Wave equation is in fact an hyperbolic type of PDE.

$$\frac{\partial^2 F}{\partial t^2} = c^2 \frac{\partial^2 F}{\partial x^2}$$

Now we can use finite difference approximation for the above equation

$$\frac{\partial^2 F}{\partial t^2} = \frac{F_j^{m-1} - 2F_j^m + F_j^{m+1}}{(\Delta t)^2}$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{F_{j-1}^m - 2F_j^m + F_{j+1}^m}{(\Delta x)^2}$$

$$\Rightarrow \frac{F_j^{m-1} - 2F_j^m + F_j^{m+1}}{(\Delta t)^2} = c^2 \frac{F_{j-1}^m - 2F_j^m + F_{j+1}^m}{(\Delta x)^2}$$

$$F_j^{m+1} = \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(F_{j-1}^m - 2F_j^m + F_{j+1}^m\right) + 2F_j^m - F_j^{m-1}$$

later we will discover that to have a stable solution we should have :

$$\boxed{\frac{c\Delta t}{\Delta x} \le 1}$$

(in the case of equality we will have the exact solution)

$$\boxed{F^{n+1} = \left(\frac{c\Delta t}{\Delta x}\right)^2 \begin{pmatrix} 1 & 0 & \cdots & & & & \\ 1 & -2 & 1 & \cdots & & & \\ 0 & 1 & -2 & 1 & \cdots & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ \vdots & & & & 1 & -2 & 1 \\ \cdots & & & & \cdots & 0 & 1 \end{pmatrix} F^{n} + 2F^{n} - F^{n-1}}$$

but to use the above equation, we need to know f_j^{-1} (to Caculate f_j^{1})

to do that we can use : $\boxed{\frac{\partial f}{\partial t} = \frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t}}$

Note that since we know $\left.\frac{\partial f}{\partial t}\right|_{t=0}$ as initial value

Then we can calculate f^{-1} like following:

$$\frac{f^{1}-f^{-1}}{\Delta t}=\dot{f}_{0} \Rightarrow \boxed{f^{-1}=f^{1}-\Delta t\,\dot{f}_{0}}$$

So to calculate the first step we can write:

$$f_{j}^{1}=\left(\frac{c\Delta t}{\Delta x}\right)^{2}\left(f_{j-1}^{0}-2f_{j}^{0}+f_{j+1}^{0}\right)+2f_{j}^{0}-\underbrace{f_{j}^{-1}}_{=f_{j}^{1}-\Delta t\,\dot{f}_{0}}$$

$$\Rightarrow \boxed{f_{j}^{1}=\frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^{2}\left(f_{j-1}^{0}-2f_{j}^{0}+f_{j+1}^{0}\right)+f_{j}^{0}-\frac{\Delta t}{2}\dot{f}_{0}}$$

Van Neumman Stability analysis:

Since in the update rule we have term f_{j}^{m-1}, we can not use Neumman method directly. instead we define variable

$$\boxed{v_{j}^{m}=f_{j}^{m-1}}$$

$$\Rightarrow \begin{cases} f_j^{m+1} = \left(\dfrac{c\Delta t}{\Delta x}\right)^2 \underbrace{\left(f_{j-1}^{m} - 2f_j^{m} + f_{j+1}^{m}\right)}_{D^2 f_j^m} + 2f_j^m - v_j^m \\ v_j^{m+1} = f_j^m \end{cases}$$

$$\Rightarrow \begin{pmatrix} f_j^{m+1} \\ v_j^{m+1} \end{pmatrix} = \overbrace{\begin{pmatrix} r^2 D^2 + 2 & -1 \\ 1 & 0 \end{pmatrix}}^{G} \begin{pmatrix} f_j^m \\ v_j^m \end{pmatrix}$$

To have an stable solution, we should have:

$\|G\| \leq 1 \longrightarrow$ all of eigen values of G must be $|\lambda_i| \leq 1$.

to analyse this condition we should calculate Fourier transform:

$$\begin{pmatrix} \tilde{f}_j^{m+1} \\ \tilde{v}_j^{m+1} \end{pmatrix} = \begin{pmatrix} -4r^2 \sin^2\left(\frac{\eta}{2}\right) + 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{f}_j^m \\ \tilde{v}_j^m \end{pmatrix}$$

$\left(D^2 \xrightarrow{\text{Fourier}} -4\sin^2\left(\frac{\eta}{2}\right) \right)$ (For more results please see the note for diffusion equ.)

$$\det(M - \lambda I) = 0$$

$$\Rightarrow \det \begin{pmatrix} -4r^2\sin^2(\eta/2) + 2 - \lambda & -1 \\ 1 & 0 - \lambda \end{pmatrix}$$

you can find the numerical solution for this in this note book

$$\Rightarrow \lambda^2 + \lambda\big(-2 + \overbrace{4r^2\sin^2(\tfrac{\eta}{2})}^{Q}\big) + 1 = 0$$

this can be solved numerically

$$\lambda^2 + \lambda(Q-2) + 1 = 0$$

$$\lambda = \frac{-(Q-2) \pm \sqrt{(Q-2)^2 - 4}}{2} \; ; \; \boxed{|\lambda| \leq 1}$$

$$\Rightarrow Q \leq 4 \Rightarrow 4r^2\sin^2\left(\frac{\eta}{2}\right) \leq 4$$

$$\Rightarrow \boxed{r \leq 1}$$

(in the case of equality we will have the exact solution)

$$\boxed{\frac{c\Delta t}{\Delta x} \leq 1}$$

(Giordano Computational physics book)