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A biased view on GLSMs

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Abstract

Four-ish lectures on GLSMs.

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Contents

1 Introduction and disclaimers

This is a review that mostly focuses on GLSMs associated to compact Calabi-Yau threefolds. It is by no means complete. Solutions to all exercises can be found in the literature. Happy to point out references.

1.1 Suggested Literature

- [1]: Witten's original paper.
- [2]: The big mirror symmetry book.
- [3]: D-branes in GLSMs with a lot of introductory material for both, mathematics and physics.
- [4]: SUSY partition functions in the context they are used here.

2 GLSM basics

2.1 Definition and phases

We will not give a complete physics description of GLSMs and be more oriented towards the mathematical definition, though this will not coincide with recent definitions given in the mathematics literature, see e.g [5].

Physically, and as first introduced in [1], a GLSM is a quantum field theory in a $1+1$-dimensional Minkowski space. As is common in string theory, we will perform a "Wick rotation" to replace space-time by a Riemann surface Σ, the *worldsheet*. For the remainder of this note we will assume that Σ has genus $g=0$. The classical action is of the form

$$S=\int_{\Sigma}\mathrm{d}^2z\,\mathcal{L}, \tag{2.1}$$

where $(z,\overline{z})$ are complex coordinates parameterising Σ. Some of the degrees of freedom will be purely (anti-)holomorphic. As is common in string theory, they will be called left- and right-moving degrees of freedom. $\mathcal{L}$ is the *Lagrangian density* which depends on the fields. The field content of the theory is rich. GLSMs are gauge theories with gauge group G and associated vector fields. For worldsheets without a boundary, theory has $\mathcal{N}=(2,2)$ supersymmetry, which means, that both, in the left- and right-moving sector each bosonic degree of freedom has two fermionic superpartners.

In the following, we will only be concerned with the scalar fields, i.e. complex or Lie-algebra valued functions of the worldsheet coordinates. We can get away with that because we will mostly be concerned with the vacua, i.e. the minimal energy states of the theory. Lorentz invariance of the ground state implies that the vacua are given in terms of configurations of scalar fields.

Exercise 2.1. *A prominent example with a non-trivial vacuum structure appears in the context of the Higgs effect. Find your favourite reference on the Higgs effect and appreciate that the (classical) vacuum is a circle which is defined by a complex scalar field taking values at the minimum of the "Mexican hat" potential.*

Even though we are going to avoid fermions and gauge fields, the GLSM is a theory for which one can write down an explicit Lagrangian, explicit (super-)symmetry transformations and gauge transformations, and show that the theory is invariant under these transformation. We refer to the literature for details.

Let us finally "define" a GLSM. We start off with the following data:

$$(G, V, \rho_V, R). \tag{2.2}$$

Here, G is a compact Lie group, the gauge group, V is a complex vector space. The coordinates ϕ_i $(i = 1, \ldots, \mathrm{dim}V)$ are complex scalar fields. They are the scalar components of the chiral multiplet. More precisely, $\phi_i(z) : \Sigma \to V$ are holomorphic with respect to the worldsheet. There are also right-moving chiral scalars $\overline{\phi}(\overline{z})$ which are antiholomorphic.

The gauge group acts on V via the matter representation ρ_V. To give more details about the representation, we introduce a maximal torus $T \subset G$ and the corresponding Lie algebras $\mathfrak{t}$ and $\mathfrak{g}$ as well as their complexifiactions $\mathfrak{t}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$ and their duals $\mathfrak{t}^*_{\mathbb{C}}$ and $\mathfrak{g}^*_{\mathbb{C}}$. Then the gauge charges Q^a_i $(a = 1, \ldots, \mathrm{rk}G)$ of the ϕ_i take values in $\mathfrak{t}^*_{\mathbb{C}}$. For later reference we also introduce a pairing $\langle \cdot, \cdot \rangle : \mathfrak{g}^*_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$. For general GLSMs, $\rho_V : G \to GL(V)$. Our main focus is on GLSMs associated to *Calabi-Yaus* (broadly defined), so we will impose the the condition $\rho_V : G \to SL(V)$. In other words, $\sum_i Q^a_i = 0$. This can be traced back to an anomaly cancellation condition related to the R-symmetries of the theory. The fact that we have $\mathcal{N} = (2, 2)$ supersymmetry implies that there is a pair of global $U(1)$ R-symmetries $U(1)_L, U(1)_R$ in the left- and right-moving sector. Upon a change of basis, this information can be recast into the pair $U(1)_V, U(1)_A$ of vector and axial $U(1)$-symmetries. The axial $U(1)$-symmetry is anomalous, i.e. not preserved at quantum level, unless the Calabi-Yau condition is satisfied. The vector $U(1)$-symmetry is non-anomalous and the $\phi_i(z)$ transform in the representation $R : U(1)_V \to GL(V)$ with weights R_i. This is part of the defining data (2.2).

The GLSM is a supersymmetric gauge theory and hence has a vector mulitiplet. The scalar fields are $\sigma \in \mathfrak{g}_{\mathbb{C}}$.

In addition, the GLSM has various parameters. There are gauge couplings e, and two parameters (ζ^a, θ^a) where $\zeta^a \in \mathbb{R}^{\mathrm{rk}G}$ are the Fayet-Iliopoulos (FI) parameter and the $\theta^a \in (\mathbb{R}/(2\pi\mathbb{Z}))^{\mathrm{rk}G}$ are called θ-angles. They combine into complex parameters

$$t^a = \zeta^a - i\theta^a \in \mathfrak{g}^*_{\mathbb{C}} \tag{2.3}$$

These are to be identified with the complexified Kähler parameters of the stringy Kähler moduli space $\mathcal{M}_K$ associated to a Calabi-Yau. By studying GLSMs at different values for these couplings, one can thus probe the Kähler moduli space of a Calabi-Yau. In particular, one can go beyond the boundary of the Kähler cone, i.e. beyond "large volume" regions of the moduli space, thereby extending the notion of "Calabi-Yau" beyond the geometric framework.

Furthermore, a GLSM can have a superpotential $W(\phi) \in \mathrm{Sym}V^*$ which is holomorphic, gauge invariant, and has R-charge 2.

$$W(g\phi) = W(\phi), \quad W(\lambda^R\phi) = \lambda^2 W(\phi), \qquad g \in G, \lambda \in \mathbb{C}^*. \tag{2.4}$$

If the vacuum of the GLSM has a geometric interpretation, eg. in terms of a Calabi-Yau manifold, the respective geometry will be compact for non-zero superpotential. In the following we will focus on cases with non-trivial W.

