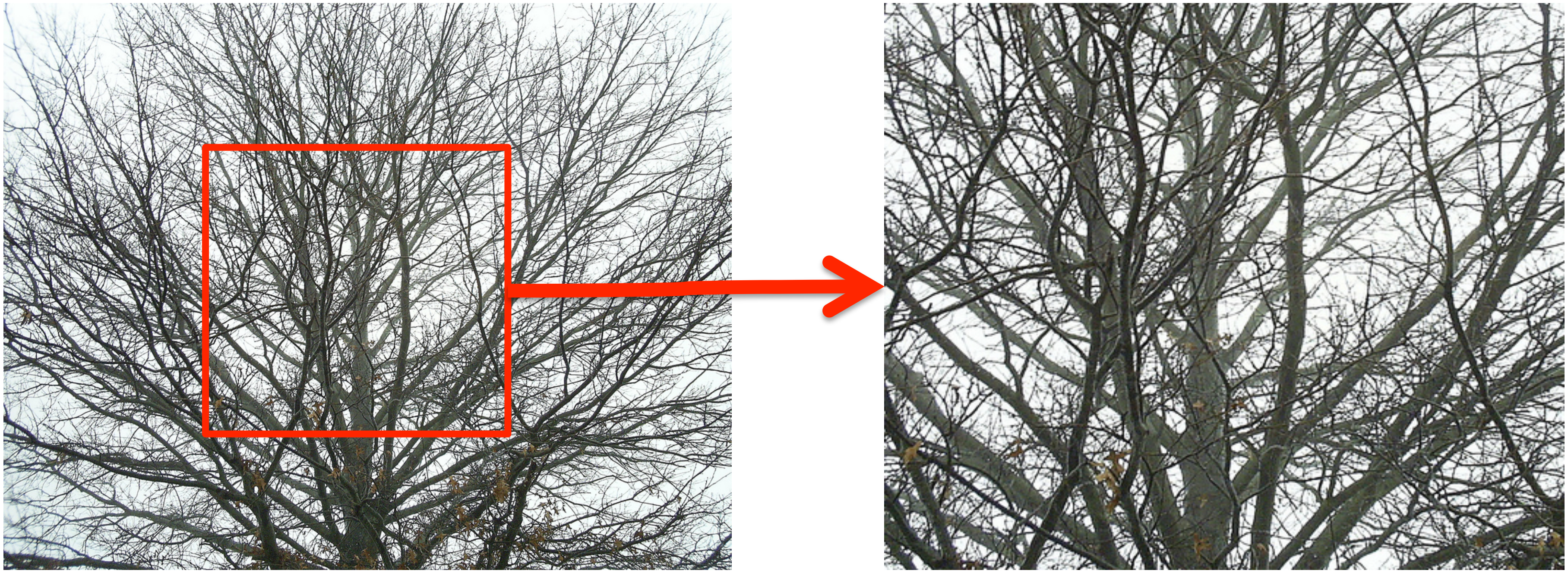
Fractals:

Objects with “self-similarity” at different scales

Trees are Fractal

Illustrating "self-similarity"



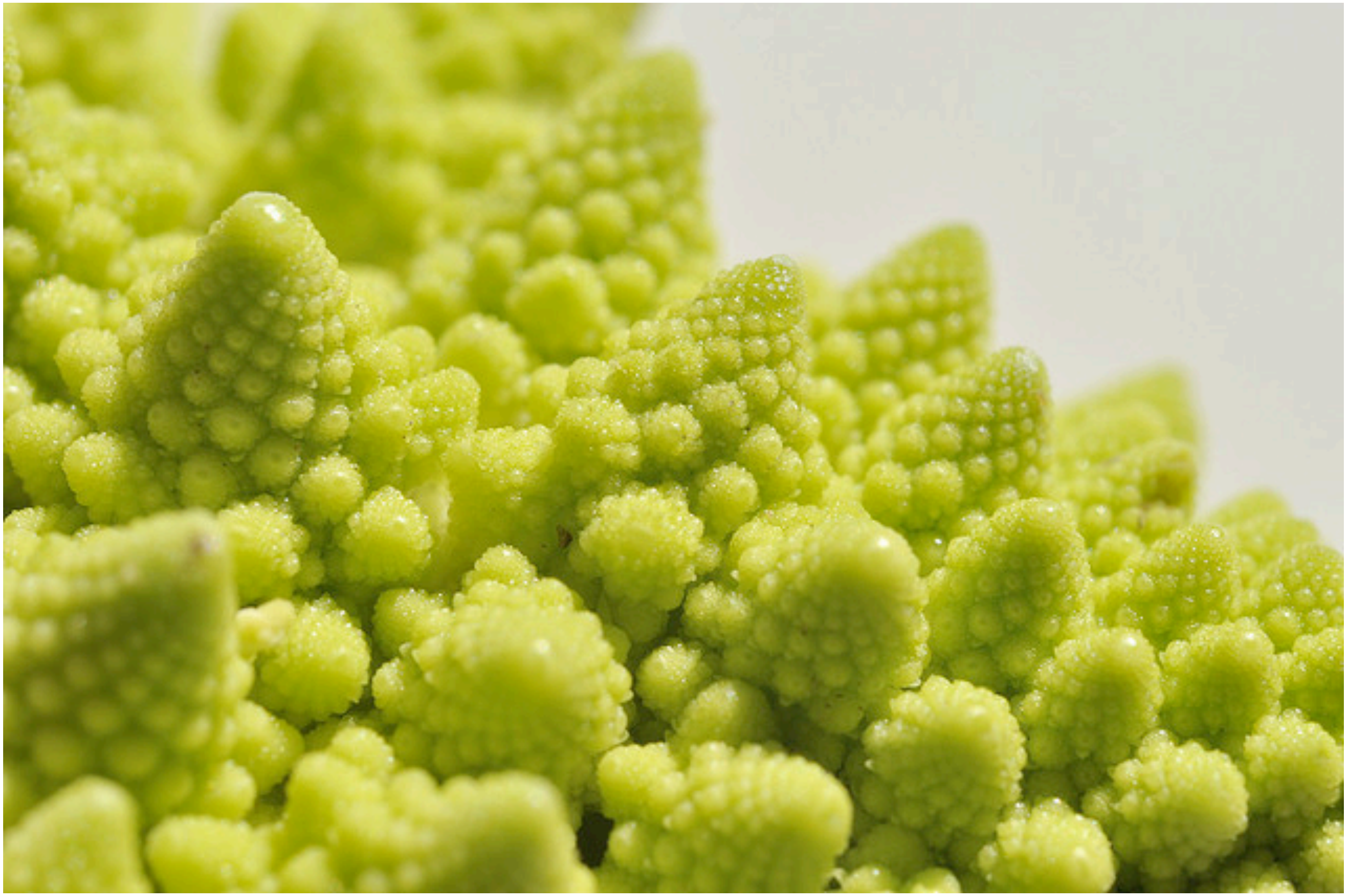
Illustrating “self-similarity”

Illustrating “self-similarity”

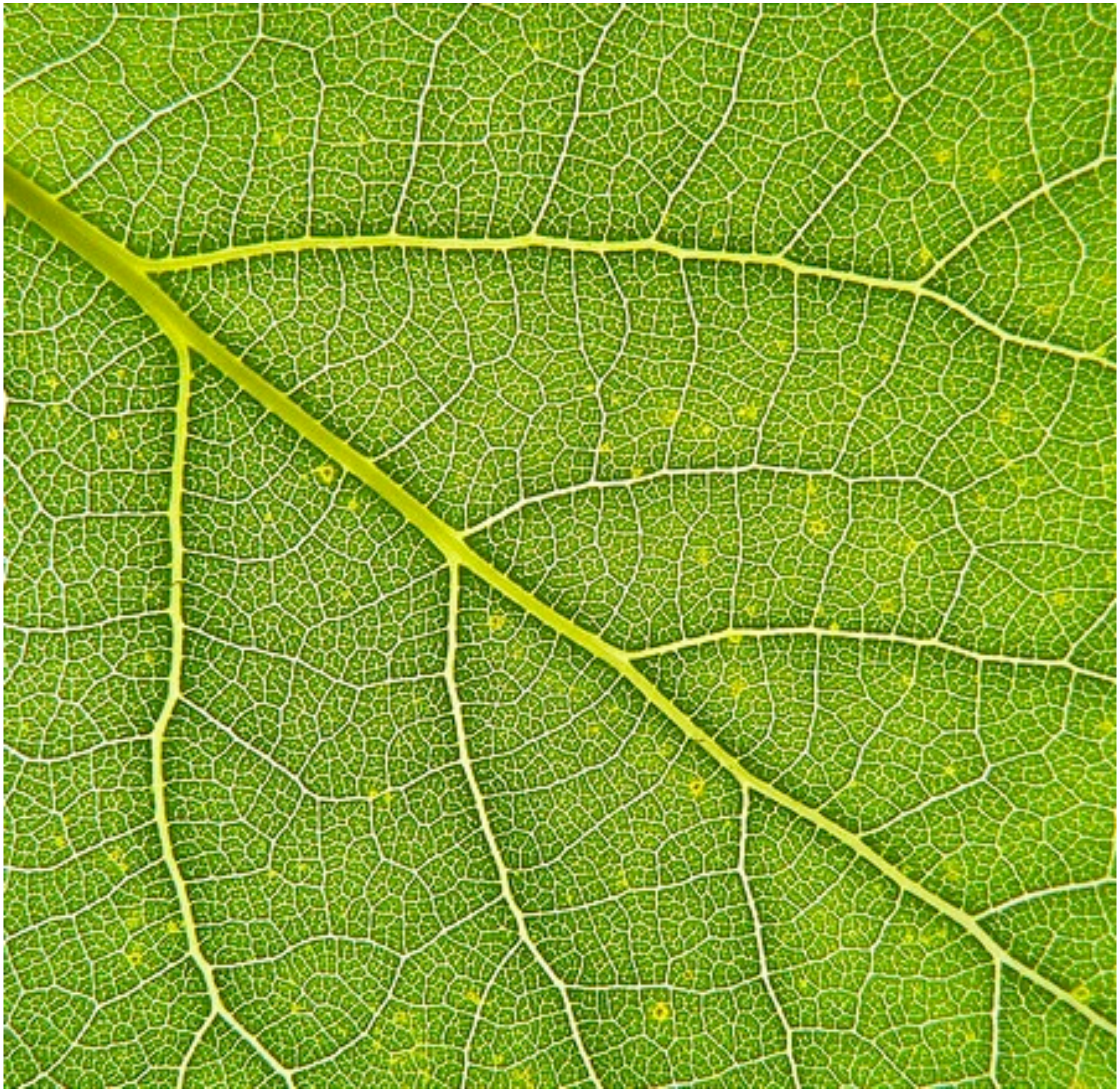
For all you mathematicians out there:

