Dynamics and Chaos:

Chaos:

- One particular type of dynamics of a system

- Defined as “sensitive dependence on initial conditions

- Deterministic chaos

- No linearity

Chaos in Nature

- Dripping faucets
- Electrical circuits
- Solar system orbits
- Weather and climate (the "butterfly effect")
- Brain activity (EEG)
- Heart activity (EKG)
- Computer networks
- Population growth and dynamics
- Financial data

What is the difference between *chaos* and *randomness*?

Notion of "deterministic chaos"

Some meanings:

Model

Isomorphism

Patterns , tendences

Analogies, metaphors

System

Let a system S a set p of n elements and a set R of interdependences:

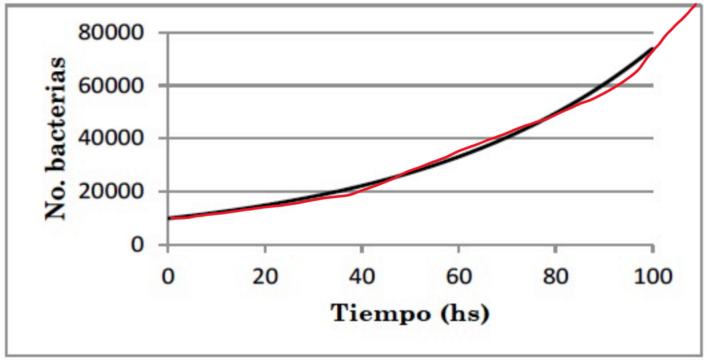
$S = \{ p, R \}$.   (1)

Let Q_i a property of the element p_i (i = 1,2,3,.., n), the system S is defined as:

$$
\begin{aligned}
dQ_1/dt &= f_1(Q_1, Q_2, \ldots, Q_n) \\
dQ_2/dt &= f_2(Q_1, Q_2, \ldots, Q_n) \\
&\vdots \\
dQ_i/dt &= f_i(Q_1, Q_2, \ldots, Q_n) \\
&\vdots \\
dQ_n/dt &= f_n(Q_1, Q_2, \ldots, Q_n)
\end{aligned}
\qquad (2)
$$

where $f_i(Q_1, Q_2, \ldots, Q_n)$ is an interdependence function between n elements

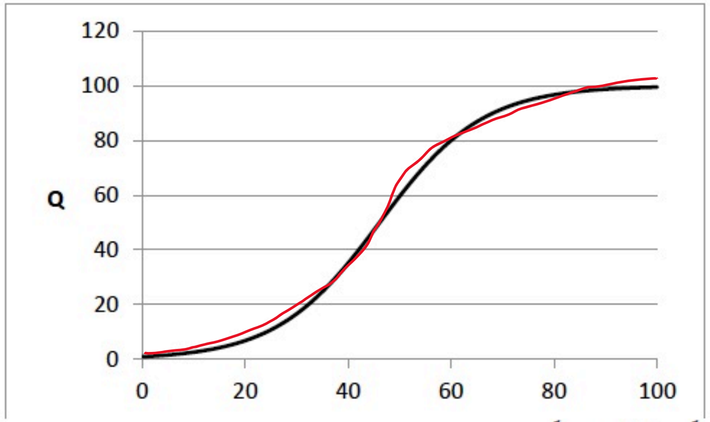
$$\frac{dQ}{dt} = f(Q) = kQ \quad (3) \qquad \boxed{Q = Q_0 e^{kt}} \quad (4)$$



