Modeling traffic flow

Urszula Maria Starowicz
University in Stavanger

This project explores the nonlinear evolution of piecewise constant initial states in conservation laws, where solutions exhibit a combination of smooth and discontinuous waves. We investigate the mechanisms leading to discontinuities by analyzing the general conservation law:

$$u_t + f(u)_x = 0,$$

and addressing fundamental questions about solution behavior. Through the Method of Characteristics (MOC), we compute solutions and derive conditions for shock formation, identifying when and why discontinuities arise, even from smooth initial data. We characterize rarefaction and compression waves, compute exact solutions for select cases, and analyze the Rankine-Hugoniot condition to determine the speed of moving discontinuities. By solving a single Riemann problem, we examine weak and similarity solutions, the entropy condition, and shock propagation. Finally, we apply these insights to a traffic flow model, comparing analytical and numerical solutions to understand real-world implications.

1 Introduction

a) A brief summary

This project explores the simplest classical traffic flow model, formulated as a nonlinear conservation law to describe the dynamics of vehicles moving along a one-dimensional road from left to right. The study aims to analyze how the vehicle density $u(x,t)$ evolves over time from an initial distribution $u_0(x)$. Some of the examples of the similar models are models showing traffic congestion patterns, particularly scenarios where a high density of vehicles leads to persistently low flux and how increase of flux removes the high density of vehicles. However, the main investigated topics in this project are how traffic flow can be optimized to reduce congestion by adjusting driving behavior—such as maintaining optimal distances to preceding vehicles—to either prevent or dissolve traffic jams.

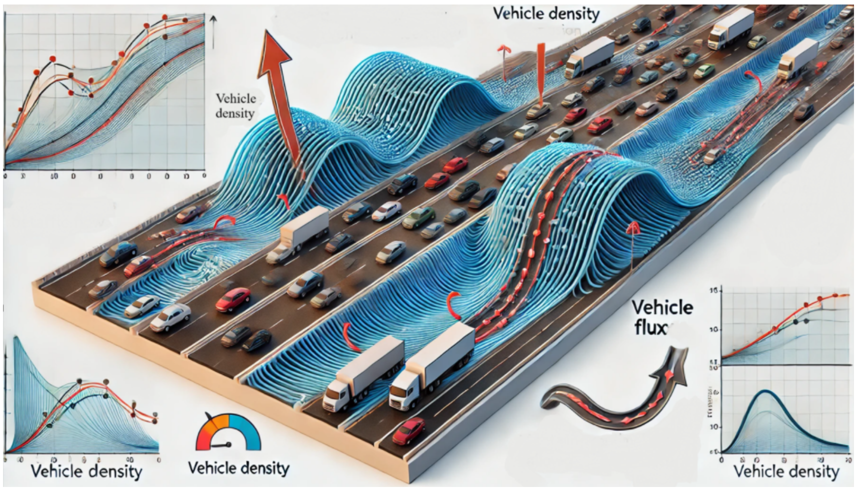

Figure 1: Nice picture to visualize situation on the road

b) **The mathematical model**

Firstly for this project is given a nonlinear scalar conservation law:

$$u_t + f(u)_x = 0, \qquad (1)$$
$$u(x, t = 0) = u_0(x)$$

where:

- u is the density of cars - [number of cars/length]
- $f(u)$ is given nonlinear function

Converting equations (1) into a mathematical model needed to the problem that describes the dynamics of vehicles moving along the road, the outcome will be a mass balance equation:

$$u_t + (V(u)u)_x = 0 \qquad (2)$$

where:

- $x \in [a, b]$ - $b - a$ is a length of the road
- $V(u)$ is a function of velocity of the vehicles
- u is the number density of vehicles.

Moreover, the number of density of vehicles is in range from 0 to 1 because it is calculated:

$$u(x, t) = \frac{\tilde{u}(x, t)}{u_{max}}$$

where:

- $\tilde{u}(x, t)$ is a density in a point
- u_{max} is a maximal possible density

Going further, it can be assumed that on the road is given constant maximal velocity V_{max}. Additionally, velocity is proportional to the expression $1 - u$ - smaller density of cars allows to drive with higher speed.

$$V(u) = V_{max}(1 - u) \tag{3}$$

Setting $V_{max} = 1$ and combining equations (3) and (5), we receive:

$$u_t + (u(1 - u))_x = 0 \tag{4}$$

for $x \in [a, b]$.

Finally, on the road vehicles move from left to right, which gives a natural boundary condition:

$$u\Big|_{x=a} = u_{in} \tag{5}$$

The initial condition is, as given in equations (1):

$$u(x, t = 0) = u_0(x)$$

Thanks to that model, we should be able to solve main problem of this project.

c) **Underlying assumption of $V(u) = V_{\mathbf{max}}(1 - u)$**

Main assumption in the project is that the problem can be modeled by mass balance equation $u_t + (V(u)u)_x = 0$. This is obtained by differentiating the function $f(u)_x$ in the equation (1) and to do so we assume that the flux function does not change the sign when differentiated to $f''(u)$. Having received equation in form:

$$u_t + f'(u)u_x = 0$$

Considering u along the path $(x(t), t)$, we obtain:

$$x'(t) = f'(u(x(t)))$$

where initial condition is $x(t = 0) = x_0$.

Having this path, we can look into varying of u:

$$\frac{d}{dt}u(x(t), t) = u_x\frac{dx}{dt} + u_t\frac{dt}{dt} = u_x f'(u) + u_t = 0$$

Knowing that u is a solution to the equation (1), it can be deducted that u is a constant along the path $(x(t), t)$. This results in that $x'(t) = f'(u(X(t), t))$ must be also constant. But for all of this equation to work the must be an assumption made that u is a smooth function. Then we can transform $x'(t) = f'(u(X(t), t) = V(u)$.

Next, we can consider should be the interpretation of the velocity. The whole project is based on the road, where cars are driven with various velocity. They are creating the flow on the road that has a given maximal velocity - it can be understand as a speed limit. This number is constant and to make the calculations easier we assumed that $V_{max} = 1$.

Moreover, it can be deducted that the busier traffic is - the density of the vehicles is higher - the slower the flow on the road will be. That will be incorporated in the model as $u_{max} - u$. Since $u \in [0, 1]$, we will have $1 - u$.

d) **Research questions**

In the introduction for the project we were present by the professor with some interesting questions, such as:

- How can this model show the mechanisms that are creating the basic traffic flow?
- How the density of cars change over time?
- Where will be the peak of the density of vehicles?
- What will be the flux of the flow on the road? Where will it be high or low?
- What is the most efficient way to drive to not create a traffic jam?

Under consideration, should also be taken issues as:

- How initial condition affect the car distribution on the road?
- How the change of the velocity impact the traffic flow?
- What are the mechanism that influence the flow?
- What should be done by drivers stuck in the traffic jam to dissolve it?

2 Methods to deal with the discontinuities in hyperbolic conservation Laws

This section focuses on the behavior of solutions to hyperbolic conservation laws, particularly the formation and evolution of discontinuities, such as shock waves. We will use the method of characteristics to analyze how solutions evolve along characteristic curves, highlighting when smooth solutions can transition into discontinuous ones. The Rankine-Hugoniot condition will be introduced to determine the speed at which shock waves propagate. We will also explore the Riemann problem, which addresses the evolution of solutions with initial discontinuities, and examine different types of solutions, including shock and continuous solutions. Finally, we will discuss the role of the entropy condition in selecting the physically relevant solutions.

2.1 MOC

Using method of characteristics, we will examine what is happening with the solution u through out the path. Mostly the focus will be put on the underlying assumption that it should be a smooth function. Lifting a bit the veil of secrecy, the solution, even with a smooth initial data $u_0(x)$, can become discontinuous and maybe later will once again will be continuous. Why will it happen and when, will be investigated in this section.

a) **Computation of a general solution $u_t + f(u)_x = 0$, given that $u(x, t = 0) = u_0(x)$**

As written in the Introduction, firstly we have to assume that the solution is smooth. The characteristic for

- $x(t)$:
$$\frac{dx(t)}{dt} = f'(u(x(t), t)) \tag{6}$$
where $x(t = 0) = x_0$.

- $u(x(t),t)$:
$$\frac{d}{dt}u(x(t),t) = u_x\frac{dx}{dt} + u_t\frac{dt}{dt} = u_x \cdot f'(u) + u_t = 0$$
In conclusion u function is constant along the path $(x(t),t)$. Hence, given that the curve $x(t)$ starts at x_0 at $t=0$ it is described by:
$$x'(t) = f'(u(x_0, t=0)) = f'(u_0(x_0))$$
And then integrating equation (12) we will get:
$$\int_{x(t=0)}^{x(t)} dx = \int_0^t f'(u_0(x_0))dt = f'(u_0(x_0))t$$
which leads to:
$$x(t) = f'(u_0(x_0))t + x_0$$

The conclusion from this derivation is that the general solution for the system of equations given in the problem task is $u(x,t) = u_0(x_0)$. Finally the solution is:

$$u(x,t) = u_0(x - f'(u(x,t)t) \tag{7}$$

To test if this solution always is correct we can check if it is the solution for the equation $u_t + f'(u)u_x = 0$. Starting with direct computing of derivatives u_t and u_x. Firstly, in equation $u(t) = \frac{\partial}{\partial t}u_0(x - f'(u(x,t)t))$ where $x - f'(u(x,t)t) = V(x,t)$. Having that we can put together those two equations and end up with $u(t) = \frac{du_0}{dV}\cdot\frac{\partial V}{\partial t}$. Knowing the relationships between two differentiated functions, namely $(u\cdot V)' = u'V + uV'$.

$$u_t = u_0'(x - f'(u(x,t)t)\frac{\partial}{\partial t}[x - f'(u(x,t)t] = -u_0(x - f'(u(x,t)t)[tf''(u(x,t))u_t + f'(u(x,t))]$$

$$u_x = u_0'(x - f'(u(x,t)t)\frac{\partial}{\partial x}[x - f'(u(x,t)t] = -u_0(x - f'(u(x,t)t)[1 - tf''(u(x,t))u_x]$$