The *classical vacua* of the theory are given by the zeros of the scalar potential[1]

$$U = |[\sigma, \overline{\sigma}]|^2 + \frac{1}{2}\left(|\langle Q, \sigma\rangle\phi|^2 + |\langle Q, \overline{\sigma}\rangle\phi|^2\right) + \frac{e^2}{2}D^2 + |F|^2, \tag{2.5}$$

where the *D-term* is

$$D = \mu(\phi) - \zeta, \tag{2.6}$$

with $\mu : V \to \mathfrak{g}^*$ the moment map. For gauge group $G = U(1)^k$, this can be written as

$$D^a = \sum_{i=1}^{\mathrm{dim}V} Q_i^a|\phi_i|^2 - \zeta^a, \qquad a = 1, \dots, k. \tag{2.7}$$

The *F-term* is

$$F = dW. \tag{2.8}$$

A few comments are in order.

- U is a sum of squares, so to find vacua, each of the summands has to be zero.
- D is a real equation. Its solution will depend on the value of the FI-parameters ζ, which should be identified with the stability parameter(s) of a GIT quotient. Note however, that from the physics perspective and upon taking into account quantum corrections, ζ is paired with the θ-angle, which complexifies it. This is essential for understanding categorical equivalences and D-brane transport from a physics perspective.

Finding solutions $U = 0$ is typically hard. Roughly, one can divide up the solutions as follows.

- All σ are zero. Then the solutions are determined by $D = 0, F = 0$. They will depend on ζ. Since the vacuum configuration breaks some of all of the gauge symmetry, this type of solutions in called[2] *Higgs branch*.
- All ϕ are zero. Then the D-term equation (2.6) implies that this can only happen at $\zeta = 0$. This is called the *Coulomb branch*, see section 2.2.
- Everything else. This is referred to as a *mixed branch*.

[1] Here we are suppressing indices: for instance, $\langle Q, \sigma\rangle\phi = \sum_{i=1}^{\mathrm{dimV}} \sum_{a=1}^{\mathrm{rk}G} Q_i^a \sigma_a \phi_i$.

[2] With some caveats for theories with non-abelian G.

We begin with discussing the Higgs branch. For $\sigma = 0$, the classical vacua are determined by the D-term and F-term equations. Formally we can write,

$$X_\zeta = \mu^{-1}(\zeta)/G \cap dW(0)^{-1} \simeq (V - F_\zeta)/G_{\mathbb{C}} \cap dW(0)^{-1} \tag{2.9}$$

We refer to F_ζ as the *deleted set*. It encodes the field configurations for which the D-term equation (2.6) does not have a solution for the chosen value of ζ. This connects to the loci where GIT quotients are ill-defined. Note also that for G abelian, $\mu^{-1}(\zeta)/G$ coincides with the definition of a toric variety in terms of a symplectic quotient. Different vacuum configurations X_ζ for different choices of ζ are referred to as *phases of a GLSM.*

Example 2.1. *The quintic. Consider a GLSM with $G = U(1)$ and $V = \mathbb{C}(-5) \oplus \mathbb{C}(1)^{\oplus 5}$. In other words, have scalar fields content $\phi = (p, x_1, \dots, x_5)$ with charges*

	p	$x_1, \dots, x_5$	FI
$U(1)$	-5	1	ζ
$U(1)_V$	$2-5\kappa$	2κ	

(2.10)

The charges add up to zero, so the GLSM is Calabi-Yau and we make the restriction $0 \leq \kappa \leq \frac{1}{5}$. The vector R-charges are chosen such that the following superpotential satisfies (2.4):

$$W = pG_5(x), \tag{2.11}$$

where $G_5(x)$ is a sufficiently generic polynomial of degree 5. The genericity condition to impose is related to the smoothness of the geometry that will appear in one of the phases. We impose the condition that the only solution of $\frac{\partial G_5}{\partial x_i} = 0$ is $x_1 = \ldots = x_5 = 0$. An example of a generic-enough superpotential is the Fermat quintic $G_5 = \sum_{i=1}^5 x_i^5$.

Since $\mathrm{rk}G = 1$*, there is one FI-theta parameter and one σ-field. We first consider the Higgs branch. There is only one D-term condition:*

$$-5|p|^2 + \sum_{i=1}^{5} |x_i|^2 = \zeta. \tag{2.12}$$

The F-term conditions are

$$G_5(x) = 0, \qquad p\frac{\partial G_5}{\partial x_i} = 0. \tag{2.13}$$

This GLSM has two phases, determined by $\zeta > 0$ and $\zeta < 0$, separated by a Coulomb branch at $\zeta = 0$.

Let us start with the $\zeta > 0$-phase. If $\zeta > 0$, the D-term equation (2.12) has no solution if $x_1 = \ldots = x_5 = 0$, so the deleted set is $F_\zeta = \{x_1 = \ldots = x_5 = 0\}$. Under the genericity assumption and excluding the deleted set, the F-term equations (2.13) will only have a solution if $p = 0$. Therefore we get the following vacuum:

$$X_{\zeta>0} = \left(\mathbb{C}^{\oplus 5} - \{x_1 = \ldots = x_5 = 0\}\right) / (x_i \sim \lambda x_i) \cap \{G_5(x) = 0\}, \qquad \lambda \in \mathbb{C}^* \tag{2.14}$$

Since the $U(1)$ is a gauge symmetry, the vacuum configurations related by a gauge transformation are equivalent, hence we have to mod out the $U(1)$. The vacuum itself therefore no longer has a $U(1)$-symmetry, hence it is spontaneously broken. The vacuum describes a degree five *hypersurface in complex projective space*[3] $\mathbb{P}^4$*. This is the famous quintic Calabi-Yau threefold. The D-term equation reduces to an equation defining a 5-sphere and the FI-parameter is a radial (volume) parameter. Considering the low-energy effective theory, i.e. turning on fluctuations of the fields about the vacuum, one can show that the theory reduces at low energies to an* $\mathcal{N} = (2,2)$ *supersymmetric non-linear sigma model with target space* $X_{\zeta>0}$*, see eg. [2, section 15.4] for a detailed discussion.*

