A Comparative Study of Graph Matching Algorithms in Computer Vision

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Abstract

The graph matching optimization problem is an essential component for many tasks in computer vision, such as bringing two deformable objects in correspondence. Naturally, a wide range of applicable algorithms have been proposed in the last decades. Since a common standard benchmark has not been developed, their performance claims are often hard to verify as evaluation on differing problem instances and criteria make the results incomparable. To address these shortcomings, we present a comparative study of graph matching algorithms. We create a uniform benchmark where we collect and categorize a large set of existing and publicly available computer vision graph matching problems in a common format. At the same time we collect and categorize the most popular open-source implementations of graph matching algorithms. Their performance is evaluated in a way that is in line with the best practices for comparing optimization algorithms. The study is designed to be reproducible and extensible to serve as a valuable resource in the future.

Our study provides three notable insights: (i) popular problem instances are exactly solvable in substantially less than 1 second, and, therefore, are insufficient for future empirical evaluations; (ii) the most popular baseline methods are highly inferior to the best available methods; (iii) despite the NP-hardness of the problem, instances coming from vision applications are often solvable in a few seconds even for graphs with more than 500 vertices.

1 Introduction

Finding correspondences between elements of two discrete sets, such as keypoints in images or vertices of 3D meshes, is a fundamental problem in computer vision and as such highly relevant for numerous vision tasks, including 3D reconstruction [49], tracking [64], shape model learning [29], and image alignment [7], among others. Graph matching [24,55,63] is a standard way to

address such problems. In graph matching, vertices of the matched graphs correspond to the elements of the discrete sets to be matched. Graph edges define the cost structure of the problem: pairs of matched vertices are penalized in addition to the vertex-to-vertex matchings. This allows to take, e.g., the underlying geometrical relationship between vertices into account, but also makes the optimization problem NP-hard.

Deep graph matching [51] is a modern learning-based approach, that combines neural networks for computing matching costs with combinatorial graph matching algorithms to find a matching. The graph matching algorithm plays a crucial role in this context, as it has to provide high-quality solutions within a limited time budget. The high demand on run-time is also due to back-propagation learning and graph matching minimization being interleaved and executed together many times during training.

Hence, our work focuses on the *optimization* part of the graph matching pipeline. The modeling and learning aspects are beyond its scope. We evaluate a range of existing *open-source* algorithms. Our study compares their performance on a diverse set of computer vision problems. The focus of the evaluation lies on both, speed and objective value of the solution.

Why do we require a benchmark? Dozens of algorithms addressing the graph matching problem have been proposed in the computer vision literature, see, e.g., the surveys [24,55,63], and references therein. Most works promise state-of-the-art performance, which is persuasively demonstrated by experimental evaluation. However, (i) results from one article are often incomparable to results from another, since different problem instances with different costs are used, even if these instances are based on the same image data; (ii) not every existing method is evaluated on all available problem instances, even if open-source code is available. Some methods, especially those with poor performance on many instances, are very popular as baselines, whereas better performing techniques are hardly considered in comparisons; (iii) new algorithms are often only evaluated on easy, small-scale problems. This does not provide any information on how these algorithms perform on larger, more difficult problem instances. For these reasons, the field of graph matching has, in our view, not developed as well as it could have done. By providing a reproducible and extensible benchmark we hope to change this in the future. Such a benchmark is of particular importance for the fast-moving field of *deep graph matching*, as it helps to select an appropriate, fast solver for the combinatorial part of the learning pipeline.

Graph matching problem. Let $\mathcal V$ and $\mathcal L$ be the two finite sets, whose elements we want to match to each other. For each pair $i, j \in V$ and each pair $s, l \in \mathcal{L}$ a cost $c_{is,jl} \in \mathbb{R}$ is given. Each pair can be interpreted as an edge between a pair of vertices of an underlying graph. This is where the term *graph matching* comes from. Note that direct vertex-i-to-vertex-s matching costs are defined by the diagonal elements $c_{is,is}$ of the resulting cost (or affinity) matrix $C = (c_{is,il})$ with $is = (i,s) \in \mathcal{V} \times \mathcal{L}$, $jl = (j, l) \in \mathcal{V} \times \mathcal{L}$. The diagonal elements are referred to as unary costs, in contrast to the pairwise costs defined by non-diagonal entries. The goal of graph matching is to find a matching, or mutual assignment, between elements of the sets ${\mathcal V}$ and ${\mathcal L}$ that minimizes the total cost for all pairs of assignments. It can be formulated as the following integer quadratic problem¹:

$$\min_{x \in \{0,1\}^{\mathcal{V} \times \mathcal{L}}} \sum_{i,j \in \mathcal{V}} \sum_{s,l \in \mathcal{L}} c_{is,jl} x_{is} x_{jl}$$
s.t.
$$\begin{cases}
\forall i \in \mathcal{V} : \sum_{s \in \mathcal{L}} x_{is} \leq 1, \text{ and} \\
\forall s \in \mathcal{L} : \sum_{i \in \mathcal{V}} x_{is} \leq 1.
\end{cases}$$
(1)

The vector x defines the matching as $x_{is} = 1$ corresponds to assigning i to s. The inequalities in (1) allow for this assignment to be *incomplete*, i.e., some elements of both sets may remain unassigned. This is in contrast to *complete* assignments, where each element of $\mathcal V$ is matched to *exactly one* element of $\mathcal L$ and vice versa. Note that complete assignments require $|\mathcal V| = |\mathcal L|$.

Relation to the quadratic assignment problem. The graph matching problem (1) is closely related to the NP-hard [50] quadratic assignment problem (QAP) [12], which is well-studied in operations research [12,15,48]. The QAP only considers complete assignments, i.e., $|\mathcal{V}| = |\mathcal{L}|$ and equality is required in the constraints in (1). In contrast, in the field of computer vision incomplete assignments are often required in the model to allow for, e.g., outliers or matching of images with different numbers of features. Still, graph matching and the QAP are polynomially reducible to each other, see supplement for a proof.

The most famous QAP benchmark is the QAPLIB [11] containing 136 problem instances. However, the benchmark problems in computer vision (CV) substantially differ from those in QAPLIB both by the feasible set

that includes incomplete assignments, as well as by the structure of the cost matrix C: (i) CV problems are usually of a general, more expressive Lawler form [42], whereas QAPLIB considers factorizable costs $c_{is,jl}$ = $f_{ij}d_{sl}$ known as Koopmans-Beckmann form. The latter allows for more efficient specialized algorithms. (ii) the cost matrix C in QAPLIB is often dense, whereas in CV problems it is typically sparse, i.e., a large number of entries in *C* are 0; (iii) for CV problem instances the cost matrix C may contain infinite costs on the diagonal to prohibit certain vertex-to-vertex mappings; (iv) QAPLIB problems are different from an optimization point of view. For instance, while the classical LP relaxation [1] is often quite loose for QAPLIB problem instances, it is tight or nearly tight for typical instances considered in CV.

Consequently, comparison results on QAPLIB and CV instances differ significantly. It is also typical for NP-hard problems that instances coming from different applied areas require different optimization techniques. Therefore, a dedicated benchmarking on the CV datasets is required.

Contributions. Our contribution is three-fold: (i) Based on open source data, we collected, categorized and generated 451 existing graph matching instances grouped into 11 datasets in a common format. Most graph matching papers use only a small subset of these datasets for evaluation. Our format provides a ready-to-use cost matrix C and does not require any image analysis to extract the costs. (ii) We collected and categorized 20 open-source graph matching algorithms and evaluated them on the above datasets. During that we adapted the cost matrix to requirements of particular algorithms where needed. For each method we provide a brief technical description. We did not consider algorithms with no publicly available open source code. (iii) To allow our benchmark to grow further, we set up a web site² with all results. Our benchmark is reproducible, extensible and follows the best practices of [6]. We will maintain its web-page in the future and welcome scientists to add problem instances as well as algorithms.

Our work significantly excels evaluations in *all* the papers introducing the algorithms we study. This implies also to the largest existing comparison [31] so far. The latter considers only 8 out of the 11 datasets and evaluates 6 algorithms out of our 20.

2 Background to algorithms

In this section we briefly review the main theoretical concepts and building blocks of the considered approaches.

¹ For sets A and B the notation $x \in A^B$ denotes a vector x whose coordinates take on values from the set A and are indexed by elements of B, i.e., each element of B corresponds to a value from A.

² The web site for the benchmark is available at https://vislearn.github.io/gmbench/.

Linearization. In case all pairwise costs are zero, the objective in (1) linearizes to $\sum_{i \in \mathcal{V}, s \in \mathcal{L}} c_{is,is} x_{is}$, turning (1) into the *incomplete linear assignment problem* (*iLAP*). A typical way of how the iLAP is obtained in existing algorithms is by considering the Taylor expansion of the objective (1) in the vicinity of a given point x. The linear term of this expansion forms the iLAP objective. The iLAP can be reduced to a complete linear assignment problem (LAP) [10], see supplement, and addressed by, e.g., Hungarian [41] or auction [8] algorithms. Below, when we refer to LAP this includes both LAP and iLAP problems.

Birkhoff polytope and permutation matrices. For $|\mathcal{V}| = |\mathcal{L}|$ and $x \in [0,1]^{\mathcal{V} \times \mathcal{L}}$, where [0,1] denotes the closed interval from 0 to 1, the constraints $\sum_{s \in \mathcal{L}} x_{is} = 1$, $\forall i \in \mathcal{V}$, and $\sum_{i \in \mathcal{V}} x_{is} = 1$, $\forall s \in \mathcal{L}$, define the set of doubly-stochastic matrices also known as Birkhoff polytope. Its restriction to binary vectors $x \in \{0,1\}^{\mathcal{V} \times \mathcal{L}}$ is called the set of permutations or permutation matrices.

Inequality to equality transformation. By adding *slack* or *dummy* variables, indexed by #, with zero cost in the objective in (1), the uniqueness constraints in (1) can be rewritten as equalities for $\mathcal{V}^{\#} := \mathcal{V} \dot{\cup} \{\#\}$, $\mathcal{L}^{\#} := \mathcal{L} \dot{\cup} \{\#\}$, and $x \in [0,1]^{\mathcal{V}^{\#} \times \mathcal{L}^{\#}}$, where $\dot{\cup}$ is the disjoint union:

$$\mathcal{B} := \left\{ x \middle| \begin{array}{l} \forall i \in \mathcal{V} \colon & \sum_{s \in \mathcal{L}^{\#}} x_{is} = 1, \quad \text{and} \\ \forall s \in \mathcal{L} \colon & \sum_{i \in \mathcal{V}^{\#}} x_{is} = 1 \end{array} \right\}. \quad (2)$$

Here, $x_{i\#} = 1$ (or $x_{\#s} = 1$) means that the node i (or label s) is unassigned. Following [65], we refer to the elements of \mathcal{B} as *doubly-semi-stochastic matrices*.

Doubly-stochastic relaxation. Replacing the *integrality constraints* $x \in \{0,1\}^{\mathcal{V} \times \mathcal{L}}$ in (1) with the respective box constraints $x \in [0,1]^{\mathcal{V} \times \mathcal{L}}$ leads to a *doubly-stochastic relaxation* of the graph matching problem.³ Despite the convexity of its feasible set, the doubly-stochastic relaxation is NP-hard because of the non-convexity of its quadratic objective in general [53].

Probabilistic interpretation. Doubly-stochastic relaxations are often motivated from a probabilistic perspective, where the individual matrix entries represent matching probabilities. An alternative probabilistic interpretation is to consider the *product graph* between \mathcal{V} and \mathcal{L} , in which the edge weights directly depend on the cost matrix \mathcal{C} . This way, graph matching can be understood as selecting reliable nodes in the product graph, e.g., by random walks [17].

Injective and bijective formulations. Assume $|\mathcal{V}| \leq |\mathcal{L}|$. A number of existing approaches consider an asymmetric formulation of the graph matching problem (1), where the uppermost constraint in (1) is exchanged for equality, i.e., $\forall i \in \mathcal{V}: \sum_{s \in \mathcal{L}} x_{is} = 1$. We call this formulation *injective*. The strict inequality case $|\mathcal{V}| < |\mathcal{L}|$ is also referred to as an *unbalanced QAP* in the

literature. Note that to address problems of the general form (1) by such algorithms, it is necessary to extend the set \mathcal{L} with $|\mathcal{V}|$ dummy elements. This is similar to the reduction from graph matching to QAP described in the supplement. Since available implementations of multiple considered algorithms are additionally restricted to the case $|\mathcal{V}| = |\mathcal{L}|$, i.e. to the classical QAP as introduced in Section 1, we adopt the term bijective to describe the corresponding algorithms and datasets. The graph matching objective Spectral relaxation. in (1) can be compactly written as $x^{T}Cx$. Instead of the uniqueness constraints in (1) the spectral relaxation considers the non-convex constraint $x^{T}x = n$. This constraint includes all matchings with exactly n assignments, which is of interest when the total number of assignments n is known, e.g., for the injective formulation where $n = |\mathcal{V}|$. The minimization of $x^T C x$ subject to $x^{T}x = n$ reduces to an eigenvector problem, i.e., finding a vector *x* corresponding to the smallest eigenvalue of the matrix C. The latter amounts to minimizing a Rayleigh quotient [30].

