Online Appendix for the Paper Entitled: Revisiting the Resource-Constrained Scheduling Problem with Identical Jobs

by

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Appendix

This appendix contains complete proofs that were either omitted or condensed in the main paper.

Proof of Proposition 2

Proof. To establish the supermodularity of f we show that

$$f(\bar{K} \cup \{j\}) - f(\bar{K}) \leq f(K \cup \{j\}) - f(K), \tag{1}$$

for any $\bar{K} \subset K \subset N$ and $j \in N \backslash K$. By the definition of f, the difference $f(\bar{K} \cup \{j\}) - f(\bar{K})$ is solely determined by the size of the sets $\bar{K} \cup \{j\}$ and $\bar{K}$. Moreover, since all jobs are identical and f represents the minimum sum of start times in a feasible schedule, $f(\bar{K} \cup \{j\}) - f(\bar{K})$ equals the start time of the last job among the $|\bar{K} \cup \{j\}|$ jobs. But the start time of that job (irrespective of whether that job is j) equals $r_0 + \rho(\bar{K} \cup \{j\})p_0 = f(\bar{K} \cup \{j\}) - f(\bar{K})$. Similarly, $f(K \cup \{j\}) - f(K) = r_0 + \rho(K \cup \{j\})p_0$. Hence, showing (1) is equivalent to showing that $\rho(\bar{K} \cup \{j\}) \leq \rho(K \cup \{j\})$. The latter follows from the fact that $\left(\bar{K} \cup \{j\}\right) \subset (K \cup \{j\})$, by noticing also that $\rho(K)$ is non-decreasing with respect to $|K|$. The submodularity of g is shown in an analogous manner. $\square$

Proof of Lemma 1

Proof. If $|N| \leq \delta$, $B(f)$ contains the single point $s_j = r_0$, for all $j \in N$. Otherwise, because $EP(f)$ is full-dimensional and the points of $B(f)$ are exactly the points of $EP(f)$ satisfying $s(N) = f(N)$, we have $\dim B(f) = |N| - 1$. The latter result also stems from the fact that f is supermodular and N is f-inseparable. $\square$

Proof of Theorem 2

Proof. The face of $B(f)$ defined by setting any of the variables s_j to r_0 is clearly proper and non-empty. The remaining part of the proof is similar to that of Theorem 1, for $j \in N \setminus K$ and $K = \{j\}$. $\square$

Proof of Theorem 4

Proof. Showing (i) reduces to proving (8), when $\lambda = 0$, for $i \in K$. This can be done as in the proof of Theorem 1 with (13) being defined as $\pi_i = a_i$. It is easy to show (ii) and (iii). $\square$

Proof of Lemma 2

Proof. First observe that θ_{b,a_0} is integer. Solving, in the above expression, with respect to θ_{a_0} while substituting terms from the equation of Remark 1, we get

$$\begin{aligned}\theta_{b,a_0} &= \sum_{i=1,\dots n} \frac{a_i + \epsilon_{b,a_i}(b - \upsilon_{b,a_i})}{b} - \frac{a_0 + \epsilon_{b,a_0}(b - \upsilon_{b,a_0})}{b} \\ &= \frac{1}{b}(\sum_{i=1,\dots n} \epsilon_{b,a_i}(b - \upsilon_{b,a_i}) - \epsilon_{b,a_0}(b - \upsilon_{b,a_0})).\end{aligned}$$

The maximum value of the right-hand side is attained if $\epsilon_{b,a_i} = 1$, $\upsilon_{b,a_i} = 1$ and $\epsilon_{b,a_0} = 0$. Therefore

$$\theta_{b,a_0} \leq n\frac{b-1}{b} = n - \frac{n}{b}$$

yielding $\theta_{b,a_0} \leq n - \lceil \frac{n}{b} \rceil$ since θ_{b,a_0} is integer. $\square$

Proof of Theorem 5

To show Theorem 5, we observe that, for any $X, Y \subseteq N$,

$$|Y| = |Y \setminus X| + |Y \cap X|, \tag{2}$$
$$|X| = |X \setminus Y| + |Y \cap X|, \tag{3}$$
$$|Y \cup X| = |Y \setminus X| + |Y \cap X| + |X \setminus Y|. \tag{4}$$