In the $\zeta < 0$*-phase, the deleted set is* $F_{\zeta<0} = \{p = 0\}$*. The F-term equations imply that* $x_1 = \ldots = x_5 = 0$*. So the vacuum is a circle given by*

$$|p| = \sqrt{-\frac{\zeta}{5}}. \tag{2.15}$$

Since the gauge group acts transitively on the circle, the vacuum reduces to a point, meaning that we can choose "unitary gauge" to fix a vacuum expectations value (VEV) for p*:*

$$\langle p\rangle = \sqrt{-\frac{\zeta}{5}}. \tag{2.16}$$

Since p *transforms as* $p \to \lambda^{-5}p$*, the vacuum is fixed if we choose* λ *to be a fifth root of unity. So the gauge symmetry is spontaneously broken to* $\mathbb{Z}_5 \subset U(1)$*. Turning on fluctuations about this vacuum, the low-energy effective theory turns out to be a* $\mathcal{N} = (2,2)$ *supersymmetric Landau-Ginzburg model with superpotential*

$$W^{\zeta<0} = \langle p\rangle G_5(x). \tag{2.17}$$

The residual discrete gauge symmetry acts non-trivially on the x*-fields, which implies that the model is actually an orbifold.*

We have discovered two low-energy effective theories. The $\zeta > 0$ *phase is a sigma model with a smooth, compact Calabi-Yau threefold as its target space. This is often called a large volume or geometric phase. The Landau-Ginzburg model is a sigma model with a non-compact target space*[4] $\mathbb{C}^5/\mathbb{Z}_5$ *with a superpotential. In a sense, the presence of the superpotential "compactifies" the model. This is known as the Landau-Ginzburg/Calabi-Yau correspondence [1].*

A final remark on the R-charge ambiguity in the definition of the quintic GLSM. When fields obtain a VEV, it should be fixed by the symmetries. This implies that the fields that get a VEV should have R-charge zero. For the geometric phase, one should thus fix the ambiguity to $\kappa = 0$ *while in the Landau-Ginzburg phase one should choose* $\kappa = \frac{1}{5}$*. In the Calabi-Yau case, the theory becomes a conformal field theory at low energies. The R-charges match with the R-charges of these CFTs.*

Exercise 2.2. *A two-parameter model. Consider a GLSM with gauge group* $U(1)^2$ *and the following field content*

	p	x_6	x_3	x_4	x_5	x_1	x_2	FI
$U(1)_1$	-4	1	1	1	1	0	0	ζ_1
$U(1)_2$	0	-2	0	0	0	1	1	ζ_2
$U(1)_V$	$2-8\kappa_1$	$2\kappa_1 - 4\kappa_2$	$2\kappa_1$	$2\kappa_1$	$2\kappa_1$	$2\kappa_2$	$2\kappa_2$	

(2.18)

[3] Since we are only dealing with complex spaces, we refrain from writing $\mathbb{CP}^4$.

[4] The $\mathbb{Z}_5$ fixes $x_1 = \ldots = x_5 = 0$, so the target space is a stack.

with $0 \leq \kappa_1 \leq \frac{1}{4}$ *and* $0 \leq \kappa_2 \leq \frac{1}{8}$ *and superpotential* $W = pG_{(4,0)}(x_1, \ldots, x_6)$ *where* $G_{(4,0)}$ *has multidegree* $(4,0)$ *with respect to the weights of the two* $U(1)$*s. Write down the D-term and F-term equations and discuss the classical phases of this model. Draw a "phase diagram", i.e. indicate the phase boundaries in the* (ζ_1, ζ_2)*-plane.*
Hint: [6]

Example 2.2. *A non-abelian model. We consider a non-abelian GLSM with* $G = U(2)$ *known as the Rødland model [7, 8]. The field content is*

	$p^1, \ldots, p^7$	$x^a_1, \ldots, x^a_7$	
$U(2)$	$\det^{-1}$	$\square$	ζ
$U(1)_V$	$2 - 2\kappa$	κ	

(2.19)

with $0 \leq \kappa \leq 1$*. The p-fields transform in the inverse determinantal representations, the* $x_i \in \mathbb{C}^2$ *transform in the fundamental representation. In other words, for* $g \in U(2)$*:*

$$p \to \det{}^{-1} g\, p, \qquad x_i \to g\, x_i. \tag{2.20}$$

We consider the GLSM with the following superpotential

$$W = \sum_{i,j,k=1}^{7} \sum_{a,b=1}^{2} A^{ij}_k p^k x^a_i \varepsilon_{ab} x^b_j = \sum_{i,j=1}^{7} A^{ij}(p)[x_i x_j], \tag{2.21}$$

where ε_{ab} *is the Levi-Civita symbol. It is easy to see that this is gauge invariant and has R-charge* 2*. We assume that the complex coefficients* A^{ij}_k *are sufficiently generic*[5] *[8]. The D-term equations are*

$$xx^\dagger - \sum_{i=1}^{7} |p^i|^2 \mathrm{id}_{2\times 2} = \zeta \mathrm{id}_{2\times 2}, \tag{2.22}$$

where $x = x^a_i$ *is a* 2×7 *matrix.*

The $\zeta > 0$*-phase can be analysed as for the quintic. The D-term equations imply that* x *has to have rank* 2*, so the deleted set is* $F_{\zeta>0} = \{\mathrm{rk}\, x < 2\}$*. The F-terms, together with the genericity condition, imply that* $p^1 = \ldots = p^7 = 0$ *so that the F-terms reduce to*

$$\sum_{i,j=1}^{7} A^{ij}_k [x_i x_j] = 0, \qquad k = 1, \ldots, 7. \tag{2.23}$$

We can identify the $[x_i x_j]$ *as the Plücker coordinates of the Grassmannian* $G(2,7)$*. The* $\zeta > 0$*-phase is thus a codimension* 7 *complete intersection in* $G(2,7)$*:*

$$X_{\zeta>0} = \{[x_i x_j] \in G(2,7) | \sum_{i,j=1}^{7} A^{ij}_k [x_i x_j] = 0\}. \tag{2.24}$$

This is a Calabi-Yau threefold: $\mathrm{dim} G(k,n) = k(n-k)$*, so for* $k = 2, n = 7$ *the complete intersection has dimension three.*

The $\zeta < 0$*-phase is also geometric. The D-term implies that* $F_{\zeta<0} = \{p^1 = \ldots = p^7 = 0\}$ *and that the vacuum manifold is given in terms of the p-field configurations constrained to*

[5] Finding suitable genericity conditions for non-abelian GLSMs is not straightforward.

lie on a $\mathbb{P}^6$. The gauge symmetry is broken to $SU(2)$, so the low-energy will be a non-abelian gauge theory, similar to strong and weak interactions. This is why this phase is often referred to as a strongly coupled phase. It complicates the physics because techniques like perturbation theory do not work. To study the low energy behaviour we turn on classical fluctuations which generate a potential of the form (2.21), where the p-fields take their VEVs. From a physics perspective, the potential that constrains the x-fields is special because it is quadratic. This means is a mass term for the x-fields and the antisymmetric 7×7-matrix matrix $A(p)$ is the mass matrix. At low energies the theory will be governed by the light states. In particular, if there are massless states, the massless degrees of freedom will be the relevant ones. Indeed, there will be massless degrees of freedom if the rank of $A(p)$ drops. Under the correct genericity assumptions, $A(p)$ cannot have rank two. A careful analysis [8] then shows that the low-energy theory is a non-linear sigma model with target

$$X_{\zeta<0} = \{(p^1, \dots, p^7) \in \mathbb{P}^6 | \mathrm{rk} A(p) = 4\}. \tag{2.25}$$

This is a Calabi-Yau threefold that is realised in terms of a on-complete intersection. The GLSMs and its two phases estabish the Pfaffian/Grassmannian correspondence.