Another way to deal with the Path following. non-convexity of the graph matching problem is pathfollowing, represented by [69] in our study. The idea is to solve a sequence of optimization problems with objective $f_{\alpha^t}(x) = (1 - \alpha^t) f_{\text{cvx}}(x) + \alpha^t f_{\text{cav}}(x)$ for α^t , $t \in 1, ..., N$, gradually growing from 0 to 1. The (approximate) solution of each problem in the sequence is used as a starting point for the next. The hope is that this iterative process, referred to as following the convexto-concave path, leads to a solution with low objective value for the whole problem. The objective for $\alpha^1 = 0$ is equal to $f_{cvx}(x)$ and is convex, therefore it can be solved to global optimality. The objective for $\alpha^N = 1$ is equal to $f_{cav}(x)$ and is concave. Its local optima over the set of doubly-stochastic matrices are guaranteed to be binary, and, therefore, feasible assignments, i.e., they satisfy all constraints of (1).

Graphical model representation. The graph matching problem can be represented in the form of a *maximum a posteriori* (*MAP*) *inference* problem for discrete graphical models [54], known also as *Markov random field* (*MRF*) *energy minimization* and closely related to *valued and weighted constraint satisfaction* problems. As several graph matching works in computer vision [31, 57, 67], use this representation, we provide it below in more detail.

Let $(\mathcal{V}, \mathcal{E})$ be an undirected graph, with the finite set \mathcal{V} introduced above being the *set of nodes* and $\mathcal{E} \subseteq \binom{\mathcal{V}}{2}$ being the set of *edges*. For convenience we denote edges $\{i, j\} \in \mathcal{E}$ simply by ij. Let the finite set \mathcal{L} introduced above be the set of *labels*. We associate with each node $i \in \mathcal{V}$ a set $\mathcal{L}_i^\# = \mathcal{L}_i \ \dot{\cup} \ \{\#\}$ with $\mathcal{L}_i \subseteq \mathcal{L}$. Like above, # stands for the *dummy label* distinct from all labels in \mathcal{L} to encode that *no label* is selected. With each label $s \in \mathcal{L}_i^\#$ in each node $i \in \mathcal{V}$ the *unary cost* $\theta_{is} := c_{is,is}$ (0 for s = #) is associated. The case $|\mathcal{L}_i| < |\mathcal{L}|$ corresponds to infinite unary costs $c_{is,is} = \infty$, $s \in \mathcal{L} \setminus \mathcal{L}_i$, as the respective

³ Strictly speaking, the term *doubly-stochastic* corresponds to the case when equality constraints are considered in (1). In [65] the inequality variant is called *doubly semi-stochastic* but we use *doubly-stochastic* in both cases.

assignments can be excluded from the very beginning. Likewise, with each edge $ij \in \mathcal{E}$ and each label pair $sl \in \mathcal{L}_i^\# \times \mathcal{L}_j^\#$, the pairwise cost $\theta_{is,jl} = c_{is,jl} + c_{jl,is}$ (0 for s or l = #) is associated. The graph $(\mathcal{V}, \mathcal{E})$ being undirected implies ij = ji and $\theta_{is,jl} = \theta_{jl,is}$. An edge ij belongs to \mathcal{E} only if there is a label pair $sl \in \mathcal{L}_i \times \mathcal{L}_j$ such that $\theta_{is,jl} \neq 0$. In this way a sparse cost matrix \mathcal{C} may translate into a sparse graph $(\mathcal{V}, \mathcal{E})$.

The problem of finding an optimal assignment of labels to nodes, equivalent to the *graph matching* problem (1), can thus be stated as

$$\min_{y \in \mathcal{Y}} \left[E(y) := \sum_{i \in \mathcal{V}} \theta_{iy_i} + \sum_{ij \in \mathcal{E}} \theta_{iy_i, jy_j} \right]
\text{s.t.} \quad \forall i, j \in \mathcal{V}, i \neq j : y_i \neq y_j \text{ or } y_i = \#$$
(3)

where \mathcal{Y} stands for the Cartesian product $\underset{i \in \mathcal{V}}{\times} \mathcal{L}_{i}^{\#}$, and $y_{i} = s$, $s \in \mathcal{L}_{i}$, is equivalent to $x_{is} = 1$ in terms of (1). Essentially, (3) corresponds to a *MAP inference problem for discrete graphical models* [54] with additional uniqueness constraints for the labels.

ILP representation and LP relaxations. Based on (3) the graph matching problem can be expressed by a linear objective subject to linear and integrality constraints by introducing variables $x_{is,jl} = x_{jl,is}$ for each pair of labels $sl \in \mathcal{L}_i^\# \times \mathcal{L}_j^\#$ in neighboring nodes $ij \in \mathcal{E}$, and enforcing the equality $x_{is,jl} = x_{is}x_{jl}$ with suitable linear constraints. An *integer linear program (ILP)* formulation of the graph matching problem (1) can then be written as:

$$\min_{\substack{x \in \{0,1\}^{\mathcal{I}} \\ s \in \mathcal{L}_{i}^{\#}}} \sum_{i \in \mathcal{V}} c_{is} x_{is} + \sum_{ij \in \mathcal{E}} (c_{is,jl} + c_{jl,is}) x_{is,jl}$$
(4)

$$\forall i \in \mathcal{V}: \sum_{s \in \mathcal{L}_{i}^{\#}} x_{is} = 1, \quad \forall s \in \mathcal{L}: \sum_{i \in \mathcal{V}} x_{is} \leq 1, \quad (5)$$

$$\forall ij \in \mathcal{E}, \ l \in \mathcal{L}_{j}^{\#} \colon \sum_{s \in \mathcal{L}_{i}^{\#}} x_{is,jl} = x_{jl} \,. \tag{6}$$

Here $\mathcal{J} = \{(i,s) \colon i \in \mathcal{V}, \ s \in \mathcal{L}_i^\#\} \cup \{(is,jl) \colon ij \in \mathcal{E}, sl \in \mathcal{L}_i^\# \times \mathcal{L}_j^\#\}$ denotes the set of coordinates of the vector x. The formulation (4)-(6) differs from the standard ILP representation for discrete graphical models by the label uniqueness constraints (5, rightmost). Substitution of the integrality constraints $x \in \{0,1\}^{\mathcal{J}}$ in (4) with the box constraints $x \in [0,1]^{\mathcal{J}}$ results in the respective LP relaxation.

3 Graph matching algorithms

Below we summarize the graph matching methods that we consider in our comparison, see Table 1 for an overview of their characteristics and references.

3.1 Primal heuristics

Linearization based. These methods are based on iterative linearizations of the quadratic objective (1) derived from its Taylor expansion.

Iterated projected fixed point (ipfp) [46] solves on each iteration the LAP obtained through linearization in the vicinity of a current, in general non-integer, assignment. Between iterations the quadratic objective is optimized along the direction to the obtained LAP solution, which yields a new, in general non-integer assignment. We evaluate two versions of ipfp which differ by their initialization: ipfpu is initialized with $x^0 \in [0,1]^{\mathcal{V} \times \mathcal{L}}$, where $x^0_{is} = 1/\sqrt{N}$ if $c_{is,is} < \infty$, and $x^0_{is} = 0$ otherwise. Here, $N := |\{is \in \mathcal{V} \times \mathcal{L} \mid c_{is,is} < \infty\}|$. ipfps starts from the result of the spectral matching sm [43] described below.

Graduated assignment (ga) [27] optimizes the doubly-stochastic relaxation. On each iteration it approximately solves the LAP obtained through linearization in the vicinity of a current, in general non-integer, assignment utilizing the Sinkhorn algorithm [40] for a given fixed temperature. The obtained approximate solution is used afterwards as the new assignment. The temperature is decreased over iterations to gradually make the solutions closer to integral.

Fast approximate quadratic programming (fw) [62] considers the Frank-Wolfe method [25] for optimizing over the set \mathcal{B} , c.f. (2). Each iteration first solves a LAP to find the optimum of the linearization at the current solution, followed by a line search in order to find the best convex combination of the current and the new solution. To obtain an integer solution, the objective of the LAP solution is evaluated in each iteration, and the lowest one among all solutions is kept. The initial LAP is based on the unary costs only. The implementation [56] we evaluate is applicable to the general Lawler form of the problem (1), in contrast to the Koopmans-Beckmann form addressed in [62].

Norm constraints based. *Spectral matching* (sm) [43] uses a spectral relaxation that amounts to a Rayleigh quotient problem [30] which can be optimized by the power iteration method. Here, each update comprises of a simple matrix multiplication and a subsequent normalization, so that x^t is iteratively updated via $x^{t+1} = -Cx^t/\|Cx^t\|_2$.

Spectral matching with affine constraints (smac) [21] is similar to sm, but additionally takes into account affine equality constraints that enforce one-to-one matchings. The resulting formulation amounts to a Rayleigh quotient problem under affine constraints, that can efficiently be computed in terms of the eigenvalue decomposition.

Max-pooling matching (mpm) [18] resembles sm, but it replaces the sum-pooling implemented in terms of the matrix multiplication -Cx in the power iteration update of SM by a max-pooling operation. With that, only

method	IQP	ILP	bijective	non-pos.	0-unary	lineariz.	norm	doubly	spectral	discret.	path fol.	fusion	duality	SGA	BCA	Matlab	ŧ
fgmd [69]	_	_	+	_	_	_	_		0,	_	+			0,		[68]	
fm [31]	·	+	ľ								Ċ	+				[OO]	[32]
fw [62]	+	ľ				+		+				Ċ					[56]
ga [27]	+		+			+		+		+						[20]	
ipfps [46]	+		+	+		+		+		+						[44]	
ipfpu [46]	+		+			+		+		+						[44]	
1sm [33]	+		+	+			+			+						[66]	
mpm [18]	+		+	+			+			+						[19]	
pm [65]	+		+	+	+			+		+						[68]	
rrwm [17]	+		+			+		+		+						[16]	
smac [21]	+		+	+			+		+	+						[20]	
sm [43]	+		+	+			+		+	+						[44]	
dd-ls(0/3/4)[59]		+											+	+			[38]
fm-bca[31]		+										+	+		+		[32]
hbp [67]		+	+										+		+	[66]	
mp(-mcf/-fw) [57]		+											+		+		[56]

Meaning of properties ('+' indicates presence): IQP: addresses IQP formulation; ILP: addresses ILP formulation; bijective: addresses bijective formulation; non-pos.: requires non-positive costs, see Remark 1; 0unary: requires zero unary costs; lineariz.: linearization-based method; norm: imposes norm-constraints; doubly: addresses doublystochastic relaxation; spectral: solves spectral relaxation; discret.: discretization as in Remark 2; path fol.: path following method; fusion: utilizes fusion; duality: Lagrange dualitybased; SGA: uses dual sub-gradient ascent; BCA: uses dual block-coordinate ascent; Matlab: implemented in Matlab [reference to code]; C++: implemented in C++ [reference to

Table 1: Method properties. Purely primal heuristics are separated from the dual methods by a horizontal line.

candidate matches with the smallest costs are taken into account.

Local sparse model (1sm) [33] solves the relaxation $\max_{x} x^{\top} C x$, s.t. $||x||_{1,2}^2 = \sum_{i=1}^{|\mathcal{V}|} \left(\sum_{k=1}^{|\mathcal{L}|} |x_{ik}|\right)^2 = 1$, $x \ge 0$. The $l_{1,2}$ -norm $||x||_{1,2}$ should encourage the solution of the above relaxation to be sparse in each row when treating x as a matrix. This resembles the sparsity property of permutation matrices, which satisfy $||x||_{1,2} = |\mathcal{V}|$.

Remark 1. All of the norm constraints based algorithms described above require non-positive costs in order to guarantee convergence of the underlying iterative techniques. This condition can be w.l.o.g. assumed for any graph matching problem. The corresponding cost transformation is described in the supplement.

Probabilistic interpretation based. Reweighted Random Walks Matching (rrwm) [17] interprets graph matching as the problem of selecting reliable nodes in an association graph, whose weighted adjacency matrix is given by -C. Nodes are selected through a random walk that starts from one node and randomly visits nodes according to a Markov transition matrix derived from the edge weights of the association graph. In order to take into account matching constraints, the authors of [17] consider a reweighted random walk strategy.