The above equalities in conjunction with Lemma 2 yield

$$\theta_{|Y|} = \left\lceil \frac{|Y \setminus X|}{\delta} \right\rceil + \left\lceil \frac{|Y \cap X|}{\delta} \right\rceil - \left\lceil \frac{|Y|}{\delta} \right\rceil, \tag{5}$$

$$\theta_{|X|} = \left\lceil \frac{|X \setminus Y|}{\delta} \right\rceil + \left\lceil \frac{|Y \cap X|}{\delta} \right\rceil - \left\lceil \frac{|X|}{\delta} \right\rceil, \tag{6}$$

$$\theta_{|Y\cup X|} = \left\lceil \frac{|Y \setminus X|}{\delta} \right\rceil + \left\lceil \frac{|Y \cap X|}{\delta} \right\rceil + \left\lceil \frac{|X \setminus Y|}{\delta} \right\rceil - \left\lceil \frac{|Y \cup X|}{\delta} \right\rceil, \tag{7}$$

where

$$\theta_{|Y|}, \theta_{|X|} \leq \quad 1, \text{ if } \delta \geq 2, \tag{8}$$

$$\theta_{|Y\cup X|} \leq \begin{cases} 2, \text{ if } \delta \geq 3, \\ 1, \text{ if } \delta = 2. \end{cases} \tag{9}$$

We can drop the ceiling operator from (5), (6), (7) by substituting terms from the equation of Remark 1. To simplify notation we will drop the δ subscript from ϵ and υ. This convention will be used throughout. Then canceling out equivalent terms from (2), (3), (4) respectively, yields

$$\theta_{|Y|} = \frac{\epsilon_{|Y\setminus X|}(\delta - \upsilon_{|Y\setminus X|}) + \epsilon_{|Y\cap X|}(\delta - \upsilon_{|Y\cap X|}) - \epsilon_{|Y|}(\delta - \upsilon_{|Y|})}{\delta}, \tag{10}$$

$$\theta_{|X|} = \frac{\epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|}) + \epsilon_{|Y\cap X|}(\delta - \upsilon_{|Y\cap X|}) - \epsilon_{|X|}(\delta - \upsilon_{|X|})}{\delta}, \tag{11}$$

$$\theta_{|Y\cup X|} = \frac{\epsilon_{|Y\setminus X|}(\delta - \upsilon_{|Y\setminus X|}) + \epsilon_{|Y\cap X|}(\delta - \upsilon_{|Y\cap X|}) + \epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|}) - \epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|})}{\delta}. \tag{12}$$

We are now ready to prove Theorem 5.

Proof. For $r_1 \geq r_0 + p_0\rho(N)$ and any $X, Y \subseteq N$ with $X \setminus Y \neq \emptyset$ and $Y \setminus X \neq \emptyset$ we must show that

$$F = g(Y) - g(Y \setminus X) - (f(X) - f(X \setminus Y)) \geq 0. \tag{13}$$

By the definition of g

$$\begin{aligned} g(Y) - g(Y \setminus X) = (|Y| - |Y \setminus X|)r_1 - p_0\rho(Y)(|Y| - \frac{\delta}{2}(\rho(Y) + 1)) \\ + p_0\rho(Y \setminus X)(|Y \setminus X| - \frac{\delta}{2}(\rho(Y \setminus X) + 1)) \end{aligned}$$

while by the definition of f

$$
\begin{aligned}
f(X) - f(X \setminus Y) = (|X| - |X \setminus Y|)r_0 + p_0\rho(X)(|X| - \frac{\delta}{2}(\rho(X) + 1)) \\
- p_0\rho(X \setminus Y)(|X \setminus Y| - \frac{\delta}{2}(\rho(X \setminus Y) + 1)).
\end{aligned}
$$

Putting it all in (13) and observing from (2), (3) that

$$
|X \cap Y| = |Y| - |Y \setminus X| = |X| - |X \setminus Y| \tag{14}
$$

yields

$$
\begin{aligned}
F &= |X \cap Y| \, (r_1 - r_0) \\
&\quad - p_0(\rho(Y) \, |Y| + \rho(X) \, |X| - \rho(Y \setminus X) \, |Y \setminus X| - \rho(X \setminus Y) \, |X \setminus Y|) \\
&\quad + p_0 \frac{\delta}{2}((\rho(Y) - \rho(Y \setminus X))(\rho(Y) + \rho(Y \setminus X) + 1) \\
&\qquad\qquad + (\rho(X) - \rho(X \setminus Y))(\rho(X) + \rho(X \setminus Y) + 1)).
\end{aligned}
$$