In physics, non-abelian GLSMs exhibit interesting new properties like the strong coupling behaviour we have just seen. Another new feature is a non-abelian duality [9] which a strong/weak coupling duality in the sense that it connects this GLSM to another one where $X_{\zeta<0}$ is realised as a weakly coupled phase and $X_{\zeta>0}$ is strongly coupled. This correspondence also relates to fancy mathematics such as non-commutative algebraic geometry and homological projective duality.

2.2 Coulomb branches

So far, we have only considered the $\sigma = 0$ case. Now we consider the other extreme where all σ-fields are non-zero. Going back to the scalar potential (2.5), vanishing of the first term implies that the Lie algebra-valued fields have to commute, i.e. they have to take values in a maximal torus of the Lie algebra: $\sigma \in \mathfrak{t}_{\mathbb{C}}$. We can achieve $U = 0$ by setting $\phi = 0$ and which will only work if at least some of the FI-parameters are zero. So the Coulomb branch only exists at the boundaries between phases. The classical analysis does not pose any restriction on the θ-angle, and hence there appears to be a true boundary between phases with no option to smoothly interpolate between them.

The physics on the Coulomb branch is also drastically different compared to the Higgs branch. Not only are the relevant fields different, there is also no compact vacuum manifold. The only restriction imposed on the σ-fields is that they that they commute, but they are not constrained to some compact subset. This changes when we consider the quantum theory. In the classical theory, there is a term in the Lagrangian that is linear in σ called the *twisted superpotential*: $\widetilde{W} = -\langle t, \sigma \rangle$. Taking into account quantum effects, one-loop corrections generate an *effective potential*

$$U_{eff} = \min_{n \in P} \frac{e_{eff}^2}{2} |t_{eff}(\sigma) + 2\pi i n|^2, \tag{2.26}$$

where $t_{eff}(\sigma) = -\partial_\sigma \widetilde{W}_{eff}(\sigma)$ and P is the weight lattice of T. The *effective potential on the Coulomb branch* is given by

$$\widetilde{W}_{eff} = -\langle t, \sigma \rangle - \sum_{i=1}^{\dim V} \langle Q_i, \sigma \rangle \left(\log \langle Q_i, \sigma \rangle - 1 \right) + i\pi \sum_{\alpha>0} \langle \alpha, \sigma \rangle, \tag{2.27}$$

where $\alpha > 0$ are the positive roots of G, see [2, section 15.5] for a derivation in the abelian case and [9] for the non-abelian case. Upon taking into account these quantum corrections, there is still a Coulomb branch but only at the critical locus of (2.27).

Exercise 2.3. *Quintic GLSM. Compute* $\widetilde{W}_{eff}$ *for the quintic GLSM and convince yourself that there is a Coulomb branch located at*

$$e^{-t} = e^{-(\zeta - i\theta)} = -\frac{1}{5^5} \qquad \leftrightarrow \qquad \zeta = 5 \log 5 \approx 8.05, \quad \theta = \pi (\mathrm{mod}\ 2\pi). \tag{2.28}$$

This is a deep result:

- The quantum effects moved the location of the Coulomb branch away from $\zeta = 0$. More importantly, the Coulomb branch is no longer a circle in the FI-θ-parameter space, but a point, or more generally, a codimension one locus in the parameter space. This means one can find paths between the phases that avoid the Coulomb branch.
- The σ-dependence has dropped out. This is special to the Calabi-Yau case. For one-parameter GLSMs that are not Calabi-Yau, like for instance the GLSM for $\mathbb{P}^4$ which is the same as the one for the quintic, but without the p-field and the potential, a σ-dependence remains, and the Coulomb branch encodes the defining relation of the quantum cohomology ring of $\mathbb{P}^4$ [6].
- Under mirror symmetry, we can make the identification $z = e^{-t}$, where z is a coordinate on the complex structure moduli space on the mirror quintic. The location of the Coulomb branch is mirror to the location of the conifold point.

Exercise 2.4. *Two-parameter GLSM. Analyse the Coulomb branch of the GLSM discussed in Exercise 2.2:*

1. *Write down* $\widetilde{W}_{eff}$ *and compute its critical locus.*
2. *Show that* e^{-t_1}, e^{-t_2} *can be expressed in terms of a variable* $w = \frac{\sigma_2}{\sigma_1}$.
3. *Ignoring the* θ*-angle directions, plot the Coulomb branch locus in the* (ζ_1, ζ_2)*-plane. What do you observe? Is anything missing?*

Exercise 2.5. *Analyse the Coulomb branch for the Rødland GLSM. Remove all the solutions that are fixed under the action of the Weyl group. How many singular points do you find? Compared to the quintic, what does this result imply for paths interpolating between the Pfaffian and the Grassmannian phase?*

On the Coulomb branch the gauge symmetry is broken to the maximal abelian subgroup $T \subset G$. If $\mathrm{rk}G > 1$, there is also the possibility that a non-maximal abelian, or even a non-abelian [10], subgroup is preserved. In these cases there can be a *mixed branch.* In this case, non-trivial subsets of the ϕ- and σ-fields are massless and govern the low-energy behaviour of the theory. See e.g. [6, 11] for some accounts and examples.

3 Branes

So far, we have implicitly assumed that the Riemann surface Σ underlying the GLSM does not have boundaries. If we consider theories that have a boundary, we have to impose boundary conditions on the fields. We could even have additional degrees of freedom that are localised on the boundary. Boundary conditions are referred to as D-branes (where "D" stands for "Dirichlet", even though we consider Dirichlet and Neumann boundary conditions).

