Correlation functions and FFT

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1 The time correlation function

We define the time autocorrelation function of a signal $x(t)$ as

$$C(\tau) = \langle xx(\tau)\rangle = \frac{1}{T}\int_0^T dt\, x(t)x(t-\tau), \tag{1}$$

wher T is the total length of the signal.

Example 1. Autocorrelation

Let us suppose the following function

$$x(t) = \sin\left(\frac{2\pi t}{T}\right) \tag{2}$$

Its time autocorrelation function over a whole period T is

$$C(\tau) = \frac{1}{T}\int_0^T dt\, \sin\left(\frac{2\pi t}{T}\right)\sin\left(\frac{2\pi(t-\tau)}{T}\right) \tag{3}$$

By using Property 32 we have

$$\begin{aligned}
C(\tau) &= \frac{1}{2}\frac{1}{T}\int_0^T dt\, \cos\left(\frac{2\pi\tau}{T}\right) - \frac{1}{2}\frac{1}{T}\int_0^T dt\, \cos\left(\frac{4\pi t}{T} - \frac{2\pi\tau}{T}\right) && (4)\\
&= \frac{1}{2}\frac{1}{T}\cos\left(\frac{2\pi\tau}{T}\right)\int_0^T dt - \frac{1}{2}\frac{1}{T}\,\frac{T}{4\pi}\sin\left(\frac{4\pi t}{T} - \frac{2\pi\tau}{T}\right)\Bigg|_0^T && (5)\\
&= \frac{1}{2}\cos\left(\frac{2\pi\tau}{T}\right) && (6)
\end{aligned}$$

Example 2. Cross correlation of two functions with different periods

Let us compute the correlation between two functions

$$x(t) = \sin\left(\frac{2\pi t}{T}\right) \qquad\qquad y(t) = \sin\left(\frac{4\pi t}{T}\right) \tag{7}$$

so that the function $y(t)$ has a period $T' = \frac{1}{2}T$, being T the period of $x(t)$. If we compute the time cross correlation between these two functions *over a period* T we have

$$\begin{aligned}
C(\tau) &= \langle xy(\tau)\rangle && (8)\\
&= \frac{1}{T}\int_0^T dt\, x(t)y(t-\tau) && (9)\\
&= \frac{1}{2}\frac{1}{T}\int_0^T dt\, \cos\left(\frac{-2\pi t}{T} + \frac{2\pi\tau}{T}\right) - \frac{1}{2}\frac{1}{T}\int_0^T dt\, \cos\left(\frac{6\pi t}{T} - \frac{2\pi\tau}{T}\right) && (10)\\
&= \frac{1}{2}\frac{1}{T}\,\frac{-T}{2\pi}\sin\left(\frac{-2\pi t}{T} - \frac{2\pi\tau}{T}\right)\Bigg|_0^T - \frac{1}{2}\frac{1}{T}\,\frac{T}{6\pi}\sin\left(\frac{6\pi t}{T} - \frac{2\pi\tau}{T}\right)\Bigg|_0^T && (11)\\
&= 0 && (12)
\end{aligned}$$

Example 3. Cross correlation of two functions with different periods

Let us suppose

$$x(t) = \sin\left(\frac{2\pi t}{T}\right) \qquad\qquad y(t) = \sin\left(\frac{3\pi t}{T}\right). \tag{13}$$

Again, that the function $y(t)$ has a period $T' = \frac{2}{3}T$, being T the period of $x(t)$. If we compute the time cross correlation between these two functions *over a period* T we have, on one hand,

$$\begin{aligned}
\frac{1}{T}\int_0^T dt\, x(t)y(t-\tau) &= \frac{1}{T}\int_0^T dt\, \sin\left(\frac{2\pi t}{T}\right)\sin\left(\frac{3\pi(t-\tau)}{T}\right) && (14)\\
&= \frac{1}{2T}\int_0^T dt\, \cos\left(\frac{-\pi t}{T}+\frac{3\pi\tau}{T}\right) - \frac{1}{2T}\int_0^T dt\, \cos\left(\frac{5\pi t}{T}-\frac{3\pi\tau}{T}\right) && (15)\\
&= -\frac{1}{2\pi}\left[\sin\left(-\pi+\frac{3\pi\tau}{T}\right)-\sin\left(\frac{3\pi\tau}{T}\right)\right] - \frac{1}{10\pi}\left[\sin\left(5\pi-\frac{3\pi\tau}{T}\right)-\sin\left(\frac{-3\pi\tau}{T}\right)\right] && (16)\\
&= \frac{1}{\pi}\sin\left(\frac{3\pi\tau}{T}\right) - \frac{1}{5\pi}\sin\left(\frac{3\pi\tau}{T}\right) && (17)
\end{aligned}$$

where we used Properties 34, 35, and 36 in the last step. We finally have, then

$$\frac{1}{T}\int_0^T dt\, x(t)y(t-\tau) = \frac{4}{5\pi}\sin\left(\frac{3\pi\tau}{T}\right) \tag{18}$$