Probabilistic matching (pm) [65] considers a probabilistic formulation of graph matching in which the quadratic objective is replaced by a relative entropy objective. It is shown that by doing so one can obtain a convex problem formulation via marginalization, which is optimized in terms of an iterative successive projection algorithm.

Remark 2. Most of the primal heuristics considered above aim to optimize the quadratic objective (1) over a continuous

set such as, e.g., the Birkhoff polytope. The resulting assignment $x \in \mathbb{R}^{V \times L}$ is, therefore, not guaranteed to be integer. As suggested in [17], to obtain an integer assignment we solve a LAP with (-x) treated as the cost matrix. We apply this procedure as a postprocessing step for ipfp, ga, sm, smac, mpm, 1sm, rrwm, and pm. Note that this postprocessing does not change an integer assignment.

Path following based. Factorized graph matching (fgmd) [69] proposes an efficient factorization of the cost matrix to speed-up computations, and is based on the convex-concave path following strategy, see Section 2. Individual problems from the path are solved with the Frank-Wolfe method [25].

Randomized generation and fusion based. Fusion moves with a greedy heuristic (fm) [31] is based on the graphical model representation and consists of two parts: A randomized greedy assignment generation, and fusion of the assignments. The randomized generator greedily fixes labels in the nodes in a way that minimizes the objective value restricted to the already fixed labels. The fusion procedure merges the current assignment with the next generated one by approximately solving an auxiliary binary MAP inference problem utilizing QPBO-I [52]. The merged solution is guaranteed to be at least as good as the two input assignments. This property guarantees monotonic improvement of the objective value.

3.2 Lagrange duality-based techniques

The methods below consider the Lagrange decompositions [28] of the graph matching problem (1) [59], or its graphical model representation (3) [31, 57, 67], and optimize the corresponding dual. The methods differ in the dual optimization and chosen primal solution reconstruction algorithms.

⁴ Non-negative in original maximization formulations

	(i) graphical model	(ii) LAP	(iii) primal
hbp	MPLP [26]	Hungarian [41]	branch & bound
mp(-mcf)	anisotropic diffu- sion [54]	network sim-	LAP
mp-fw	∫sion [54]	∫plex [2]	fw
fm-bca	MPLP++ [60]	BCA	fm

Algorithms used for optimizing the *graphical model* and *LAP* part, as well as technique used to obtain a *primal* solution. For *LAP* in the *primal* column the solution of the LAP subproblem is reused as feasible assignment. Instead of solving the LAP subproblem, fm-bca performs a series of *BCA* steps wrt. the LAP dual variables.

Table 2: Characterization of dual BCA algorithms.

Block-coordinate methods (hbp, mp-*, fm-bca). The works [31, 57, 67] employ a block-coordinate ascent (BCA) technique to optimize the dual problem obtained by relaxing the coupling (6) and label uniqueness constraints (5, rightmost). Since the dual is piece-wise linear, BCA algorithms may not attain the dual optimum, but may get stuck in a sub-optimal fixed point [9, 54].

Although the elementary operations performed by these algorithms are very similar, their convergence speed and attained fixed points differ drastically. In a nutshell, these methods decompose the problem (3) into the graphical model without uniqueness constraints, and the LAP problem, as described, e.g., in [31]. Dual algorithms reparametrize the problem making it more amenable to primal techniques [54]. Table 2 gives an overview of the evaluated combinations for (i) optimizing the dual of the graphical model, (ii) optimizing the LAP, and (iii) obtaining the primal solution from the reparametrized costs⁵, which influence the practical performance of BCA solvers. Additionally, mp-mcf and mp-fw tighten the relaxation by considering triples of graph nodes as subproblems.

Subgradient method (dd-ls*). The algorithms denoted as dd-ls* with * being 0, 3 or 4 represent different variants of a dual subgradient optimization method [59]. The variant dd-1s0 addresses the relaxation equivalent to a symmetrized graphical model formulation, see supplement for a description. This is achieved by considering the Lagrange decomposition of the problem into two graphical models, with V and $\mathcal L$ being the set of nodes, respectively, and a LAP subproblem. The graphical models are further decomposed into acyclic ones, i.e. trees, solvable by dynamic programming, see, e.g., [54, Ch.9]. The tree decomposition is not described in [59], and we reconstructed it based on the source code [38] and communication with the authors. As we observed it to be more efficient than the *max-flow* subproblems suggested in the paper [59] the latter were not used in our evaluation.

Variants dd-1s3 and dd-1s4 tighten the relaxation of dd-1s0 by considering *local subproblems* of both graphical models in the decomposition. These are obtained by reducing the node sets $\mathcal V$ and $\mathcal L$ to 3 or respectively 4 elements inducing a connected subgraph of the graphical model, see [59] for details.

4 Benchmark

Datasets. The 11 datasets we collected for evaluation of the graph matching algorithms stem from applications in computer vision and bio-imaging. All existing graph matching papers use only a subset of these datasets for evaluation purposes. Together these datasets contain 451 problem instances. Table 3 gives an overview of their characteristics. We modified costs in several datasets to make them amenable to some algorithms, see supplement. Our modification results in a constant shift of the objective value for each feasible assignment, and, therefore, does not influence the quality of the solution.

Below we give a brief description of each dataset. Along with the standard computer vision datasets with small-sized problems, hotel, house-dense/sparse, car, motor and opengm with $|\mathcal{V}|$ up to 52, our collection contains the middle-sized problems flow, with $|\mathcal{V}|$ up to 126, and the large-scale worms and pairs problems with $|\mathcal{V}|$ up to 565.

Wide baseline matching (hotel, house-dense/sparse) is based on a series of images of the same synthetic object with manually selected landmarks from different viewing angles based on the work by [14]. For hotel and house-dense we use the same models as in [57] published in [58]. house-sparse consists of the same image pairs as house-dense, but the cost structure is derived following the approach of [67] that results in significantly sparser problem instances. Graphs with the landmarks as nodes are obtained by Delaunay triangulation. The costs are set to $c_{is,jl} = -\exp(-(d_{ij}-d_{sl})^2/2500)A_{ij}^1A_{sl}^2$ where d_{ij} , d_{sl} are Euclidean distances between two landmarks and $A^1 \in \{0,1\}^{\mathcal{V}\times\mathcal{V}}$, $A^2 \in \{0,1\}^{\mathcal{L}\times\mathcal{L}}$ are adjacency matrices of the corresponding graphs. The unary costs are zero.

Keypoint matching (car, motor) contains *car* and *motor*bike images from the PASCAL VOC 2007 Challenge [23] with the features and costs from [47]. We use the instances available from [32].

Large displacement flow (flow) was introduced by [3] for key point matching on scenes with large motion. We use the instances from [32] which use keypoints and costs as in [57].

OpenGM matching (opengm) is a set of non-rigid point matching problems by [39], now part of the *OpenGM* Benchmark [36]. We use the instances from [32].

 $^{^5\,\}mbox{\it Reparametrized}$ costs are also known as $\mbox{\it reduced}$ costs, e.g., in the simplex tableau.

dataset	#inst.	#opt.	bijective	injective	non-pos.	0-unary	<u> </u>	\lambda \lambda	density (%)	diagonal dens. (%)	data
caltech-large [17]	9	1		+	+	+	36-219	0.4-2.3	0.55	18.1	[16]
caltech-small[17]	21	12		+	+	+	9-117	0.4-3	0.99	26.8	[16]
car [23,47]	30	30	+		+		19-49	1	2.9	100	[45]
flow [3,57]	6	6					48-126	≈ 1	0.39	15.8	[4, 5]
hotel [14,59]	105	105	+				30	1	12.8	100	[13, 58]
house-dense [14,59]	105	105	+				30	1	12.6	100	[13, 58]
house-sparse [14,67]	105	105	+		+	+	30	1	1.5	100	[13]
motor [23,47]	20	20	+		+		15-52	1	3.8	100	[45]
opengm [36,39]	4	4	+				19-20	1	74.8	100	[37]
pairs [31,34]	16	0					511-565	≈ 1	0.0019	3.7	[32]
worms [34]	30	28					558	≈ 2.4	0.00038	1.6	[35]

Meaning of properties:

#inst.: number of problem instances; #opt.: number of known optima; bijective/injective: bi-/injective assignment is assumed; non-pos.: all costs are non-positive; 0-unary: datasets with zero unary costs; $|\mathcal{V}|$: number of elements in \mathcal{V} ; $|\mathcal{L}|/|\mathcal{V}|$: ratio of the number of elements in \mathcal{L} to the number of elements in \mathcal{V} ; density (%): percentage of non-zero elements in the cost matrix \mathcal{C} ; diag. dens. (%): percentage of non-infinite elements on the diagonal of \mathcal{C} ; data: [references] to problem instances, images, feature coordinates or ground truth.

Table 3: Dataset properties. A '+' indicates that all problem instances of the dataset have the respective property.

The caltech dataset was proposed in [17]. The data available at the project page [16] contains the *mutual projection error* matrix $D = (d_{is,jl})$, lists of possible assignments, and partial ground truth. We reconstructed the dataset from this data. Unary costs are set to zero. Pairwise costs for pairs of possible assignments are set to $c_{is,jl} = -\max(50 - d_{is,jl}, 0)$. We divided the dataset into caltech-small and caltech-large, where all instances with more than 40000 non-zero pairwise costs are considered as large.

Worm atlas matching (worms) has the goal to annotate nuclei of *C. elegans*, a famous model organism used in developmental biology, by assigning nuclei names from a known atlas of the organism. A detailed description can be found in [34]. We use the instances obtained from [32] which are originally from [35].

Worm-to-worm matching (pairs) directly matches the cell nuclei of individual *C. elegans* worms to each other. The resulting models are much coarser than those of the worms dataset. We consider the same 16 problem instances as [31] using the models from [32].

Evaluation metrics. For *fixed-time* performance evaluation [6] we restrict run-time $(1, 10, 100 \, \text{s})$ and evaluate attained objective values E, lower bound D and, for datasets with ground truth available, accuracy *acc*. We also report the number of optimally solved instances per dataset.

For *fixed-target* performance evaluation [6] we measure the time $t_s(p)$ until each solver s solves the problem p within an optimality tolerance of 0.1%. For instances with unknown optimum, we consider the best achieved objective value across all methods as optimum as suggested in [6]. The performance ratio to the best solver is computed by $r_s(p) = \frac{t_s(p)}{\min\{t_s(p):\forall s\}}$. We create a performance profile [6, 22] by computing $\rho_s(\tau) = \frac{1}{|P|} \cdot |\{r_s(p) \leq \tau : \forall p\}|$ for each solver s where |P| denotes the total number of problem instances. Intuitively, $\rho_s(\tau)$ is the probability of solver s being at most t times slower than the fastest solver.

5 Empirical Results

Fixed-time evaluation presented in Table 4 addresses small problem instances, whereas Table 5 addresses mid-size and large problem instances. The performance profile for fixed-target evaluation is presented in Figure 1. More detailed results are available in the supplement. Results have been obtained by taking the minimum run-time across five trials on an AMD EPYC 7702 2.0 GHz processor. Randomized alogrithms were made deterministic by fixing their random seed (fm and fm-bca). We equally treat Matlab and C++ implementations, in spite of the apparent efficiency considerations. The reason for this is that the solution quality of *all* Matlab algorithms in our evaluation is inferior to the C++ techniques, even if run-time is ignored.

For small problems we show results for 1 second in Table 4, as the best methods already solve almost all instances to optimality within this time. The best methods on these datasets are fm, fm-bca and dd-ls0. dd-ls3/4 have higher costs per iteration, and require more than 1 second to arrive at the solution quality of dd-ls0. The other dual BCA-based methods perform almost as good on all but the opengm dataset, which seems to be the most difficult dataset amongs the one in Table 4. Apart from fm pure primal heuristics are unable to compete with duality-based techniques. The comparison of the results for house-dense and house-sparse shows that most of the primal heuristics perform much better on sparse problems.