Fractal versus *Fractal-Like*

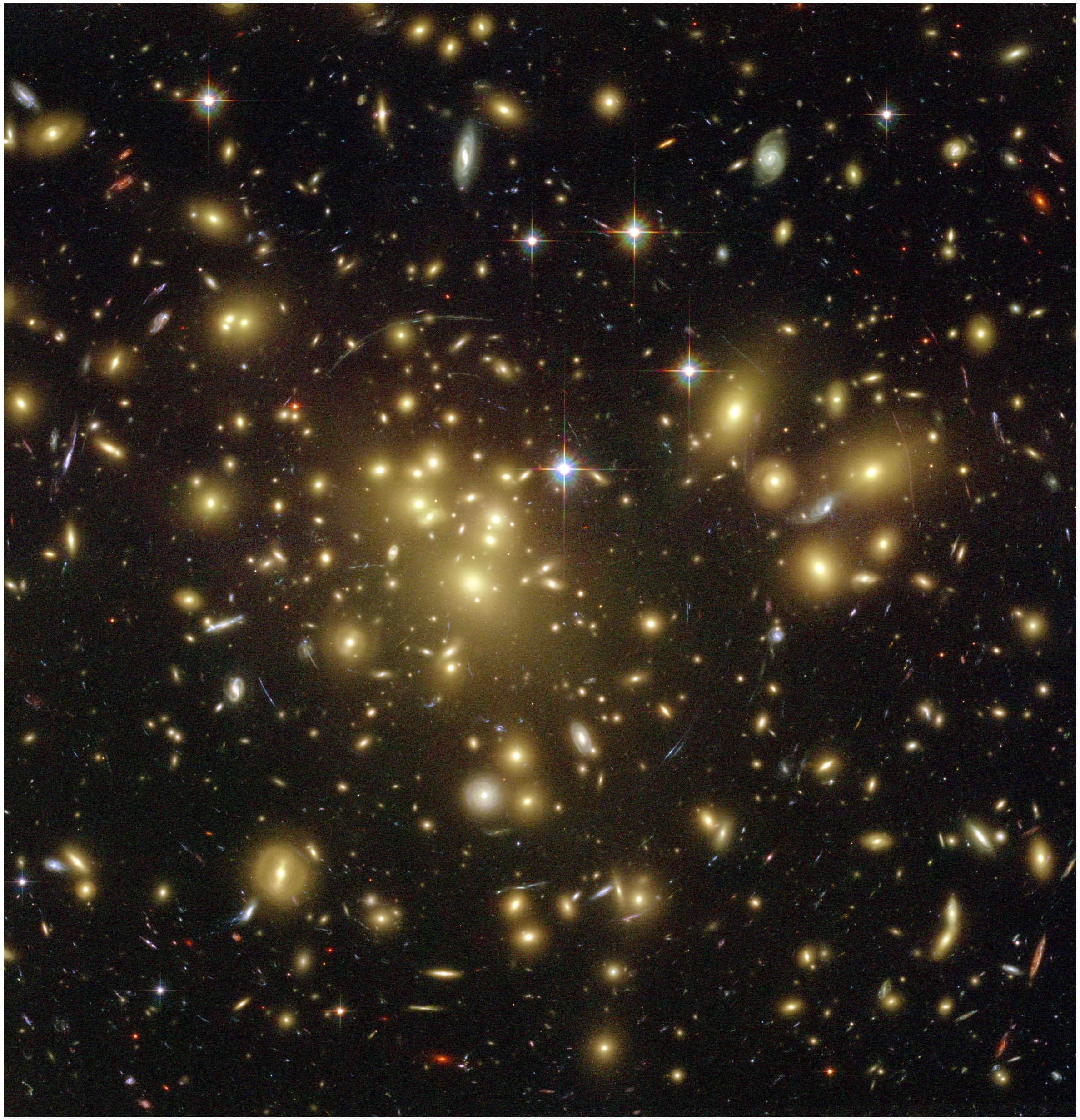
Broccoli is Fractal



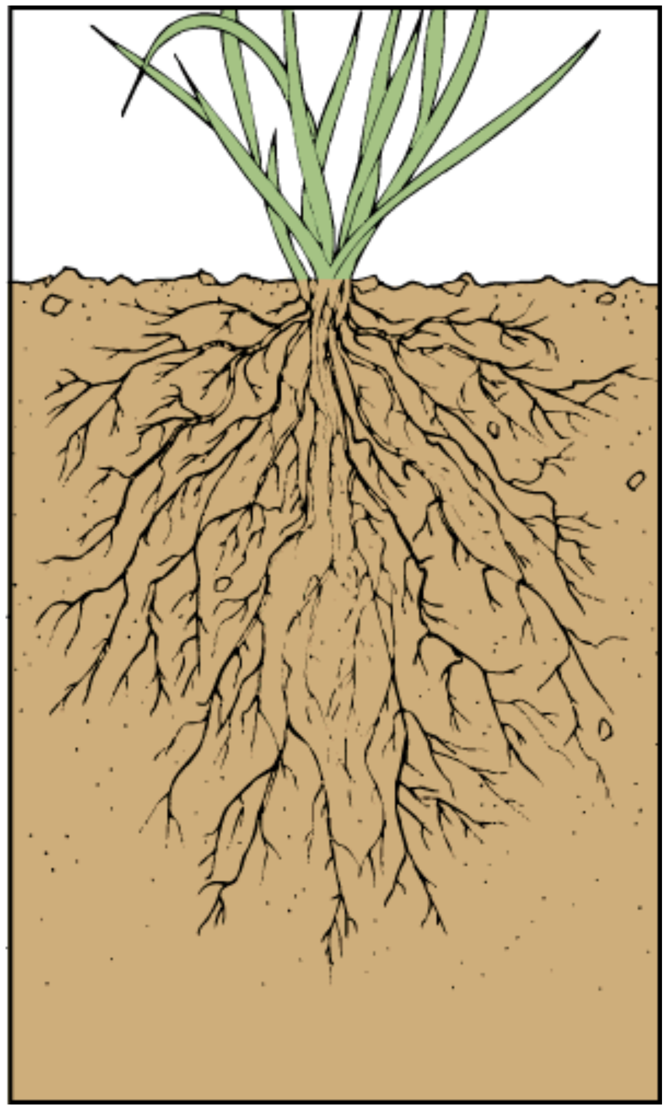
Leaf Veins are Fractal



Galaxy Clusters are Fractal

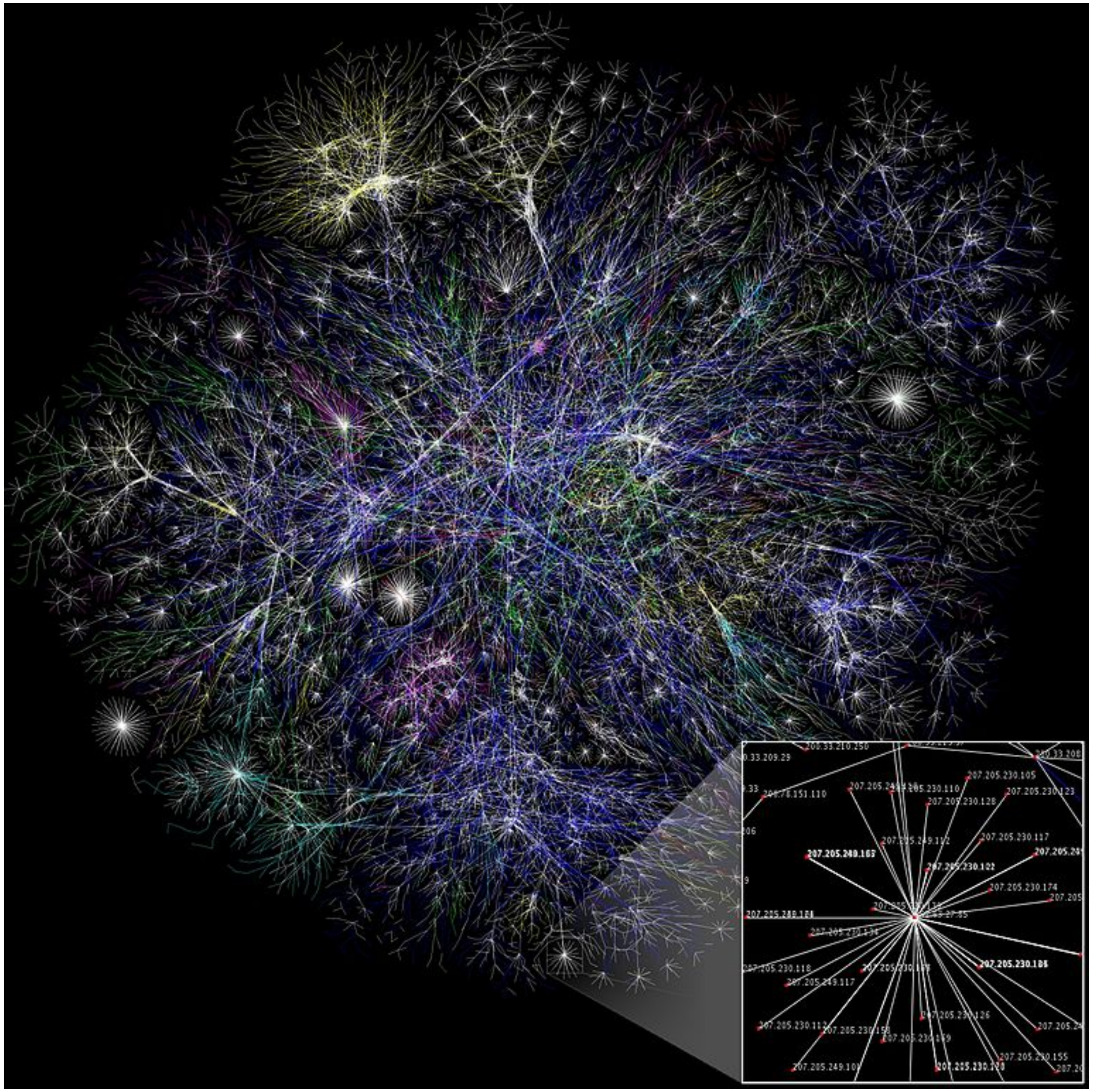


Plant Roots are Fractal



Mountain Ranges are Fractal

The World-Wide-Web is Fractal



A Bit of History

Benoit Mandelbrot, 1924−2010

Many mathematicians have studied the notions of self-similarity, and of “fractional dimension” and what an object with a fractional dimension would look like.

The term *fractal*, to describe such objects, was coined by the mathematician Benoit Mandelbrot, from the Latin root for “fractured”.

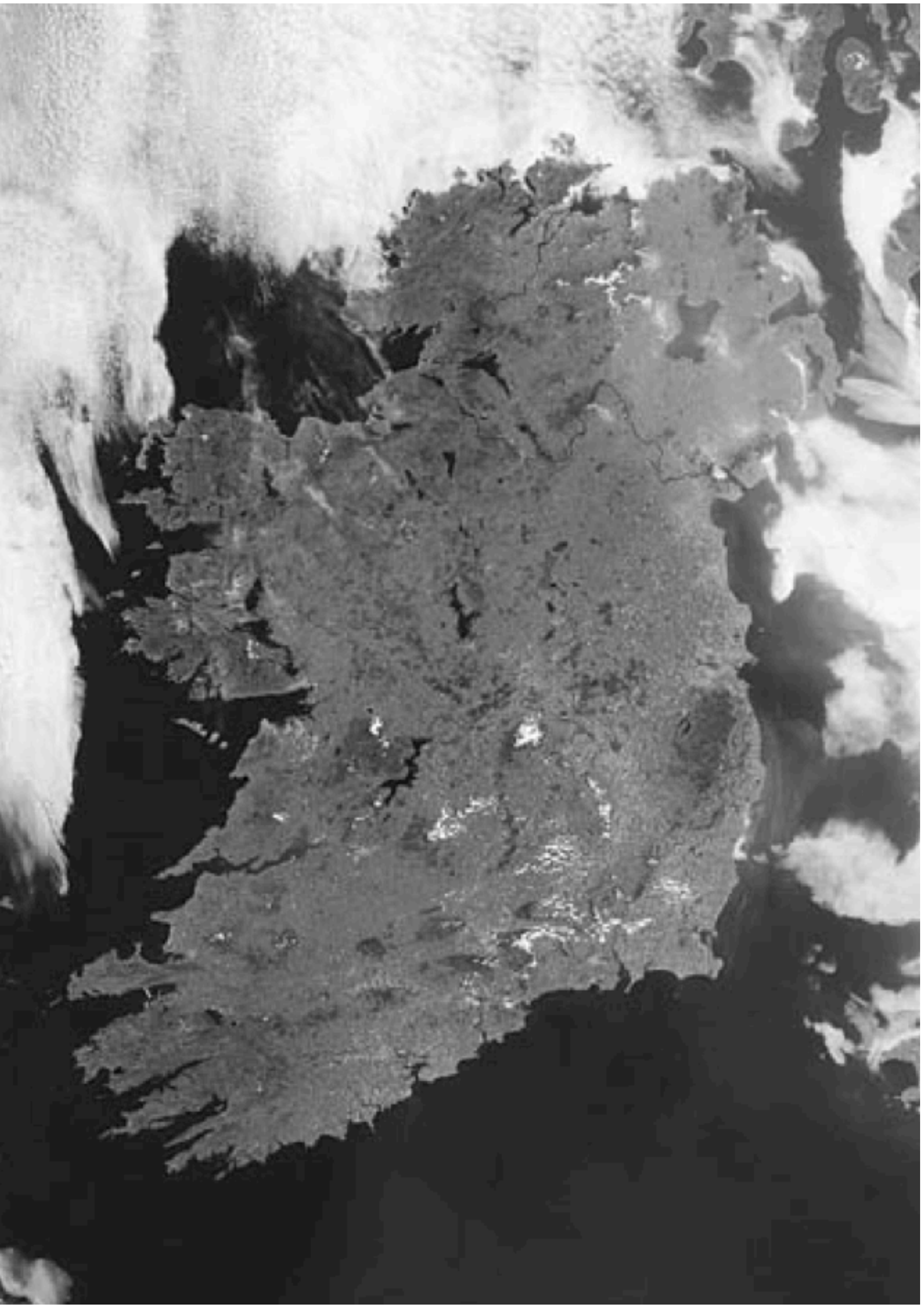
Mandelbrot’s goal was to develop a mathematical “theory of roughness” to better describe the natural world.

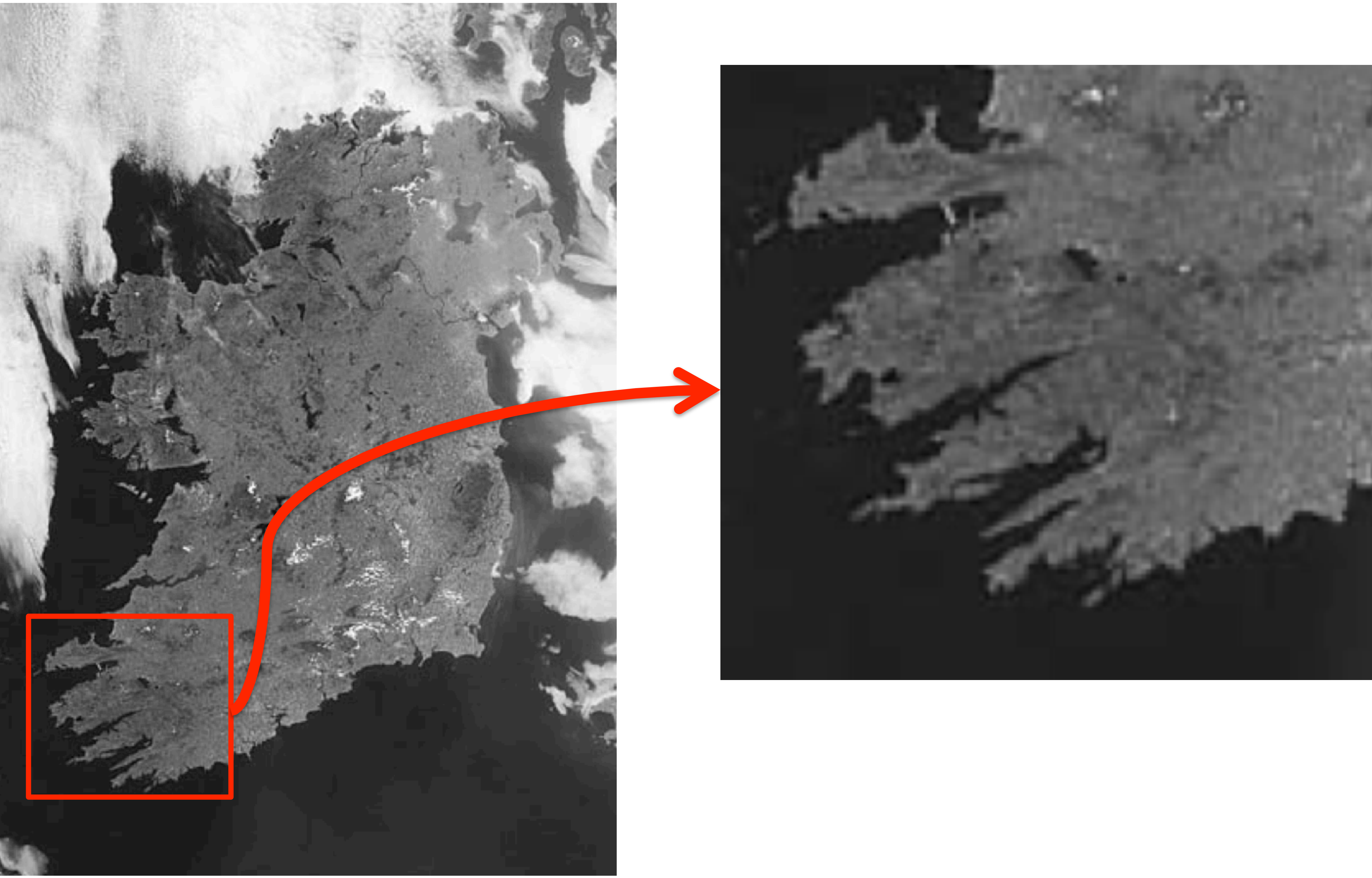
He brought together the work of different mathematicians in different fields to create the field of *Fractal Geometry.*

