Crank_Nicolson method for Diffusion

Crank Nicolsom method can produce better approximation and more stable solution to partial differential equ.
C_N method in fact approximates the derivities at time $t+\frac{\Delta t}{2}$.

$$\frac{\partial f}{\partial t}\left(x, t+\Delta t/2\right) = \frac{f_j^{n+1} - f_j^n}{\Delta t}$$

$$\frac{\partial^2 f}{\partial x^2}\left(x, t+\frac{\Delta t}{2}\right) = \frac{1}{2}\left(\frac{f_{j-1}^n - 2f_j^n + f_{j+1}^n}{(\Delta x)^2} + \frac{f_{j-1}^{n+1} - 2f_j^{n+1} + f_{j+1}^{n+1}}{(\Delta x)^2}\right)$$

$$\Rightarrow \quad \frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$$

$$\Rightarrow \quad f_j^{n+1} - f_j^n = \overbrace{\frac{D\Delta t}{2\Delta x^2}}^{r}\left(f_{j-1}^n - 2f_j^n + f_{j+1}^n + f_{j-1}^{n+1} - 2f_j^{n+1} + f_{j+1}^{n+1}\right)$$

$$\Rightarrow \quad \overbrace{\left(\frac{2}{r}+2\right)}^{\alpha} f_j^{n+1} - f_{j-1}^{n+1} - f_{j+1}^{n+1} = \overbrace{\left(\frac{2}{r}-2\right)}^{\beta} f_j^n + f_{j-1}^n + f_{j+1}^n$$

as you can see this method is implicit, so we need to solve simultaneous equations at each time step.

$$\underbrace{\begin{pmatrix} 1 & 0 & \cdots & & & \\ -1 & \alpha & -1 & \cdots & & \\ 0 & -1 & \alpha & -1 & \cdots & \\ & & \ddots & \ddots & \ddots & \\ \cdots & & & -1 & \alpha & -1 \\ \cdots & & & & 0 & 1 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} F_0^{n+1} \\ F_1^{n+1} \\ F_2^{n+1} \\ \vdots \\ F_{m-1}^{n+1} \\ F_m^{n+1} \end{pmatrix}}_{F^{n+1}} = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & & & \\ 1 & \beta & 1 & \cdots & & \\ 0 & 1 & \beta & 1 & \cdots & \\ & & \ddots & \ddots & \ddots & \\ \cdots & & & 1 & \beta & 1 \\ \cdots & & & & 0 & 1 \end{pmatrix} \begin{pmatrix} F_0^{n} \\ F_1^{n} \\ F_2^{n} \\ \vdots \\ F_{m-1}^{n} \\ F_m^{n} \end{pmatrix}}_{\tilde{F}^n}$$

$$\Rightarrow \quad M F^{n+1} = \tilde{F}^n \Rightarrow \boxed{F^{n+1} = M^{-1} \tilde{F}^n}$$

Since the matrices we are dealing with are tridiagonal we can use the efficient algorithms designed for these sparse matrices.

Von Neumman Stability analisys:

let's start with the update rule:

$$\boxed{\alpha F_j^{n+1} - F_{j-1}^{n+1} - F_{j+1}^{n+1} = \beta F_j^n - F_{j-1}^n - F_{j+1}^n}$$

as I have fully covered in explicit FDA for Diffusion, we can write inverse Fourier transform as:

$$\mathcal{E}[j\Delta x, n\Delta t] = \sum_{h=0}^{N-1} E[h,n] \exp\left(\frac{i2\pi hj}{N}\right) = \mathcal{E}_j^n$$

now we can isert above equation in update rule. Note that as you can see we should insert a whole sum ($\sum \{\}$) instead of each term in the update rule. but since every thing is linear, we can use just one term in sum (in fact the equality must hold for each element is the sum)

$$\mathcal{E}_j^{n\pm1} = \sum_{h=0}^{N-1} E[h, n\pm1] \exp\left(\frac{i2\pi hj}{N}\right)$$

$$\mathcal{E}_{j\pm1}^{n} = \sum_{h=0}^{N-1} E[h,n] \exp\left(\frac{i2\pi h(j\pm1)}{N}\right)$$

$$\Rightarrow E[h,n]\exp\left(\frac{i2\pi hj}{N}\right) = \square_j^n \;;\; \exp\left(\frac{2\pi h}{N}\right) = e^{i\xi}$$

$\Rightarrow$ plug in the update rule:

$$\alpha \square_j^{n+1} - \square_j^{n+1} e^{i\xi} - \square_j^{n+1} e^{-i\xi} = \beta \square_j^n + \square_j^n e^{i\xi} + \square_j^n e^{-i\xi}$$

$$\Longrightarrow G = \frac{\square_j^{n+1}}{\square_j^{n}}$$ amplification factor

$$\Rightarrow G = \frac{\beta + e^{i\xi} + e^{-i\xi}}{\alpha - e^{i\xi} - e^{-i\xi}} \quad \star$$

$$\alpha = \frac{2}{r} + 2$$

$$\beta = \frac{2}{r} - 2$$

$$\sin^2\left(\xi/2\right) = \left(\frac{e^{i\xi/2} + e^{-i\xi/2}}{2i}\right)^2 = -\frac{1}{4}\left(e^{i\xi} + e^{-i\xi} + 2\right)$$

$$\Rightarrow \boxed{e^{i\xi} + e^{-i\xi} = -4\sin^2\left(\xi/2\right) - 2} \quad \star\star$$

inserting $\star$ in $\star\star$

$$G = \frac{\frac{2}{r} - 2 - 2 - 4\sin^2(\xi/2)}{\frac{2}{r} + 2 + 2 + 4\sin^2(\xi/2)} = \frac{1 - \underbrace{2r\left(1 + \sin^2(\xi/2)\right)}_{\eta}}{1 + \underbrace{2r\left(1 + \sin^2(\xi/2)\right)}_{\eta}} \qquad \eta \geq 0$$

$$|G| \leq 1 \Rightarrow \left|\frac{1-\eta}{1+\eta}\right| \leq 1$$

$$\begin{cases} \dfrac{1-\eta}{1+\eta} \leq 1 \Rightarrow 1-\eta \leq 1+\eta \Rightarrow \eta \geq 0 \quad ☑ \\[2ex] \dfrac{1-\eta}{1+\eta} \geq -1 \Rightarrow 1-\eta \geq -1-\eta \Rightarrow 1 \geq -1 \quad ☑ \end{cases}$$

So $|G|$ is always less than one

So the scheme is unconditionally stable!