ECE 550/650 – Intro to Quantum Computing

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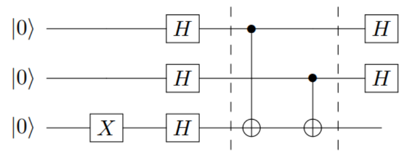
Outline of the course

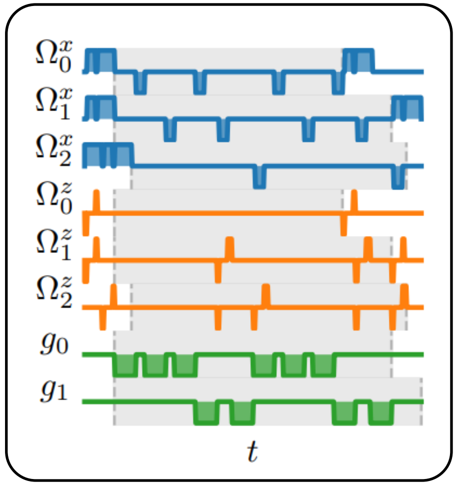
- Quantum Optics
  - What is interference (classical vs. single particle)
  - Superposition of states
  - Measurement and measurement basis
- Atomic physics
  - Spin states in magnetic fields and spin transitions
  - Transitions between atomic states (Rabi oscillations of qubits)
- Single qubits
  - Single qubit gates (electro-magnetic pulses, RF, MW, phase)
  - Error sources (dephasing, spontaneous decay)
  - Ramsey pulses and Spin echo pulse sequences
  - Calibration (finding resonance and verifying pulse time and amplitudes)
- Two qubit gates
  - Two qubit interactions – gate speed vs. error rates
  - Entanglement – correlation at a distance
  - Bell states and the Bell basis
  - XX gates, Controlled Phase gates, Swap




```
qc = QubitCircuit(3)
qc.add_gate("X", targets=2)
qc.add_gate("SNOT", targets=0)
qc.add_gate("SNOT", targets=1)
qc.add_gate("SNOT", targets=2)

# Oracle function f(x)
qc.add_gate(
    "CNOT", controls=0, targets=2)
qc.add_gate(
    "CNOT", controls=1, targets=2)

qc.add_gate("SNOT", targets=0)
qc.add_gate("SNOT", targets=1)
```

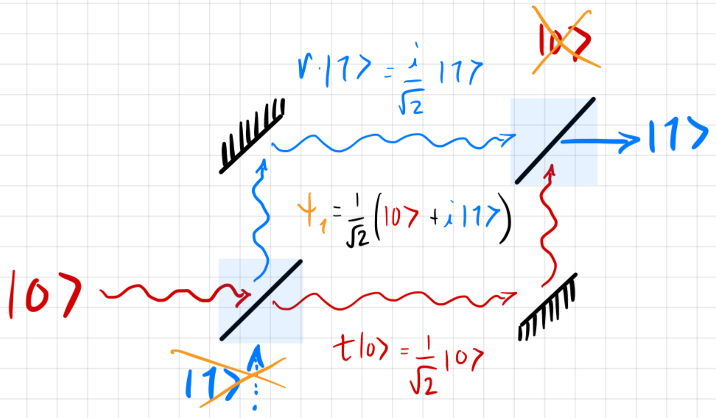


- Quantum Hardware
  - Photonics – nonlinear phase shifts
  - Transmons – charge noise, SWAP gate
- Quantum Circuits
  - Single and two qubit gates
  - Hadamard gate , CNOT gate
- Quantum Algorithms
  - Amplitude amplification
  - Grover's Search
  - Oracle - Deutsch Jozsa
  - Bernstein Vazirani
  - Quantum Fourier Transform and period finding
  - Shor's algorithm

If time permits

- Error Correction
  - Repetition codes
  - Color Codes
  - Surface code

MZI with Matrices



$$\psi_0 = \begin{bmatrix} E_{in1} \\ E'_{in2} \end{bmatrix} = \begin{bmatrix} E'_{in1} \\ 0 \end{bmatrix} = E_{in}|0\rangle \qquad |0\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|\psi_1\rangle = \boxed{\text{Beam Splitter}_1}\,|\psi_0\rangle = \begin{bmatrix} t & r \\ r & t \end{bmatrix}|\psi_0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}|\psi_0\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} E_{in} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1\cdot E_{in} + i\cdot 0 \\ i\cdot E_{in} + 1\cdot 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} E_{in} \\ iE_{in} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\cdot E_{in}\begin{bmatrix} 1 \\ i \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}}E_{in}(|0\rangle + i|1\rangle)}_{\text{Superposition State}} = \frac{1}{\sqrt{2}}E_{in}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$\psi = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|\psi_2\rangle = \boxed{\text{Beam Splitter}_2}\,\boxed{\text{Beam Splitter}_1}\,|\psi_0\rangle = \boxed{\text{Beam Splitter}_2}\,|\psi_1\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1\cdot 1 + i\cdot i \\ i\cdot 1 + 1\cdot i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 - 1 \\ i + i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 2i \end{bmatrix} = \frac{2}{2}\begin{bmatrix} 0 \\ i \end{bmatrix}$$

$$|\psi_2\rangle = i\begin{bmatrix} 0 \\ 1 \end{bmatrix} = i|1\rangle \checkmark$$

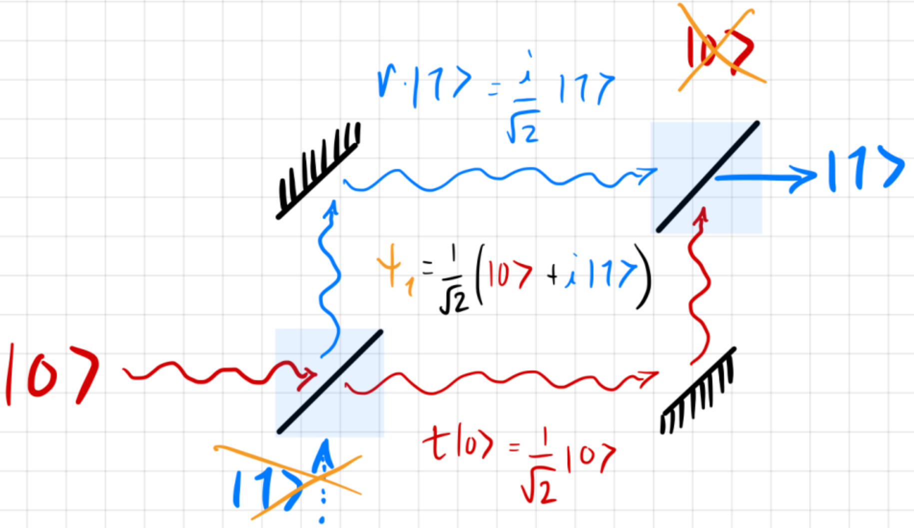
$$I = E_{in}'^{\,2}$$

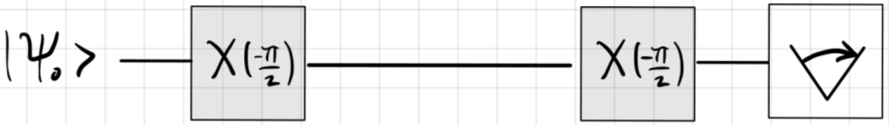
$$\langle\psi_2|\psi_2\rangle = -i\langle 1| \cdot i|1\rangle = -(-1)\langle 1|1\rangle = 1 \checkmark$$

$$|\psi_2\rangle = \boxed{X(\tfrac{-\pi}{2})}\,\boxed{X(\tfrac{-\pi}{2})}\,|\psi_0\rangle = \boxed{X(\tfrac{-\pi}{2})}\,|\psi_1\rangle$$

$$= i|1\rangle$$

MZI Quantum Circuit



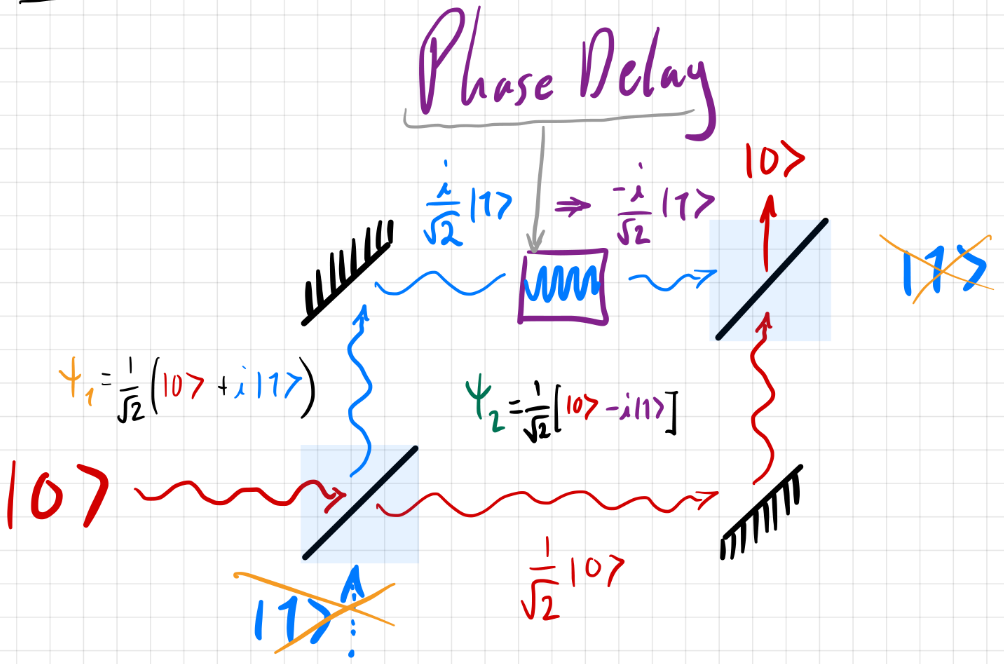


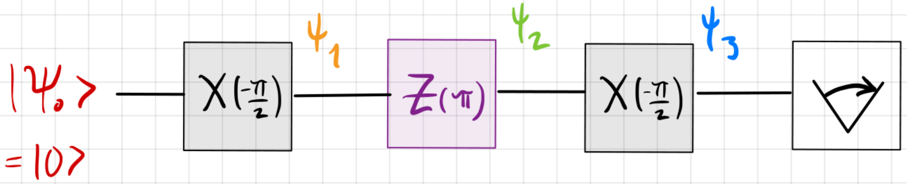
$|\psi_0\rangle = |0\rangle$

$|\psi_1\rangle = X(\frac{-\pi}{2})|\psi_0\rangle$
$= \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

