Correlation functions and FFT

J. A. de la Torre

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1 The time correlation function

We define the time autocorrelation function of a signal $x(t)$ as

$$C(\tau) = \langle xx(\tau) \rangle = \frac{1}{T} \int_0^T dt\, x(t)x(t-\tau). \tag{1}$$

Autocorrelation

Let us suppose the following signal

$$x(t) = \sin\left(\frac{2\pi t}{T}\right) \tag{2}$$

Its time autocorrelation function is

$$C(\tau) = \frac{1}{T} \int_0^T dt\, \sin\left(\frac{2\pi t}{T}\right) \sin\left(\frac{2\pi(t-\tau)}{T}\right) \tag{3}$$

By using Property 25 we have

$$C(\tau) = \frac{1}{2}\frac{1}{T} \int_0^T dt\, \cos\left(\frac{2\pi\tau}{T}\right) - \frac{1}{2}\frac{1}{T} \int_0^T dt\, \cos\left(\frac{4\pi t}{T} - \frac{2\pi\tau}{T}\right) \tag{4}$$

$$= \frac{1}{2}\frac{1}{T} \cos\left(\frac{2\pi\tau}{T}\right) \int_0^T dt - \frac{1}{2}\frac{1}{T}\frac{T}{4\pi} \sin\left(\frac{4\pi t}{T} - \frac{2\pi\tau}{T}\right)\Big|_0^T \tag{5}$$

$$= \frac{1}{2} \cos\left(\frac{2\pi\tau}{T}\right) \tag{6}$$

Cross correlation

Let us compute the correlation between two signals

$$x(t) = \sin\left(\frac{2\pi t}{T}\right) \tag{7}$$

$$y(t) = \sin\left(\frac{4\pi t}{T}\right) \tag{8}$$

so that

$$C(\tau) = \langle xy(\tau) \rangle \tag{9}$$

$$= \frac{1}{T} \int_0^T dt\, x(t)y(t-\tau) \tag{10}$$

$$= \frac{1}{2}\frac{1}{T} \int_0^T dt\, \cos\left(\frac{-2\pi t}{T} + \frac{2\pi\tau}{T}\right) - \frac{1}{2}\frac{1}{T} \int_0^T dt\, \cos\left(\frac{6\pi t}{T} - \frac{2\pi\tau}{T}\right) \tag{11}$$

$$= \frac{1}{2}\frac{1}{T}\frac{-T}{2\pi} \sin\left(\frac{-2\pi t}{T} - \frac{2\pi\tau}{T}\right)\Big|_0^T - \frac{1}{2}\frac{1}{T}\frac{T}{6\pi} \sin\left(\frac{6\pi t}{T} - \frac{2\pi\tau}{T}\right)\Big|_0^T \tag{12}$$

$$= 0 \tag{13}$$

Another cross correlation

Let us suppose

$$x(t) = \sin\left(\frac{2\pi t}{T}\right) \qquad\qquad y(t) = \sin\left(\frac{3\pi t}{T}\right) \tag{14}$$

We compute, on one hand

$$\frac{1}{T}\int_0^T dt\, x(t)y(t-\tau) = \frac{1}{T}\int_0^T dt\, \sin\left(\frac{2\pi t}{T}\right)\sin\left(\frac{3\pi(t-\tau)}{T}\right) \tag{15}$$

$$= \frac{1}{2T}dt\,\cos\left(\frac{-\pi t}{T} + \frac{3\pi\tau}{T}\right) - \frac{1}{2T}dt\,\cos\left(\frac{5\pi t}{T} - \frac{3\pi\tau}{T}\right) \tag{16}$$

$$= -\frac{1}{2\pi}\left[\sin\left(-\pi + \frac{3\pi\tau}{T}\right) - \sin\left(\frac{3\pi\tau}{T}\right)\right] - \frac{1}{10\pi}\left[\sin\left(5\pi - \frac{3\pi\tau}{T}\right) - \sin\left(\frac{-3\pi\tau}{T}\right)\right] \tag{17}$$

$$= \frac{1}{\pi}\sin\left(\frac{3\pi\tau}{T}\right) - \frac{1}{5\pi}\sin\left(\frac{3\pi\tau}{T}\right) \tag{18}$$

where we used Properties 26, 27, and 28 in the last step. We finally have, then

$$\frac{1}{T}\int_0^T dt\, x(t)y(t-\tau) = \frac{4}{5\pi}\sin\left(\frac{3\pi\tau}{T}\right) \tag{19}$$

Note that this is **different** from, on the other hand,

$$\frac{1}{T}\int_0^T dt\, x(t-\tau)y(t) = \frac{1}{T}\int_0^T dt\, \sin\left(\frac{2\pi(t-\tau)}{T}\right)\sin\left(\frac{3\pi t}{T}\right) \tag{20}$$

$$= \frac{1}{2T}dt\,\cos\left(\frac{-\pi t}{T} - \frac{2\pi\tau}{T}\right) - \frac{1}{2T}dt\,\cos\left(\frac{5\pi t}{T} - \frac{2\pi\tau}{T}\right) \tag{21}$$

$$= -\frac{1}{2\pi}\left[\sin\left(-\pi - \frac{2\pi\tau}{T}\right) - \sin\left(-\frac{2\pi\tau}{T}\right)\right] - \frac{1}{10\pi}\left[\sin\left(5\pi - \frac{2\pi\tau}{T}\right) - \sin\left(\frac{-2\pi\tau}{T}\right)\right] \tag{22}$$

$$= -\frac{1}{\pi}\sin\left(\frac{2\pi\tau}{T}\right) - \frac{1}{5\pi}\sin\left(\frac{2\pi\tau}{T}\right) \tag{23}$$

where we again used Properties 26, 27, and 28 in the last step. We finally have, then

$$\frac{1}{T}\int_0^T dt\, x(t-\tau)y(t) = -\frac{6}{5\pi}\sin\left(\frac{2\pi\tau}{T}\right) \tag{24}$$

Trigonometric relations

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] \tag{25}$$

$$\sin(x-\pi) = -\sin x \tag{26}$$

$$\sin(\pi - x) = \sin x \tag{27}$$

$$\sin(x) = -\sin(-x) \tag{28}$$