Radial Oscillations in the Solenoid

This note describes modifying the Figure of Merit to make a trade-off between radial oscillations and rotation in the solenoid. The issue addressed here is finding a quantity that can be evaluated in any plane that characterizes the size of the oscillations. We will do this by expressing the oscillating state in the solenoid in terms of constants of motion, whose values measure the size of the oscillations. Because we deal with constants of motion the evaluation can be made in any plane.

First consider the general conserved quantity

$$\frac{d}{dz}\left[\mathbf{E}\cdot\mathbf{Q}+\frac{L^2}{2}-\frac{1}{2}\mathbf{P}\cdot\mathbf{P}\right]=0.$$

We take the entering flat beam to have all moments zero except the following:

$$E_+(0), E_-(0), Q_+(0), Q_-(0).$$

Thus, when the beam arrives in the solenoid ($z=z_s$) and has been made round by the transformer we have from the constant of motion, J_0

$$E_+(z_s)Q_+(z_s)-\frac{1}{2}P_+^2(z_s)=J_0=E_+(0)Q_+(0)+E_-(0)Q_-(0)-\frac{1}{2}L_s^2.$$

Here L_s is the Larmor frame angular momentum in the solenoid, which is locally constant.

Assuming the beam has been made round in the solenoid, the dynamics in the solenoid involves only those variables with subscript "+". The equations for the symmetric moments in the solenoid give

$$\frac{d}{dz}Q_+=P_+$$
$$\frac{d}{dz}P_+=E_+-\frac{1}{2}k_\Omega^2Q_++\cdots$$

Introducing, $Q_+=a^2/2$, the above can be rewritten

$$a\frac{d^2a}{dz^2}=\left(E_+-\frac{P_+^2}{2Q_+}\right)-\frac{1}{4}k_\Omega^2a^2+\cdots.$$

Using the constant of motion, J_0, we can replace E_+ and we arrive at the standard envelope equation for a cylindrical beam in a solenoidal field with space charge.

$$\frac{d^2a}{dz^2} = \frac{2J_0}{a^3} - \frac{1}{4}k_\Omega^2 a + \Lambda / a .$$

Here J_0 plays the role of a conserved emittance. Radial force balance is achieved when the right side vanishes.

$$0 = \frac{2J_0}{a_0^2} - \frac{1}{4}k_\Omega^2 a_0^2 + \Lambda = \frac{J_0}{Q_{+0}} - \frac{1}{2}k_\Omega^2 Q_{+0} + \Lambda$$

This can be solved for Q_{+0} , which is the beam radius that satisfies radial force balance and no oscillations in radius. That is, a beam with $P_+=0$, and $Q_+= Q_{+0}$ will remain constant in z in the solenoid.

$$Q_{+0} = \frac{1}{k_\Omega^2}\left[\sqrt{2J_0 k_\Omega^2 + \Lambda^2} - \Lambda\right]$$

The case of radial force balance is special in that it requires rotation in the solenoid, which may adversely affect cooling of a copropagating beam. To explore the trade-off between rotation and oscillation we solve the envelope equation to characterize the oscillating state. The envelope equation admits a second constant of motion

$$J_1 = \frac{1}{2}\left(\frac{da}{dz}\right)^2 + \frac{J_0}{a^2} + \frac{1}{8}k_\Omega^2 a^2 - \Lambda\left[\ln(a/a_0) + \frac{1}{2}\right]$$

Restoring the variables Q_+, P_+ this becomes

$$2J_1 = \frac{P_+^2}{2Q_+} + \frac{J_0}{Q_+} + \frac{1}{2}k_\Omega^2 Q_+ - \Lambda\left[\ln(Q_+/Q_{+0}) + 1\right]$$

We show below that

$$J_1 \geq J_0 / Q_{+0}$$

Thus, $J_1 - J_0 / Q_{+0}$ can be used to characterize the size of the radial osillations.

Notes to show $J_1 \geq J_0 / Q_{+0}$.