$|\psi_2\rangle = X(-\pi/2)|\psi_1\rangle$
$= |1\rangle$

MZI w/ Phase shift Quantum Circuit





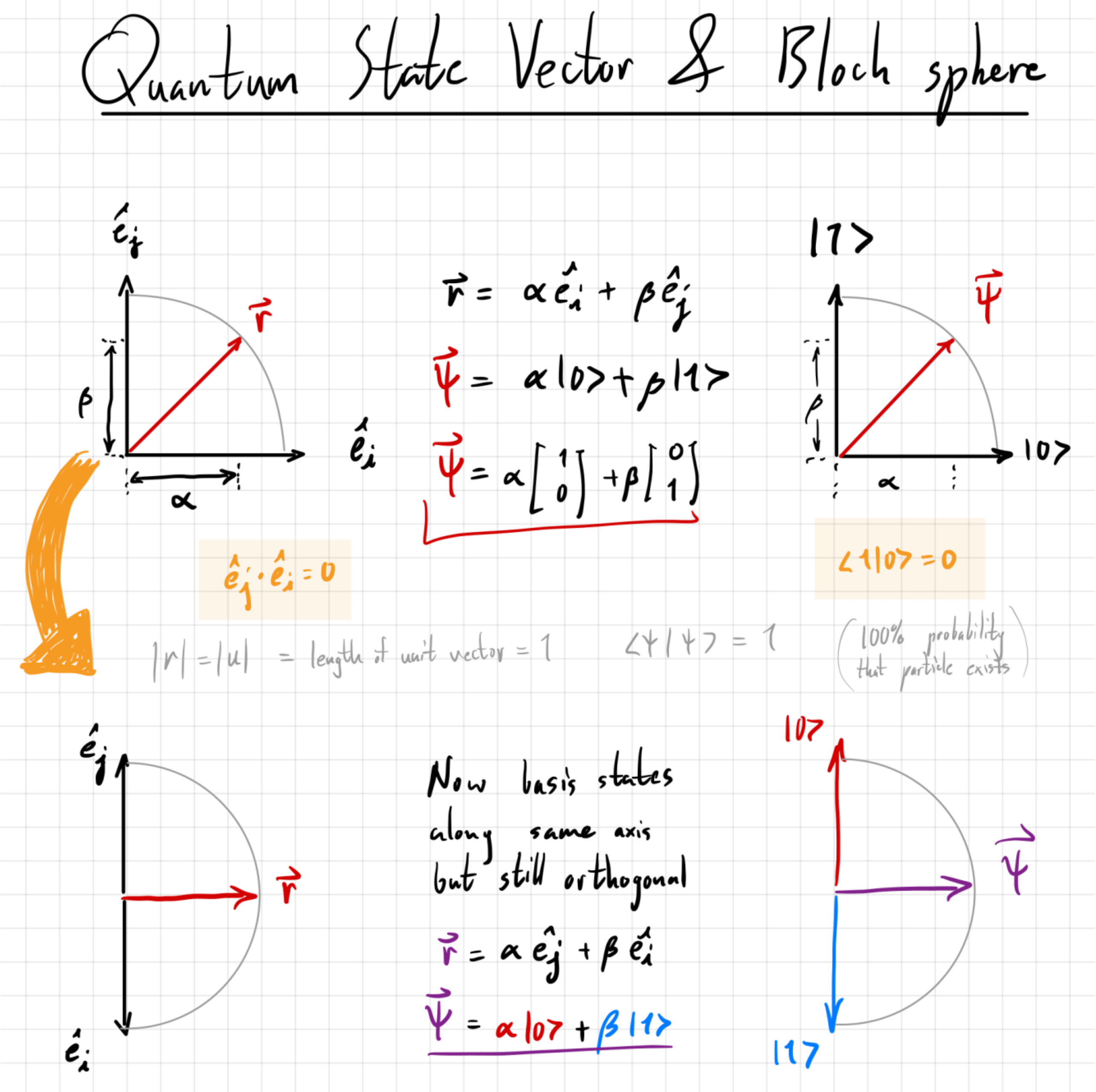
$|\psi_1\rangle = X(\frac{-\pi}{2})|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

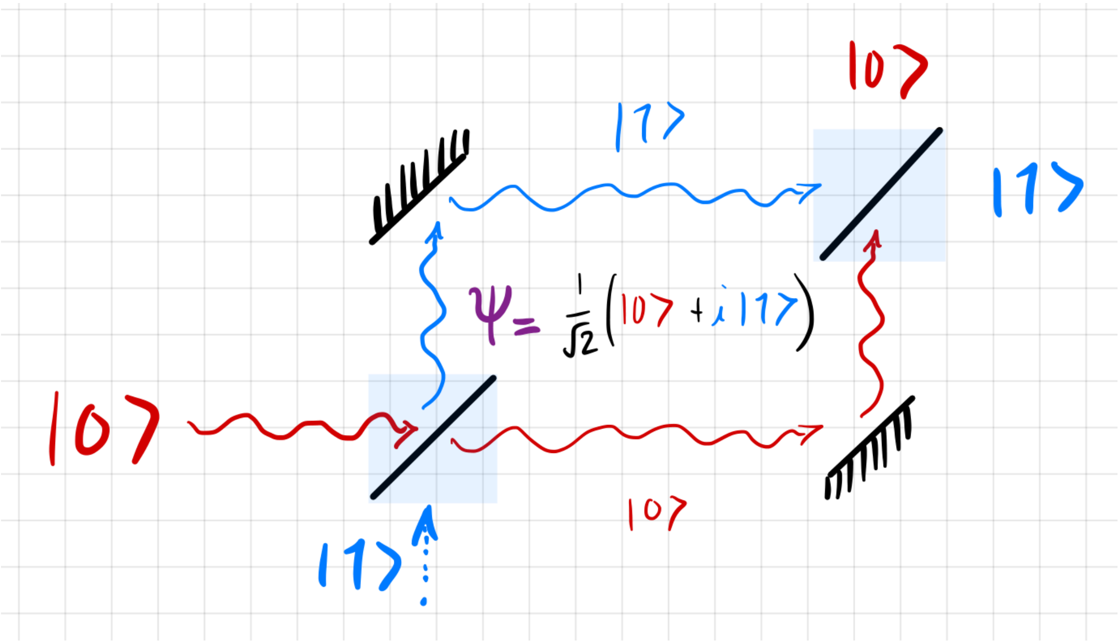
$|\psi_2\rangle = Z(\pi)|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

Flips sign (π phase, 180°)

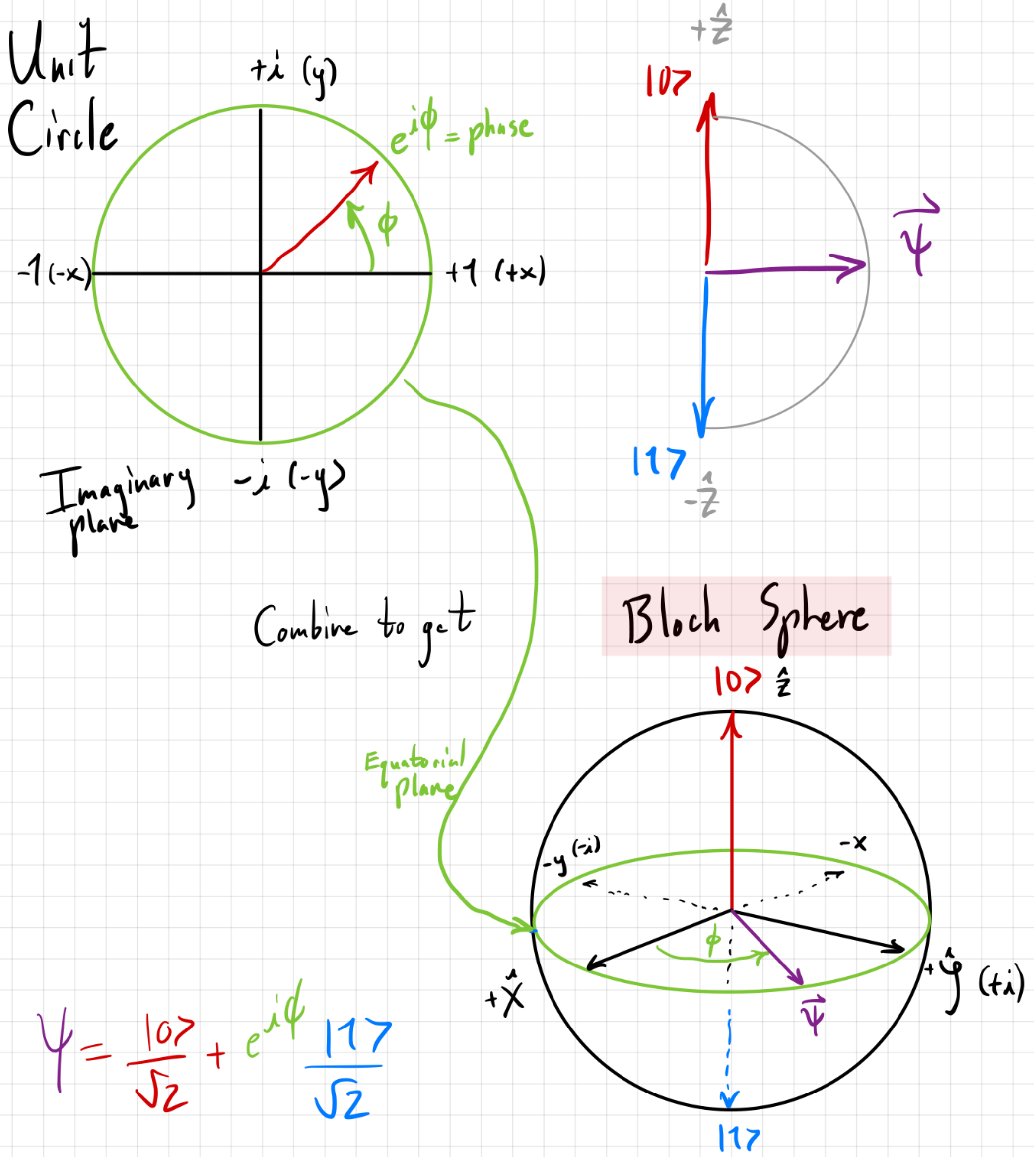
$|\psi_3\rangle = X(-\pi)|\psi_2\rangle = |0\rangle$

