General Physics I

2024 Fall Semester for Physics Major

Lecture 11: Two-Body Interacting Systems and Examples

清大諮商中心

Daw-Wei Wang

Physics Department NTHU

(I) Center of Mass Motion of the Two-Body System

1. Since **the 3rd law is about the interaction between two objects,** different from the 1st and 2nd laws, we should consider two-body system with the equations of motion:

$$\mathbf{F}_1 = \mathbf{F}_1^{ext} + \mathbf{F}_{2\to 1} = \frac{d\,\mathbf{p}_1}{dt} = m_1 \frac{d\,\mathbf{v}_1}{dt}$$
$$\mathbf{F}_2 = \mathbf{F}_2^{ext} + \mathbf{F}_{1\to 2} = \frac{d\,\mathbf{p}_2}{dt} = m_2 \frac{d\,\mathbf{v}_2}{dt} \qquad \mathbf{F}_{1\to 2} = -\,\mathbf{F}_{2\to 1}$$

2. Adding them together, we find that the total external force can have lead to the following equation of motion of the center of mass position:

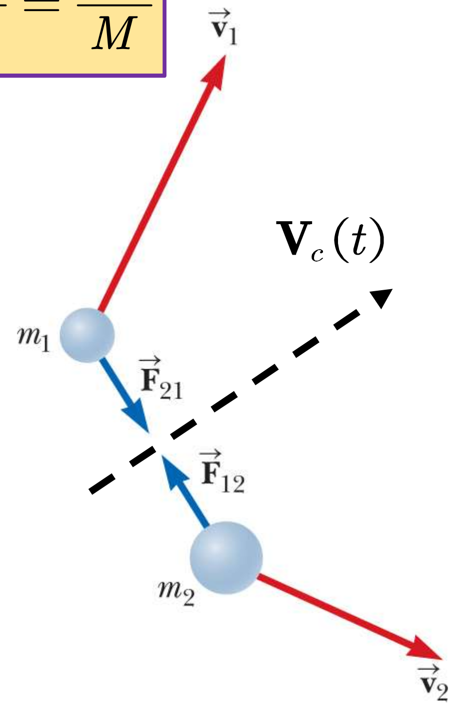
$$\mathbf{F}_{tot}^{ext} = \mathbf{F}_1^{ext} + \mathbf{F}_2^{ext} = \frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) \equiv \frac{d}{dt}\mathbf{P}_{tot}$$
$$= \frac{d}{dt}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2) \equiv M\frac{d}{dt}\mathbf{V}_c \qquad \boxed{M = m_1 + m_2}$$
$$= \frac{d^2}{dt^2}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2) \equiv (m_1 + m_2)\frac{d^2}{dt^2}\mathbf{R}_c$$

Here the Center of Mass Position, $\mathbf{R}_c$, and C.M. velocity can be defined as following:

$$\boxed{\mathbf{R}_c \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}}$$

$$\boxed{\mathbf{V}_c \equiv \frac{d\,\mathbf{R}_c}{dt} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} = \frac{\mathbf{P}_{tot}}{M}}$$

4. The advantage is: **We can treat the whole system as a "single" particle** with the mass $M = m_1 + m_2$, under the influence of a total external force, $\boldsymbol{F}_{tot}^{ext}$, without dealing with the effects of internal interactions.



(II) Relative Motion of Two-Body System

1. However, the internal motion $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ between these two particles can be still important. The way to investigate it is to study the relative part. **For simplicity, we consider the case without external field**, i.e.

$$\mathbf{F}_{2\to1} = m_1 \frac{d\,\mathbf{v}_1}{dt} \qquad \mathbf{F}_{1\to2} = -\,\mathbf{F}_{2\to1} = m_2 \frac{d\,\mathbf{v}_2}{dt}$$

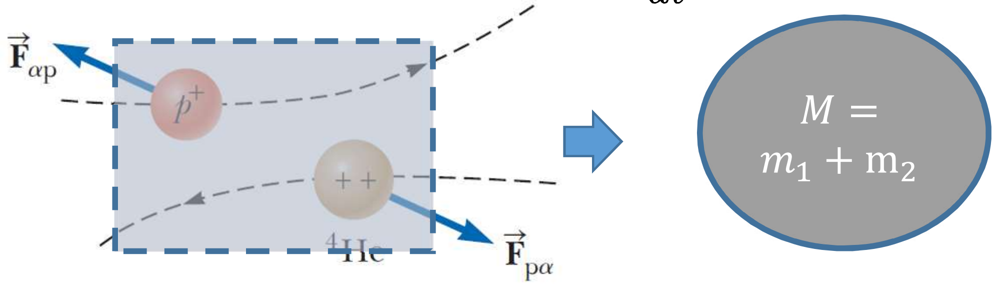
Therefore $$m_2 \frac{d\,\mathbf{v}_2}{dt} = -\,m_1 \frac{d\,\mathbf{v}_1}{dt}$$

$$\frac{d\,\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{v}_1 - \mathbf{v}_2) = \left(1 + \frac{m_1}{m_2}\right)\frac{d\,\mathbf{v}_1}{dt} = \frac{m_1}{\mu}\frac{d\,\mathbf{v}_1}{dt}$$