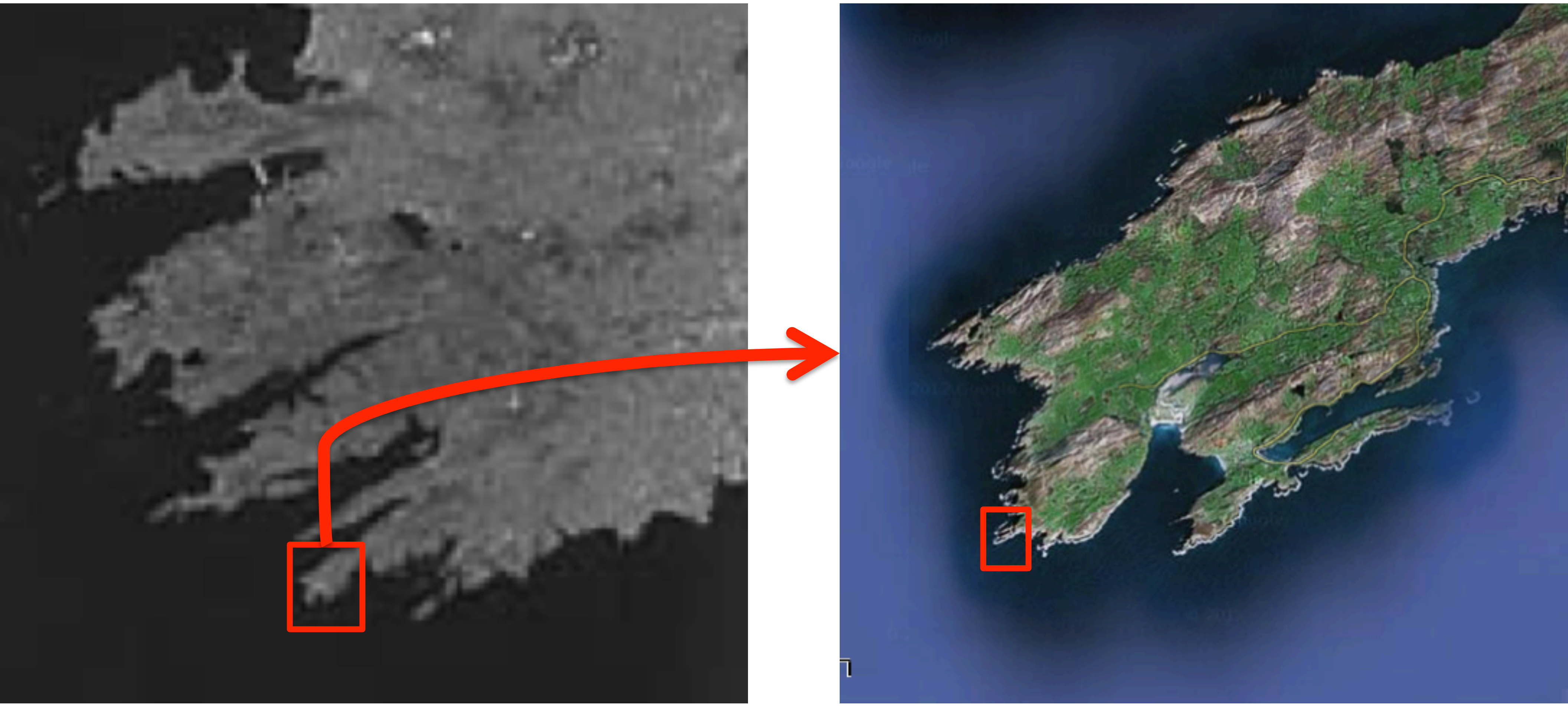
Mandelbrot’s example:
Measuring the length of the coastline of Great Britain

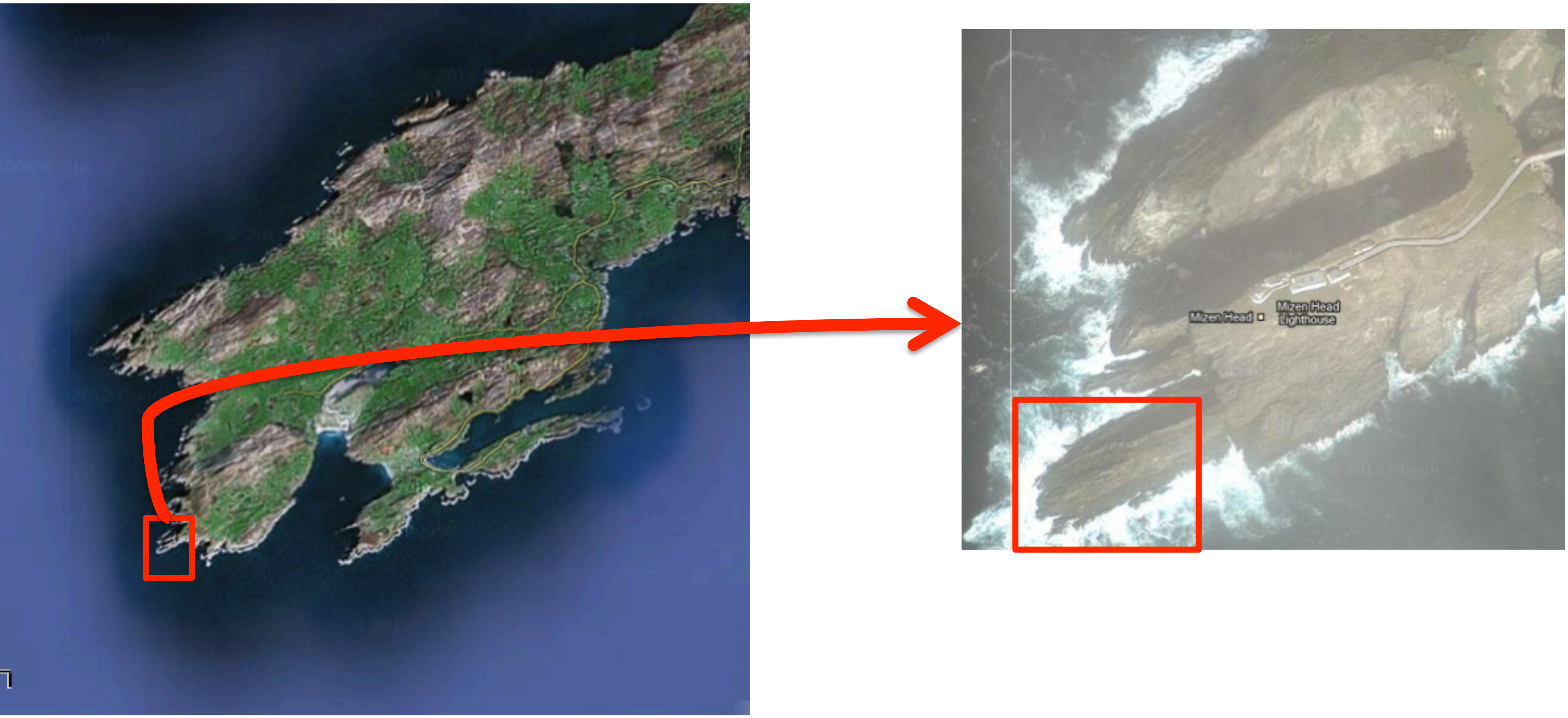
What size ruler should you use?