Bloch Sphere - Superposition

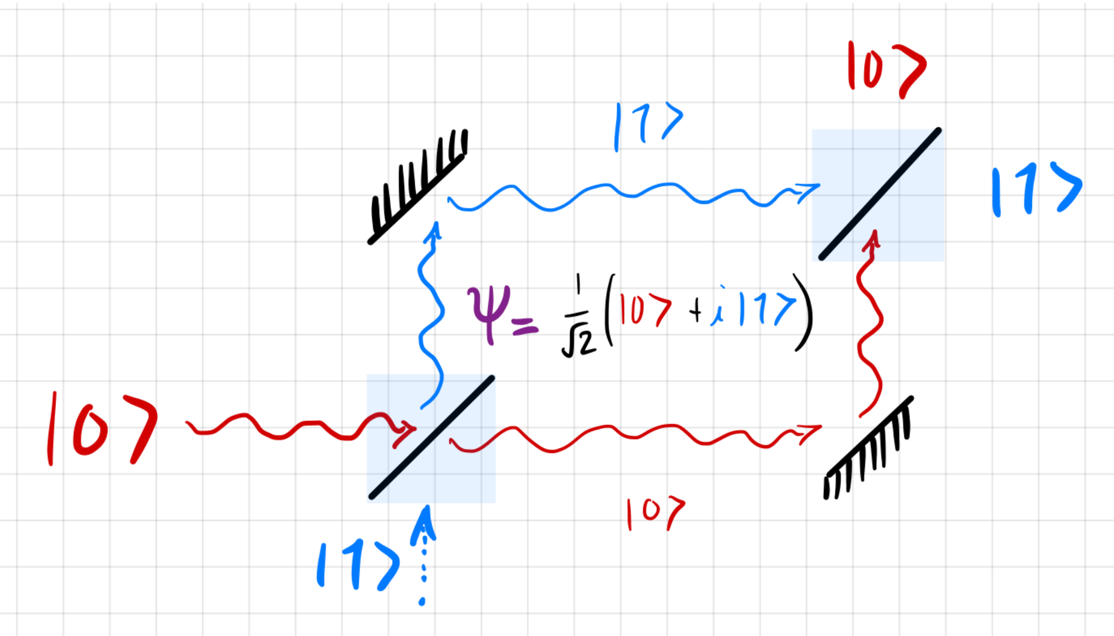




Bloch Sphere



$$\psi = \frac{|0\rangle}{\sqrt{2}} + e^{i\phi}\frac{|1\rangle}{\sqrt{2}}$$



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Quantum States - Dirac & 'Bra' 'Ket' notation

$|0\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad$ $|1\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad$ Column Vectors

$|\psi\rangle \equiv$ Ket

$\langle 0| \equiv \begin{bmatrix} 1 & 0 \end{bmatrix}$ $\qquad$ $\langle 1| \equiv \begin{bmatrix} 0 & 1 \end{bmatrix}$ $\qquad$ Row Vectors

$\langle \psi| =$ 'Bra'

$\langle \psi | \psi \rangle$ 'Bra-Ket'

Braket Notation Tip:

1 ≡ One $\quad$ 7 ≡ Seven $\quad$ ✓ $|1\rangle = |\text{one}\rangle$, $|1>$ = hard to read

Eleven greater than?

Quantum States - Dirac & 'Bra' 'Ket' notation

$|0\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad |1\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Column Vectors

$|\psi\rangle \equiv$ Ket

$\langle 0| \equiv [1 \;\; 0]$ $\quad \langle 1| \equiv [0 \;\; 1]$

Row Vectors

$\langle \psi|$ = 'Bra'

$\langle \psi | \psi \rangle$ 'Bra-Ket'

Braket Notation Tip:

$1 \equiv$ One $\quad 7 \equiv$ Seven $\quad$ ✓ $|1\rangle = |\text{one}\rangle$, $|1\rangle$ = hard to read

Eleven greater than?

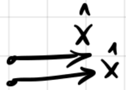
Inner Product (aka Dot Product)

$\langle 0|0\rangle = [1 \; 0] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\cdot 1 + 0\cdot 0 = 1$

$\langle 1|0\rangle = [0 \; 1] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0\cdot 1 + 1\cdot 0 = 0$

Overlap Integral $\quad \langle \psi|\psi\rangle = \int \psi^*(\vec{r})\, \psi(\vec{r})\, d\vec{r}$

$\langle \hat{x}|\hat{x}\rangle = \hat{x}\cdot\hat{x} = \int \hat{x}^*\cdot\hat{x}\, dr = 1$



Perfect overlap

$\langle \hat{x}|\hat{y}\rangle = \hat{x}\cdot\hat{y} = \int \hat{x}\cdot\hat{y}\, dr = 0$

No overlap

Inner Product (aka Dot Product)

$$\langle 0|0\rangle = [1\ 0]\begin{bmatrix}1\\0\end{bmatrix} = 1\cdot 1 + 0\cdot 0 = 1$$

$$\langle 1|0\rangle = [0\ 1]\begin{bmatrix}1\\0\end{bmatrix} = 0\cdot 1 + 1\cdot 0 = 0$$

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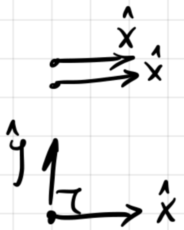
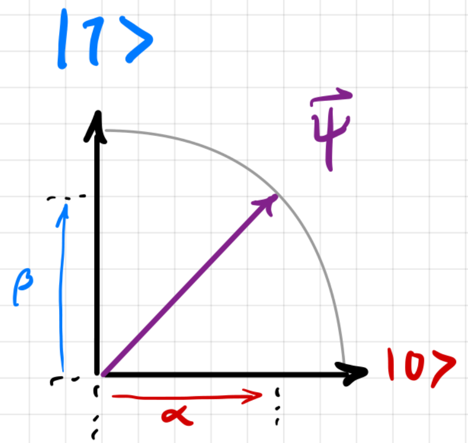


$$\langle \hat{x}|\hat{y}\rangle = \hat{x}\cdot\hat{y} = \int \hat{x}\cdot\hat{y}\,dr = 0$$

No overlap

$$\langle 0|0\rangle = 1$$

$$\langle 0|1\rangle = 0$$

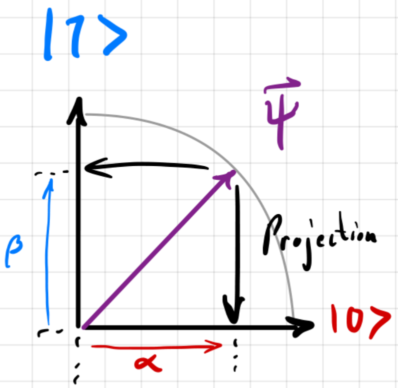


$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$= \alpha\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\end{bmatrix}$$

$$= \begin{bmatrix}\alpha\\\beta\end{bmatrix}$$

$$\langle\psi| = [\alpha^*\ \beta^*]$$



$$\langle\psi|0\rangle = [\alpha\ \beta]\begin{bmatrix}1\\0\end{bmatrix} = \alpha$$

$$\langle\psi|1\rangle = [\alpha\ \beta]\begin{bmatrix}0\\1\end{bmatrix} = \beta$$

$$\langle r|\hat{x}\rangle = \vec{r}\cdot\hat{x}$$

Inner Product (aka Dot Product)

$$\langle 0|0\rangle = [1\ 0]\begin{bmatrix}1\\0\end{bmatrix} = 1\cdot 1 + 0+0 = 1$$

$$\langle 1|0\rangle = [0\ 1]\begin{bmatrix}1\\0\end{bmatrix} = 0\cdot 1 + 1\cdot 0 = 0$$

Overlap Integral $\qquad \langle \psi|\psi\rangle = \int \psi^*(\vec{r})\,\psi(\vec{r})\,d\vec{r}$

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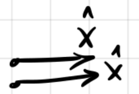

$$\langle \hat{x}|\hat{y}\rangle = \hat{x}\cdot\hat{y} = \int \hat{x}\cdot\hat{y}\,dr = 0$$

No overlap

Measurement

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[\,|0\rangle + |1\rangle\,\right]$$

$$\langle\psi|\psi\rangle = \frac{1}{\sqrt{2}}\left[\langle 0| + \langle 1|\right]\frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

$$= \frac{1}{2}\left[\underbrace{\langle 0|0\rangle}_{=1} + \underbrace{\langle 0|1\rangle}_{=0} + \underbrace{\langle 1|0\rangle}_{=0} + \underbrace{\langle 1|1\rangle}_{=1}\right]$$

$$= \frac{1}{2}(1+1) = \frac{2}{2} = 1 \checkmark$$

$$P(\psi) = |\psi|^2 = 1^2 = 1$$

$$P(|0\rangle) = \langle\psi|0\rangle\langle 0|\psi\rangle$$

$$\frac{1}{\sqrt{2}}\left[\langle 0| + \cancel{\langle 1|}\right]|0\rangle \cdot \langle 0|\cdot\frac{1}{\sqrt{2}}\left[|0\rangle + \cancel{|1\rangle}\right]$$

$$= \frac{1}{2}\left[1\cdot 1\right] = 1/2$$

Measurement

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle \right]$$

$$\langle\psi|\psi\rangle = \frac{1}{\sqrt{2}}\left[\langle 0| + \langle 1|\right]\frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

$$= \frac{1}{2}\left[\underbrace{\langle 0|0\rangle}_{=1} + \underbrace{\langle 0|1\rangle}_{=0} + \underbrace{\langle 1|0\rangle}_{=0} + \underbrace{\langle 1|1\rangle}_{=1}\right]$$

$$= \frac{1}{2}(1+1) = \frac{2}{2} = 1 \checkmark$$

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$$= \frac{1}{2}\left[1 \cdot 1\right] = 1/2$$

Outer Products

$$|0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$|1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|0\rangle\langle 0| + |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{1} \quad \text{(Identity)}$$

Outer Products

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Transition to other states

$$|0\rangle\langle 1| = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$$

$$|1\rangle\langle 0| = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

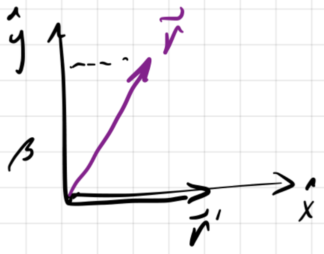
$$|0\rangle\langle 1| + |1\rangle\langle 0| = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

off Diagonals

$$\vec{r} = \alpha\hat{x} + \beta\hat{y}$$

$$\vec{r}\,' = |\hat{x}\rangle\underbrace{\langle\hat{y}|\vec{r}\rangle}_{=\beta} = \vec{r}\,'$$

$$= \beta|\hat{x}\rangle$$



Transition to other states

$$|0\rangle\langle 1| = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$$

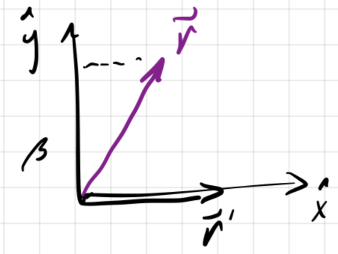
$$|1\rangle\langle 0| = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

$$|0\rangle\langle 1| + |1\rangle\langle 0| = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$ off Diagonals

$$\vec{r} = \alpha\hat{x} + \beta\hat{y}$$

$$\vec{r}\,' = |\hat{x}\rangle\underbrace{\langle\hat{y}|\vec{r}\rangle}_{=\beta} = \vec{r}\,'$$

$$= \beta|\hat{x}\rangle$$



$$|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi_1\rangle = \left[|0\rangle\langle 1| + |1\rangle\langle 0|\right]|\psi_0\rangle$$

$$= \left[|0\rangle\langle 1| + |1\rangle\langle 0|\right]\left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$= |0\rangle\langle 1|\alpha|0\rangle + |0\rangle\langle 1|\beta|1\rangle + |1\rangle\langle 0|\alpha|0\rangle + |1\rangle\langle 0|\beta|1\rangle$$