$k = 0.02$, $Qo = 10000$

Crecimiento de bacterias

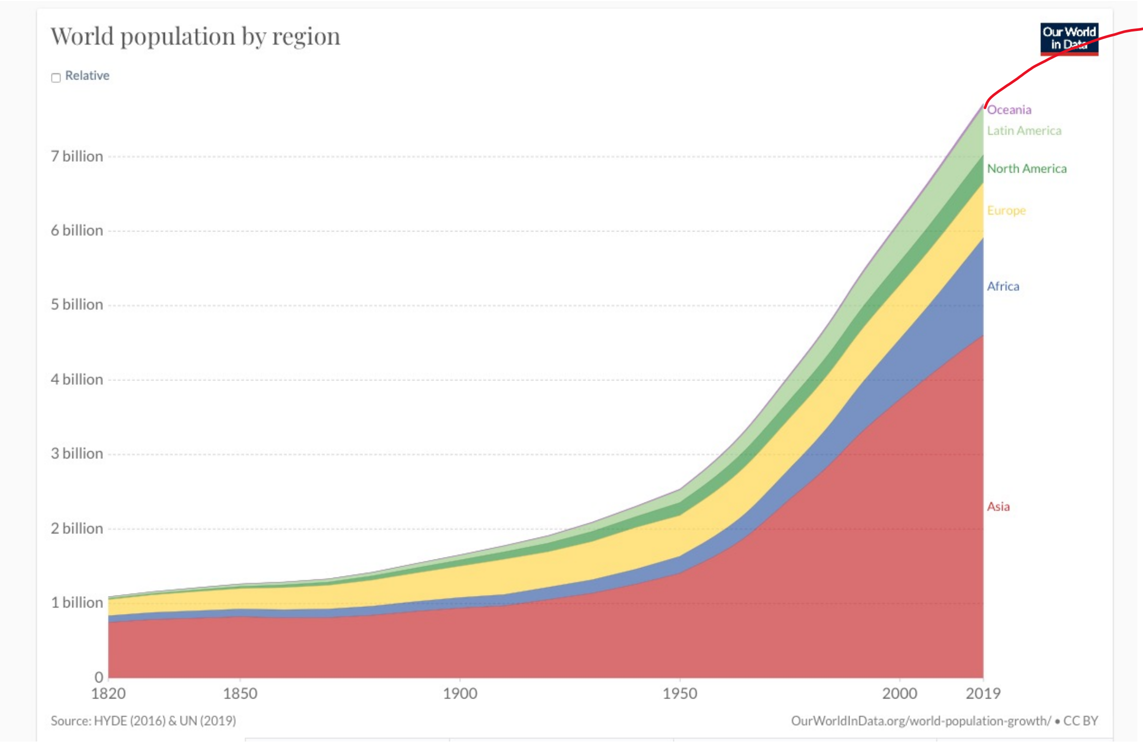
$$dQ/dt = k_1 Q + k_2 Q^2 \quad (5) \qquad Q = k_1 Q_0 e^{k_1 t} / (1 - k_2 Q_0 e^{k_1 t}) \quad (6)$$



$k_1 = 0.1$, $k_2 = -0.01$ y $Q_0 = 1$.