Substituting $|Y|$, $|X|$ from (2), (3) respectively, we get

$$
\begin{aligned}
F &= |X \cap Y| \, (r_1 - r_0) \\
&\quad - p_0((\rho(Y) + \rho(X)) \, |X \cap Y| \\
&\quad + (\rho(Y) - \rho(Y \setminus X)) \, |Y \setminus X| + (\rho(X) - \rho(X \setminus Y)) \, |X \setminus Y|) \\
&\quad + p_0 \frac{\delta}{2}((\rho(Y) - \rho(Y \setminus X))(\rho(Y) + \rho(Y \setminus X) + 1) \\
&\qquad\qquad + (\rho(X) - \rho(X \setminus Y))(\rho(X) + \rho(X \setminus Y) + 1))
\end{aligned}
$$

yielding

$$
\begin{aligned}
F &= |X \cap Y| \, (r_1 - r_0) \\
&\quad - p_0((\rho(Y) + \rho(X)) \, |X \cap Y|) \\
&\quad + p_0((\rho(Y) - \rho(Y \setminus X))(\frac{\delta}{2}(\rho(Y) + \rho(Y \setminus X) + 1) - |Y \setminus X|) \\
&\qquad\qquad + (\rho(X) - \rho(X \setminus Y))(\frac{\delta}{2}(\rho(X) + \rho(X \setminus Y) + 1) - |X \setminus Y|)).
\end{aligned} \tag{15}
$$

Because

$$
\begin{aligned}
|Y \setminus X| &= \frac{|Y \setminus X|}{2} + \frac{|Y \setminus X|}{2} \\
&= \frac{|Y \setminus X|}{2} + \frac{|Y|}{2} - \frac{|X \cap Y|}{2},
\end{aligned}
$$

(due to (2)), we have that

$$\begin{aligned}&\frac{\delta}{2}(\rho(Y)+\rho(Y\setminus X)+1)-|Y\setminus X|\\&=\frac{1}{2}(\delta\left\lceil\frac{|Y|}{\delta}\right\rceil-|Y|)+\frac{1}{2}(\delta\left\lceil\frac{|Y\setminus X|}{\delta}\right\rceil-|Y\setminus X|)+\frac{|X\cap Y|}{2}-\frac{\delta}{2},\end{aligned}$$

while substituting terms in brackets from the equation of Remark 1, we obtain

$$\begin{aligned}&\frac{\delta}{2}(\rho(Y)+\rho(Y\setminus X)+1)-|Y\setminus X|\\&=\frac{1}{2}((|X\cap Y|-\delta)+\delta(\epsilon_{|Y|}+\epsilon_{|Y\setminus X|})-\epsilon_{|Y|}\upsilon_{|Y|}-\epsilon_{|Y\setminus X|}\upsilon_{|Y\setminus X|})\end{aligned}\tag{16}$$

In an analogous manner, we obtain

$$\begin{aligned}&\frac{\delta}{2}(\rho(X)+\rho(X\setminus Y)+1)-|X\setminus Y|\\&=\frac{1}{2}((|X\cap Y|-\delta)+\delta(\epsilon_{|X|}+\epsilon_{|X\setminus Y|})-\epsilon_{|X|}\upsilon_{|X|}-\epsilon_{|X\setminus Y|}\upsilon_{|X\setminus Y|}).\end{aligned}\tag{17}$$

Plugging (16), (17) in (15) and performing further substitutions from the equation of Remark 1, (2)–(7) and (12), we derive

