Beam envelope equations for non-symmetric focusing systems

In this note we present a system of equations that describes the evolution of the spatial moments of a charged particle beam distribution in the presence of a combination of transverse forces. These include the Lorentz forces of a spatially varying solenoidal magnetic field, a superposition of arbitrarily oriented quadrupole magnetic fields, and a model of the self-consistent fields due to the beam's charge and current density. Due to the absence of symmetry implied by the general system of forces, the system of equations is more complicated than the traditional envelope equations for which only the RMS radii of the beam in two transverse directions are followed. Here the system will consist of 10 moment evolution equations and an accompanying conservation law.

The underlying assumption that will be made is that the beam particles' trajectories are well described by the paraxial equations of motion for which transverse forces are linear in the particles' displacements from the axis, or are linearly proportional to a particle's transverse velocity. The final equations will describe the beam's evolution in the Larmor frame defined by the applied solenoidal magnetic field. We begin the calculation in the lab frame, transform to the Larmor frame, and complete the derivation in the Larmor frame.

To start, we write for the evolution of the transverse particle displacements in a Cartesian coordinate system in the lab frame,

$$\begin{aligned} x_1'' &= k_x + k_\Omega y_1' + \frac{1}{2}k_\Omega' y_1 \\ y_1'' &= k_y - k_\Omega x_1' - \frac{1}{2}k_\Omega' x_1 \end{aligned} . \qquad (1)$$

Here (x_1, y_1) are a particle's transverse displacements in the lab frame, a prime denotes differentiation with respect to the axial coordinate z, and

$$k_\Omega(z) = \frac{qB_z(z)}{mc\gamma v_z}, \qquad (2)$$

is the spatial gyration rate due to the axial magnetic field. (Note about signs: if the beam consists of electrons, q = -e, and if $B_z > 0$ the $k_\Omega < 0$.) The quantities $\left(k_x, k_y\right)$ are due to the transverse forces from the quadrupoles and space charge, and will be defined subsequently.

Our next step is to transform the variables to their Cartesian representation in a rotating frame,

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $\phi(z)$ is an axially dependent rotation phase that will also be defined subsequently. Substituting () in () and multiplying by the inverse of the rotation matrix appearing in () results in the system

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} - (\phi')^2 \begin{pmatrix} x \\ y \end{pmatrix} \text{-} k_\Omega \phi' \begin{pmatrix} x \\ y \end{pmatrix}$$

$$+ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left(2\phi' \begin{pmatrix} x' \\ y' \end{pmatrix} + \phi'' \begin{pmatrix} x \\ y \end{pmatrix} + k_\Omega \begin{pmatrix} x' \\ y' \end{pmatrix} + \frac{k'_\Omega}{2} \begin{pmatrix} x \\ y \end{pmatrix} \right).$$

$$= \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{pmatrix} k_x \\ k_y \end{pmatrix}$$

Notice, if we now choose

$$\phi' = -\frac{k_\Omega}{2},$$

this defines the local Larmor frame, and the last term on the left vanishes. This leaves the following set of equations for individual particle motion in the Larmor frame,

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} + \left(\frac{k_\Omega}{2} \right)^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{pmatrix} k_x \\ k_y \end{pmatrix}.$$

Moments

We wish to obtain equations for the average of products of the four variables (x, x', y, y'). There are 16 ordered products. However, order does not matter, leaving 10 independent products. We choose combinations of these products that distinguish x-y symmetric and non-symmetric motion. For spatial moments we choose

$$\mathbf{Q} = \begin{pmatrix} Q_+ \\ Q_- \\ Q_x \end{pmatrix} = \begin{pmatrix} \langle x^2 + y^2 \rangle / 2 \\ \langle x^2 - y^2 \rangle / 2 \\ \langle xy \rangle \end{pmatrix}.$$

Here the angle brackets mean average over the beam distribution function. Accompanying these spatial moments are momentum like moments

$$\mathbf{P} = \frac{d}{dz}\mathbf{Q} = \begin{pmatrix} P_+ \\ P_- \\ P_x \end{pmatrix} = \begin{pmatrix} \langle xx' + yy' \rangle \\ \langle xx' - yy' \rangle \\ \langle yx' + xy' \rangle \end{pmatrix}.$$

The angular momentum is in the same group

$$L = \langle xy' - yx' \rangle.$$

The group of 10 moments is completed by three energy-like moments

$$\mathbf{E} = \begin{pmatrix} E_+ \\ E_- \\ E_x \end{pmatrix} = \begin{pmatrix} \langle x'^2 + y'^2 \rangle \\ \langle x'^2 - y'^2 \rangle \\ 2\langle y'x' \rangle \end{pmatrix}.$$

The evolution of these moments is given by the following system of equations

$$\begin{aligned} \frac{d}{dz}\mathbf{Q} &= \mathbf{P} \\ \frac{d}{dz}\mathbf{P} &= \mathbf{E} + \mathbf{O}\cdot\mathbf{Q} \\ \frac{d}{dz}\mathbf{E} &= \mathbf{O}\cdot\mathbf{P} + \mathbf{N}L \\ \frac{d}{dz}L &= -\mathbf{N}^\dagger\cdot\mathbf{Q} \end{aligned}.$$

Here the matrix **O** and vector **N** are defined as follows.

$$\mathbf{O} = -\frac{k_\Omega^2}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\sum_{quads} K_q \begin{bmatrix} 0 & c_q & -s_q \\ c_q & 0 & 0 \\ -s_q & 0 & 0 \end{bmatrix} + \frac{4\Lambda}{ab}\begin{bmatrix} 1 & c_\alpha\Delta & s_\alpha\Delta \\ c_\alpha\Delta & 1 & 0 \\ s_\alpha\Delta & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{N} = 2\sum_{Quads} K_q \begin{pmatrix} 0 \\ s_q \\ c_q \end{pmatrix} - \frac{4\Lambda}{ab} \begin{pmatrix} 0 \\ s_\alpha \Delta \\ -c_\alpha \Delta \end{pmatrix}$$

In the equations above the following expressions and notation have been introduced. Each quadrupole magnet has field strength given in the lab frame by

$$\begin{aligned} B_{qx} &= B_q'(z)\left[\sin(2\psi_q)x_1 - \cos(2\psi_q)y_1\right] \\ B_{qy} &= -B_q'(z)\left[\cos(2\psi_q)x_1 + \sin(2\psi_q)y_1\right] \end{aligned},$$

where $B_q'(z)$ defines the axial profile of the field and the angle ψ_q defines the orientation. The lab frame acceleration factors for each quadrupole are given by

$$\begin{pmatrix} k_{xq} \\ k_{yq} \end{pmatrix} = \frac{q}{m\gamma c v_z} \begin{pmatrix} -B_{qy} \\ B_{qx} \end{pmatrix} = \frac{qB_q'(z)}{m\gamma c v_z} \begin{pmatrix} \cos(2\psi_q)x_1 + \sin(2\psi_q)y_1 \\ \sin(2\psi_q)x_1 - \cos(2\psi_q)y_1 \end{pmatrix}.$$

For a particle in the Cartesian plane $x_1 = r\cos\theta, y_1 = r\sin\theta$, the acceleration in it's local radial direction is

$$k_{qr} = \cos\theta k_{qx} + \sin\theta k_{qy} = K_q(z) r\cos\left(2\psi_q - 2\theta\right),$$

where

$$K_q(z) = \frac{qB_q'(z)}{m\gamma c v_z}.$$

If $K_q > 0$ the magnet will be defocusing when $\theta = \psi_q, \psi_q + \pi$, and focusing when $\theta = \psi_q + \pi/2, \psi_q - \pi/2$. If $K_q < 0$ these are reversed.

This leads to the expressions in the Larmor frame where

$$\left(s_q, c_q\right) = \left(\sin(2\phi - 2\psi_q), \cos(2\phi - 2\psi_q)\right).$$

The space charge term is calculated in the Larmor frame. Here it is assumed that the charge density distribution is uniform inside an ellipse with major radii (*a(z), b(z)*) and making an angle $\alpha(z)$ with respect to the x-axis in the Larmor frame. The values of a, b, and α are determined by the three spatial moments (Q_+, Q_-, Q_x). The result is the following

$$\frac{4c_\alpha \Delta}{ab} = -\frac{Q_-}{\{Q_+ + Q_\Delta\} Q_\Delta} \qquad \text{(SC1)}$$

$$\frac{4s_\alpha \Delta}{ab} = -\frac{Q_x}{\{Q_+ + Q_\Delta\} Q_\Delta} \qquad \text{(SC2)}$$

$$\frac{4}{ab} = \frac{1}{Q_\Delta} \qquad \text{(SC3)}$$

Here

$$Q_\Delta = \left[Q_+^2 - \left(Q_-^2 + Q_x^2 \right) \right]^{1/2}. \qquad \text{(SC4)}$$

Here

$$\Lambda = \frac{1}{4\pi} \frac{cZ_0 I}{m v_z^3 \gamma_0^3 / e},$$

with Z_0 = 377 Ohms, and I is the beam current.

These will be discussed subsequently in more detail.

Constant of motion

The equations can be combined to show the following conservation relation. This represents the conservation of total emmittance.

$$\frac{d}{dz}\left[\mathbf{E}\cdot\mathbf{Q} + \frac{L^2}{2} - \frac{1}{2}\mathbf{P}\cdot\mathbf{P} \right] = 0.$$

Scaling of equations

Let us suppose that we have found a solution to the moment equations in the form of functions

$$\mathbf{Q}_0(z_0), \mathbf{P}_0(z_0), \mathbf{E}_0(z_0), \mathbf{O}_0(z_0), \mathbf{N}_0(z_0), L_0(z_0) \ .$$

It can then be verified that a scaled solution is

$$\mathbf{Q}(z) = \lambda \mathbf{Q}_0(\varepsilon z)$$
$$\mathbf{P}(z) = \lambda\varepsilon\, \mathbf{P}_0(\varepsilon z)$$
$$\mathbf{E}(z) = \lambda\varepsilon^2\, \mathbf{E}_0(\varepsilon z)$$
$$\mathbf{O}(z) = \varepsilon^2 \mathbf{O}_0(\varepsilon z)$$
$$\mathbf{N}(z) = \varepsilon^2 \mathbf{N}_0(\varepsilon z)$$
$$L(z) = \lambda\varepsilon\, L_0(\varepsilon z)$$

Here ε, λ are arbitrary constants. If $\varepsilon > 1$ the scaled solution is shorter in spatial length than the original solution, and if $\varepsilon < 1$ the scaled solution is longer than the original.