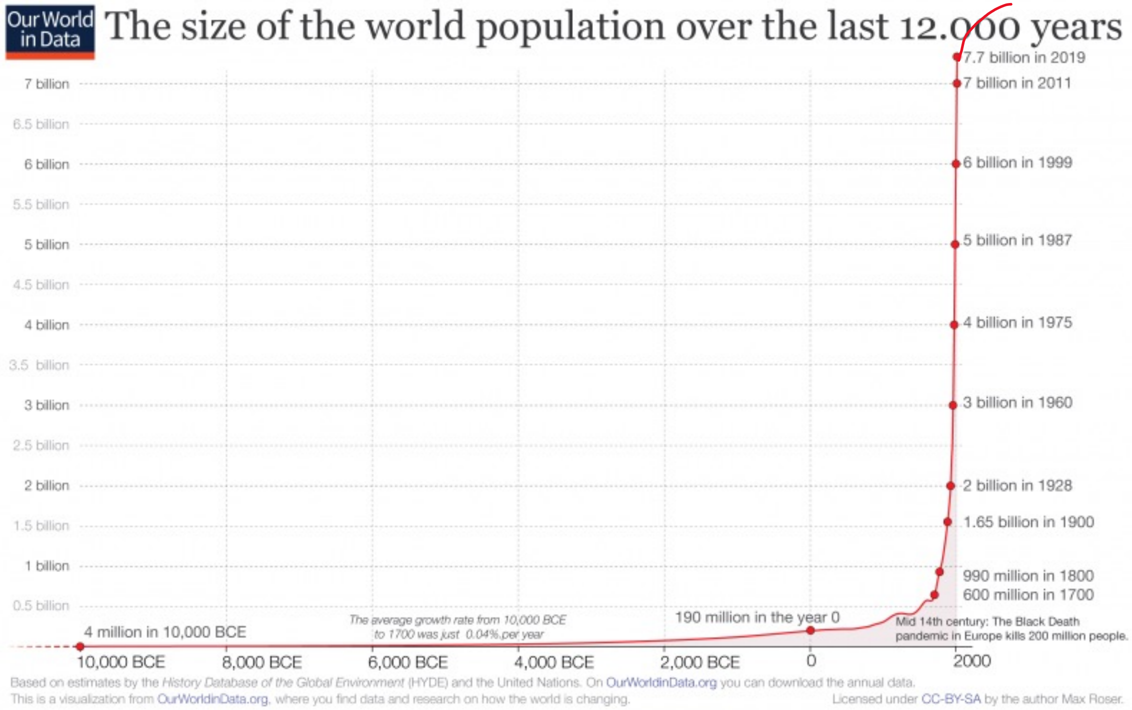
Crecimiento limitado. Curva sigmoide

2.2. Iteration Concept



https://ourworldindata.org/grapher/world-population-by-world-regions-post-1820

World Population and Grow Rate



https://ourworldindata.org/world-population-growth

Reproduction happens over and over again

Exponential Population Growth Model

$birthrate = 2$

$n = population,$

$n_0 = initial\ population,$

$n_1 = population\ at\ year\ 1,$

$n_2 = population\ at\ yaer\ 2\ \ldots$

$\ldots$

$n_t = population\ at\ year\ t,$

$birthrate = number\ of\ offspring\ produced\ each\ year,$

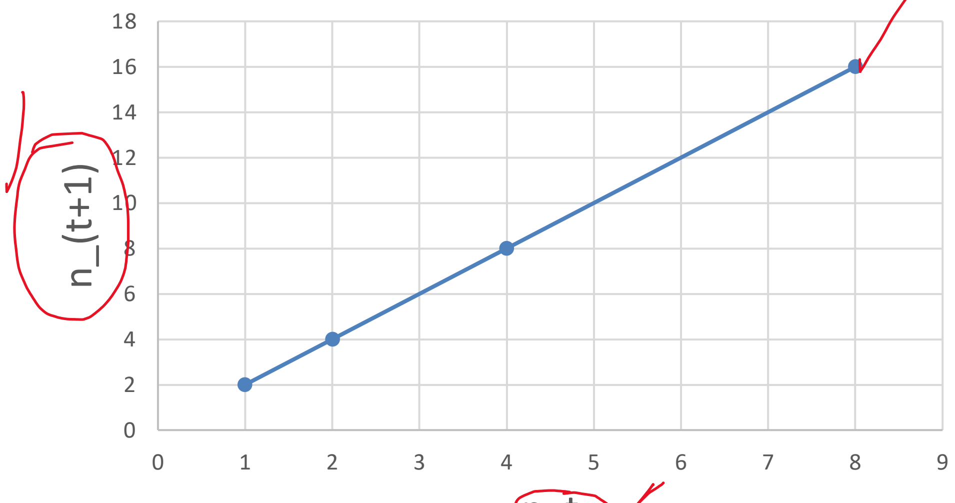
$n_1 = birthrate * n_0,$

$n_2 = birthrate * n_1\ \ldots$

$\ldots$

$$\boxed{n_{t+1} = birthrate * n_t}$$

	Year t	n_t	
3	0	1	