Add the force balance relation to the expression for *2J₁*

$$2J_1 = \frac{P_+^2}{2Q_+} + J_0\left(\frac{1}{Q_+} + \frac{1}{Q_{+0}}\right) + \frac{1}{2}k_\Omega^2\left(Q_+ - Q_{+0}\right) - \Lambda\left[\ln(Q_+/Q_{+0})\right]$$

$$2J_1 = \frac{P_+^2}{2Q_+} + \frac{J_0}{Q_{+0}}\left(1 + \frac{1}{1+(Q_+ - Q_{+0})/Q_{+0}}\right) + \frac{1}{2}k_\Omega^2\left(Q_+ - Q_{+0}\right) - \Lambda\left[\ln(Q_+/Q_{+0})\right]$$

Define $x = (Q_+ - Q_{+0})/Q_{+0}$

$$2J_1 = \frac{P_+^2}{2Q_+} + \frac{J_0}{Q_{+0}}\left(1 + \frac{1}{1+x}\right) + \frac{1}{2}k_\Omega^2 Q_{+0}x - \Lambda\left[\ln(1+x)\right]$$

Now use

$$0 = \frac{J_0}{Q_{+0}} - \frac{1}{2}k_\Omega^2 Q_{+0} + \Lambda$$

$$2J_1 = \frac{P_+^2}{2Q_+} + \frac{J_0}{Q_{+0}}\left(1 + \frac{1}{1+x} + x\right) - \Lambda\left[\ln(1+x) - x\right]$$

Now for -1<x

$$1 + x + \left(1+x\right)^{-1} > 2,$$

$$x - \ln(1+x) > 0$$

So $J_1 \geq J_0 / Q_{+0}$.

Figures of merit

Our current figure of merit is

$$F = \frac{1}{2}\left[\left|\mathbf{P}\right|^2 + k_0^2\left(Q_-^2 + Q_x^2\right) + k_0^{-2}\left(E_-^2 + E_x^2\right)\right] + \frac{1}{2}\left[k_0^{-2}\left(E_+ - \frac{1}{2}k_\Omega^2 Q_+ + \Lambda\right)^2 + \left(2E_+Q_+ - L^2\right)^2\right]$$

The terms that need to be modified are the first and the last two. The first thing we should do is remove the term P_+^2. This should still lead to a solution where the beam is round, that is, all the “x” and “-“ moments vanish. What to do next is not totally obvious. One possibility is force relaxation to an oscillating solution with a particular J_1 value. The difficulty here is that J_1 depends in a complicated way on all the moments, so it will be a problem doing all the derivatives. Then there is the issue of the rotation in the solenoid, maybe we should be minimizing the E_+ moment in the lab frame,

$$E_{+,Lab} = E_+ + 2f'^2 Q_+ + 2f'L$$

$$f' = -\frac{k_\Omega}{2}$$

So let’s try

$$F = \frac{1}{2}\left[P_x^2 + P_-^2 + k_0^2\left(Q_-^2 + Q_x^2\right) + k_0^{-2}\left(E_-^2 + E_x^2\right)\right] + \frac{1}{2}\left[\varepsilon_1 k_0^{-2} E_{+,Lab}^2 + \varepsilon_2\left(J_1 - J_0/Q_{+0}\right)\right]$$

The two epsilons are somewhat adjustable.

Try first: KISS, Keep it Simple Stupid

$$F = \frac{1}{2}\left[P_-^2 + P_x^2 + k_0^2\left(Q_-^2 + Q_x^2\right) + k_0^{-2}\left(E_-^2 + E_x^2\right)\right] + \frac{\varepsilon_1}{2}\left[k_0^{-2}\left(E_+ - \frac{1}{2}k_\Omega^2 Q_+ + \Lambda\right)^2 + P_+^2\right] + \frac{\varepsilon_2}{2}\left(L + 2f'Q_+\right)^2$$

where

$$L_{Lab} = L + 2f'Q_+$$

Constants of Motion

In the paper by K-J Kim, Eqs.(31-33) describe general relations betweeb the properties of beams that have been transformed. These relations were also invoked in referee B’s report. Here it is shown that these relations can be expected to be satisfied by our simulations.

The sigma matrix for the 4-D phase space corresponding to transverse displacements of beam particles is given by

$$\Sigma_T = \begin{bmatrix} \Sigma(XX) & \Sigma(YX) \\ \Sigma(XY) & \Sigma(YY) \end{bmatrix},$$

Where

$$\Sigma\left(XX\right)=\begin{bmatrix} \langle xx\rangle & \langle \dot{x}x\rangle \\ \langle x\dot{x}\rangle & \langle \dot{x}\dot{x}\rangle \end{bmatrix},\quad \Sigma\left(XY\right)=\begin{bmatrix} \langle xy\rangle & \langle \dot{x}y\rangle \\ \langle x\dot{y}\rangle & \langle \dot{x}\dot{y}\rangle \end{bmatrix},$$

$$\Sigma\left(YX\right)=\begin{bmatrix} \langle yx\rangle & \langle \dot{y}x\rangle \\ \langle y\dot{x}\rangle & \langle \dot{y}\dot{x}\rangle \end{bmatrix},\quad \Sigma\left(YY\right)=\begin{bmatrix} \langle yy\rangle & \langle \dot{y}y\rangle \\ \langle y\dot{y}\rangle & \langle \dot{y}\dot{y}\rangle \end{bmatrix}$$