$\langle 1|0\rangle = 0$ $\langle 1|1\rangle = 1$ $\langle 0|0\rangle = 1$ $\langle 0|1\rangle = 0$

$$= \beta|0\rangle + \alpha|1\rangle$$ (swapped $|0\rangle$ & $|1\rangle$)

$$= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}0\cdot\alpha + 1\cdot\beta\\1\cdot\alpha + 0\cdot\beta\end{bmatrix} = \begin{bmatrix}\beta\\\alpha\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \sigma_x$$

Measurement

1. **Project $|\Psi\rangle$ onto $|0\rangle$ with Outer Product**
2. **Measure overlap with $|\Psi\rangle$**

$$P(|0\rangle) = \langle\psi|0\rangle\langle 0|\psi\rangle$$

$$\frac{1}{\sqrt{2}}\left[\langle 0| + \cancel{\langle 1|}\right]|0\rangle \cdot \langle 0| \cdot \frac{1}{\sqrt{2}}\left[|0\rangle + \cancel{|1\rangle}\right]$$

$$= \frac{1}{2}\,[1\cdot 1] = 1/2$$

Outer Products

$$|0\rangle\langle 0| = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

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$$|0\rangle\langle 0| + |1\rangle\langle 1| = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \mathbb{1} \quad (\text{Identity})$$

Beam Splitter has two parts

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\hat{\mathbb{1}}} + \frac{i}{\sqrt{2}}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\hat{\sigma}_x} = \underbrace{\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}}_{\text{Transmissin}} + \underbrace{\begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix}}_{\text{Reflection}}$$

Transmission $\Rightarrow$ Identity $\cdot \frac{1}{\sqrt{2}}$ = Amplitude to stay in the same mode (same state)

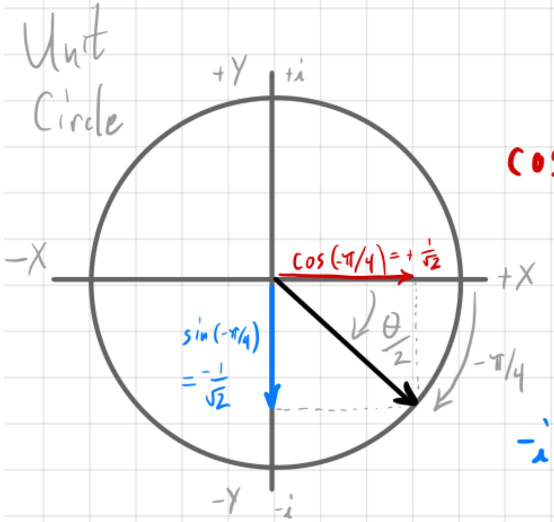
Reflection $\Rightarrow$ $\underbrace{\text{Off Diagonals} \cdot \frac{1}{\sqrt{2}} = \text{Amplitude to couple to the other mode (other state)}}_{\text{Couples states to each other (Outer Products)}}$

Beam Splitter has two parts

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix} = \frac{1}{\sqrt{2}}\underbrace{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}_{\hat{\mathbb{1}}} + \frac{i}{\sqrt{2}}\underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}_{\hat{\sigma}_x} = \overbrace{\begin{pmatrix}t & 0\\ 0 & t\end{pmatrix}}^{\text{Transmission}} + \overbrace{\begin{pmatrix}0 & r\\ r & 0\end{pmatrix}}^{\text{Reflection}}$$

$$= \cos(-\pi/4)\,\hat{\mathbb{1}} - i\sin(-\pi/4)\,\hat{\sigma}_x = e^{-i\underbrace{\pi/2}_{\text{Twice the angle!!!}}\cdot\,\hat{\sigma}_x/2}$$

Axis 'steals' factor of 1/2

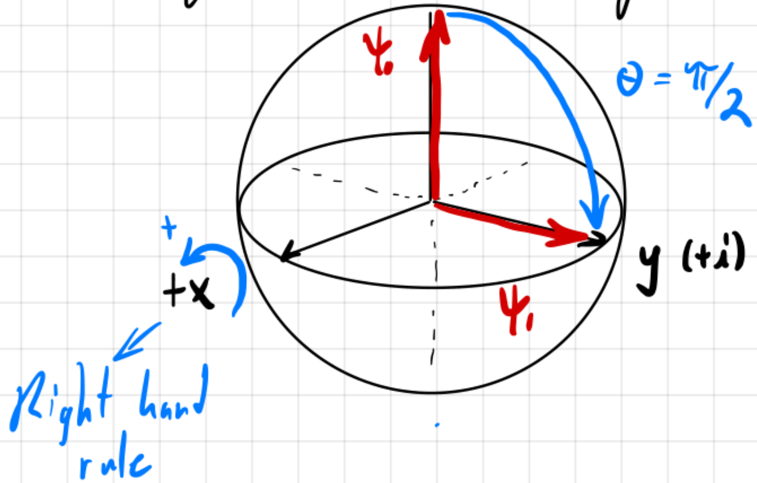


$\cos(-\pi/4) = +\frac{1}{\sqrt{2}}$

$\sin(-\pi/4) = \frac{-1}{\sqrt{2}}$

$-i\cdot\sin(-\pi/4) = \frac{+i}{\sqrt{2}}$ ✓

Equivalent to rotations on a sphere



Euler Formula for Matrices: $e^{-i\theta \cdot \hat{\sigma}_i/2} = \cos(\theta/2) \cdot \hat{\mathbb{1}} - i\sin(\theta/2) \cdot \hat{\sigma}_i$

$$X(\theta = \pi/2) = e^{-i\theta \cdot \hat{\sigma}_x/2} = \cos(\theta/2)\,\hat{\mathbb{1}} - i\sin(\theta/2)\,\hat{\sigma}_x$$

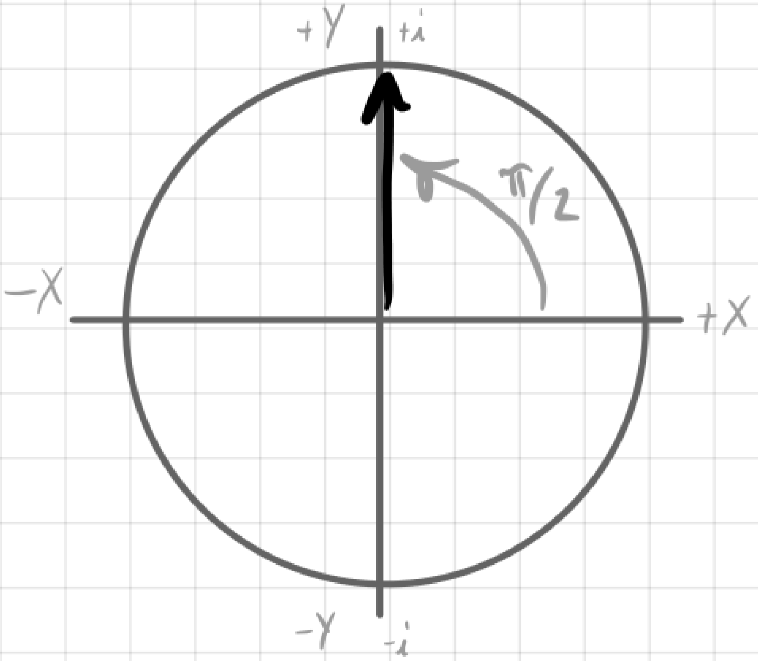
$$= \cos(\pi/4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - i\sin(\pi/4)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - i\left(\frac{-1}{\sqrt{2}}\right)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \boxed{BS}\ \checkmark$$

Euler Formula for Matrices: $e^{-i\theta\cdot\hat{\sigma}_i/2} = \cos(\theta/2)\cdot\hat{\mathbb{1}} - i\sin(\theta/2)\cdot\hat{\sigma}_i$

$$Z(\theta=\pi) = e^{-i\theta\,\hat{\sigma}_z/2} = \cos(\pi/2)\,\hat{\mathbb{1}} - i\sin(\pi/2)\,\hat{\sigma}_z$$



$$= \cancel{0\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix}} - i(1)\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$= 0 - i\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$= \begin{bmatrix}-i&0\\0&+i\end{bmatrix} = \begin{bmatrix}e^{i\theta/2}&0\\0&e^{-i\theta/2}\end{bmatrix}$$

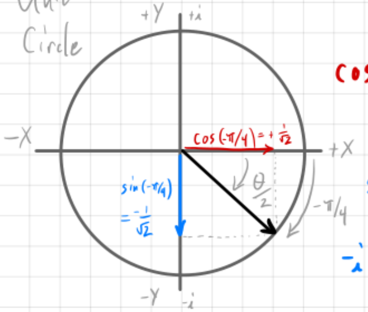
Beam Splitter has two parts

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix} = \frac{1}{\sqrt{2}}\underbrace{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}_{\hat{\mathbb{1}}} + \frac{i}{\sqrt{2}}\underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}_{\hat{\sigma}_x} = \underbrace{\begin{pmatrix}t & 0\\ 0 & t\end{pmatrix}}_{\text{Transmission}} + \underbrace{\begin{pmatrix}0 & r\\ r & 0\end{pmatrix}}_{\text{Reflection}}$$

$$= \cos(-\pi/4)\,\hat{\mathbb{1}} - i\sin(-\pi/4)\,\hat{\sigma}_x = e^{\overbrace{-i\pi/2}^{\text{Twice the angle!!!}}\cdot\hat{\sigma}_x/2}$$

Axis 'steals' factor of 1/2

Unit Circle

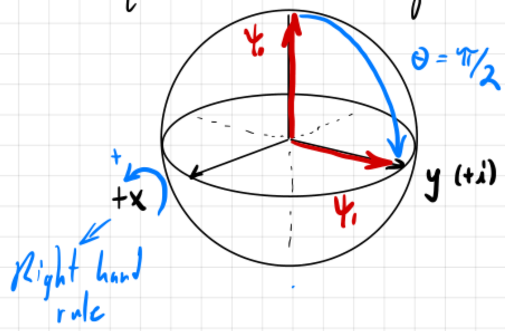


$\cos(-\pi/4) = +\frac{1}{\sqrt{2}}$

$\sin(-\pi/4) = \frac{-1}{\sqrt{2}}$

$-i\cdot\sin(-\pi/4) = \frac{+i}{\sqrt{2}}$ ✓

Equivalent to rotations on a sphere



Euler Formula for Matrices: $e^{-i\theta\cdot\hat{\sigma}_i/2} = \cos(\theta/2)\cdot\hat{\mathbb{1}} - i\sin(\theta/2)\cdot\hat{\sigma}_i$

$$X(\theta=\pi/2) = e^{-i\theta\cdot\hat{\sigma}_x/2} = \cos(\theta/2)\,\hat{\mathbb{1}} - i\sin(\theta/2)\,\hat{\sigma}_x$$

$$= \cos(\pi/4)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - i\sin(\pi/4)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - i\left(\frac{-1}{\sqrt{2}}\right)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix} = \boxed{BS}$$ ✓

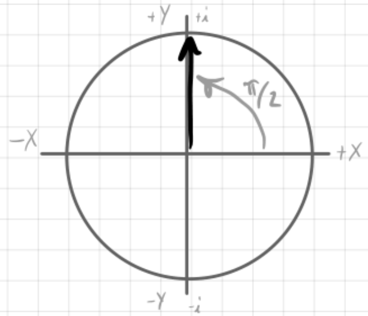
Euler Formula for Matrices: $e^{-i\theta\cdot\hat{\sigma}_i/2} = \cos(\theta/2)\cdot\hat{\mathbb{1}} - i\sin(\theta/2)\cdot\hat{\sigma}_i$

$$Z(\theta=\pi) = e^{-i\theta\,\hat{\sigma}_z/2} = \cos(\pi/2)\,\hat{\mathbb{1}} - i\sin(\pi/2)\,\hat{\sigma}_z$$

$$= \cancel{0\cdot\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}} - i(1)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$= 0 - i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$= \begin{bmatrix}-i & 0\\ 0 & +i\end{bmatrix} = \begin{bmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{bmatrix}$$



recall $e^{i\pi/2} = +i$, $e^{-i\pi/2} = -i$

$$\underbrace{e^{+i\pi/2}}_{\substack{\text{Global phase}\\ \text{(rotate both states)}}}\begin{bmatrix}-i & 0\\ 0 & +i\end{bmatrix} = \begin{bmatrix}e^{+i\pi/2 - i\pi/2} & 0\\ 0 & e^{i\pi/2 + i\pi/2}\end{bmatrix}$$

$$= \begin{bmatrix}e^{0} & 0\\ 0 & e^{i\pi}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

π phase shift!