3	1	2	
3	2	4	
3	3	8	27
	…		
	t	2^t	



Linear behaviour

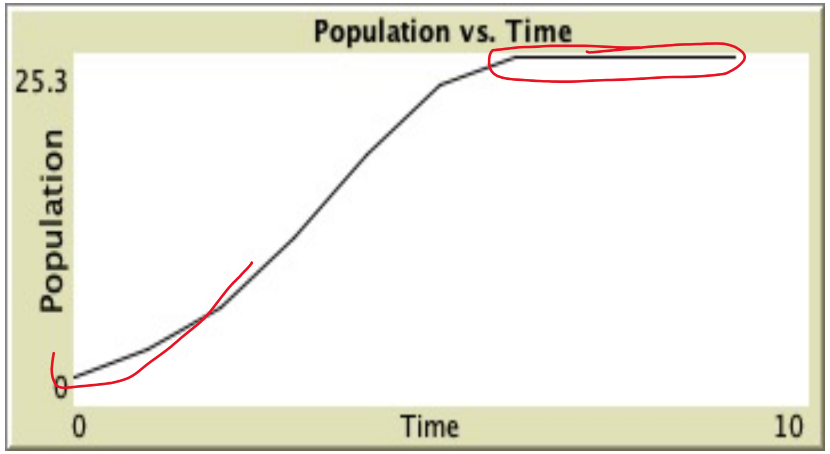
Logistic Growth Model

$$n_{t+1} = birthrate \ * [n_t - died\ offspring\ due\ to\ overcrowding],$$

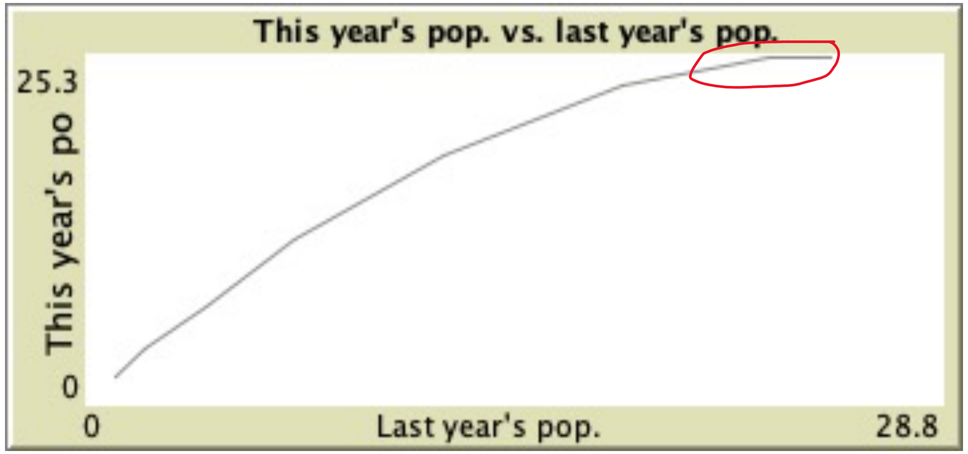
$$died\ offspring\ due\ to\ overcrowding = \frac{n_t^2}{\max population}$$

Extended Model:

$$n_{t+1} = (birthrate - deathrate) * [n_t - \frac{n_t^2}{\max population}\ ]$$



"Logistic Model" of Pierre Verhulst



NO linear behaviour

Logistic Growth Model

$$n_{t+1} = \underbrace{(birthrate - deathrate)} * [n_t - \frac{n_t^2}{\max population}]$$

$$\mathrm{R} = (birthrate - deathrate)$$

$$\mathrm{K} = \max population$$

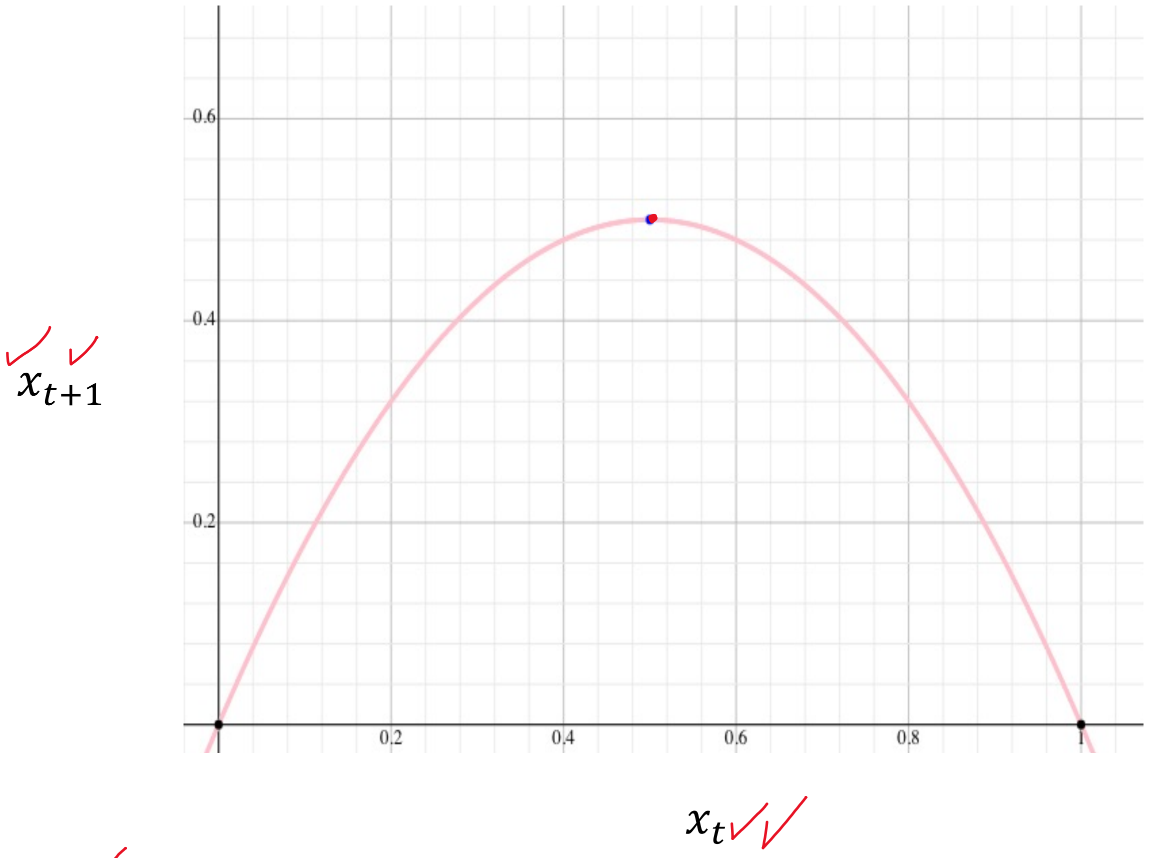
$$n_{t+1} = R\left[n_t - \frac{n_t^2}{K}\right]$$

