Full Derivation of Right-Invariant Dynamics

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I. Right Invariant Dynamics for World-centric Estimator

The state we are estimating includes the robot's pose, velocity and the position of contacts measured in the world frame. It also includes a vector of IMU biases.

$$
\begin{aligned}
\mathbf{X}_t &\triangleq \begin{bmatrix} \mathbf{R}_t & \mathbf{v}_t & \mathbf{p}_t & \mathbf{d}_t \\ \mathbf{0}_{1,3} & 1 & 0 & 0 \\ \mathbf{0}_{1,3} & 0 & 1 & 0 \\ \mathbf{0}_{1,3} & 0 & 0 & 1 \end{bmatrix} \\
\boldsymbol{\theta}_t &\triangleq \begin{bmatrix} \mathbf{b}_t^g \\ \mathbf{b}_t^a \end{bmatrix}
\end{aligned} \tag{1}
$$

Using a "strap-down" inertial model, the continuous-time dynamics can be written as:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}_t &= \begin{bmatrix} \mathbf{R}_t\left(\tilde{\boldsymbol{\omega}}_t - \mathbf{b}_t^g\right)_\times & \mathbf{R}_t(\tilde{\mathbf{a}}_t - \mathbf{b}_t^a) + \mathbf{g} & \mathbf{v}_t & \mathbf{0}_{3,1} \\ \mathbf{0}_{1,3} & 0 & 0 & 0 \\ \mathbf{0}_{1,3} & 0 & 0 & 0 \\ \mathbf{0}_{1,3} & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \mathbf{R}_t & \mathbf{v}_t & \mathbf{p}_t & \mathbf{d}_t \\ \mathbf{0}_{1,3} & 1 & 0 & 0 \\ \mathbf{0}_{1,3} & 0 & 1 & 0 \\ \mathbf{0}_{1,3} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (\mathbf{w}_t^g)_\times & \mathbf{w}_t^a & \mathbf{0}_{3,1} & \boldsymbol{h}_R(\tilde{\boldsymbol{\alpha}}_t)\mathbf{w}_t^v \\ \mathbf{0}_{1,3} & 0 & 0 & 0 \\ \mathbf{0}_{1,3} & 0 & 0 & 0 \\ \mathbf{0}_{1,3} & 0 & 0 & 0 \end{bmatrix} \\
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\theta}_t &= \begin{bmatrix} \mathbf{w}_t^{bg} \\ \mathbf{w}_t^{ba} \end{bmatrix}
\end{aligned} \tag{2}
$$