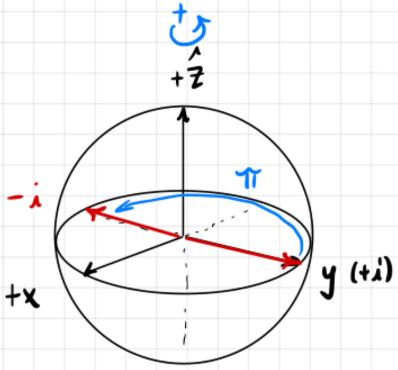
$$\psi_1 = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$

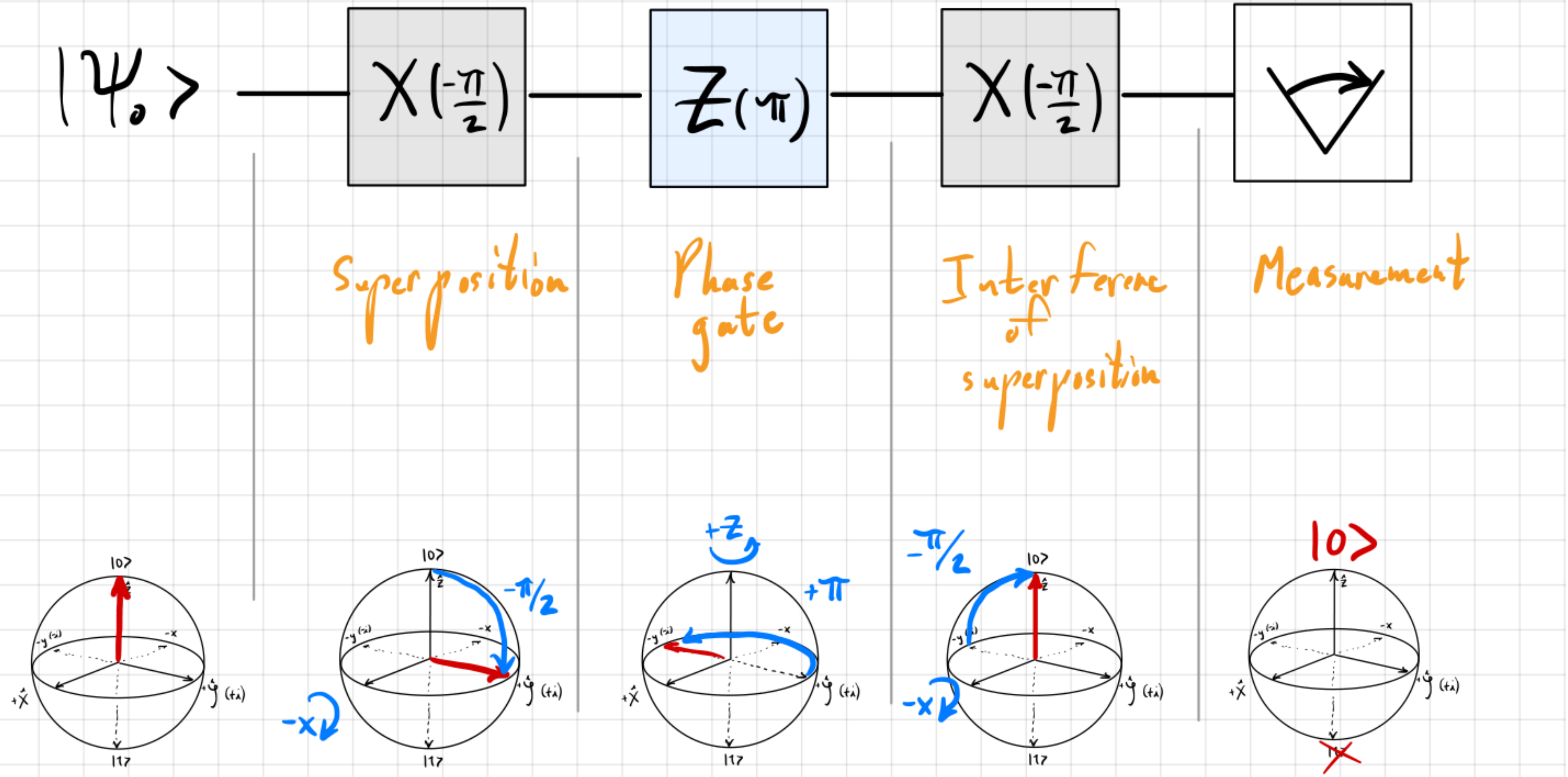
$$|\psi_2\rangle = Z(\pi)\cdot\frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$

$$= \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ +i\end{bmatrix}$$

Flip

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$





$\psi_0 = |0\rangle$

$\psi_1 = X(-\pi/2)\,\psi_0$

$\psi_1 = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$

$\psi_2 = Z(\pi)\,\psi_1$

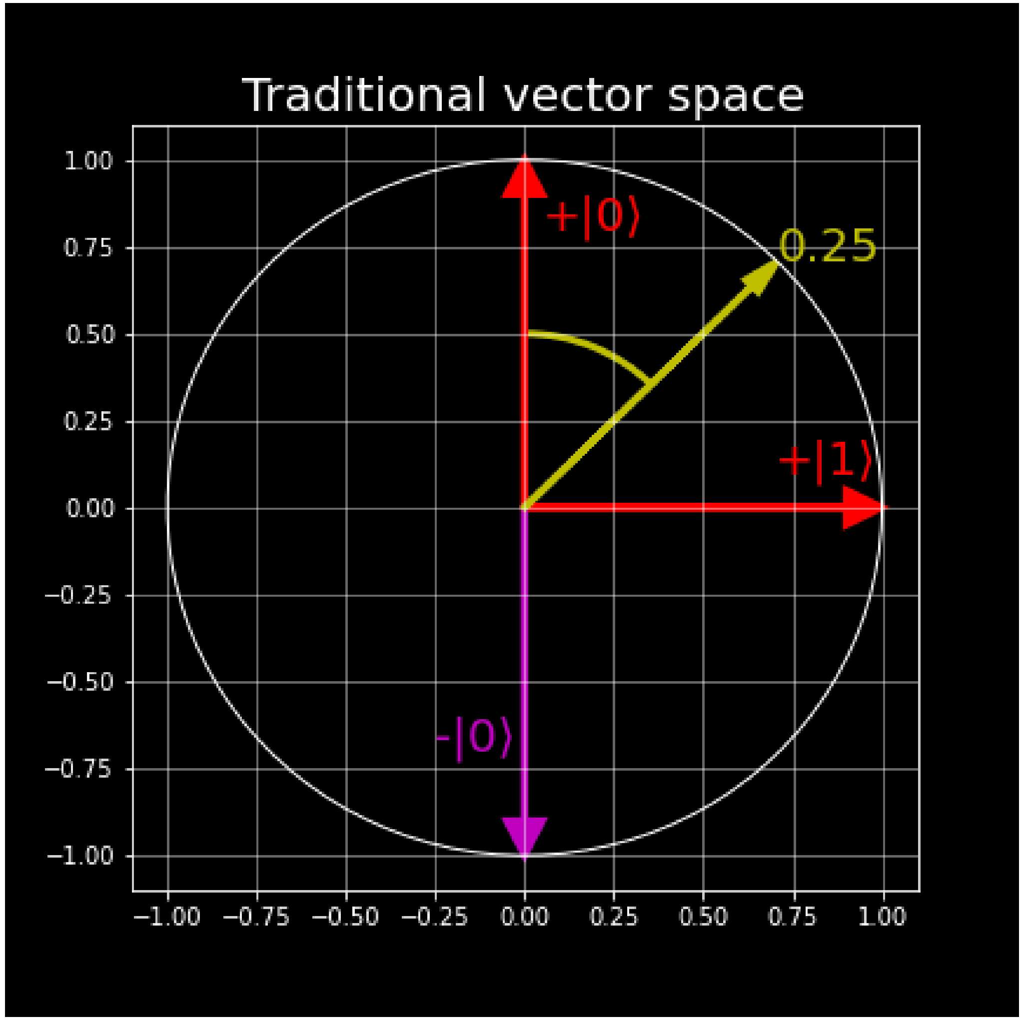
$\psi_2 = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$