Which concludes in:

$$f(u)_x = f'(u)u_x = u_0'(x - f'(u(x,t)t)[f'(u) - tf''(u)\cdot u_x \cdot f'(u)]$$

where $u_x \cdot f'(u) = f(u)_x$. Putting all of this into one final equation, we are getting:

$$u_t + f(u)_x = -u_0'(x - f'(u(x,t)t)\cdot[f'(u) + tf''(u)u_t] + u_0'(x - f'(u(x,t)t)[f'(u) - tf''(u)\cdot f(u)_x]$$

$$u_t + f(u)_x = -u_0(x - f'(u(x,t)t)[u_t tf''(u) + tf''(u)\cdot f(u)_x]$$
$$u_t + f(u)_x = -u_0(x - f'(u(x,t)t)\cdot t\cdot f''(u)[u_t + f(u)_x]$$

And for asserting if our solution is always true we got the equation:

$$[u_t + f(u)_x]\cdot(1 + u_0(x - f'(u(x,t)t))\cdot t\cdot f''(u)) = 0 \tag{8}$$

where from the beginning of this project we know that the condition that has to be met so the computed solution is general and always works is $u_t + f(u)_x = 0$. We are sure that this is true when $(1 + u_0(x - f'(u(x,t)t))\cdot t\cdot f''(u) \neq 0$. Unfortunately when $(1 + u_0(x - f'(u(x,t)t))\cdot t\cdot f''(u) = 0$ the statement $u_t + f(u)_x = 0$ cannot be concluded. Further on in the project we will investigate what is happening then.

b) **Computation of a formula that provides deeper understanding on when discontinuity is formatted**

To get to it let's go back to the unique solution $u(x,t) = u_0(x - f'(u_0(x_0))t) = u_0(x - f'(u(x,t))t)$.

$$u_x = \frac{\partial}{\partial x} u_0(V(x,t)) = u_0'(V)\frac{\partial V}{\partial x}$$

Since a characteristic variable is $x - f'(u(x,t))t = V(x,t)$, let's differentiate it with respect to x:

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(x - f'(u(x,t))t) = 1 - f''(u(x,t))u_x t$$

Thus,

$$u_x = u_0'(V)(1 - f''(u(x,t))u_x t)$$

Doing a little bit of calculus and solving it for u_x:

$$u_x + u_0'(x - f'(u(x,t))t) \cdot f''(u(x,t))u_x t = u_0'(x - f'(u(x,t))t)$$

$$u_x(1 + u_0'(x - f'(u(x,t))t) \cdot f''(u(x,t))t) = u_0'(x - f'(u(x,t))t)$$

Finally we are getting:

$$u_x = \frac{u_0'(x_0)}{1 + u_0'(x_0) \cdot f''(u_0(x_0))t} \tag{9}$$

This derivation essentially shows how the spatial derivative u_x evolves over time in a nonlinear wave equation, where characteristics shift according to $f'(u)$. The denominator captures the influence of the nonlinearity, leading to the possibility of shock formation when the denominator goes to zero.

Having the formula for u_x we can compute when the blow up will happen, known as the **breaking time**. It will be when the discontinuity is formed for the first time. This will happen when the denominator will be equal to 0, due to division by zero.

$$min_{x_0}[u_0'(x_0) \cdot f''(u_0(x_0))]t = -1$$

Hence, the breaking time T_b is the value of the variable t is given by:

$$T_b = \frac{-1}{min_{x_0}[u_0'(x_0) \cdot f''(u_0(x_0))]}. \tag{10}$$

To fix in memory, let us apply this formula to the example - Burger's equation. The task is to estimate when a discontinuity is first formed when we consider $u_t + f(u)_x = 0$, where $u(x, t = 0) = u_0(x)$ and additionally we are given information that $f(u) = u^2$.

Inserting $f(u) = u^2$ to the main equation:

$$u_t + (\frac{1}{2}u^2)_x = 0 \tag{11}$$

We for sure known that $f'(u) = u$. The characteristics are determined by:

$$x(t) = u_0(x_0)t + x_0 \tag{12}$$

Moreover, we consider the initial data:

$$u_0(x) = \begin{cases} 2x, & \text{if } 0 \le x \le \frac{1}{2} \\ 2(1-x), & \text{if } \frac{1}{2} < x \le 1 \end{cases}$$

Converting it to the form of the equation (12):

$$x = x_0 + t \begin{cases} 2x_0, & \text{if } 0 \le x_0 \le \frac{1}{2} \\ 2(1-x_0), & \text{if } \frac{1}{2} < x_0 \le 1 \end{cases}$$

And let's simplify it:

$$x = \begin{cases} x_0(1+2t), & \text{if } 0 \le x_0 \le \frac{1}{2} \\ x_0(1-2t)+2t, & \text{if } \frac{1}{2} < x_0 \le 1 \end{cases}$$

And solving it for x_0:

$$x_0 = \begin{cases} \frac{x}{1+2t}, & \text{if } 0 \le \frac{x}{1+2t} \le \frac{1}{2} \\ \frac{x-2t}{1-2t}, & \text{if } \frac{1}{2} < \frac{x-2t}{1-2t} \le 1 \end{cases}$$

Next, let's combine those equation with $u(x,t) = u_0(x_0)$ with the assumption that characteristics do not cross:

$$u(x,t) = \begin{cases} \frac{2x}{1+2t}, & \text{if } 0 \le x \le \frac{1}{2} + t \\ \frac{2(1-x)}{1-2t}, & \text{if } \frac{1}{2} + t < x \le 1. \end{cases}$$

That is, in view of initial condition from equation (1):

$$u(x,t) = u_0(x_0) = \begin{cases} u_0 \frac{x}{1+2t}, & \text{if } 0 \le \frac{x}{1+2t} \le \frac{1}{2} \\ u_0 \frac{x-2t}{1-2t}, & \text{if } \frac{1}{2} < \frac{x-2t}{1-2t} \le 1 \end{cases}$$

At the figure below is shown couple of different times of this solution for the problem.

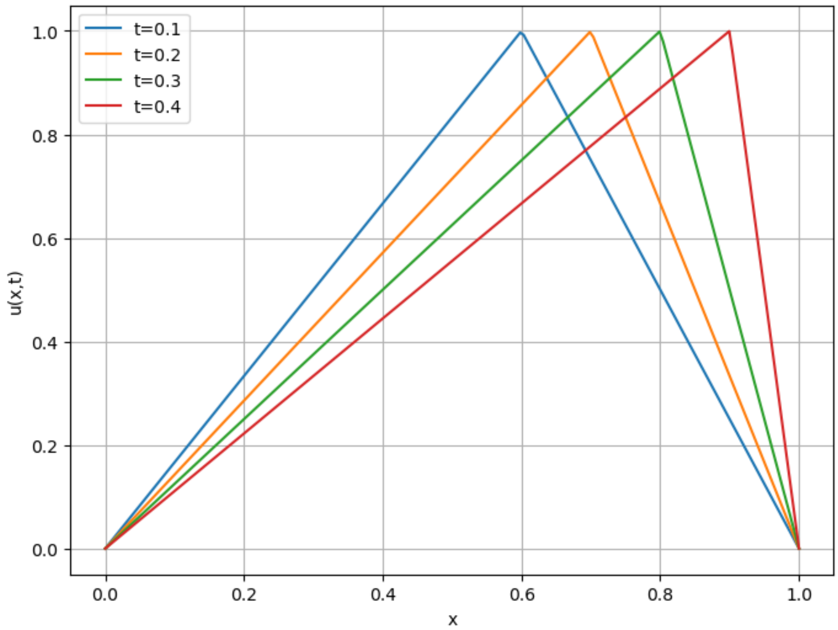

Figure 2: Solution of Burgers Equation

On the figure rarefaction waves are observed on the left side - $x \in [0, \frac{1}{2)}$. When x is bigger than $\frac{1}{2}$ then the characteristics stop spreading and starting to get closer to each other - they are compression waves. Rarefaction and compression waves will be explain in some time!

Before we assumed that the solution want blow up and we do not know what is happening when $x = \frac{1}{2}$, so this should be investigated. Since t is positive, the only option for discontinuity to occur is when:

$$1 - 2t = 0$$

$$t = \frac{1}{2}^{-}$$

c) **Rarefaction wave and compression wave**

Before introducing next complicated equations, let go back in time to the middle school mathematics classes and remind how the function formula looks like:

$$y = ax + b$$

This equation may look a bit odd among the level of complication of the previous task, but it is, at least in my understanding, the easiest way to comprehend the problem of waves. So, let's explain each part of the formula: y - how far along the function will go, a - slope or gradient, x how far up the function will go and b - value of y when $x = 0$. Now looking back on the function that we have computed for the x_0.

$$x_0 = x - f'(u_0(x_0))t$$

$$x = f'(u_0(x_0))t + x_0$$

In our equation we can liken variables form it to the general formula for the function. Variable the describes how far along the function will go - y - is x, on the vertical ax variable we have t and the initial value we have x_0. Up to this point everything is pretty much self explanatory. However, a - the slope in our case is a function $f'(u_0(x_0))$, this means that it will change. The way that it will be changing is determined by that if the function is increasing or decreasing. Firstly assume that $f''(u) > 0$.

- Increasing function Function $f'(u)$ is increasing when:

  $$u_0(x_1) < u_0(x_2)$$

  if

  $$x_1 < x_2$$

  and then the function:

  $$f'(u_0(x_1)) < f'(u_0(x_2))$$

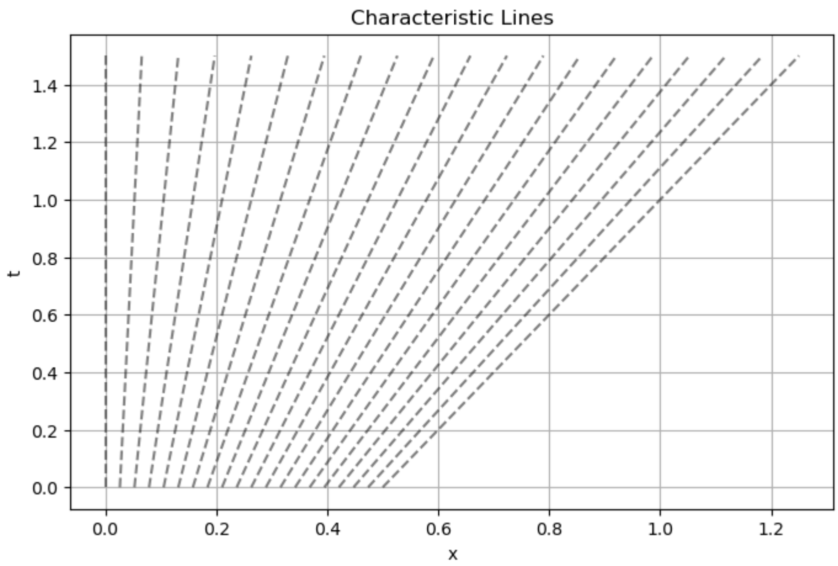

Figure 3: Rarefaction waves

On the plot above we can see that the slope is becoming sharper and sharper because the coefficient next to the variable t is getting bigger. Those characteristics are spreading and they are called a **rarefaction waves**. In this case characteristics will never cross, thus the jump will never occur.

