MATH1003

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15/4/2021
21/5/2021

MATH1003
Intermediate Calculus

Course Description

- Definite integrals. Fundamental theorem of calculus. Applications of definite integrals. Approximations of definite integrals. Sequence and series. Power series and Taylor series.

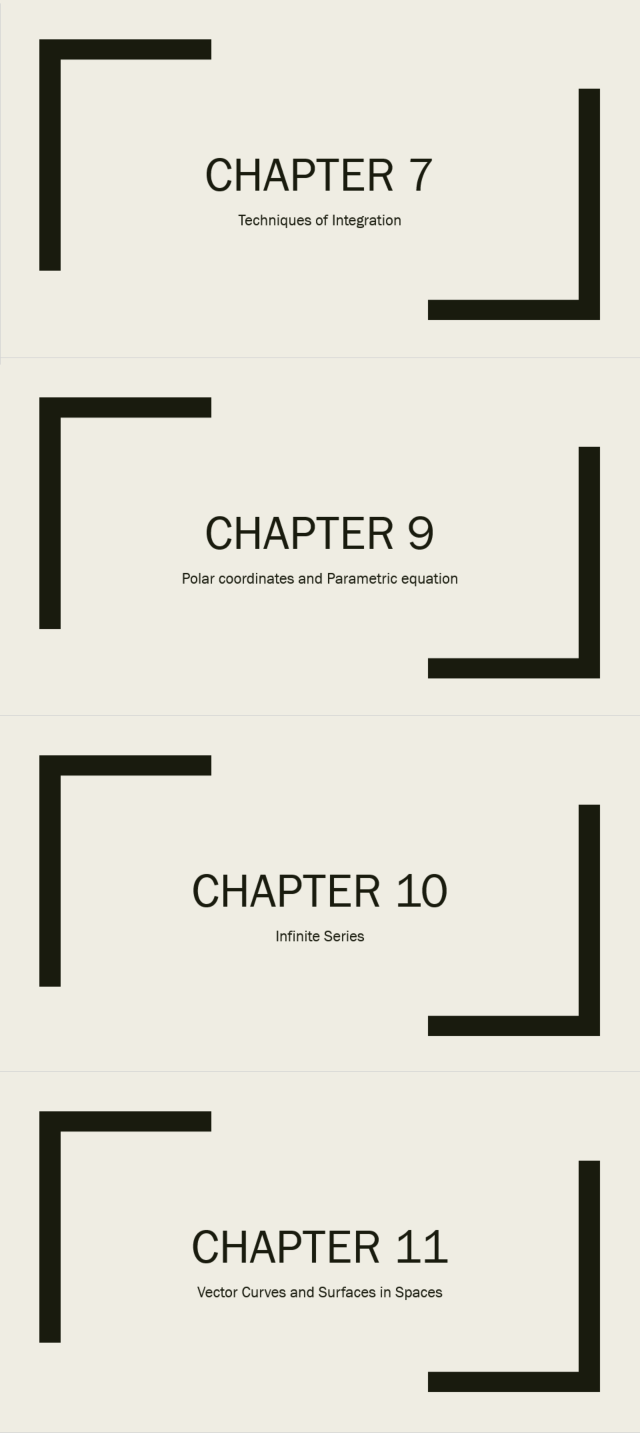
Prerequisite

- GEST1004

- MATH2000 ECEN3001

Chapters

- *Chapter 7: Techniques of Integration*
- *Chapter 9: Polar Coordinates*
- *Chapter 10: Infinite Series （Test, Taylor）*
- *Chapter 11: Vectors, Curves and Surfaces in Space*



GEST1004

Differentiation

- $\frac{dy}{dx} = \frac{dy}{\cancel{dt}} \cdot \frac{\cancel{dt}}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

Substitution 换元法

$\int \frac{lnx}{x} dx$

- *Solution:*
- *Let* $u = lnx$
- $du = \frac{1}{x} dx$
- $\int \frac{lnx}{x} dx = \int u du = \frac{1}{2} u^2 + C = \frac{1}{2} ln^2 x + C$

Integral

#9.5A **Integral Computations with Parametric Curves**

(a) The area under the curve:

$$A = \int_a^b y dx$$

(b) The volume of revolution around the $x -$ axis (**Disk Method**):

$$V_x = \int_a^b \pi y^2 dx$$

(c) The volume of revolution around the $y -$ axis (**Shell's Method**):

$$V_y = \int_a^b 2\pi x y dx$$

(d) The arc length of the curve:

$$\boldsymbol{s} = \int_0^s ds = \int_{x=a}^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Notes:

$$(ds)^2 = (dx)^2 + (dy)^2$$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(e) The area of the surface of revolution around the $x -$ axis:

$$\boldsymbol{S_x} = \int_0^s 2\pi y ds = \int_{x=a}^b 2\pi y \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

(f) The area of the surface of revolution around the $y -$ axis:

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Differentiation

- $\frac{dy}{dx} = \frac{dy}{\cancel{dt}} \cdot \frac{\cancel{dt}}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

CHAPTER 7

Techniques of Integration

Some of Fundamental Theorems

- $F' = f = G' \;\rightarrow\; F = G + C \qquad F = x,\;\; G = x + 1$
- $\int f(x)dx = F(x) + C \;\longleftrightarrow\; F'(x) = f(x) \qquad F = \frac{1}{2}x^2 + 1 \quad \mathrm{f = x}$
- ……

Indefinite Integrals of Common Integrands:

$$sin^2x + cos^2x = 1$$
$$cot^2x + 1 = csc^2x$$
$$tan^2x + 1 = sec^2x$$

Polynomial Function:

$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$ for $n \neq -1$ $\qquad$ $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$

Trigonometric Function:

$\frac{d}{dx}sinx = cosx$ $\qquad$ $\int cosx\,dx = sinx + C$

$\frac{d}{dx}cosx = -sinx$ $\qquad$ $\int sinx\,dx = -cosx + C$

$\frac{d}{dx}tanx = sec^2x$ $\qquad$ $\int sec^2x\,dx = tanx + C$

$\frac{d}{dx}cscx = -cscx \cdot cotx$ $\qquad$ $\int cscx \cdot cotx\,dx = -cscx + C$

$\frac{d}{dx}secx = secx \cdot tanx$ $\qquad$ $\int secx \cdot tanx\,dx = secx + C$

$\frac{d}{dx}cotx = -csc^2x$ $\qquad$ $\int csc^2x\,dx = -cotx + C$

Inverse Trigonometric Function:

$\frac{d}{dx}sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$ $\qquad$ $\int \frac{1}{\sqrt{1-x^2}}dx = sin^{-1}x + C$

Assumed $1 - x^2 > 0$. That is $-1 < x < 1$

$\frac{d}{dx}cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$ $\qquad$ $\int \frac{1}{\sqrt{1-x^2}}dx = -cos^{-1}x + C$

Assumed $1 - x^2 > 0$. That is $-1 < x < 1$

$\frac{d}{dx}tan^{-1}x = \frac{1}{1+x^2}$ $\qquad$ $\int \frac{1}{1+x^2}dx = tan^{-1}x + C$

The Exponential Function:

$\frac{d}{dx}e^x = e^x$ $\qquad$ $\int e^x\,dx = e^x + C$

$\frac{d}{dx}a^x = a^x \cdot lna$ $\qquad$ $\int a^x\,dx = \frac{a^x}{lna} + C$

Assumed $a > 0$ and $a \neq 1$

The Logarithmic Function:

$\frac{d}{dx}lnx = \frac{1}{x}$ for $x > 0$ $\qquad$ $\int \frac{1}{x}dx = lnx + C$ (Assumed $x > 0$)

$\frac{d}{dx}ln|x| = \frac{1}{x}$ for $x \neq 0$ $\qquad$ $\int \frac{1}{x}dx = ln|x| + C$ (Assumed $x \neq 0$)

Usually we fix the natural domain of x to include as many as possible / as much as possible.