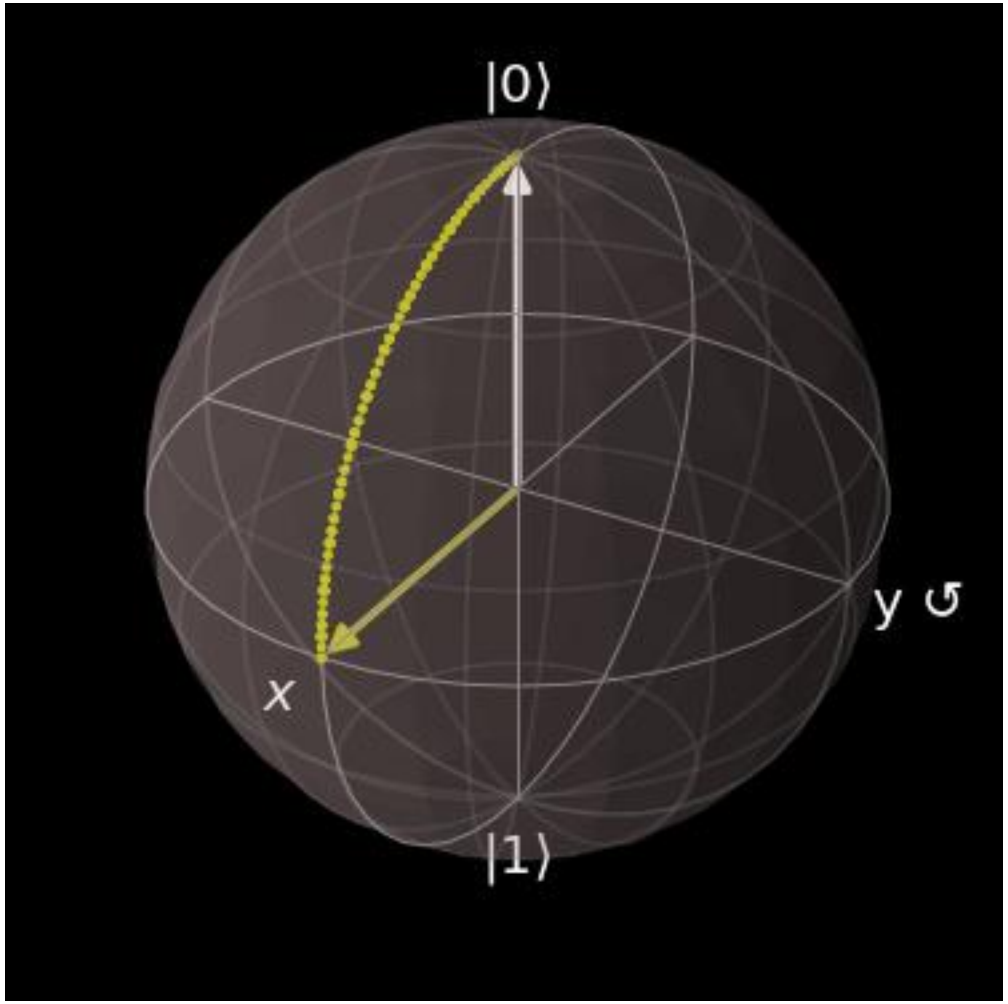
$\psi_3 = X(-\pi/2)\,\psi_2$

$\psi_3 = |0\rangle$

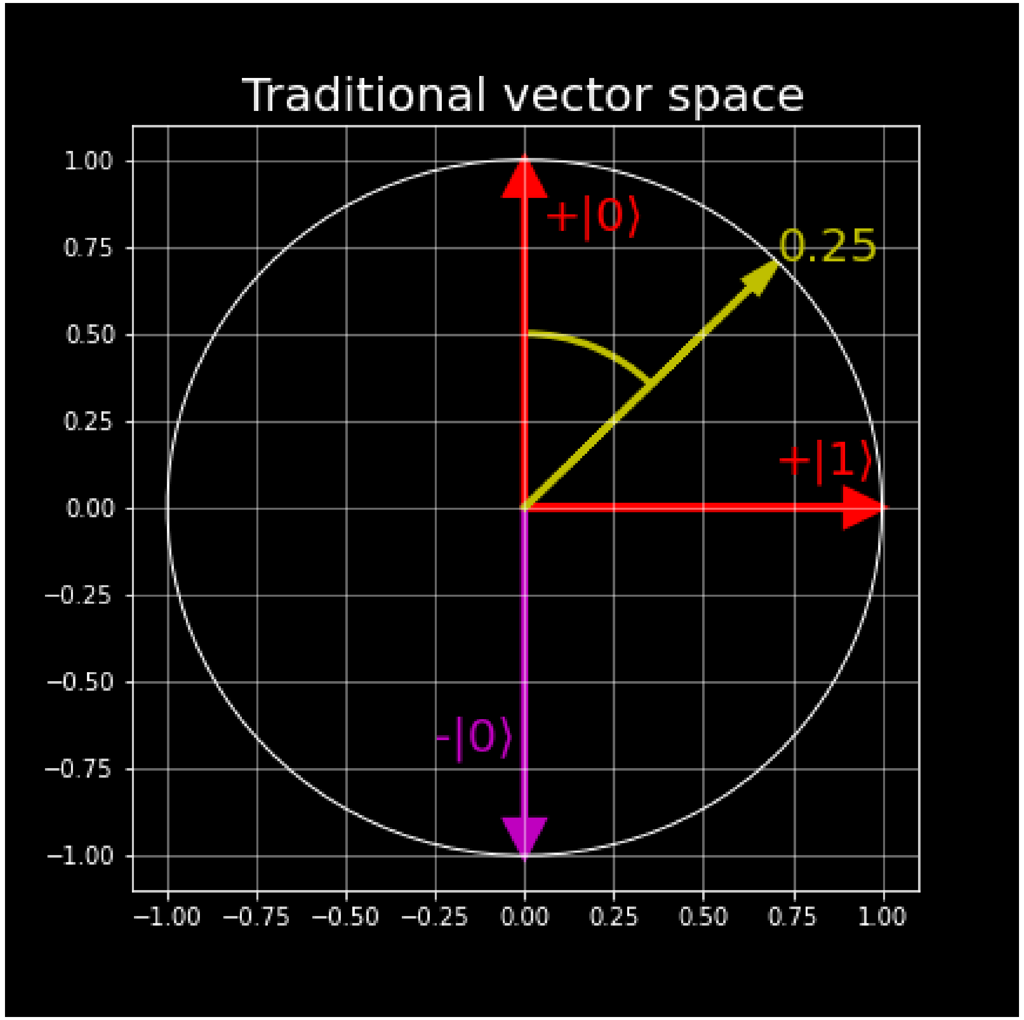
$|\psi|^2 = \langle 0|0\rangle = 1$

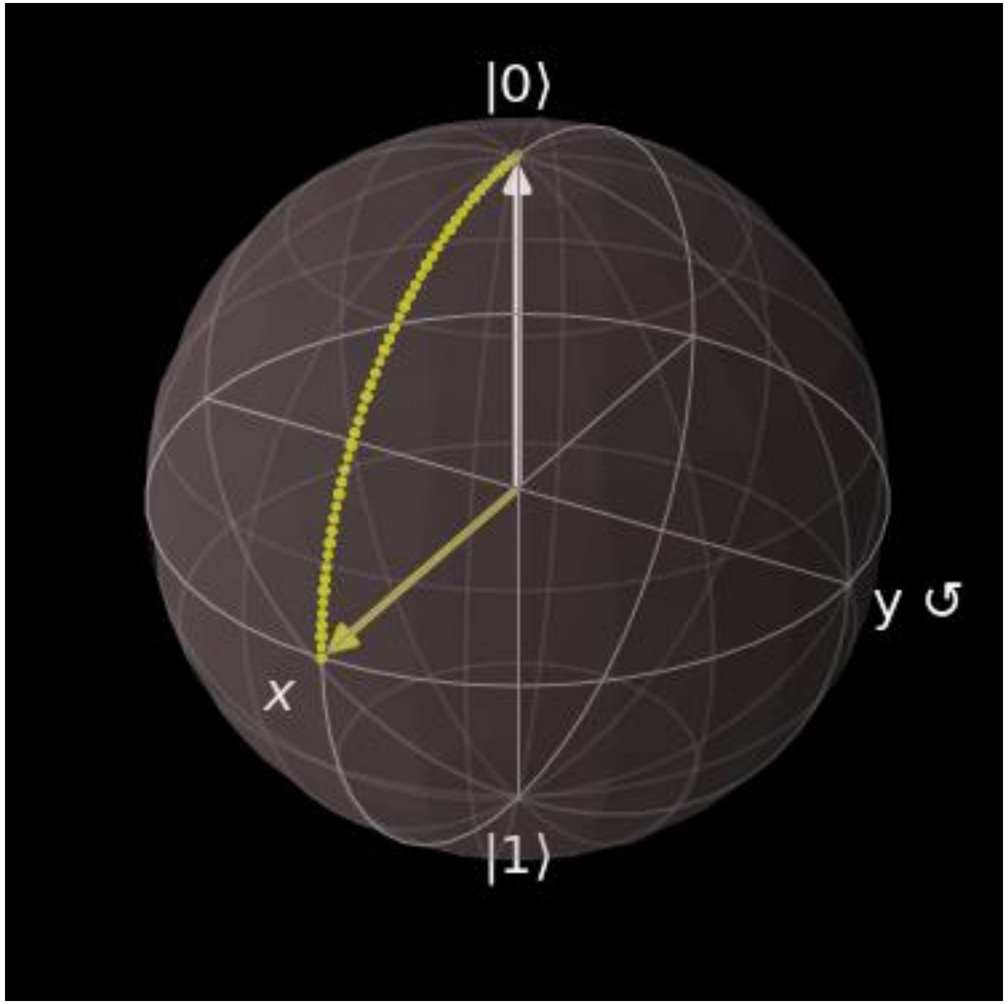
Cartesian 2D Coordinates vs. Bloch Sphere