$$\frac{n_{t+1}}{K} = R\left[\frac{n_t}{k} - \frac{n_t^2}{K^2}\right], where\ x_t = \frac{n_t}{K}$$

$$\boxed{x_{t+1} = R\,[x_t - x_t^2\,]\ \text{“}Logistic\ Map\text{”}}$$

Logistic Map

$$x_{t+1} = R\,[x_t - x_t^2\,],$$



$$suppose\ R = 2, x_0 = 0.2$$

$$x_1 = 2\,(0.2 - 0.2^2) = 0.32$$

$$x_2 = 2\,(0.32 - 0.32^2) = 0.4352$$

$$x_3 = 2\,(0.4352 - 0.4352^2) = 0.49160192$$

$$x_4 = 2\,(0.49160192 - 0.49160192^2) = 0.499858945$$

$$x_5 = 2\,(0.499858945\ - 0.499858945\ ^2) = 0.499999961$$

$$x_6 = 2\,(0.499999961\ - 0.499999961\ ^2) \approx 0.5$$

Fixed point attractor

Lord Robert May
b. 1936

Mitchell Feigenbaum
b. 1944

"The fact that the simple and deterministic equation [i.e., the Logistic Map] can possess dynamical trajectories which look like some sort of random noise has disturbing practical implications. It means, for example, that apparently erratic fluctuations in the census data for an animal population need not necessarily betoken either the vagaries of an unpredictable environment or sampling errors; they may simply derive from a rigidly deterministic population growth relationship...Alternatively, it may be observed that in the chaotic regime, arbitrarily close initial conditions can lead to trajectories which, after a sufficiently long time, diverge widely. This means that, even if we have a simple model in which all the parameters are determined exactly, long-term prediction is nevertheless impossible"

—— Robert May, 1976

Chaos: Seemingly random behavior with *sensitive dependence on initial conditions*