If we use the right invariant error, $\boldsymbol{\eta}_t^r = \bar{\mathbf{X}}_t \mathbf{X}_t^{-1} = \exp(\boldsymbol{\xi}_t)$, along with the parameter error, $\boldsymbol{\zeta}_t \triangleq \bar{\boldsymbol{\theta}}_t - \boldsymbol{\theta}_t$, then the log-linear error dynamics can be derived as:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{R}}_t \mathbf{R}_t^\mathsf{T}\right) &= \bar{\mathbf{R}}_t\left(\tilde{\boldsymbol{\omega}}_t - \bar{\mathbf{b}}_t^g\right)_\times \mathbf{R}_t^\mathsf{T} + \bar{\mathbf{R}}_t\left(\mathbf{R}_t\left(\tilde{\boldsymbol{\omega}}_t - \mathbf{b}_t^g - \mathbf{w}_t^g\right)_\times\right)^\mathsf{T} \\
&= \bar{\mathbf{R}}_t\left(\tilde{\boldsymbol{\omega}}_t - \bar{\mathbf{b}}_t^g\right)_\times \bar{\mathbf{R}}_t^\mathsf{T}\boldsymbol{\eta}_t^R - \bar{\mathbf{R}}_t\left(\tilde{\boldsymbol{\omega}}_t - \mathbf{b}_t^g - \mathbf{w}_t^g\right)_\times \bar{\mathbf{R}}_t^\mathsf{T}\boldsymbol{\eta}_t^R \\
&= \bar{\mathbf{R}}_t\left(-\boldsymbol{\zeta}_t^g + \mathbf{w}_t^g\right)_\times \bar{\mathbf{R}}_t^\mathsf{T}\boldsymbol{\eta}_t^R \\
&= \left(\bar{\mathbf{R}}_t\left(-\boldsymbol{\zeta}_t^g + \mathbf{w}_t^g\right)\right)_\times \boldsymbol{\eta}_t^R \\
&\approx \left(\bar{\mathbf{R}}_t\left(-\boldsymbol{\zeta}_t^g + \mathbf{w}_t^g\right)\right)_\times \left(\mathbf{I} + \left(\boldsymbol{\xi}_t^R\right)_\times\right) \\
&\approx \left(\bar{\mathbf{R}}_t\left(\mathbf{w}_t^g - \boldsymbol{\zeta}_t^g\right)\right)_\times
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{v}}_t - \bar{\mathbf{R}}_t \mathbf{R}_t^\mathsf{T}\mathbf{v}_t\right) &= \bar{\mathbf{R}}_t(\tilde{\mathbf{a}}_t - \bar{\mathbf{b}}_t^a) + \mathbf{g} - \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{R}}_t \mathbf{R}_t^\mathsf{T}\right)\mathbf{v}_t + \bar{\mathbf{R}}_t \mathbf{R}_t^\mathsf{T}\left(\mathbf{R}_t(\tilde{\mathbf{a}}_t - \mathbf{b}_t^a - \mathbf{w}_t^a) + \mathbf{g}\right)\right) \\
&= \bar{\mathbf{R}}_t(\tilde{\mathbf{a}}_t - \bar{\mathbf{b}}_t^a) + \mathbf{g} - \left(\bar{\mathbf{R}}_t\left(-\boldsymbol{\zeta}_t^g + \mathbf{w}_t^g\right)\right)_\times \boldsymbol{\eta}_t^R \mathbf{v}_t - \bar{\mathbf{R}}_t(\tilde{\mathbf{a}}_t - \mathbf{b}_t^a - \mathbf{w}_t^a) - \bar{\mathbf{R}}_t \mathbf{R}_t^\mathsf{T}\mathbf{g} \\
&= \bar{\mathbf{R}}_t(\tilde{\mathbf{a}}_t - \bar{\mathbf{b}}_t^a) + \mathbf{g} - \left(\bar{\mathbf{R}}_t\left(-\boldsymbol{\zeta}_t^g + \mathbf{w}_t^g\right)\right)_\times (\bar{\mathbf{v}}_t - \boldsymbol{\eta}_t^v) - \bar{\mathbf{R}}_t(\tilde{\mathbf{a}}_t - \mathbf{b}_t^a - \mathbf{w}_t^a) - \boldsymbol{\eta}_t^R\mathbf{g} \\
&= \bar{\mathbf{R}}_t(-\boldsymbol{\zeta}_t^a + \mathbf{w}_t^a) + \mathbf{g} - \boldsymbol{\eta}_t^R\mathbf{g} - \left(\bar{\mathbf{R}}_t\left(-\boldsymbol{\zeta}_t^g + \mathbf{w}_t^g\right)\right)_\times (\bar{\mathbf{v}}_t - \boldsymbol{\eta}_t^v) \\
&\approx \bar{\mathbf{R}}_t(-\boldsymbol{\zeta}_t^a + \mathbf{w}_t^a) + \mathbf{g} - \left(\mathbf{I} + \left(\boldsymbol{\xi}_t^R\right)_\times\right)\mathbf{g} - \left(\bar{\mathbf{R}}_t\left(-\boldsymbol{\zeta}_t^g + \mathbf{w}_t^g\right)\right)_\times (\bar{\mathbf{v}}_t - \boldsymbol{\xi}_t^v) \\
&\approx (\mathbf{g})_\times \boldsymbol{\xi}_t^R + \bar{\mathbf{R}}_t(\mathbf{w}_t^a - \boldsymbol{\zeta}_t^a) + (\bar{\mathbf{v}}_t)_\times \bar{\mathbf{R}}_t\left(\mathbf{w}_t^g - \boldsymbol{\zeta}_t^g\right)