$$F=\big(|X\cap Y|\,(r_1-r_0)-|X\cap Y|\,p_0\rho(Y\cup X)\big)+F'p_0,$$

where

$$\begin{aligned}F'=\;&\epsilon_{|Y\cap X|}\frac{\delta-\upsilon_{|Y\cap X|}}{\delta}(\epsilon_{|Y\setminus X|}(\delta-\upsilon_{|Y\setminus X|})+\epsilon_{|X\setminus Y|}(\delta-\upsilon_{|X\setminus Y|})+\epsilon_{|Y\cap X|}(\delta-\upsilon_{|Y\cap X|})-\delta)\\&+\frac{|X\cap Y|}{\delta}\epsilon_{|Y\cup X|}(\delta-\upsilon_{|Y\cup X|})+\frac{\delta}{2}(\theta_{|Y|}(\theta_{|Y|}+1)+\theta_{|X|}(\theta_{|X|}+1))\\&-\theta_{|Y|}(\epsilon_{|Y\setminus X|}(\delta-\upsilon_{|Y\setminus X|})+\epsilon_{|Y\cap X|}(\delta-\upsilon_{|Y\cap X|}))\\&-\theta_{|X|}(\epsilon_{|X\setminus Y|}(\delta-\upsilon_{|X\setminus Y|})+\epsilon_{|Y\cap X|}(\delta-\upsilon_{|Y\cap X|}))\end{aligned}\tag{18}$$

Notice that the first bracket evaluates to a non-negative quantity if $r_1-r_0\geq p_0\rho(N)$ since $\rho(N)\geq\rho(Y\cup X)$. Thus, it remains to show that F' also evaluates to a non-negative quantity.

Case 1 $\theta_{|X|}=\theta_{|Y|}=\theta_z$, *where* $\theta_z\in\{0,1\}$.

$$\begin{aligned}F'=\;&(\epsilon_{|Y\cap X|}\frac{\delta-\upsilon_{|Y\cap X|}}{\delta}-\theta_z)(\epsilon_{|Y\setminus X|}(\delta-\upsilon_{|Y\setminus X|})+\epsilon_{|X\setminus Y|}(\delta-\upsilon_{|X\setminus Y|})+\epsilon_{|Y\cap X|}(\delta-\upsilon_{|Y\cap X|}))\\&-\delta\epsilon_{|Y\cap X|}\frac{\delta-\upsilon_{|Y\cap X|}}{\delta}+\frac{|X\cap Y|}{\delta}\epsilon_{|Y\cup X|}(\delta-\upsilon_{|Y\cup X|})\\&+\delta\theta_z(\theta_z+1)-\theta_z\epsilon_{|Y\cap X|}(\delta-\upsilon_{|Y\cap X|}).\end{aligned}\tag{19}$$

Subcase 1.1 $\theta_z = 0$.

(12) implies that

$$\begin{aligned}&\delta\theta_{|Y\cup X|} + \epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|})\\ &= \epsilon_{|Y\setminus X|}(\delta - \upsilon_{|Y\setminus X|}) + \epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|}) + \epsilon_{|Y\cap X|}(\delta - \upsilon_{|Y\cap X|})\end{aligned}$$

and thus

$$\begin{aligned}F' &= \epsilon_{|Y\cap X|}\frac{\delta - \upsilon_{|Y\cap X|}}{\delta}(\delta\theta_{|Y\cup X|} + \epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|}) - \delta)\\ &\frac{|X\cap Y|}{\delta}\epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|})\\ &= \frac{\epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|})}{\delta}(|X\cap Y| + \epsilon_{|Y\cap X|}(\delta - \upsilon_{|Y\cap X|}))\\ &+ \epsilon_{|Y\cap X|}(\delta - \upsilon_{|Y\cap X|})(\theta_{|Y\cup X|} - 1). \qquad (20)\end{aligned}$$

If $\epsilon_{|Y\cap X|} = 0$ then (20) implies $F' \geq 0$. Otherwise, we will show that $F' \geq 0$. Hence, let $\epsilon_{|Y\cap X|} = 1$. If $\epsilon_{|Y\cup X|} = 0$ then $\theta_{|Y\cup X|}$ must be greater than or equal to one (implying $F' \geq 0$) since if $\theta_{|Y\cup X|} = 0$ (12) yields

$$\epsilon_{|Y\setminus X|}(\delta - \upsilon_{|Y\setminus X|}) + (\delta - \upsilon_{|Y\cap X|}) + \epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|}) = 0$$

which cannot be true as $(\delta - \upsilon_{|Y\cap X|}) \geq 1$ and the remaining terms of the right-hand side are nonnegative. If $\epsilon_{|Y\cup X|} = 1$ then (20) becomes

$$\begin{aligned}F' &= (\delta - \upsilon_{|Y\cup X|})(\frac{|X\cap Y| - \upsilon_{|Y\cap X|}}{\delta} + 1)\\ &+ (\delta - \upsilon_{|Y\cap X|})(\theta_{|Y\cup X|} - 1).\end{aligned}$$