Looking at the expressions for the matrices $\mathbf{O}, \mathbf{N}$ there are a number of conditions that must be satisfied to give the required ε^2 scaling. The solenoidal field contribution requires,

$$k_\Omega(z) = \varepsilon k_{\Omega 0}(\varepsilon z).$$

This means a shorter solution requires a stronger solenoidal field. The scaled solution gives for the phase

$$\phi(z) = \phi_0(\varepsilon z).$$

Thus, the values of the phase in the locations of the quadrupoles are preserved under the scaling. For the quadrupoles, strict application of the scaling gives

$$K_q(z) = \varepsilon^2 K_{q0}(\varepsilon z).$$

However, in the thin lens approximation only the integrated value of the quadrupole field matters

$$\bar{K} = \int dz K_q(z) = \varepsilon \int \varepsilon\, dz K_{q0}(\varepsilon z) = \varepsilon \bar{K}_0 \ .$$

In this approximation the strength of the quadrupole field also scales inversely with length.

Space charge contribution to the **O** and **N** matrices can also be preserved. We note that these contributions scale as

$$O, N \propto \Lambda / Q.$$

So to preserve the solution we require

$$O \propto \Lambda / Q = \Lambda / \left(\varepsilon\lambda Q_0\right) = \varepsilon^2 O_0 \propto \varepsilon^2 \Lambda_0 / Q_0$$

As a result the current parameter scales as

$$\Lambda = \lambda \varepsilon^3 \Lambda_0 .$$

So scaling length or amplitude requires changing beam current to maintain space charge influence.

Transformation from Larmor to fixed frame.

Variables with subscript *c* are moments in a fixed Cartesian frame.

$$\begin{pmatrix} Q_{+C} \\ Q_{-C} \\ Q_{XC} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi & -\sin 2\phi \\ 0 & \sin 2\phi & \cos 2\phi \end{bmatrix} \begin{pmatrix} Q_+ \\ Q_- \\ Q_X \end{pmatrix}$$

$$\begin{pmatrix} P_{+c} \\ P_{-c} \\ P_{xc} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi & -\sin 2\phi \\ 0 & \sin 2\phi & \cos 2\phi \end{bmatrix} \begin{pmatrix} P_+ \\ P_- \\ P_x \end{pmatrix} + 2\phi' \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin 2\phi & -\cos 2\phi \\ 0 & \cos 2\phi & -\sin 2\phi \end{bmatrix} \begin{pmatrix} Q_+ \\ Q_- \\ Q_x \end{pmatrix}$$

$$\begin{pmatrix} E_{+c} \\ E_{-c} \\ E_{xc} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi & -\sin 2\phi \\ 0 & \sin 2\phi & \cos 2\phi \end{bmatrix} \begin{pmatrix} E_+ \\ E_- \\ E_x \end{pmatrix} + 2\phi' \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin 2\phi & -\cos 2\phi \\ 0 & \cos 2\phi & -\sin 2\phi \end{bmatrix} \begin{pmatrix} P_+ \\ P_- \\ P_x \end{pmatrix}$$

$$+2\phi'^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos 2\phi & \sin 2\phi \\ 0 & -\sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{pmatrix} Q_+ \\ Q_- \\ Q_x \end{pmatrix} + 2\phi' L \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$L_c = L + 2\phi' \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q_+ \\ Q_- \\ Q_x \end{pmatrix} = L + 2\phi' Q_+$$

Inverse transformation

Note

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi & \sin 2\phi \\ 0 & -\sin 2\phi & \cos 2\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi & -\sin 2\phi \\ 0 & \sin 2\phi & \cos 2\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$\begin{pmatrix} Q_+ \\ Q_- \\ Q_X \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi & \sin 2\phi \\ 0 & -\sin 2\phi & \cos 2\phi \end{bmatrix} \begin{pmatrix} Q_{+C} \\ Q_{-C} \\ Q_{XC} \end{pmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi & \sin 2\phi \\ 0 & -\sin 2\phi & \cos 2\phi \end{bmatrix} \begin{pmatrix} P_{+c} \\ P_{-c} \\ P_{xc} \end{pmatrix} = \begin{pmatrix} P_+ \\ P_- \\ P_x \end{pmatrix} + 2\phi' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} Q_+ \\ Q_- \\ Q_x \end{pmatrix}$$

$$L_c = L + 2\phi' \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q_+ \\ Q_- \\ Q_x \end{pmatrix} = L + 2\phi' Q_+$$

Finally,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi & \sin 2\phi \\ 0 & -\sin 2\phi & \cos 2\phi \end{bmatrix} \begin{pmatrix} E_{+c} \\ E_{-c} \\ E_{xc} \end{pmatrix} = \begin{pmatrix} E_+ \\ E_- \\ E_x \end{pmatrix} + 2\phi' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} P_+ \\ P_- \\ P_x \end{pmatrix}$$

$$+2\phi'^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} Q_+ \\ Q_- \\ Q_x \end{pmatrix} + 2\phi' L \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Represent the space charge force

The space charge matrix can be represented several ways.

$$\mathbf{O}\cdot\mathbf{Q}=\frac{4\Lambda}{ab}\begin{bmatrix} 1 & c_\alpha\Delta & s_\alpha\Delta \\ c_\alpha\Delta & 1 & 0 \\ s_\alpha\Delta & 0 & 1 \end{bmatrix}\begin{pmatrix} Q_+ \\ Q_- \\ Q_x \end{pmatrix}=\frac{4\Lambda}{ab}\begin{pmatrix} Q_+ + c_\alpha\Delta Q_- + s_\alpha\Delta Q_x \\ c_\alpha\Delta Q_+ + Q_- \\ s_\alpha\Delta Q_+ + Q_x \end{pmatrix}$$

where

$$\frac{4c_\alpha\Delta}{ab}=-\frac{Q_-}{\left\{Q_+ + \left[Q_+^2-\left(Q_-^2+Q_x^2\right)\right]^{1/2}\right\}\left[Q_+^2-\left(Q_-^2+Q_x^2\right)\right]^{1/2}}$$

$$\frac{4s_\alpha\Delta}{ab}=-\frac{Q_x}{\left\{Q_+ + \left[Q_+^2-\left(Q_-^2+Q_x^2\right)\right]^{1/2}\right\}\left[Q_+^2-\left(Q_-^2+Q_x^2\right)\right]^{1/2}}$$

$$\frac{4}{ab}=\frac{1}{\left[Q_+^2-\left(Q_-^2+Q_x^2\right)\right]^{1/2}}$$

Note the following

$$\frac{\partial}{\partial Q_{-,x}}\ln\left\{Q_+ + \left[Q_+^2-\left(Q_-^2+Q_x^2\right)\right]^{1/2}\right\}=\frac{-1}{\left\{Q_+ + \left[Q_+^2-\left(Q_-^2+Q_x^2\right)\right]^{1/2}\right\}}\frac{Q_{-,x}}{\left[Q_+^2-\left(Q_-^2+Q_x^2\right)\right]^{1/2}}$$

$$\frac{\partial}{\partial Q_+}\ln\left\{Q_+ + \left[Q_+^2-\left(Q_-^2+Q_x^2\right)\right]^{1/2}\right\}=\frac{1}{\left\{Q_+ + \left[Q_+^2-\left(Q_-^2+Q_x^2\right)\right]^{1/2}\right\}}\left\{1+\frac{Q_+}{\left[Q_+^2-\left(Q_-^2+Q_x^2\right)\right]^{1/2}}\right\}$$

$$=\frac{1}{\left[Q_+^2-\left(Q_-^2+Q_x^2\right)\right]^{1/2}}$$

Define

$$Q_\Delta=\left[Q_+^2-\left(Q_-^2+Q_x^2\right)\right]^{1/2}$$

Let

$H=\Lambda\ln\left(Q_+ + Q_\Delta\right)$.

$$\frac{\partial H}{\partial Q_{-,x}} = \frac{-\Lambda Q_{-,x}}{Q_\Delta\left(Q_+ + Q_\Delta\right)}$$

$$\frac{\partial H}{\partial Q_+} = \frac{\Lambda}{Q_\Delta}$$

Thus the space charge contribution to the **O** matrix is

$$\mathbf{O} = \begin{bmatrix} \frac{\partial H}{\partial Q_+} & \frac{\partial H}{\partial Q_-} & \frac{\partial H}{\partial Q_x} \\ \frac{\partial H}{\partial Q_-} & \frac{\partial H}{\partial Q_+} & 0 \\ \frac{\partial H}{\partial Q_x} & 0 & \frac{\partial H}{\partial Q_+} \end{bmatrix} = \frac{\Lambda}{Q_\Delta} \begin{bmatrix} 1 & \frac{-Q_-}{\left(Q_+ + Q_\Delta\right)} & \frac{-Q_x}{\left(Q_+ + Q_\Delta\right)} \\ \frac{-Q_-}{\left(Q_+ + Q_\Delta\right)} & 1 & 0 \\ \frac{-Q_x}{\left(Q_+ + Q_\Delta\right)} & 0 & 1 \end{bmatrix}.$$

Dotting this matrix on the right with the vector **P** gives a vector $\mathbf{R_P}$

$$\mathbf{R}_P = \mathbf{O}\cdot\mathbf{P} = \begin{pmatrix} \frac{\partial H}{\partial Q_+} P_+ + \frac{\partial H}{\partial Q_-} P_- + \frac{\partial H}{\partial Q_x} P_x \\ \frac{\partial H}{\partial Q_-} P_+ + \frac{\partial H}{\partial Q_+} P_- \\ \frac{\partial H}{\partial Q_x} P_+ + \frac{\partial H}{\partial Q_+} P_x \end{pmatrix} = \mathbf{R}_P\left(Q_+, Q_-, Q_x \,|\, \mathbf{P}\right).$$

$$\mathbf{R}_P = \begin{pmatrix} \frac{\partial H}{\partial Q_+} P_+ + \frac{\partial H}{\partial Q_-} P_- + \frac{\partial H}{\partial Q_x} P_x \\ \frac{\partial H}{\partial Q_-} P_+ + \frac{\partial H}{\partial Q_+} P_- \\ \frac{\partial H}{\partial Q_x} P_+ + \frac{\partial H}{\partial Q_+} P_x \end{pmatrix} = \frac{\Lambda}{Q_\Delta} \begin{pmatrix} P_+ - \frac{Q_-}{Q_\Delta + Q_+} P_- - \frac{Q_x}{Q_\Delta + Q_+} P_x \\ -\frac{Q_-}{Q_\Delta + Q_+} P_+ + P_- \\ -\frac{Q_x}{Q_\Delta + Q_+} P_+ + P_x \end{pmatrix}$$

This will be used in formulating adjoint equations.