The first **reported** [K-J Kim] constant of motion is

$$-\frac{1}{2}T_R\left\{J_4\Sigma J_4\Sigma\right\},$$

Where

$$J_4=\begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Evaluate $J_4\Sigma$

$$J_4\Sigma=\begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \Sigma(XX) & \Sigma(YX) \\ \Sigma(XY) & \Sigma(YY) \end{bmatrix}=\begin{bmatrix} J\Sigma(XX) & J\Sigma(YX) \\ J\Sigma(XY) & J\Sigma(YY) \end{bmatrix}$$

$$\left(J_4\Sigma\right)^2=\begin{bmatrix} J\Sigma(XX) & J\Sigma(YX) \\ J\Sigma(XY) & J\Sigma(YY) \end{bmatrix}\begin{bmatrix} J\Sigma(XX) & J\Sigma(YX) \\ J\Sigma(XY) & J\Sigma(YY) \end{bmatrix}$$

$$\left(J_4\Sigma\right)^2=\begin{bmatrix} \left(J\Sigma(XX)\right)^2+J\Sigma(YX)J\Sigma(XY) & J\Sigma(XX)J\Sigma(YX)+J\Sigma(YX)J\Sigma(YY) \\ J\Sigma(XY)J\Sigma(XX)+J\Sigma(YY)J\Sigma(XY) & J\Sigma(XY)J\Sigma(YX)+\left(J\Sigma(YY)\right)^2 \end{bmatrix}$$

$$T_R\left\{\left(J_4\Sigma\right)^2\right\}=T_R\left\{\left(J\Sigma(XX)\right)^2+J\Sigma(YX)J\Sigma(XY)\right\}+T_R\left\{J\Sigma(XY)J\Sigma(YX)+\left(J\Sigma(YY)\right)^2\right\}$$

$$J\Sigma(XX)=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \langle xx\rangle & \langle \dot{x}x\rangle \\ \langle x\dot{x}\rangle & \langle \dot{x}\dot{x}\rangle \end{bmatrix}=\begin{bmatrix} \langle \dot{x}x\rangle & \langle \dot{x}\dot{x}\rangle \\ -\langle xx\rangle & -\langle \dot{x}x\rangle \end{bmatrix}$$

$$\left(J\Sigma(XX)\right)^2 = \begin{bmatrix} \langle \dot{x}x \rangle & \langle \dot{x}\dot{x} \rangle \\ -\langle xx \rangle & -\langle \dot{x}x \rangle \end{bmatrix} \begin{bmatrix} \langle \dot{x}x \rangle & \langle \dot{x}\dot{x} \rangle \\ -\langle xx \rangle & -\langle \dot{x}x \rangle \end{bmatrix} = \begin{bmatrix} \langle \dot{x}x \rangle^2 - \langle xx \rangle \langle \dot{x}\dot{x} \rangle & 0 \\ 0 & \langle \dot{x}x \rangle^2 - \langle xx \rangle \langle \dot{x}\dot{x} \rangle \end{bmatrix}$$

$$T_R\left\{\left(J\Sigma(XX)\right)^2\right\} = 2\left(\langle \dot{x}x \rangle^2 - \langle xx \rangle \langle \dot{x}\dot{x} \rangle\right)$$

Likewise

$$T_R\left\{\left(J\Sigma(YY)\right)^2\right\} = 2\left(\langle y\dot{y} \rangle^2 - \langle yy \rangle \langle \dot{y}\dot{y} \rangle\right)$$

$$J\Sigma(YX) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \langle yx \rangle & \langle \dot{y}x \rangle \\ \langle y\dot{x} \rangle & \langle \dot{y}\dot{x} \rangle \end{bmatrix} = \begin{bmatrix} \langle y\dot{x} \rangle & \langle \dot{y}\dot{x} \rangle \\ -\langle yx \rangle & -\langle \dot{y}x \rangle \end{bmatrix}$$