- Decreasing function

  Function $f'(u)$ is decreasing when:

$$u_0(x_1) > u_0(x_2)$$

  if

$$x_1 < x_2$$

  and then the function:

$$f'(u_0(x_1)) > f'(u_0(x_2))$$

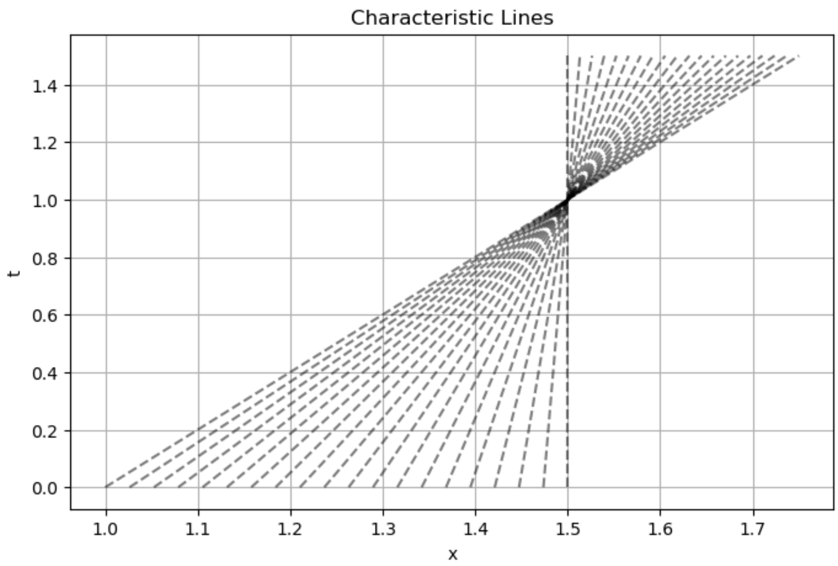

Figure 4: Compression waves

The opposite case, as shown above, is when the slope is getting smaller. The characteristics are approaching each other and will intersect at some time - the jump will occur. This behavior of the characteristics is referred to as **compression waves**.

d) **Example** $f(u) = \frac{1}{4}u^2$

$$u_t + (\frac{1}{4}u^2)_x = 0 \tag{13}$$

Firstly, derivative $f'(u) = \frac{1}{2}u$. The general solution is $u(x, t = 0) = u_0(x)$ and characteristics are determined similarly as before:

$$x(t) = \frac{1}{2}u_0(x_0)t + x_0 \tag{14}$$

The same as in the Burgers' equation the solution transforms into $u(x, t) = u_0(x_0)$, where x_0 is determined by $x = x_0 + \frac{1}{2}u_0(x_0)t$. Next we are given initial data in the exercise:

$$u_0(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} < x \leq 1 \\ 3 - 2x & \text{if } 1 < x \leq \frac{3}{2} \end{cases}$$

Thus, combining with formula for x:

$$x = x_0 + \tfrac{1}{2}t \begin{cases} 2x, & \text{if } 0 \leq x_0 \leq \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} < x_0 \leq 1 \\ 3 - 2x & \text{if } 1 < x_0 \leq \frac{3}{2} \end{cases}$$

Let's see how it looks like:

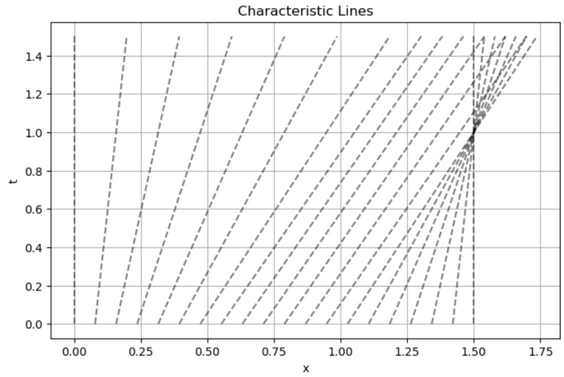

Figure 5: Characteristic lines

Lines on the left were $x \in [0, \frac{1}{2}]$ are spreading, this means that there are rarefaction waves and the solution will not blow up. When $x \in (\frac{1}{2}, 1]$ the lines are parallel to each other, so blow up also will not happen. However when $x \in (1, \frac{3}{2}]$ the characteristics are getting closer together. To go further with the solving let's see how x_0 will look like:

$$x_0 = \begin{cases} \frac{x}{1+t}, & \text{if } 0 \leq \frac{x}{1+t} \leq \frac{1}{2} \\ x - \frac{1}{2}t, & \text{if } \frac{1}{2} < x - \frac{1}{2}t \leq 1 \\ \frac{x - \frac{3}{2}t}{1-t} & \text{if } 1 < \frac{x - \frac{3}{2}t}{1-t} \leq \frac{3}{2} \end{cases}$$

Combining with $u(x,t) = u_0(x_0)$ - assuming that characteristics do not cross:

$$u(x,t) = \begin{cases} u_0 \frac{x}{1+t}, & \text{if } 0 \leq \frac{x}{1+t} \leq \frac{1}{2} \\ u_0(x - \frac{1}{2}t), & \text{if } \frac{1}{2} < x - \frac{1}{2}t \leq 1 \\ u_0 \frac{x - \frac{3}{2}t}{1-t} & \text{if } 1 < \frac{x - \frac{3}{2}t}{1-t} \leq \frac{3}{2} \end{cases}$$

That is, in view of initial data:

$$u(x,t) = \begin{cases} \frac{2x}{1+t}, & \text{if } 0 \leq x \leq \frac{1}{2}(1+t) \\ 1, & \text{if } \frac{1}{2}(1+t) < x \leq 1 + \frac{1}{2}t \\ 3 - 2(\frac{x - \frac{3}{2}t}{1-t}) & \text{if } 1 + \frac{1}{2}t < x \leq \frac{3}{2} \end{cases}$$

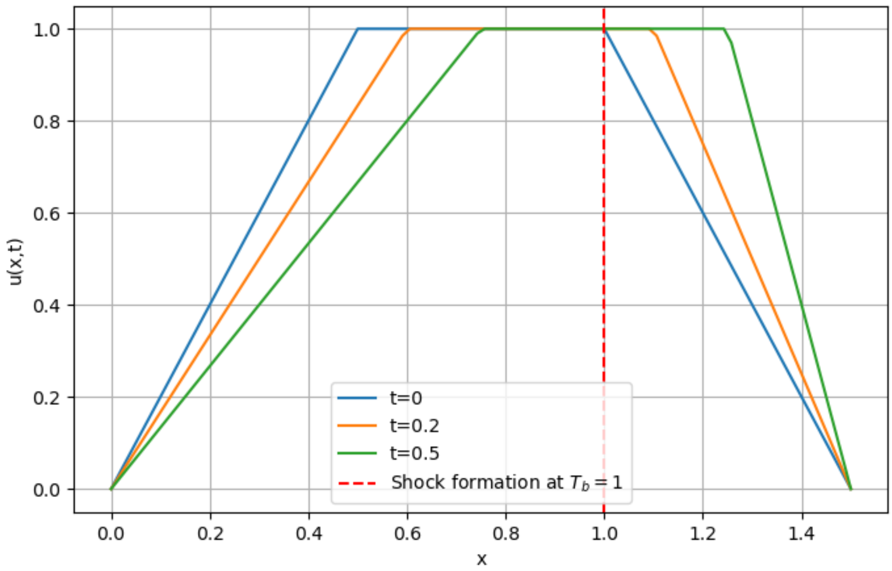

Figure 6: Plot of solution computed by the equation system for different times

Once again we assumed that the solution will not blow up, but we should investigate it. We can see that the dangerous moment is when $1 - t = 0$, so $t = 1$. Knowing that t is positive we should check for the breaking time according to the equation (10). Let's start with computing $f''(u)$.

$$f'(u) = \frac{1}{2}u$$

$$f''(u) = \frac{1}{2}$$

Next, compute $u_0'(x_0)$ from the initial data:

$$u_0'(x) = \begin{cases} 2, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0, & \text{if } \frac{1}{2} < x \leq 1 \\ -2 & \text{if } 1 < x \leq \frac{3}{2} \end{cases}$$

Now let's compute the $min_{x_0}[u_0'(x_0) \cdot f''(u_0(x_0))]t$:

$$u_0'(x_0) \cdot f''(u_0(x_0)) = \begin{cases} 2 \cdot \frac{1}{2} = 1, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 \cdot \frac{1}{2} = 0, & \text{if } \frac{1}{2} < x \leq 1 \\ -2 \cdot \frac{1}{2} = -1 & \text{if } 1 < x \leq \frac{3}{2} \end{cases}$$

Minimal value from the above equation system is -1. Having that we can compute T_b:

$$T_b = \frac{-1}{min_{x_0}[u_0'(x_0) \cdot f''(u_0(x_0))]} = \frac{-1}{-1} = 1.$$

The breaking time is $T_b = 1$.

e) **Comparison of discrete version of the problem $u_t + f(u)_x = 0$, given that $u(x, t = 0) = u_0(x)$ with the exact solution from the previous task**

To develop the numerical solution the Rusanov scheme was used. The Rusanov scheme is a numerical method for solving hyperbolic conservation laws using a finite volume approach. It approximates fluxes at cell interfaces while ensuring stability through numerical dissipation.