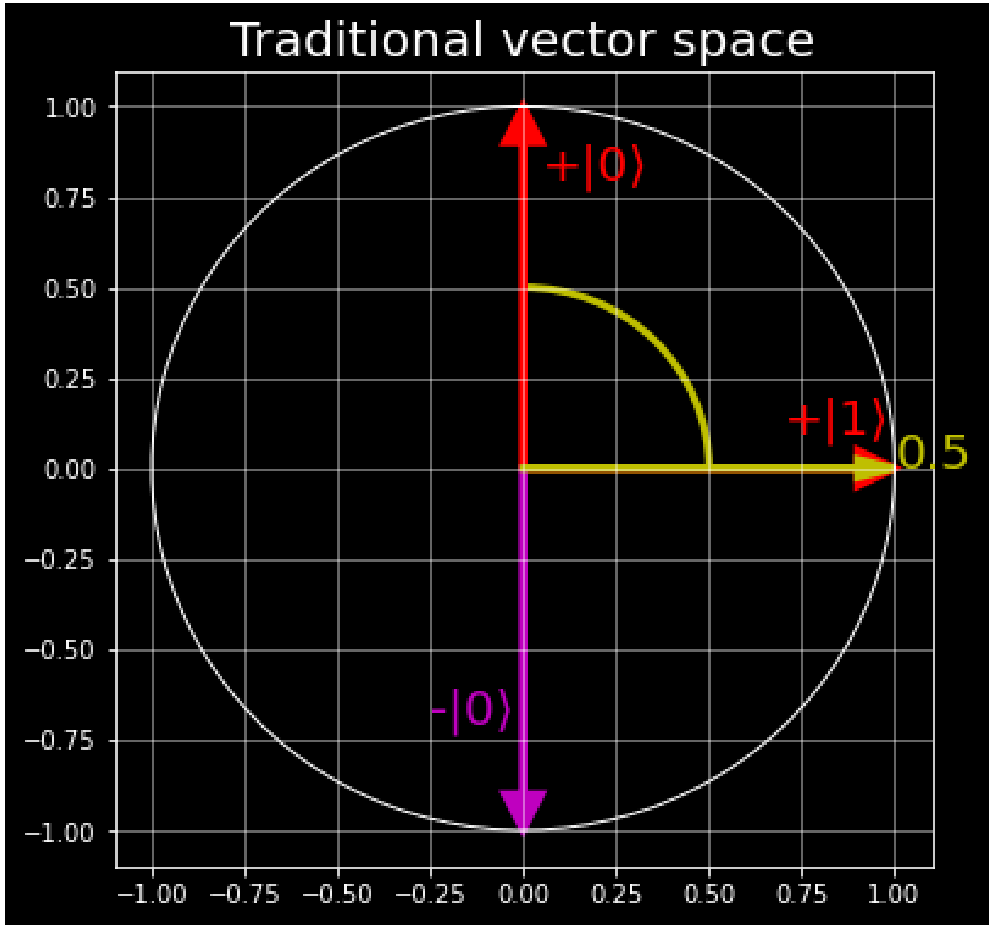


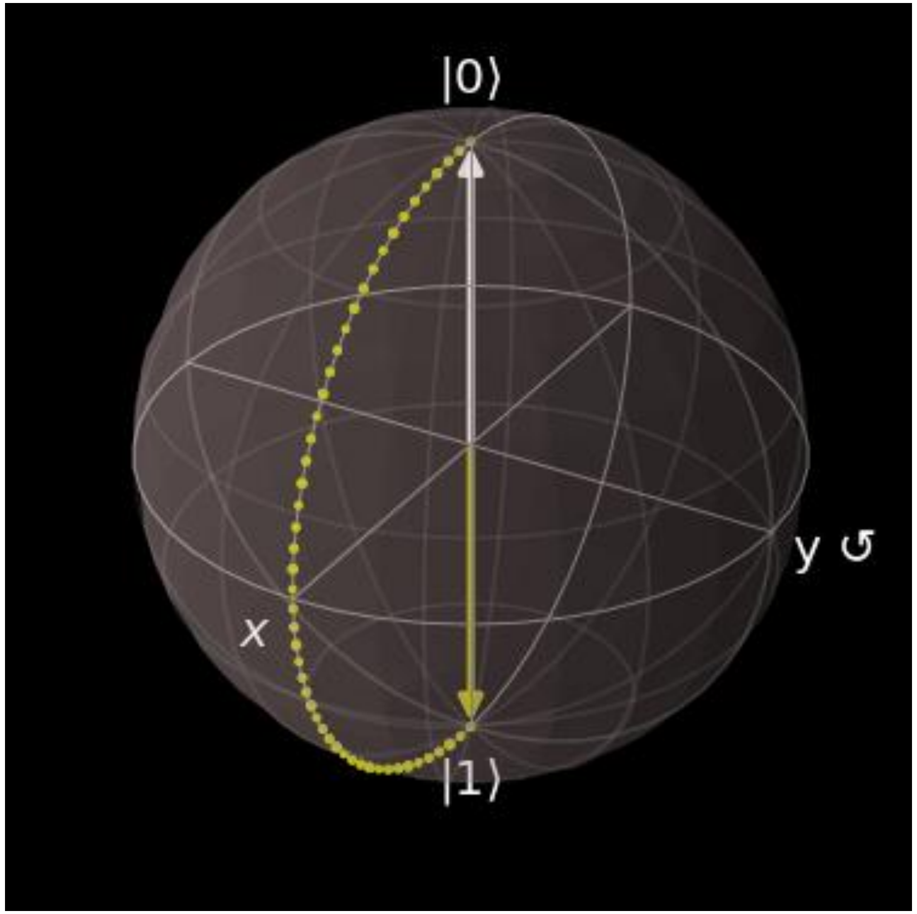
Cartesian 2D Coordinates vs. Bloch Sphere





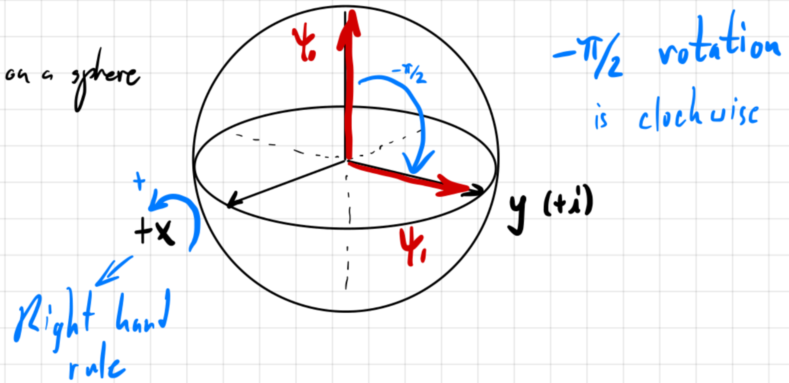
Cartesian 2D Coordinates vs. Bloch Sphere





Euler Formula for Matrices: $e^{i\theta\cdot\hat{\sigma}_i/2} = \cos(\theta/2)\cdot\hat{\mathbb{1}} + i\sin(\theta/2)\cdot\hat{\sigma}_i$

Equivalent to rotations on a sphere



$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} = \cos(-\pi/4)\,\hat{\mathbb{1}} + i\sin(-\pi/4)\,\hat{\sigma}_x$$

Axis 'steals' factor of 1/2

$$= e^{-i\underbrace{\pi/2}_{\text{Twice the angle!!!}}\cdot\hat{\sigma}_x/2}$$

Coupling states with rotation

How do we smoothly transition between these two states? We've seen that we can construct operators using outerproducts but the NOT operator is discrete, flipping the state entirely.

Naively, we may suppose that rotating the state vector about this sphere will transition us from one state (down) to the other state (up) smoothly. And we'd be right!

We shall see later that this is true more explicitly, but it does turn out that coupling these two states together is equivalent to a rotation about the Bloch sphere.

Generally for any axis P = {X,Y,Z}

$$R_p(\theta) = e^{(-i\theta P/2)} = \cos(\theta/2)I - i\sin(\theta/2)P$$

Which follows from Euler's formula:

$e^{i\theta} = \cos\theta + i\sin\theta$

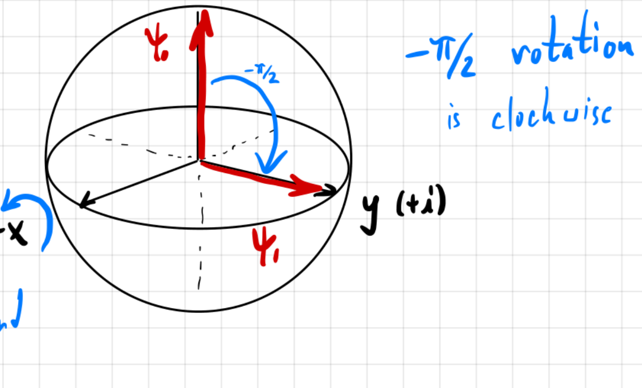
We can choose to rotate states on the Bloch sphere about any of the axes (X,Y,Z). However, if we want to rotate from one state (in the Z basis) to another state(in the Z basis) then we must pick an axis other than Z (since that will trivially not change the probability of finding the qubit in each state. To rotate the state from one to the other we must rotate about either the 'X axis' or the 'Y axis'.

Generically for any axis P = {X,Y,Z} on the Bloch Sphere:

$$R_p(\theta) = e^{(-i\theta P/2)} = \cos(\theta/2)I - i\sin(\theta/2)P$$

Euler Formula for Matrices: $e^{i\theta\cdot\hat{\sigma}_i/2} = \cos(\theta/2)\cdot\hat{\mathbb{1}} + i\sin(\theta/2)\cdot\hat{\sigma}_i$

Equivalent to rotations on a sphere



$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} = \cos(-\pi/4)\,\hat{\mathbb{1}} + i\sin(-\pi/4)\,\hat{\sigma}_x$$

Axis 'steals' factor of 1/2

$$= e^{-i\underbrace{\pi/2}_{\text{Twice the angle!!!}}\cdot\hat{\sigma}_x/2}$$

Arbitrary angle of rotation

We can also rotate the state about the other axes of the Bloch Sphere.

Generically for any axis P = {X,Y,Z} on the Bloch Sphere:

$$R_p(\theta) = e^{(-i\theta P/2)} = \cos(\theta/2)I - i\sin(\theta/2)P$$

Pauli Matrices

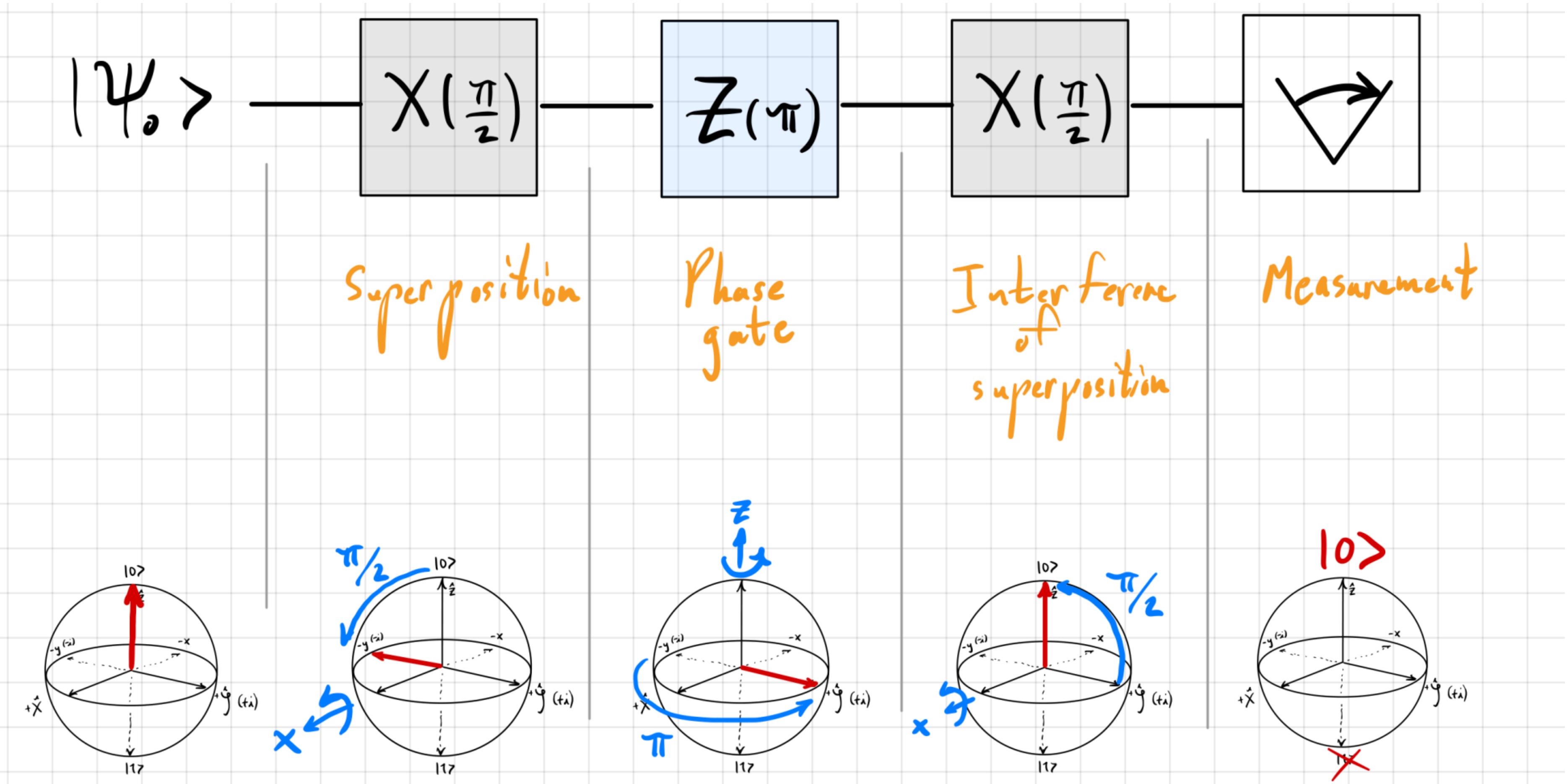
$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