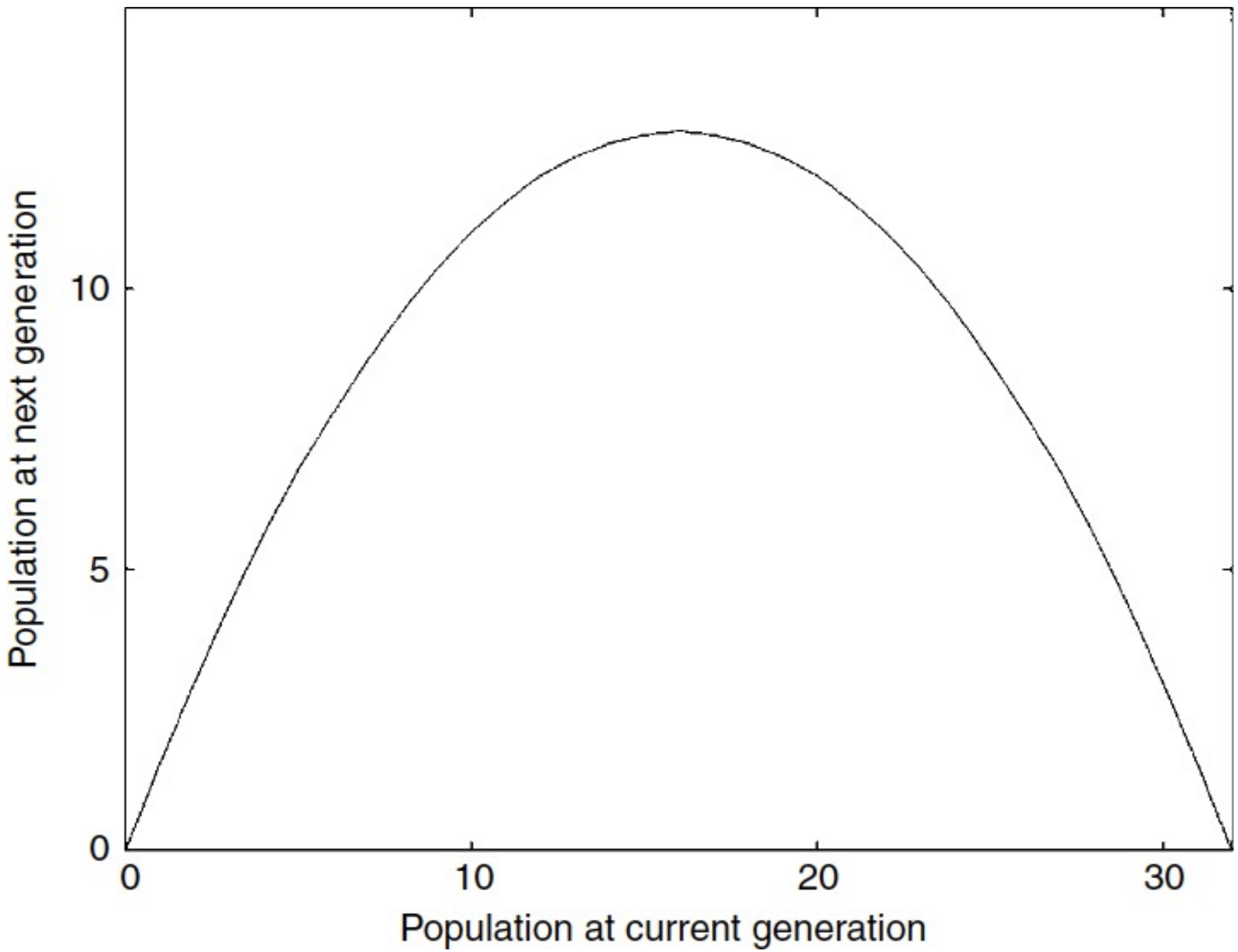
Logistic map: A simple, completely deterministic equation that, when iterated, can display chaos (depending on the value of *R*).

Deterministic chaos: Perfect prediction, *a la* Laplace's deterministic "clockwork universe", is impossible, even in principle, if we're looking at a chaotic system.