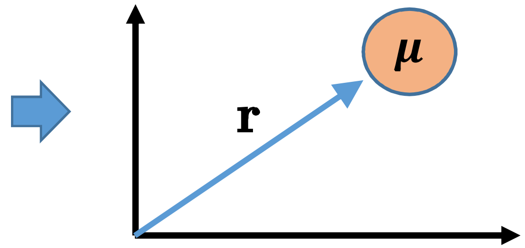
$$\boxed{\mathbf{F}_{2\to1} = m_1 \frac{d\,\mathbf{v}_1}{dt} = \mu \frac{d\,\mathbf{v}}{dt}} \qquad \boxed{\mu \equiv \frac{m_1 m_2}{m_1 + m_2}}$$

2. We therefore could summarize that in general, the two-body system can be **separated into two independent coordinates:**

C.M. coordinate: $$\mathbf{F}_{tot}^{ext} = M \frac{d^2 \mathbf{R}_c}{dt^2}$$



Relative coordinate: $$\mathbf{F}_{int} = \mu \frac{d^2 \mathbf{r}}{dt^2}$$



3. An advantage is that we could separate the total kinetic energy into two parts for the C.M. and relative coordinates respectively.

$$T_{CM} = \frac{M}{2}\mathbf{V}_c^2, \quad T_{rel} = \frac{\mu}{2}\mathbf{v}^2 \qquad \mathbf{v} \equiv \mathbf{v}_1 - \mathbf{v}_2 \qquad \mathbf{V}_c \equiv \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}$$

$$T_{tot} = \frac{m_1}{2}\mathbf{v}_1^2 + \frac{m_2}{2}\mathbf{v}_2^2 = \frac{M}{2}\mathbf{V}_c^2 + \frac{\mu}{2}\mathbf{v}^2 = T_{CM} + T_{rel}$$

(III) Momentum and Conservation in Two-Body Collision Problems

1. For simplicity, here we just consider the system without external force. The full two-body collision can be described by either of the following two pictures:

(1) Two-Body in Laboratory Coordinates:

$$\mathbf{F}_{int} = m_1 \frac{d\mathbf{v}_1}{dt} \qquad -\mathbf{F}_{int} = m_2 \frac{d\mathbf{v}_2}{dt}$$

(2) C.M. and Relative Coordinates:

$$\boxed{\mathbf{F}_{tot}^{ext} = 0 = M\frac{d^2\mathbf{R}_c}{dt^2}} \qquad \boxed{\mathbf{F}_{int} = \mu \frac{d^2\mathbf{r}}{dt^2}}$$

where

$$\mathbf{F}_{int}(|\mathbf{r}|) \equiv \mathbf{F}_{2\to1}(|\mathbf{r}_1 - \mathbf{r}_2|)$$

$$\mathbf{R}_c \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2} \qquad M \equiv m_1 + m_2$$

$$\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2 \qquad \mu \equiv \frac{m_1 m_2}{m_1+m_2}$$

2. In principle, if we know the interaction form for the internal force, $\mathbf{F}_{int}$, we could solve the relative motion **by setting the center of mass coordinate to be at origin. It can be understood as "scattering problem", and will be discussed in the future.**

3. However, even if we **do not know the details of interacting process**, but just the results long time before and after interaction, we could still have some results available according to the conservation laws.

$$0 = M\frac{d^2\mathbf{R}_c}{dt^2} = M\frac{d\mathbf{V}_c}{dt} \Rightarrow \mathbf{V}_c = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1+m_2} = const.$$

$$m_1\mathbf{v}_1^i + m_2\mathbf{v}_2^i = m_1\mathbf{v}_1^f + m_2\mathbf{v}_2^f$$

$$m_1\mathbf{v}_1^f - m_1\mathbf{v}_1^i = \Delta\mathbf{p}_1 = -\Delta\mathbf{p}_2 = -\left(m_2\mathbf{v}_2^f - m_2\mathbf{v}_2^i\right)$$

Note that, above result is based on two laws:

(1) by the 2nd law with $\mathbf{F}_{tot}^{ext} = 0$, and

(2) by the 3rd law with $\mathbf{F}_{1\to2} = \mathbf{F}_{2\to1}$.