The numerical flux at the interface $x_{j+\frac{1}{2}}$ is computed as:

$$F^n_{j+\frac{1}{2}} = \frac{1}{2}\left(f(U^n_j) + f(U^n_{j+1})\right) - \frac{1}{2}s_{j+\frac{1}{2}}(U^n_{j+1} - U^n_j),$$

where $f(U)$ is the flux function, and $s_{j+\frac{1}{2}}$ is the dissipation coefficient, that is determined by the maximum wave speed:

$$s_{j+\frac{1}{2}} = \max\left|\frac{df}{dU}(U^n_j), \frac{df}{dU}(U^n_{j+1})\right|.$$

This ensures local adaptation to the problem's wave speeds. The solution is updated using:

$$U^{n+1}_j = U^n_j - \frac{\Delta t}{\Delta x}\left(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}\right),$$

where Δt and Δx are the time step and spatial resolution. This formulation maintains conservation while reducing numerical dissipation.

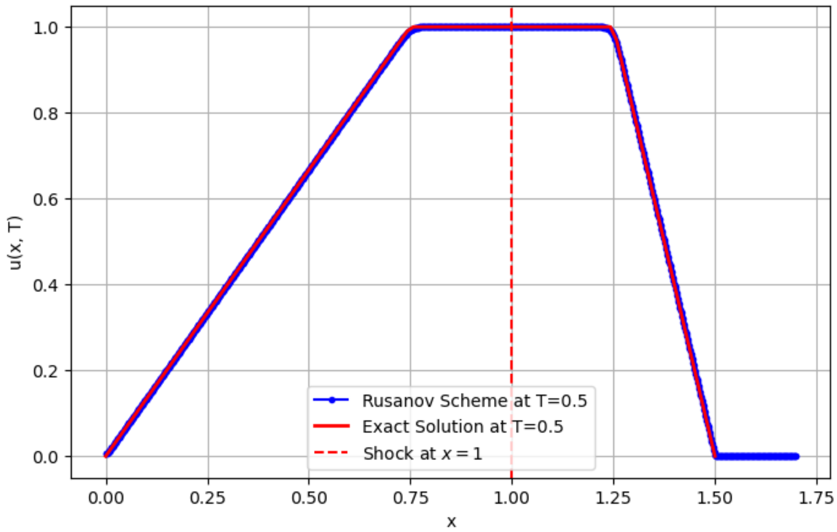

Figure 7: Comparison of the exact and numerical solution

The comparison of the exact and numerical solution was shown above at the time $t = 0.1$. As we can see the scheme is suited perfectly to the exact solution. The grid in this case was 1000.

2.2 Rankine-Hugoniot condition

The Rankine-Hugoniot condition is crucial for understanding shock waves in conservation laws, particularly in the equation $u_t + f(u)_x = 0$, where discontinuities arise. It determines the speed of the shock wave, relating it to the flux function $f(u)$ and the values of the solution on either side of the shock, u_l and u_r. By integrating the equation over a space-time region, we derive the shock speed formula:

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

This condition is essential for modeling the propagation of shock waves in various physical systems.

a) **Condition described by Rankine-Hugoniot**

Having a solution of $u_t + f(u)_x = 0$ that possesses a jump. The Rankine-Hugoniot condition is used to determine the speed of it - a shock wave in a conservation law. It expresses the jump condition across a discontinuity.

To establish that lets assume that shock $(u_l(t), u_r(t))$ - that represent a discontinuity - evolved in the solution and it will propagate with a certain speed $s(t)$, which maybe will change with time. To investigate it lets suppose that the shock is moving inside a small time period $[t_1, t_1 + \Delta t]$over which the shock speed $s = -\frac{\Delta x}{\Delta t}$ is essentially a constant. Firstly, in the rectangle $[t_1, t_1 + \Delta t] \times [x_1, x_1 + \Delta x]$ we have a line that splits it into two roughly similar triangles.

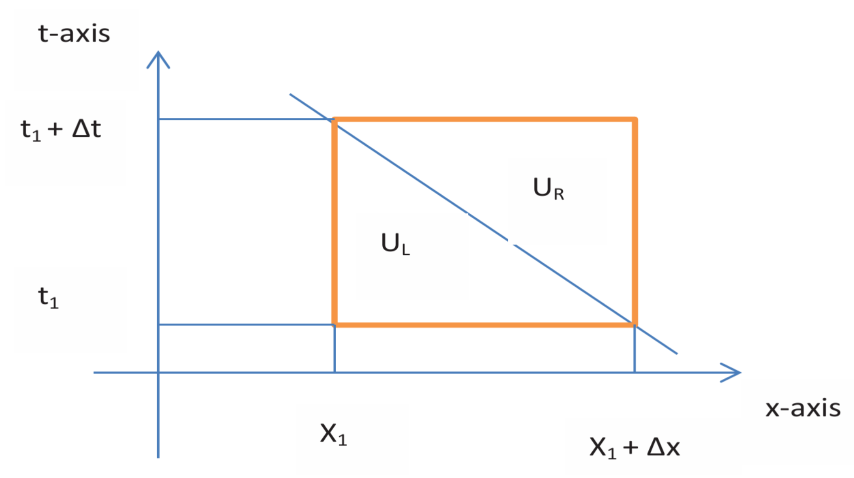

Figure 8: Illustration of a shock moving from the *"Intro-ConsLaw-2021"*

In conclusion:

- The Rankine-Hugoniot condition tells us how fast a shock wave moves relative to the fluid.
- If the shock speed s is positive, the shock moves rightward; if s is negative, it moves leftward.
- If s is equal to the characteristic wave speed, the discontinuity becomes a contact discontinuity.

b) **Main steps**

To establish the path that describes the moving jump, we need to go through some computation. Starting with integration over the space, of rectangle R, the well-known equation $u_t + f(u)_x = 0$:

$$\iint_R [u_t + f(u)_x = 0] dxdt = 0$$

$$\int_{t_1}^{t_1+\Delta t} \int_{x_1}^{x_1+\Delta x} (u_t) dxdt + \int_{t_1}^{t_1+\Delta t} \int_{x_1}^{x_1+\Delta x} (f(u)_x) dxdt = 0$$

$$\int_{t_1}^{t_1+\Delta t} [\frac{d}{dt}(\int_{x_1}^{x_1+\Delta x} u(x,t)dx)dt] + \int_{t_1}^{t_1+\Delta t} [f(u(x_1+\Delta x, t)) - f(u(x_1,t))]dt = 0$$

$$\int_{x_1}^{x_1+\Delta x} [u_l - u_r]dx + \int_{t_1}^{t_1+\Delta t} [f(u_r) - f(u_l)]dt = 0$$

Using that u is close to being constant along each adge of the R.

$$\Delta x[u_l - u_r] + \Delta t[f(u_r) - f(u_l)] = 0$$

$$\Delta x[u_l - u_r] = \Delta t[f(u_l) - f(u_r)]$$

And finally we achieve the formula for the speed of the shock:

$$s = \frac{\Delta x}{\Delta t} = \frac{f(u_l) - f(u_r)}{u_l - u_r} \tag{15}$$

2.3 Solving a Riemann problem

The Riemann problem is a fundamental concept in the study of hyperbolic partial differential equations (PDEs), especially in the context of fluid dynamics and conservation laws. It involves solving a system of hyperbolic conservation laws with an initial condition that consists of two constant values separated by a discontinuity. The problem is typically framed as a set of equations that govern the evolution of the unknown function over time, subject to initial data that has a jump discontinuity. In this section, we explore different types of solutions to the Riemann problem, including continuous and shock solutions, as well as weak solutions and their physical significance. We also examine how the entropy condition helps select the physically relevant solutions in cases with multiple weak solutions.

a) **The Riemann problem**

The Riemann problem is a fundamental initial value problem in the study of hyperbolic partial differential equations (PDEs), particularly in fluid dynamics and conservation laws. It consists of solving a system of hyperbolic conservation laws with a piecewise constant initial condition that has a single discontinuity.

A general Riemann problem is given by a hyperbolic conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

with an initial condition:

$$u(x,0) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases} \tag{16}$$

where:

- $u(x,t)$ is the unknown solution,
- $f(u)$ is the flux function,
- u_L and u_R are the left and right initial states.

The solution to a Riemann problem typically consists of shock waves, rarefaction waves, or contact discontinuities, depending on the properties of the system.

- **Shock solution**: When characteristics converge, forming a discontinuity that moves with a certain speed.
- **Similarity solution**: When characteristics spread apart, forming a smooth, self-similar solution.

b) **Similarity and shock solution**

A) Continuous Solution | Rarefaction wave solution | Similarity solution
In this solution we have two characteristics associated with $x = 0$:

$$x = f'(u_l)t$$

$$x = f'(u_r)t$$

Assumed that the velocity $f'(u_l)$ is lower than velocity $f'(u_r)$ for $u_l < u_r$, then these will not cross and we will be looking for a continuous solution with the Riemann initial condition to the equation $u_t + f(u)_x = 0$. The solution that we are looking for is depending on similarity solution - $\frac{x}{t}$. This is defined by:

$$u(x,t) := v(\frac{x}{t})$$

Then derive condition to make sure that u is the solution.

$$u_t(x,t) = -\frac{x}{t^2}v'(\frac{x}{t})$$

$$f(u)_x = f'(v(\frac{x}{t}))u_x = f'(v(\frac{x}{t}))v'(\frac{x}{t})\frac{1}{t}$$

Hence, knowing that u is the solution we have the equation $u_t + f(u)_x = 0$ in form:

$$-\frac{x}{t^2}v'(\frac{x}{t}) + f'(v(\frac{x}{t}))v'(\frac{x}{t})\frac{1}{t} = 0$$

After some easy computation and assumption that $v'(\frac{x}{t}) \neq 0$, when $v'(\frac{x}{t}) = 0$ - which happens only when v is constant so $u_l = u_r$, we got:

$$f'(v(\frac{x}{t})) = \frac{x}{t}$$

Knowing that f' is monotonically increasing we can get the inverse $(f')^{-1}$:

$$v(\frac{x}{t}) = (f')^{-1}(\frac{x}{t})$$

Given the characteristic curve associated with u_l for $x = f'(u_l)t$, we got:

$$v(\frac{x}{t}) = (f')^{-1}(f'(u_l)) = u_l$$

Similarly, for $x = f'(u_r)t$ we got:

$$v(\frac{x}{t}) = u_r$$

In conclusion, for $f'(u_l) \leq \frac{x}{t} \leq f'(u_r)$ we have:

$$u(x,t) = v(x,t) = \begin{cases} u_l, & \text{if } \frac{x}{t} \leq f'(u_l) \\ (f')^{-1}(\frac{x}{t}), & \text{if } f'(u_l) < \frac{x}{t} < f'(u_r) \\ u_r & \text{if } f'(u_r) \leq \frac{x}{t} \end{cases} \tag{17}$$

