GIT, Stacks, and Quasimap Theory

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The goal of these lectures is to explain the utility of quasimaps in modern enumerative geometry. Briefly, this utility comes from "wall-crossing theorems" that arise from viewing two geometric objects (Deligne-Mumford stacks) as open subobjects of something larger (an Artin stack).

We will start by studying geometric invariant theory (GIT) quotients as examples of complex varieties (or stacks) that are highly amenable to computation. Conveniently, they are also examples of global quotient stacks, and will provide our toehold for understanding stacks in more generality. Quasimap theory mixes the concepts of GIT and stacks in fascinating ways, and we will see ‘walls'' arise in various aspects of the theory. It is a rich story with diverse applications.

Annotated suggested reading:
For each of the three topics of the lectures, I give a condensed source and an expanded source.

Condensed sources:

- King, "Moduli of Representations of Finite Dimensional Algebras," Section 2 (2 pages). This passage contains the definitions of affine GIT that we will use in this course.
- Alper, " Stacks and Moduli"pp. 13-18. This excerpt from Alper's book discusses the moduli stack of triangles. Rather than focusing on stacks as moduli, we will focus first on stacks as geometric spaces, taking an approach complementary to Alper's. A second short source for stacks is Fantechi's, "Stacks for Everybody" (10 pages). Again the approach is orthogonal to ours, but you might try to understand Fantechi's example of vector bundles at some level.
- Webb, Quasimaps and Some Examples of Stacks for Everybody," (16 pages). This article summarizes the construction of quasimap moduli spaces and invariants with lots of examples.

Expanded sources:

- Hoskins, Geometric Invariant Theory and Symplectic Quotients. These notes provide a rigorous introduction to GIT *and* to symplectic reduction, which turns out to be an analogous theory. Confusingly, the setup considered in this and most other texts is not exactly the one that we use in this course and that is summarized in King's paper above. The relationship between the two appears in Section 4.6 of Hoskins notes. Hoskin's paper "Moduli Spaces and Geometric Invariant Theory: Old and New Perspectives" may also be helpful for understanding the relationship, especially Section 2.8.

- Alper, Stacks and Moduli. This is an online in-progress textbook on the theory of algebraic stacks.
- Ciocan-Fontanine—Kim(--Maulik), "Moduli Stacks of Stable Toric Quasimaps", "Stable Quasimaps to GIT Quotients" and "Wall-Crossing in Genus Zero Quasimap Theory and Mirror Maps". This series of three papers are the original sources for quasimap theory.