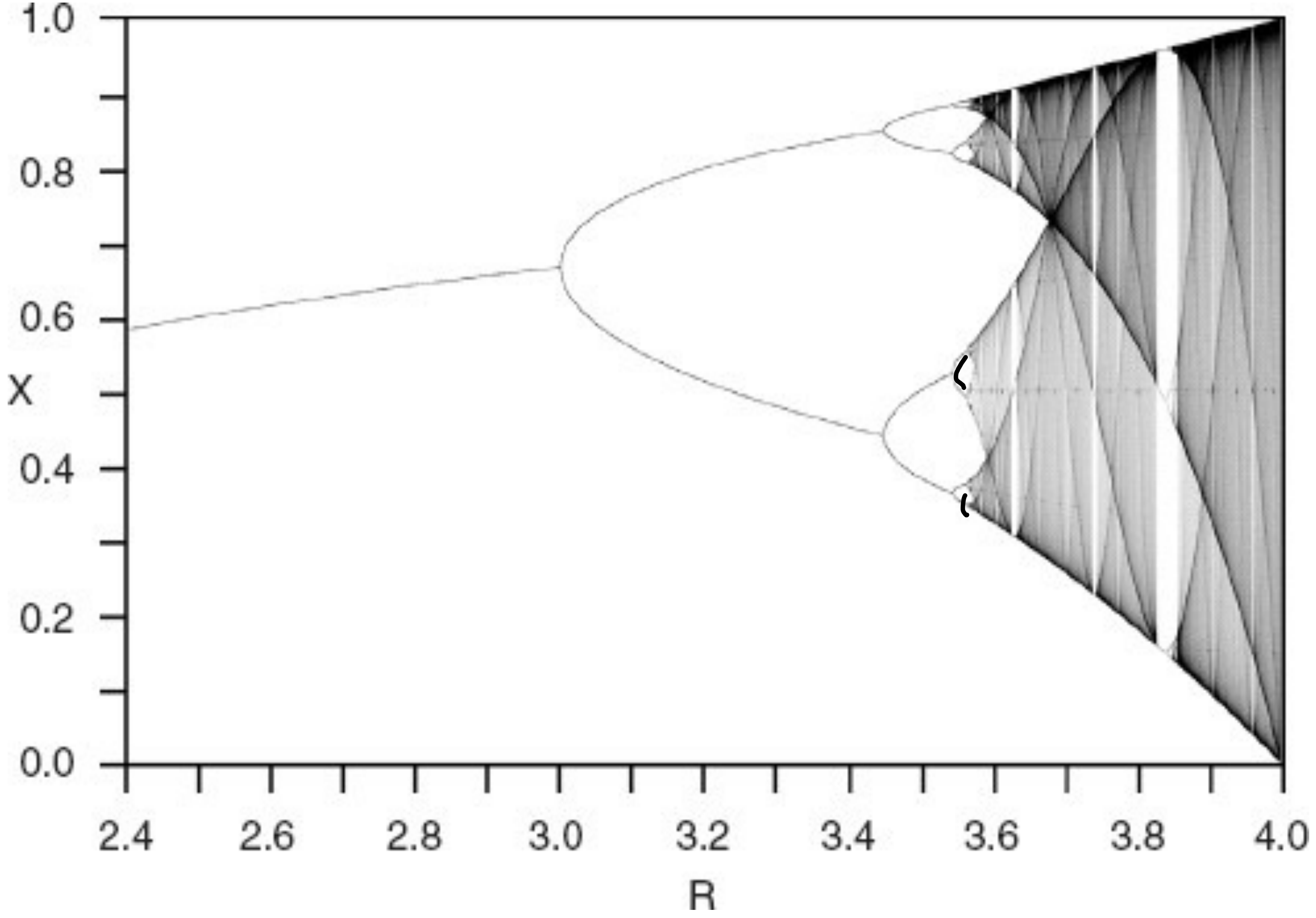
Universality in Chaos

While chaotic systems are not predictable in detail, a wide class of chaotic systems has highly predictable, “universal” properties.

A *Unimodal* (“one humped”) Map



Logistic Map Bifurcation Diagram



Bifurcations in the Logistic Map

$R_1 \approx 3.0$: period 2

$R_2 \approx 3.44949$ period 4

$R_3 \approx 3.54409$ period 8

$R_4 \approx 3.564407$ period 16

$R_5 \approx 3.568759$ period 32

$R_\infty \approx 3.569946$ period ∞ (onset of chaos)

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$R_\infty \approx 3.569946$ period ∞ (chaos)

Rate at which distance between bifurcations is shrinking:

$$\frac{R_2 - R_1}{R_3 - R_2} = \frac{3.44949 - 3.0}{3.54409 - 3.44949} = 4.75147992$$