For larger problems the most representative times shown in Table 5 are 1, 10 and 100 seconds, depending on the dataset. Again, the duality-based methods and the fm heuristic lead the table. The fm-bca method consistently attains the best or close to best objective and accuracy values on all datasets, whereas its lower bound is often worse than the lower bounds obtained by the mp-* and dd-ls* methods. In contrast, most of the primal heuristics as well as hbp fail, and, for brevity, are omitted in Table 5.

	h	otel(1	s)	hous	se-dens	se (1s)	hous	se-sı	parse(1s)	С	ar (1	s)	mo	tor(1s)	open	ıgm (1s)	cal	tech-s	mall(1s)
	opt	Ε	acc	opt	Ε	acc	opt	Ε	acc	opt	E	acc	opt	Ε	acc	opt	Ε	opt	Ε	acc
fgmd	0	—,	*	0	_	*	0		*	0	_	_*	0	_	_*	0	*	0	-	*
fm	97	-4292	100	100	-3778	100	100	-67	100	77	-69	88	90	-63	93	100	-171	52	-8906	58
fw	97	-4288	99	100	-3778	100	0	0	0	7	-63	63	20	-58	70	0	-152	0	0	0
ga	0	947	15	0	3491	8	100	-67	100	57	-68	84	55	-62	90	50	-167	0	-	 *
ipfps	0	1051	14	0	3654	8	100	-67	100	10	-65	80	25	-61	85	0	-95	19	-8983	67
ipfpu	0	1062	15	0	3659	8	100	-67	100	7	-60	69	15	-58	77	0	-86	10	-8829	62
lsm	0	—	*	0	_	_*	46	-65	96	0	-51	52	10	-52	64	0	-67	0		<u></u> *
mpm	43	-2585	78	0	1260	53	0	-60	90	7	_	_*	5	_	_*	0	-94	0	-	 *
pm	0	775	33	0	3262	18	0	-54	83	0	-35	23	0	-35	32	0	-83	0	-6510	51
rrwm	0	744	15	0	2895	10	100	-67	100	37	-68	87	50	-62	89	0	-154	5	-	 *
sm	0	1086	13	0	3789	9	96	-67	100	7	-63	76	40	-60	87	0	-101	0	-3932	36
smac	1	-1571	61	0	2817	31	37	-44	63	0	-52	52	10	-52	66	0	-84	0	-6196	42
dd-1s0	100	-4293	100	100	-3778	100	100	-67	100	97	-69	91	100	-63	97	50	-160	43	-7414	58
dd-1s3	100	-4293	100	100	-3778	100	96	-66	100	47	-57	74	65	-57	87	0	-118	33	-6842	57
dd-ls4	98	-4291	100	92	-3763	99	18	-56	86	3	-49	59	30	-52	78	0	-105	24	-6332	54
fm-bca	100	-4293	100	100	-3778	100	100	-67	100	93	-69	92	100	-63	97	75	-170	38	-8927	62
hbp	97	—	*	98	_	*	100	-67	100	77	_	_*	95	_	_*	0	*	0	-	_*
mp	93	-4280	99	99	-3777	100	100	-67	100	80	-69	92	90	-63	96	0	-57	14	-7967	59
mp-fw	99	-4292	100	100	-3778	100	100	-67	100	90	-69	91	95	-63	98	0	-150	33	-8886	60
mp-mcf	90	-4245	98	31	-3542	89	100	-67	100	87	-69	91	90	-63	98	0	-57	5	-7882	60

opt: optimally solved instances (%); E: average best objective value; acc: average accuracy corresponding to best objective (%)
—*: method yields no solution for at least one problem instance within the given time interval.

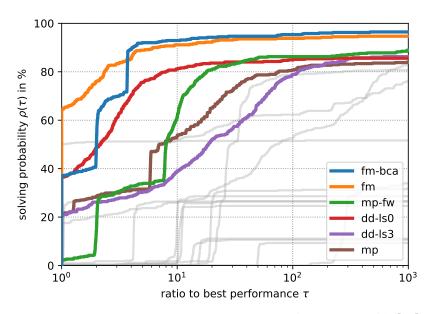
Table 4: Fixed-time evaluation of small problem instances. Maximal run-time per problem instance is 1 second. Boldface marks best values, except for accuracy since algorithm do not optimize it explicitly and do not have access to ground truth. Horizontal line separates purely primal from duality-based methods. Accuracy omitted for opengm as no ground truth available. Dual bounds omitted as most problems are solved optimally.

		flow(1s)		worr	ns (1s)		caltec	h-large	(10s)	caltec	h-large	(100s)	pai	rs (10s)		paiı	rs (100s	;)
	opt	Ε	D	opt	E	D	acc	E	D	acc	E	D	acc	E	D	acc	E	D	acc
fm	83	-2838	-3436	93	-48457	-55757	89	-34117	-142829	52	-34125	-142829	52	-65625	-76418	54	-65825	-76418	55
fw	67	-2828	-	0	-46974	_	81	0	-	0	0	_	0	-65797	-	54	-65802	-	54
dd-1s0	33	-2345	-2968	0	60443	-163870	26	-32973	-35007	51	-33539	-34959	52	-61482	-73521	41	-62974	-67306	57
dd-1s3	17	-2059	-3030	0	64017	-160520	24	-28653	-42079	49	-33552	-34914	49	-61638	-73528	41	-62426	-67599	50
dd-ls4	0	-2062	-3090	0	65731	-160409	24	-25599	-62120	46	-30148	-38880	51	-61634	-74053	41	-61634	-70214	41
fm-bca	83	-2838	-2898	93	-48460	-48514	89	-34040	-48223	51	-34073	-48217	51	-65567	-70163	55	-65913	-69003	58
mp	33	-2628	-2887	0		_*		-32017	-46070	48	-32069	-46066	48	-64150	-68255	57	-64380	-68136	57
mp-fw	83	_	_*	0		*		-34237	-48882	51	-34277	-45923	51		*			_*	
mp-mcf	33	-2521	-2892	0		_*		-30362	-46630	47	-30737	-43833	47	-63990	-68318	56	-64174	-68053	57

opt, E, acc, -*: same as in Table 4;

D: best attained lower bound if applicable, i.e., for dual methods, otherwise –

Table 5: Fixed-time evaluation of mid-size and large problem instances. Only the best performing algorithms are shown. Notation is the same as in Table 4. For each dataset the maximum allowed run-time per instance is given in parentheses. For flow no ground truth is available, so the column *acc* is omitted. For caltech-large and pairs no global optima are known, and the column *opt* is omitted.



Methods solving the largest number of instances, see $\rho(\tau=10^3)$, are highlighted in color. Other methods are shown as "ghosts", i.e., unlabeled in gray. fm is the best solver in 65% of all cases, see $\rho(\tau=1)$. fm-bca outperforms fm when the allowed performance ratio is increased to $\tau \geq 3.7$. Overall, fm-bca solves $\approx 97\%$ and fm solves $\approx 95\%$ of all instances. Following are duality-based methods like mp-fw and dd-1s0.

Figure 1: Run time performance profile [22] across all 451 instances.

Algorithms dd-1s3/4 consider tighter relaxations than dd-1s0, but are slower, therefore lose in the competition on short time intervals. However, they have the ability to attain the best lower bounds given longer runs ($\gg 100$ s).

There is a significant performance gap between the closely related hbp, mp-* and fm-bca methods. Foremost, this is explained by the method for reconstructing the primal solution: The fm algorithm used in the fm-bca solver is solid also as a stand-alone technique, and significantly outperforms the fw and LAP heuristics used in the mp-* algorithms. The branch-and-bound solver used in hbp is quite slow and does not scale well. The second reason for different performance of these methods is the specific BCA algorithm used for the underlying discrete graphical model, c.f. Table 2. According to the recent study [61], which provides a unified treatment of the dual BCA methods for dense⁶ graphical models, MPLP++ performs best, followed by anisotropic diffusion and MPLP as the slowest method. Table 5 shows that there is no solution suitable for every purpose: The speed of fm-bca comes at the price of a looser lower bound. Nonetheless, combining a primal heuristic with a dual optimizer consistently improves upon the results obtained by the heuristic alone. This holds for fm, but the effect is even more pronounced for the fw and LAP heuristics.

The fixed-target evaluation in Figure 1 confirms that the fm and fm-bca method are amongst the best performing solvers. While fm-bca uses fm as primal heuristics with additional dual BCA updates, the overhead of the latter is visible. After increasing the allowed performance ratio for fm-bca to a factor of 3.7, we can expect better solutions than fm alone. Other top performers are duality-based algorithms with mp-fw and dd-ls0 being the closest followers.

6 Conclusions

Our evaluation shows that: (i) Most instances from the popular datasets hotel, house, car and motor can be solved to optimality in well below a second by several optimization techniques. opengm can also be solved to optimality in under a second, although it turns out to be hard for many methods. Therefore, we argue that these datasets alone are not sufficient anymore to empirically show efficiency of new algorithms. The most difficult in our collection are the datasets caltech-* and pairs. For a comprehensive evaluation of new methods more datasets are required. (ii) The most popular comparison baselines like ipfp, ga, rrwm, pm, sm, smac, 1sm, mpm and fgmd are not competitive, and, therefore, comparison to these alone should not anymore be considered as sufficient. (iii) The most efficient methods are duality-based techniques equipped with efficient primal heuristics. In particular, the fm/fm-bca method currently attains the best or nearly best objective values for most problem instances in the shortest time. (iv) Although being NPhard in general, the graph matching problem can be often efficiently solved in computer vision practice. For many of the considered datasets, including those with $|\mathcal{L}| > 1000$ and $|\mathcal{V}| > 500$, a reasonable approximate solution can be attained in less than a second.

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⁶ Most of the considered graphical models are dense in terms of [61].

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Supplementary Material – A Comparative Study of Graph Matching Algorithms in Computer Vision

A1 Equivalence Proofs

Theorem 1. The graph matching problem and the quadratic assignment problem (QAP) are polynomially reducible to each other.

While the formal proof can be found below, let us develop an intuition for the problem behind Theorem 1. In general, the main difference between graph matching and the QAP is that while the QAP requires complete matchings, graph matching allows for incomplete matchings, i.e., not every element of the first set has to be assigned to one of the second and vice versa. So the equivalence construction mainly has to deal with these side constraints.

When transitioning from graph matching to the QAP, we go from incomplete to complete matchings. Therefore, the idea is to extend the two sets, usually called ${\mathcal V}$ and ${\mathcal L}$, by "dummy" elements to which all previously unassigned elements can be assigned without changing the total cost, see Figure A1 for an illustration.

On the other hand, when going from the QAP to graph matching, we switch from a complete to an incomplete matching. Here, the idea is to shift the cost structure in such a way that the cost difference of any two complete matchings stays the same, while at the same time guaranteeing that any incomplete matching can be improved by completing it, see Figure A2 for an illustration.

Proof. We will prove Theorem 1 by construction.

Reduction of graph matching to QAP. Consider the graph matching problem, c.f. (1),

$$\min_{\substack{x \in \{0,1\}^{\mathcal{V} \times \mathcal{L}} \\ s,l \in \mathcal{L}}} \sum_{\substack{i,j \in \mathcal{V} \\ s,l \in \mathcal{L}}} c_{is,jl} x_{is} x_{jl}$$
(A1)

subject to
$$\sum_{s \in \mathcal{L}} x_{is} \le 1$$
 for all $i \in \mathcal{V}$, (A2)
$$\sum_{i \in \mathcal{V}} x_{is} \le 1$$
 for all $s \in \mathcal{L}$

for finite sets V and \mathcal{L} , and cost function $c: (V \times \mathcal{L})^2 \to \mathbb{R}$

Let M be the disjoint union of \mathcal{V} and \mathcal{L} , $M := \mathcal{V} \dot{\cup} \mathcal{L}$, and let $w : M^4 \to \mathbb{R}$ be defined as

$$w_{is,jl} = w(i,s,j,l) := \begin{cases} c_{is,jl}, & \text{if } i,j \in \mathcal{V}, s,l \in \mathcal{L}, \\ 0, & \text{otherwise.} \end{cases}$$

We can then formulate the QAP

$$\min_{\substack{y \in \{0,1\}^{M \times M} \\ s,l \in M}} \sum_{\substack{i,j \in M \\ s,l \in M}} w_{is,jl} \, y_{is} \, y_{jl} \tag{A3}$$

subject to
$$\sum_{s \in M} y_{is} = 1$$
 for all $i \in M$, (A4)
$$\sum_{i \in M} y_{is} = 1 \text{ for all } s \in M.$$

Let y^* be a solution of (A3) satisfying (A4), i.e.,

$$y^{\star} \in \underset{y \in \{0,1\}^{M \times M}}{\min} \sum_{\substack{i,j \in M \\ s,l \in M}} w_{is,jl} y_{is} y_{jl}$$
s. t.
$$\sum_{s \in M} y_{is} = 1 \quad \text{for all } i \in M,$$

$$\sum_{i \in M} y_{is} = 1 \quad \text{for all } s \in M.$$

Now let $x^* \in \{0,1\}^{\mathcal{V} \times \mathcal{L}}$ be defined as $x_{is}^* = y_{is}^*$ for all $i \in \mathcal{V}$, $s \in \mathcal{L}$. It remains to show that x^* then is a solution of (A1) satisfying (A2).