$$J\Sigma(XY) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \langle xy \rangle & \langle y\dot{x} \rangle \\ \langle x\dot{y} \rangle & \langle \dot{y}\dot{x} \rangle \end{bmatrix} = \begin{bmatrix} \langle \dot{y}x \rangle & \langle \dot{y}\dot{x} \rangle \\ -\langle yx \rangle & -\langle y\dot{x} \rangle \end{bmatrix}$$

$$J\Sigma(YX)J\Sigma(XY) = \begin{bmatrix} \langle y\dot{x} \rangle & \langle \dot{y}\dot{x} \rangle \\ -\langle yx \rangle & -\langle \dot{y}x \rangle \end{bmatrix} \begin{bmatrix} \langle \dot{y}x \rangle & \langle \dot{y}\dot{x} \rangle \\ -\langle yx \rangle & -\langle y\dot{x} \rangle \end{bmatrix} = \begin{bmatrix} \langle y\dot{x} \rangle \langle \dot{y}x \rangle - \langle \dot{y}\dot{x} \rangle \langle yx \rangle & 0 \\ 0 & -\langle yx \rangle \langle \dot{y}\dot{x} \rangle + \langle \dot{y}x \rangle \langle y\dot{x} \rangle \end{bmatrix}$$

$$T_R\left\{J\Sigma(YX)J\Sigma(XY)\right\} = 2\left[\langle y\dot{x} \rangle \langle \dot{y}x \rangle - \langle \dot{y}\dot{x} \rangle \langle yx \rangle\right]$$
$$T_R\left\{J\Sigma(XY)J\Sigma(YX)\right\} = 2\left[\langle y\dot{x} \rangle \langle \dot{y}x \rangle - \langle \dot{y}\dot{x} \rangle \langle yx \rangle\right]$$

Thus,

$$-\frac{1}{2}T_R\left\{\left(J_4\Sigma\right)^2\right\} = \left(\langle xx \rangle \langle \dot{x}\dot{x} \rangle - \langle \dot{x}x \rangle^2\right) + \left(\langle yy \rangle \langle \dot{y}\dot{y} \rangle - \langle y\dot{y} \rangle^2\right) - 2\left[\langle y\dot{x} \rangle \langle \dot{y}x \rangle - \langle \dot{y}\dot{x} \rangle \langle yx \rangle\right]$$

Now introduce our variables:

$$\langle xx \rangle = Q_+ + Q_-, \quad \langle yy \rangle = Q_+ - Q_-, \quad \langle xy \rangle = Q_x$$
$$\langle \dot{x}\dot{x} \rangle = \frac{1}{2}\left(E_+ + E_-\right), \quad \langle \dot{y}\dot{y} \rangle = \frac{1}{2}\left(E_+ - E_-\right), \quad \langle \dot{x}\dot{y} \rangle = \frac{1}{2}E_x$$
$$\langle x\dot{x} \rangle = \frac{1}{2}\left(P_+ + P_-\right), \quad \langle y\dot{y} \rangle = \frac{1}{2}\left(P_+ - P_-\right), \quad \langle x\dot{y} \rangle = \frac{1}{2}\left(P_x + L\right), \quad \langle \dot{x}y \rangle = \frac{1}{2}\left(P_x - L\right)$$

$$\left(\langle xx \rangle \langle \dot{x}\dot{x} \rangle - \langle \dot{x}x \rangle^2\right) = \left(Q_+ + Q_-\right)\frac{1}{2}\left(E_+ + E_-\right) - \frac{1}{4}\left(P_+ + P_-\right)^2$$

$$\left(\langle yy\rangle\langle \dot{y}\dot{y}\rangle - \langle y\dot{y}\rangle^2\right) = \left(Q_+ - Q_-\right)\frac{1}{2}\left(E_+ - E_-\right) - \frac{1}{4}\left(P_+ - P_-\right)^2$$

$$\left(\langle xx\rangle\langle \dot{x}\dot{x}\rangle - \langle \dot{x}x\rangle^2\right) + \left(\langle yy\rangle\langle \dot{y}\dot{y}\rangle - \langle y\dot{y}\rangle^2\right) = Q_+E_+ + Q_-E_- - \frac{1}{2}\left(P_=^2 + P_-^2\right)$$

$$2\left[\langle y\dot{x}\rangle\langle \dot{y}x\rangle - \langle \dot{y}\dot{x}\rangle\langle yx\rangle\right] = 2\left[\frac{1}{4}\left(P_x - L\right)\left(P_x + L\right) - \frac{1}{2}E_xQ_x\right]$$