Adjoint Equations for Linear Perturbations

First consider a linearization of the system

$$\frac{d}{dz}\delta\mathbf{Q}^{(X)} = \delta\mathbf{P}^{(X)}$$
$$\frac{d}{dz}\delta\mathbf{P}^{(X)} = \delta\mathbf{E} + \mathbf{O}\cdot\delta\mathbf{Q}^{(X)} + \delta\mathbf{O}^{(X)}\cdot\mathbf{Q}$$
$$\frac{d}{dz}\delta\mathbf{E}^{(X)} = \mathbf{O}\cdot\delta\mathbf{P}^{(X)} + \mathbf{N}\delta L^{(X)} + \delta\mathbf{O}^{(X)}\cdot\mathbf{P} + \delta\mathbf{N}^{(X)}L$$
$$\frac{d}{dz}\delta L^{(X)} = -\mathbf{N}^{\dagger}\cdot\delta\mathbf{Q}^{(X)} - \delta\mathbf{N}^{\dagger(X)}\cdot\mathbf{Q}$$

where the superscript (X) signifies that it is a true perturbation, i.e. the result of changing the focusing system: the solenoidal and quadrupole magnetic fields. The matrices **O** and **N** are perturbed due to two effects. One set of perturbations is due to changes in the solenoidal and quadrupole magnetic fields. These are being varied in order to optimize the configuration. A second set of perturbations is due to changes is the space charge terms. These changes are expressed through the changes in the spatial moments, $\delta\mathbf{Q}^{(X)}$.

Now consider a second perturbation with superscript (Y) for which we don't perturb the matrices, but include a forcing function in the energy equation. The reason for this forcing function will become apparent subsequently. These are the adjoint equations that will actually be solved.

$$\frac{d}{dz}\delta\mathbf{Q}^{(Y)} = \delta\mathbf{P}^{(Y)}$$
$$\frac{d}{dz}\delta\mathbf{P}^{(Y)} = \delta\mathbf{E}^{(Y)} + \mathbf{O}\cdot\delta\mathbf{Q}^{(Y)}$$
$$\frac{d}{dz}\delta\mathbf{E}^{(Y)} = \mathbf{O}\cdot\delta\mathbf{P}^{(Y)} + \mathbf{N}\delta L^{(Y)} + \delta\dot{\mathbf{E}}^{(Y)}$$
$$\frac{d}{dz}\delta L^{(Y)} = -\mathbf{N}^{\dagger}\cdot\delta\mathbf{Q}^{(Y)}$$

We then form the following

$$\begin{aligned}
&\delta\mathbf{P}^{(Y)}\cdot\frac{d}{dz}\delta\mathbf{P}^{(X)} + \delta\mathbf{P}^{(X)}\cdot\frac{d}{dz}\delta\mathbf{P}^{(Y)} - \delta\mathbf{Q}^{(X)}\cdot\frac{d}{dz}\delta\mathbf{E}^{(Y)} - \delta\mathbf{E}^{(Y)}\cdot\frac{d}{dz}\delta\mathbf{Q}^{(X)} \\
&-\delta\mathbf{Q}^{(Y)}\cdot\frac{d}{dz}\delta\mathbf{E}^{(X)} - \delta\mathbf{E}^{(X)}\cdot\frac{d}{dz}\delta\mathbf{Q}^{(Y)} - \delta L^{(Y)}\frac{d}{dz}\delta L^{(X)} - \delta L^{(X)}\cdot\frac{d}{dz}\delta L^{(Y)} = . \\
&= \frac{d}{dz}\left\{\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{P}^{(X)} - \delta\mathbf{Q}^{(X)}\cdot\delta\mathbf{E}^{(Y)} - \delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{E}^{(X)} - \delta L^{(Y)}\delta L^{(X)}\right\} \equiv \frac{d}{dz}\delta\varepsilon
\end{aligned}$$

Next we evaluate the terms

$$
\begin{aligned}
&\frac{d}{dz}\delta\varepsilon \\
&=\delta\mathbf{P}^{(Y)}\cdot\left\{\delta\mathbf{E}^{(X)}+\mathbf{O}\cdot\delta\mathbf{Q}^{(X)}+\delta\mathbf{O}^{(X)}\cdot\mathbf{Q}\right\}-\delta\mathbf{Q}^{(X)}\cdot\left\{\mathbf{O}\cdot\delta\mathbf{P}^{(Y)}+\mathbf{N}\delta L^{(Y)}+\delta\dot{\mathbf{E}}^{(Y)}\right\} \\
&-\delta\mathbf{E}^{(X)}\cdot\delta\mathbf{P}^{(Y)}-\delta L^{(Y)}\left(-\mathbf{N}^{\dagger}\cdot\delta\mathbf{Q}^{(X)}-\delta\mathbf{N}^{\dagger(X)}\cdot\mathbf{Q}\right) \\
&+\delta\mathbf{P}^{(X)}\cdot\left\{\delta\mathbf{E}^{(Y)}+\mathbf{O}\cdot\delta\mathbf{Q}^{(Y)}\right\}-\delta\mathbf{Q}^{(Y)}\cdot\left\{\mathbf{O}\cdot\delta\mathbf{P}^{(X)}+\mathbf{N}\delta L^{(X)}+\delta\mathbf{O}^{(X)}\cdot\mathbf{P}+\delta\mathbf{N}^{(X)}L\right\} \\
&-\delta\mathbf{E}^{(Y)}\cdot\delta\mathbf{P}^{(X)}-\delta L^{(X)}\left(-\mathbf{N}^{\dagger}\cdot\delta\mathbf{Q}^{(Y)}\right)
\end{aligned}
$$

Cancel terms

$$
\begin{aligned}
&\frac{d}{dz}\varepsilon=\left\{\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{O}^{(X)}\cdot\mathbf{Q}-\delta\mathbf{Q}^{(X)}\cdot\delta\dot{\mathbf{E}}^{(Y)}+\delta L^{(Y)}\delta\mathbf{N}^{\dagger(X)}\cdot\mathbf{Q}\right\} \\
&+\left\{-\delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{O}^{(X)}\cdot\mathbf{P}-\delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{N}^{(X)}L\right\}
\end{aligned}.
$$

Let's consider that the quantities, $\delta\mathbf{N}^{(X)}$ and $\delta\mathbf{O}^{(X)}$. As mentioned, these will consist of two parts. One part comes from changes in the solenoidal and quadrupole fields, and another part comes from changes in the space charge field. The changes in the space charge contributions are proportional to $\delta\mathbf{Q}^{(X)}$.

Writing the changes as a sum of contributions from changes in the focusing configuration and changes in the space charge gives,

$$
\delta\mathbf{O}^{(X)}\cdot\mathbf{Q}=\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{Q}+\mathbf{M}_{Q}\cdot\delta\mathbf{Q}^{(X)},
$$

$$
\delta\mathbf{O}^{(X)}\cdot\mathbf{P}=\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{P}+\mathbf{M}_{P}\cdot\delta\mathbf{Q}^{(X)},
$$

and

$$
\delta\mathbf{N}^{(X)}=\delta\mathbf{N}_{Q,B}{}^{(X)}+\mathbf{M}_{N}\cdot\delta\mathbf{Q}^{(X)}.
$$

We then have

$$
\begin{aligned}
&\frac{d}{dz}\varepsilon=\left\{\delta\mathbf{P}^{(Y)}\cdot\mathbf{M}_{Q}\cdot\delta\mathbf{Q}^{(X)}+\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{Q}-\delta\mathbf{Q}^{(X)}\cdot\delta\dot{\mathbf{E}}^{(Y)}+\delta L^{(Y)}\mathbf{Q}\cdot\mathbf{M}_{N}\cdot\delta\mathbf{Q}^{(X)}+\delta L^{(Y)}\mathbf{Q}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)}\right\} \\
&+\left\{-\delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{P}-\delta\mathbf{Q}^{(Y)}\cdot\mathbf{M}_{P}\cdot\delta\mathbf{Q}^{(X)}-\delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)}L-\delta\mathbf{Q}^{(Y)}\cdot\mathbf{M}_{N}\cdot\delta\mathbf{Q}^{(X)}L\right\}
\end{aligned}
$$

The next step is to pick $\delta\dot{\mathbf{E}}^{(Y)}$ to cancel all terms proportional to the unknown $\delta\mathbf{Q}^{(X)}$.

$$
0=\left\{\delta\mathbf{P}^{(Y)}\cdot\mathbf{M}_{Q}\cdot\delta\mathbf{Q}^{(X)}-\delta\mathbf{Q}^{(X)}\cdot\delta\dot{\mathbf{E}}^{(Y)}+\delta L^{(Y)}\mathbf{Q}\cdot\mathbf{M}_{N}\cdot\delta\mathbf{Q}^{(X)}-\delta\mathbf{Q}^{(Y)}\cdot\mathbf{M}_{P}\cdot\delta\mathbf{Q}^{(X)}-\delta\mathbf{Q}^{(Y)}\cdot\mathbf{M}_{N}\cdot\delta\mathbf{Q}^{(X)}L\right\}
$$

or

$$
\delta\dot{\mathbf{E}}^{(Y)}=\left\{\delta\mathbf{P}^{(Y)}\cdot\mathbf{M}_{Q}+\delta L^{(Y)}\mathbf{Q}\cdot\mathbf{M}_{N}-\delta\mathbf{Q}^{(Y)}\cdot\mathbf{M}_{P}-\delta\mathbf{Q}^{(Y)}\cdot\mathbf{M}_{N}L\right\}.
$$

This quantity then enters the adjoint equations.

This leaves for the adjoint relation

$$\frac{d}{dz}\varepsilon=\left\{\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{Q}+\delta L^{(Y)}\mathbf{Q}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)}\right\}+\left\{-\delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{P}-\delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)}L\right\}.$$

Integrating over z from initial to final point.

$$\left(\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{P}^{(X)}-\delta\mathbf{Q}^{(X)}\cdot\delta\mathbf{E}^{(Y)}-\delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{E}^{(X)}-\delta L^{(Y)}\delta L^{(X)}\right)\Big|_{z=z_i}^{z=z_f}=$$

$$\int_{z_i}^{z_f}dz\left\{\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{Q}+\delta L^{(Y)}\mathbf{Q}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)}-\delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{P}-\delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)}L\right\}$$

The adjoint variables multiplying the changes in the focusing matrices then give the sensitivity of the quantity in the left to changes in these matrices.