B) Discontinuous solution | Shock solution

In opposite to the previous one, we are investigating the case when the velocity $f'u_l)$ is higher than $f'(u_r)$, when $u_l > u_r$. Then the two characteristics will cross and we should look for the discontinuous solution of $u_t + f(u)_x = 0$ using Riemann initial data. Doing than, we should ensure that the jump satisfies the Rankine-Hugoniot condition. To achieve that the solution should be following this:

$$u(x,t) = \begin{cases} u_l, & \text{if } \frac{x}{t} \leq s \\ u_r & \text{if } s < \frac{x}{t} \end{cases} \tag{18}$$

where s is the speed/slope of the discontinuity, that was described in the subsection Task 2.2.

c) **Weak solution**

A weak solution is a type of solution to a differential equation that may not be differentiable in the classical sense but still satisfies the equation in an integral or distributional sense. Weak solutions are essential in mathematical analysis, particularly in partial differential equations (PDEs), where classical solutions may not exist due to irregularities in the data or coefficients. Thanks to this property, we are able to use them to allow discontinuous solutions, but it still cannot satisfy he equation $u_t + f(u)_x = 0$ in classical sense. However, we should remember that there can be multiple weak solutions for one initial data.

Starting as usually with the main equation of this project: $u_t + f(u)_x = 0$ and we let the rectangle $R = [x_1, x_2] \times [t_1, t_2]$ and integrate by the area, we get:

$$\iint_{R_1} [u_t + f(u)_x] dxdt = 0$$

Using R:

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} [u_t + f(u)_x] dxdt = 0$$

Alternatively, we can resign from using R and write it in the form:

$$\int_0^{\infty} \int_{-\infty}^{\infty} [u_t + f(u)_x] \phi(x,t) dxdt = 0$$

where $\phi(x,t)$ is a function:

$$\phi(x,t) = \begin{cases} 1, & \text{if } (x,t) \in R \\ 0 & \text{if } (x,t) \notin R \end{cases}$$

Lets generalize this notion by using u is a weak solution if the (34785634875623)

We can now extend this concept by defining u as a weak solution if equation (34785634875623) holds for any smooth function $\phi(x,t)$ with compact support, meaning φ is zero outside a bounded region in both space and time. Since $\phi(x,t)$ is smooth, we can apply integration by parts in both space and time, leading to the following integral equality:

$$\int_0^\infty \int_{-\infty}^\infty [u_t + f(u)_x]\phi \, dx \, dt = \int_{-\infty}^\infty u(x,0)\phi(x,0) \, dx. \tag{19}$$

Next, we will explicitly present examples of weak solutions, focusing on Riemann problems. Specifically, we will demonstrate that both a rarefaction wave solution and a shock solution can be weak solutions. Additionally, we will show that a given problem may have multiple weak solutions, illustrating that different solutions can arise from the same initial data.

d) **Entropy condition**

Starting again with the conservation law:

$$u_t + f(u)_x = 0,$$

where $f'(u) > 0$, a discontinuity propagating with a speed given by the Rankine-Hugoniot condition:

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r}$$

is said to satisfy the Lax entropy condition if:

$$f'(u_l) > s > f'(u_r). \tag{20}$$

From this condition, we observe the following: - If $u_l < u_r$, then $f(u_l) < f(u_r)$, meaning a discontinuity satisfying the Rankine-Hugoniot condition cannot exist. - If $u_l > u_r$, then $f(u_l) > f(u_r)$, meaning a discontinuity satisfying the Rankine-Hugoniot condition is possible.

This implies that only a *decreasing* jump (i.e., a shock) is allowed for a weak solution to satisfy the Lax entropy condition. In other words, the entropy condition ensures that physically meaningful discontinuities, known as shocks, are correctly identified and included in the set of admissible weak solutions. This criterion is essential when constructing weak solutions to Riemann problems.

The Lax entropy condition is necessary because weak solutions to hyperbolic conservation laws are not always unique. Without an additional constraint, such as the entropy condition, an infinite family of weak solutions may exist, making it impossible to determine a physically relevant or mathematically unique solution.

The entropy condition serves as a selection principle: it restricts the set of weak solutions to those that respect the natural flow of information in a physical system. This ensures that the chosen weak solution is the one that correctly models real-world phenomena, such as shock waves in fluid dynamics or traffic flow discontinuities.

Thus, the Lax entropy condition plays a fundamental role in distinguishing physically meaningful solutions from nonphysical ones, ultimately guaranteeing the uniqueness and correctness of the weak solution.

e) **Solve the given Riemann problem**

$$f(u) = 2u(1-u) \tag{21}$$

with initial data:

$$u_0(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{3}{4}, & \text{if } x \in [0,1] \\ 0 & \text{if } x > 1 \end{cases}$$

The characteristic velocity is determined by:

$$f'(u) = 2(1-2u). \tag{22}$$

First lets consider the Riemann problem at $x = 0$. Then we can focus on only those initial conditions :

$$u_0(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{3}{4}, & \text{if } x \geq 0 \end{cases}$$

Our $u_l = 0$ and $u_r = \frac{3}{4}$, which means that in this case $u_l < u_r$.

$$f'(u_l) = 2(1-2u) = 2(1-2*0) = 2$$
$$f'(u_r) = 2(1-2*\frac{3}{4}) = -1$$
$$s = \frac{f(u_l)-f(u_r)}{u_l-u_r} = \frac{[2*0(1-0)]-[2*\frac{3}{4}(1-\frac{3}{4})]}{0-\frac{3}{4}} = \frac{1}{2}$$

And now knowing that $f'(u_l) < f'(u_r)$ - the velocity $f'(u_l)$ is higher than the velocity $f'(u_r)$ and the characteristics will cross at some time. Hence, we will look for the discontinuous solution that satisfies the Rankine-Hugoniot condition. To achieve that we will follow this solution:

$$u(x,t) = \begin{cases} u_l, & \text{if } \frac{x}{t} \leq s \\ u_r, & \text{if } \frac{x}{t} > s \end{cases}$$

$$u(x,t) = \begin{cases} 0, & \text{if } x \leq \frac{1}{2}t \\ \frac{3}{4}, & \text{if } x > \frac{1}{2}t \end{cases}$$

After solving the first problem lets go on the the second one located at $x = 1$. There our initial conditions look like this:

$$u_0(x) = \begin{cases} \frac{3}{4}, & \text{if } x \leq 1 \\ 0, & \text{if } x > 1 \end{cases}$$

Here we can see that opposite to the first problem $u_l = \frac{3}{4} > u_r = 0$.

$$f'(u_l) = 2(1 - 2 * \frac{3}{4}) = -1$$

$$f'(u_r) = 2(1 - 2 * 0) = 2$$

And in this case $f'(u_l) < f'(u_r)$ that means that the velocity of $f'(u_l)$ is lower than the velocity $f'(u_r)$. Looking at those information we can tell that we should look for the rarefaction wave solution in the form of:

$$u(x,t) = \begin{cases} u_l, & \text{if } \frac{x}{t} \leq f'(u_l) \\ (f')^{-1}(\frac{x}{t}), & \text{if } f'(u_l) < \frac{x}{t} < f'(u_r) \\ u_r, & \text{if } \frac{x}{t} \geq f'(u_r) \end{cases}$$

Firstly, lets compute this $(f')^{-1}(\frac{x}{t})$.

$$\frac{x-1}{t} = 2(1 - 2u)$$

$$u = \frac{1}{2}(1 - \frac{x-1}{2t})$$

Remembering that in this case our x starts at 1, then our solution will look like this:

$$u(x,t) = \begin{cases} \frac{3}{4}, & \text{if } x \leq 1 - t \\ \frac{1}{2}(1 - \frac{x-1}{2t}), & \text{if } 1 - t < x < 1 + 2t \\ 0, & \text{if } x \geq 2t \end{cases}$$

Finally, combining solution for the two problems we will have the full solution in the form of:

$$u(x,t) = \begin{cases} 0, & \text{if } x \leq \frac{1}{2}t \\ \frac{3}{4}, & \text{if } \frac{1}{2}t < x \leq 1 - t \\ \frac{1}{2}(1 - \frac{x-1}{2t}), & \text{if } 1 - t < x < 1 + 2t \\ 0, & \text{if } x \geq 1 + 2t \end{cases}$$

This problem can be interpreted, in understanding of the traffic flow model, as the part of the road - from point 0 to point 1 - with the density of the cars $\frac{3}{4}$. This means that there is some maximum amount of the cars 1 and there we have 'only' $\frac{3}{4}$ of this maximum amount. And this problem represents what will happen with the density 0 before and after the consider part of the road.

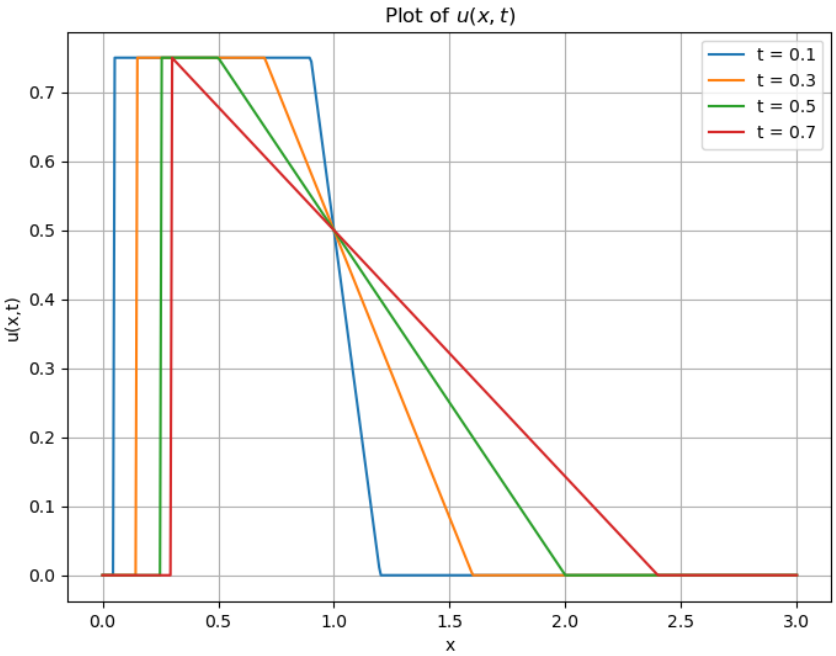

Figure 9: Exact solution in different times

To find the valid time period for this solution, we need to look at the interaction between the shock and the rarefaction wave:

The shock moves at speed $s = \frac{1}{2}$, so the shock is valid for $t \leq 2$.