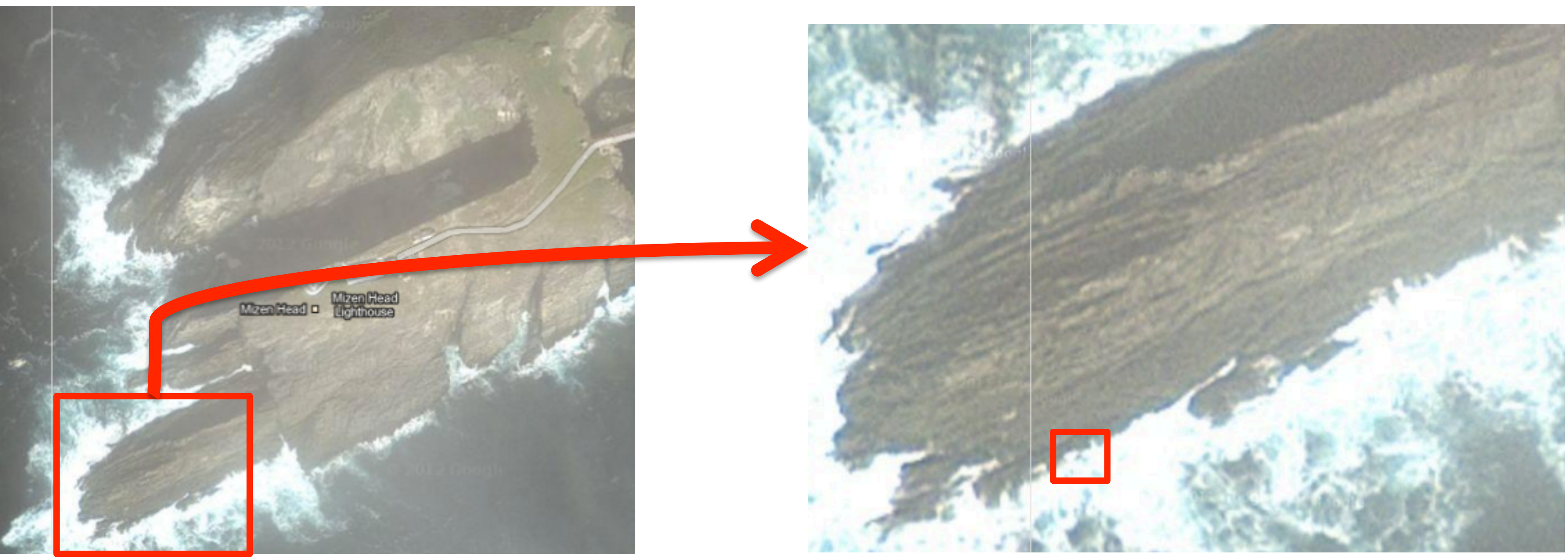
Ireland

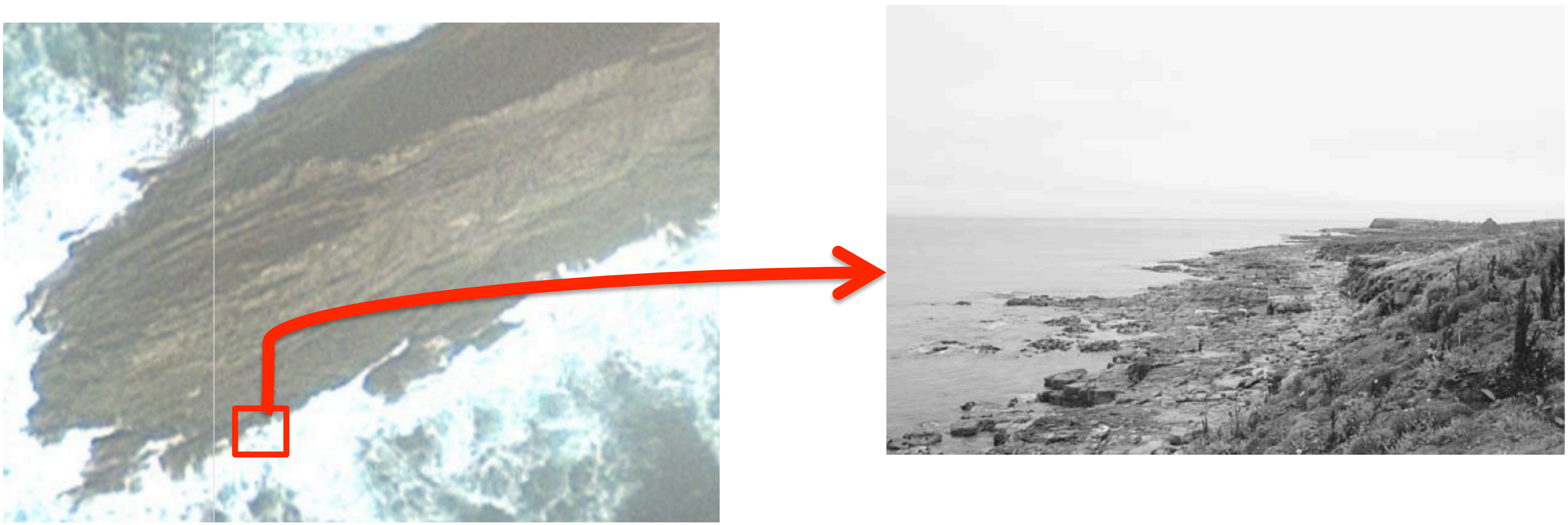







Mizen Head
Mizen Head Lighthouse


Mizen Head
Mizen Head
Lighthouse



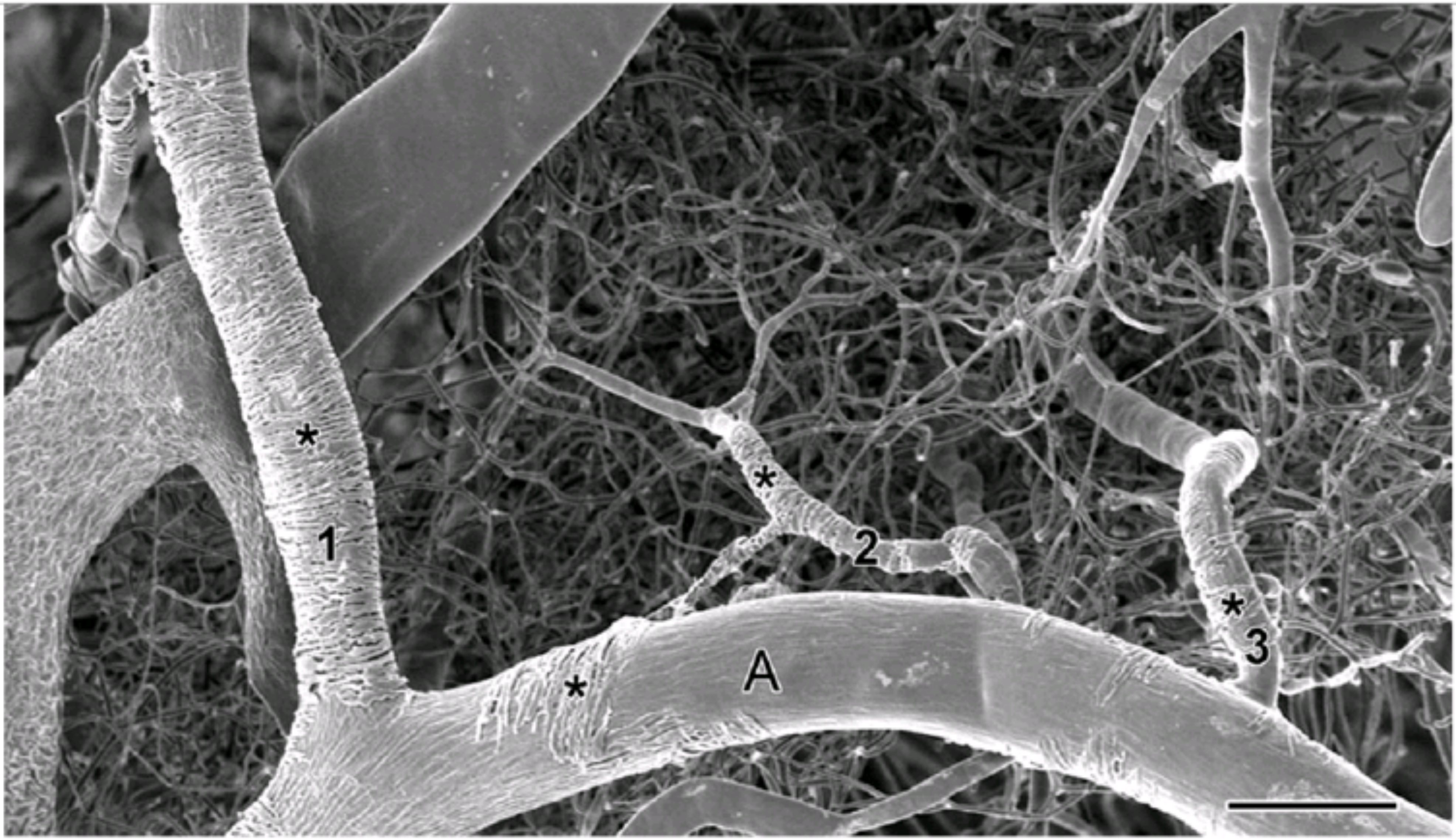
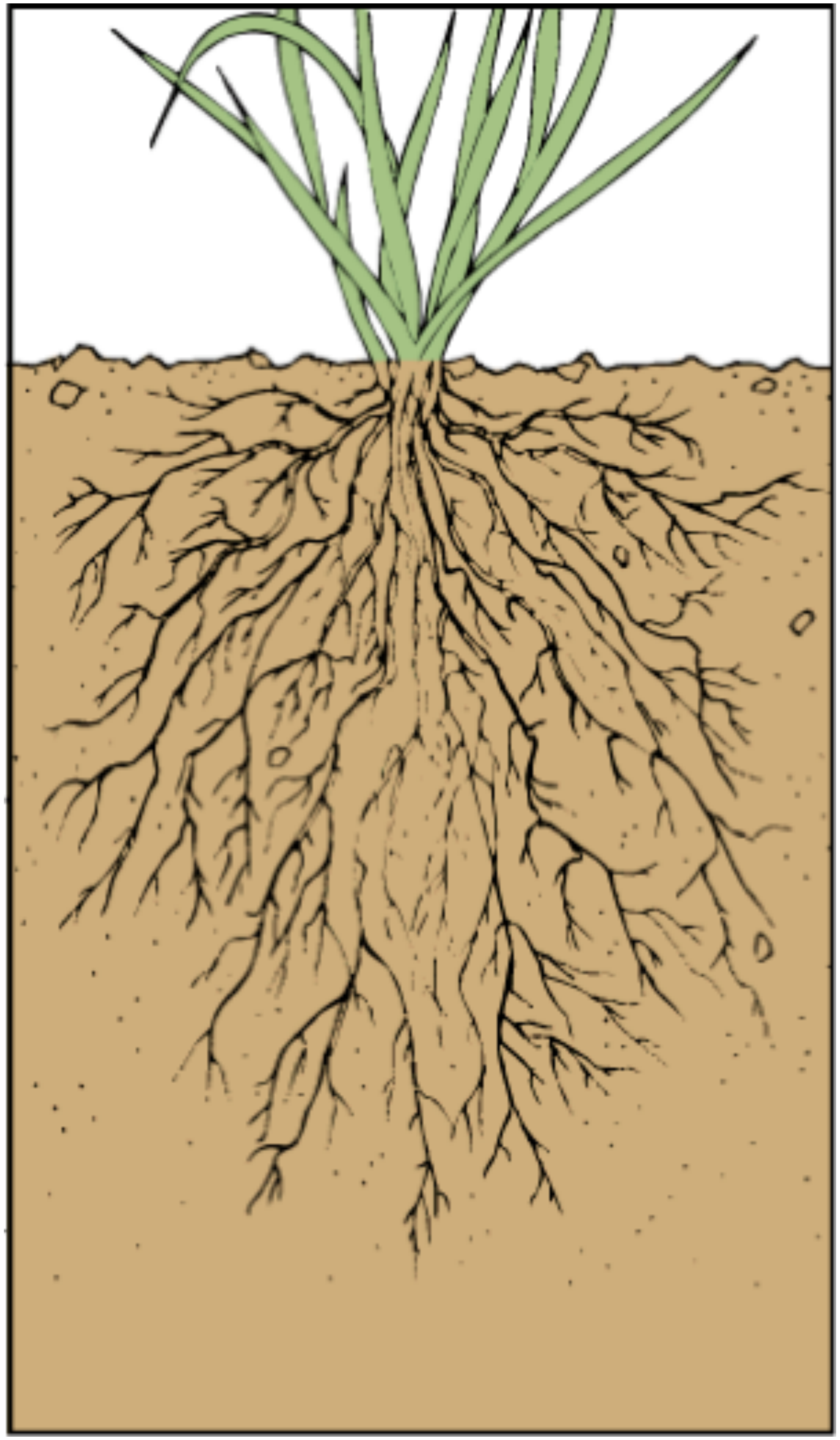


Figure 6. Scanning electron micrograph revealing vasculature within the area corresponding to the maximum acoustically evoked intrinsic signal. The arteries (A) and vein (V) can be clearly distinguished. 1, 2, 3: three types of arterial collateral vessels (see text). Note evidence of smooth muscle banding (asterisk symbols) on arteriole walls. Bar = 100 μm.



http://static.ddmcdn.com/gif/willow/root-info2.gif

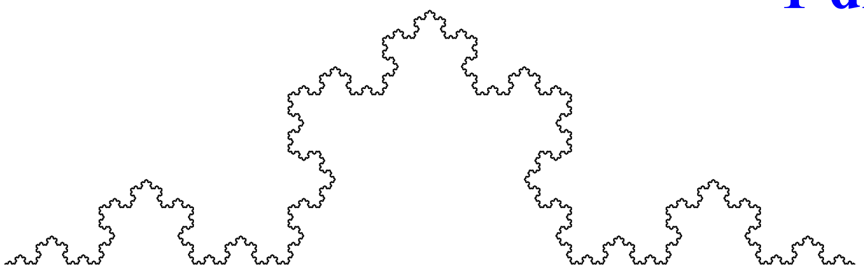
http://images.sciencedaily.com/2012/05/120503120130.jpg

Mandelbrot's example:
Measuring the length of the coastline of Great Britain

What size ruler should you use?

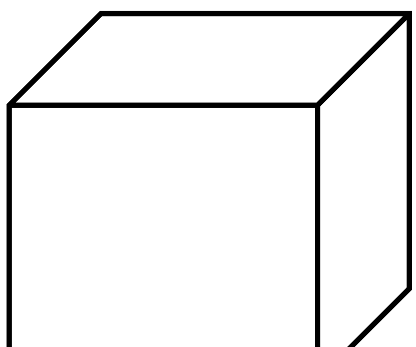
Dimension:
"Extension in a given direction"

1-dimensional



In between 1 and 2 dimensional!

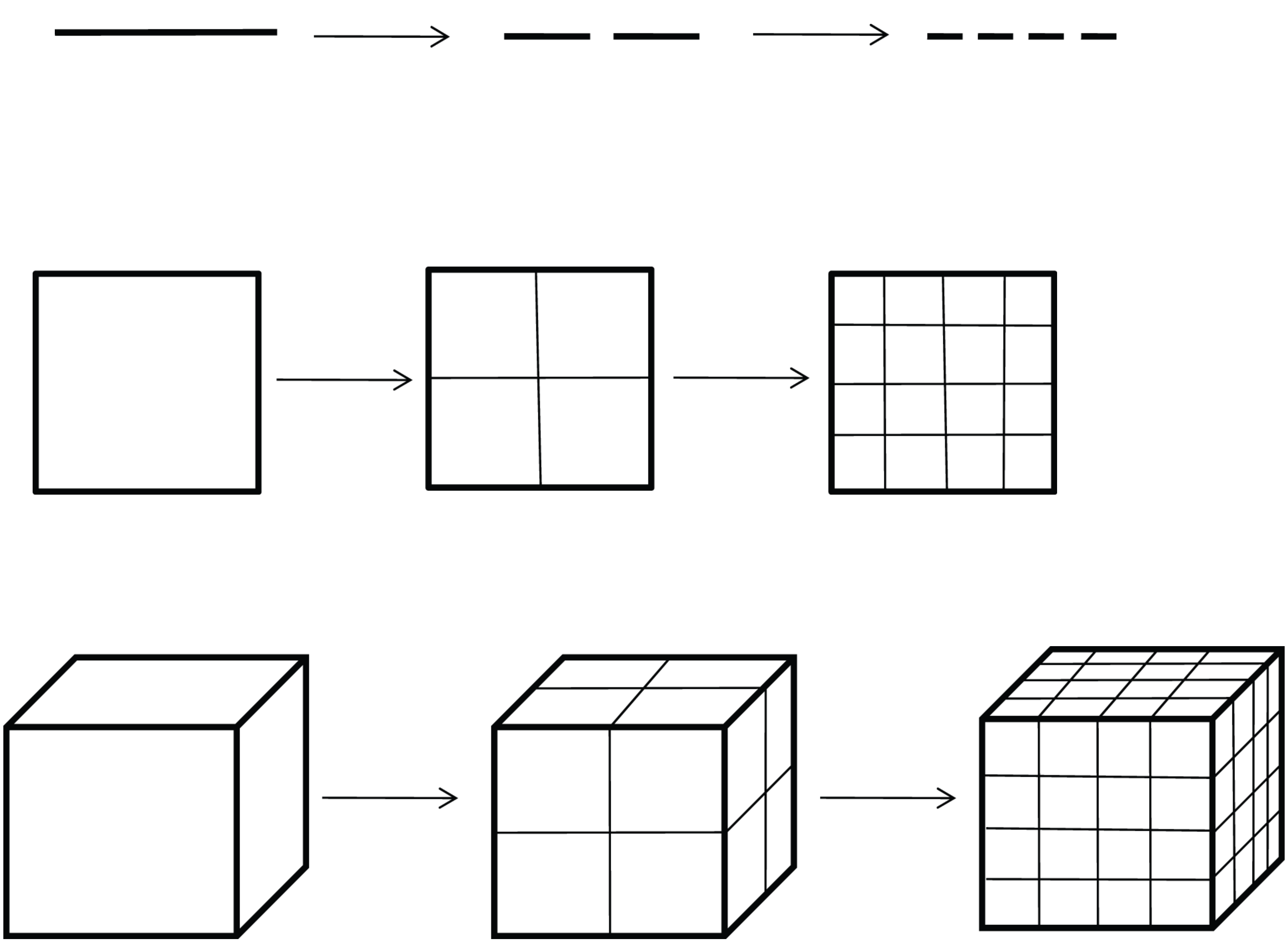
2-dimensional



3-dimensional

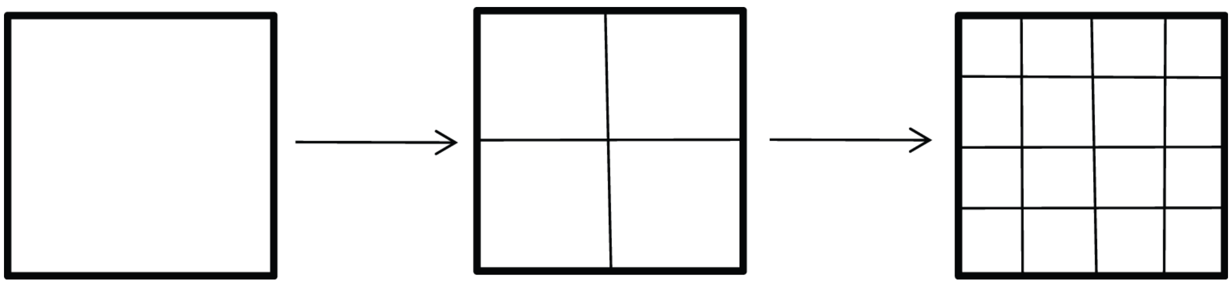
Characterizing Dimension

What happens when you continually bisect (cut in two equal halves) the sides of lines, squares, cubes, etc.?