Example

To see how this works, consider the case where a flat to round transition is being evaluated and the entering flat beam has a given set of parameters. This means the changes in the moments with superscript (X) are all zero at $z=z_i$. We then want to pick the values of the perturbed adjoint moments at $z=z_f$, those with superscript (Y), so that the left side evaluated at $z = z_f$ looks like a change in figure of merit. Suppose we want to arrive ($z=z_f$) in the solenoidal magnetic field with a perfectly round beam of constant radius in the soleneoid. Our target Q vector is thus,

$$\mathbf{Q}_T=\begin{pmatrix} Q_{+0} \\ 0 \\ 0 \end{pmatrix}.$$

Our target momentum variable is **P**=0. Our target energy variable is, consistent with $d\mathbf{P}/dz=0$,

$$\mathbf{E}_T=\begin{pmatrix} E_{+0} \\ 0 \\ 0 \end{pmatrix},$$

where

$$E_{+0}=\left(\frac{k_\Omega^2}{2}-\frac{4\Lambda}{ab}\right)Q_{+0}=\frac{k_\Omega^2}{2}Q_{+0}-\Lambda .$$

Let the entering flat beam parameters be given be given by subscript i. Let us further suppose that the entering beam has no angular momentum. The initial constant of motion is

$$J_i=\mathbf{E}_i\cdot\mathbf{Q}_i-\frac{1}{2}\mathbf{P}_i\cdot\mathbf{P}_i .$$

The final value of angular momentum, L_0, then satisfies

$$J_f=E_{+0}Q_{+0}+\frac{1}{2}L_o^2=J_i .$$

If in the solenoid particles rotate with a radially varying rotation rate $\Omega(r)$ in the Larmor frame then

$$E_{+0}=\left\langle \delta x'^2+\delta y'^2\right\rangle+\left\langle \Omega^2 r^2\right\rangle .$$

The angular momentum in this case is

$$L_0=\left\langle \Omega r^2\right\rangle .$$

We thus have the inequality

$$E_{+0}\geq\left\langle \Omega^2 r^2\right\rangle\geq\left\langle \Omega r^2\right\rangle^2/\left\langle r^2\right\rangle=L_0^2/\left(2Q_{+0}\right).$$

A figure of merit might be

$$F=\frac{1}{2}\left[\left|\mathbf{P}\right|^2+k_0^2\left(Q_-^2+Q_x^2\right)+k_0^{-2}\left(E_-^2+E_x^2\right)+k_0^{-2}\left(E_+-\frac{1}{2}k_\Omega^2 Q_++\Lambda\right)^2+\left(2E_+Q_+-L_0^2\right)^2\right].$$

The goal would be to minimize this function in the solenoid ($z=z_f$). Minimizing the first term makes the shape constant in the solenoid. Minimizing the second term makes the beam round. Minimizing the third term lowers the beam temperature. Minimizing the fourth term also lowers the beam temperature and keeps the beam radius constant ($\dot{P}_+=0$). Minimizing the last term makes the rotation laminar and rigid. Here all variables are evaluated at z_f which in this case is in the solenoid.

The variation of this figure of merit is

$$\delta F=\delta\mathbf{Q}^{(X)}\cdot\frac{\partial F}{\partial\mathbf{Q}}+\delta\mathbf{P}^{(X)}\cdot\mathbf{P}+\delta\mathbf{E}^{(X)}\cdot\frac{\partial F}{\partial\mathbf{E}}+\delta L^{(X)}\frac{\partial F}{\partial L}.$$

Comparing with the adjoint final quantity

$$\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{P}^{(X)}-\delta\mathbf{Q}^{(X)}\cdot\delta\mathbf{E}^{(Y)}-\delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{E}^{(X)}-\delta L^{(Y)}\delta L^{(X)},$$

we should then set at z_f (the final point in the solenoid)

$$\delta\mathbf{P}^{(Y)}(z_f)=\mathbf{P}(z_f)=\left.\frac{\partial F}{\partial\mathbf{P}}\right|_{z_f}$$

$$-\delta\mathbf{E}^{(Y)}(z_f)=\left.\frac{\partial F}{\partial\mathbf{Q}}\right|_{z_f}$$

$$-\delta\mathbf{Q}^{(Y)}(z_f)=\left.\frac{\partial F}{\partial\mathbf{E}}\right|_{z_f}$$

$$\delta L^{(y)}(z_f)=\left.\frac{\partial F}{\partial L}\right|_{z_f}$$

These are the “initial conditions” used to integrate the perturbed moment equations (superscript (Y)) backward from z_f to z_i. To do this you will need to have available the unperturbed moments as functions of z. You could save these from the initial calculation that started flat and ended in the solenoid, or you could reintegrate backwards the reference solution, starting with the final values of the moments in the solenoid from the previous run and integrating back to the input where the beam is flat. During this backward integration you need to store all values of the adjoint variables so you can evaluate the integral,

$$\int_{z_i}^{z_f}dz\left\{\delta\mathbf{P}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{Q}+\delta L^{(Y)}\mathbf{Q}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)}-\delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{O}_{Q,B}{}^{(X)}\cdot\mathbf{P}-\delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{N}_{Q,B}{}^{(X)}L\right\}$$

for different instances of the magnetic field profiles.

The matrices $\delta\mathbf{O}_{Q,B}{}^{(X)},\delta\mathbf{N}_{Q,B}{}^{(X)}$ are evaluated by perturbing the strength, orientation, or location of the quadrupoles or solenoid. In the case of perturbation of the strength of a quadrupole the matrices are defined with K_q replaced by

$$\delta K_q(z)=\varepsilon_K K_q(z),$$

where ε_K is a fractional change in the strength of the quadrupole. If the position of the quadrupole is changed then K_q is replaced by

$$\delta K_q(z) = d_q \frac{\partial}{\partial z} K_q(z).$$

Similarly, perturbing the orientation of the quadrupole introduces

$$\left(\delta s_q, \delta c_q\right) = -2 d_\psi \left(\cos(2\phi - 2\psi_q), -\sin(2\phi - 2\psi_q)\right).$$

Perturbing the solenoidal field also modifies the **O** and **N** matrices. This perturbation can be evaluated by differentiating $k_\Omega(z)$ where is appears. In principle this also shifts the phase $\phi(z)$. If this phase is defined to be zero in the quadrupole region, then this change of phase can be ignored.

Variations of space charge term

The variation in the space charge contribution to matrix **O** and vector $\mathbf{R}_P$ can be found by differentiation,

$$\delta\mathbf{O}^{(X)}\cdot\mathbf{P}=\left(\delta Q_+^{(X)}\frac{\partial}{\partial Q_+}+\delta Q_-^{(X)}\frac{\partial}{\partial Q_-}+\delta Q_x^{(X)}\frac{\partial}{\partial Q_x}\right)\begin{pmatrix}\frac{\partial H}{\partial Q_+}P_+ +\frac{\partial H}{\partial Q_-}P_- +\frac{\partial H}{\partial Q_x}P_x\\ \frac{\partial H}{\partial Q_-}P_+ +\frac{\partial H}{\partial Q_+}P_-\\ \frac{\partial H}{\partial Q_x}P_+ +\frac{\partial H}{\partial Q_+}P_x\end{pmatrix}.$$

$$=\left(\delta Q_+^{(X)}\frac{\partial}{\partial Q_+}+\delta Q_-^{(X)}\frac{\partial}{\partial Q_-}+\delta Q_x^{(X)}\frac{\partial}{\partial Q_x}\right)\mathbf{R}_P$$

This will lead to terms of the form

$$\delta\mathbf{Q}^{(Y)}\cdot\delta\mathbf{O}^{(X)}\cdot\mathbf{P}=\delta\mathbf{Q}^{(Y)}\cdot\mathbf{M}_P\cdot\delta\mathbf{Q}^{(X)}$$

Here inserting the expressions for the elements

$$\mathbf{R}_P=\frac{\Lambda}{Q_\Delta}\begin{pmatrix}P_+ -\frac{Q_-}{Q_\Delta+Q_+}P_- -\frac{Q_x}{Q_\Delta+Q_+}P_x\\ -\frac{Q_-}{Q_\Delta+Q_+}P_+ +P_-\\ -\frac{Q_x}{Q_\Delta+Q_+}P_+ +P_x\end{pmatrix}$$

Similarly define $\mathbf{R_Q}$ **and** $\mathbf{M_Q}$**.**

$$\mathbf{R}_Q=\mathbf{O}\cdot\mathbf{Q}=\begin{pmatrix}\frac{\partial H}{\partial Q_+}Q_+ +\frac{\partial H}{\partial Q_-}Q_- +\frac{\partial H}{\partial Q_x}Q_x\\ \frac{\partial H}{\partial Q_-}Q_+ +\frac{\partial H}{\partial Q_+}Q_-\\ \frac{\partial H}{\partial Q_x}Q_+ +\frac{\partial H}{\partial Q_+}Q_x\end{pmatrix}=\mathbf{R}_Q\left(Q_+,Q_-,Q_x\,|\,\mathbf{Q}\right)$$

and

$$\mathbf{M}_Q = \begin{bmatrix} \frac{\partial R_{Q+}}{\partial Q_+} & \frac{\partial R_{Q+}}{\partial Q_-} & \frac{\partial R_{Q+}}{\partial Q_x} \\ \frac{\partial R_{Q-}}{\partial Q_+} & \frac{\partial R_{Q-}}{\partial Q_-} & \frac{\partial R_{Q-}}{\partial Q_x} \\ \frac{\partial R_{Qx}}{\partial Q_+} & \frac{\partial R_{Qx}}{\partial Q_-} & \frac{\partial R_{Qx}}{\partial Q_x} \end{bmatrix}$$

Here it is understood that the derivatives in $\mathbf{M}_Q$ fall only on H.

Evaluate $\mathbf{M}_P$, $\mathbf{M}_Q$, $\mathbf{M}_N$

$$\mathbf{R}_P = \frac{\Lambda}{Q_\Delta}\begin{pmatrix} P_+ - \frac{Q_-}{Q_\Delta + Q_+}P_- - \frac{Q_x}{Q_\Delta + Q_+}P_x \\ -\frac{Q_-}{Q_\Delta + Q_+}P_+ + P_- \\ -\frac{Q_x}{Q_\Delta + Q_+}P_+ + P_x \end{pmatrix}$$

$$\mathbf{M}_P = \begin{pmatrix} V_{+1} \\ V_{-1} \\ V_{x1} \end{pmatrix}\begin{pmatrix} U_{+1} & U_{-1} & U_{x1} \end{pmatrix} + \begin{pmatrix} V_{+2} \\ V_{-2} \\ V_{x2} \end{pmatrix}\begin{pmatrix} U_{+2} & U_{-2} & U_{x2} \end{pmatrix}$$
$$+ \begin{pmatrix} V_{+3} \\ V_{-3} \\ V_{x3} \end{pmatrix}\begin{pmatrix} U_{+3} & U_{-3} & U_{x3} \end{pmatrix} + \begin{pmatrix} V_{+4} \\ V_{-4} \\ V_{x4} \end{pmatrix}\begin{pmatrix} U_{+4} & U_{-4} & U_{x4} \end{pmatrix}$$

$$\mathbf{M}_P = \mathbf{V}_1\mathbf{U}_1^T + \mathbf{V}_2\mathbf{U}_2^T + \mathbf{V}_3\mathbf{U}_3^T + \mathbf{V}_4\mathbf{U}_4^T$$

$$\mathbf{V}_1 = -\frac{\Lambda}{Q_\Delta^2}\begin{pmatrix} P_+ - \dfrac{Q_-}{Q_\Delta + Q_+}P_- - \dfrac{Q_x}{Q_\Delta + Q_+}P_x \\ -\dfrac{Q_-}{Q_\Delta + Q_+}P_+ + P_- \\ -\dfrac{Q_x}{Q_\Delta + Q_+}P_+ + P_x \end{pmatrix}$$

$$\mathbf{U}_1^T = \begin{pmatrix} \dfrac{\partial Q_\Delta}{\partial Q_+} & \dfrac{\partial Q_\Delta}{\partial Q_-} & \dfrac{\partial Q_\Delta}{\partial Q_x} \end{pmatrix} = \frac{1}{Q_\Delta}\begin{pmatrix} Q_+ & -Q_- & -Q_x \end{pmatrix}$$

$$\mathbf{V}_2 = \frac{\Lambda}{Q_\Delta\left(Q_\Delta + Q_+\right)^2}\begin{pmatrix} Q_-P_- + Q_xP_x \\ Q_-P_+ \\ Q_xP_+ \end{pmatrix}$$

$$\mathbf{U}_2^T = \begin{pmatrix} \dfrac{\partial\left(Q_\Delta + Q_+\right)}{\partial Q_+} & \dfrac{\left(Q_\Delta + Q_+\right)}{\partial Q_-} & \dfrac{\partial\left(Q_\Delta + Q_+\right)}{\partial Q_x} \end{pmatrix} = \mathbf{U}_1^T + \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{V}_3 = -\frac{\Lambda}{Q_\Delta\left(Q_\Delta + Q_+\right)}\begin{pmatrix} P_- \\ P_+ \\ 0 \end{pmatrix}, \quad \mathbf{V}_4 = -\frac{\Lambda}{Q_\Delta\left(Q_\Delta + Q_+\right)}\begin{pmatrix} P_x \\ 0 \\ P_+ \end{pmatrix}$$

$$\mathbf{U}_3^T = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{U}_4^T = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