Clearly $F' \geq 0$ if $\theta_{|Y\cup X|} \geq 1$. Thus assume $\theta_{|Y\cup X|} = 0$ and because $\frac{|X\cap Y| - \upsilon_{|Y\cap X|}}{\delta} \geq 0$

$$F' \geq (\delta - \upsilon_{|Y\cup X|}) - (\delta - \upsilon_{|Y\cap X|}).$$

Observe that if $\theta_{|Y\cup X|} = 0$ then (12) yields

$$(\delta - \upsilon_{|Y\cup X|}) = (\delta - \upsilon_{|Y\cap X|}) + \epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|}) + \epsilon_{|Y\setminus X|}(\delta - \upsilon_{|Y\setminus X|})$$

and because the two last terms of the right-hand side are nonnegative, we have that

$$(\delta - \upsilon_{|Y\cup X|}) \geq (\delta - \upsilon_{|Y\cap X|})$$

implying that $F' \geq 0$.

Subcase 1.2 $\theta_z = 1$.

(19) yields

$$\begin{aligned} F' &= \delta\theta_{|Y\cup X|}(\epsilon_{|Y\cap X|}\frac{\delta - \upsilon_{|Y\cap X|}}{\delta} - 1) \\ &+ 2(\delta - \epsilon_{|Y\cap X|}(\delta - \upsilon_{|Y\cap X|})) \\ &+ \frac{\epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|})}{\delta}(|X\cap Y| + \epsilon_{|Y\cap X|}(\delta - \upsilon_{|Y\cap X|}) - \delta). \end{aligned} \tag{21}$$

If $\epsilon_{|Y\cap X|} = 0$ then (21) becomes

$$F' = (2 - \theta_{|Y\cup X|})\delta + \epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|})(\frac{|X\cap Y|}{\delta} - 1). \tag{22}$$

In this case, $\delta \mid |X\cap Y|$ implying that $\frac{|X\cap Y|}{\delta} = k$, where $k \geq 0$ and integer. For $k \geq 1$, the above implies that $F' \geq 0$. Next assume that $k = 0$ implying that $|X\cap Y| = 0$. Then (4) becomes $|Y\cup X| = |Y\setminus X| + |X\setminus Y|$, (7) $\left\lceil\frac{|Y\setminus X|}{\delta}\right\rceil + \left\lceil\frac{|X\setminus Y|}{\delta}\right\rceil = \left\lceil\frac{|Y\cup X|}{\delta}\right\rceil + \theta_{|Y\cup X|}$. In this case, Lemma 2 implies $\theta_{|Y\cup X|} \leq 1$. Thus (22) yields

$$F' \geq \delta - \epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|}) \geq 1,$$

since $\delta - 1 \geq \upsilon_{|Y\cup X|} \geq 1$.

If $\epsilon_{|Y\cap X|} = 1$ then (21) becomes

$$\begin{aligned} F' = &\frac{\epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|})}{\delta}(|X\cap Y| - \upsilon_{|Y\cap X|}) \\ &- \theta_{|Y\cup X|}\upsilon_{|Y\cap X|} + 2\upsilon_{|Y\cap X|} \end{aligned}$$

which implies that $F' \geq 0$ since $|X\cap Y| - \upsilon_{|Y\cap X|} \geq 0$ and $\theta_{|Y\cup X|} \leq 2$.

Case 2 $\theta_{|Y|} = 1 - \theta_{|X|}$

Without loss of generality assume that $\theta_{|X|} = 0$ yielding $\theta_{|Y|} = 1$. (10) leads to

$$\delta + \epsilon_{|Y|}(\delta - \upsilon_{|Y|}) = \epsilon_{|Y\setminus X|}(\delta - \upsilon_{|Y\setminus X|}) + \epsilon_{|Y\cap X|}(\delta - \upsilon_{|Y\cap X|}).$$

That implies

$$\epsilon_{|Y\setminus X|} = \epsilon_{|Y\cap X|} = 1 \tag{23}$$

because the left-hand side is greater than or equal to δ whereas each of the terms of the right-hand side evaluates to at most $\delta - 1$. Thus (10) yields

$$\epsilon_{|Y|}(\delta - \upsilon_{|Y|}) = \delta - \upsilon_{|Y\setminus X|} - \upsilon_{|Y\cap X|} \geq 0 \tag{24}$$

since the left-hand side is non-negative.