Show that $\frac{d}{dx}sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

Proof:

Let $y = sin^{-1}x$. That is, $siny = x$.

$$1 = \frac{d}{dx}x = \frac{d}{dx}siny = \left(\frac{d}{dy}siny\right)\cdot\left(\frac{dy}{dx}\right) = cosy\cdot\frac{dy}{dx}$$

$$\frac{d}{dx}sin^{-1}x = \frac{dy}{dx} = \frac{1}{cosy} = \frac{1}{\sqrt{1-sin^2y}} = \frac{1}{\sqrt{1-x^2}}$$

(Assumed $siny \geq 0$)

Show that $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

Proof:

Let $y = \cos^{-1}x$. That is, $\cos y = x$.

$$1 = \frac{d}{dx}x = \frac{d}{dx}\cos y = \left(\frac{d}{dy}\cos y\right)\cdot\left(\frac{dy}{dx}\right) = -\sin y \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}\cos^{-1}x = \frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

(Assumed $\cos y \geq 0$)

Show that $\frac{d}{dx}tan^{-1}x = \frac{1}{1+x^2}$

Proof:

Let $y = tan^{-1}x$. That is, $tany = x$.

$$1 = \frac{d}{dx}x = \frac{d}{dx}tany = \left(\frac{d}{dy}tany\right)\cdot\left(\frac{dy}{dx}\right) = sec^2y\cdot\frac{dy}{dx}$$

$$\frac{d}{dx}tan^{-1}x = \frac{dy}{dx} = \frac{1}{sec^2y} = \frac{1}{1+tan^2y} = \frac{1}{1+x^2}$$

Show that $\boldsymbol{sin^{-1}x = -cos^{-1}x + \frac{\pi}{2}}$ for $\mathbf{-1 < x < 1}$

Proof:

Let $y(x) = sin^{-1}x + cos^{-1}x$ for $-1 < x < 1$

$y'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0$ for $-1 < x < 1$

$y(0) = sin^{-1}0 + cos^{-1}0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}$

So, $y(x) = y(0)$ for $-1 < x < 1$

That is, $sin^{-1}x = -cos^{-1}x + \frac{\pi}{2}$ for $-1 < x < 1$

1.Simple Rule for Indefinite Integral:

Rule:

- $\int (a \cdot u(x) + b \cdot v(x)) \, \mathrm{dx} = \mathrm{a} \cdot \int u(x) \, \mathrm{dx} + b \cdot \int v(x) \, \mathrm{dx}$

Example:

- $\int 2x + 3x^2 \, \mathrm{dx} = 2 \int x \, \mathrm{dx} + 3 \int x^2 \, \mathrm{dx}$

2. Integration by parts: 分部积分

Rule:

- $\int udv + \int vdu = uv + C$

Example:

- $\int xde^x + \int e^x dx = xe^x + C$

2. Integration by parts: (Proof)

- *Chain Rule:* $\frac{d}{dx}uv = u \cdot \frac{d}{dx}v + v\frac{d}{dx}u$

- $u \cdot v + C = \int \left(\frac{d}{dx}u \cdot v\right) dx = \int u \cdot \frac{d}{dx}vdx + \int v \cdot \frac{d}{dx}udx = \int udv + \int vdu$

Example:

- $x \cdot e^x + C = \int xde^x + \int e^x dx$
- $\int xe^x dx$

2. Integration by parts:
←反对幂指三←

- $\int udv + \int vdu = uv + C$

Example:

- $\int xe^x dx = \int xde^x = x \cdot e^x - \int e^x dx + C = x \cdot e^x - e^x + C = (x-1)e^x + C$

3.Substitution OR Integration by parts?

Substitution:

- $\int \frac{lnx}{x} dx$
- $\int \frac{x}{1+x^2} dx$
- $\int \frac{sinx}{cos^5x} dx$

Integration by parts :

- $\int xe^{-x} dx$
- $\int xlnxdx$
- $\int x^2 sinxdx$

4.Trick I

(trigonometric function)

- $\int tanxdx \quad \int cotxdx \quad \int secxdx \quad \int cscxdx$
- $\int sin3x \cdot cos5xdx$

$$2sinA \cdot cosB = sin(A + B) + sin(A - B)$$
$$2cosA \cdot cosB = cos(A + B) + cos(A - B)$$
$$-2sinA \cdot sinB = cos(A + B) - cos(A - B)$$

- $\int sec^n xdx$

4.Trick II

(trigonometric function)

- $\int sin^m x \cdot cos^n xdx$

Case 1: At least one of m and n is odd $\int sin^5 x \cdot cos^4 xdx$ $\boldsymbol{u = sinx/cosx}$

Case 2: Both of m and n are even $\int sin^2 x \cdot cos^4 xdx$ $\rightarrow \boldsymbol{cos2x}$

Case 3: One is 0 and the other one is even $\int cos^4 xdx$ $\rightarrow \boldsymbol{cos2x}$

4.Trick III
(trigonometric function)

- $\int tan^m x \cdot sec^n x dx$

Case 1.1: m is odd, n≠0	$\int tan^3 x \cdot \sec^3 x dx$	$\boldsymbol{u = secx}$
Case 1.2: m is odd, n=0	$\int tan^3 x dx$	$\boldsymbol{tan^2 x = sec^2 x - 1}$
Case 2.1: n is even, m≠0	$\int tan^2 x \cdot sec^4 x dx$	$\boldsymbol{u = tanx}$
Case 2.2: n is even, m=0	$\int sec^4 x dx$	$\boldsymbol{sec^2 x = tan^2 x + 1}$
Case 3.1: m is even and n is odd	$\int tan^2 x \cdot secxdx$	
		$\boldsymbol{tan^2 x = sec^2 x - 1}$
Case 3.2: m is even, n=0	$\int tan^4 x \cdot \sec^3 x dx$	

4.Trick IV
(trigonometric function)

$$\int \frac{1}{\sqrt{9+16x-4x^2}}dx$$

- ① $a^2 - b^2x^2$ $\quad sin\theta = \frac{ax}{b}$ $\quad \int \frac{x^3}{\sqrt{1-x^2}}dx$
- ② $a^2 + b^2x^2$ $\quad tan\theta = \frac{ax}{b}$ $\quad \int \frac{1}{(4x^2+9)^2}dx$
- ③ $b^2x^2 - a^2$ $\quad sec\theta = \frac{ax}{b}$ $\quad \int \frac{\sqrt{x^2-25}}{x}dx$