First note that x^* satisfies the constraints (A2) since for all $i \in \mathcal{V}$

$$\sum_{s \in \mathcal{L}} x_{is}^{\star} = \sum_{s \in \mathcal{L}} y_{is}^{\star}$$

$$\leq \sum_{s \in M} y_{is}^{\star} \quad \text{as } \mathcal{L} \subseteq M \text{ and } y^{\star} \in \{0, 1\}^{M \times M},$$

$$\leq 1 \qquad \text{as } y^{\star} \text{ satisfies (A4)}.$$

Analogously, $\sum_{i \in \mathcal{V}} x_{is}^{\star} \leq 1$ for all $s \in \mathcal{L}$. Suppose now that x^{\star} is not a solution of (A1), i.e.,

Suppose now that x^* is not a solution of (A1), i.e. there exists $x' \in \{0,1\}^{V \times \mathcal{L}}$ satisfying (A2) with

$$\sum_{\substack{i,j \in \mathcal{V} \\ s,l \in \mathcal{L}}} c_{is,jl} x'_{is} x'_{jl} < \sum_{\substack{i,j \in \mathcal{V} \\ s,l \in \mathcal{L}}} c_{is,jl} x^{\star}_{is} x^{\star}_{jl}.$$

We can then define $y' \in \{0, 1\}^{M \times M}$ by

$$y_{is}' := \begin{cases} x_{is}', & \text{if } i \in \mathcal{V}, s \in \mathcal{L}, \\ x_{si}', & \text{if } i \in \mathcal{L}, s \in \mathcal{V}, \\ 1, & \text{if } i \in \mathcal{V}, s = i, \text{ and } \sum_{l \in \mathcal{L}} x_{il}' < 1, \\ 1, & \text{if } i \in \mathcal{L}, s = i, \text{ and } \sum_{j \in \mathcal{V}} x_{js}' < 1, \\ 0, & \text{otherwise.} \end{cases}$$

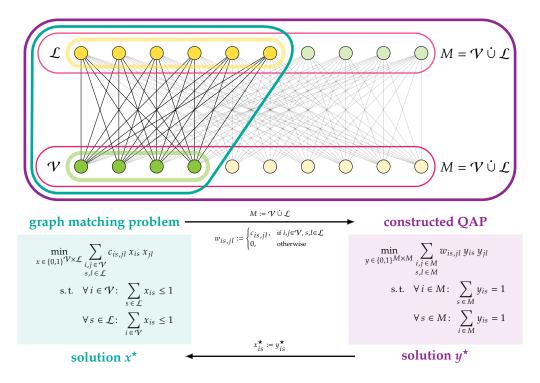


Figure A1: From graph matching to the QAP. The notation corresponds to the notation used in the proof of Theorem 1. The idea behind reducing graph matching to quadratic assignment is to enable completion of the possibly incomplete matchings feasible for graph matching. This is done by providing "dummy" nodes to which those nodes can be matched that would otherwise be left unassigned. As potentially all nodes could be unassigned, we provide a corresponding dummy for each.

Observe that y' satisfies the constraints (A4) since for all $i \in M$

$$\begin{split} \sum_{s \in M} y'_{is} &= \sum_{s \in \mathcal{V}} y'_{is} + \sum_{s \in \mathcal{L}} y'_{is} \\ &= \begin{cases} y'_{ii} + \sum_{s \in \mathcal{L}} x'_{is} & \text{if } i \in \mathcal{V} \\ \sum_{s \in \mathcal{V}} x'_{si} + y'_{ii} & \text{if } i \in \mathcal{L} \end{cases} \\ &= \begin{cases} 1 - \sum_{l \in \mathcal{L}} x'_{il} + \sum_{s \in \mathcal{L}} x'_{is} & \text{if } i \in \mathcal{V} \\ \sum_{s \in \mathcal{V}} x'_{si} + 1 - \sum_{j \in \mathcal{V}} x'_{ji} & \text{if } i \in \mathcal{L} \end{cases} \\ &= 1 \end{split}$$

and, analogously, $\sum_{i \in M} y'_{is} = 1$ for all $s \in M$. Then we obtain

$$\begin{split} \sum_{\substack{i,j \in M \\ s,l \in M}} w_{is,jl} \, y_{is}' \, y_{jl}' \\ &= \sum_{\substack{i,j \in \mathcal{V} \\ s,l \in \mathcal{L}}} c_{is,jl} \, y_{is}' \, y_{jl}' \quad \text{by definition of } w, \\ &= \sum_{\substack{i,j \in \mathcal{V} \\ s,l \in \mathcal{L}}} c_{is,jl} \, x_{is}' \, x_{jl}' \quad \text{by definition of } y', \\ &< \sum_{\substack{i,j \in \mathcal{V} \\ s,l \in \mathcal{L}}} c_{is,jl} \, x_{is}^{\star} \, x_{jl}^{\star} \quad \text{due to the choice of } x', \end{split}$$

$$= \sum_{\substack{i,j \in \mathcal{V} \\ s,l \in \mathcal{L}}} c_{is,jl} y_{is}^{\star} y_{jl}^{\star} \quad \text{by definition of } x^{\star},$$

$$= \sum_{\substack{i,j \in M \\ s,l \in M}} w_{is,jl} y_{is}^{\star} y_{jl}^{\star} \quad \text{by definition of } w,$$

$$\leq \sum_{\substack{i,j \in M \\ s,l \in M}} w_{is,jl} y_{is}' y_{jl}' \quad \text{since } y^{\star} \text{ solution of (A3)},$$

which is a contradiction. Hence, x^* is a solution of (A1).

Therefore, graph matching is polynomially reducible to the quadratic assignment problem as the size of the constructed QAP is polynomial in the size of the initial graph matching problem, and each solution of the QAP directly induces a solution to the graph matching problem.

Reduction of QAP to graph matching. Consider the OAP

$$\min_{x \in \{0,1\}^{M \times M}} \sum_{\substack{i,j \in M \\ s,l \in M}} w_{is,jl} \, x_{is} \, x_{jl} \tag{A5}$$

subject to
$$\sum_{s \in M} x_{is} = 1$$
 for all $i \in M$, (A6)
$$\sum_{i \in M} x_{is} = 1$$
 for all $s \in M$,

for a finite set M and cost function $w: M^4 \to \mathbb{R}$.

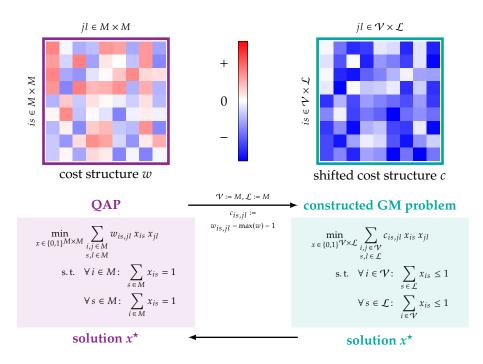


Figure A2: From the QAP to graph matching (GM). The notation corresponds to the notation used in the proof of Theorem 1. The idea behind reducing quadratic assignment to graph matching is to make sure that while the requirement for complete matchings is dropped, a better objective value can always be obtained when completing a given incomplete matching. This is guaranteed by shifting the cost structure to incur negative costs for all assignments.

We can now formulate the graph matching problem

$$\min_{\substack{x \in \{0,1\}^{M \times M} \\ s,l \in M}} \sum_{\substack{i,j \in M \\ s,l \in M}} c_{is,jl} \, x_{is} \, x_{jl} \tag{A7}$$

subject to
$$\sum_{s \in M} x_{is} \le 1$$
 for all $i \in M$, (A8)
$$\sum_{i \in M} x_{is} \le 1 \text{ for all } s \in M,$$

where $\mathcal{V} = \mathcal{L} = M$, and $c_{is,jl} := w_{is,jl} - \max(w) - 1$ for all $i, j, s, l \in M$. Note that by definition $c_{is,jl} < 0$ for all $i, j, s, l \in M$.

Let x^* be a solution of (A7) satisfying (A8). We now want to prove that x^* also satisfies (A6) and is a solution of (A5).

Suppose that x^* does not satisfy (A6), i.e., there exists $i' \in M$ with $\sum_{s \in M} x_{i's}^* < 1$. Then there also exists $s' \in M$ with $\sum_{i \in M} x_{is'}^* < 1$. Observe that $x_{i's}^* = 0$ for all $s \in M$, and $x_{is'}^* = 0$ for all $i \in M$ since $x^* \in \{0,1\}^{M \times M}$.

We can now define x' as

$$x'_{is} = \begin{cases} 1, & \text{if } i = i', s = s', \\ x^{\star}_{is'} & \text{otherwise.} \end{cases}$$

Due to the choice of i' and s', x' also satisfies the constraints (A8). Furthermore,

$$\sum_{\substack{i,j \in M \\ s,l \in M}} c_{is,jl} \, x'_{is} \, x'_{jl}$$

$$= \sum_{\substack{i,j \in M \\ s,l \in M}} c_{is,jl} \, x_{is}^{\star} \, x_{jl}^{\star} + \sum_{\substack{i,s \in M \\ is \neq i's'}} c_{is,i's'} \, x_{is}^{\star} \\
+ \sum_{\substack{j,l \in M \\ jl \neq i's'}} c_{i's',jl} \, x_{jl}^{\star} + c_{i's',i's'} \\
< \sum_{\substack{i,j \in M \\ i,j \in M}} c_{is,jl} \, x_{is}^{\star} \, x_{jl}^{\star},$$

since $c_{is,jl} < 0$ for all $i, j, s, l \in M$. This contradicts x^* being a solution of (A7). Hence, x^* satisfies (A6).

Suppose now that x^* is not a solution of (A5), i.e., there exists $\overline{x} \in \{0,1\}^{M \times M}$ satisfying (A6) with

$$\sum_{\substack{i,j \in M \\ s,l \in M}} w_{is,jl} \, \overline{x}_{is} \, \overline{x}_{jl} < \sum_{\substack{i,j \in M \\ s,l \in M}} w_{is,jl} \, x_{is}^{\star} \, x_{jl}^{\star}.$$

Note that \overline{x} also satisfies (A8) as it satisfies (A6).

Due to the constraints (A6), x^* and \overline{x} have exactly |M| nonzero entries, hence when summing over all $i, j, s, l \in M$, both, $x_{is}^* x_{jl}^*$ and $\overline{x}_{is} \overline{x}_{jl}$, are nonzero exactly $|M|^2$ times. Therefore, we obtain

$$\sum_{\substack{i,j \in M \\ s,l \in M}} w_{is,jl} \, \overline{x}_{is} \, \overline{x}_{jl} - |M|^2 (\max(w) + 1)$$

$$< \sum_{\substack{i,j \in M \\ s,l \in M}} w_{is,jl} \, x_{is}^{\star} \, x_{jl}^{\star} - |M|^2 (\max(w) + 1)$$

$$\sum_{\substack{i,j \in M \\ s,l \in M}} \left(w_{is,jl} - \max(w) - 1 \right) \overline{x}_{is} \overline{x}_{jl}$$

$$< \sum_{\substack{i,j \in M \\ s,l \in M}} \left(w_{is,jl} - \max(w) - 1 \right) x_{is}^{\star} x_{jl}^{\star}$$

$$\sum_{\substack{i,j \in M \\ s,l \in M}} c_{is,jl} \overline{x}_{is} \overline{x}_{jl} < \sum_{\substack{i,j \in M \\ s,l \in M}} c_{is,jl} x_{is}^{\star} x_{jl}^{\star}$$

This would contradict x^* being a solution of (A7). Thus, x^* is a solution of (A5).

This proves that the quadratic assignment problem is polynomially reducible to graph matching as the size of the constructed graph matching problem is polynomial in the size of the initial QAP. Each solution of the graph matching problem directly yields a solution of the QAP.

Remark A1. Note that a polynomial reduction from the QAP to graph matching can also be obtained by shifting only the unary costs by a sufficiently large constant, i.e., by defining

$$c_{is,jl} = \begin{cases} w_{is,jl} - K, & if \ jl = is, i, j, s, l \in M \\ w_{is,jl}, & otherwise. \end{cases}$$

for sufficiently large K. This is relevant in practice as it only changes the diagonal of the cost matrix, and not the full matrix. In this way it does not as heavily influence the sparsity of the cost structure.

The statements in the proof of Theorem 1 hold in particular if all quadratic terms incur a cost of zero, so we can reduce the objective function to its linear component, i.e., the objective function

$$\sum_{\substack{i,j \in \mathcal{V} \\ s,l \in \mathcal{I}}} c_{is,jl} \, x_{is} \, x_{jl}$$

of the graph matching problem, c.f. (A1) in the proof above, can be reduced to

$$\sum_{\substack{i\in\mathcal{V}\\s\in\mathcal{L}}}c_{is}\,x_{is},$$

which corresponds to the incomplete linear assignment problem. Similarly, we can replace the objective function

$$\sum_{\substack{i,j \in M \\ s,l \in M}} w_{is,jl} \, y_{is} \, y_{jl}$$

of the quadratic assignment problem, c.f. (A5), by

$$\sum_{i,s\in M} w_{is}\,y_{is},$$

which corresponds to the objective of the linear assignment problem. With this in mind, the corollary below directly follows from Theorem 1:

Corollary 1. The incomplete linear assignment problem and the linear assignment problem (LAP) are polynomially reducible to each other.