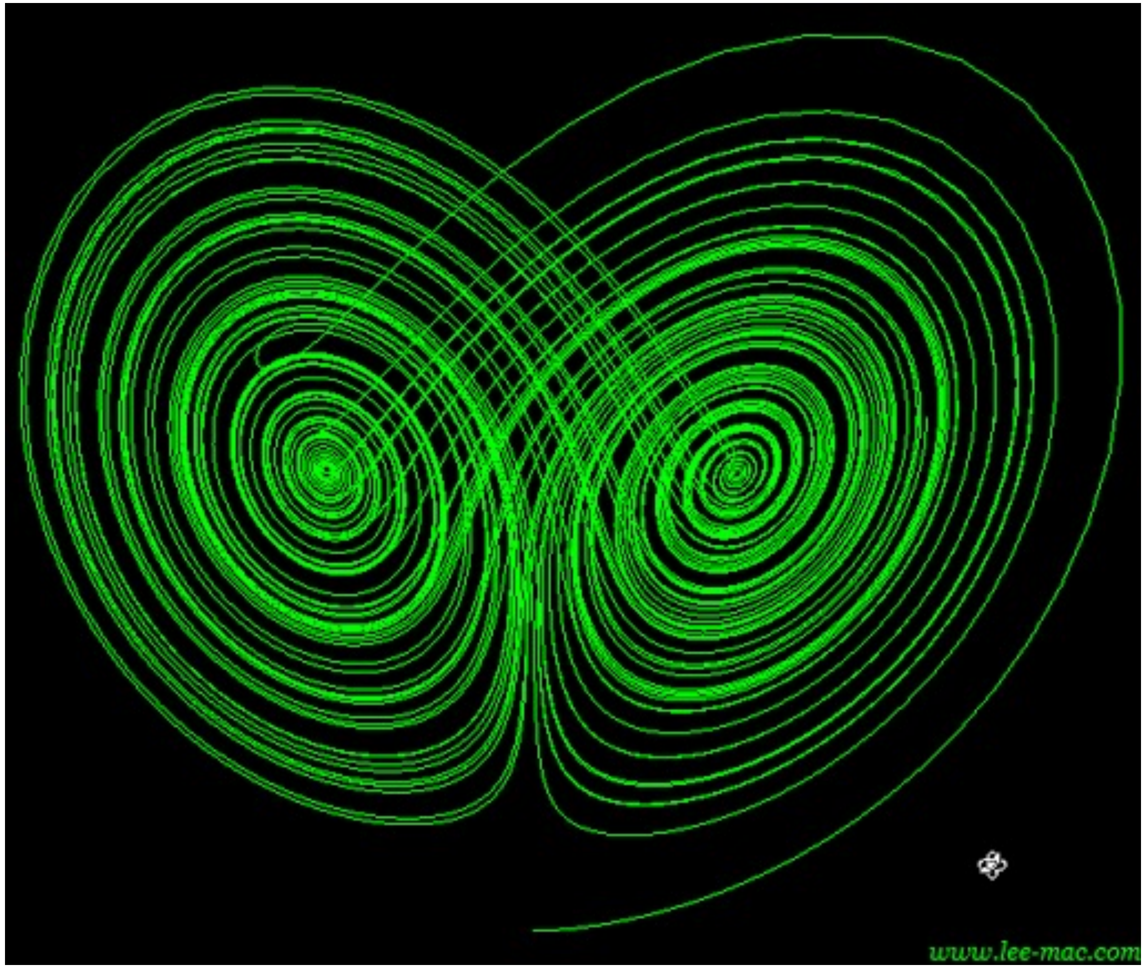
$$\frac{R_3 - R_2}{R_4 - R_3} = \frac{3.54409 - 3.44949}{3.564407 - 3.54409} = \boxed{4.65619924}$$

$$\frac{R_4 - R_3}{R_5 - R_4} = \frac{3.564407 - 3.54409}{3.568759 - 3.564407} = \boxed{4.66842831}$$

$$\vdots$$

$$\lim_{n\to\infty}\left(\frac{R_{n+1} - R_n}{R_{n+2} - R_{n+1}}\right) \approx \boxed{4.6692016....}$$

Lorenz Attractor



$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

SELECT ALL

```
(defun c:lorenz ( / a b c h n x x0 y y0 z z0 )
    (setq
        n  10000 ;; Iterations
        h  0.01  ;; Increment
        a  10.0  ;; sigma
        b  28.0  ;; rho
        c  (/ 8.0 3.0) ;; beta

        x0 0.1   ;;
        y0 0.0   ;; Initial values
        z0 0.0   ;;
    )
    (entmake '((0 . "POLYLINE") (70 . 8)))
    (repeat n
        (setq
            x  (+ x0 (* h a (- y0 x0)))
            y  (+ y0 (* h (- (* x0 (- b z0)) y0)))
            z  (+ z0 (* h (- (* x0 y0) (* c z0))))
            x0 x
            y0 y
            z0 z
        )
        (entmake (list '(0 . "VERTEX") '(70 . 32) (list 10 x y z)))
    )
    (entmake '((0 . "SEQEND")))
    (princ)
)
```

http://ww.lee-mac.com/attractors.html

Amazingly, at almost exactly the same time, the same constant was independently discovered (and mathematically derived by) another research team, the French mathematicians Pierre Collet and Charles Tresser.

Summary
Significance of dynamics and chaos for complex systems

- Complex, unpredictable behavior from simple, deterministic rules

- Dynamics gives us a vocabulary for describing complex behavior

- There are fundamental limits to detailed prediction

- At the same time there is universality: “Order in Chaos”