# Stability Analysis For ODEs and PDEs

We will be deriving all the intuitions based on ~~the~~ a 2D model but every thing Can be generalized to ND.

$$\frac{dx}{dt} = f(x,y)$$

$$\frac{dy}{dt} = g(x,y)$$

We Can write the system of ODEs above in a vector format So every thing will be generalizable to ND.

$$\Phi = \begin{pmatrix} x \\ y \end{pmatrix} \quad ; \quad F = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

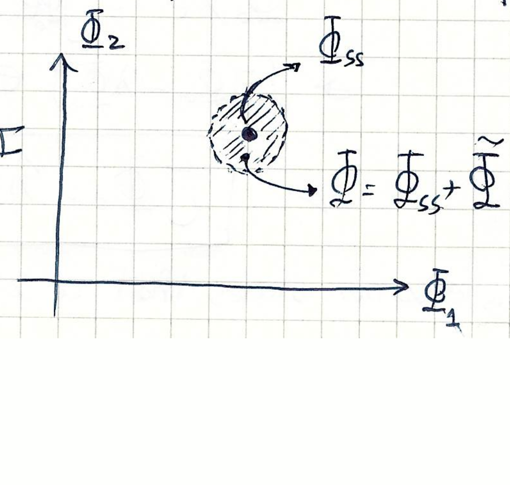
$$\Rightarrow \boxed{\frac{d\Phi}{dt} = F(\Phi)} \quad (\mathrm{I})$$

Suppose that there is a steady state Solution for this dynamical system. i.e. there is a $\Phi_{ss}$ such that $F(\Phi_{ss}) = 0$.

at steady state $\Rightarrow \dfrac{d\Phi}{dt} = F(\Phi_{ss}) = 0$

Now what matters to us is the behaviour of the system around the steady state. So let's see how the equation (I) will look like around the steady state. A point very close to steady State Can be written as:

$$\boxed{\Phi = \Phi_{ss} + \tilde{\Phi}} \quad \mathrm{II}$$



Let's try to see if ODE$_s$ look simple around ~~$\Phi_{ss}$~~ or not. To know that we simply plug in II in I and do some simplification through approximation

$$\frac{d}{dt}\left(\Phi_{ss}+\tilde{\Phi}\right)=F\left(\Phi+\tilde{\Phi}\right)$$

By assuming $\tilde{\Phi}$ to be very close to $\Phi$ we can use Taylor's expansion to write:

$$\frac{d\tilde{\Phi}}{dt}=\cancelto{0}{F(\Phi_{ss})}+J\tilde{\Phi}$$

In which $J$ is the jacobian matrix for $F$.

$$\Rightarrow \boxed{\frac{d\Phi}{dt}=J\Phi} \quad (III)$$

We droped the "$\sim$" sign for the last equation.

So the original ODE (I) will look like (III) if evaluated at points very close to steady state solution.

A candidate solution for III can be:

$$\Phi=\Phi_0 e^{\lambda t} \quad (IV)$$

So by inserting IV in III we will have

$$\Phi_0\lambda e^{\lambda t}=J\Phi_0 e^{\lambda t}$$

$$\Rightarrow (J\Phi_0-\lambda\Phi_0)\,e^{\lambda t}=0$$

$$\Rightarrow \boxed{J\Phi_0=\lambda\Phi_0} \quad (V)$$

This equation states that a solution of the form IV is a solution

For (IV) in which $\Phi_0$ is eigenvector and $\lambda$ is eigenvalue of the Jacobian matrix $J$. So the general solution around SS can be written as:

$$\Phi = \sum_i |\lambda_i\rangle e^{\lambda_i t}$$

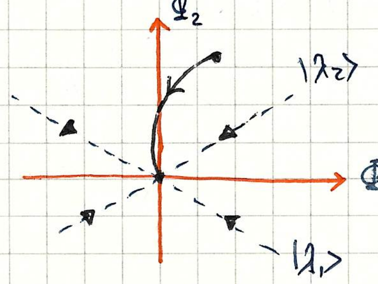
and since $J$ is a 2x2 matrix, so we can write:

$$\boxed{\Phi = |\lambda_1\rangle e^{\lambda_1 t} + |\lambda_2\rangle e^{\lambda_2 t}} \quad (VI)$$

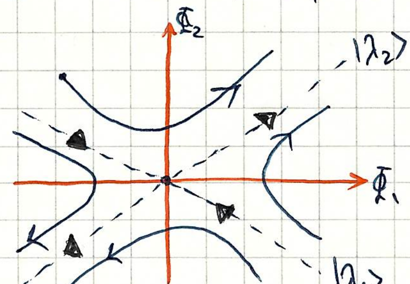
Since $\lambda_1$ and $\lambda_2$ are eigenvalues for $J$, then they are the roots of the following characteristic equation:

$$\lambda^2 - Tr(J)\lambda + det(J) = 0$$

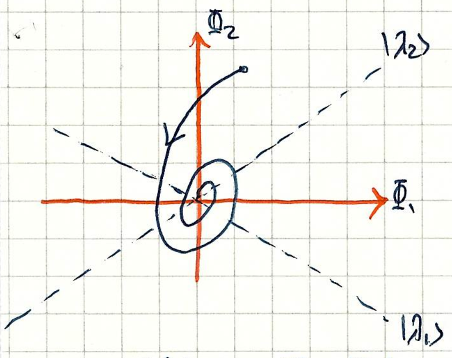
From VI it is clear that we can observe the following behaviours based on the values for $\lambda_1$ and $\lambda_2$