(IV) Energy Conservation in Two-Body Interacting Problem

1. For energy conservation, however, it is not that straightforward as the momentum conservation, depending on the work done by the internal force during the collision.

$$W_{int}^{i\to f} = \int_i^f \mathbf{F}_{int} \cdot d\mathbf{r} = \mu \int_i^f \frac{d\mathbf{v}}{dt} \cdot d\mathbf{r}$$

$$= \mu \int_i^f \mathbf{v} \cdot d\mathbf{v} = \frac{\mu}{2}\left[(\mathbf{v}^f)^2 - (\mathbf{v}^i)^2\right]$$

$$T_{rel}^f - T_{rel}^i = \frac{\mu}{2}(\mathbf{v}^f)^2 - \frac{\mu}{2}(\mathbf{v}^i)^2 = W_{int}^{i\to f}$$

2. On the other hand, we could also separate the internal force to be a conservative one and a non-conservative one (friction), so that the work can be obtained to be

$$W_{int}^{i\to f} = \int_i^f \mathbf{F}_{int} \cdot d\mathbf{r} = \int_i^f (\mathbf{F}_{con} + \mathbf{f}_k) \cdot d\mathbf{r}$$

$$= (-U_{rel}^f) - (-U_{rel}^i) - H_{int}^{i\to f}$$

3. Combine the above two results together, we find the following general form of energy conservation during the two-body collision:

$$W_{int}^{i\to f} = T_{rel}^f - T_{rel}^i = (-U_{rel}^f) - (-U_{rel}^i) - H_{int}^{i\to f}$$

$$\boxed{\underbrace{T_{rel}^f + U_{int}^f}_{\equiv E_{rel}^f} = \underbrace{T_{rel}^i + U_{int}^i}_{\equiv E_{rel}^i} - H_{int}^{i\to f}}$$

Note that, this is true even for the system with an external field, because this is related to relative coordinate only.

4. The total energy of the CM motion should be calculated separately from $\mathbf{F}_{tot}^{ext} = M d\mathbf{V}_c/dt$

$$\Rightarrow T_{CM}^f - T_{CM}^i = \frac{M}{2}(\mathbf{V}_c^f)^2 - \frac{M}{2}(\mathbf{V}_c^i)^2 = W_{ext}^{i\to f}$$

Again, we could also separate the work done by the external force to be a conservative one (➔ become potential energy) and the one by friction, but not show them here to avoid confusion.

(V) Several Special Cases to Study

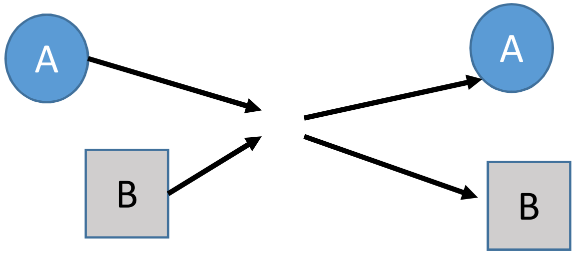
$$\underbrace{T^f_{rel} + U^f_{int}}_{\equiv E^f_{rel}} = \underbrace{T^i_{rel} + U^i_{int}}_{\equiv E^i_{rel}} - H^{i\to f}_{int}$$

1. Case I (Elastic Scattering) :

Internal potential is unchanged. No heat is generated.

➔ *The total relative Kinetic Energy is also conserved)*

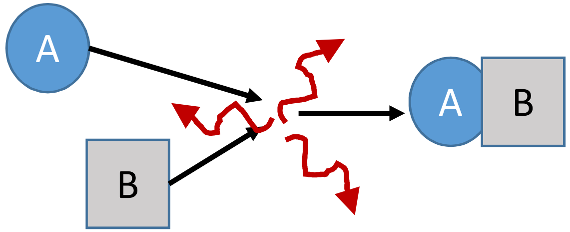
$$U^f_{int} = U^i_{int} \qquad H^{i\to f}_{int} = 0 \qquad \boxed{T^f_{rel} = T^i_{rel}}$$



2. Case II (Perfect Inelastic Scattering without potential change) :

Internal potential is unchanged. Heat is generated. Final velocities are the same ➔ *The total relative Kinetic Energy is not conserved*

$$U^f_{int} = U^i_{int} \qquad H^{i\to f}_{int} \neq 0 \qquad \boxed{T^f_{rel} = T^i_{rel} - H^{i\to f}_{int}}$$

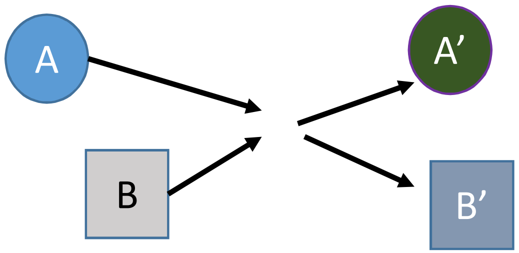


3. Case III (Inelastic Scattering with potential change) :

Internal potential is changed. No heat is generated.