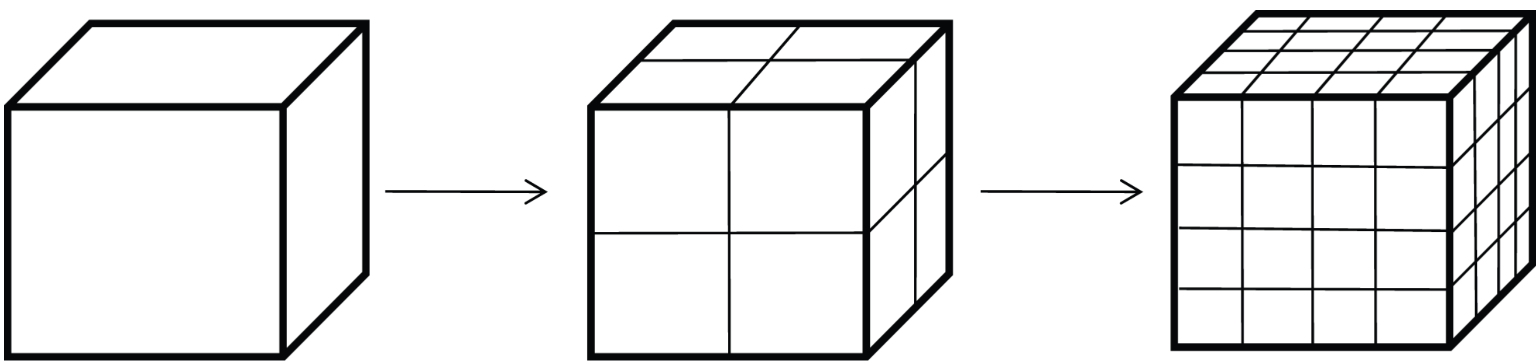


Dimension 1: Each level is made up of two 1/2-sized copies of previous level

Dimension 2: Each level is made up of four 1/4-sized copies of previous level



Dimension 3: Each level is made up of eight 1/8-sized copies of previous level

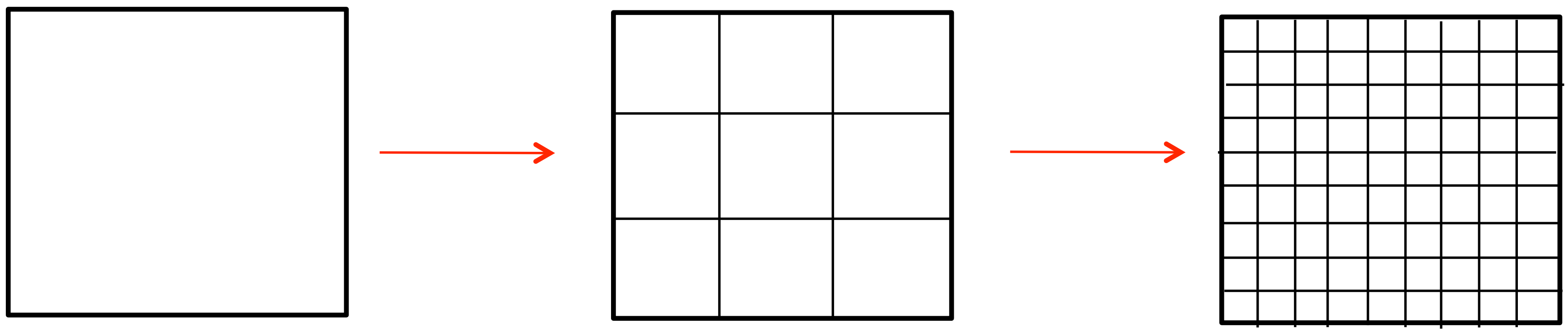


Dimension 4: Each level is made up of sixteen 1/16-sized copies of previous level

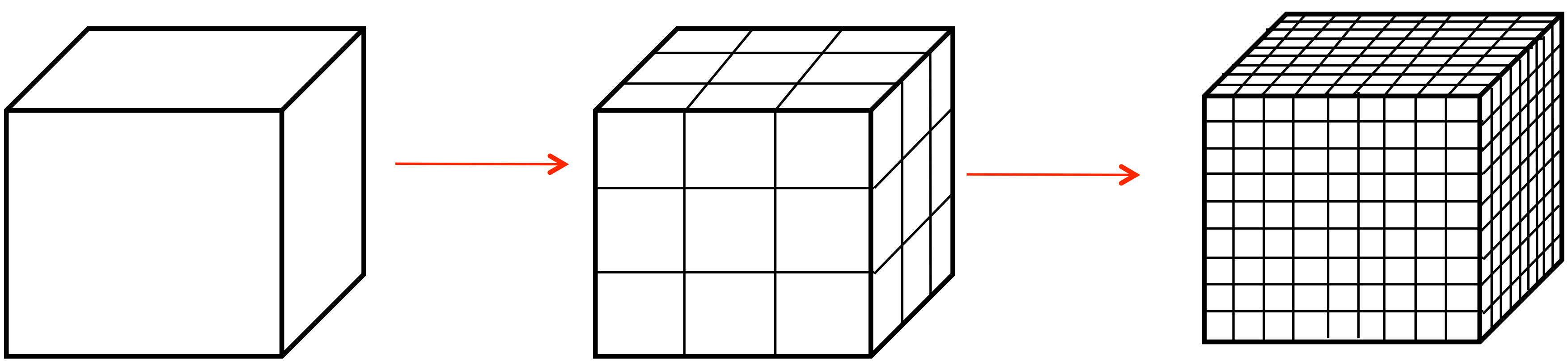
Dimension 20: Each level is made up of ??

Trisecting sides

Dimension 1: Each level is made up of three 1/3-sized copies of previous level



Dimension 2: Each level is made up of nine 1/9-sized copies of previous level



Dimension 3: Each level is made up of 27 1/27-sized copies of previous level

~~Trisecting~~ sides

M-secting

Dimension 1: Each level is made up of ~~three 1/3-sized copies~~ of previous level

M 1/M-sized copies

Dimension 2: Each level is made up of ~~nine 1/9-sized copies~~ of previous level

M^2 $1/M^2$-sized copies

Dimension D: Each level is made up of M^D $1/M^D$-sized copies of previous level

Dimension 3: Each level is made up of ~~27 1/27-sized copies~~ of previous level

M^3 $1/M^3$-sized copies

Definition of dimension

Create a geometric structure from a given *D*-dimensional object (e.g., line, square, cube, etc) by repeatedly dividing the length of its sides by a number *M*.

Then each level is made up of M^D copies of the previous level.

Call the number of copies *N*.

Then $N = M^D$.

We have:

$$\log N = D \log M$$

$$D = \log N / \log M$$

$\log N = D \log M$

$D = \log N / \log M$

Dimension 1: $N = 2$, $M = 2$, $D = \log 2 / \log 2 = 1$

$N = 3$, $M = 3$, $D = \log 3 / \log 3 = 1$

Dimension 2: $N = 4$, $M = 2$, $D = \log 4 / \log 2 = 2$

$N = 9$, $M = 3$, $D = \log 9 / \log 3 = 2$

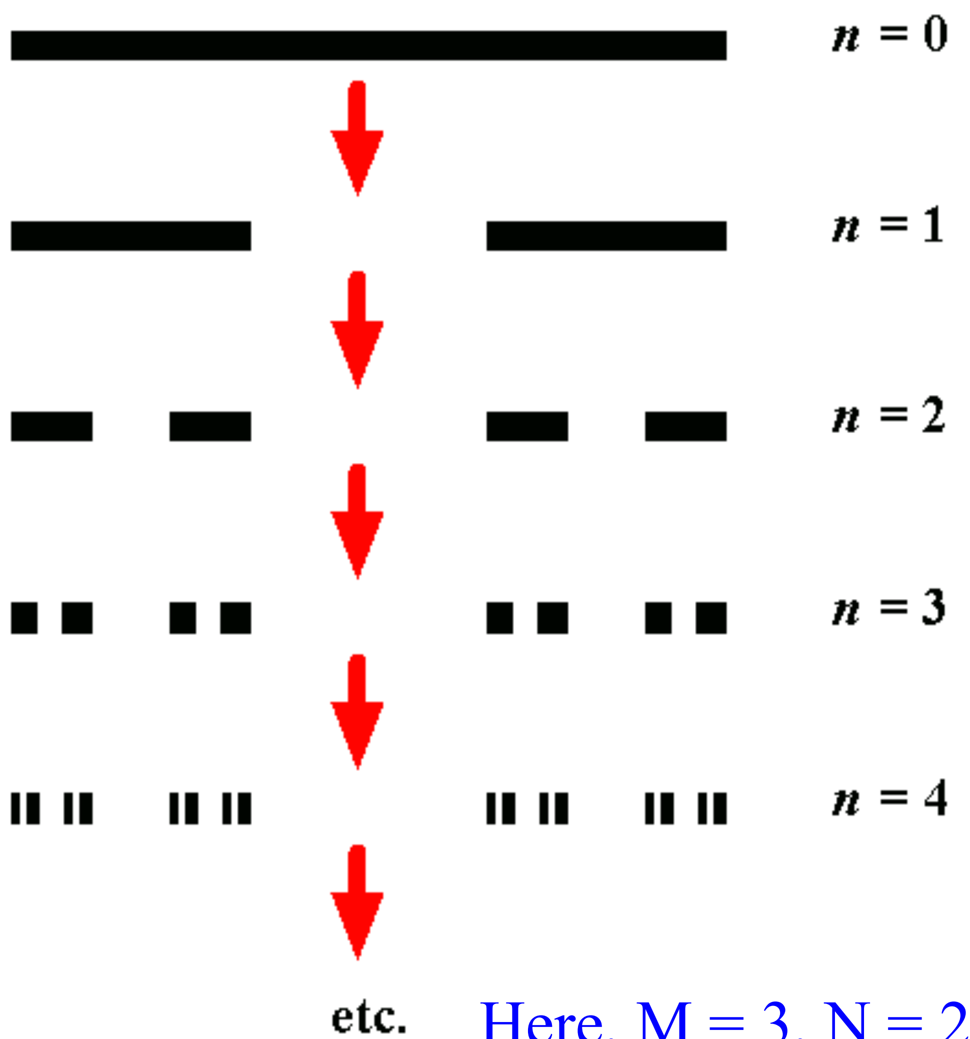
Koch curve: Here, $N = 4$, $M = 3$

So Fractal Dimension = $\log 4 / \log 3$

≈ 1.26

A measure of how the increase in number of copies scales with the decrease in size of the segment -- Roughly – the density of the self-similarity.

