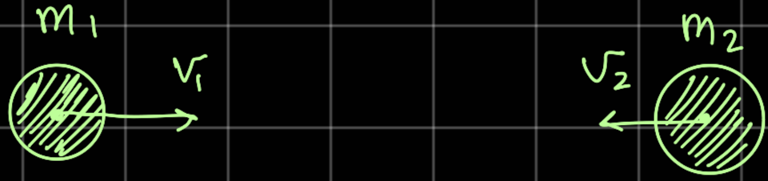
To Solve the 2D elastic collision of two balls both of which are moving, we need to first calculate the 1D problem.



$\Downarrow$ after collision

$$P^{\text{Before}} = P^{\text{after}} \Rightarrow m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$$

$$E^{\text{Before}} = E^{\text{after}} \Rightarrow \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$$

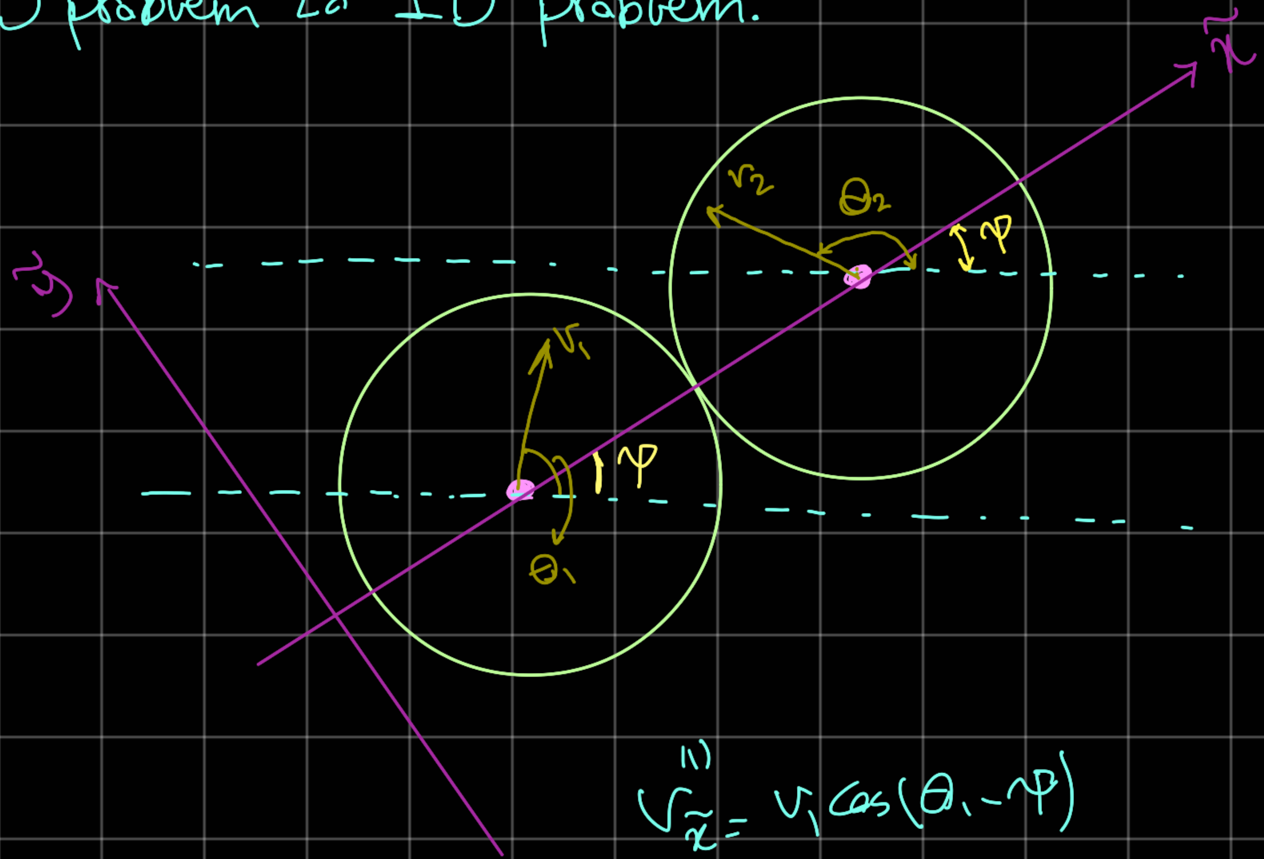
$$\Rightarrow \begin{cases} m_1 (v_1^2 - u_1^2) = m_2 (u_2^2 - v_2^2) & (\mathrm{I}) \\ m_1 (v_1 - u_1) = m_2 (u_2 - v_2) & (\mathrm{II}) \end{cases}$$

$\underline{\text{deviding I by II}}$ $\left\{ \; v_1 + u_1 = v_2 + u_2 \right.$

$\frac{a^2 - b^2}{a - b} = a + b$

$$\Rightarrow \begin{cases} v_1 - v_2 = u_2 - u_1 \\ m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \end{cases} \Rightarrow \begin{cases} u_1 = \dfrac{2 m_2 v_2 + (m_1 - m_2) v_1}{m_1 + m_2} \\ u_2 = \dfrac{2 m_1 v_1 + (m_2 - m_1) v_2}{m_1 + m_2} \end{cases}$$

Now using some intuitive tricks we can convert the 2D problem to 1D problem.