➔ *The total relative Kinetic Energy is not conserved*

$$U^f_{int} \neq U^i_{int} \qquad H^{i\to f}_{int} = 0 \qquad \boxed{T^f_{rel} = T^i_{rel} - \underbrace{(U^f_{int} - U^i_{int})}_{\Delta U_{int}}}$$



(VI) Conservation of Momentum and Energy in the Laboratory Frame

1. Although we have separate the two-body coordinates in to the C.M. frame and relative frame to make the physics clearer, **it is still more convenient to re-express it in the laboratory (observer's) frame,** because physical quantities can be measured directly.

2. For simplicity, we just consider the situation without external force, so that the kinetic energy is conserved in the C.M. Frame, i.e.

$$\boxed{T^f_{CM}} = \boxed{T^i_{CM}} \Rightarrow \frac{M}{2}(\mathbf{V}^f_c)^2 = \frac{M}{2}(\mathbf{V}^i_c)^2$$

3. Using the results in the relative frame,

$$\boxed{T^f_{rel} + U^f_{int}} = \boxed{T^i_{rel} + U^i_{int} - H^{i\to f}_{int}}$$

and the known identity:

$$T_{tot} = \frac{m_1}{2}\mathbf{v}_1^2 + \frac{m_2}{2}\mathbf{v}_2^2 = \frac{M}{2}\mathbf{V}_c^2 + \frac{\mu}{2}\mathbf{v}^2 = T_{CM} + T_{rel}$$

we obtain

$$\boxed{\underbrace{T^f_{tot} + U^f_{int}}_{\equiv E^f_{lab}} = \underbrace{T_{tot} + U^i_{int} - H^{i\to f}_{int}}_{\equiv E^i_{lab}}}$$

4. Therefore, in the laboratory frame without external forces, the conservation of total energy leads to

$$\begin{aligned} T^f_{tot} + U^f_{int} &= \frac{m_1}{2}(\mathbf{v}^f_1)^2 + \frac{m_2}{2}(\mathbf{v}^f_2)^2 + U^f_{int} \\ &= \frac{m_1}{2}(\mathbf{v}^i_1)^2 + \frac{m_2}{2}(\mathbf{v}^i_2)^2 + U^i_{int} - H^{i\to f}_{int} \\ &= T^i_{tot} + U^i_{int} - H^{i\to f}_{int} \end{aligned}$$

5. In the laboratory frame without external forces, the conservation of total momentum is

$$M\mathbf{V}^f_c = M\mathbf{V}^i_c \Rightarrow m_1\mathbf{v}^f_1 + m_2\mathbf{v}^f_2 = m_1\mathbf{v}^i_1 + m_2\mathbf{v}^i_2$$

6. In the textbook or many other books, one may use different index position, say $\mathbf{v}^i_1 \to \mathbf{v}_{1i}$ $\mathbf{v}^f_1 \to \mathbf{v}_{1f}$

$$\frac{m_1}{2}\mathbf{v}^2_{1f} + \frac{m_2}{2}\mathbf{v}^2_{2f} + U^f_{int} = \frac{m_1}{2}\mathbf{v}^2_{1i} + \frac{m_2}{2}\mathbf{v}^2_{2i} + U^i_{int} - H^{i\to f}_{int}$$

$$m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f} = m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}$$

(VII) Special Case I: Elastic Collision in 1D

1. Considering 1D motion first, we could have the following equations immediately ($U_{int}^{i/f} = H_{int}^{i/f} = 0$),

$$m_1 v_{1f} + m_2 v_{2f} = m_1 v_{1i} + m_2 v_{2i}$$

$$\frac{m_1}{2} v_{1f}^2 + \frac{m_2}{2} v_{2f}^2 = \frac{m_1}{2} v_{1i}^2 + \frac{m_2}{2} v_{2i}^2$$

2. There are two equations for the two unknown answers: v_{1f} and v_{2f} , the solution is shown to be (see the Ch 9 of the textbook for more details)