Setting $\theta_{|X|} = 0$ and $\epsilon_{|Y\cap X|} = 1$ in (11) yields

$$\delta - \upsilon_{|Y\cap X|} + \epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|}) = \epsilon_{|X|}(\delta - \upsilon_{|X|})$$

implying that

$$\epsilon_{|X|} = 1 \tag{25}$$

since $\delta - \upsilon_{|Y\setminus X|} + \epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|}) \geq 1$. (11) yields

$$\epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|}) = \upsilon_{|Y\cap X|} - \upsilon_{|X|} \geq 0 \tag{26}$$

because the left-hand side is non-negative.

(18) becomes

$$\begin{aligned}F' = &-\frac{\upsilon_{|Y\cap X|}}{\delta}(\delta - \upsilon_{|Y\setminus X|} - \upsilon_{|Y\cap X|})\\ &\frac{\delta - \upsilon_{|Y\cap X|}}{\delta}\epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|})\\ &+ \frac{|X\cap Y|}{\delta}\epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|}).\end{aligned} \tag{27}$$

(12) yields

$$\epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|}) = (\delta - \upsilon_{|Y\setminus X|} - \upsilon_{|Y\cap X|}) + (\upsilon_{|Y\cap X|} - \upsilon_{|X|}) + \delta(1 - \theta_{|Y\cup X|}).$$

Substituting in (27), we obtain

$$\begin{aligned}F' &= -\upsilon_{|Y\cap X|}(\delta - \upsilon_{|Y\setminus X|} - \upsilon_{|Y\cap X|})\\ &\frac{\delta - \upsilon_{|Y\cap X|}}{\delta}\epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|})\\ &+ \frac{|X\cap Y|}{\delta}((\delta - \upsilon_{|Y\setminus X|} - \upsilon_{|Y\cap X|}) + (\upsilon_{|Y\cap X|} - \upsilon_{|X|}) + \delta(1 - \theta_{|Y\cup X|}))\\ &= (\delta - \upsilon_{|Y\setminus X|} - \upsilon_{|Y\cap X|})(\frac{|X\cap Y| - \upsilon_{|Y\cap X|}}{\delta})\\ &+ \frac{\delta - \upsilon_{|Y\cap X|}}{\delta}\epsilon_{|X\setminus Y|}(\delta - \upsilon_{|X\setminus Y|})\\ &+ \frac{|X\cap Y|}{\delta}(\upsilon_{|Y\cap X|} - \upsilon_{|X|}) + |X\cap Y|\,(1 - \theta_{|Y\cup X|}).\end{aligned}$$

Because of $\delta - \upsilon_{|Y\setminus X|} - \upsilon_{|Y\cap X|} \geq 0$ (by (24)), $\frac{|X\cap Y| - \upsilon_{|Y\cap X|}}{\delta} \geq 0$ and $\upsilon_{|Y\cap X|} - \upsilon_{|X|} \geq 0$ (by (26)), we have that $F' \geq 0$ if $\theta_{|Y\cup X|} \leq 1$. Assume that $\theta_{|Y\cup X|} = 2$. It is easy to see that (12) and (11) yield

$$\delta\theta_{|Y\cup X|} = \delta\theta_{|X|} + \epsilon_{|X|}(\delta - \upsilon_{|X|}) + \epsilon_{|Y\setminus X|}(\delta - \upsilon_{|Y\setminus X|}) - \epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|}).$$

In this case $\theta_{|X|} = 0$, $\epsilon_{|X|} = 1$ (by (25)), $\epsilon_{|Y\setminus X|} = 1$ (by (23)) and $\theta_{|Y\cup X|} = 2$ yielding

$$\begin{aligned} 2\delta &= 2\delta - \upsilon_{|X|} - \upsilon_{|Y\setminus X|} - \epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|}) \Rightarrow \\ \epsilon_{|Y\cup X|}(\delta - \upsilon_{|Y\cup X|}) &= -(\upsilon_{|X|} + \upsilon_{|Y\setminus X|}) \end{aligned}$$

leading to a contradiction since the left-hand side is strictly non negative and the right-hand side is negative ((25) and (23) imply that $\upsilon_{|X|}, \upsilon_{|Y\setminus X|} \geq 1$). Thus it can only be $\theta_{|Y\cup X|} \leq 1$ and therefore $F' \geq 0$. $\square$