The presence of a boundary necessarily breaks supersymmetry[6]. There are special classes of D-branes that preserve some amount of supersymmetry. In the case of the $\mathcal{N} = (2, 2)$ theories we consider there are two types of D-branes that preserve half of they supersymmetry. They are called A-branes and B-branes. Mathematically, D-branes are objects in certain categories. For example, for the case of non-linear sigma models with Calabi-Yau target space X, A-branes wrap (special) Lagrangian submanifolds on X and are described in terms of objects of the Fukaya category of X, while B-branes wrap holomorphic cycles on X and are described in terms of objects in the bounded derived category of coherent sheaves on X. See Aspinwall's iconic lectures on D-branes on Calabi-Yaus [12] for a proper discussion.

A-branes in GLSMs are poorly understood. An account on most of what is known can be found in [13]. B-branes in GLSMs are a rich topic. An exhaustive discussion for abelian GLSMs has been given in the seminal work[7] of Herbst, Hori and Page [3]. We will not give the physics derivation of the data that characterises a D-branes. Roughly, this is what happens: Computing a variation of the GLSM action under B-type supersymmetry, leads to boundary terms. Some of them can be cancelled by imposing appropriate boundary conditions on the fields, but the presence of the superpotential leads to a boundary term of the form

$$\delta_B S = -\mathrm{Re} \int_{\partial\Sigma} dx^0 \overline{\epsilon} \psi_i \frac{\partial W}{\partial \phi_i}, \tag{3.1}$$

where $\overline{\epsilon}$ is the fermionic parameter of the supersymmetry transformation and the ψ_i are worldsheet fermions. Such a boundary term also occurs in Landau-Ginzburg theories, where this issue is known as the Warner problem [14]. As it turns out, choosing $W|_{\partial\Sigma} = \mathrm{const}$ is too constraining, and one is forced to introduce new degrees of freedom that are localised on the boundary. In the GLSM, this can be achieved by inserting a Wilson-line-like operator in the path integral. The variation of the extended action then cancels the Warner term upon imposing certain restrictions on the boundary degrees of freedom. This leads to the definition of the B-branes in GLSMs we are about to give.

3.1 B-branes in GLSMs

A B-brane $\mathcal{B}$ in a GLSM is given by the data

$$\mathcal{B} = (M, Q, \rho, r_*). \tag{3.2}$$

Here, M is a $\mathrm{Sym}V^*$-module that comes equipped with a $\mathbb{Z}_2$-grading so that we can decompose it as $M = M^0 \oplus M^1$. In physics, this is sometimes called the Chan-Paton space. Next, Q is

[6] Computing the anticommutator of two SUSY generators gives the generator of translations. The presence of a boundary breaks translation invariance in the transverse directions, and consequently parts of supersymmetry.

[7] The paper has 265 pages, which is actually quite short, given the immense amount of incredibly deep insights.

an odd endomorphism on M with the condition that it is a *matrix factorisation of the GLSM superpotential*. This means that Q is a $\dim M \times \dim M$ matrix with polynomial entries in the fields ϕ satisfying

$$Q^2 = W \cdot \mathrm{id}_M. \tag{3.3}$$

Gauge invariance extends to the theory with boundary. The representation $\rho : G \to GL(M)$ is determined by the condition

$$\rho(g)^{-1} Q(g\phi)\rho(g) = Q(\phi). \tag{3.4}$$

The weights q of ρ will be referred to as the *brane charges*. How the brane transforms under R-symmetry is determined by the matrix factorisation condition (3.3), which requires Q to have R-charge 1:

$$\lambda^{r_*} Q(\lambda^R \phi)\lambda^{-r_*} = \lambda Q(\phi). \tag{3.5}$$

The weights of this representation are referred to as R-charges of the brane. The $\mathbb{Z}_2$-grading is linked to the R-grading.

Following [3], the data defining $\mathcal{B}$ can be cast into *complexes of Wilson line branes*. This means we can write $M = \oplus \mathcal{W}(q_a)_{r_i}$ where the $\mathcal{W}(q_a)_{r_i}$ are irreducible components of M with gauge charges q_a and R-charges r_i. The brane is then characterised by a (twisted) complex of Wilson line branes where the entries in (3.3) define forward and backwards maps between even and odd graded components of $\mathcal{W}$.

Canonical constructions of matrix factorisations involve Clifford algebras, which turns M into a Clifford module. Suppose we can write $W = \sum_{i=1}^{N} f_i \cdot g_i$. Then we can write

$$Q = \sum_{i=1}^{N} f_i \eta_i + g_i \overline{\eta}_i, \tag{3.6}$$

where $(\eta_i, \overline{\eta}_i)$ can be represented as $2^N \times 2^N$ matrices that satisfy a Clifford algebra:

$$\{\eta_i, \overline{\eta}_j\} = \delta_{ij}, \qquad \{\eta_i, \eta_j\} = \{\overline{\eta}_i, \overline{\eta}_j\} = 0. \tag{3.7}$$

Then we can build M by introducing a vacuum (highest weight state) $|0\rangle$ and declare the η_i to be annihilation operators: $\eta_i|0\rangle = 0$. Then M can be written as

$$M = |0\rangle \bigoplus \oplus_i \overline{\eta}_i|0\rangle \bigoplus \oplus_{i<j} \overline{\eta}_i \overline{\eta}_j |0\rangle \bigoplus \ldots \bigoplus \overline{\eta}_1 \ldots \overline{\eta}_N |0\rangle \tag{3.8}$$

The gauge and R-charges of (f_i, g_i) fix the gauge and R-charges of the Clifford generators, making it easy to determine the gauge and R-charges of the components of M and translating the information into the Wilson line brane notation.

In general, one cannot expect to have a full classification of matrix factorisations, but there are some canonical constructions.