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right) v_{2i}$$

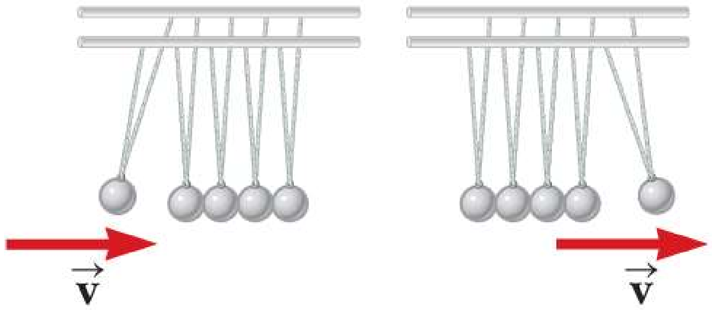
$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2i}$$

3. Special situation I ($m_1 = m_2$):

$$v_{1f} = v_{2i}$$

$$v_{2f} = v_{1i}$$

Exchange velocities



4. Special situation II ($m_1 \gg m_2$):

$$v_{1f} \approx v_{1i}$$

$$v_{2f} \approx 2v_{1i} - v_{2i}$$

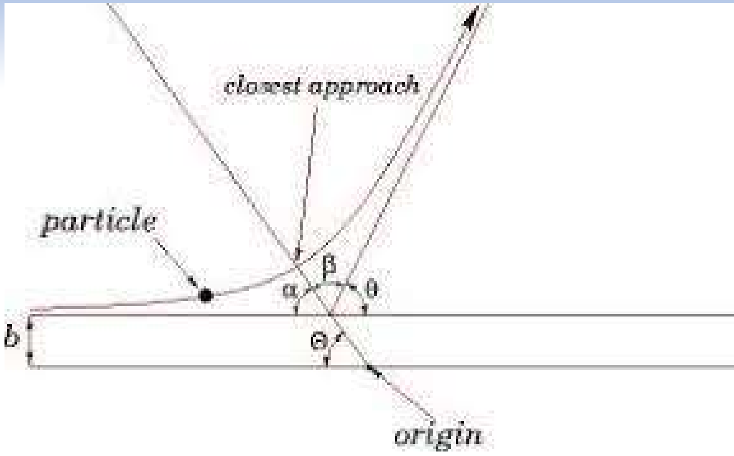
The heavier does not change velocity, but the lighter change its velocity significantly

https://www.royalenfield.com/us/en/motorcycles/

https://www.isuzucv.com/en/index

(VIII) Special Case I: Elastic Collision in 2D or 3D

1. **The angular momentum conservation makes sure that the scattering in 3D without an external force must be within a 2D plane.** We could see this from the equation of the motion:



$$\mathbf{L} = \mu \mathbf{r} \times \mathbf{v} \qquad \mathbf{F}_{int}(|\mathbf{r}|) = F_{int}(r)\hat{\mathbf{r}} = \mu \frac{d\mathbf{v}}{dt}$$

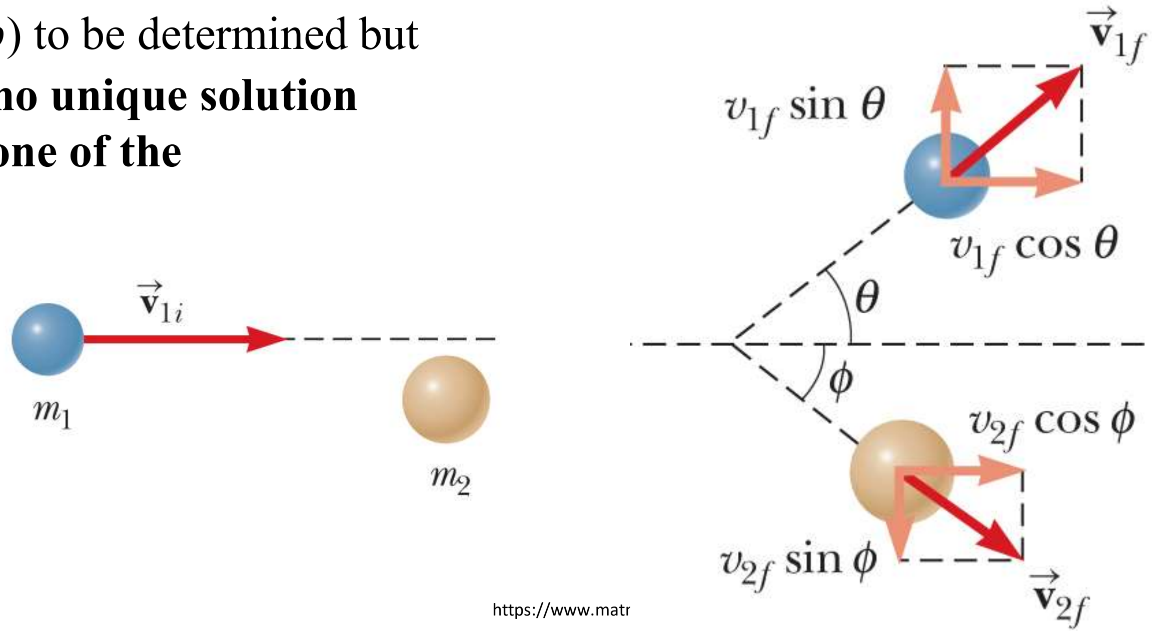
$$\Rightarrow \frac{d}{dt}\mathbf{L} = \mu \frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mu \mathbf{r} \times \frac{d\mathbf{v}}{dt} = \mu \mathbf{v} \times \mathbf{v} + \mathbf{r} \times F(r)\hat{\mathbf{r}} = 0$$

