Correlation functions and FFT

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November 18, 2016

1 The time correlation function

We define the time correlation function of a signal $x(t)$ as

$$C(\tau) = \langle xx(\tau)\rangle = \frac{1}{T}\int_0^T dt\, x(t)x(t-\tau). \tag{1}$$

Let us suppose the following signal

$$x(t) = \sin\left(\frac{2\pi t}{T}\right) \tag{2}$$

The time correlation function is

$$C(\tau) = \frac{1}{T}\int_0^T dt\ \sin\left(\frac{2\pi t}{T}\right)\sin\left(\frac{2\pi(t-\tau)}{T}\right) \tag{3}$$

We use the property

$$\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right) \tag{4}$$

so that

$$\begin{aligned}
C(\tau) &= \frac{1}{2}\frac{1}{T}\int_0^T dt\ \cos\left(\frac{2\pi\tau}{T}\right) - \frac{1}{2}\frac{1}{T}\int_0^T dt\ \cos\left(\frac{4\pi t}{T} - \frac{2\pi\tau}{T}\right) && (5)\\
&= \frac{1}{2}\frac{1}{T}\cos\left(\frac{2\pi\tau}{T}\right)\int_0^T dt - \frac{1}{2}\frac{1}{T}\ \frac{T}{4\pi}\sin\left(\frac{4\pi t}{T} - \frac{2\pi\tau}{T}\right)\Bigg|_0^T && (6)\\
&= \frac{1}{2}\cos\left(\frac{2\pi\tau}{T}\right) && (7)
\end{aligned}$$

Let us compute the correlation between two signals

$$\begin{aligned}
x(t) &= \sin\left(\frac{2\pi t}{T}\right) && (8)\\
y(t) &= \sin\left(\frac{4\pi t}{T}\right) && (9)
\end{aligned}$$

so that

$$
\begin{aligned}
C(\tau) &= \langle xy(\tau)\rangle && (10)\\
&= \frac{1}{T}\int_0^T dt\, x(t)y(t-\tau) && (11)\\
&= \frac{1}{2}\frac{1}{T}\int_0^T dt\,\cos\left(\frac{-2\pi t}{T}+\frac{2\pi\tau}{T}\right) - \frac{1}{2}\frac{1}{T}\int_0^T dt\,\cos\left(\frac{6\pi t}{T}-\frac{2\pi\tau}{T}\right) && (12)\\
&= \frac{1}{2}\frac{1}{T}\,\frac{-T}{2\pi}\sin\left(\frac{-2\pi t}{T}-\frac{2\pi\tau}{T}\right)\Bigg|_0^T - \frac{1}{2}\frac{1}{T}\,\frac{T}{6\pi}\sin\left(\frac{6\pi t}{T}-\frac{2\pi\tau}{T}\right)\Bigg|_0^T && (13)\\
&= 0 && (14)
\end{aligned}
$$

2 Time correlation function using Fourier Transform

Formally, time correlation function is defined as

$$
C(\tau) = \langle xx(\tau)\rangle = \frac{1}{T}\int_0^T dt\, x(t)x(t-\tau). \qquad (15)
$$

2.1 The continous Fourier Transform

2.2 The discrete Fourier Transform

$$
\begin{aligned}
X_k &= \sum_{j=0}^{N-1} x_j e^{-2\pi ijk/N} && (16)\\
x_j &= \sum_{k=0}^{N-1} X_k e^{2\pi ijk/N} && (17)
\end{aligned}
$$

2.3 Fast Fourier Transform