Example 3.1. *For any W, we can write down a trivial matrix factorisation*

$$Q = 1 \cdot \eta + W \cdot \overline{\eta} = \begin{pmatrix} 0 & 1 \\ W & 0 \end{pmatrix}, \tag{3.9}$$

where we have made an explicit choice for Clifford matrices in the second equality. Since $(1, W)$ are gauge invariant and have R-charge $(0, 2)$, the Clifford algebra generators $(\eta, \overline{\eta})$ have *gauge charge $(0,0)$ and R-charge $(1,-1)$, respectively. The Clifford module is $M = |0\rangle \oplus \overline{\eta}|0\rangle$. We have the freedom to choose the gauge and R-charges of the vacuum to be (q,r) with $q,r \in \mathbb{Z}$. This choice amounts to a choice of overall normalisation of (ρ, r_*) which is not fixed by (3.4), (3.5). Then the matrix factorisation (3.9) determines infinitely many GLSM branes $\mathcal{B}_{q,r}$ represented by the following complexes of Wilson line branes:*

$$\mathcal{W}(q)_r \underset{1}{\overset{W}{\rightleftarrows}} \mathcal{W}(q)_{r+1} \tag{3.10}$$

Example 3.2. *Quintic GLSM. Consider the quintic GLSM with superpotential $W = pG_5(x)$ and choose the R-charge ambiguity $\kappa = 0$ such that p has R-charge 2 and the x_i have R-charge 0.*

1. *The matrix factorisation*
$$Q = G_5\eta + p\overline{\eta} \tag{3.11}$$
leads to the following complex of Wilson line branes:
$$\mathcal{W}(q)_r \underset{G_5}{\overset{p}{\rightleftarrows}} \mathcal{W}(q+5)_{r+1} \tag{3.12}$$

2. *The matrix factorisation*
$$Q = \sum_{i=1}^{5} x_i\eta_i + p\frac{1}{5}\frac{\partial G_5}{\partial x_i}\overline{\eta}_i \tag{3.13}$$
leads to the following Koszul-type complex of Wilson line branes:
$$\mathcal{W}(q)_r \rightleftarrows \mathcal{W}(q+1)^{\oplus 5}_{r+1} \rightleftarrows \mathcal{W}(q+2)^{\oplus 10}_{r+2} \rightleftarrows \ldots \rightleftarrows \mathcal{W}(q+5)_{r+5} \tag{3.14}$$

Exercise 3.1. *Consider the quintic GLSM with Fermat-type superpotential.*

$$W = p(x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5). \tag{3.15}$$

1. *Can you find matrix factorisations that exist for this superpotential but not for generic $G_5(x)$?*
   Hint: *Try polynomial division.*

2. *Consider branes on the quintic that are intersections of linear equations. Find examples of D0- and D2-branes on the quintic and guess lifts to GLSM branes.*

A natural question to ask is how GLSM branes relate to branes in the respective phases. This assumes that we know how to describe B-branes in a given phase, which may not necessarily be true. Examples of phases where we know how to describe B-branes are non-linear sigma models and Landau-Ginzburg models, and recently so-called hybrid models [15, 16]. There are prescriptions on how to relate GLSM branes to branes in the phases, see [3, Section 10] for the case of Landau-Ginzburg and non-linear sigma model branes.

Note that the brane data we have given so far only refers to the chiral fields ϕ. One also has to impose boundary conditions on the σ-fields. It was argued in [3] that for abelian GLSMs, that the boundary condition for sigma is that it has to be restricted to a Lagrangian submanifold[8] $\gamma \subset \mathfrak{t}_{\mathbb{C}}$.

[8]There are some caveats for the non-abelian case [17].

3.2 D-brane transport

We have established that one can transverse between phases along paths that avoid the Coulomb branch loci at the phase boundaries. Extending the correspondences between phases to the level of branes leads to equivalences of D-brane categories associated to the respective phases. The most well-known one is the Orlov correspondence [18], which is stated as an equivalence between the derived category of coherent sheaves (B-branes in non-linear sigma models) and the category of matrix factorisations (B-branes in Landau-Ginzburg models). In [3], the GLSM was used to give a "physics proof" of this correspondence.

The first observation we make is that, due to the presence of the Coulomb branch, which is a singular point in the moduli space, there are infinitely many inequivalent paths between two phases, depending on how we avoid the Coulomb branch: the result will be different depending on how many times and with which orientation we circle around the singularity. So we should expect there to be infinitely many correspondences, one per homotopy class of paths. We expect the following picture[9]:

$$
\begin{tikzcd}
 & MF_G(W) \arrow[ddl, "\pi_{X_1}"'] \arrow[ddr, "\pi_{X_2}"] & \\
 & \cup & \\
 & \mathcal{T}_{w_i} \arrow[dl, "\cong"] \arrow[dr, "\cong"'] & \\
D(X_1) \arrow[rr, leftrightarrow, "\cong"'] & & D(X_2)
\end{tikzcd}
\tag{3.16}
$$

At the top we have the category of GLSM branes that we denote by $MF_G(W)$. At the bottom there are two categories associated to phases X_1 and X_2. As we will explain shortly, $MF_G(W)$ is much bigger than the categories associated to the phases, but there are infinitely many subcategories $\mathcal{T}_{w_i}$, called *window categories* which are equivalent to the categories of both phases, thus establishing and equivalence between the $D(X_1)$ and $D(X_2)$.

The window categories contain the information about the choice of paths in the moduli space. Each homotopy class of paths comes with its own window category. So, how are these categories characterised and how does one obtain the information? The answer to the second question is again encoded in the Coulomb branch. In the presence of a boundary, there is a boundary effective potential on the Coulomb branch. For abelian Calabi-Yau GLSMs with $G = U(1)$ this can be found in [3], where also a generalisation to higher rank abelian G can be found:

$$
V_{eff} = \frac{1}{2\pi}\left(\zeta + \sum_{i=1}^{\mathrm{dim}V} Q_i \log|Q_i|\right) - \left(\frac{\theta}{2\pi} + q\right)\mathrm{Re}(\sigma) + \frac{1}{4}\sum_{i=1}^{\mathrm{dim}V} |Q_i \mathrm{Re}(\sigma)| \tag{3.17}
$$

Note that this depends on the FI-paramter, the theta angle and the brane charge q. There is only one brane charge, because this expression should be associated to a single Wilson line brane. The potential is sensitive to the path which is parameterised by (ζ, θ). Potentials in physical theories have to be bounded from below, because otherwise the theory is "unstable". The Coulomb branch is where the first term is zero, so the condition for V_{eff} to be bounded

[9] This is more complicated for non-abelian GLSMs such as the Rødland model [19, 20, 17].

from below is determined by the last two terms. On examples like the quintic, each path is homotopic to a straight path with constant theta angle that takes values in an open interval of with 2π between two copies of the singularity. For a given range of theta angle associated to a choice of path, the allowed range of brane charges q is restricted by the condition on V. This answers the question how window categories $\mathcal{T}_{w_i}$ are characterised: given a range of θ associated to a path, the representation $\rho(g)$ is restricted to have certain weights only. This is the *grade restriction rule.*

Example 3.3. *For the quintic GLSM, the grade restriction rule is*

$$-\frac{5}{2} < \frac{\theta}{2\pi} + q < \frac{5}{2}. \tag{3.18}$$

Choosing, for instance, a window w where $\theta \in (-5\pi, -3\pi)$, the allowed brane charges are

$$w: \qquad q \in \{0, 1, 2, 3, 4\}. \tag{3.19}$$

For a GLSM brane to be in the subcategory T_w, the gauge charges of all its Wilson line brane components have to be in the window.