2. In general, there will 4 quantities $(v_{1f}, v_{2f}, \theta, \varphi)$ to be determined but only three equations from conservational law, i.e. **no unique solution and one has to know the interaction or at least one of the observational results.**

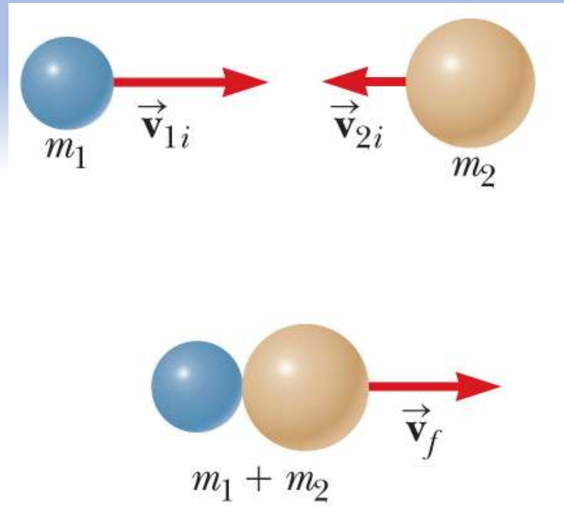
$$m_1 v_{1f} \sin\theta + m_2 v_{2f} \sin\phi = 0$$

$$m_1 v_{1f} \cos\theta + m_2 v_{2f} \cos\phi = m_1 v_{1i} + m_2 v_{2i}$$

$$\frac{m_1}{2} v_{1f}^2 + \frac{m_2}{2} v_{2f}^2 = \frac{m_1}{2} v_{1i}^2 + \frac{m_2}{2} v_{2i}^2$$



(IX) Special Case II: Perfect Inelastic Collision



1. For this inelastic collision, we have $\mathbf{v}_{1f} = \mathbf{v}_{2f} = \mathbf{v}_f$ and $H_{int}^{i/f} \neq 0$

$$(m_1 + m_2)\mathbf{v}_f = m_1 \mathbf{v}_{1i} + m_2 \mathbf{v}_{2i}$$

$$\frac{1}{2}(m_1 + m_2)\mathbf{v}_f^2 = \frac{m_1}{2}\mathbf{v}_{1i}^2 + \frac{m_2}{2}\mathbf{v}_{2i}^2 - H_{int}^{i \to f}$$

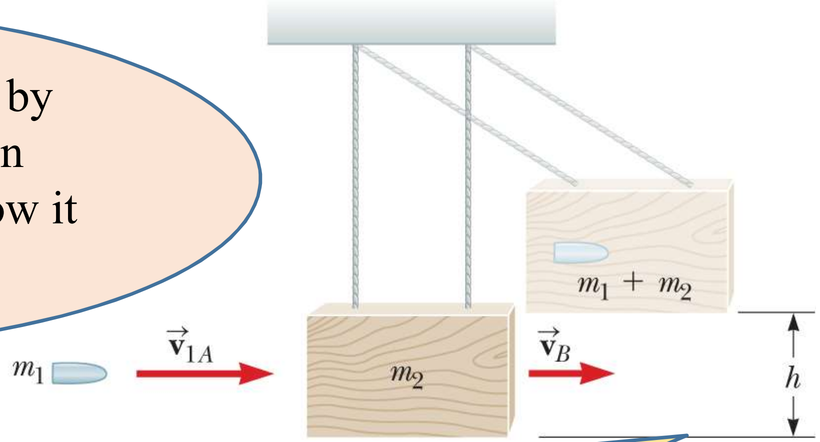
2. We can see that the momentum conservation could determine the final velocity directly, while the energy conservation gives $H_{int}^{i/f} \neq 0$

