Correlation functions and FFT

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April 6, 2017

1 The time correlation function

We define the time autocorrelation function of a signal $x(t)$ as

$$C(\tau) = \langle xx(\tau)\rangle = \frac{1}{T}\int_0^T dt\, x(t)x(t-\tau). \tag{1}$$

Autocorrelation

Let us suppose the following signal

$$x(t) = \sin\left(\frac{2\pi t}{T}\right) \tag{2}$$

Its time autocorrelation function is

$$C(\tau) = \frac{1}{T}\int_0^T dt\, \sin\left(\frac{2\pi t}{T}\right)\sin\left(\frac{2\pi(t-\tau)}{T}\right) \tag{3}$$

By using Property 25 we have

$$\begin{aligned}
C(\tau) &= \frac{1}{2}\frac{1}{T}\int_0^T dt\, \cos\left(\frac{2\pi\tau}{T}\right) - \frac{1}{2}\frac{1}{T}\int_0^T dt\, \cos\left(\frac{4\pi t}{T}-\frac{2\pi\tau}{T}\right) && (4)\\
&= \frac{1}{2}\frac{1}{T}\cos\left(\frac{2\pi\tau}{T}\right)\int_0^T dt - \frac{1}{2}\frac{1}{T}\,\frac{T}{4\pi}\sin\left(\frac{4\pi t}{T}-\frac{2\pi\tau}{T}\right)\Bigg|_0^T && (5)\\
&= \frac{1}{2}\cos\left(\frac{2\pi\tau}{T}\right) && (6)
\end{aligned}$$

Cross correlation

Let us compute the correlation between two signals

$$\begin{aligned}
x(t) &= \sin\left(\frac{2\pi t}{T}\right) && (7)\\
y(t) &= \sin\left(\frac{4\pi t}{T}\right) && (8)
\end{aligned}$$

so that

$$\begin{aligned}
C(\tau) &= \langle xy(\tau)\rangle && (9)\\
&= \frac{1}{T}\int_0^T dt\, x(t)y(t-\tau) && (10)\\
&= \frac{1}{2}\frac{1}{T}\int_0^T dt\, \cos\left(\frac{-2\pi t}{T}+\frac{2\pi\tau}{T}\right) - \frac{1}{2}\frac{1}{T}\int_0^T dt\, \cos\left(\frac{6\pi t}{T}-\frac{2\pi\tau}{T}\right) && (11)\\
&= \frac{1}{2}\frac{1}{T}\,\frac{-T}{2\pi}\sin\left(\frac{-2\pi t}{T}-\frac{2\pi\tau}{T}\right)\Bigg|_0^T - \frac{1}{2}\frac{1}{T}\,\frac{T}{6\pi}\sin\left(\frac{6\pi t}{T}-\frac{2\pi\tau}{T}\right)\Bigg|_0^T && (12)\\
&= 0 && (13)
\end{aligned}$$

Another cross correlation

Let us suppose

$$x(t) = \sin\left(\frac{2\pi t}{T}\right) \qquad\qquad y(t) = \sin\left(\frac{3\pi t}{T}\right) \tag{14}$$

We compute, on one hand

$$\begin{aligned}
\frac{1}{T}\int_0^T dt\, x(t)y(t-\tau) &= \frac{1}{T}\int_0^T dt\, \sin\left(\frac{2\pi t}{T}\right)\sin\left(\frac{3\pi(t-\tau)}{T}\right) && (15)\\
&= \frac{1}{2T}\int_0^T dt\, \cos\left(\frac{-\pi t}{T}+\frac{3\pi\tau}{T}\right) - \frac{1}{2T}\int_0^T dt\, \cos\left(\frac{5\pi t}{T}-\frac{3\pi\tau}{T}\right) && (16)\\
&= -\frac{1}{2\pi}\left[\sin\left(-\pi+\frac{3\pi\tau}{T}\right)-\sin\left(\frac{3\pi\tau}{T}\right)\right] - \frac{1}{10\pi}\left[\sin\left(5\pi-\frac{3\pi\tau}{T}\right)-\sin\left(\frac{-3\pi\tau}{T}\right)\right] && (17)\\
&= \frac{1}{\pi}\sin\left(\frac{3\pi\tau}{T}\right) - \frac{1}{5\pi}\sin\left(\frac{3\pi\tau}{T}\right) && (18)
\end{aligned}$$

where we used Properties 26, 27, and 28 in the last step. We finally have, then

$$\frac{1}{T}\int_0^T dt\, x(t)y(t-\tau) = \frac{4}{5\pi}\sin\left(\frac{3\pi\tau}{T}\right) \tag{19}$$

Note that this is **different** from, on the other hand,

$$\begin{aligned}
\frac{1}{T}\int_0^T dt\, x(t-\tau)y(t) &= \frac{1}{T}\int_0^T dt\, \sin\left(\frac{2\pi(t-\tau)}{T}\right)\sin\left(\frac{3\pi t}{T}\right) && (20)\\
&= \frac{1}{2T}\int_0^T dt\, \cos\left(\frac{-\pi t}{T}-\frac{2\pi\tau}{T}\right) - \frac{1}{2T}\int_0^T dt\, \cos\left(\frac{5\pi t}{T}-\frac{2\pi\tau}{T}\right) && (21)\\
&= -\frac{1}{2\pi}\left[\sin\left(-\pi-\frac{2\pi\tau}{T}\right)-\sin\left(-\frac{2\pi\tau}{T}\right)\right] - \frac{1}{10\pi}\left[\sin\left(5\pi-\frac{2\pi\tau}{T}\right)-\sin\left(\frac{-2\pi\tau}{T}\right)\right] && (22)\\
&= -\frac{1}{\pi}\sin\left(\frac{2\pi\tau}{T}\right) - \frac{1}{5\pi}\sin\left(\frac{2\pi\tau}{T}\right) && (23)
\end{aligned}$$

where we again used Properties 26, 27, and 28 in the last step. We finally have, then

$$\frac{1}{T}\int_0^T dt\, x(t-\tau)y(t) = -\frac{6}{5\pi}\sin\left(\frac{2\pi\tau}{T}\right) \tag{24}$$

Trigonometric relations

$$\begin{aligned}
\sin a \sin b &= \frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right] && (25)\\
\sin(x-\pi) &= -\sin x && (26)\\
\sin(\pi-x) &= \sin x && (27)\\
\sin(x) &= -\sin(-x) && (28)
\end{aligned}$$