The end result: $$-\frac{1}{2}T_R\left\{\left(J_4\Sigma\right)^2\right\} = Q_+E_+ + Q_-E_- + E_xQ_x + \frac{L^2}{2} - \frac{1}{2}\left(P_=^2 + P_-^2 + P_x^2\right)$$

Determinants of Sigma Matrix

The determinant of the sigma matrix is also conserved. This will give a second relation among the moments at the entrance of the transformer and in the solenoid. We will assume that in the solenoid the beam is cylindrically symmetric. The nonzero moments in the solenoid are Q_+, P_+, E_+, L

$$\langle xx\rangle = Q_+,\quad \langle yy\rangle = Q_+,\quad \langle xy\rangle = 0$$
$$\langle \dot{x}\dot{x}\rangle = \frac{1}{2}E_+,\quad \langle \dot{y}\dot{y}\rangle = \frac{1}{2}E_+,\quad \langle \dot{x}\dot{y}\rangle = 0$$
$$\langle x\dot{x}\rangle = \frac{1}{2}P_+,\quad \langle y\dot{y}\rangle = \frac{1}{2}P_+,\quad \langle x\dot{y}\rangle = \frac{1}{2}L,\quad \langle \dot{x}y\rangle = -\frac{1}{2}L.$$

Thus, in the solenoid

$$\Sigma_T = \frac{1}{2}\begin{bmatrix} 2Q_+ & P_+ & 0 & L \\ P_+ & E_+ & -L & 0 \\ 0 & -L & 2Q_+ & P_+ \\ L & 0 & P_+ & E_+ \end{bmatrix}$$

Then the determinant is

$$Det\{2\Sigma_T\} = 2Q_+\begin{vmatrix} E_+ & -L & 0 \\ -L & 2Q_+ & P_+ \\ 0 & P_+ & E_+ \end{vmatrix} - P_+\begin{vmatrix} P_+ & -L & 0 \\ 0 & 2Q_+ & P_+ \\ L & P_+ & E_+ \end{vmatrix} - L\begin{vmatrix} P_+ & E_+ & -L \\ 0 & -L & 2Q_+ \\ L & 0 & P_+ \end{vmatrix}$$

$$\begin{vmatrix} E_+ & -L & 0 \\ -L & 2Q_+ & P_+ \\ 0 & P_+ & E_+ \end{vmatrix} = E_+\left(2Q_+E_+ - P_+^2\right) + L\left(-L_+E_+\right) = E_+\left(2Q_+E_+ - P_+^2 - L^2\right)$$

$$\begin{vmatrix} P_+ & -L & 0 \\ 0 & 2Q_+ & P_+ \\ L & P_+ & E_+ \end{vmatrix} = P_+\left(2Q_+E_+ - P_+^2\right) + L\left(-LP_+\right) = P_+\left(2Q_+E_+ - P_+^2 - L^2\right)$$

$$\begin{vmatrix} P_+ & E_+ & -L \\ 0 & -L & 2Q_+ \\ L & 0 & P_+ \end{vmatrix} = P_+\left(-LP_+\right) + L\left(E_+2Q_+ - L^2\right) = L\left(2E_+Q_+ - L^2 - P_+^2\right)$$

This gives in the solenoid

$$Det\{2\Sigma_T\} = \left(2Q_+E_+ - P_+^2 - L^2\right)\left(2Q_+E_+ - P_+^2 - L^2\right) = \left(2Q_+E_+ - P_+^2 - L^2\right)^2$$

The nonzero moments for the entering flat beam are

$$Q_+, Q_-, E_+, E_-$$

$$\langle xx\rangle = Q_+ + Q_-, \quad \langle yy\rangle = Q_+ - Q_-, \quad \langle xy\rangle = 0$$
$$\langle \dot{x}\dot{x}\rangle = \frac{1}{2}\left(E_+ + E_-\right), \quad \langle \dot{y}\dot{y}\rangle = \frac{1}{2}\left(E_+ - E_-\right), \quad \langle \dot{x}\dot{y}\rangle = 0$$
$$\langle x\dot{x}\rangle = 0, \quad \langle y\dot{y}\rangle = 0, \quad \langle x\dot{y}\rangle = 0, \quad \langle \dot{x}y\rangle = 0$$

$$\Sigma_T = \frac{1}{2}\begin{bmatrix} 2\left(Q_+ + Q_-\right) & 0 & 0 & 0 \\ 0 & E_+ + E_- & 0 & 0 \\ 0 & 0 & 2\left(Q_+ - Q_-\right) & 0 \\ 0 & 0 & 0 & E_+ - E_- \end{bmatrix}$$