A2 Symmetry of the Formulations

Note that formulations (3) and (4)-(6) are asymmetric, since the sets ${\mathcal V}$ and ${\mathcal L}$ play different roles in the formulations, in contrast to the symmetric formulation (1). Although swapping the roles of V and \mathcal{L} does not influence the optimal solutions, it influences the problem structure, such as the sparsity of the resulting graphical model, as well as the tightness of the corresponding LP relaxation. Symmetrized formulations based on the graphical model representation use two graphical models, with the role of ${\mathcal V}$ and ${\mathcal L}$ being swapped in one of them [57]. These graphical models, with variables denoted correspondingly by x and \hat{x} , are coupled by additional constraints which enforce either the unary variables to be equal in both representations, i.e., $x_{is} = \hat{x}_{si}$, or the pairwise variables, i.e., $x_{is,jl} = \hat{x}_{si,lj}$. The first type of coupling constraints is computationally cheaper, whereas the second one leads to a tighter LP relaxation equivalent to the standard one considered in operations research [1]. In our experimental evaluation the second type of constraints is implicitly used by dd-1s* algorithms.

A3 Data Format (dd-format)

As a unified data format for all datasets we use the one introduced by [59]. It encodes a sparse representation of the graph matching problem (1) in plain text. Due to its origin we refer to it as *dd-* or *dual decomposition* data format. It is used by dd-ls*, as well as fm, fm-bca and mp-* algorithms as a native input. Table A1 presents an overview of the file format and shows a description of of all input line types.

With other algorithms suitable converters are used. The one for the graphical model representation (3) is readily available at [32]. For Matlab algorithms a matrix representation of (1) is required. We implemented this conversion by ourselves and will make the code available.

A4 Cost Transformations

Different methods have different requirements for the cost matrix, see columns *bijective*, *non-positive* and *zero unary* in Table 1. For datasets where the costs do not fulfill those requirements, see equivalent columns in Table 3, the costs have to be transformed to make them amenable to those algorithms. The transformations are chosen to in a way such that (i) the cost difference between any two matchings stays the same; and (ii) the

input line	description
c comment line	comments are ignored
p <#V> <#L> <#A> <#E>	prologue: $ \mathcal{V} $, $ \mathcal{L} $, number of assignments and edges to follow
a <id> <i> <s> <cost></cost></s></i></id>	specifies possible assignment $i \to s$ for $i \in \mathcal{V}$, $s \in \mathcal{L}$ (unary terms $c_{is,is}x_{is}$)
e <id1> <id2> <cost></cost></id2></id1>	specifies edge between two assignments (pairwise terms $(c_{is,jl} + c_{jl,is})x_{is}x_{jl}$)
i0 <i> <x> <y></y></x></i>	optional: coordinate of the node $i \in \mathcal{V}$ as a point in the left image
i1 <s> <x> <y></y></x></s>	optional: coordinate of the node $s \in \mathcal{L}$ as a point in the right image
n0 <i> <j></j></i>	optional: specifies that points $i,j\in\mathcal{V}$ are neighbors in the left image
n1 <s> <l></l></s>	optional: specifies that points $s,l\in\mathcal{L}$ are neighbors in the right image

Table A1: Short specification of the dd file format. The format originated in [59]. It encodes a sparse representation of the graph matching problem (1). Unary costs are encoded as *assignments* and pairwise costs are encoded as *edges* between two assignments. Any lines marked as optional or comments are ignored when building the graph matching problem. They were introduced in [59] to ease visualization of matching problems between two images.

/* initialize */

```
Algorithm 1: Transform into non-positive costs.
```

Input: Graph matching problem (V, \mathcal{L}, c) **Output:** Equivalent non-positive cost vector c'

/* initialize */
$$c' \leftarrow (0...0)$$

/* first, shift unary costs */
for $i \in \mathcal{V}$ and $s \in \mathcal{L}$ do

$$\begin{array}{c} \alpha \leftarrow \max\{c_{is'} \mid s' \in \mathcal{L} \text{ and } c_{is'} \neq \infty\} \\ c'_{is} \leftarrow c_{is} - \max\{0, \alpha\} \end{array}$$

/* second, shift pairwise costs */
for $i, j \in \mathcal{V}, i < j \text{ and } s, t \in \mathcal{L} \text{ do}$

$$\alpha \leftarrow \max\{c_{is',jl'} \mid s',l' \in \mathcal{L}\} \\ c'_{is,jl} \leftarrow c_{is,jl} - \max\{0, \alpha\}$$

sparsity of the cost matrix is preserved as much as possible. In the following we describe each of the three transformations. The full implementation is part of our published source code for the benchmark. Without loss of generality we assume the cost matrix to be upper triangular, i.e. $c_{is,jl} = 0$ for $i, j \in \mathcal{V}$, $s, l \in \mathcal{L}$ and i > j.

Bijective Costs. Non-bijective problem instances are transformed into the bijective ones in a way similar to the described in the proof of Theorem 1 and illustrated in Fig. A1.

Non-positive Costs. Here we transform the cost matrix into a form where all *finite* costs are non-positive. Note that infinite costs will stay positive, as they are identified by corresponding algorithms and prohibit selection of the associated matchings. Algorithm 1 shows the transformation which shifts the costs of all bijective assignments by a fixed constant. Therefore, for non-bijective instances we first transform them into the bijective form as described in previous paragraph.

Remove Unary Costs. The pm algorithm does not take into account unary costs, see column *zero unary* in Table 1. We use Algorithm 2 to transform unary costs into pairwise costs.

Algorithm 2: Transform unary into pairwise costs.

Input: Graph matching problem (V, \mathcal{L}, c) **Output:** Equivalent cost vector c' with empty diagonal

$$\begin{array}{l} \textbf{for } i,j \in \mathcal{V} \text{ and } s,l \in \mathcal{L} \textbf{ do} \\ \\ c'_{is,jl} \leftarrow \begin{cases} c_{is,jl} & \text{if } (i,s) \neq (j,l) \\ 0 & \text{otherwise} \end{cases} \\ /^* \text{ count how many additional cost entries are } \\ \text{necessary; remember decision with least number } \\ \text{of additional entries in array } J */ \\ \textbf{for } i \in \mathcal{V} \text{ and } s \in \mathcal{L} \textbf{ do} \\ \\ & J_{is} \leftarrow \underset{i \in \mathcal{U}}{\text{arg min}} |\{l \in \mathcal{L} \mid c_{is,jl} = 0 \text{ and } i \neq j, s \neq l\}| \end{cases}$$

For the maximization algorithms ipfpu, ipfps, ga, rrwm, pm, sm, smac, lsm, mpm, fgmd and hbp the sign of the elements in the cost matrix is flipped additionally after performing all other operations.

A5 Accuracy computation

For hotel, house-sparse, house-dense, car and motor the complete ground truth assignments are known. The accuracy is computed as the number of correctly assigned nodes over n_V . The ground truth for caltech-small, caltech-large, worms and pairs is only partial i.e. not every node has a ground truth label. This is taken into account by computing the accuracy as the number of correctly assigned nodes over

method	command line
dd-ls0	tkrgmlineartreemax-iter N
dd-1s3	tkrgmlinearlocal 2treemax-iter N
dd-ls4	tkrgmlinearlocal 3treemax-iter N
fw	<pre>graph_matching_frank_wolfe_text_input input.dd</pre>
mp	<pre>graph_matching_mp -i input.ddroundingReparametrization uniform:0.5</pre>
mp-mcf	<pre>graph_matching_mp_tightening -i input.ddtightentightenInterval 50 \</pre>
	tightenIteration 200tightenConstraintsPercentage 0.01 $ackslash$
	tightenReparametrization uniform:0.5graphMatchingRounding mcf
mp-fw	<pre>graph_matching_mp_tightening -i input.ddtightentightenInterval 50 \</pre>
	tightenIteration 200tightenConstraintsPercentage 0.01 $ackslash$
	tightenReparametrization uniform:0.5graphMatchingRounding fw
fm	qap_ddmax-batches Nbatch-size Ogenerate 1
fm-bca	qap_ddmax-batches Nbatch-size 10generate 10

Table A2: Command line parameters for all C++ methods.

the number of nodes with available ground truth label.

A6 Run Time Measurement

To make the comparison as fair as possible, we exclude preprocessing steps and the time it takes to load the graph matching files into the solver. This means that the following steps are excluded from our run-time measurements: (i) Conversion process from the dd file format into the input format of the solvers; (ii) Loading the file from storage into memory; (iii) Parsing the input file; and (iv) Dataset transformation if needed.

Practically, we start the timer directly before the solver starts the optimization routine. For optimization methods that have not included this functionality, we have modified the methods accordingly.

A7 Reproducibility

Original Source Code. References with links to the original source code can be found in column *Matlab* and *C*++ in Table 1. Note that for the dd-ls* methods we use the same command line wrapper as used in [31]. The wrapper allows to select the problem decomposition at run-time without need for recompilation. The specific command lines for the C++ programs (see column C++ in Table 1) that have been used in the benchmark are listed in Table A2.

Matlab Wrappers. We incorporated the implementations of the Matlab algorithms in our own wrappers in order to be able to load the datasets and to make the output standardized. No parameters have to be specified.

Reproducible Containers. For all methods we build Linux containers so that the environment as well as the specific version of the solvers are reproducible. The container files contain instructions for downloading,

processing and building a fixed version of the corresponding methods. The source code of each method is fetched from the Internet. For the case that source code is no longer online, we preserve a historical copy of the repositories. The container build scripts will pin each method to a specific version or commit id to preserve reproducibility.

HPC Scripts. For usage in high-performance compute clusters (HPC clusters) we provide a small script to transform the container images into singluarity image. Most HPC clusters provide tools to easily run these singularity images. Additionally, we provide SLURM scheduling scripts so that the benchmark can be reproduced easily and quickly.

Project Web Site. The wrappers, container files and other scripts are publicly available on the project web site, see https://vislearn.github.io/gmbench/.

A8 Detailed Evaluation Results

More detailed evaluation results are provided on the following pages. Figure A3 shows run-time performance profiles similar to Figure 1 but for each dataset separately. Tables A3 to A13 show the average results of each method on each dataset for time limits 1s, 10s, 100s and 300s similar to Table 4 and Table 5.

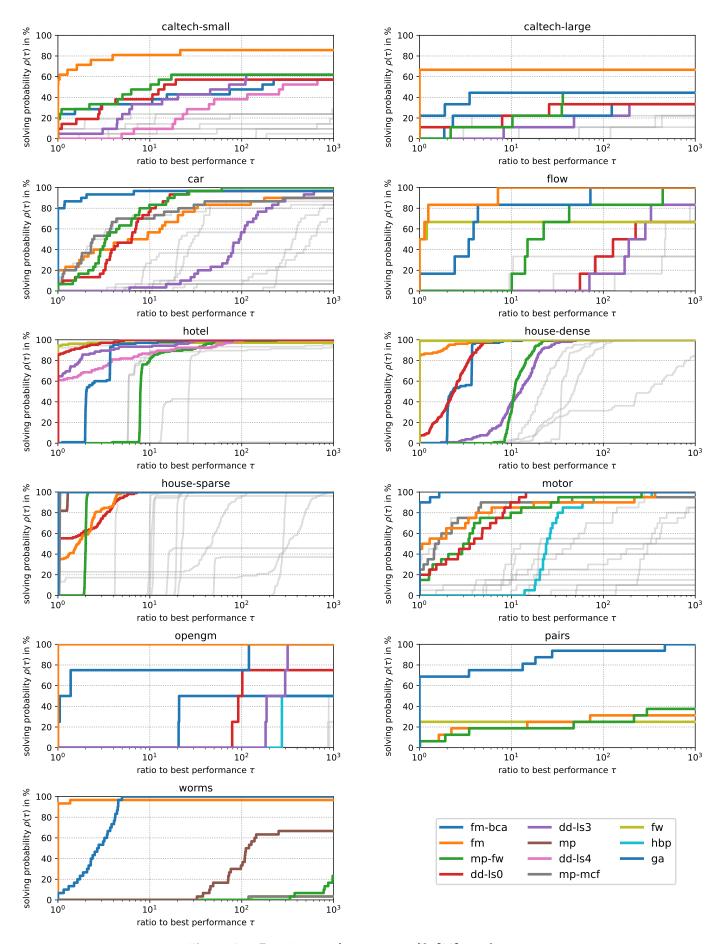


Figure A3: Run time performance profile [22] per dataset.