Evaluate $\mathbf{M_Q}$ Replace P_{+-x} by Q_{+-x} after differentiation

$$\mathbf{M}_Q = \mathbf{W}_1\mathbf{U}_1^T + \mathbf{W}_2\mathbf{U}_2^T + \mathbf{W}_3\mathbf{U}_3^T + \mathbf{W}_4\mathbf{U}_4^T$$

$$\mathbf{W}_1 = -\frac{\Lambda}{Q_\Delta^2}\begin{pmatrix} Q_+ - \frac{Q_-}{Q_\Delta + Q_+} Q_- - \frac{Q_x}{Q_\Delta + Q_+} Q_x \\ -\frac{Q_-}{Q_\Delta + Q_+} Q_+ + Q_- \\ -\frac{Q_x}{Q_\Delta + Q_+} Q_+ + Q_x \end{pmatrix} = -\frac{\Lambda}{Q_\Delta}\begin{pmatrix} 1 \\ \frac{Q_-}{Q_\Delta + Q_+} \\ \frac{Q_x}{Q_\Delta + Q_+} \end{pmatrix}$$

$$\mathbf{W}_2 = \frac{\Lambda}{Q_\Delta \left(Q_\Delta + Q_+\right)^2}\begin{pmatrix} Q_- Q_- + Q_x Q_x \\ Q_- Q_+ \\ Q_x Q_+ \end{pmatrix}$$

$$\mathbf{W}_3 = -\frac{\Lambda}{Q_\Delta \left(Q_\Delta + Q_+\right)}\begin{pmatrix} Q_- \\ Q_+ \\ 0 \end{pmatrix}, \quad \mathbf{W}_4 = -\frac{\Lambda}{Q_\Delta \left(Q_\Delta + Q_+\right)}\begin{pmatrix} Q_x \\ 0 \\ Q_+ \end{pmatrix}$$

Evaluate $\mathbf{M}_N$

$$\mathbf{N} = 2\sum_{Quads} K_q \begin{pmatrix} 0 \\ s_q \\ c_q \end{pmatrix} - \frac{4\Lambda}{ab}\begin{pmatrix} 0 \\ s_\alpha \Delta \\ -c_\alpha \Delta \end{pmatrix} = \mathbf{N}_q + \Lambda \begin{pmatrix} 0 \\ \frac{Q_x}{Q_\Delta \left(Q_+ + Q_\Delta\right)} \\ -\frac{Q_-}{Q_\Delta \left(Q_+ + Q_\Delta\right)} \end{pmatrix}$$

$$\mathbf{X}_1 = -\Lambda \begin{pmatrix} 0 \\ \frac{Q_x}{Q_\Delta^2 \left(Q_+ + Q_\Delta\right)} \\ -\frac{Q_-}{Q_\Delta^2 \left(Q_+ + Q_\Delta\right)} \end{pmatrix}, \quad \mathbf{X}_2 = -\Lambda \begin{pmatrix} 0 \\ \frac{Q_x}{Q_\Delta \left(Q_+ + Q_\Delta\right)^2} \\ -\frac{Q_-}{Q_\Delta \left(Q_+ + Q_\Delta\right)^2} \end{pmatrix}$$

$$\mathbf{X}_3 = \Lambda \begin{pmatrix} 0 \\ 0 \\ -\dfrac{1}{Q_\Delta\left(Q_+ + Q_\Delta\right)} \end{pmatrix}, \quad \mathbf{X}_4 = \Lambda \begin{pmatrix} 0 \\ \dfrac{1}{Q_\Delta\left(Q_+ + Q_\Delta\right)} \\ 0 \end{pmatrix}$$

$$\mathbf{M}_N = \mathbf{X}_1\mathbf{U}_1^T + \mathbf{X}_2\mathbf{U}_2^T + \mathbf{X}_3\mathbf{U}_3^T + \mathbf{X}_4\mathbf{U}_4^T$$

Special Case +/- 45 degree quads – no space charge

If $\psi_q = \pm\pi/4$ then the moments split into two groups in the absence of space charge. Further, there are then three constants of motion, not just one. Note in this case in the **O** and **N** matrices c_q = 0 and s_q = +/-1. The first group of variables that evolve as a group is $\left(Q_-, P_-, E_-, L\right)$. The governing equations are

$$\frac{d}{dz}Q_- = P_-$$
$$\frac{d}{dz}P_- = E_-$$
$$\frac{d}{dz}E_- = 2K_q s_q L$$
$$\frac{d}{dz}L = -2K_q s_q Q_-$$

The associated constant of motion is

$$J_- = E_- Q_- + \frac{1}{2}L^2 - \frac{1}{2}P_-^2.$$

The second group of variables is $\left(Q_+, Q_x, P_+, P_x, E_+, E_x\right)$. The evolution equations are

$$\frac{d}{dz}Q_+ = P_+ \qquad \frac{d}{dz}Q_x = P_x$$
$$\frac{d}{dz}P_+ = E_+ - 2K_q s_q Q_x \qquad \frac{d}{dz}P_x = E_x - 2K_q s_q Q_+$$
$$\frac{d}{dz}E_+ = -2K_q s_q P_x \qquad \frac{d}{dz}E_x = -2K_q s_q P_+$$

Notice that this group can be further divided by introducing sum and difference variables

$$Q_{s,d} = Q_+ \pm Q_x$$
$$P_{s,d} = P_+ \pm P_x$$
$$E_{s,d} = E_+ \pm E_x$$

The evolution equations for these are

$$\frac{d}{dz} Q_{s,d} = P_{s,d}$$
$$\frac{d}{dz} P_{s,d} = E_{s,d} \mp 2K_q s_q Q_{s,d}$$
$$\frac{d}{dz} E_{s,d} = \mp 2K_q s_q P_{s,d}$$

There are two constants of motion associated with these equations

$$J_{s,d} = E_{s,d} Q_{s,d} - \frac{1}{2} P_{s,d}^2$$

Special Case Unrotated Quadrupoles

Round to flat transition

There are a total of 10 linear, first order differential equations requiring 10 initial or final conditions. At the same time there are 10 independent solutions. Let us assume the following boundary conditions. All three components of *P* should vanish both at the round and flat ends of the configuration (6 conditions). The angular momentum should vanish at the flat end of the configuration. (1 condition), Q_x should vanish at both ends (2 conditions). If we set the size of the beam in the round end, that would count as the tenth condition, and it would seem that the boundary conditions could be satisfied for any configuration of the quads. However, the above counting does not take into consideration that the equations separate into three uncoupled groups.

Let's first list the constants of motion. Here we denote a value in the flat region with subscript f and a value in the round region with subscript r.

$$J_- = E_{-r} Q_{-r} + \frac{1}{2} L_r^2 = E_{-f} Q_{-f}$$
$$J_{s,d} = E_{s,d,r} Q_{+,r} = E_{s,d,f} Q_{+,f}$$

Given that we desire to have $Q_{-r} = 0$, it appears that the degree of flatness and the emmitance are directly related to the angular momentum. With Q_x vanishing at both ends the transverse spatial extent of the beam and the emmitance are also linked.

Let's consider representing the solutions of the three separate systems parameterized by their initial conditions.

Let

$$Q_{s,d}(z|E_{s,d}(0),Q_+(0))$$
$$P_{s,d}(z|E_{s,d}(0),Q_+(0))$$
$$E_{s,d}(z|E_{s,d}(0),Q_+(0))$$

be solutions for the sum and difference coordinates with the initial conditions

$$Q_{s,d}(0|E_{s,d}(0),Q_+(0)) = Q_+(0)$$
$$P_{s,d}(0|E_{s,d}(0),Q_+(0)) = 0 \qquad .$$
$$E_{s,d}(0|E_{s,d}(0),Q_+(0)) = E_{s,d}(0)$$

We can construct a linear combination of these for two different values of $E_{s,d}(0)$ such $Q_{s,d}(0)=Q_+(0)$ and that $P_{s,d}(z_f)=0$. In general these sum and difference solutions will have different values of $Q_{s,d}(z_f)$ implying $Q_x(z_f)$ is not zero. This shows that it is not possible to satisfy all the boundary conditions without adjusting the quads. It is then necessary to vary the parameters of the quads until $Q_s(z_f)=Q_d(z_f)$.

Next consider the equations for the variables with the "-" subscripts. Construct solutions

$$Q\_(z|E\_(0),L(0))$$
$$P\_(z|E\_(0),L(0))$$
$$E\_(z|E\_(0),L(0)),$$
$$L(z|E\_(0),L(0))$$

that satisfy

$$
\begin{aligned}
&Q_{-}(0|E_{-}(0),L(0))=0\\
&P_{-}(0|E_{-}(0),L(0))=0\\
&E_{-}(0|E_{-}(0),L(0))=E_{-}(0)\\
&L(0|E_{-}(0),L(0))=L(0)
\end{aligned}
.
$$

We need to find a linear combination of these such that

$$
\begin{aligned}
&P_{-}(z_f)=0\\
&L(z_f)=0
\end{aligned}
$$

This should be achievable for any quad parameters. Further, since it is a linear system there is a multiplicative factor that can be used to modify the solution. This can be used to vary the parameters of the flat beam.