$$Det\{2\Sigma_T\} = 4\left(Q_+^2 - Q_-^2\right)\left(E_+^2 - E_-^2\right)$$

We will now use the two constants of motion, u= upstream, s = solenoid

$$Det\left\{2\mathrm{S}_T\right\} = 4\left(Q_{+u}^2 - Q_{-u}^2\right)\left(E_{+u}^2 - E_{-u}^2\right) = \left(2Q_{+s}E_{+s} - P_{+s}^2 - L^2\right)^2$$

$$-\frac{1}{2}T_R\left\{\left(J_4\mathrm{S}\right)^2\right\} = Q_{+u}E_{+u} + Q_{-u}E_{-u} = Q_{+s}E_{+s} + \frac{L^2}{2} - \frac{1}{2}\left(P_{+s}^2\right)$$

Restore Cartesian variables upstream

$$Det\left\{2\Sigma_T\right\} = 16\left\langle xx\right\rangle_u\left\langle yy\right\rangle_u\left\langle \dot{x}\dot{x}\right\rangle_u\left\langle \dot{y}\dot{y}\right\rangle_u = \left(2Q_{+s}E_{+s} - P_{+s}^2 - L^2\right)^2$$

$$-\frac{1}{2}T_R\left\{\left(J_4\Sigma\right)^2\right\} = \left(\left\langle xx\right\rangle_u\left\langle \dot{x}\dot{x}\right\rangle_u + \left\langle yy\right\rangle_u\left\langle \dot{y}\dot{y}\right\rangle_u\right) = Q_{+s}E_{+s} + \frac{L^2}{2} - \frac{1}{2}\left(P_{+s}^2\right)$$

These relations should be verified numerically.

$$4\left\langle xx\right\rangle_u\left\langle yy\right\rangle_u\left\langle \dot{x}\dot{x}\right\rangle_u\left\langle \dot{y}\dot{y}\right\rangle_u = \left[Q_{+s}E_{+s} - \frac{P_{+s}^2}{2} - \frac{L^2}{2}\right]^2 \equiv \varepsilon_{th,s}^2$$

$$\left(\left\langle xx\right\rangle_u\left\langle \dot{x}\dot{x}\right\rangle_u + \left\langle yy\right\rangle_u\left\langle \dot{y}\dot{y}\right\rangle_u\right) = Q_{+s}E_{+s} + \frac{L^2}{2} - \frac{1}{2}\left(P_{+s}^2\right) = \varepsilon_{th,s} + L^2$$

$$\left(\left\langle xx\right\rangle_u\left\langle \dot{x}\dot{x}\right\rangle_u + \left\langle yy\right\rangle_u\left\langle \dot{y}\dot{y}\right\rangle_u\right)^2 = \left(\varepsilon_{th,s} + L^2\right)^2$$

$$\left(\left\langle xx\right\rangle_u\left\langle \dot{x}\dot{x}\right\rangle_u + \left\langle yy\right\rangle_u\left\langle \dot{y}\dot{y}\right\rangle_u\right)^2 - 4\left\langle xx\right\rangle_u\left\langle yy\right\rangle_u\left\langle \dot{x}\dot{x}\right\rangle_u\left\langle \dot{y}\dot{y}\right\rangle_u = \left(\varepsilon_{th,s} + L^2\right)^2 - \varepsilon_{th,s}^2$$

$$\left(\left\langle xx\right\rangle_u\left\langle \dot{x}\dot{x}\right\rangle_u - \left\langle yy\right\rangle_u\left\langle \dot{y}\dot{y}\right\rangle_u\right)^2 = \left(\varepsilon_{th,s} + L^2\right)^2 - \varepsilon_{th,s}^2$$

This gives

$$2\left\langle xx\right\rangle_u\left\langle \dot{x}\dot{x}\right\rangle_u = \left(\varepsilon_{th,s} + L^2\right) + \sqrt{\left(\varepsilon_{th,s} + L^2\right)^2 - \varepsilon_{th,s}^2}$$

$$2\left\langle yy\right\rangle_u\left\langle \dot{y}\dot{y}\right\rangle_u = \left(\varepsilon_{th,s} + L^2\right) - \sqrt{\left(\varepsilon_{th,s} + L^2\right)^2 - \varepsilon_{th,s}^2}$$