The rotations of the Bloch sphere about the Cartesian axes in the Bloch basis are then given by:

$$R_x(\theta) = e^{(-i\theta X/2)} = \cos(\theta/2)I - i\sin(\theta/2)X = \begin{bmatrix}\cos\theta/2 & -i\sin\theta/2\\ -i\sin\theta/2 & \cos\theta/2\end{bmatrix}$$

$$R_y(\theta) = e^{(-i\theta Y/2)} = \cos(\theta/2)I - i\sin(\theta/2)Y = \begin{bmatrix}\cos\theta/2 & -\sin\theta/2\\ \sin\theta/2 & \cos\theta/2\end{bmatrix}$$

$$R_z(\theta) = e^{(-i\theta Z/2)} = \cos(\theta/2)I - i\sin(\theta/2)Z = \begin{bmatrix}e^{-i\theta/2} & 0\\ 0 & e^{i\theta/2}\end{bmatrix}$$



$\psi_0 = |0\rangle$

$\psi_1 = X(\pi/2)\,\psi_0$

$\psi_1 = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$

$\psi_2 = Z(\pi)\,\psi_1$

$\psi_2 = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$

$\psi_3 = X(\pi/2)\,\psi_2$

$\psi_3 = |0\rangle$

$|\psi|^2 = \langle 0|0\rangle = 1$

Example - rotations about the X basis

$$R_x(\theta) = e^{(-i\theta X/2)} = \cos(\theta/2)I - i\sin(\theta/2)X = \begin{pmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

If we want to rotate by π about X then this becomes

$$R_x(\pi) = e^{(-i\pi X/2)} = \cos(\pi/2)I - i\sin(\pi/2)X = -iX = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(Notice that this rotation causes the vector to become complex.)

We'll see below that this evolution (rotation) of the state (vector) is dictated by the time dependent Schrodinger equation. But first let's try a rotation.

If we rotate by $-\frac{\pi}{2}$:

$$R_x(-\pi/2) = e^{i\frac{\pi}{2}\frac{X}{2}} = \cos(-\pi/4)I - i\sin(-\pi/4)X = \frac{I}{\sqrt{2}} + \frac{iX}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

Which is our beam splitter!

We see that we actually have been applying a *negative* rotation about the X axis. The 'right hand rule' tells us that counter clockwise about the x axis is positive, so clockwise is negative.

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Time Dependent Schrödinger Eq.

$$i\hbar \frac{d\vec{\Psi}}{dt} = E \cdot \vec{\Psi}$$

Describes h
control th
$\Rightarrow$ How we

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

$\hat{H} =$

General Solution:

$$|\psi(t)\rangle = \underbrace{\boxed{e^{-i\hat{H}t/\hbar}}}_{\text{Rotation!!!}} \cdot \underbrace{|\psi(t=0)\rangle}_{\text{Initial state}}$$

Arbitrary angle of rotation

We can also rotate the state about the other axes of the Bloch Sphere.

Generically for any axis P = {X,Y,Z} on the Bloch Sphere:

$$\boxed{R_p(\theta) = e^{(-i\theta P/2)}} = \cos(\theta/2)I - i\sin(\theta/2)P$$

Pauli Matrices

X,Y = State (coupling) **Z = Energy/Phase**

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The rotations of the Bloch sphere about the Cartesian axes in the Bloch basis are then given by:

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$$R_y(\theta) = e^{(-i\theta Y/2)} = \cos(\theta/2)I - i\sin(\theta/2)Y = \begin{bmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

$$R_z(\theta) = e^{(-i\theta Z/2)} = \cos(\theta/2)I - i\sin(\theta/2)Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

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State Measurement

Measurement is an important concept in quantum mechanics. Imagine that we want to measure the qubit state after a rotation to verify that we have rotated it to another state. What do we **expect** to measure (what is the expectation value)?

We know quantum states are quantized. An electron can only ever be in one state or the other and a photon can only ever be in one cavity or the other. However, it can have a probability of being in both one *and* the other before we measure it.

It is like a coin that can only be heads or tails once it falls (never landing on edge) but while it is in the air has some probability of being both. If the qubit is 'flipped' into an equal superposition of up and down (like a coin) it will be up 50% of the time and down 50% of the time but it can only ever land heads/tails (up/down).

On the Bloch sphere we can see that the probability of being in each state is related to the projection of the state vector along the z axis. If the state (vector) is pointing up then it is more likely to be measured up. If it is pointing down then down. And if it is sideways (no component in the z direction) then it is in an equal superposition of up and down.

However, we need some observable or measureable value to determine which state we were in. For the electron this observable is the spin. The spin project operator is :

$$S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\langle s_z \rangle = \langle \Psi | S_z | \Psi \rangle$$

It projects the state onto the Z basis and multiplies by $\pm\hbar/2$ depending on the state. Now the superposition tells us the probability that we'll get $\pm\hbar/2$.

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Measurement and Expectation Values

Expectation value of spin:
$\langle S_z \rangle = \langle \Psi | \mathrm{S_z} | \Psi \rangle$

Recall photons:
$\langle I \rangle = \langle \vec{\mathrm{E}} | \vec{\mathrm{E}} \rangle$

Plank's Constant
$$\frac{h}{2\pi} = \hbar = 1 \times 10^{-34}\, J \cdot s$$
Energy (Joules) = $\hbar\omega = hf$

$$\hat{S}_z \equiv \frac{\hbar}{2}\hat{\sigma}_z \equiv \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\langle 0 | S_z | 0 \rangle = \frac{\hbar}{2}(1 \quad 0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}(1 \quad 0)\begin{pmatrix} 1 * 1 + 0 * 0 \\ 0 * 1 + (-1) * 0 \end{pmatrix} = \frac{\hbar}{2}(1 \quad 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}$$

$$\langle 1 | S_z | 1 \rangle = \frac{\hbar}{2}(0 \quad 1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}(0 \quad 1)\begin{pmatrix} 1 * 0 + 0 * 1 \\ 0 * 0 + (-1) * 1 \end{pmatrix} = \frac{\hbar}{2}(0 \quad 1)\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \frac{-\hbar}{2}$$

Pauli Matrices (Spin Matrices)

$$\hat{\sigma}_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{\sigma}_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Projection Operators

$$\hat{\sigma}_z \equiv |0\rangle\langle 0| - |1\rangle\langle 1| \qquad \hat{\sigma}_x \equiv |0\rangle\langle 1| + |1\rangle\langle 0| \qquad \hat{\sigma}_y \equiv -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$

Changes phase **Changes state** **Changes state and phase!**

Pauli Matrices and rotations about the Bloch Sphere

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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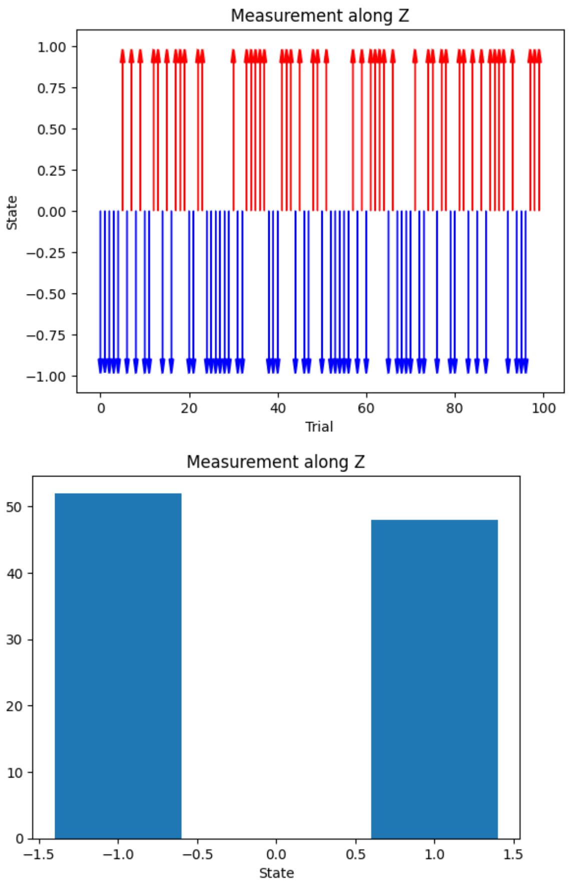
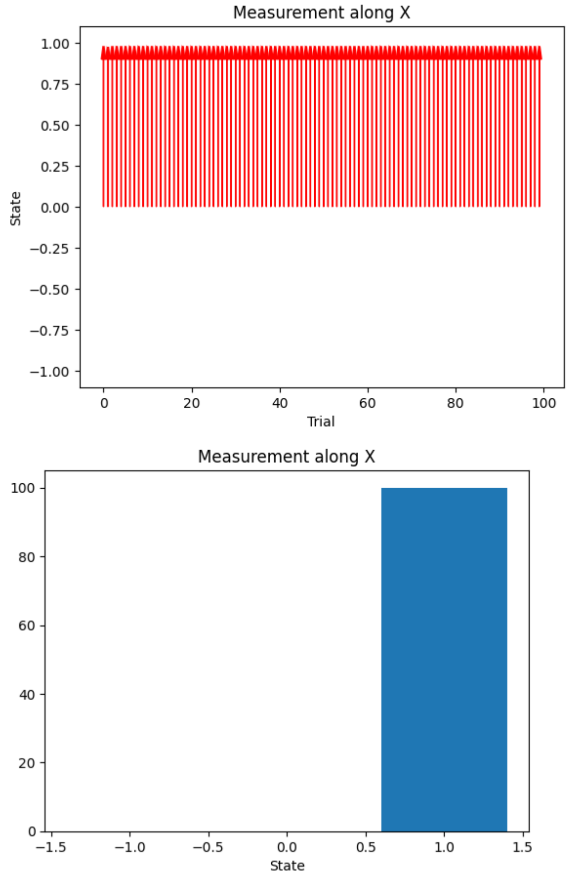
Measurement Basis

- Measure |+⟩ along Z:



- Measure |+⟩ along X:



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$$\hat{S}_z \equiv \frac{\hbar}{2}\hat{\sigma}_z \equiv \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\langle + | S_z | + \rangle = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 * \frac{1}{\sqrt{2}} + 0 * \frac{1}{\sqrt{2}} \\ 0 * \frac{1}{\sqrt{2}} + (-1) * \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = 0$$