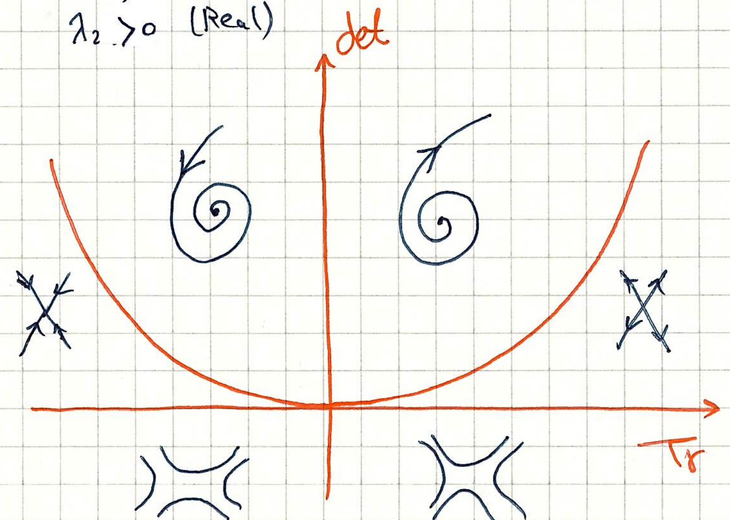

$\lambda_1 > 0$ (Real)
$\lambda_2 > 0$ (Real)




$\lambda_1 < 0$ (Real)
$\lambda_2 > 0$ (Real)




Real$(\lambda_1) < 0$   Imag$(\lambda_1) \neq 0$
Real$(\lambda_2) < 0$   Imag$(\lambda_2) \neq 0$



# Stability analysis for PDEs

The analysis we be similar to the ODE Case. The only difference is that Jacobian will be Changed to a modified Jacobian.
Consider the following system.

$$\frac{\partial x}{\partial t} = f(x,y) + D_x \frac{\partial^2 x}{\partial x^2}$$

$$\frac{\partial y}{\partial t} = g(x,y) + D_y \frac{\partial^2 y}{\partial x^2}$$

We Can make it more Compact:

$$\Phi = \begin{pmatrix} x \\ y \end{pmatrix} ; \; F(\Phi) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} ; \; D = \begin{pmatrix} D_x & 0 \\ 0 & D_y \end{pmatrix}$$

$$\Rightarrow \boxed{\dot{\Phi} = F(\Phi) + D\Phi''} \quad (\mathrm{I})$$

Now suppose that there is a homogeneous steady state in which

$$\boxed{\dot{\Phi} = F(\Phi_{ss}) = 0} \quad (\mathrm{II}) \quad \text{(well-mixed model)}$$

So when we are very close to the steady state we can write:

$$\boxed{\Phi = \Phi_{ss} + \tilde{\Phi}} \quad (\mathrm{III})$$

By inserting III in I we will have

$$\dot{\tilde{\Phi}} = \cancel{F(\Phi_{ss})}^{\,0} + J\tilde{\Phi} + D\tilde{\Phi}''$$

and by dropping "~" signs we will have:

$$\boxed{\dot{\Phi} = J\Phi + D\Phi''} \quad (\mathrm{IV})$$

Equation (IV) means that when we are very close to HSS, then the original system (I) will look like IV.

let's try a solution of the form:

$$\boxed{\Phi = \Phi_0 e^{\lambda t} e^{iqx}} \quad (V)$$

By inserting (V) in (VI) we will have:

$$\lambda \Phi_0 e^{\lambda t} e^{iqx} = J \Phi_0 e^{\lambda t} e^{iqx} - D q^2 \Phi_0 e^{\lambda t} e^{iqx}$$

$$\Rightarrow \left( J\Phi_0 - \lambda \Phi_0 - q^2 D \Phi_0 \right) e^{\lambda t} e^{iqx} = 0$$

$$\Rightarrow \boxed{\left( J - q^2 D \right) \Phi_0 = \lambda \Phi_0}$$

By calling $J - q^2 D$ as $\tilde{J}$ (modified Jacobian) we can write:

$$\boxed{\tilde{J} \Phi_0 = \lambda \Phi_0}$$

* So all of our past findings for ODE stability analysis hold for PDE stability analysis as well with a difference that in the case of PDE we are dealing with modified Jacobian.