The rarefaction wave propagates between $1-t$ and $1+2t$, so for this to be valid, the maximum value of t occurs when $1 + 2t \geq 2$, which gives $t \leq \frac{1}{2}$.

Thus, the valid time period for the combined solution is:

$$0 \leq t \leq \frac{1}{2}.$$

This time period ensures that the shock and rarefaction wave do not interfere, and the solution remains valid for the given initial conditions.

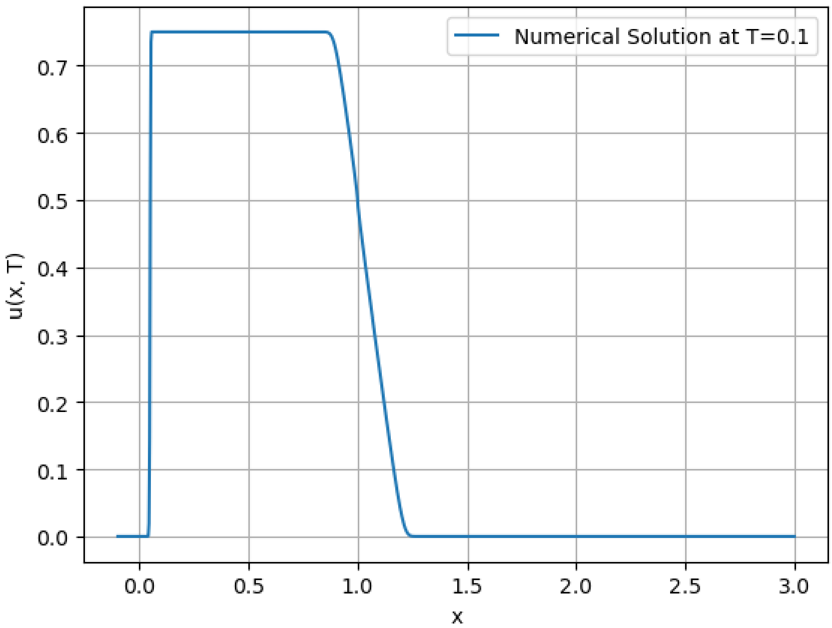

Figure 10: Numerical solution at $t = 0.1$

By trying different grid values, it was estimate that around 800 grid cells is enough to achieve results similar to the exact solution.

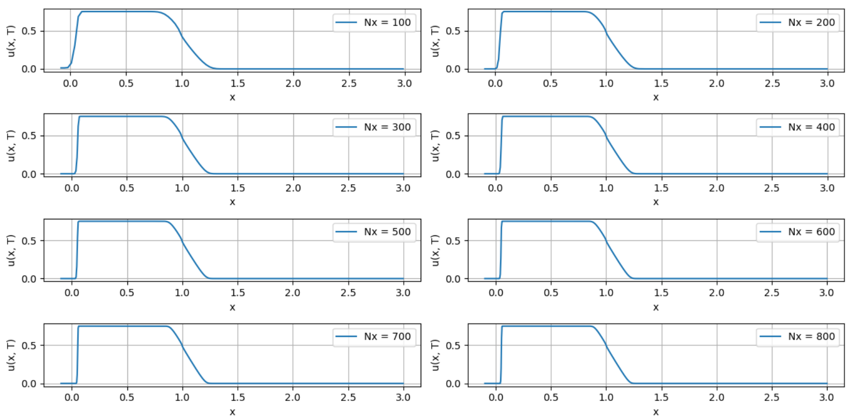

Figure 11: Plots of different grid values Nx

In the context of traffic flow, this solution models how a traffic jam (shock) forms and spreads, while the rarefaction wave models the smooth expansion of a low-density region (vehicles leaving the jam). The overall behavior suggests the transition of a traffic system from a congested state to a less congested one, with the shock and rarefaction wave representing two different mechanisms of traffic flow: one where cars are suddenly bunched together, and one where cars are spreading out to lower densities.

3 Investigations of a traffic flow model

In this section, we analyze the evolution of vehicle density $u(x,t)$ over time using the traffic flow model introduced earlier. The governing equation is derived from the continuity equation and the velocity-density relation $V(u) = V_{\max}(1-u)$, leading to the nonlinear conservation law:

$$u_t + (u(1-u))_x = 0.$$

The initial vehicle distribution is defined piecewise, with density jumps at specific locations along the road. By examining the characteristic speeds $f'(u) = 1 - 2u$, we determine whether each discontinuity results in a shock wave or a rarefaction wave. For rarefaction waves, we derive continuous solutions using the inverse of the characteristic equation, while for shocks, we apply the Rankine-Hugoniot condition to ensure mass conservation. The resulting solutions describe how traffic density propagates and evolves over time, providing insight into congestion patterns. Each transition is analyzed separately to construct a complete picture of the traffic flow dynamics.

a) **The density of the vehicles $u(x,t)$ through out the the time with initial condition u_0**

Using the mathematical model developed in the Task 1 - *Introduction*:

$$V(u) = V_{max}(1-u)$$

Setting $V_{max} = 1$ and combining equations (3) and (5), we receive:

$$u_t + (u(1-u))_x = 0$$

for $x \in [a,b]$.

Finally, on the road vehicles move from left to right, which gives a natural boundary condition:

$$u\Big|_{x=a} = u_{in}$$

The initial condition is:

$$u(x, t=0) = u_0(x)$$

For the simulation of the traffic flow we will consider two different initial distributions. First one is:

$$u_0(x) = \begin{cases} 0.2, & \text{if } x \in [-0.5, 0] \\ 0.4, & \text{if } x \in (0, 0.5] \\ 0.6, & \text{if } x \in (0.5, 1.5] \\ 0.8, & \text{if } x \in (1.5, 2.5] \\ 0.9, & \text{if } x \in (2.5, 3.5] \end{cases}$$

Lets start with the characteristic of the function $u_t + (u(1-u))_x = 0$:

$$f'(u) = 1 - 2u$$

And then check for discontinuities in every change in distribution of cars. Zooming into the first change at $x = 0$, we are looking on:

$$u_0(x) = \begin{cases} 0.2, & \text{if } x \in [-0.5, 0] \\ 0.4, & \text{if } x \in (0, 0.5] \end{cases}$$

There our $u_l = 0.2 < 0.4 = u_r$ and after computing speed:

$$f'(u_l) = 1 - 2u = 1 - 2 * 0.2 = 0.6$$

$$f'(u_r) = 1 - 2 * 0.4 = 0.2$$

we can see that given that the speed of the function on the left is higher than this on the right $f'(u_l) > f'(u_r)$, this means that the lines will be spreading and the rarefaction wave will be formed. The solution for this part will then look like that:

$$u(x,t) = \begin{cases} 0.2, & \text{if } \frac{x}{t} \leq f'(u_l) \\ (f')^{-1}(\frac{x}{t}), & \text{if } f'(u_l) < \frac{x}{t} < f'(u_r) \\ 0.4, & \text{if } \frac{x}{t} \leq f'(u_r) \end{cases}$$

Introducing inverse derivative of the function, we got:

$$\frac{x}{t} = 1 - 2u$$

$$u = 0.5 - \frac{x}{2t}$$

The solution for this part is:

$$u(x,t) = \begin{cases} 0.2, & \text{if } \frac{x}{t} \leq 0.6 \\ 0.5 - \frac{x}{2t}, & \text{if } 0.6 < \frac{x}{t} < 0.2 \\ 0.4, & \text{if } \frac{x}{t} \leq 0.2 \end{cases}$$

$$u(x,t) = \begin{cases} 0.2, & \text{if } x \leq 0.6t \\ 0.5 - \frac{x}{2t}, & \text{if } 0.6t < x < 0.2t \\ 0.4, & \text{if } x \leq 0.2t \end{cases}$$

During the second change at $x = 0.5$ we are following the same steps.

$$u_0(x) = \begin{cases} 0.4, & \text{if } x \in (0, 0.5] \\ 0.6, & \text{if } x \in (0.5, 1.5] \end{cases}$$

There our $u_l = 0.4 < 0.6 = u_r$ and after computing speed:

$$f'(u_l) = 1 - 2 * 0.2 = 0.2$$

$$f'(u_r) = 1 - 2 * 0.6 = -0.2$$

In this case the characteristics will cross and the jump will occur, making us to compute speed:

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r}$$

$$s = \frac{0.4(1 - 0.4) - 0.6(1 - 0.6)}{0.4 - 0.6} = \frac{0.24 - 0.24}{-0.2} = 0$$

Hence, we will look for the discontinuous solution that satisfies the Rankine-Hugoniot condition. To achieve that we will follow this solution:

$$u(x,t) = \begin{cases} u_l, & \text{if } \frac{x-0.5}{t} \leq s \\ u_r, & \text{if } \frac{x-0.5}{t} > s \end{cases}$$

$$u(x,t) = \begin{cases} 0.4, & \text{if } x \leq 0.5 \\ 0.6, & \text{if } x > 0.5 \end{cases}$$

The third change that occur at $x = 1.5$:

$$u_0(x) = \begin{cases} 0.6, & \text{if } x \in (0.5, 1.5] \\ 0.8, & \text{if } x \in (1.5, 2.5] \end{cases}$$