Note that this is **different** from, on the other hand,

$$\begin{aligned}
\frac{1}{T}\int_0^T dt\, x(t-\tau)y(t) &= \frac{1}{T}\int_0^T dt\, \sin\left(\frac{2\pi(t-\tau)}{T}\right)\sin\left(\frac{3\pi t}{T}\right) && (19)\\
&= \frac{1}{2T}\int_0^T dt\, \cos\left(\frac{-\pi t}{T}-\frac{2\pi\tau}{T}\right) - \frac{1}{2T}\int_0^T dt\, \cos\left(\frac{5\pi t}{T}-\frac{2\pi\tau}{T}\right) && (20)\\
&= -\frac{1}{2\pi}\left[\sin\left(-\pi-\frac{2\pi\tau}{T}\right)-\sin\left(-\frac{2\pi\tau}{T}\right)\right] - \frac{1}{10\pi}\left[\sin\left(5\pi-\frac{2\pi\tau}{T}\right)-\sin\left(\frac{-2\pi\tau}{T}\right)\right] && (21)\\
&= -\frac{1}{\pi}\sin\left(\frac{2\pi\tau}{T}\right) - \frac{1}{5\pi}\sin\left(\frac{2\pi\tau}{T}\right) && (22)
\end{aligned}$$

where we again used Properties 34, 35, and 36 in the last step. We finally have, then

$$\frac{1}{T}\int_0^T dt\, x(t-\tau)y(t) = -\frac{6}{5\pi}\sin\left(\frac{2\pi\tau}{T}\right) \tag{23}$$

Example 4. Cross correlation of two function with the same period

Let us suppose

$$x(t) = \sin\left(\frac{2\pi t}{T}\right) \qquad\qquad y(t) = \cos\left(\frac{2\pi t}{T}\right) = \sin\left(\frac{2\pi t}{T}+\frac{\pi}{2}\right) \tag{24}$$

We compute, on one hand

$$\begin{aligned}
\frac{1}{T}\int_0^T dt\, x(t)y(t-\tau) &= \frac{1}{T}\int_0^T dt\, \sin\left(\frac{2\pi t}{T}\right)\cos\left(\frac{2\pi(t-\tau)}{T}\right) && (25)\\
&= \frac{1}{2T}\int_0^T dt\, \sin\left(\frac{2\pi\tau}{T}\right) + \frac{1}{2T}\int_0^T dt\, \sin\left(\frac{4\pi t}{T}-\frac{2\pi\tau}{T}\right) && (26)\\
&= \frac{1}{2}\sin\left(\frac{2\pi\tau}{T}\right) && (27)
\end{aligned}$$

where we used Property 33.

Note that this is **different** from, on the other hand,

$$\frac{1}{T}\int_0^T dt\, x(t-\tau)y(t) = \frac{1}{T}\int_0^T dt\, \sin\left(\frac{2\pi(t-\tau)}{T}\right)\cos\left(\frac{2\pi t}{T}\right) \tag{28}$$

$$= \frac{1}{2T}\int_0^T dt\, \sin\left(-\frac{2\pi\tau}{T}\right) + \frac{1}{2T}\int_0^T dt\, \sin\left(\frac{4\pi t}{T} - \frac{2\pi\tau}{T}\right) \tag{29}$$

$$= \frac{1}{2}\sin\left(\frac{-2\pi\tau}{T}\right) \tag{30}$$

$$= -\frac{1}{2}\sin\left(\frac{2\pi\tau}{T}\right) \tag{31}$$

where we again used Properties 33 and 36 in the last step.

Trigonometric relations

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] \tag{32}$$

$$\sin a \cos b = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right] \tag{33}$$

$$\sin(x-\pi) = -\sin x \tag{34}$$

$$\sin(\pi - x) = \sin x \tag{35}$$

$$\sin(x) = -\sin(-x) \tag{36}$$