Cantor set



Here, M = 3, N = 2

So Fractal Dimension = log 2 / log 3 ≈ .63

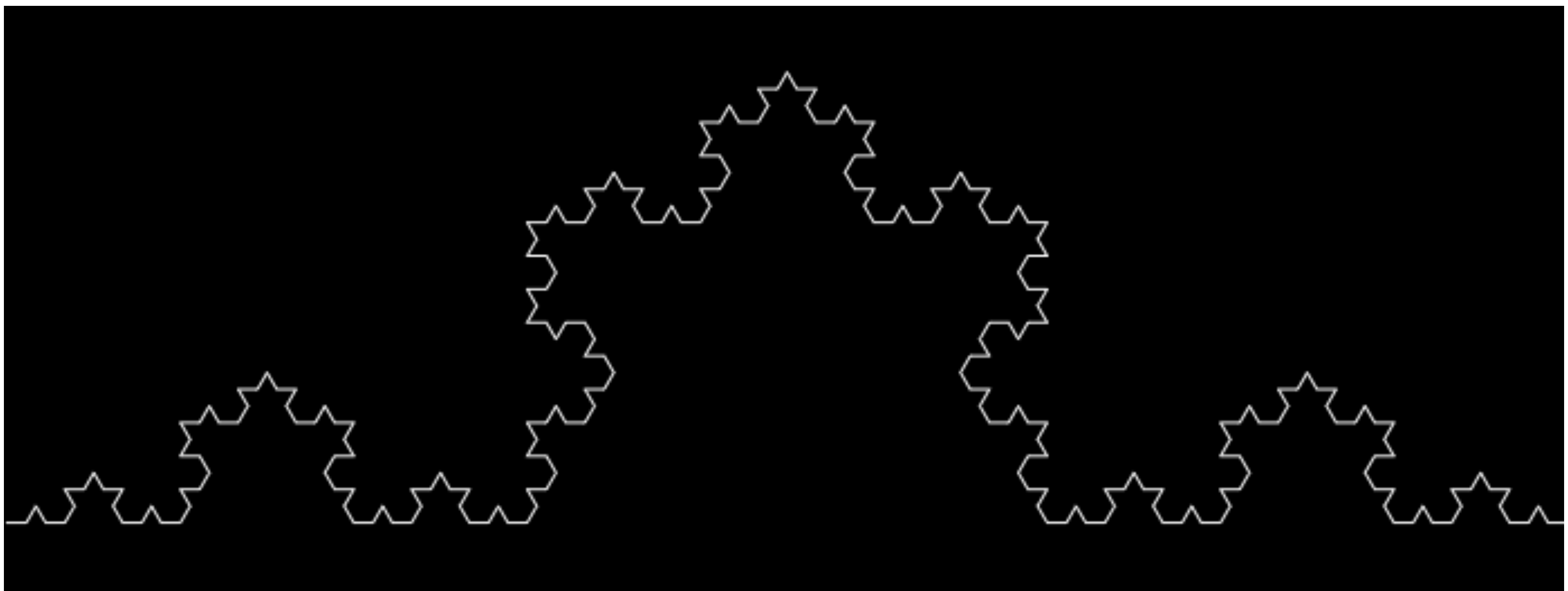
Fractal Dimension

N = number of copies of previous level = 4

M = reduction factor from previous level = 3

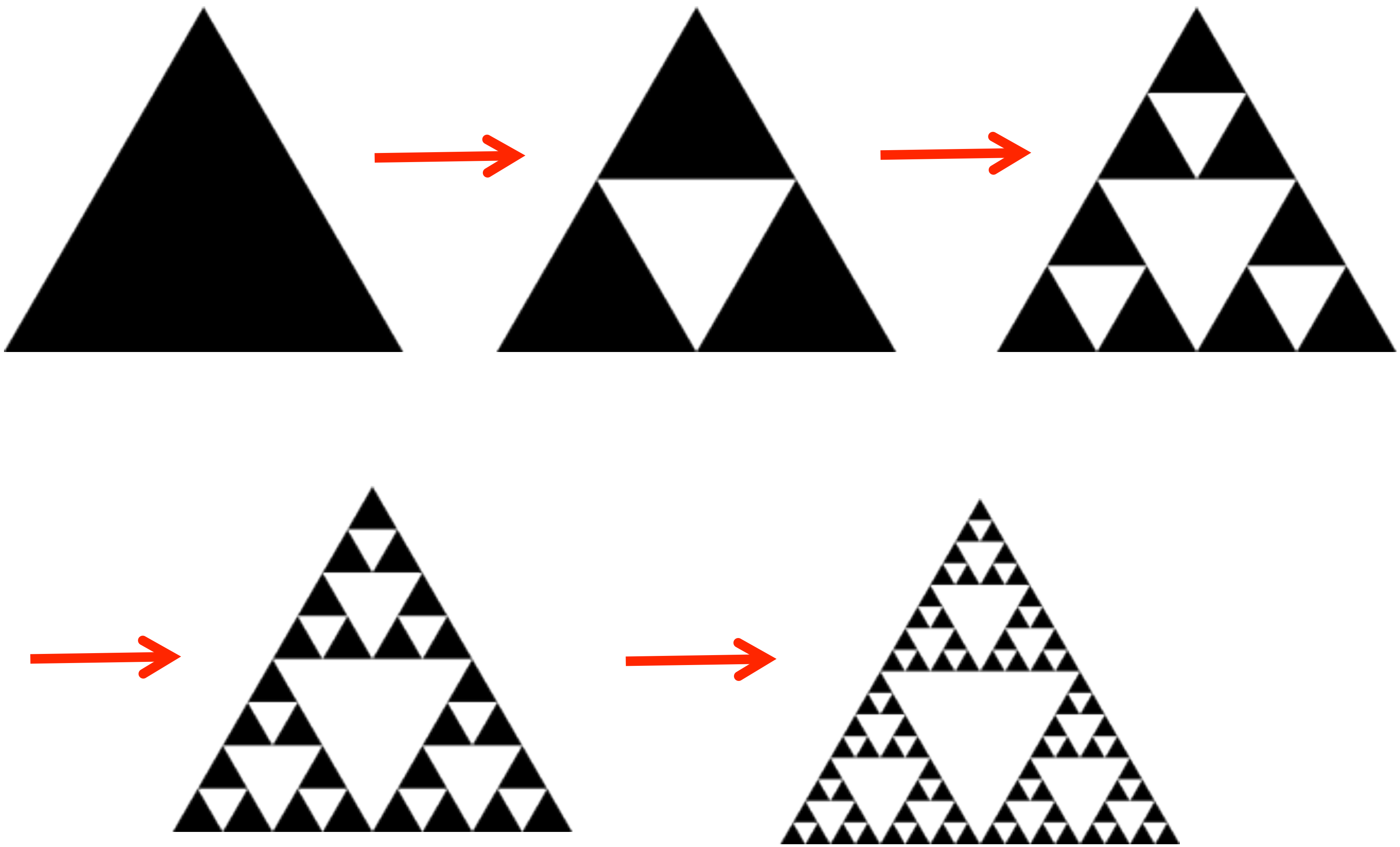
Dimension $D = \log N / \log M = \log 4 / \log 3 \approx 1.26$

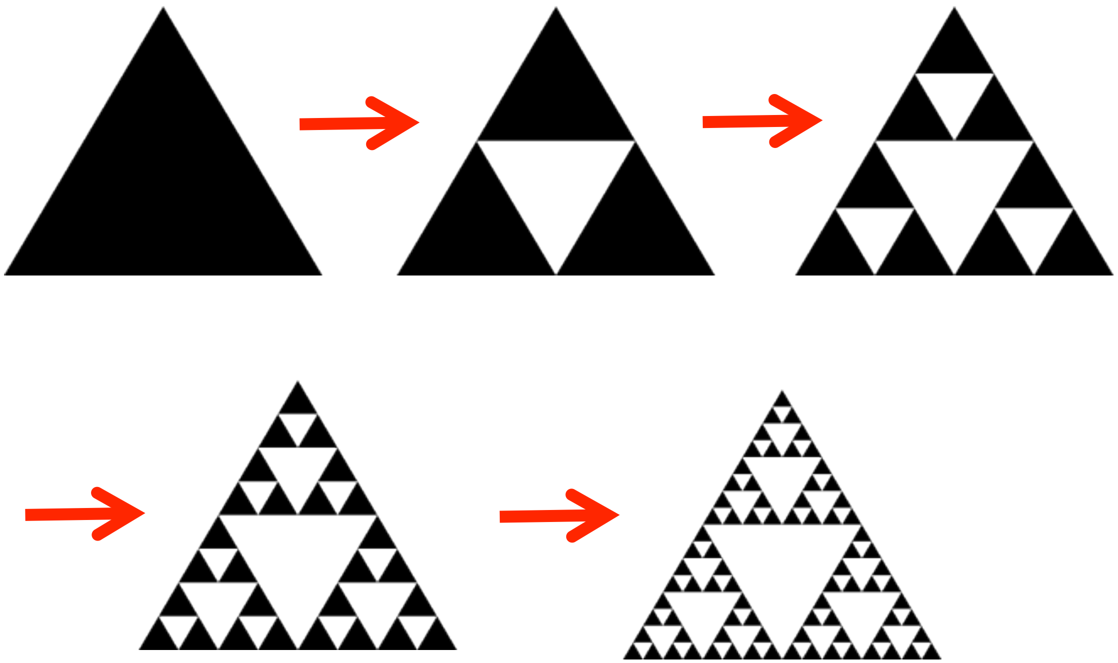
This version of fractal dimension is called *Hausdorff Dimension*, after the German mathematician Felix Hausdorff



Sierpinski Triangle

(Waclaw Sierpinski, 1916)





Fractal Dimension:

$$D = \frac{\log\ (\text{number of copies of previous level})}{\log\ (\text{reduction factor of side from previous level})}$$

Approximate dimension of Cauliflower

Fractal Structure of a White Cauliflower

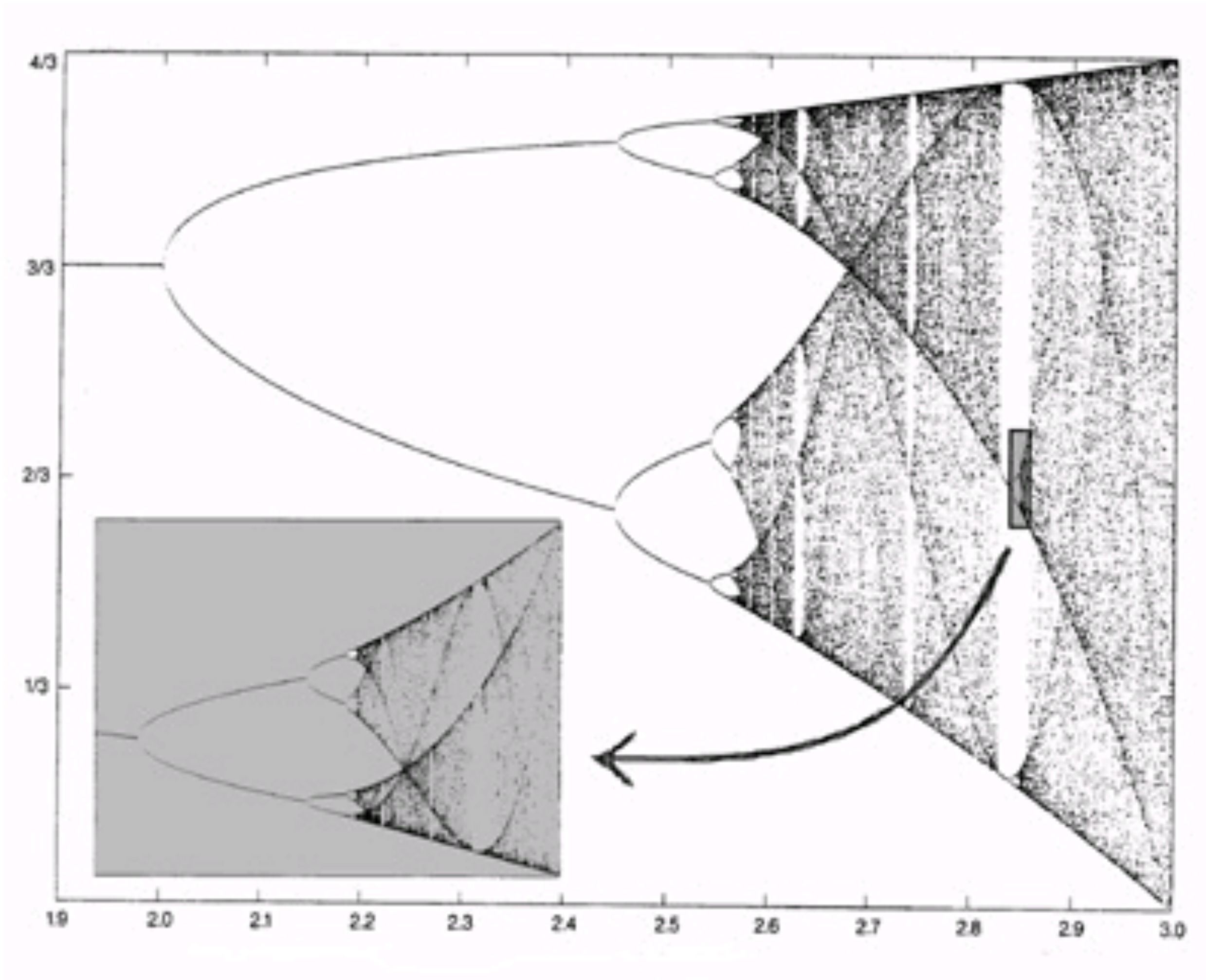
Sang-Hoon Kim*

Division of Liberal Arts, Mokpo National Maritime University, Mokpo 530-729 and Institute for Condensed Matter Theory, Chonnam National University, Gwangju 500-757

(Received 17 September 2004)

The fractal st[illegible] the box-counting method on its cross section[illegible] ± 0.02, independent of the direction. From[illegible] of a cauliflower is about 2.8. The vertical cro[illegible] set of a rectangular tree. We discuss the con[illegible] the vertical cross section has an angle of 67° in our model.