Not that such a heat generated by collision is made by the friction force. We do NOT have to know it in our problem !!

$$\mathbf{v}_f = \frac{m_1 \mathbf{v}_{1i} + m_2 \mathbf{v}_{2i}}{m_1 + m_2}$$

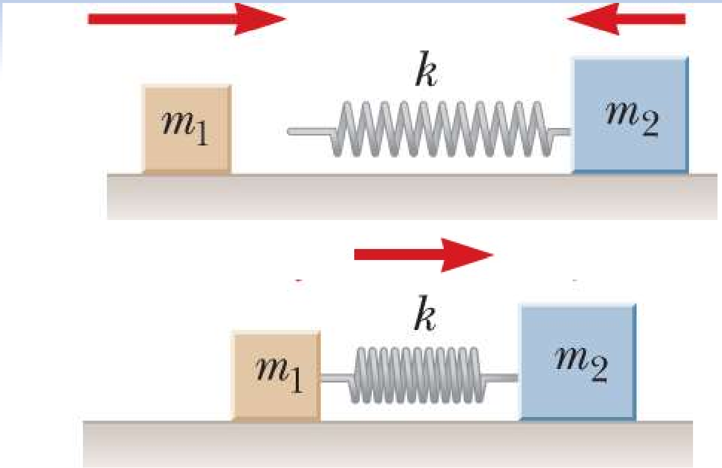
$$H_{int}^{i \to f} = \frac{m_1}{2}\mathbf{v}_{1i}^2 + \frac{m_2}{2}\mathbf{v}_{2i}^2 - \frac{1}{2}(m_1 + m_2)\mathbf{v}_f^2$$

$$= \frac{m_1}{2}\mathbf{v}_{1i}^2 + \frac{m_2}{2}\mathbf{v}_{2i}^2 - \frac{1}{2}\frac{(m_1 \mathbf{v}_{1i} + m_2 \mathbf{v}_{2i})^2}{m_1 + m_2}$$



The height is completely determined by the final total kinetic energy after collision.

(X) Special Case III: Inelastic Collision due to the Internal Potential Change in 1D



1. For this inelastic collision, **we have internal potential change, so that $U^f_{int} \neq U^i_{int}$** . Assuming no friction of heat generation.

$$(m_1 + m_2)\mathbf{v}_f = m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} \qquad \Delta U_{int} \equiv U^f_{int} - U^i_{int}$$

$$\frac{1}{2}(m_1 + m_2)\mathbf{v}_f^2 = \frac{m_1}{2}\mathbf{v}_{1i}^2 + \frac{m_2}{2}\mathbf{v}_{2i}^2 - \Delta U_{int}$$

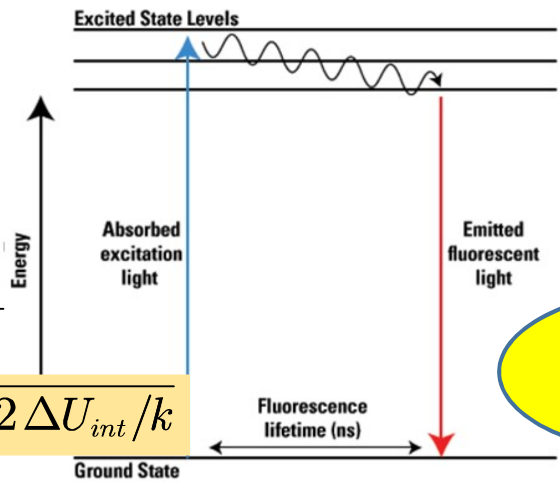
2. Assuming the two objects will be locked by the spring, what will be the final velocity and the largest compression of the spring ? What is the oscillation frequency after the collision ?

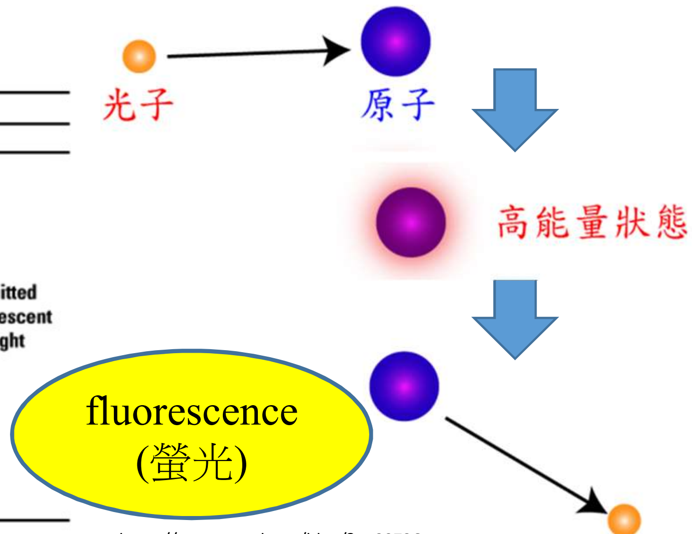
$$\mathbf{v}_f = \frac{m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}}{m_1 + m_2} \qquad \omega = \sqrt{k/\mu}$$

