac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n \rightarrow \infty} b_n = L$		Sum = $\left( \sum_{n=1}^a b_n \right) - L$
<i>p</i> -Series	$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n) dn$ is convergent	$\int_1^{\infty} a(n) dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n \rightarrow \infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n \rightarrow \infty} a_n/b_n > 0$	Also diverges if $\lim_{n \rightarrow \infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$	$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n \rightarrow \infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n \rightarrow \infty} (a_n)^{1/n} < 1$	$\lim_{n \rightarrow \infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n \rightarrow \infty} (a_n)^{1/n} = \infty$