4.Trick V
(fraction)

Improper fractions ⟶ Simple proper fractions ⟶ Partial fractions

$$\int \frac{x^4}{x^2+4x+4}dx \qquad = \int x^2 - 4x + 12dx - \int \frac{32x+48}{x^2+4x+4}dx$$

$$\int \frac{32x+48}{(x+2)^2}dx \qquad = \int \frac{A}{x+2}dx + \int \frac{B}{(x+2)^2}dx \qquad = \int \frac{32}{x+2}dx + \int \frac{-16}{(x+2)^2}dx$$

Exercise time!

$$\int (x - x^2)^{\frac{3}{2}} dx \qquad \int \frac{x^2}{\sqrt{9 - 4x^2}} dx$$

$$\int sin^{-1} x dx \qquad \int e^{2x} \cdot sin3x dx$$

$$\int x^3 \sqrt{1 - x^2} dx \qquad \int \frac{18x + 6}{9x^2 + 6x + 5} dx$$

7.1 – #7.7 **Harder Problems**

Evaluate the indefinite integrals:

(a) $\displaystyle\int \frac{5+2\ln x}{x(1+\ln x)^2}\,dx$

(b) $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$

(c) $\displaystyle\int \sqrt{1+e^x}\,dx$

(d) $\displaystyle\int \frac{e^{3x}}{e^{2x}+4}\,dx$

(e) $\displaystyle\int x\cdot\sqrt{\frac{1-x^2}{1+x^2}}\,dx$

(f) $\displaystyle\int \ln(1+x+x^2)\,dx$

(g) $\displaystyle\int \frac{1}{1+\sin x}\,dx$

(h) $\displaystyle\int \frac{\tan x}{\sec x+1}\,dx$

Solutions:

(a)

$$\begin{aligned}
&\int \frac{5+2\ln x}{x(1+\ln x)^2}\,dx\\
&=\int \frac{5+2\ln x}{(1+\ln x)^2}\cdot\frac{1}{x}\,dx\\
&=\int \frac{2u+3}{u^2}\,du\\
&=2\int \frac{1}{u}\,du+3\int u^{-2}\,du\\
&=2\ln|u|+3\cdot\frac{u^{-1}}{-1}+C\\
&=2\ln|u|-\frac{3}{u}+C\\
&=2\ln|1+\ln x|-\frac{3}{1+\ln x}+C
\end{aligned}$$

Let $u=1+\ln x$. Then, $du=\frac{1}{x}dx$.

$5+2\ln x=5+2(u-1)=2u+3$

(b)

$$\begin{aligned}
&\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx\\
&=\int \frac{u^3}{1+u^2}\cdot 6u^5\,du\\
&=6\int \frac{u^8}{\cdots}\,du
\end{aligned}$$

Let $u=x^{\frac{1}{6}}$ (That is, $u^2=\sqrt[3]{x}$)Then, $u^6=x$. So, $6u^5du=dx$.

By Long Division, $u^8=(1+u^2)(u^6-u^4+u^2-1)+1$

5. Improper integrals 反常积分 ~~瑕积分~~

- $\int_a^b f(x)dx$

5. Improper integrals 反常积分

Converge or Diverge?

收敛 发散

$\int_{-\infty}^{0} \frac{1}{\sqrt{1-x}} dx$ *converges or diverges? If converges, find the value.*

Solution: For $M > 0$

$$\int_{-M}^{0} \frac{1}{\sqrt{1-x}} dx = -2(1-x)^{\frac{1}{2}}|_{x=-M}^{0} = -2\left(1-(1+M)^{\frac{1}{2}}\right) \quad \to \infty \; as \; M \to \infty$$

$$\therefore \int_{-\infty}^{0} \frac{1}{\sqrt{1-x}} dx \;\; \text{diverges}$$

$$\therefore \int_{-\infty}^{\infty} |x| e^{-x^2} dx \;\; converges\; to\; 1$$

5. Improper integrals 反常积分

Converge or Diverge?

收敛 *发散*

$$\lim_{M\to\infty}\int_{-M}^{M} f(x)dx \neq \int_{-\infty}^{\infty} f(x)dx$$

For example:

$\lim_{M\to\infty}\int_{-M}^{M} xdx$ *Converges to 0*

$\int_{\infty}^{\infty} xdx$ *Diverges*

6.Useless things…(at least for now)

- *Gamma function* $\Gamma(t) = \int_0^\infty x^{t-1} \cdot e^{-x}\, dx$ $\Gamma(n)\ converges\ to\ (n-1)!$
- *Beta function* $B(x,y) = \int_0^1 t^{x-1} \cdot (1-t)^{y-1} dt\ for\ x,y > 0$
- *Escape velocity*

$$W_r = \int_R^r \frac{GMm}{x^2} dx \to \frac{GMm}{R}\ as\ r \to \infty$$

$$\frac{1}{2} mv^2 = \frac{GMm}{R}$$

$$v = \sqrt{\frac{2GM}{R}}$$

CHAPTER 9

Polar coordinates and Parametric equation

1.Polar coordinates

- $\rho = f(\theta) \qquad (\rho, \theta)$
- 极径$\rho \ \in R$ 偏转角$\theta \in [0,2\pi]$
- $\left(1, \frac{\pi}{3}\right) = (-1, \frac{4\pi}{3})$
- $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \quad \rho^2 = x^2 + y^2$

- For example:

$\rho = 1$

$\rho = 4\cos\theta$

2.Area of Polar coordinates

Area of sector:

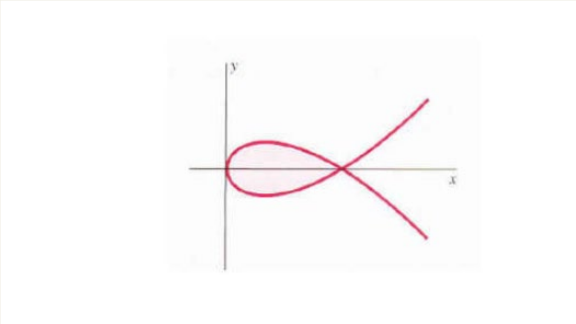
$$A = \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2}\theta \cdot r^2$$

Area of polar coordinates graph:

$$A = \frac{1}{2}\int_{\theta_2}^{\theta_1} \rho^2 d\theta$$

3.Parametric equation

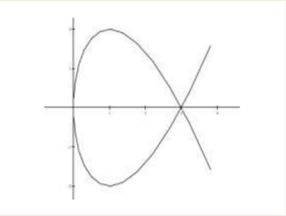
- $\begin{cases} x = \frac{1-t^2}{1+t^2} \\ y = \frac{2t}{1+t^2} \end{cases}$ $(t = tan\frac{\theta}{2} \rightarrow x^2 = cos^2\theta, y^2 = sin^2\theta)$
- $\begin{cases} x = tcosa \\ y = tsina \end{cases}$ (t is parameter) $\mathrm{y = x \cdot tan}a$
- $\begin{cases} x = t^2 \cdot \sqrt{3} \\ y = 3t - \frac{1}{3}t^3 \end{cases}$ *Find the area of the region rounded by the part of the cure for which* $-3 \leq t \leq 3$
- ……



- Find the area inside the circle $r = 2sin\theta$ and outside the circle $r = 1$
- Given the curve $\mathrm{x} = t^2, \mathrm{y} = t^3 - 3t \; for \; t \in R. Find$:

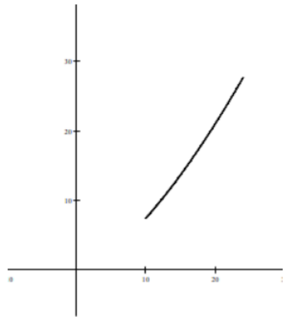
(a) the points on the curve where the tangent line is horizontal

(b) the slope of each tangent line at any point where the curve intersects the x- axis



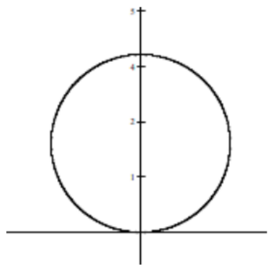
Question 5

Given the curve $x = 2t$, $y = \frac{2}{3}t^{\frac{3}{2}}$ for $5 \leq t \leq 12$.
Find the arc length of the curve.