$$\Delta U_{int} = \frac{m_1}{2}\mathbf{v}_{1i}^2 + \frac{m_2}{2}\mathbf{v}_{2i}^2 - \frac{1}{2}(m_1 + m_2)\mathbf{v}_f^2$$

$$= \frac{m_1}{2}\mathbf{v}_{1i}^2 + \frac{m_2}{2}\mathbf{v}_{2i}^2 - \frac{1}{2}\frac{(m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i})^2}{m_1 + m_2}$$

$$= \frac{\mu}{2}(\mathbf{v}_{1i} - \mathbf{v}_{2i})^2 = \frac{1}{2}k\Delta x^2 \qquad \Delta x_{max} = \sqrt{2\Delta U_{int}/k}$$

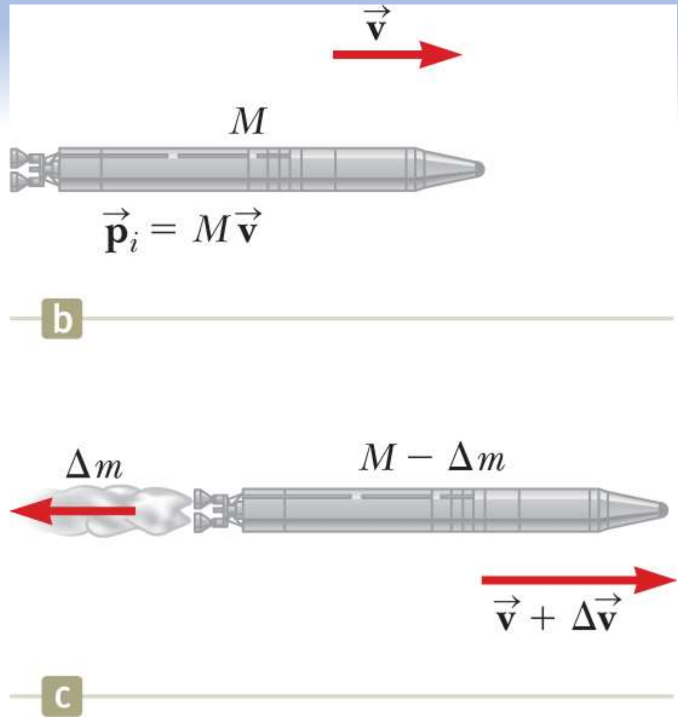




https://case.ntu.edu.tw/blog/?p=39736

(XI) Rocket Equation

1. For a rocket flying in the space without gravity, it is accelerated by ejecting its fuel through explosion. Assuming the total mass is M and the initial velocity is v, the fuel of mass Δm is ejected from the body of rocket with a relative velocity v_e.



Initial momentum: $\mathbf{p}_i = M\mathbf{v}$

Final momentum: $\mathbf{p}_f = (M - \Delta m)(\mathbf{v} + \Delta\mathbf{v}) + \Delta m(\mathbf{v} - \mathbf{v}_e)$

Momentum conservation ➔ $M\mathbf{v} = (M - \Delta m)(\mathbf{v} + \Delta\mathbf{v}) + \Delta m(\mathbf{v} - \mathbf{v}_e)$

$$0 = M\Delta\mathbf{v} - \Delta m\,\mathbf{v}_e + O(\Delta^2)$$

$$\frac{\Delta\mathbf{v}}{\mathbf{v}_e} = \frac{\Delta m}{M} = -\frac{\Delta M}{M}$$

because $\Delta M = -\Delta m < 0$

$$\int_{\mathbf{v}_i}^{\mathbf{v}_f} \frac{d\mathbf{v}}{\mathbf{v}_e} = -\int_{M_i}^{M_f} \frac{\Delta M}{M}$$

$$\mathbf{v}_f = \mathbf{v}_i - \mathbf{v}_e \ln\left(\frac{M_f}{M_i}\right) = \mathbf{v}_i + \mathbf{v}_e \ln\left(\frac{M_i}{M_f}\right)$$

That is why it is the most challenging part to send human beings to the Mars. They cannot return unless one could generate enough fuel for the return rocket on the Mars itself.