	(caltech-	small(1s)	C	altech-	small (10s	3)	Ca	altech-s	mall (100	s)	Ca	altech-s	mall (300	s)
	opt	Е	D	acc	opt	Е	D	acc	opt	Е	D	acc	opt	Е	D	acc
fgmd	0		*		14		*		29		*		43		*	
fm	52	-8906	-43926	58	57	-9040	-43926	62	57	-9050	-43926	63	57	-9050	-43926	63
fw	0	0	_	0	0	0	_	0	0	0	_	0	0	0	_	0
ga	0		_*		5		_*		5		_*		5		_*	
ipfps	19	-8983	_	67	19	-8983	_	67	19	-8983	_	67	19	-8983	_	67
ipfpu	10	-8829	_	62	10	-8829	_	62	10	-8829	_	62	10	-8829	_	62
lsm	0		_*		0	0	_	0	0	0	_	0	0	0	_	0
mpm	0		_*		0		_*		0		_*		0		_*	
pm	0	-6510	_	51	0	-6510	_	51	0	-6510	-	51	0	-6510	_	51
rrwm	5		_*		5	-8848	_	61	5	-8848	-	61	5	-8848	_	61
sm	0	-3932	_	36	0	-3932	_	36	0	-3932	-	36	0	-3932	_	36
smac	0	-6196	_	42	0	-6196	_	42	0	-6196	-	42	0	-6196	_	42
dd-ls0	43	-7414	-10658	58	48	-8251	-9502	60	48	-8251	-9502	60	48	-8251	-9502	60
dd-1s3	33	-6842	-15159	57	52	-7862	-9904	60	52	-8357	-9466	61	52	-8357	-9466	61
dd-ls4	24	-6332	-18033	54	38	-6919	-13407	53	52	-7878	-9810	60	52	-8324	-9443	61
fm-bca	38	-8927	-12827	62	43	-8943	-12797	62	48	-8953	-12797	61	52	-8960	-12797	61
hbp	0		_*		5		_*		10		_*		10		_*	
mp	14	-7967	-12191	59	14	-8026	-12183	60	19	-8095	-12182	62	19	-8095	-12182	62
mp-fw	33	-8886	-12793	60	38	-9052	-12244	65	38	-9056	-11217	65	43	-9057	-10818	64
mp-mcf	5	-7882	-12380	60	10	-8030	-11737	59	14	-8324	-10987	57	14	-8435	-10731	57

Table A3: Detailed fixed-time evaluation for dataset caltech-small.

	cal	tech-large	(1s)	calt	ech-large((10s)	calt	ech-large(1	100s)	calt	ech-large(3	300s)
	Е	D	acc	Е	D	acc	Е	D	acc	E	D	acc
fgmd		_*			_*			_*			_*	
fm	-33972	-142829	52	-34117	-142829	52	-34125	-142829	52	-34125	-142829	52
fw	0	-	0	0	-	0	0	-	0	0	_	0
ga		_*			_*			_*			_*	
ipfps		_*			_*		-33998	-	51	-33998	-	51
ipfpu		_*		-34216	_	52	-34216	-	52	-34216	_	52
lsm		_*			_*		0	-	0	0	_	0
mpm		_*			_*			_*			_*	
pm	-29106	-	43	-29106	-	43	-29106	-	43	-29106	_	43
rrwm		_*			_*			_*			_*	
sm		_*		-14423	-	28	-14423	-	28	-14423	_	28
smac		_*		-24183	-	39	-24183	-	39	-24183	_	39
dd-ls0	-26236	-56071	48	-32973	-35007	51	-33539	-34959	52	-33539	-34959	52
dd-1s3	-25226	-72766	44	-28653	-42079	49	-33552	-34914	49	-33557	-34911	49
dd-ls4	-25268	-79327	46	-25599	-62120	46	-30148	-38880	51	-32266	-35577	51
fm-bca	-33758	-48582	51	-34040	-48223	51	-34073	-48217	51	-34082	-48217	51
hbp		_*			_*			_*			_*	
mp	-31315	-46170	48	-32017	-46070	48	-32069	-46066	48	-32074	-46066	48
mp-fw	-34227	-51526	51	-34237	-48882	51	-34277	-45923	51	-34287	-44424	51
mp-mcf	-29813	-48168	45	-30362	-46630	47	-30737	-43833	47	-31230	-42564	48

opt: optimally solved instances (%);
 E: average best objective value;
 D: average best lower bound if applicable;
 acc: average accuracy corresponding to best objective (%);
 —*: method yields no solution for at least one problem instance

Table A4: Detailed fixed-time evaluation for dataset caltech-large.

		car	(1s)			car	(10s)			car(100s)			car	(300s)	
	opt	Е	D	acc	opt	Е	D	acc	opt	Е	D	acc	opt	Е	D	acc
fgmd	0		*		83		_*		83	-69	_	89	83	-69	_	89
fm	77	-69	-107	88	90	-69	-107	89	90	-69	-107	89	90	-69	-107	89
fw	7	-63	_	63	7	-63	_	63	7	-63	-	63	7	-63	_	63
ga	57	-68	_	84	57	-68	_	84	57	-68	-	84	57	-68	_	84
ipfps	10	-65	_	80	10	-65	_	80	10	-65	-	80	10	-65	_	80
ipfpu	7	-60	_	69	7	-60	_	69	7	-60	-	69	7	-60	_	69
lsm	0	-51	_	52	0	-51	_	52	0	-51	-	52	0	-51	_	52
mpm	7		_*		7	-57	_	65	7	-57	-	65	7	-57	_	65
pm	0	-35	-	23	0	-35	_	23	0	-35	-	23	0	-35	_	23
rrwm	37	-68	_	87	37	-68	_	87	37	-68	-	87	37	-68	_	87
sm	7	-63	_	76	7	-63	_	76	7	-63	-	76	7	-63	_	76
smac	0	-52	_	52	0	-52	_	52	0	-52	-	52	0	-52	_	52
dd-ls0	97	-69	-69	91	97	-69	-69	91	97	-69	-69	91	97	-69	-69	91
dd-1s3	47	-57	-75	74	87	-68	-70	90	97	-69	-69	91	97	-69	-69	91
dd-ls4	3	-49	-78	59	57	-59	-73	77	87	-67	-70	89	97	-69	-69	92
fm-bca	93	-69	-71	92	97	-69	-71	92	97	-69	-71	92	97	-69	-71	92
hbp	77		*		87		_*		87	-69	-73	91	87	-69	-73	91
mp	80	-69	-71	92	83	-69	-71	92	87	-69	-71	92	87	-69	-71	92
mp-fw	90	-69	-71	91	97	-69	-70	91	100	-69	-70	91	100	-69	-70	91
mp-mcf	87	-69	-70	91	90	-69	-70	91	93	-69	-70	91	93	-69	-70	91

Table A5: Detailed fixed-time evaluation for dataset car.

		flow(1s)			flow(10s)			flow(100s)			flow(300s)	
	opt	Е	D	opt	Е	D	opt	Е	D	opt	Е	D
fgmd	0	_	*	0	_	_*	0	_	_*	0	_	_*
fm	83	-2838	-3436	100	-2840	-3436	100	-2840	-3436	100	-2840	-3436
fw	67	-2828	_	67	-2828	_	67	-2828	_	67	-2828	_
ga	0	_	.*	0	_	_*	0	-2469	_	0	-2469	_
ipfps	0	-766	_	0	-766	_	0	-766	_	0	-766	_
ipfpu	0	-512	_	0	-512	_	0	-512	_	0	-512	_
lsm	0	_	.*	0	_	_*	0	31042	_	0	31042	_
mpm	0	_	.*	0	_	_*	0	_	_*	0	_	_*
pm	0	-32	_	0	-32	_	0	-32	_	0	-32	_
rrwm	0	_	.*	0	-226	_	0	-226	_	0	-226	_
sm	0	-139	_	0	-139	_	0	-139	_	0	-139	_
smac	0	0	_	0	0	_	0	0	_	0	0	_
dd-ls0	33	-2345	-2968	67	-2819	-2854	67	-2819	-2854	67	-2819	-2854
dd-1s3	17	-2059	-3030	83	-2821	-2847	83	-2834	-2844	83	-2834	-2844
dd-ls4	0	-2062	-3090	67	-2767	-2863	83	-2835	-2843	83	-2835	-2843
fm-bca	83	-2838	-2898	100	-2840	-2879	100	-2840	-2878	100	-2840	-2878
hbp	0	_	.*	0	_	_*	0	_	_*	0	_	_*
mp	33	-2628	-2887	33	-2674	-2882	50	-2690	-2881	50	-2690	-2881
mp-fw	83	_	*	83	_	_*	83	-2838	-2870	83	-2838	-2862
mp-mcf	33	-2521	-2892	33	-2719	-2869	50	-2749	-2854	50	-2789	-2851

opt: optimally solved instances (%); E: average best objective value; D: average best lower bound if applicable; acc: average accuracy corresponding to best objective (%); —*: method yields no solution for at least one problem instance

Table A6: Detailed fixed-time evaluation for dataset flow.

		hote	1(1s)			hotel	L (10s)			hotel	(100s)			hotel	(300s)	
	opt	Е	Ď	acc	opt	Е	Ď	acc	opt	Е	D D	acc	opt	Е	D D	acc
fgmd	0		_*		0		_*		96	-4283	_	98	96	-4283	_	98
fm	97	-4292	-4406	100	97	-4292	-4406	100	97	-4292	-4406	100	97	-4292	-4406	100
fw	97	-4288	_	99	97	-4288	_	99	97	-4288	-	99	97	-4288	_	99
ga	0	947	_	15	0	947	_	15	0	947	-	15	0	947	_	15
ipfps	0	1051	_	14	0	1051	_	14	0	1051	-	14	0	1051	_	14
ipfpu	0	1062	_	15	0	1062	_	15	0	1062	-	15	0	1062	_	15
lsm	0		_*		0	1729	_	12	0	1729	-	12	0	1729	_	12
mpm	43	-2585	_	78	43	-2585	_	78	43	-2585	-	78	43	-2585	_	78
pm	0	775	_	33	0	775	_	33	0	775	-	33	0	775	_	33
rrwm	0	744	_	15	0	744	_	15	0	744	-	15	0	744	_	15
sm	0	1086	_	13	0	1086	_	13	0	1086	-	13	0	1086	_	13
smac	1	-1571	_	61	1	-1571	_	61	1	-1571	-	61	1	-1571	_	61
dd-ls0	100	-4293	-4294	100	100	-4293	-4294	100	100	-4293	-4294	100	100	-4293	-4294	100
dd-1s3	100	-4293	-4294	100	100	-4293	-4293	100	100	-4293	-4293	100	100	-4293	-4293	100
dd-ls4	98	-4291	-4297	100	100	-4293	-4293	100	100	-4293	-4293	100	100	-4293	-4293	100
fm-bca	100	-4293	-4300	100	100	-4293	-4300	100	100	-4293	-4300	100	100	-4293	-4300	100
hbp	97		_*		100	-4293	-4305	100	100	-4293	-4305	100	100	-4293	-4305	100
mp	93	-4280	-4299	99	96	-4285	-4299	99	98	-4289	-4299	100	98	-4289	-4299	100
mp-fw	99	-4292	-4306	100	100	-4293	-4299	100	100	-4293	-4296	100	100	-4293	-4295	100
mp-mcf	90	-4245	-4303	98	93	-4264	-4298	99	95	-4274	-4296	99	97	-4277	-4295	99

Table A7: Detailed fixed-time evaluation for dataset hotel.