Question 6

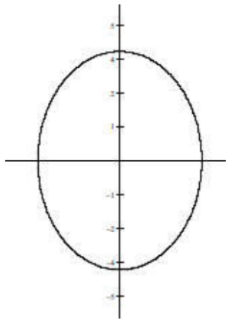
Given the curve $r = 4sin\theta$ for $0 \leq \theta \leq \pi$.
Find the area of the surface of revolution generated by revolving the curve around the $x-$ axis.



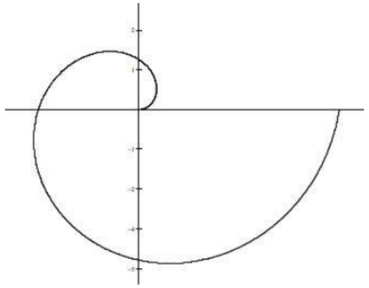
Question 7

Write an equation of the line tangent to the given curve at the indicated point:

(a) the curve $x = 3sint$, $y = 4cost$, at $t = \frac{\pi}{4}$

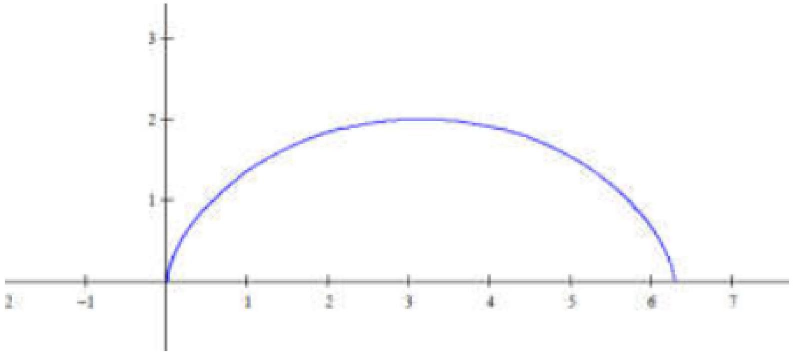


(b) the curve $r = \theta$ at $\theta = \frac{\pi}{2}$



Example 2:

Find the area under and arc length of the cycloidal arch with parametric equations $x = a(t - sint)$ and $y = a(1 - cost)$ for $0 \leq t \leq 2\pi$ and $a > 0$.



CHAPTER 10

Infinite Series

1.Fundamental theorems

- *1. if* $\lim_{n\to\infty} a_n = L$ *then* $\lim_{n\to\infty} a_{n+k} = L$ *(k is a real number)*
- *2. if* $\lim_{n\to\infty} a_{2n} = L,\ \lim_{n\to\infty} a_{2n-1} = L$ *then* $\lim_{n\to\infty} a_n = L$
- *3. if* $\lim_{n\to\infty} a_n = A,\ \lim_{n\to\infty} b_n = B\ ,\ \ c_n = a_n ?\, b_n,\ \ \lim_{n\to\infty} c_n = A ?\, B$
- *Squeeze rule*
- $a_n = f(n), \lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x)$ *(x is a real number) [it is for L'Hopital's Rule]*
- *For* $-1 < r < 1, \lim_{n\to\infty} r^n = 0$

2.Monotonic theorem

- *Monotonic Increasing* *Sequence that is* *Bounded Above* *converges*
- *Monotonic Decreasing* *Sequence that is* *Bounded Below* *converges*

3.Nested Radicals and Continued Fractions

- $\sqrt{q + p\sqrt{q + \sqrt{\cdots}}} = p + \frac{q}{p + \frac{q}{\cdots}}$ $\qquad a_{n+1} = \sqrt{q + pa_n}, \quad a_{n+1} = p + \frac{q}{a_n}$
- *Proof*

4.Sum sequence

- $S_n = \sum_{k=1}^{n} a_k$
- $\{S_n\}$ *is called sum sequence/series*
- *Sometimes we write the series as* $\sum_{n=1}^{\infty} a_n$ *or* $\sum a_n$

5.The geometric series (几何级数/等比数列)

- $a_n = a_0 \cdot r^{(n-1)}$
- $S_n = \sum a_n = \frac{a \cdot (1-r^n)}{1-r}$
- *Theorem:*
- If $|r| < 1, \sum_{n=1}^{\infty} a_n \, converges$ to $\frac{a}{1-r}$
- If $|r| > 1, \sum_{n=1}^{\infty} a_n \, diverges$
- *Exercise:*
- *Rewrite* $0.287\dot{5}\dot{6}$ *as a geometric series*

6.Nth term test for diverges
第n项审敛法/常数项级数审敛法

- *If* $\lim\limits_{n\to\infty} a_n \neq 0$ *or* $\lim\limits_{n\to\infty} a_n$ *doesn't exist as a real number, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*
- *Remark:*

$\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim\limits_{n\to\infty} a_n = 0$

7.Harmonic series

- $\sum_{n=1}^{\infty} \frac{1}{n}$
- *Converges or diverges?*
- *Can we apply nth term test on it?*
- *Diverge*
- *if p>1, ∑1np converges and if p<=1 it diverges for p∈R+*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \; for\ p > 0\ is\ called\ \boldsymbol{p - series}$$

8.Important Theorem about the Convergence or Divergence of Series

Suppose k is a positive integer.
$\{a_n\}$ and $\{b_n\}$ $are\ sequences\ of\ real\ number\ AND$

$a_n = b_n$ *for* $n \geq k$

Then,

(a) $\sum_{n=1}^{\infty} a_n$ *converges* $\Leftrightarrow \sum_{n=1}^{\infty} b_n$ *converges*

(b) $\sum_{n=1}^{\infty} a_n$ *diverges* $\Leftrightarrow \sum_{n=1}^{\infty} b_n$ *diverges*

Conclusion:

We only care the "tail" of the series.

