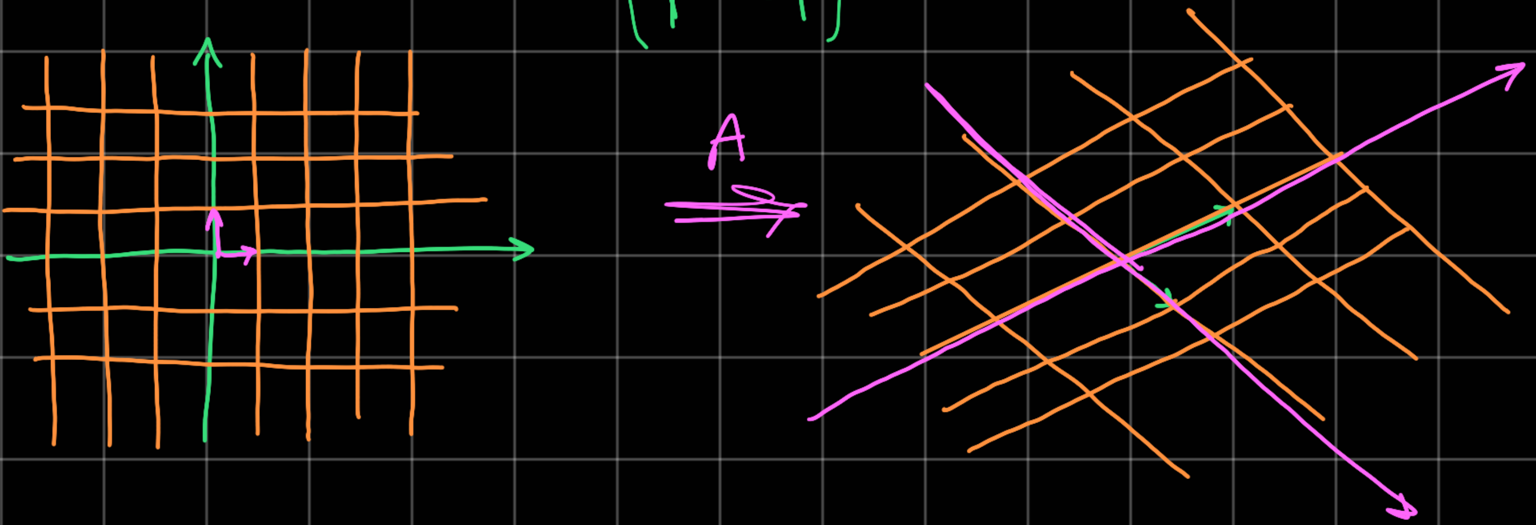
Consider the following transformation

$$x' = ax + by \qquad y' = cd + dy \;\rightarrow\; \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \boxed{X' = AX}$$

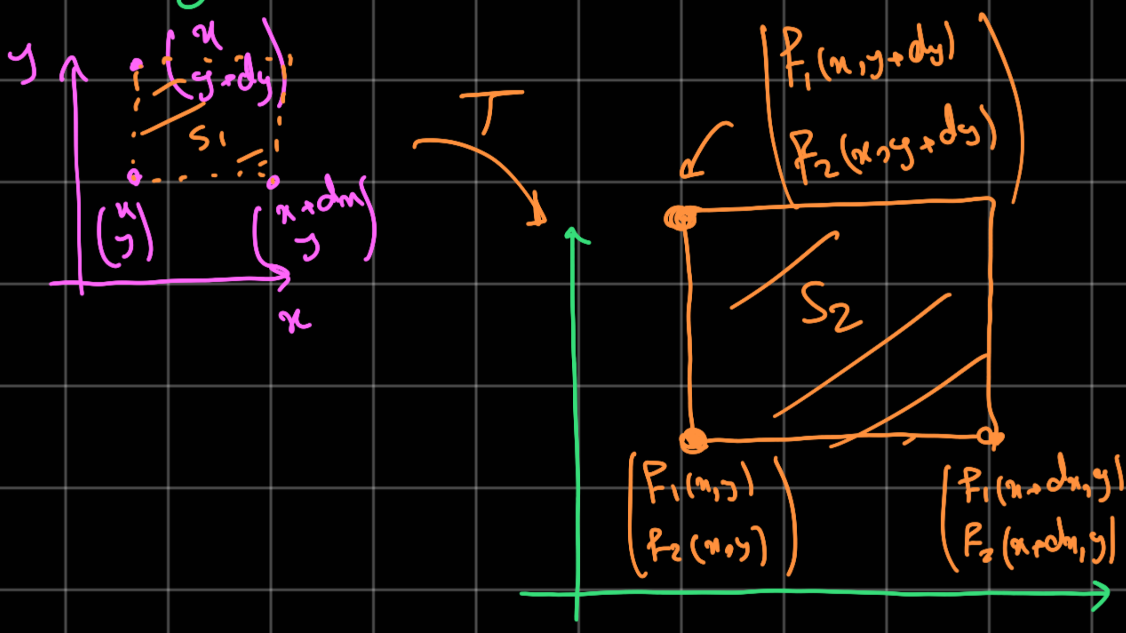
For example $A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$



as you can see, the element of Area has been changed. but How much ?!

lets consider a general 2D transformation:

$$x' = f_1(x,y) \qquad y' = f_2(x,y)$$



$$S_1 = |\vec{dx} \times \vec{dy}| = \left| \begin{pmatrix} dx \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ dy \\ 0 \end{pmatrix} \right| = dx\,dy$$

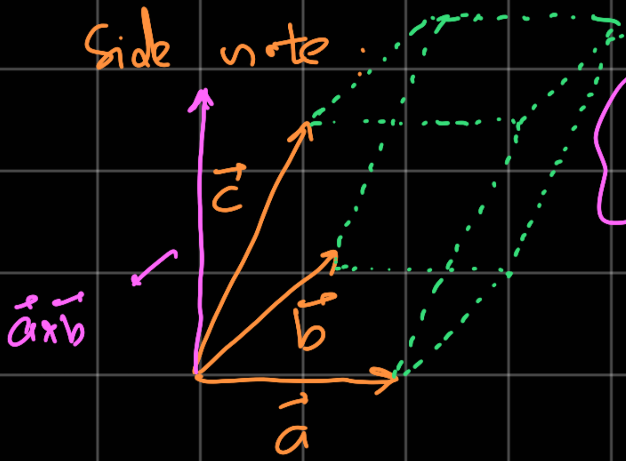
$$S_2 = |\vec{dx}' \times \vec{dy}'| = \left| \begin{pmatrix} \frac{\partial f_1}{\partial x} dx' \\ \frac{\partial f_2}{\partial x} dx' \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{\partial f_1}{\partial y} dy' \\ \frac{\partial f_2}{\partial y} dy' \\ 0 \end{pmatrix} \right| = \det \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} dx'dy'$$

Note that $f(x+dx, y) - f(x,y) = \frac{\partial f}{\partial x} dx$

and generalization for N Dimensions

$$dx_1 dx_2 \dots dx_n = \det \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{vmatrix} dx_1' dx_2' \dots dx_n'$$

Side note:



$$V = (\vec{a} \times \vec{b}) \cdot \vec{c}$$