Symmetric Triplet

First consider the group of variables

$$
\begin{aligned}
&\frac{d}{dz}Q_{-}=P_{-}\\
&\frac{d}{dz}P_{-}=E_{-}\\
&\frac{d}{dz}E_{-}=2K_q s_q L\\
&\frac{d}{dz}L=-2K_q s_q Q_{-}
\end{aligned}
$$

We will examine to see if these can be made to satisfy all boundary conditions for a symmetric triplet. At z=0 we have

$$
\begin{aligned}
&Q_{-}(0)=Q_1\\
&P_{-}(0)=0\\
&E_{-}(0)=E_1\\
&L(0)=0
\end{aligned}
$$

Quadrupoles are located at z=0, z=d, and z=2d. We will treat these in the thin lens limit. The variables L and E_{-} will suffer jumps at the locations of the quadrupoles, while Q_{-} and P_{-} will be continuous functions. We will adopt the convention that the values L_n and E_n apply to the values of these variables as they enter the nth quadrupole, and Q_n and P_n apply at the nth quadrupole.

Integrating the angular momentum gives the recursion relation

$$L_{n+1} = L_n - I_n Q_n .$$

Where $I_n = \int_{-\infty}^{\infty} 2K_q(z) s_q \, dz$

Integrating the energy variable in conjunction with the angular momentum variable gives

$$E_{n+1} = E_n + I_n L_n - \frac{1}{2} I_n^2 Q_n .$$

Solving the momentum equation

$$P_{n+1} = P_n + dE_{n+1} ,$$

And for the Q variables

$$Q_{n+1} = Q_n + \frac{d}{2}\left(P_{n+1} + P_n\right).$$

We thus have a set of recursion relations that will determine the variable values at each quadrupole.

$$L_{n+1} = L_n - I_n Q_n$$
$$E_{n+1} = E_n + I_n L_n - \frac{1}{2} I_n^2 Q_n$$
$$P_{n+1} = P_n + dE_{n+1}$$
$$Q_{n+1} = Q_n + \frac{d}{2}\left(P_{n+1} + P_n\right)$$

We verify that these recursion relations conserve the prescribed quantity
At a Quad

$$Q_n(E_{n+1} - E_n) = Q_n\left(I_n L_n - \frac{1}{2} I_n^2 Q_n \right)$$
$$\frac{1}{2}\left(L_{n+1}^2 - L_n^2\right) = \frac{1}{2}\left(\left(L_n - I_n Q_n\right)^2 - L_n^2\right) = \frac{1}{2}\left(\left(I_n Q_n\right)^2 - 2L_n I_n Q_n\right)$$

These sum to zero

Between Quads

$$E_{n+1}\left(Q_{n+1}-Q_n\right)-\frac{1}{2}\left(P_{n+1}^2-P_n^2\right)=E_{n+1}\frac{d}{2}\left(P_{n+1}+P_n\right)-\frac{1}{2}\left(dE_{n+1}\right)\left(P_{n+1}+P_n\right)=0$$

Checks

Returning to the conditions at the exit.

We have as final conditions $P_3 = Q_3 = 0$. The final values of E_- and L are determined by jumps at the third quadrupole,

$$L_f = L_3 - I_3 Q_3 = L_3$$

$$E_f = E_3 + I_3 L_3 - \frac{1}{2} I_3^2 Q_3 = E_3 + I_3 L_3$$

As a consequence, after satisfying the conditions $P_3 = Q_3 = 0$, we must use the above to find the values of L_3 and E_3.

Working backward we find relations for the subscript 2 variables

$$Q_3 = Q_2 + \frac{d}{2} P_2 = 0$$

$$P_3 = P_2 + dE_3 = 0$$

$$E_3 = E_2 + I_2 L_2 - \frac{1}{2} I_2^2 Q_2$$

$$L_3 = L_2 - I_2 Q_2$$

Note that the strength of quadrupole 3 does not enter in satisfying the boundary conditions. Eliminating E3 gives

$$0 = Q_2 + \frac{d}{2} P_2$$

$$0 = P_2 + dE_2 + dI_2 L_2 - \frac{d}{2} I_2^2 Q_2$$

Working forward from z=0 gives

$$L_2 = -I_1 Q_1$$

$$E_2 = E_1 - \frac{1}{2} I_1^2 Q_1$$

$$P_2 = dE_2 = dE_1 - \frac{d}{2} I_1^2 Q_1$$

$$Q_2 = Q_1 + \frac{d}{2} P_2 = \left(1 - \frac{d^2}{4} I_1^2 \right) Q_1 + \frac{d^2}{2} E_1$$

or

$$L_2 = -I_1 Q_1$$

$$E_2 = E_1 - \frac{1}{2} I_1^2 Q_1$$

$$P_2 = dE_1 - \frac{d}{2} I_1^2 Q_1$$

$$Q_2 = \left(1 - \frac{d^2}{4} I_1^2 \right) Q_1 + \frac{d^2}{2} E_1$$

Substituting the subscript 2 variables into the subscript 3 relations

$$Q_3 = 0 = \left(1 - \frac{d^2}{4} I_1^2 \right) Q_1 + \frac{d^2}{2} E_1 + \frac{d}{2} \left(dE_1 - \frac{d}{2} I_1^2 Q_1 \right)$$

$$P_3 = 0 = dE_1 - \frac{d}{2} I_1^2 Q_1 + d \left(E_1 - \frac{1}{2} I_1^2 Q_1 \right) + dI_2 \left(-I_1 Q_1 \right)$$

$$-\frac{d}{2} I_2^2 \left(\left(1 - \frac{d^2}{4} I_1^2 \right) Q_1 + \frac{d^2}{2} E_1 \right)$$

$$L_3 = L_2 - I_2 Q_2 = -I_1 Q_1 - I_2 \left[\left(1 - \frac{d^2}{4} I_1^2 \right) Q_1 + \frac{d^2}{2} E_1 \right]$$

$$E_3 = E_2 + I_2 L_2 - \frac{1}{2} I_2^2 Q_2 = E_1 - \frac{1}{2} I_1^2 Q_1 - I_2 I_1 Q_1 - \frac{1}{2} I_2^2 \left[\left(1 - \frac{d^2}{4} I_1^2 \right) Q_1 + \frac{d^2}{2} E_1 \right]$$

Or

$$Q_3 = 0 = \left(1 - \frac{d^2}{2} I_1^2\right) Q_1 + d^2 E_1$$
$$P_3 = 0 = \left(-I_1^2 - I_2 I_1 - \frac{1}{2} I_2^2 \left(1 - \frac{d^2}{4} I_1^2\right)\right) Q_1 + \left(2 - \frac{d^2 I_2^2}{4}\right) E_1$$
$$L_3 = -\left[I_1 + I_2\left(1 - \frac{d^2}{4} I_1^2\right)\right] Q_1 - I_2 \frac{d^2}{2} E_1$$
$$E_3 = \left(1 - \frac{1}{4} I_2^2 d^2\right) E_1 - \left[I_2 I_1 + \frac{1}{2} I_1^2 + \frac{1}{2} I_2^2 \left(1 - \frac{d^2}{4} I_1^2\right)\right] Q_1$$

Since Q1 is arbitrary the first two conditions put a constraint on the triplet variables, and eventually on the allowed value of E_1. We first introduce the notation

$$X_1 = dI_1, \qquad X_2 = dI_2$$
$$X_1 = -\sqrt{1+\sqrt{2}}, \quad X_2 = \frac{2\sqrt{2}}{\sqrt{1+\sqrt{2}}}$$
$$X_1^2 = 1 + \sqrt{2}$$
$$X_2^2 = \frac{8}{1+\sqrt{2}}$$
$$X_1 X_2 = -2\sqrt{2}$$

The specific values are what is expected based on the triplet conditions. Introducing this notation.

$$Q_3 = 0 = \left(1 - \frac{1}{2} X_1^2\right) Q_1 + d^2 E_1$$
$$P_3 = 0 = \left(-X_1^2 - X_2 X_1 - \frac{1}{2} X_2^2 \left(1 - \frac{1}{4} X_1^2\right)\right) Q_1 + \left(2 - \frac{X_2^2}{4}\right) d^2 E_1$$
$$dL_3 = -\left[X_1 + X_2\left(1 - \frac{1}{4} X_1^2\right)\right] Q_1 - X_2 \frac{d^2}{2} E_1$$
$$d^2 E_3 = \left(1 - \frac{1}{4} X_2^2\right) E_1 - \left[X_2 X_1 + \frac{1}{2} X_1^2 + \frac{1}{2} X_2^2 \left(1 - \frac{1}{4} X_1^2\right)\right] Q_1$$

$$\left(1-\frac{1}{2}X_1^2\right)=1-\frac{1+\sqrt{2}}{2}=\frac{1-\sqrt{2}}{2}$$
$$\left(2-\frac{X_2^2}{4}\right)=2-\frac{1}{4}\frac{8}{1+\sqrt{2}}=2-\frac{2}{1+\sqrt{2}}=\frac{2\sqrt{2}}{1+\sqrt{2}}$$
$$\left(-X_1^2-X_2X_1-\frac{1}{2}X_2^2\left(1-\frac{1}{4}X_1^2\right)\right)=\left\{-\left(1+\sqrt{2}\right)+2\sqrt{2}-\frac{1}{2}\frac{8}{1+\sqrt{2}}\left(1-\frac{1+\sqrt{2}}{4}\right)\right\}$$
$$=\sqrt{2}-1-\frac{3-\sqrt{2}}{1+\sqrt{2}}=\frac{\left(1+\sqrt{2}\right)\left(\sqrt{2}-1\right)-3+\sqrt{2}}{1+\sqrt{2}}=\frac{1-3+\sqrt{2}}{1+\sqrt{2}}=\frac{-2+\sqrt{2}}{1+\sqrt{2}}$$

Returning to the Q3 and P3 relations, the determinant is

$$Det=\left(1-\frac{1}{2}X_1^2\right)\left(2-\frac{X_2^2}{4}\right)-\left(-X_1^2-X_2X_1-\frac{1}{2}X_2^2\left(1-\frac{1}{4}X_1^2\right)\right)$$

$$\left(\frac{1-\sqrt{2}}{2}\right)\left(\frac{2\sqrt{2}}{1+\sqrt{2}}\right)-\frac{-2+\sqrt{2}}{1+\sqrt{2}}=\frac{\sqrt{2}-2}{1+\sqrt{2}}-\frac{-2+\sqrt{2}}{1+\sqrt{2}}=0$$

Thus, the triplet conditions satisfy the requirement that both Q3 and P3 can be zero at the exit.
The Q3 relation can now be used to fix the value of E1

$$\frac{d^2E_1}{Q_1}=-\left(1-\frac{1}{2}X_1^2\right)=\frac{\sqrt{2}-1}{2}$$

This is an import relation. It states that for a specific injected Q1=Q- there is a specific value of E- needed to satisfy the desired exit constraints.