9.Integral Test for positive-term series 积分审敛法

- *Suppose* $a_n = f(n) > 0 for\ n = 1,2,3, \ldots$
- *The series* $\sum_{n=1}^{\infty} a_n = converges \Leftrightarrow \int_1^{\infty} f(t)dt\ converges$
- *The series* $\sum_{n=1}^{\infty} a_n = diverges \Leftrightarrow \int_1^{\infty} f(t)dt\ diverges$

Integral Test Remainder Estimate

9.Integral Test for positive-term series 积分审敛法

- *Suppose that* $f: R \to R$ *is a function such that:*
- *1.* $f(t) > 0\ for\ any\ t \geq 1\ AND$
- *2.* f *is continuous on*$[1, \infty)\ AND$
- *3.* f *is strictly decreasing on* $[1, \infty)$
- *That is, for any* $\alpha, \beta \in [1, \infty)$ *with* $\alpha < \beta, we\ must\ have\ f(\alpha) > f(\beta)$
- *Then the series*
- $\sum_{n=1}^{\infty} f(n) converges \Leftrightarrow\ the\ improper\ integral\ \int_1^{\infty} f(t) dt\ converges.$

10. Comparison Test for Positive-Term Series
比较审敛法

- Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive $-$ term series, $a_n \leq b_n$
- Then, b_n converges $\Rightarrow a_n$ converges;
- a_n diverges $\Rightarrow b_n$ diverges

11. Limit Comparison Test for Positive-Term Series

- *Suppose* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are positive-term series,* $\lim_{n\to\infty} \frac{a_n}{b_n} = L$
- *Case1: L is a positive real number:*

$\sum_{n=1}^{\infty} a_n \ \ converges \Leftrightarrow \sum_{n=1}^{\infty} b_n \ converge, \sum_{n=1}^{\infty} a_n \ \ diverges \Leftrightarrow \sum_{n=1}^{\infty} b_n \ diverge$

- *Case2: L=0:*

$\sum_{n=1}^{\infty} a_n \ \ diverges \Rightarrow \sum_{n=1}^{\infty} b_n \ diverge$

- *Case3: L=+∞ :*

$\sum_{n=1}^{\infty} b_n \ \ diverges \Rightarrow \sum_{n=1}^{\infty} a_n \ diverge$

12. Rearrangement and Grouping

- …

13.Alternating Series

- $\sum_{n=1}^{\infty} a_n$
- $b_n = (-1)^{n+1} a_n$
- $c_n = (-1)^n a_n$

14.Ratio Test & Root Test

- $\{a_n\}$
- $\rho = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \quad \rho = \lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$

Case 1: $0 \leq \rho < 1$, $\sum_{n=1}^{\infty} a_n$ converges absolutely

Case 2: $\rho > 1$, $\sum_{n=1}^{\infty} a_n$ diverges

Case 3: $\rho = 1$, no conclusion

15.Power series

- $u_n(x) = a_n\, x^n \qquad (u_0(x) = a_0)$
- $a_0 + \sum_{n=1}^{\infty} a_n\, x^n \qquad \sum_{n=0}^{\infty} a_n\, x^n (\mathrm{x} \neq 0)$
- $a_0 + \sum_{n=1}^{\infty} a_n\, (x - x_0\ )^n$
- Convergence radius
- Convergence interval

15.Power series

- $u_n(x) = a_n\, x^n$
- Convergence radius
- Convergence interval

Example 1

Find the radius of convergence and the interval of convergence for $\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n} x^n$

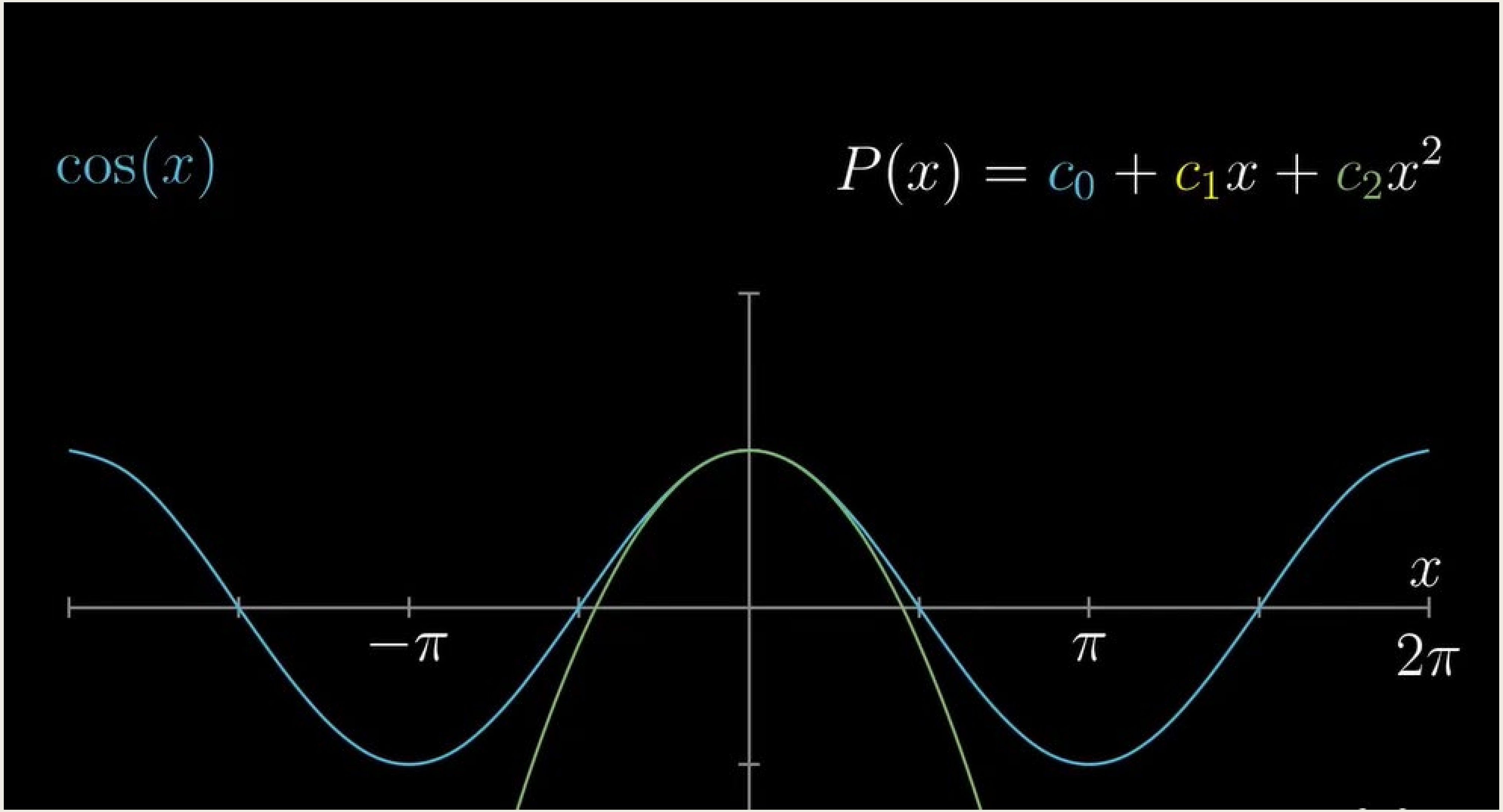
Sometimes, we write $\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n} x^n$ as $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$.