There our $u_l = 0.6 < 0.8 = u_r$ and after computing speed:

$$f'(u_l) = 1 - 2 * 0.6 = -0.2$$

$$f'(u_r) = 1 - 2 * 0.8 = -0.6$$

In this case the shock will not occur and functions will form a rarefaction waves, this means that we are going by the same steps from the first change. The solution for this part looks like:

$$u(x,t) = \begin{cases} 0.6, & \text{if } \frac{x-1.5}{t} \leq -0.2 \\ 0.5 - \frac{x-1.5}{2t}, & \text{if } -0.2 < \frac{x-1.5}{t} < -0.6 \\ 0.8, & \text{if } \frac{x-1.5}{t} \leq -0.6 \end{cases}$$

$$u(x,t) = \begin{cases} 0.6, & \text{if } x \leq -0.2t + 1.5 \\ 0.5 - \frac{x-1.5}{2t}, & \text{if } -0.2t + 1.5 < x < -0.6t + 1.5 \\ 0.8, & \text{if } x \leq -0.6t + 1.5 \end{cases}$$

The last change happens at $x = 2.5$.

$$u_0(x) = \begin{cases} 0.8, & \text{if } x \in (1.5, 2.5] \\ 0.9, & \text{if } x \in (2.5, 3.5] \end{cases}$$

There our $u_l = 0.8 < 0.9 = u_r$ and after computing speed:

$$f'(u_l) = 1 - 2 * 0.8 = -0.6$$

$$f'(u_r) = 1 - 2 * 0.9 = -0.8$$

The rarefaction waves, this means that we are going by the same steps from the previous change. The solution for this part looks like:

$$u(x,t) = \begin{cases} 0.8, & \text{if } \frac{x-2.5}{t} \leq -0.6 \\ 0.5 - \frac{x-2.5}{2t}, & \text{if } -0.6 < \frac{x-1.5}{t} < -0.8 \\ 0.9, & \text{if } \frac{x-2.5}{t} \leq -0.8 \end{cases}$$

$$u(x,t) = \begin{cases} 0.8, & \text{if } x \leq -0.6t + 2.5 \\ 0.5 - \frac{x-2.5}{2t}, & \text{if } -0.6t + 2.5 < x < -0.8t + 2.5 \\ 0.9, & \text{if } x \leq -0.8t + 2.5 \end{cases}$$

Combining all solution we receive the very complex solution:

$$u(x,t) = \begin{cases} 0.2, & \text{if } x \leq 0.6t, \\ 0.5 - \frac{x}{2t}, & \text{if } 0.6t < x < 0.2t, \\ 0.4, & \text{if } 0.2t \leq x \leq 0.5, \\ 0.6, & \text{if } 0.5 < x \leq -0.2t + 1.5, \\ 0.5 - \frac{x-1.5}{2t}, & \text{if } -0.2t + 1.5 < x < -0.6t + 1.5, \\ 0.8, & \text{if } -0.6t + 1.5 \leq x \leq -0.6t + 2.5, \\ 0.5 - \frac{x-2.5}{2t}, & \text{if } -0.6t + 2.5 < x < -0.8t + 2.5, \\ 0.9, & \text{if } x \geq -0.8t + 2.5. \end{cases}$$

The simulation of this problem looks like this:

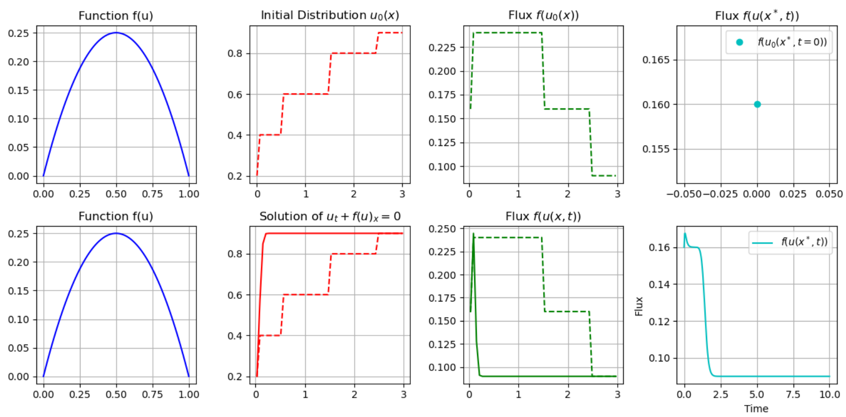

Figure 12: Upper row: The initial state. Lower row: The state at time $T = 10$

In the second initial distribution we will follow exactly the same steps for each occurring change.

$$u_0(x) = \begin{cases} 0.2, & \text{if } x \in [-0.5, 0] \\ 0.4, & \text{if } x \in (0, 0.5] \\ 0.6, & \text{if } x \in (0.5, 1.5] \\ 0.8, & \text{if } x \in (1.5, 2.5] \\ 0.5, & \text{if } x \in (2.5, 3.5] \end{cases}$$

As we can see only difference from the previous case in in the fourth change at $x = 2.5$, where for the period $x \in (2.5, 3.5]$ instead of previous value $u_r = 0.9$ we have $u_r = 0.5$. Lets then just reconsider this part and leave the previous ones as they were computed before.

$$u_0(x) = \begin{cases} 0.8, & \text{if } x \in (1.5, 2.5] \\ 0.5, & \text{if } x \in (2.5, 3.5] \end{cases}$$

There our $u_l = 0.8 < 0.9 = u_r$ and after computing speed:

$$f'(u_l) = 1 - 2 * 0.8 = -0.6$$

$$f'(u_r) = 1 - 2 * 0.5 = 0$$

This time the speed $f'(u_r) = 0$ this means that it does not move. The left function is spreading, creating the rarefaction waves.

$$u(x,t) = \begin{cases} 0.8, & \text{if } \frac{x-2.5}{t} \leq -0.6 \\ 0.5 - \frac{x-2.5}{2t}, & \text{if } -0.6 < \frac{x-1.5}{t} < 0 \\ 0.5, & \text{if } \frac{x-2.5}{t} \leq 0 \end{cases}$$

$$u(x,t) = \begin{cases} 0.8, & \text{if } x \leq -0.6t + 2.5 \\ 0.5 - \frac{x-2.5}{2t}, & \text{if } -0.6t + 2.5 < x < 2.5 \\ 0.5, & \text{if } x \leq 2.5 \end{cases}$$

And then our final solution will look like this:

$$u(x,t) = \begin{cases} 0.2, & \text{if } x \leq 0.6t, \\ 0.5 - \frac{x}{2t}, & \text{if } 0.6t < x < 0.2t, \\ 0.4, & \text{if } 0.2t \leq x \leq 0.5, \\ 0.6, & \text{if } 0.5 < x \leq -0.2t + 1.5, \\ 0.5 - \frac{x-1.5}{2t}, & \text{if } -0.2t + 1.5 < x < -0.6t + 1.5, \\ 0.8, & \text{if } -0.6t + 1.5 \leq x \leq -0.6t + 2.5, \\ 0.5 - \frac{x-2.5}{2t}, & \text{if } -0.6t + 2.5 < x < 2.5, \\ 0.5, & \text{if } x \geq 2.5. \end{cases}$$

Having run the simulation, those are received plots:

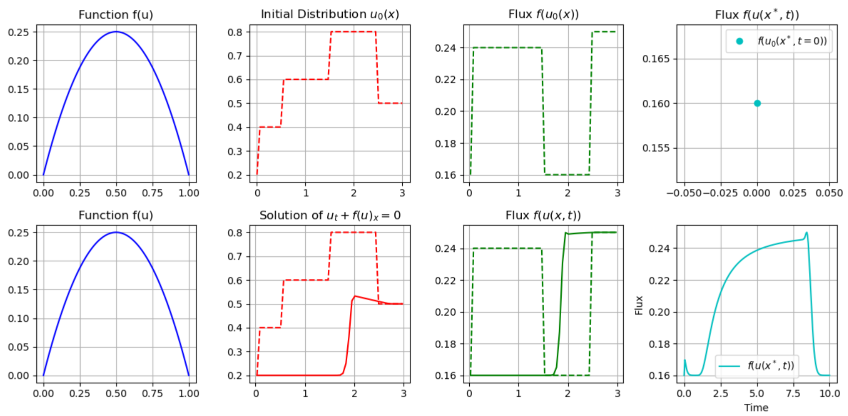

Figure 13: Upper row: The initial state. Lower row: The state at time $T = 10$

Compering those two sightly different simulations. The difference is only at the last part of the road, in the first simulation the density of cars is 90% of the road capability and in the second on the same part of the road the density is only 50%. The first case shows that after some time the distribution of the cars from initial step-up-like condition changes to the pack road on the whole length up to 90%. We can imagine that the traffic jam was created. On the other hand, when on the last part of the road density was only smaller, the traffic jam did not occur. There was a little jump in the density of the cars maybe up to 53% but the rest of the road was half packed or even empty.

b) **Mechanisms that dictate the traffic flow**

According to the common sense, the further away from other car we are, the less chance we have to get stuck in the traffic jam. But how far is far enough? It can be safely assumed that if we do not see on the horizon - without the fog - in front of us and behind any car, then we will not be stuck in the traffic jam. Knowing, that lets do not simulate distance between the cars bigger than horizon line - which is compute by $3.57 * \sqrt{h}$, where h is persons height. Assuming that the average height of the person sitting in a car is 1.5 m, we will not consider distance bigger than 4.37 km between cars. But setting jokes aside, lets check what is happening with different distributions on the road. First simulation shows how the road will look like if drivers will drive in groups - on parts high density and in between the groups low density.

$$u_0(x) = \begin{cases} 0.1, & \text{if } x \in [-0.5, 0] \\ 0.9, & \text{if } x \in (0, 0.5] \\ 0.1, & \text{if } x \in (0.5, 1.5] \\ 0.9, & \text{if } x \in (1.5, 2.5] \\ 0.1, & \text{if } x \in (2.5, 3.5] \end{cases}$$