$$v_{\tilde{x}}^{(1)} = v_1 \cos(\theta_1 - \varphi)$$

$$v_{\tilde{y}}^{(1)} = v_1 \sin(\theta_1 - \varphi)$$

$$v_{\tilde{x}}^{(2)} = v_2 \cos(\theta_2 - \varphi)$$

$$v_{\tilde{y}}^{(2)} = v_2 \sin(\theta_2 - \varphi)$$

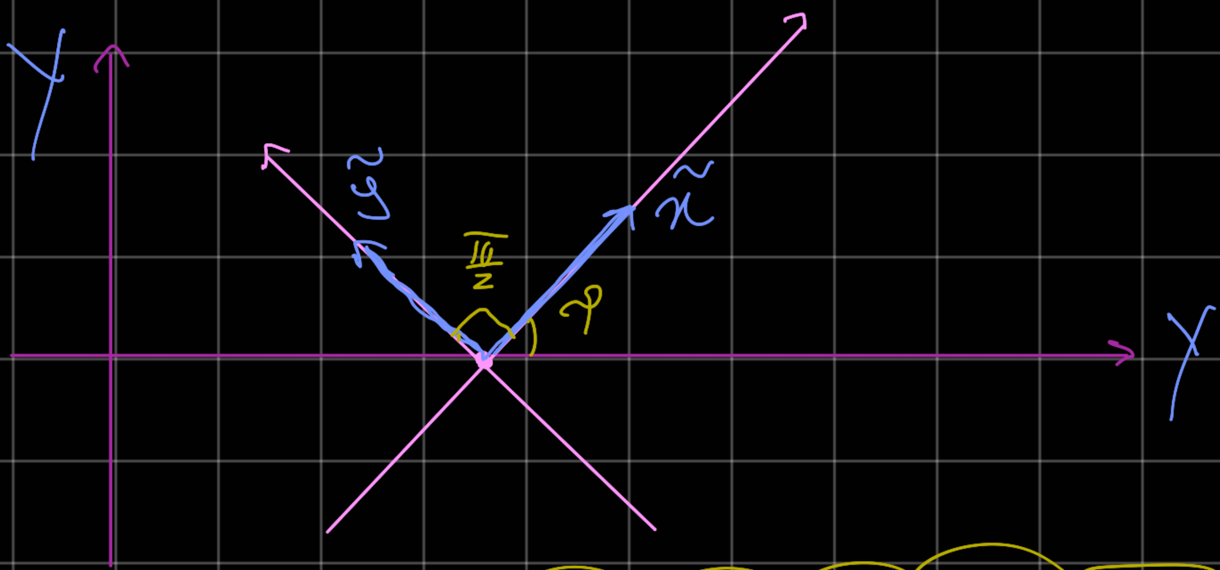
After the collision, the components prependicular to the collision line will feel no force and thus these component will not change (we are assuming they are not spining they will not spin after collision (the surface is slipery)

$$\Rightarrow \begin{cases} u_{\tilde{y}}^{(1)} = v_1 \sin(\theta_1 - \varphi) \\ u_{\tilde{y}}^{(2)} = v_2 \sin(\theta_2 - \varphi) \end{cases}$$

The components parallel to the collision line will act like a 1D collision

$$\Rightarrow \begin{cases} u^{(1)}_{\tilde{x}} = \dfrac{(m_1 - m_2) v_1 \cos(\theta_1 - \varphi) + 2m_2 v_2 \cos(\theta_2 - \varphi)}{m_1 + m_2} \\ u^{(2)}_{\tilde{x}} = \dfrac{(m_2 - m_1) v_2 \cos(\theta_2 - \varphi) + 2m_1 v_1 \cos(\theta_1 - \varphi)}{m_1 + m_2} \end{cases}$$

$\Rightarrow$ Conversion from rotated frame to original



$$\boxed{\begin{aligned} X &= \tilde{x}\cos\varphi + \tilde{y}\cos\left(\tfrac{\pi}{2} + \varphi\right) \\ Y &= \tilde{x}\sin\varphi + \tilde{y}\sin\left(\tfrac{\pi}{2} + \varphi\right) \end{aligned}}$$

So the velocities after the collision will be:

$$v_x^{(1)} = \frac{(m_1-m_2)v_1\cos(\theta_1-\varphi)+2m_2v_2\cos(\theta_2-\varphi)}{m_1+m_2}\cos\varphi + v_1\sin(\theta_1-\varphi)\cos\left(\frac{\pi}{2}+\varphi\right)$$

$$v_x^{(1)} = \frac{(m_1-m_2)v_1\cos(\theta_1-\varphi)+2m_2v_2\cos(\theta_2-\varphi)}{m_1+m_2}\sin(\varphi) + v_1\sin(\theta_1-\varphi)\sin\left(\frac{\pi}{2}+\varphi\right)$$

$$v_x^{(1)} = \frac{(m_2-m_1)v_2\cos(\theta_2-\varphi)+2m_1v_1\cos(\theta_1-\varphi)}{m_1+m_2}\cos\varphi + v_2\sin(\theta_2-\varphi)\cos\left(\frac{\pi}{2}+\varphi\right)$$

$$v_x^{(1)} = \frac{(m_2-m_1)v_2\cos(\theta_2-\varphi)+2m_1v_1\cos(\theta_1-\varphi)}{m_1+m_2}\sin(\varphi) + v_2\sin(\theta_2-\varphi)\sin\left(\frac{\pi}{2}+\varphi\right)$$

(see wikipedia elastic collision)