None of the branes in Example 3.2 satisfy the grade restriction rule. This means that these branes, while they may be well-defined lifts of branes in a certain phase, are not globally defined along any path in the moduli space. This does not make them bad branes from the GLSM perspective. For example, consider the brane (3.12) with $q = 0$. We can guess some properties of this brane which can be confirmed rigorously by the methods of [3]. One of the maps encoded in the matrix factorisation is the hypersurface equation $G_5(x)$ of the quintic, so one can expect the the large volume limit of the GLSM brane is the structure sheaf $\mathcal{O}_X$ of the quintic, i.e. a special D6-brane that wraps the Calabi-Yau. The backwards map, p, in (3.12) on the other hand encodes the deleted set $F_{\zeta<0} = \{p = 0\}$ in the Landau-Ginzburg phase, so one should expect that this GLSM brane does not exist in the Landau-Ginzburg phase. This naive reasoning turns out to be correct[10]. This is bad news: the Landau-Ginzburg image of "something" should not be "nothing". What do we learn from this?

- There are non-grade-restricted branes that can be perfectly fine in one phase but cannot be meaningfully analytically continued beyond the phase boundary.
- There is a notion of an *empty brane.* These are GLSM branes that are localised on the deleted set F_ζ associated to a given phase. They do not exist in the respective phase.

What do we do if we have GLSM lift of an interesting brane in phase X_1 and we want to know its image in phase X_2 when we transport it along a chosen path, but our GLSM lift of the brane is not grade restricted? The answer is, we have to replace our chosen GLSM lift by a different GLSM lift that is the window category associated to our path of choice and that gives the same brane in X_1. This will give the correct analytic continuation of this brane to the phase X_2. Mathematics tells us that this is always possible. How do we do this in practice? We make use of the empty branes and the fact that B-brane categories are *triangulated.* In physics terms this means that we can compute bound states of D-branes by open strings via tachyon condensation. Mathematically, this amounts to employing the *cone*

[10]Let me reiterate: This will not work for arbitrary objects. We are considering very special branes here.

construction. The open string states are the morphsims between objects of the category. This is very easy for matrix factorisations. An bound state of two brane $\mathcal{B}_1$ and $\mathcal{B}_2$ qith matrix factorisations Q_1 and Q_2 by an open string state ψ going from $\mathcal{B}_1$ to $\mathcal{B}_2$ is given by a matrix factorisation

$$Q = \begin{pmatrix} Q_1 & \psi \\ 0 & Q_2 \end{pmatrix}, \tag{3.20}$$

where ψ is defined by

$$Q_1\psi + \psi Q_2 = 0, \qquad \psi \neq Q_1\phi - \phi Q_2 \tag{3.21}$$

for some ϕ.

Since there are empty branes associated to each phase, binding them to a GLSM brane will give a different GLSM brane but will not change the brane in the phase. This also explains why the category of GLSM branes is bigger than the category of branes in the phases. Thus, by binding empty branes in a certain way, one can put any branes into any given window. This has a very explicit procedure at the level of matrix factorisations and an exercise in homological algebra when working with complexes of Wilson line branes. See [3, Section 10] for many examples and [21] for an outline of an algorithm.

Example 3.4. *We can grade-restrict (3.12) by binding a copy of the large volume empty brane (3.13) as follows:*

$$\begin{tikzcd}
 & & \mathcal{W}(0)_0 \arrow[r, shift left] & \mathcal{W}(5)_1 \arrow[l, shift left] \arrow[dr, "1"] & \\
\mathcal{W}(0)_{-3} \arrow[r, shift left] & \mathcal{W}(1)^{\oplus 5}_{-2} \arrow[l, shift left] \arrow[r, shift left] & \mathcal{W}(2)^{\oplus 10}_{-1} \arrow[l, shift left] \arrow[r, shift left] & \dots \arrow[l, shift left] \arrow[r, shift left] & \mathcal{W}(5)_2 \arrow[l, shift left]
\end{tikzcd} \tag{3.22}$$

The identity elements connected by the identity map correspond to a trivial component of the brane and can be removed. The remaining complex of Wilson line branes is grade-restricted. See [3] for details.

With these tools, we can transport any brane in any phase, provided branes in this phase are understood, to any other phase. This is a very explicit realisation of an equivalence of categories. We can determine the fate of any object along any path. This has many interesting applications, in particular when combined with the hemisphere partition function discussed in Section 4.1. One example is to determine autoequivelences of the categories associated to the phases that are related to monodromies.

4 Partition functions

In the past ten years, new results have become available that make it possible to computing non-perturbative corrections in the theories associated to the phases by means of GLSMs. Previously, the main tool to compute such quantum corrections was mirror symmetry. The GLSM methods also work for Calabi-Yaus whose mirrors are not known and to phases that are not geometric.

These new results come from *supersymmetric localisation* which makes it possible, in the presence of supersymmetry and under additional conditions, to solve the path-integral exactly. The result is an ordinary complex integral that can be solved to obtain results in the phases. For $\mathcal{N} = (2, 2)$ GLSMs, possibly with B-branes, supersymmetric localisation works for certain types of Riemann surface Σ.

- The *sphere partition function* Z_{S^2} [22, 23, 24] computes the exact Kähler potential on the stringy Kähler moduli space.
- The *torus partition function* Z_{T^2} [25, 26] computes the elliptic genus.
- The *hemisphere partition function* $Z_{D^2}(\mathcal{B})$ [27, 28, 4] computes the exact central charge of $\mathcal{B}$
- The *annulus partition function* $Z_{ann}(\mathcal{B}_1, \mathcal{B}_2)$ [4] computes the open Witten index of two brane $\mathcal{B}_1, \mathcal{B}_2$.

In [24], it was first observed that these partition function encode information relevant to Calabi-Yau spaces. It was shown how to compute Gromov-Witten/Gopakumar-Vafa invariants via the sphere partition function. In the following we will discuss the hemisphere partition function in more detail. The information about enumerative invariants can also be extracted from it, along with information about B-branes.