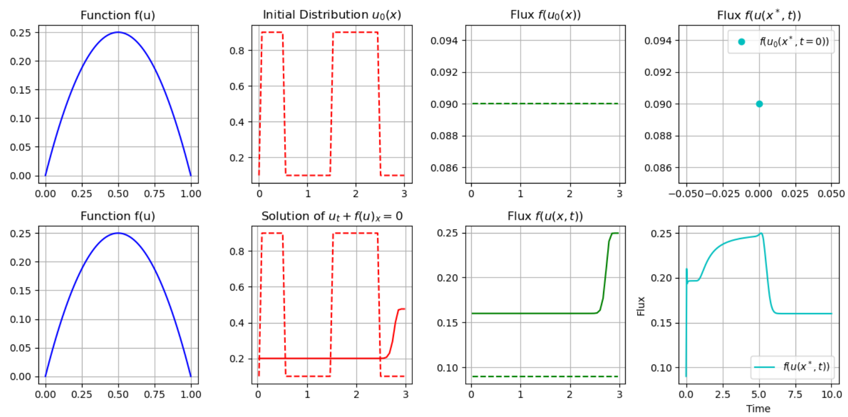

Figure 14: Groups of cars with break in between

As we can see keeping distances between cars does not lead to traffic jams. Density is distributed equally low on the nearly the whole length of the road. Knowing that, first advise for avoiding traffic jams is keeping distance between cars, which we already establish in the informal introduction to the subsection. But what will happen if we will be stuck in

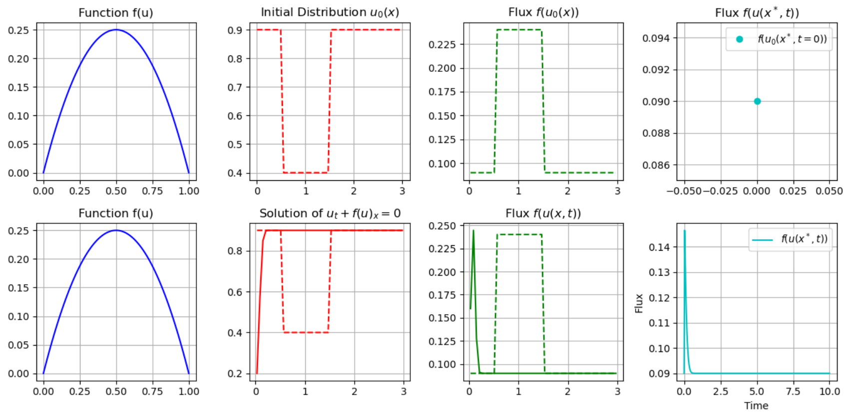

Figure 15: Low density in the middle part

the very busy road and decide to slow down and introduce only one distance between our car and the string of cars in front of us. Does the decision of only one driver make the difference? To check that lets assume that the driver is in the middle of the considered part of the road and initial distribution looks like this:

$$u_0(x) = \begin{cases} 0.9, & \text{if } x \in [-0.5, 0] \\ 0.9, & \text{if } x \in (0, 0.5] \\ 0.4, & \text{if } x \in (0.5, 1.5] \\ 0.9, & \text{if } x \in (1.5, 2.5] \\ 0.9, & \text{if } x \in (2.5, 3.5] \end{cases}$$

As we can see the decision of the only one driver in the middle of the road to keep the distance does not make it any better for the context of dissolving the traffic jam. But what if the driver that is the only one to trying to lesser the density of the cars on the road will be at the end part of the considered road?

$$u_0(x) = \begin{cases} 0.9, & \text{if } x \in [-0.5, 0] \\ 0.9, & \text{if } x \in (0, 0.5] \\ 0.9, & \text{if } x \in (0.5, 1.5] \\ 0.9, & \text{if } x \in (1.5, 2.5] \\ 0.4, & \text{if } x \in (2.5, 3.5] \end{cases}$$

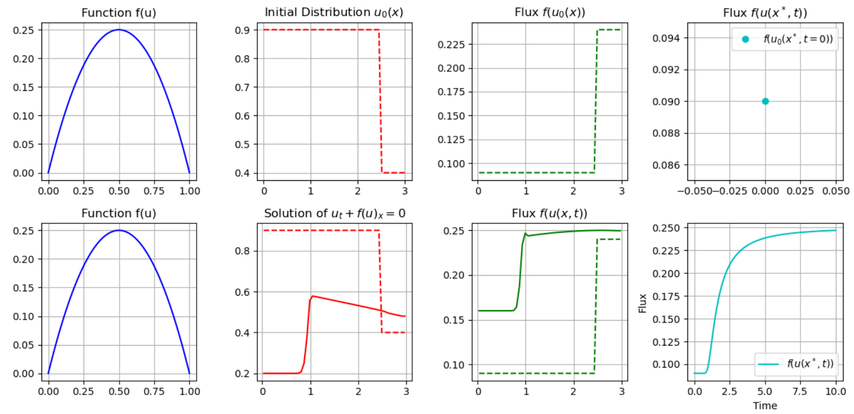

Figure 16: Low density at the end of the road

In this case we can see that decision of this driver makes the difference in consideration to the dissolving the traffic jam. That means that slowing down when being in the traffic jam does nothing to the bettering the traffic, but driving faster if there is a lot of cars behind the driver and not a lot in front of him will not lead to the traffic jam.

Next, fast and a bit boring case, done just due to the formalities, is the situation when all of the cars keep the same distance between each other and does not change anything. Initial distribution was assumed to be:

$$u_0(x) = \begin{cases} 0.4, & \text{if } x \in [-0.5, 0] \\ 0.4, & \text{if } x \in (0, 0.5] \\ 0.4, & \text{if } x \in (0.5, 1.5] \\ 0.4, & \text{if } x \in (1.5, 2.5] \\ 0.4, & \text{if } x \in (2.5, 3.5] \end{cases}$$

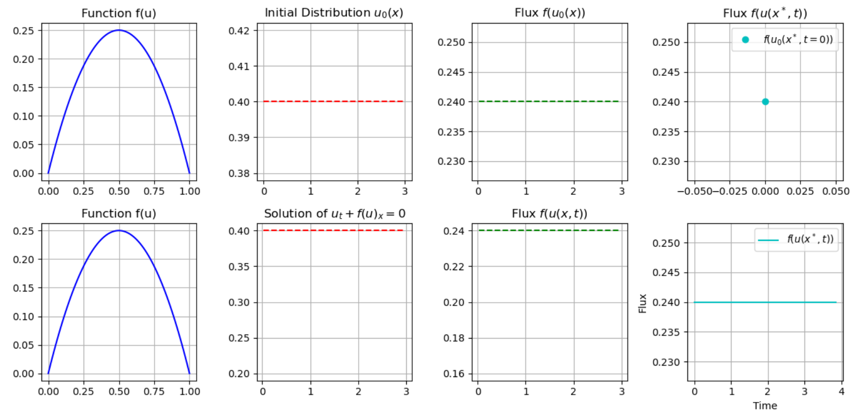

Figure 17: The same density all the road

There is no surprise, if the drivers keep the same - maybe safe distance advised by the law - the traffic jam will not occur.

The final scenario considers a situation where all drivers on the road cooperate, rather than just a single aware driver. In this case, when traffic density begins to increase, drivers at the beginning of the monitored road section slow down to reduce congestion. Meanwhile, drivers at the front, noticing a buildup of cars in their rear-view mirrors, accelerate to help distribute the flow more evenly. To simulate that the initial distribution - after actions taken by the aware drivers - is:

$$u_0(x) = \begin{cases} 0.4, & \text{if } x \in [-0.5, 0] \\ 0.6, & \text{if } x \in (0, 0.5] \\ 0.9, & \text{if } x \in (0.5, 1.5] \\ 0.6, & \text{if } x \in (1.5, 2.5] \\ 0.4, & \text{if } x \in (2.5, 3.5] \end{cases}$$

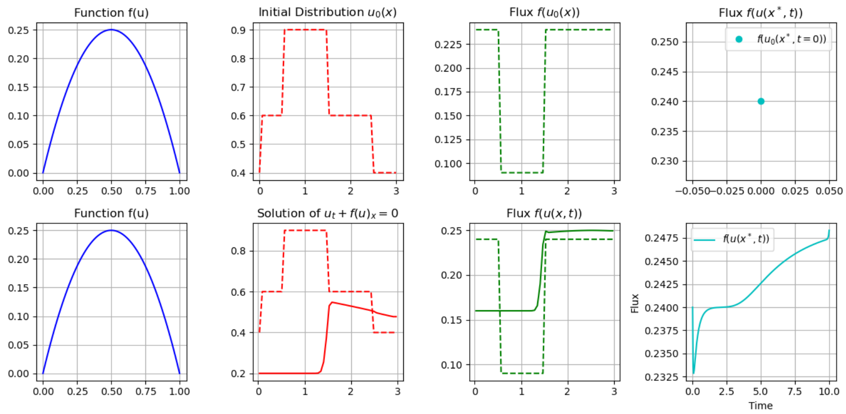

Figure 18: Aware drivers

Working together makes a difference! Multiple drivers trying to resolve the creating traffic jam for sure will succeed.

c) **Different uses of the model in the real life**

One of the uses that was mentioned in the *Numerical methods for conservation laws and related equations. Lecture Notes, UiO* was the enhanced oil recovery. It is a process used to extract oil from sub-surface reservoirs in permeable rocks. The primary stage involves drilling and extracting oil by applying pressure, recovering only 20 to 30 percent of the oil. The secondary stage injects water into the rock bed, displacing the oil for extraction. This process is modeled using two-phase flow (water and oil) in porous media, assuming a one-dimensional reservoir. The oil and water volume fractions (S_o and S_w) satisfy the equation

$$S_o + S_w = 1.$$

The evolution of these phases follows the conservation laws, with phase velocities described by Darcy's law. The oil saturation is governed by the scalar conservation law, which incorporates the mobilities and total flow rate:

$$\frac{\partial S_o}{\partial t} + \frac{q(S_o)^2}{(S_o)^2 + (1 - S_o)^2} \frac{\partial}{\partial x} = 0.$$

4 References

1. R.J. LeVeque *Finite Volume Methods for Hyperbolic Problems.* Cambridge Texts in Applied Mathematics, 2002.

2. S. Evje *Basic Theory for Nonlinear Conservation Laws.* Lecture Notes, UiS, 2021.

3. S. Mishra, U. Skre Fjordholm, R. Abgrall *Numerical methods for conservation laws and related equations.* Lecture Notes, UiO.

4. S. Evje *Modeling Traffic Flow.* Project description, UiS, 2025.