We focus now on the exiting values of L3 and E3

$$dL_3=-\left[X_1+X_2\left(1-\frac{1}{4}X_1^2\right)\right]Q_1-X_2\frac{d^2}{2}E_1$$
$$d^2E_3=\left(1-\frac{1}{4}X_2^2\right)E_1-\left[X_2X_1+\frac{1}{2}X_1^2+\frac{1}{2}X_2^2\left(1-\frac{1}{4}X_1^2\right)\right]Q_1$$

Replacing E1using its relation to Q1,

$$dL_3 = -\left[X_1 + X_2\left(1-\frac{1}{4}X_1^2\right)\right]Q_1 + \frac{1}{2}X_2\left(1-\frac{1}{2}X_1^2\right)Q_1$$

$$d^2E_3 = -\left(1-\frac{1}{4}X_2^2\right)\left(1-\frac{1}{2}X_1^2\right)Q_1 - \left[X_2X_1 + \frac{1}{2}X_1^2 + \frac{1}{2}X_2^2\left(1-\frac{1}{4}X_1^2\right)\right]Q_1$$

Evaluation of coefficients on attached document

$$\frac{1}{2}X_2\left(1-\frac{1}{2}X_1^2\right) - \left[X_1 + X_2\left(1-\frac{1}{4}X_1^2\right)\right] = \frac{2}{\sqrt{1+\sqrt{2}}}$$

$$-\left(1-\frac{1}{4}X_2^2\right)\left(1-\frac{1}{2}X_1^2\right) - \left[X_2X_1 + \frac{1}{2}X_1^2 + \frac{1}{2}X_2^2\left(1-\frac{1}{4}X_1^2\right)\right] = 2$$

$$dL_3 = \frac{2}{\sqrt{1+\sqrt{2}}}Q_1$$

$$d^2E_3 = 2Q_1$$

$$d^2E_f = d^2E_3 + dI_3dL_3 = 2Q_-(0) + X_3\frac{2}{\sqrt{1+\sqrt{2}}}Q_-(0)$$

Summary of "-" moments:

At z=0 (entrance to triplet)

Q_- is given
L = 0
P_- = 0

$$\frac{d^2E_-}{Q_-} = \frac{\sqrt{2}-1}{2}$$

At exit of triplet

Q- = 0
P- = 0

$$dL_3 = \frac{2}{\sqrt{1+\sqrt{2}}}Q_1$$

$$d^2E_3 = 2Q_1$$

$$d^2E_f = d^2E_3 + dI_3dL_3 = 2Q_-(0) + X_3\frac{2}{\sqrt{1+\sqrt{2}}}Q_-(0)$$

The second group of variables is $\left(Q_+, Q_x, P_+, P_x, E_+, E_x\right)$. The evolution equations are

$$\frac{d}{dz}Q_+ = P_+ \qquad \frac{d}{dz}Q_x = P_x$$
$$\frac{d}{dz}P_+ = E_+ - 2K_q s_q Q_x \qquad \frac{d}{dz}P_x = E_x - 2K_q s_q Q_+$$
$$\frac{d}{dz}E_+ = -2K_q s_q P_x \qquad \frac{d}{dz}E_x = -2K_q s_q P_+$$

Notice that this group can be further divided by introducing sum and difference variables

$$Q_{s,d} = Q_+ \pm Q_x$$
$$P_{s,d} = P_+ \pm P_x$$
$$E_{s,d} = E_+ \pm E_x$$

The evolution equations for these are

$$\frac{d}{dz}Q_{s,d} = P_{s,d}$$
$$\frac{d}{dz}P_{s,d} = E_{s,d} \mp 2K_q s_q Q_{s,d}$$
$$\frac{d}{dz}E_{s,d} = \mp 2K_q s_q P_{s,d}$$

There are two constants of motion associated with these equations

$$J_{s,d} = E_{s,d} Q_{s,d} - \frac{1}{2} P_{s,d}^2$$

The initial conditions we wish to impose are

$$Q_{s,d}(0) = Q_+ \pm Q_x(0) = Q_+(0)$$
$$P_{s,d}(0) = P_+ \pm P_x = 0$$
$$E_{s,d}(0) = E_+ \pm E_x = E_+(0)$$

We expect at this point to find that once the conditions at z=L are imposed there is a unique value of $E_+(0)$. However, since $E_x(0)=0$, the same value must apply to the s and d solution.

The final conditions we apply are

$$
\begin{aligned}
&Q_{s,d}(L)=Q_+\pm Q_x=Q_+(L)\\
&P_{s,d}=P_+\pm P_x=0 \qquad .\\
&E_{s,d}=E_+\pm E_x=E_+(L)
\end{aligned}
$$

Here, what we require is that the sum and difference solutions, $Q_{s,d}$ and $E_{s,d}$, are equal at z=L.

These two systems can be solved as follows.

Let variables with subscript n be the values just before the nth quadrupole. Then we have

$$
\begin{aligned}
&E_{n+1}=E_n+\Delta E_n\\
&\Delta E_n=\mp I_nP_n+\frac{1}{2}I_n^2Q_n\\
&P_{n+1}=P_n+\Delta P_n+dE_{n+1}\\
&\Delta P_n=\mp I_nQ_n\\
&Q_{n+1}=Q_n+\frac{d}{2}\left(P_n+\Delta P_n+P_{n+1}\right)
\end{aligned}
$$

The same formulas may be used to give the values of variables after the last quadrupole (n=3), we just need to set d = 0 for this case.

Solving for n+1 variables in terms of n variables gives
****

$$
\begin{aligned}
&Q_{n+1}=Q_n\left(1+\mp I_nd+\frac{d^2}{4}I_n^2\right)+P_nd\left(1+\frac{1}{2}\left(\mp I_nd\right)\right)+\frac{d^2}{2}E_n\\
&E_{n+1}=E_n+\mp I_nP_n+\frac{1}{2}I_n^2Q_n\\
&P_{n+1}=P_n\left(1+\mp I_nd\right)+\mp I_nQ_n\left(1+\frac{1}{2}\left(\mp I_nd\right)\right)+dE_n\\
&Q_{n+1}=Q_n\left(1+\frac{1}{2}X_n\right)^2+P_nd\left(1+\frac{1}{2}X_n\right)+\frac{1}{2}d^2E_n\\
&d^2E_{n+1}=d^2E_n+X_ndP_n+\frac{1}{2}X_n^2Q_n\\
&dP_{n+1}=dP_n\left(1+X_n\right)+Q_nX_n\left(1+\frac{1}{2}X_n\right)+d^2E_n
\end{aligned}
$$

Final values of variables are given in terms of n=3 values

$$E_f = E_3 + \Delta E_3$$
$$\Delta E_3 = \mp I_3 P_3 + \frac{1}{2} I_3^2 Q_3$$
$$P_f = 0 = P_3 + \Delta P_3$$
$$\Delta P_3 = \mp I_3 Q_3$$
$$P_3 = - \mp I_3 Q_3$$
$$\Delta E_3 = \mp I_3 P_3 + \frac{1}{2} I_3^2 Q_3 = -\frac{1}{2} I_3^2 Q_3$$
$$E_f = E_3 - \frac{1}{2} I_3^2 Q_3$$

$$d^2 E_f = d^2 E_3 - \frac{1}{2} X_3^2 Q_3$$
$$dP_f = dP_3 + Q_3 X_3 = 0$$
$$Q_f = Q_3$$

****

The initial conditions are $P_1 = 0$, Q_1, E_1 for both + and – variables. Working from the input,

$$E_2 = E_1 + \frac{1}{2} I_1^2 Q_1$$
$$P_2 = \left(\mp I_1 + \frac{d}{2} I_1^2 \right) Q_1 + dE_1$$
$$Q_2 = \left(1 + \mp I_1 d + \frac{1}{4} d^2 I_1^2 \right) Q_1 + \frac{d^2}{2} E_1$$

Simplifying coefficients

$$Q_2 = Q_1 \left(1 + \frac{1}{2} X_1 \right)^2 + \frac{1}{2} d^2 E_1$$
$$d^2 E_2 = d^2 E_1 + \frac{1}{2} X_1^2 Q_1$$
$$dP_2 = +Q_1 X_1 \left(1 + \frac{1}{2} X_1 \right) + d^2 E_1$$

There is now an implicit sign flip in the definition

$X_n = \mp I_n d$.
Advancing to 3

$$Q_3 = Q_2\left(1 + \mp I_2 d + \frac{d^2}{4} I_2^2\right) + P_2 d\left(1 + \frac{1}{2}\left(\mp I_2 d\right)\right) + \frac{d^2}{2} E_2$$

$$E_3 = E_2 + \mp I_2 P_2 + \frac{1}{2} I_2^2 Q_2$$

$$P_3 = P_2\left(1 + \mp I_2 d\right) + \mp I_2 Q_2\left(1 + \frac{1}{2}\left(\mp I_2 d\right)\right) + dE_2$$

Simplifying

$$Q_3 = Q_2\left(1 + \frac{1}{2} X_2\right)^2 + P_2 d\left(1 + \frac{1}{2} X_2\right) + \frac{1}{2} d^2 E_2$$

$$d^2 E_3 = d^2 E_2 + X_2 dP_2 + \frac{1}{2} X_2^2 Q_2$$

$$dP_3 = dP_2\left(1 + X_2\right) + Q_2 X_2\left(1 + \frac{1}{2} X_2\right) + d^2 E_2$$

Rewriting in terms of subscript 1 variables (See attached notes)

$$dP_3 = Q_1\left\{X_1\left(1 + \frac{1}{2} X_1\right)\left(1 + X_2\right) + \left(1 + \frac{1}{2} X_1\right)^2 X_2\left(1 + \frac{1}{2} X_2\right) + \frac{1}{2} X_1^2\right\}$$

$$+ d^2 E_1\left\{1 + \frac{1}{2} X_2\left(1 + \frac{1}{2} X_2\right) + \left(1 + X_2\right)\right\}$$

$$d^2 E_3 = d^2 E_1\left[1 + X_2 + \frac{1}{4} X_2^2\right] + Q_1\left[\frac{1}{2} X_1^2 + X_2 X_1\left(1 + \frac{1}{2} X_1\right) + \frac{1}{2} X_2^2\left(1 + \frac{1}{2} X_1\right)^2\right]$$

$$Q_3 = \left\{\left(1 + \frac{1}{2} X_2\right)^2\left(1 + \frac{1}{2} X_1\right)^2 + \left(1 + \frac{1}{2} X_2\right) X_1\left(1 + \frac{1}{2} X_1\right) + \frac{1}{4} X_1^2\right\}$$

$$+ d^2 E_1\left\{\frac{1}{2}\left(1 + \frac{1}{2} X_2\right)^2 + \left(1 + \frac{1}{2} X_2\right) + \frac{1}{2}\right\}$$