16. Taylor Series

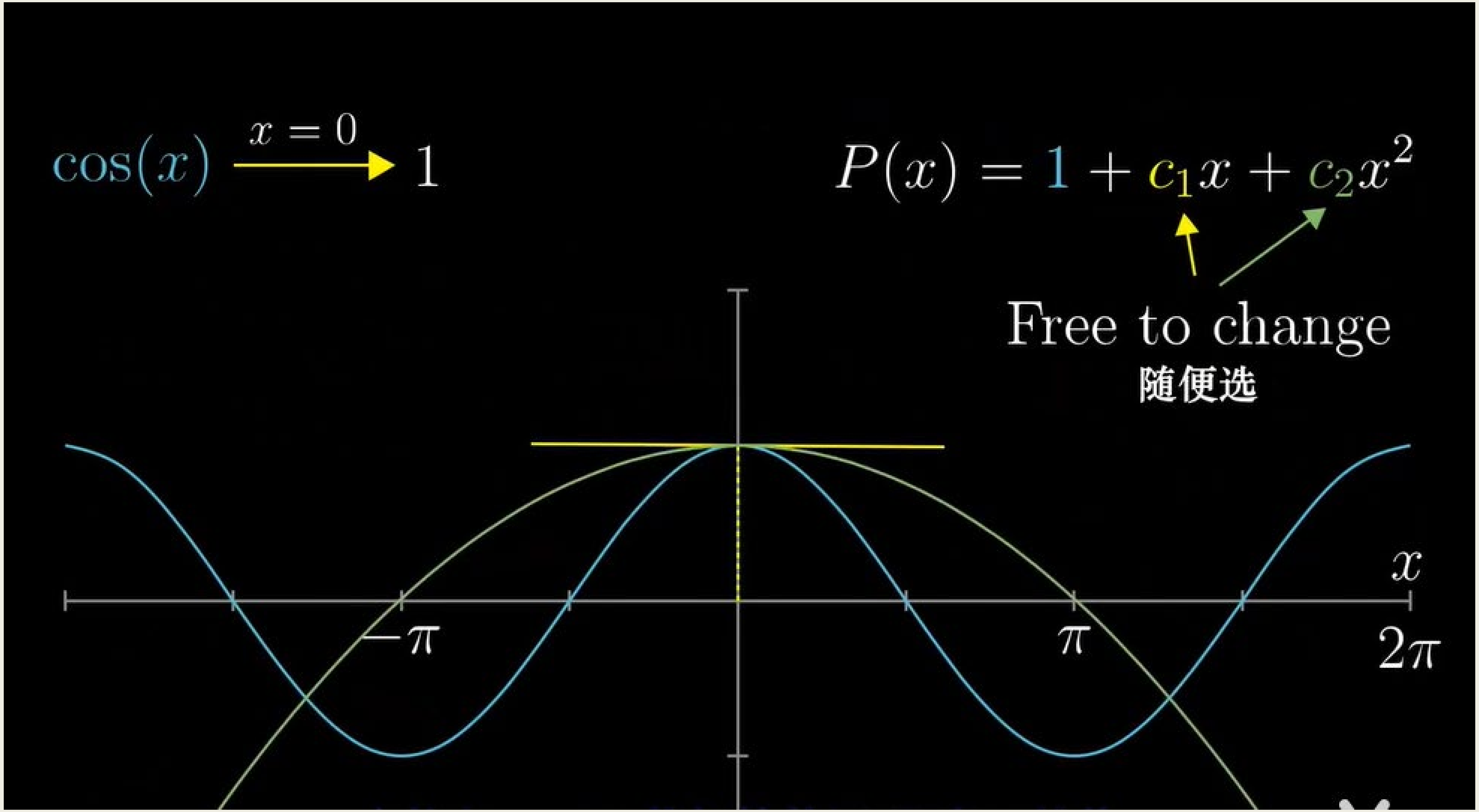
- $f(a)+\sum_{n=1}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^{n}$

- Why Taylor?

16. Taylor Series



16. Taylor Series



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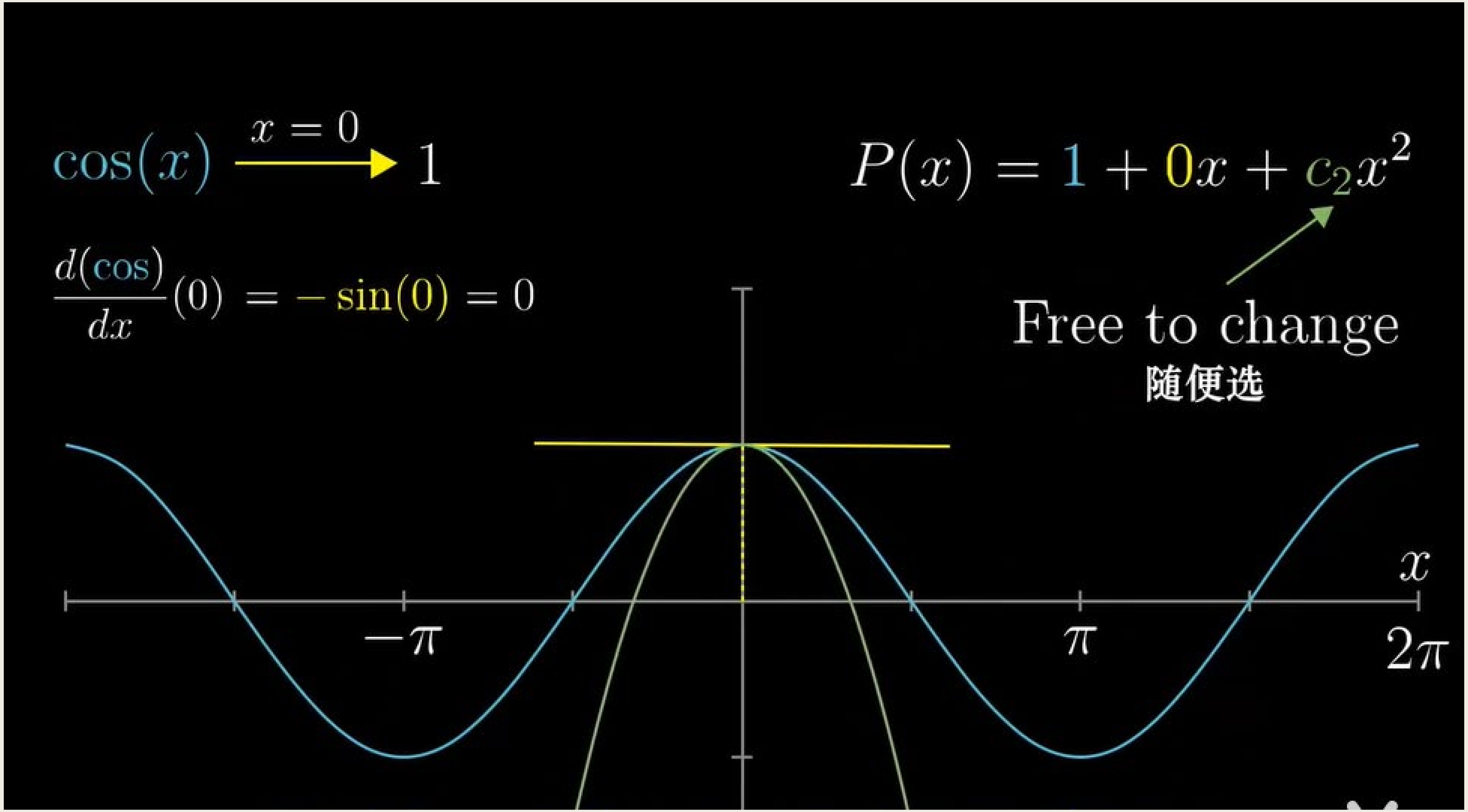