4.1 Hemisphere

The hemisphere partition function for a Calabi-Yau GLSM is given by

$$Z_{D^2}(\mathcal{B}) = C \int_\gamma d^{\mathrm{rk}G}\sigma \prod_{\alpha>0} \langle \alpha, \sigma \rangle \sinh(\langle \alpha, \sigma \rangle) \prod_{i=1}^{\dim V} \Gamma\left(i\langle Q_i, \sigma \rangle + \frac{R_i}{2} \right) e^{i\langle t, \sigma \rangle} f_{\mathcal{B}}(\sigma), \tag{4.1}$$

where C is an undetermined overall normalisation constant, and $\gamma \subset \mathfrak{t}_{\mathbb{C}}$ is a middle-dimensional integration contour that has to be chosen such that the integral converges. The only point where information about the brane enters is the *brane factor* $f_{\mathcal{B}}(\sigma)$. It is defined as

$$f_{\mathcal{B}}(\sigma) = \mathrm{tr}_M e^{i\pi r_*} \rho \left(e^{2\pi\sigma} \right) \tag{4.2}$$

Exercise 4.1. *Write down the definition of the hemisphere partition function for the quintic GLSM.*

Exercise 4.2. *Compute the brane factors for the branes discussed in Examples 3.1 and 3.2.*

How one evaluates (4.1) depends on the phase by virtue of the term $e^{i\langle t, \sigma \rangle}$, which determines the convergence of the integral and depends on the FI-parameter and hence the phase.

Exercise 4.3. *Consider the hemisphere partition function for the quintic GLSM and choose γ to be the real line.*

1. *For which values of σ does the integrand have poles? Plot them in the σ-plane.*
2. *How do you have to close the integration contour to get a convergent result in the $\zeta > 0$- and $\zeta < 0$-phases?*

The hemisphere partition function computes the *exact central charge* of a D-brane. The leading terms can be understood as the Ramond-Ramond charges, or Chern-characters, of the D-brane. The $\mathcal{O}(e^{-|t|})$-terms are worldsheet instanton corrections.

Exercise 4.4. *1. Argue that the hemisphere partition function for the brane (3.12) is zero in the Landau-Ginzburg phase, thereby confirming that it is an empty brane.*

2. Argue that the hemisphere partition function for the brane (3.13) is zero in the Calabi-Yau phase, thereby confirming that it is an empty brane.

3. (A bit more work!) Evaluate the hemisphere partition function for the brane (3.12) in the Calabi-Yau phase. Identify the topological characteristics of the quintic, thereby confirming that the brane becomes the structure sheaf of the quintic.
Hint*: You should find:*

$$Z_{D^2}^{\zeta\gg0} = \frac{H^3}{6}\varpi_3 + \frac{c_2\cdot H}{24}\varpi_1 + i\frac{\zeta(3)c_3}{(2\pi)^3}\varpi_0, \tag{4.3}$$

where $\varpi_j = \left(\frac{\log z}{2\pi i}\right)^j + \mathcal{O}(z)$, $t = -\log z + 5\pi i$, H is the hyperplane class, and c_i are the Chern classes. In fact, the ϖ_i are the periods of the mirror quintic.

Let us discuss some more about the structure of the hemisphere partition function when evaluated in the phases. This will help extract further useful information. In [29], we conjectured that the hemisphere partition function, evaluated in any[11] phase has a universal structure in terms of data associated to the low-energy effective theory in the phase:

$$Z_{D^2}^{phases}(\mathcal{B}) = \langle \mathrm{ch}(\mathfrak{B}), \widehat{\Gamma}\cdot I\rangle, \tag{4.4}$$

where:

- $\langle\cdot,\cdot\rangle$ is a topological pairing on a subspace of the bulk state space of the low-energy theory. In the case of non-linear sigma models, the states are elements of $H^{2k}(X)$ of the even cohomology of the Calabi-Yau. In Landau-Ginzburg orbifolds, the state space is determined in terms of *twisted sectors* [30, 31]. More generally, the states are elements of the (a,c)-chiral ring [32] on the conformal field theory on the worldsheet. They can be characterised by their left and right R-charges.
- $\mathrm{ch}(\mathfrak{B})$ is the Chern character of the image $\mathfrak{B}$ of $\mathcal{B}$ in the phase. It is a vector that can be expanded in terms of a basis of the state space.
- $\widehat{\Gamma}$ is the Gamma class, a characteristic class associated to the Gamma function. See e.g. [33] for a review in the context of GLSMs.
- I is the I-function, which can also be expanded in terms of the state space. Under mirror symmetry, its components are the elements of the period vector of the mirror.

Depending on the example and the phases, these expressions may be unknown. Assuming that (4.4) is true, one can find conjectural expressions for each of them.

The information given here is enough to compute enumerative invariants, even in non-geometric settings such as Landau-Ginzburg models where it is not clear what the enumerative problem is. By Givental's formalism [34] for mirror symmetry, the I-function is related to the J-function, which is the generating function of genus-zero correlation functions where the insertions are states of the state space. In a geometric setting, these correlation functions are

[11] There are some caveats here.

related to Gromov-Witten invariants. In the context of Landau-Ginzburg models orbifolds they are known as FJRW invariants [35]. The change of coordinates is given in terms of a quotient of components of the I-function which are determined by the R-charges of the states. In a mirror symmetry context, this change of coordinates is the mirror map.

5 Further topics

This review has been highly Biased towards GLSMs associated to compact Calabi-Yau threefolds. Here is a, most likely incomplete, overview of what we have not covered.

- **Non-Calabi-Yau GLSMs**. They are well-understood but have notable differences compared to the Calabi-Yau case. One of the main features of the Calabi-Yau case is that there worldsheet theory is a conformal field theory at every point in the moduli space. The moduli correspond to exactly marginal operators in the CFT. In the non-Calabi-Yau case this is no longer true. There is renormalisation group flow. The FI-theta parameters get renormalised. This affects the existence of phases. Also the Coulomb branch plays a different role.

- **Non-abelian GLSMs**. There are intresting new features such as Hori's duality [9], strongly coupled phases, relations to homological projective duality and non-commutative algebraic geometry, and new Calabi-Yaus that do not show up in known classifications.

- **Mirror symmetry**. This is known as Hori-Vafa mirror symmetry [36]. Recently, there has been a proposal for the extension to the non-abelian case [37]. Roughly speaking, GLSM mirror symmetry exchanges Coulomb and Higgs branches.

- **3D GLSMs**. They exist, exhibit mirror symmetry and one can compute partition functions. Recently, a connection between 3D partition functions and quantum K-theory has been established [38, 39]. I don't know much about this yet, but it is really interesting.

- $(0,2)$ **GLSMs**. They also exist and are related to heterotic string theory. I know next to nothing about them, sorry.

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