$D \approx 2.8$

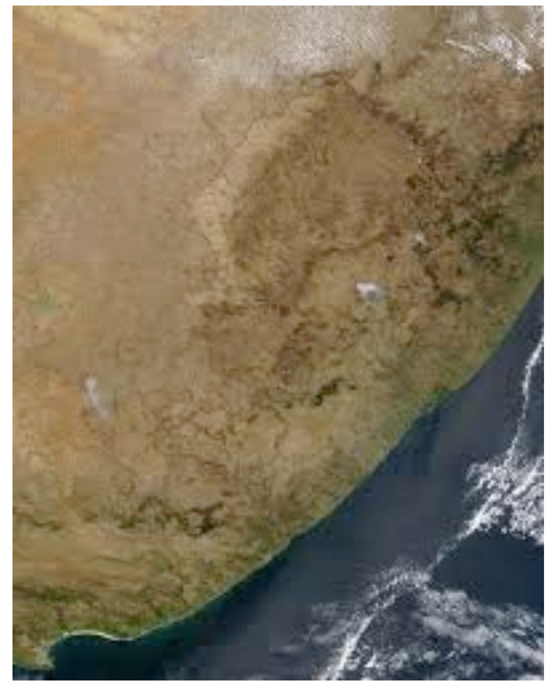


Fractal dimension ≈ 0.538

Approximate fractal dimension of coastlines
(Shelberg, Moellering, and Lam, 1982)

West Coast of Great Britain:
$D \approx 1.25$

Coast of Australia:
$D \approx 1.13$

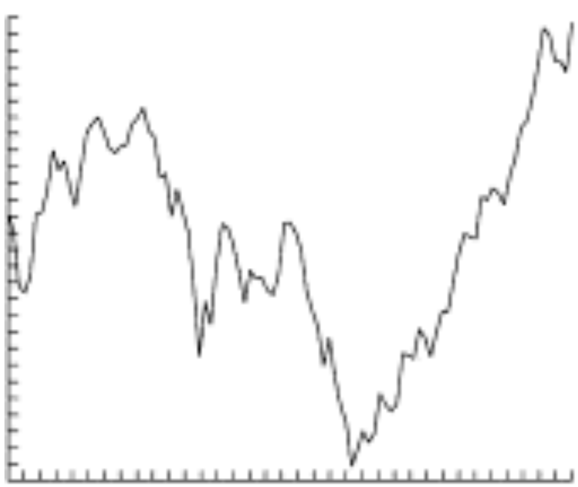


Coast of South Africa:
$D \approx 1.02$

Fractal dimension of stock prices

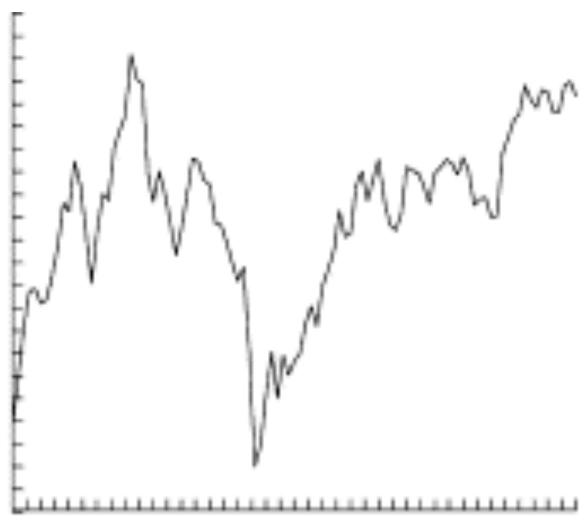
(J. A. Skjeltorp, Scaling in the Norwegian stock market, Physica A, 2000)

Oslo stock exchange general index



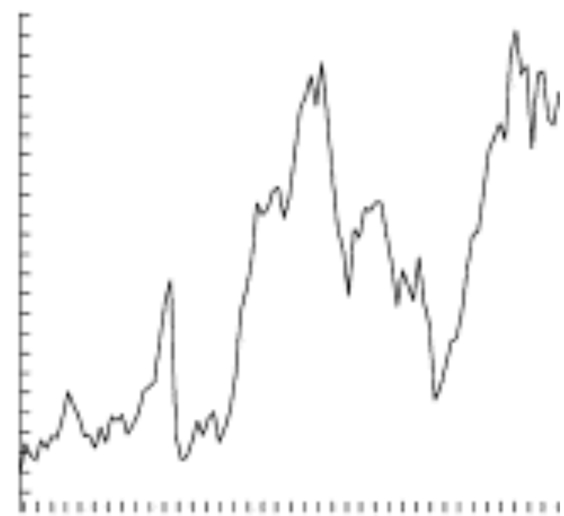
(a)

100-day
daily price record



(b)

100-week
weekly price record



(c)

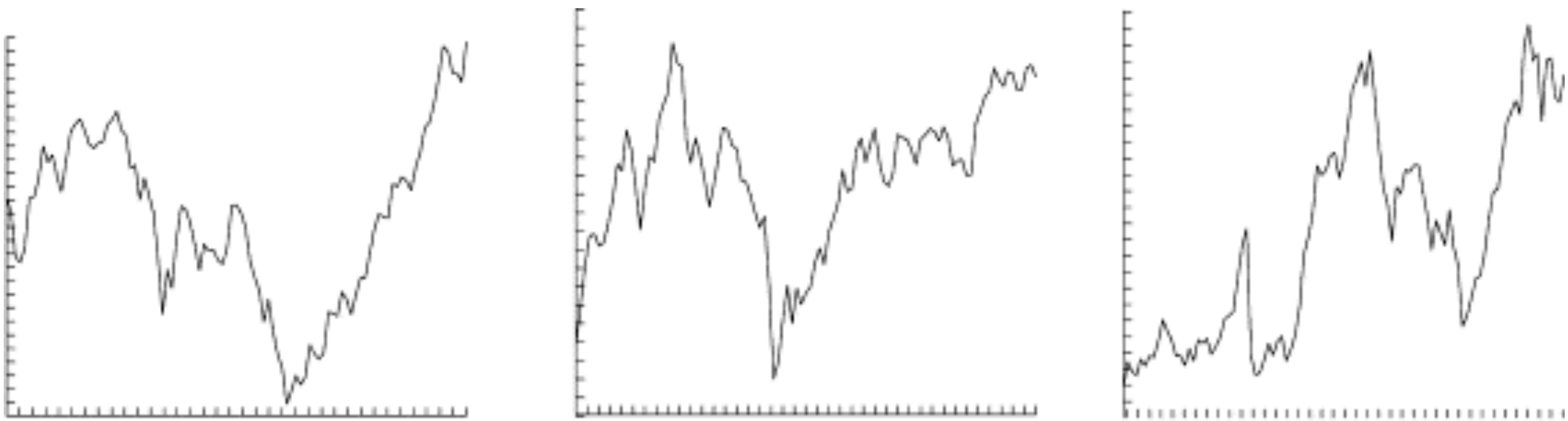
100-month
monthly price record

Self-similarity and detail at different temporal scales

Fractal dimension of stock prices

(J. A. Skjeltorp, Scaling in the Norwegian stock market, Physica A, 2000)

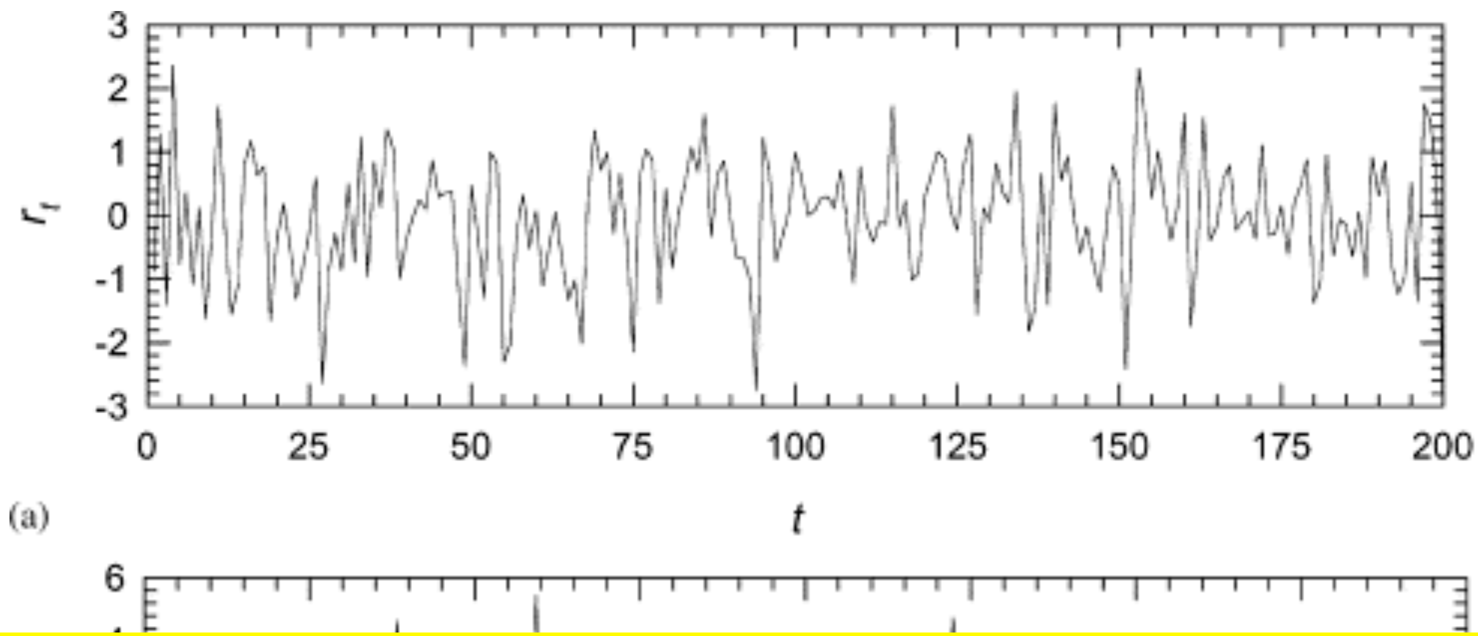
Oslo stock exchange general index



Question: Are stock prices following a "random walk"?

Project: Compare fractal dimension of stock prices with fractal dimension of "random walk"

Random Walks



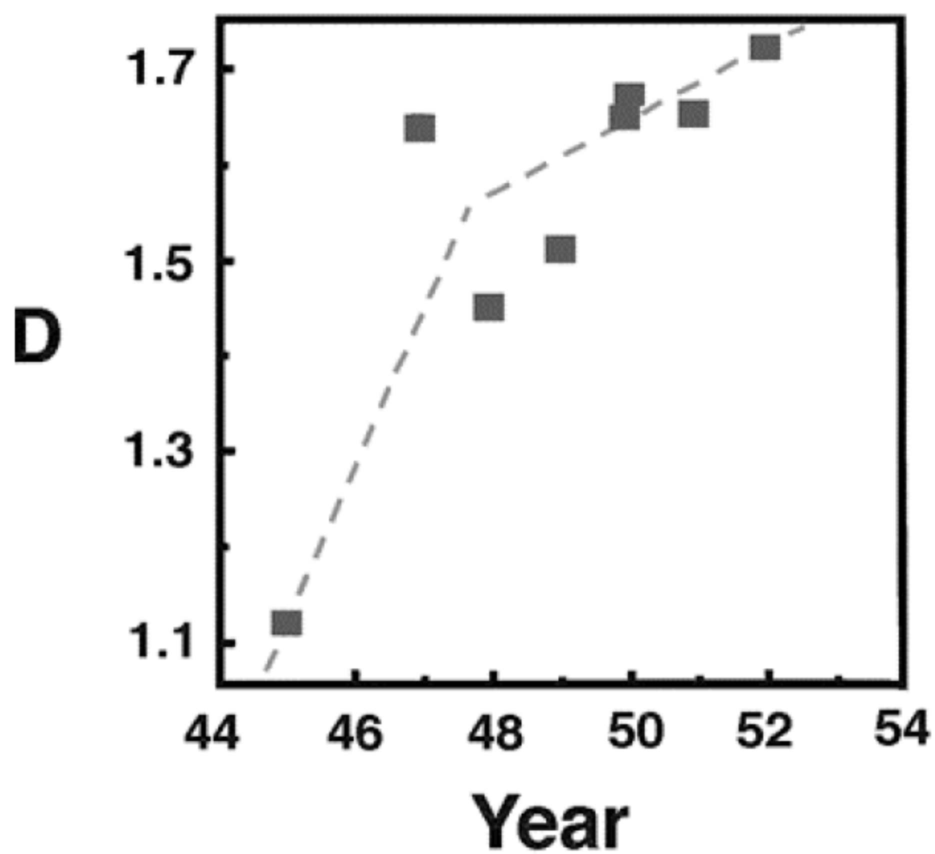
Result: The answer (after some complicated mathematics) is "no": stock prices are not following a random walk.

But **note**: There are a lot of caveats in applying fractal analysis to time series such as these.