Remember, these must hold for both signs of the X_n

Write odd and even versions

$$dP_3 = Q_1\left\{X_1^2 + 2X_1 X_2 + \frac{1}{2} X_2^2 + \frac{1}{8} X_1^2 X_2^2\right\} + d^2 E_1\left\{2 + \frac{1}{4} X_2^2\right\}$$

$$dP_3 = Q_1\left\{X_1\left(1+\frac{1}{2}X_1X_2\right)+\frac{1}{2}X_1X_2^2+X_2\left(1+\frac{1}{4}X_1^2\right)\right\}+d^2E_1\left\{\frac{3}{2}X_2\right\}$$

$$Q_3 = Q_1\left\{X_2\left(1+\frac{1}{4}X_1^2\right)+X_1\left(1+\frac{1}{4}X_2^2\right)+X_1\left(1+\frac{1}{4}X_2X_1\right)\right\}+d^2E_1X_2$$

$$Q_3 = Q_1\left\{\left(1+\frac{1}{4}X_2^2\right)\left(1+\frac{1}{4}X_1^2\right)+\frac{3}{2}X_2X_1+\frac{3}{4}X_1^2\right\}+d^2E_1\left\{\frac{1}{8}X_2^2+2\right\}$$

$$d^2E_3 = d^2E_1\left[X_2\right]+Q_1\left[\frac{1}{2}X_2X_1^2+\frac{1}{2}X_2^2X_1\right]$$

$$d^2E_3 = d^2E_1\left[1+\frac{1}{4}X_2^2\right]+Q_1\left[\frac{1}{2}X_1^2+X_2X_1+\frac{1}{2}X_2^2\left(1+\frac{1}{4}X_1^2\right)\right]$$

****

The same Q3 = Q_+ must satisfy both even and odd versions. That can only happen if the odd version gives 0 for Q3

$$0 = Q_1\left\{X_2\left(1+\frac{1}{4}X_1^2\right)+X_1\left(1+\frac{1}{4}X_2^2\right)+X_1\left(1+\frac{1}{4}X_2X_1\right)\right\}+d^2E_1X_2$$

The value of Q_3 is then determined by the even version

$$Q_3 = Q_1\left\{\left(1+\frac{1}{4}X_2^2\right)\left(1+\frac{1}{4}X_1^2\right)+\frac{3}{2}X_2X_1+\frac{3}{4}X_1^2\right\}+d^2E_1\left\{\frac{1}{8}X_2^2+2\right\}$$

P3 is odd, so the even version of P3 must give zero

$$0 = Q_1\left\{X_1^2+2X_1X_2+\frac{1}{2}X_2^2+\frac{1}{8}X_1^2X_2^2\right\}+d^2E_1\left\{2+\frac{1}{4}X_2^2\right\}$$

The odd version determines the strength of Quadrupole 3

$$dP_3 = Q_1\left\{X_1\left(1+\frac{1}{2}X_1X_2\right)+\frac{1}{2}X_1X_2^2+X_2\left(1+\frac{1}{4}X_1^2\right)\right\}+d^2E_1\left\{\frac{3}{2}X_2\right\}=-Q_3X_3$$

The E_f and E_3 are both even so the odd version must give zero

$$0 = d^2E_1\left[X_2\right]+Q_1\left[\frac{1}{2}X_2X_1^2+\frac{1}{2}X_2^2X_1\right]$$

This turns out to be redendant because of the conservation relation with $P_{s,d}=0$

$E_{s,d}Q_{s,d}(z=0)=E_{s,d}Q_{s,d}(z=L)$.

This guarantees that if P=0 at the exit and Ex =0 at the entrance, then Ex = 0 at the exit. It can also be used to determine E+ at the exit if Q+ is known at the exit,

$E_+(z=L)=E_+(z=0)Q_+(z=0)/Q_+(z=L)$.

The even P and odd Q 2 by 2 system is

$$0 = Q_1\left\{X_1^2 + 2X_1X_2 + \frac{1}{2}X_2^2 + \frac{1}{8}X_1^2X_2^2\right\} + d^2E_1\left\{2 + \frac{1}{4}X_2^2\right\}$$

$$0 = Q_1\left\{X_2\left(1+\frac{1}{4}X_1^2\right) + X_1\left(1+\frac{1}{4}X_2^2\right) + X_1\left(1+\frac{1}{4}X_2X_1\right)\right\} + d^2E_1X_2$$

The coefficients are

$$\left\{X_1^2 + 2X_1X_2 + \frac{1}{2}X_2^2 + \frac{1}{8}X_1^2X_2^2\right\} = \frac{-\sqrt{2}}{1+\sqrt{2}}$$

$$X_2\left(1+\frac{1}{4}X_1^2\right) + X_1\left(1+\frac{1}{4}X_2^2\right) + X_1\left(1+\frac{1}{4}X_2X_1\right) = \frac{\sqrt{2}-2}{\sqrt{1+\sqrt{2}}}$$

$$\left\{2 + \frac{1}{4}X_2^2\right\} = 2\sqrt{2}$$

$$X_2 = \frac{2\sqrt{2}}{\sqrt{1+\sqrt{2}}}$$

The determinant is

$$\left\{X_1^2 + 2X_1X_2 + \frac{1}{2}X_2^2 + \frac{1}{8}X_1^2X_2^2\right\}X_2 - \left\{2 + \frac{1}{4}X_2^2\right\}\left\{X_2\left(1+\frac{1}{4}X_1^2\right) + X_1\left(1+\frac{1}{4}X_2^2\right) + X_1\left(1+\frac{1}{4}X_2X_1\right)\right\}$$
$$= 0$$

See notes

This gives

$$0 = Q_1\left\{\frac{-\sqrt{2}}{1+\sqrt{2}}\right\} + d^2E_1\left\{2\sqrt{2}\right\}$$

$$d^2E_1 = Q_1\left\{\frac{1}{2+2\sqrt{2}}\right\}$$

Calculate Q3, Ef, X3. The value of Q3 is determined by

$$Q_3 = Q_1\left\{\left(1+\frac{1}{4}X_2^2\right)\left(1+\frac{1}{4}X_1^2\right) + \frac{3}{2}X_2X_1 + \frac{3}{4}X_1^2\right\} + d^2E_1\left\{\frac{1}{8}X_2^2 + 2\right\}$$

$$\left\{\left(1+\frac{1}{4}X_2^2\right)\left(1+\frac{1}{4}X_1^2\right) + \frac{3}{2}X_2X_1 + \frac{3}{4}X_1^2\right\} = \frac{1+\sqrt{2}}{2\left(1+\sqrt{2}\right)} = \frac{1}{2}$$

$$\left\{\frac{1}{8}X_2^2 + 2\right\} = \frac{3+2\sqrt{2}}{1+\sqrt{2}}$$

$$Q_3 = Q_1\frac{1}{2} + d^2E_1\frac{3+2\sqrt{2}}{1+\sqrt{2}}$$

$$= Q_1\left\{\frac{1}{2} + \frac{3+2\sqrt{2}}{1+\sqrt{2}}\left\{\frac{1}{2+2\sqrt{2}}\right\}\right\} = Q_1$$

Calculate the required Quad3 strength

$$dP_3 = Q_1\left\{X_1\left(1+\frac{1}{2}X_1X_2\right)+\frac{1}{2}X_1X_2^2 + X_2\left(1+\frac{1}{4}X_1^2\right)\right\} + d^2E_1\left\{\frac{3}{2}X_2\right\} = -Q_3X_3$$

$$\left\{X_1\left(1+\frac{1}{2}X_1X_2\right)+\frac{1}{2}X_1X_2^2 + X_2\left(1+\frac{1}{4}X_1^2\right)\right\} = -\frac{1}{\sqrt{1+\sqrt{2}}}\left\{2-\frac{5}{2}\sqrt{2}\right\}$$

$$\frac{3}{2}X_2 = \frac{3\sqrt{2}}{\sqrt{1+\sqrt{2}}}$$

$$d^2E_1 = Q_1\left\{\frac{1}{2+2\sqrt{2}}\right\}$$

$$-Q_3X_3 = Q_1\frac{1}{\sqrt{1+\sqrt{2}}}\left\{-2+\frac{5}{2}\sqrt{2}\right\} + \frac{3\sqrt{2}}{\sqrt{1+\sqrt{2}}}Q_1\left\{\frac{1}{2+2\sqrt{2}}\right\}$$

$$= \frac{Q_1}{\sqrt{1+\sqrt{2}}}\left(1+\sqrt{2}\right)$$

$$= -X_1Q_3$$

$$X_3 = X_1 = -\frac{\sqrt{2}+1}{\sqrt{1+\sqrt{2}}},$$

Checks

SUMMARY of TRIPLET

Summary of solutions for Q+, Qx, P+, Px, E+, Ex

At z=0

$$Q_+ = Q_+(0)$$
$$Q_x(0) = 0$$
$$P_+ = P_x = 0$$
$$d^2E_+ = Q_+(0)\left\{\frac{1}{2+2\sqrt{2}}\right\}$$
$$E_x(0) = 0$$

At *z=L=2d*

$$Q_+ = Q_+(0)$$
$$Q_x(L) = 0$$
$$P_+(L) = P_x(L) = 0$$
$$d^2E_+(L) = Q_+(0)\left\{\frac{1}{2+2\sqrt{2}}\right\}$$
$$E_x(L) = 0$$

Summary of Q-,P-,E-,L moments:

At z=0 (entrance to triplet)

Q_- is given
L = 0
P_- = 0

$$\frac{d^2E_-}{Q_-} = \frac{\sqrt{2}-1}{2}$$

At exit of triplet

Q- = 0
P- = 0

$$dL = \frac{2}{\sqrt{1+\sqrt{2}}}Q_-(0)$$
$$d^2E_3 = 2Q_-(0)$$
$$d^2E_-(L) = d^2E_3 + dI_3dL_3 = 2Q_-(0) + X_3\frac{2}{\sqrt{1+\sqrt{2}}}Q_-(0) = 0$$

There seems to be a factor of two missing in the Q- relations. From the conservation relation

$$J_- = E_- Q_- + \frac{1}{2}L^2 - \frac{1}{2}P_-^2 = \text{constant},$$

One would expect that

$$\frac{d^2}{2}L^2(L) = d^2 E_-(0)Q_-(0) = \frac{\sqrt{2}-1}{2}Q_-^2(0)$$

Instead we find

$$\frac{d^2L^2}{2} = \frac{2}{1+\sqrt{2}}Q_-^2(0)$$

$$\frac{\sqrt{2}-1}{2} = \frac{\left(\sqrt{2}-1\right)\left(1+\sqrt{2}\right)}{2\left(1+\sqrt{2}\right)} = \frac{1}{2\left(1+\sqrt{2}\right)}$$

We have used the following

$$d = \frac{\beta_s}{2\sqrt{1+\sqrt{2}}},\ q_1 = q_3 = -\frac{\sqrt{2}+1}{\beta_s},\ q_2 = \frac{2\sqrt{2}}{\beta_s}$$

$$X_1 = -\frac{\sqrt{2}+1}{\sqrt{1+\sqrt{2}}}, \quad X_2 = \frac{2\sqrt{2}}{\sqrt{1+\sqrt{2}}}$$

Introduce

$$X_1 = \mp dI_1$$
$$X_2 = \mp dI_2$$