		house-d	ense (1s)		house-dense (10s)				ŀ	nouse-de	nse (100s	;)	house-dense (300s)				
	opt	Е	D	acc	opt	Е	D	acc	opt	Е	D	acc	opt	Е	D	acc	
fgmd	0		*		0		*		77	-3542	_	89	77	-3542	_	89	
fm	100	-3778	-4078	100	100	-3778	-4078	100	100	-3778	-4078	100	100	-3778	-4078	100	
fw	100	-3778	_	100	100	-3778	_	100	100	-3778	-	100	100	-3778	_	100	
ga	0	3491	_	8	0	3491	_	8	0	3491	-	8	0	3491	_	8	
ipfps	0	3654	_	8	0	3654	_	8	0	3654	-	8	0	3654	_	8	
ipfpu	0	3659	_	8	0	3659	_	8	0	3659	_	8	0	3659	_	8	
lsm	0		_*		0	2391	_	18	0	2391	_	18	0	2391	_	18	
mpm	0	1260	_	53	0	1260	_	53	0	1260	_	53	0	1260	_	53	
pm	0	3262	_	18	0	3262	_	18	0	3262	_	18	0	3262	_	18	
rrwm	0	2895	_	10	0	2895	_	10	0	2895	_	10	0	2895	_	10	
sm	0	3789	_	9	0	3789	_	9	0	3789	-	9	0	3789	_	9	
smac	0	2817	_	31	0	2817	_	31	0	2817	-	31	0	2817	_	31	
dd-ls0	100	-3778	-3779	100	100	-3778	-3779	100	100	-3778	-3779	100	100	-3778	-3779	100	
dd-ls3	100	-3778	-3779	100	100	-3778	-3778	100	100	-3778	-3778	100	100	-3778	-3778	100	
dd-ls4	92	-3763	-3795	99	100	-3778	-3778	100	100	-3778	-3778	100	100	-3778	-3778	100	
fm-bca	100	-3778	-3779	100	100	-3778	-3778	100	100	-3778	-3778	100	100	-3778	-3778	100	
hbp	98		*		100	-3778	-3806	100	100	-3778	-3806	100	100	-3778	-3806	100	
mp	99	-3777	-3782	100	99	-3777	-3780	100	99	-3777	-3780	100	99	-3777	-3780	100	
mp-fw	100	-3778	-3843	100	100	-3778	-3791	100	100	-3778	-3779	100	100	-3778	-3778	100	
mp-mcf	31	-3542	-3825	89	85	-3732	-3783	98	99	-3776	-3778	100	100	-3778	-3778	100	

opt: optimally solved instances (%);
 E: average best objective value;
 D: average best lower bound if applicable;
 acc: average accuracy corresponding to best objective (%);
 —*: method yields no solution for at least one problem instance

Table A8: Detailed fixed-time evaluation for dataset house-dense.

	l	nouse-sp	parse (1s	5)	h	ouse-sp	arse (10:	s)	h	ouse-spa	arse (100)s)	house-sparse (300s)				
	opt	Е	D	acc	opt	Е	D	acc	opt	Е	D	acc	opt	E	D	acc	
fgmd	0		*		100	-67	_	100	100	-67	_	100	100	-67	_	100	
fm	100	-67	-79	100	100	-67	-79	100	100	-67	-79	100	100	-67	-79	100	
fw	0	0	_	0	0	0	_	0	0	0	-	0	0	0	_	0	
ga	100	-67	_	100	100	-67	_	100	100	-67	-	100	100	-67	_	100	
ipfps	100	-67	_	100	100	-67	_	100	100	-67	-	100	100	-67	_	100	
ipfpu	100	-67	_	100	100	-67	_	100	100	-67	-	100	100	-67	_	100	
lsm	46	-65	_	96	46	-65	_	96	46	-65	-	96	46	-65	_	96	
mpm	0	-60	_	90	0	-60	_	90	0	-60	-	90	0	-60	_	90	
pm	0	-54	_	83	0	-54	_	83	0	-54	-	83	0	-54	_	83	
rrwm	100	-67	_	100	100	-67	_	100	100	-67	-	100	100	-67	_	100	
sm	96	-67	_	100	96	-67	_	100	96	-67	-	100	96	-67	_	100	
smac	37	-44	_	63	37	-44	_	63	37	-44	_	63	37	-44	_	63	
dd-ls0	100	-67	-67	100	100	-67	-67	100	100	-67	-67	100	100	-67	-67	100	
dd-1s3	96	-66	-67	100	100	-67	-67	100	100	-67	-67	100	100	-67	-67	100	
dd-ls4	18	-56	-72	86	100	-67	-67	100	100	-67	-67	100	100	-67	-67	100	
fm-bca	100	-67	-67	100	100	-67	-67	100	100	-67	-67	100	100	-67	-67	100	
hbp	100	-67	-67	100	100	-67	-67	100	100	-67	-67	100	100	-67	-67	100	
mp	100	-67	-67	100	100	-67	-67	100	100	-67	-67	100	100	-67	-67	100	
mp-fw	100	-67	-67	100	100	-67	-67	100	100	-67	-67	100	100	-67	-67	100	
mp-mcf	100	-67	-67	100	100	-67	-67	100	100	-67	-67	100	100	-67	-67	100	

Table A9: Detailed fixed-time evaluation for dataset house-sparse.

		moto	r(1s)			moto	r (10s)			motor	(100s)		motor (300s)				
	opt	Е	D	acc	opt	Е	D	acc	opt	Е	D	acc	opt	Ε	D	acc	
fgmd	0		*		85	-63	_	97	85	-63	_	97	85	-63	_	97	
fm	90	-63	-94	93	100	-63	-94	97	100	-63	-94	97	100	-63	-94	97	
fw	20	-58	_	70	20	-58	_	70	20	-58	-	70	20	-58	_	70	
ga	55	-62	_	90	55	-62	_	90	55	-62	-	90	55	-62	_	90	
ipfps	25	-61	_	85	25	-61	_	85	25	-61	-	85	25	-61	_	85	
ipfpu	15	-58	_	77	15	-58	_	77	15	-58	-	77	15	-58	_	77	
lsm	10	-52	_	64	10	-52	_	64	10	-52	-	64	10	-52	_	64	
mpm	5		_*		5	-50	_	56	5	-50	-	56	5	-50	_	56	
pm	0	-35	_	32	0	-35	_	32	0	-35	-	32	0	-35	_	32	
rrwm	50	-62	_	89	50	-62	_	89	50	-62	-	89	50	-62	_	89	
sm	40	-60	_	87	40	-60	_	87	40	-60	-	87	40	-60	_	87	
smac	10	-52	_	66	10	-52	_	66	10	-52	-	66	10	-52	_	66	
dd-ls0	100	-63	-63	97	100	-63	-63	97	100	-63	-63	97	100	-63	-63	97	
dd-1s3	65	-57	-65	87	100	-63	-63	97	100	-63	-63	97	100	-63	-63	97	
dd-ls4	30	-52	-68	78	70	-60	-64	93	100	-63	-63	97	100	-63	-63	97	
fm-bca	100	-63	-63	97	100	-63	-63	97	100	-63	-63	97	100	-63	-63	97	
hbp	95		_*		100	-63	-65	97	100	-63	-65	97	100	-63	-65	97	
mp	90	-63	-63	96	95	-63	-63	98	95	-63	-63	98	95	-63	-63	98	
mp-fw	95	-63	-63	98	100	-63	-63	97	100	-63	-63	97	100	-63	-63	97	
mp-mcf	90	-63	-63	98	95	-63	-63	99	100	-63	-63	97	100	-63	-63	97	

opt: optimally solved instances (%);
 E: average best objective value;
 D: average best lower bound if applicable;
 acc: average accuracy corresponding to best objective (%);
 —*: method yields no solution for at least one problem instance

Table A10: Detailed fixed-time evaluation for dataset motor.

		opengm(1s))		opengm(10s	s)	C	pengm (100s	s)	opengm(300s)			
	opt	Е	D	opt	E	D	opt	Е	D	opt	E	D	
fgmd	0	_	_*	50	_	_*	75	-166	-	75	-166	_	
fm	100	-171	-192	100	-171	-192	100	-171	-192	100	-171	-192	
fw	0	-152	-	0	-152	_	0	-152	-	0	-152	-	
ga	50	-167	-	50	-167	_	50	-167	-	50	-167	-	
ipfps	0	-95	-	0	-95	_	0	-95	-	0	-95	-	
ipfpu	0	-86	-	0	-86	_	0	-86	-	0	-86	-	
lsm	0	-67	-	0	-67	_	0	-67	-	0	-67	-	
mpm	0	-94	-	0	-94	_	0	-94	-	0	-94	-	
pm	0	-83	-	0	-83	_	0	-83	-	0	-83	-	
rrwm	0	-154	-	0	-154	_	0	-154	-	0	-154	-	
sm	0	-101	-	0	-101	_	0	-101	-	0	-101	-	
smac	0	-84	-	0	-84	_	0	-84	-	0	-84	-	
dd-ls0	50	-160	-172	75	-161	-171	75	-161	-171	75	-161	-171	
dd-1s3	0	-118	-185	100	-171	-171	100	-171	-171	100	-171	-171	
dd-ls4	0	-105	-214	25	-141	-178	100	-171	-171	100	-171	-171	
fm-bca	75	-170	-177	100	-171	-177	100	-171	-177	100	-171	-177	
hbp	0	-	_*	50	-164	-185	50	-164	-185	50	-164	-185	
mp	0	-57	-192	0	-57	-192	0	-57	-192	0	-57	-192	
mp-fw	0	-150	-192	0	-150	-192	0	-150	-192	0	-150	-192	
mp-mcf	0	-57	-192	0	-57	-192	0	-57	-192	0	-57	-192	

Table A11: Detailed fixed-time evaluation for dataset opengm.

		pairs(1s)			pairs(10s)		I	pairs (100s)		pairs (300s)					
	E	D	acc	Е	D	acc	E	D	acc	Е	D	acc			
fgmd		_*			_*			_*			_*				
£m	-64812	-76418	51	-65625	-76418	54	-65825	-76418	55	-65870	-76418	56			
fw	-65722	-	54	-65797	_	54	-65802	-	54	-65802	-	54			
ga		_*			_*			_*			_*				
ipfps		_*			_*		-35115	-	7	-35115	_	7			
ipfpu		_*			_*		-35666	-	7	-35666	_	7			
lsm		_*			_*			_*			_*				
mpm		_*			_*			_*		<u></u> *					
pm		_*			_*			_*			*				
rrwm		_*			_*			_*			_*				
sm		_*			_*			_*		-196	-	0			
smac		_*			_*			_*			_*				
dd-ls0	-61482	-76497	41	-61482	-73521	41	-62974	-67306	57	-63454	-67114	60			
dd-1s3	-61638	-75656	41	-61638	-73528	41	-62426	-67599	50	-64556	-66704	60			
dd-ls4	-61634	-75852	41	-61634	-74053	41	-61634	-70214	41	-62894	-67352	53			
fm-bca		_*		-65567	-70163	55	-65913	-69003	58	-65958	-68909	58			
hbp		_*			_*			_*			_*				
mp		_*		-64150	-68255	57	-64380	-68136	57	-64418	-68130	57			
mp-fw		_*			_*			_*		-65794	-69623	55			
mp-mcf		*		-63990	-68318	56	-64174	-68053	57	-64208	-67748	57			

opt: optimally solved instances (%); E: average best objective value; D: average best lower bound if applicable; acc: average accuracy corresponding to best objective (%); —*: method yields no solution for at least one problem instance

Table A12: Detailed fixed-time evaluation for dataset pairs.

		worms	s (1s)			worms	s (10s)			worms	(100s)		worms (300s)				
	opt	Е	D	acc	opt	Е	D	acc	opt	Е	D	acc	opt	E	D	acc	
fgmd	0		*		0		_*		0		_*		0		_*		
fm	93	-48457	-55757	89	93	-48458	-55757	89	93	-48461	-55757	89	93	-48461	-55757	89	
fw	0	-46974	_	81	3	-48032	_	85	3	-48038	-	85	3	-48038	_	85	
ga	0		_*		0		_*		0		_*		0		_*		
ipfps	0		_*		0		_*		0	-1147	-	1	0	-1147	_	1	
ipfpu	0		_*		0		_*		0	0	-	0	0	0	_	0	
lsm	0		_*		0		_*		0		_*		0		_*		
mpm	0		*		0		_*		0		*		0		*		
pm	0		*		0		_*		0		*		0		_*		
rrwm	0		*		0		_*		0		*		0		_*		
sm	0		*		0		_*		0	-6453	-	6	0	-6453	_	6	
smac	0		*		0		_*		0		*		0		*		
dd-1s0	0	60443	-163870	26	0	50517	-148830	25	0	-3982	-58449	58	0	-43824	-48682	87	
dd-1s3	0	64017	-160520	24	0	49257	-144842	24	0	11744	-71523	46	0	-40882	-48943	85	
dd-ls4	0	65731	-160409	24	0	58300	-153566	25	0	31066	-109434	33	0	12795	-75088	44	
fm-bca	93	-48460	-48514	89	93	-48464	-48498	89	93	-48464	-48498	89	93	-48464	-48498	89	
hbp	0		*		0		_*		0		*		0		_*		
mp	0		*		67	-48391	-48498	89	70	-48393	-48497	89	70	-48393	-48496	89	
mp-fw	0		*		3		_*		57	-48402	-48759	88	83	-48435	-48631	88	
mp-mcf	0		*		3	-47942	-48588	88	3	-48027	-48558	88	3	-48057	-48552	88	

Table A13: Detailed fixed-time evaluation for dataset worms.