The Visual Complexity of Jackson Pollock's Drip Paintings

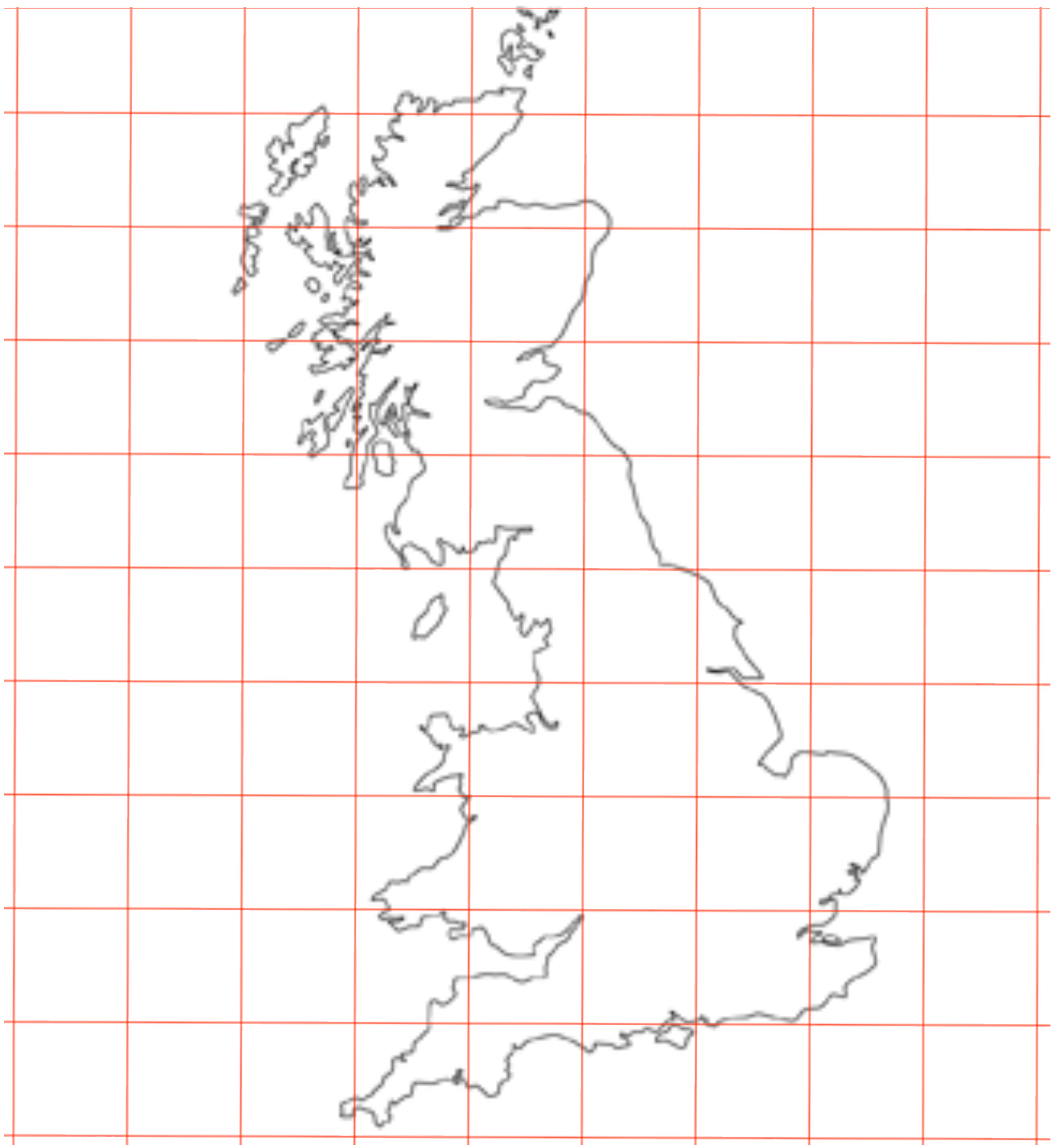
(Taylor, Spehar, Clifford, and Newell, 2008)

Evolution of "complexity" (fractal dimension) of Pollock's paintings:



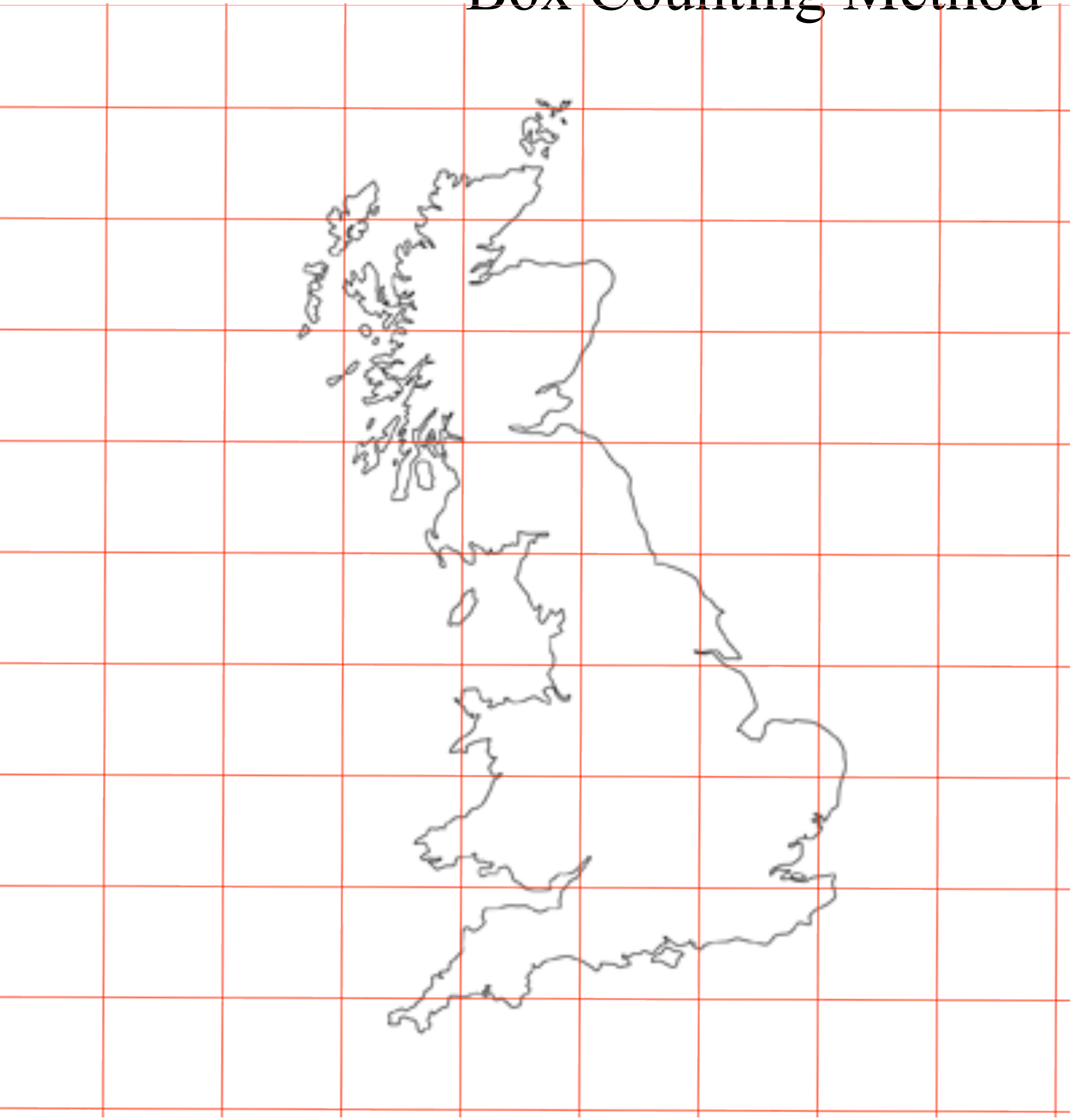
Box Counting Method

Box Counting Method



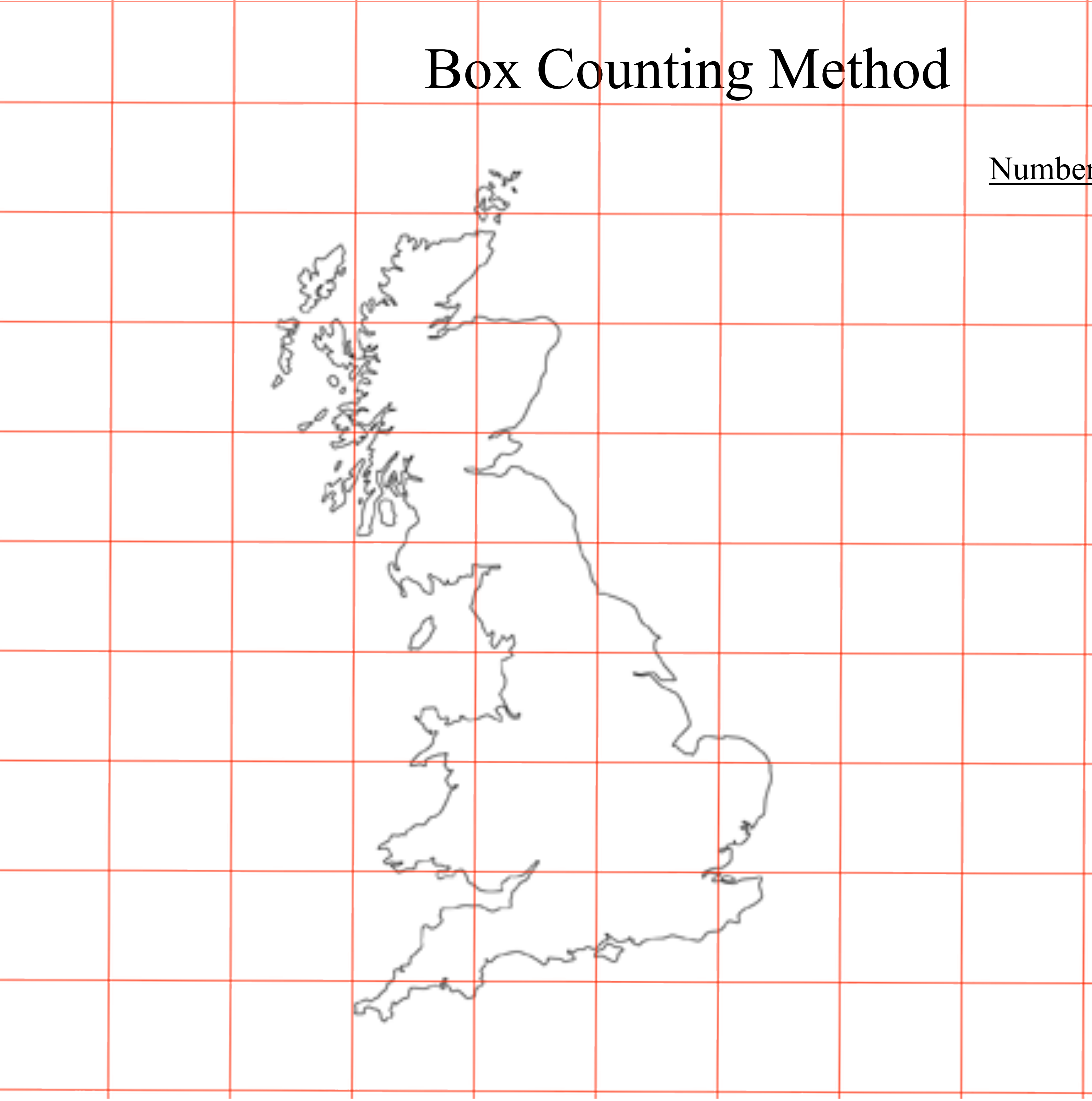
Number of boxes	Length of side
36	10

Box Counting Method



Number of boxes	Length of side
36	10
34	11

Box Counting Method



Number of boxes	Length of side
36	10
34	11
27	12

Hausdorff dimension and Box-Counting Dimension

Hausdorff dimension:

$$\log N = D \log M,$$

where N = number of copies of figure from previous level, and M = size reduction factor of a side of the previous level.

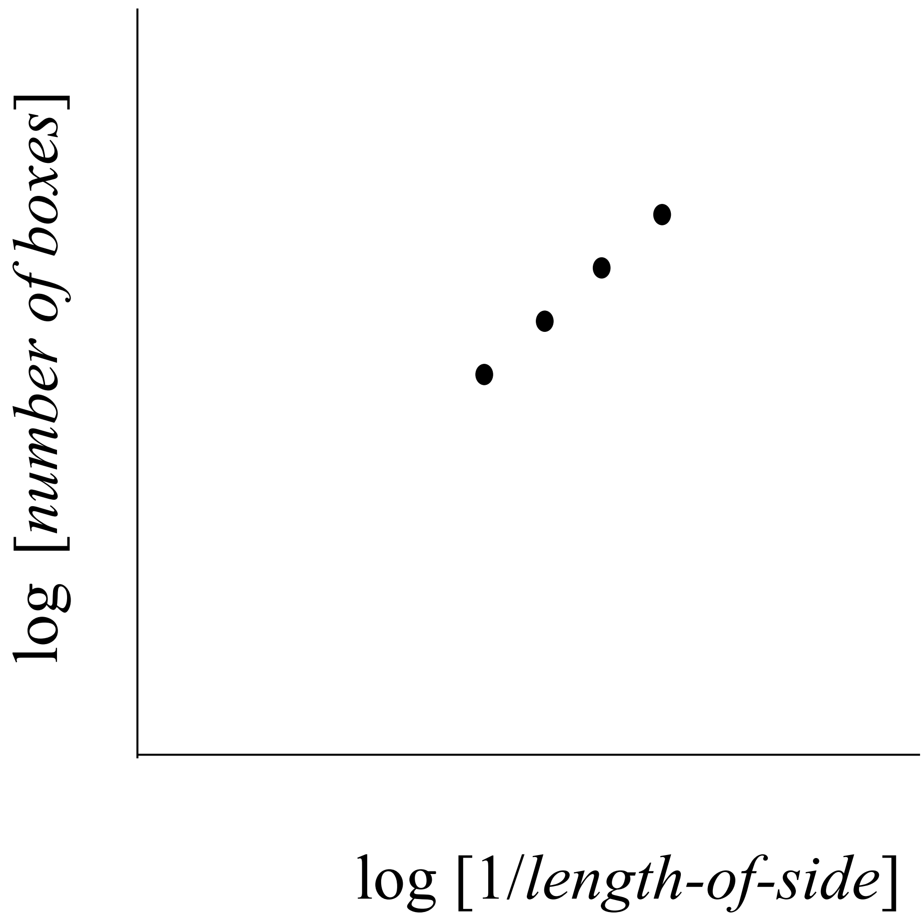
For box-counting, this can be approximated by

$$\log\ [\textit{number of boxes}] = D \log\ [1/\textit{length-of-side}]$$

D is called the ***Box-Counting Dimension***

(For details, see Chapter 4 of Fractal Explorer, linked from our Course Materials page.)

$$\log\ [\textit{number of boxes}] = D \log\ [1/\textit{box-size}]$$



$$\log\ [\textit{number of boxes}] = D \log\ [1/\textit{box-size}]$$