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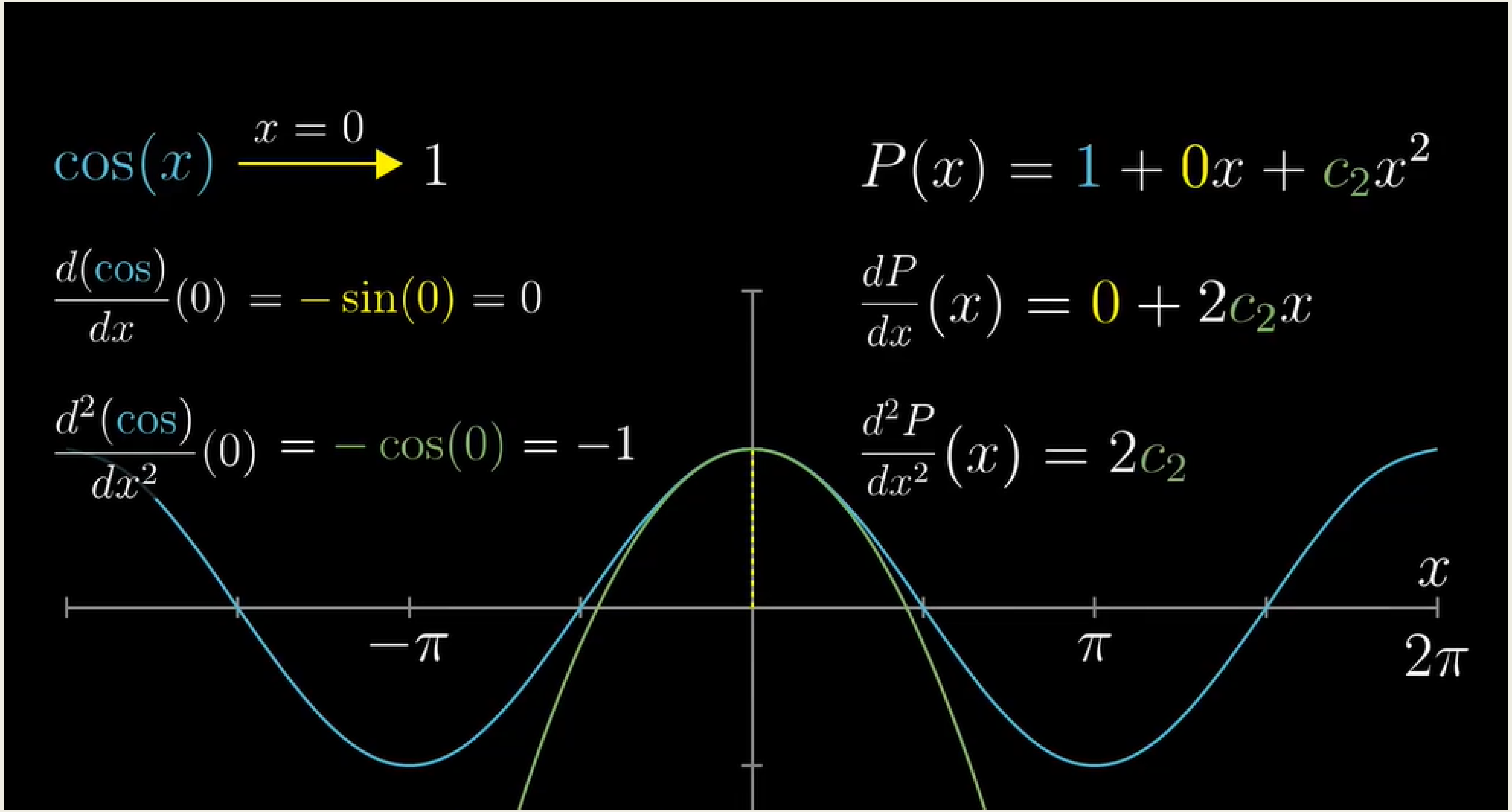

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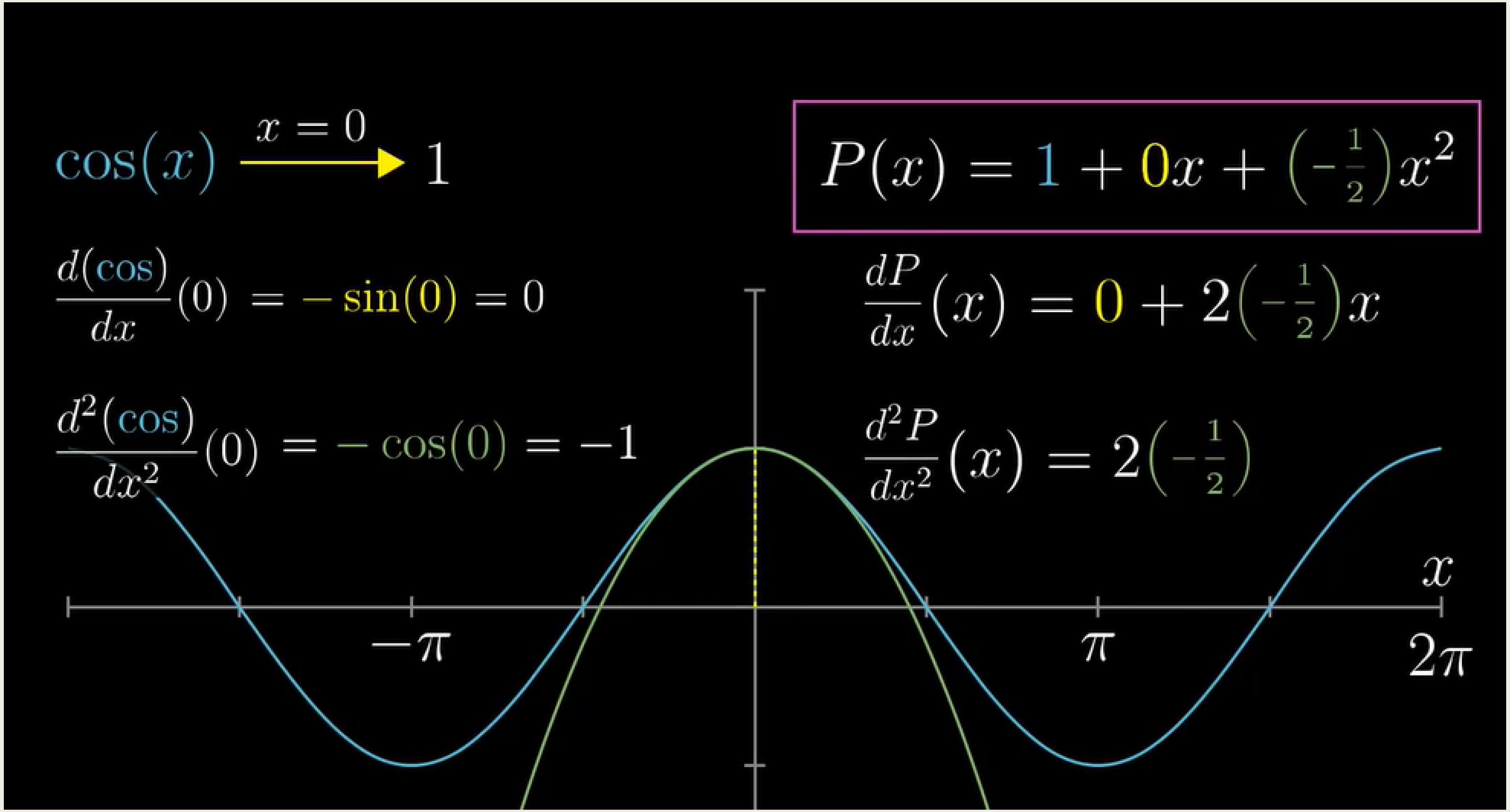
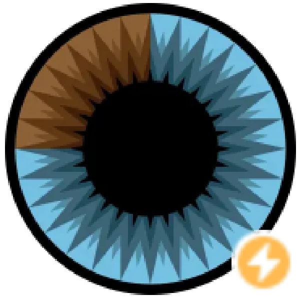