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{p}}_t - \bar{\mathbf{R}}_t\mathbf{R}_t^\mathsf{T}\mathbf{p}_t\right) &= \bar{\mathbf{v}}_t - \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{R}}_t\mathbf{R}_t^\mathsf{T}\right)\mathbf{p}_t + \bar{\mathbf{R}}_t\mathbf{R}_t^\mathsf{T}\mathbf{v}_t\right) \\
&= \bar{\mathbf{v}}_t - \left(\bar{\mathbf{R}}_t\left(-\boldsymbol{\zeta}_t^g + \mathbf{w}_t^g\right)\right)_\times \boldsymbol{\eta}_t^R\mathbf{p}_t - \boldsymbol{\eta}_t^R\mathbf{v}_t \\
&= \bar{\mathbf{v}}_t - \left(\bar{\mathbf{R}}_t\left(-\boldsymbol{\zeta}_t^g + \mathbf{w}_t^g\right)\right)_\times \left(\bar{\mathbf{p}}_t - \boldsymbol{\eta}_t^p\right) - \left(\bar{\mathbf{v}}_t - \boldsymbol{\eta}_t^v\right) \\
&\approx \bar{\mathbf{v}}_t - \left(\bar{\mathbf{R}}_t\left(-\boldsymbol{\zeta}_t^g + \mathbf{w}_t^g\right)\right)_\times \left(\bar{\mathbf{p}}_t - \boldsymbol{\xi}_t^p\right) - \left(\bar{\mathbf{v}}_t - \boldsymbol{\xi}_t^v\right) \\
&\approx \boldsymbol{\xi}_t^v + \left(\bar{\mathbf{p}}_t\right)_\times \bar{\mathbf{R}}_t\left(\mathbf{w}_t^g - \boldsymbol{\zeta}_t^g\right)
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{d}}_t - \bar{\mathbf{R}}_t\mathbf{R}_t^\mathsf{T}\mathbf{d}_t\right) &= -\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\mathbf{R}}_t\mathbf{R}_t^\mathsf{T}\right)\mathbf{d}_t + \bar{\mathbf{R}}_t\mathbf{R}_t^\mathsf{T}\left(-\mathbf{R}_t\,\boldsymbol{h}_R(\tilde{\boldsymbol{\alpha}}_t)\mathbf{w}_t^v\right)\right) \\
&= -\left(\bar{\mathbf{R}}_t\left(-\boldsymbol{\zeta}_t^g + \mathbf{w}_t^g\right)\right)_\times \boldsymbol{\eta}_t^R\mathbf{d}_t + \bar{\mathbf{R}}_t\mathbf{R}_t^\mathsf{T}\mathbf{R}_t\,\boldsymbol{h}_R(\tilde{\boldsymbol{\alpha}}_t)\mathbf{w}_t^v \\
&= -\left(\bar{\mathbf{R}}_t\left(-\boldsymbol{\zeta}_t^g + \mathbf{w}_t^g\right)\right)_\times \left(\bar{\mathbf{d}}_t - \boldsymbol{\eta}_t^d\right) + \bar{\mathbf{R}}_t\,\boldsymbol{h}_R(\tilde{\boldsymbol{\alpha}}_t)\mathbf{w}_t^v \\
&\approx -\left(\bar{\mathbf{R}}_t\left(-\boldsymbol{\zeta}_t^g + \mathbf{w}_t^g\right)\right)_\times \left(\bar{\mathbf{d}}_t - \boldsymbol{\xi}_t^d\right) + \bar{\mathbf{R}}_t\,\boldsymbol{h}_R(\tilde{\boldsymbol{\alpha}}_t)\mathbf{w}_t^v \\
&\approx \left(\bar{\mathbf{d}}_t\right)_\times \bar{\mathbf{R}}_t\left(\mathbf{w}_t^g - \boldsymbol{\zeta}_t^g\right) + \bar{\mathbf{R}}_t\,\boldsymbol{h}_R(\tilde{\boldsymbol{\alpha}}_t)\mathbf{w}_t^v
\end{aligned} \tag{6}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boldsymbol{\xi}_t^r \\ \boldsymbol{\zeta}_t\end{bmatrix} = \begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\bar{\mathbf{R}}_t & \mathbf{0} \\
(\mathbf{g})_\times & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\left(\bar{\mathbf{v}}_t\right)_\times \bar{\mathbf{R}}_t & -\bar{\mathbf{R}}_t \\
\mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & -\left(\bar{\mathbf{p}}_t\right)_\times \bar{\mathbf{R}}_t & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\left(\bar{\mathbf{d}}_t\right)_\times \bar{\mathbf{R}}_t & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix}
\begin{bmatrix}\boldsymbol{\xi}_t^r \\ \boldsymbol{\zeta}_t\end{bmatrix} + \begin{bmatrix}\mathrm{Ad}_{\bar{\mathbf{X}}_t} & \mathbf{0}_{12\times 6} \\ \mathbf{0}_{6\times 12} & \mathbf{I}_6\end{bmatrix}
\begin{bmatrix}\mathbf{w}_t^g \\ \mathbf{w}_t^a \\ \mathbf{0} \\ \boldsymbol{h}_R(\tilde{\boldsymbol{\alpha}}_t)\mathbf{w}_t^v \\ \mathbf{w}_t^{bg} \\ \mathbf{w}_t^{ba}\end{bmatrix} \tag{7}
$$