16. Taylor Series

- $f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$
- Taylor Polynomial
- Nth degree Taylor Polynomial
- Maclaurin Series
- Binomial Series $(x+1)^k$, Maclaurin Series

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n$$



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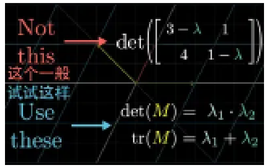

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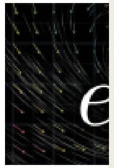

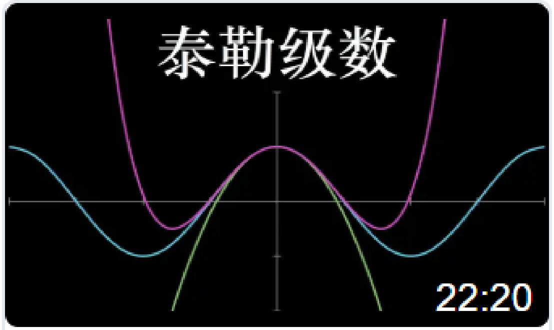

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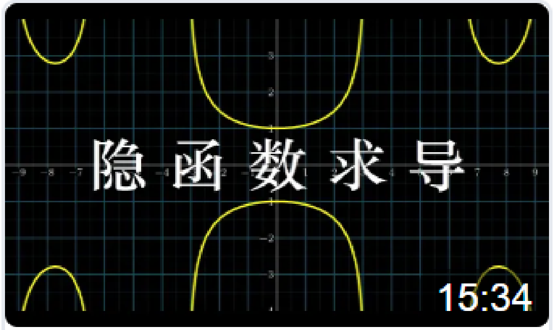

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Test

Nth Term Test for diverge

Integral Test

Comparison Test for Positive-Term series

Limit Comparison Test for Positive-Term series

Alternating Series Test

Ratio Test

Root Test

CHAPTER 11

Vector Curves and Surfaces in Spaces

1.Cross product

- $\vec{u} = (1,0,0); \vec{v} = (0,1,0)$
- $|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot sin\theta$
- How about direction?

- $\vec{u} \times \vec{v}$? $\vec{v} \times \vec{u}$

2. Matrix

- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot d - c \cdot b$

- $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
- $= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \cdots + \cdots$

- $= a_1 \cdot b_2 \cdot c_3 + b_1 \cdot c_2 \cdot a_3 + c_1 \cdot a_2 \cdot b_3 - a_3 \cdot b_2 \cdot c_1 - b_3 \cdot c_2 \cdot a_1 - c_3 \cdot a_2 \cdot b_1$

- $\vec{u} = (1,2,3); \vec{v} = (4,5,6)$
- $\vec{u} \times \vec{v}$?

3. Equation of line

Different Forms of equation for a line in Space

Point $\boldsymbol{P_0}(\boldsymbol{x_0}, \boldsymbol{y_0}, \boldsymbol{z_0})$

Direction Vector $\boldsymbol{v} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$

General Point $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$

$\overrightarrow{P_0P}$ is parallel to $\boldsymbol{v}$

$\overrightarrow{P_0P} = t\boldsymbol{v}$ for a real number t

$(\boldsymbol{x} - \boldsymbol{x_0}, \boldsymbol{y} - \boldsymbol{y_0}, \boldsymbol{z} - \boldsymbol{z_0}) = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) - (\boldsymbol{x_0}, \boldsymbol{y_0}, \boldsymbol{z_0}) = \overrightarrow{OP} - \overrightarrow{OP_0}$

$= \overrightarrow{P_0P} = t\boldsymbol{v} = t(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$

So, $\boldsymbol{x} - \boldsymbol{x_0} = t\,\boldsymbol{a}$, $\boldsymbol{y} - \boldsymbol{y_0} = t\,\boldsymbol{b}$ and $\boldsymbol{z} - \boldsymbol{z_0} = t\,\boldsymbol{c}$

$$\begin{cases} \boldsymbol{x} - \boldsymbol{x_0} = t\,\boldsymbol{a} \\ \boldsymbol{y} - \boldsymbol{y_0} = t\,\boldsymbol{b} \text{ for any } t \in R \\ \boldsymbol{z} - \boldsymbol{z_0} = t\,\boldsymbol{c} \end{cases}$$

Parametric Form of the line passing through the point $\boldsymbol{P_0}(\boldsymbol{x_0}, \boldsymbol{y_0}, \boldsymbol{z_0})$ with direction vector $\boldsymbol{v} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$

$$\frac{\boldsymbol{x} - \boldsymbol{x_0}}{\boldsymbol{a}} = \frac{\boldsymbol{y} - \boldsymbol{y_0}}{\boldsymbol{b}} = \frac{\boldsymbol{z} - \boldsymbol{z_0}}{\boldsymbol{c}}$$

Symmetric Form of the line passing through the point $\boldsymbol{P_0}(\boldsymbol{x_0}, \boldsymbol{y_0}, \boldsymbol{z_0})$ with direction vector $\boldsymbol{v} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$

(Assumed $a \neq 0$, $b \neq 0$ and $c \neq 0$)

4. Equation of plane

General Form of equation for a plane in Space

Point $\boldsymbol{P_0(x_0, y_0, z_0)}$

Normal Vector $\boldsymbol{n = (a, b, c)}$

General Point $\boldsymbol{P(x, y, z)}$

$\overrightarrow{P_0P}$ is perpendicular to $\boldsymbol{n}$

$\overrightarrow{P_0P} * \boldsymbol{n} = 0$

$\big((x, y, z) - (x_0, y_0, z_0)\big) * (a, b, c) = 0$

$(x - x_0, y - y_0, z - z_0) * (a, b, c) = 0$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$ax + by + cz = d$ where $d = ax_0 + by_0 + cz_0$

Q&A

- 考试加油嗷

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