Random Variables

A random variable (X) is in fact a function from sample space to set E (usually $\mathbb{R}$)

$$X : \Omega \to E$$

So random variable is a function that assigns a number (if $E = \mathbb{R}$) to every element in the sample space Ω.

$\Rightarrow$ E is descrete $\Rightarrow P(X=a) = P(\{\omega \in \Omega \mid X(\omega) = a\})$

$\Rightarrow$ E is Continues $\Rightarrow P(X \in [a,b]) = P(\{\omega \in \Omega \mid a \leq X(\omega) \leq b\})$

Most of the times we can assign a distribution fuenction to a random variable :

Comulative distribution function (CDF) $\Rightarrow F_X(x) = P(X \leq x)$

Probability distribution function (PDF) $\Rightarrow f_X(x) = \frac{d}{dx}(F_X(x))$

Intiutivley, $f_X(x)dx$ means the probability that X is in range $[x, x+dx]$

Examples for random variables : Height, Weight, Sum of faces of 3 dices, ...

Function of random variables:

Applying a function f on the outputs of random variable will produce another random variable

$$Y = f(X)$$

given that $F_X(x)$, $f_X(x)$ are Cumulative and density functions Let's Calculate $F_Y(x)$ and $f_Y(x)$.

$$F_y(x) = P(Y \leq x) = P(f(X) \leq x) = \begin{cases} P(X \leq f^{-1}(x)) & \text{if } f \text{ is monotonically increasing} \\ P(X \geq f^{-1}(x)) & \text{if } f \text{ is monotonically decreasing} \end{cases}$$

Knowing $P(X \geq f^{-1}(x)) = 1 - F_X(f^{-1}(x))$, we ~~gonna~~ continue the derivation for the first case and the second can easily be calculated.

$$F_y(x) = P(X \leq f^{-1}(y)) = F_X(f^{-1}(y))$$

$$\Rightarrow \boxed{F_y(x) = F_X(f^{-1}(y))}$$

$$f_y(x) = \frac{dF_y(x)}{dx} = \frac{dF_X(f^{-1}(y))}{df^{-1}(y)} \times \frac{df^{-1}(y)}{dy} = f_X(f^{-1}(y)) \times \frac{df^{-1}(y)}{dy}$$

$$\Rightarrow \boxed{f_y(x) = f_X(f^{-1}(y)) \times \frac{df^{-1}(y)}{dy}}$$

Example 1 (from random variable wikipedia page):

Find $F_Y(y)$ and $f_Y(y)$ of $Y = X^2$

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$$
$$= F_X(-\sqrt{y}) - F_X(\sqrt{y})$$ ☑

$$f_Y(y) = \frac{dF_Y(y)}{dy} = +\frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) - f_X(-\sqrt{y})\right)$$ ☑

Example 2: (From random variable wikipedia page)

Suppose X is a random variable with $F_X(x)$ as follows

$$F_X(x) = P(X \leq x) = \frac{1}{(1+e^{-x})^{\theta}}$$

Consider $Y = \log(1+e^{-X})$. Find $F_Y(y)$ and $f_Y(y)$

Answer:

$$Y = g(X) = \log(1+e^{-X})$$

$$\Rightarrow e^{-X} = \exp(g(X)) - 1$$

$$\Rightarrow X = -\ln(-1 + \exp(g(X)))$$

$$\Rightarrow g^{-1}(x) = -\ln(-1+\exp(x)) \Rightarrow \text{monotonically decreasing}$$

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \geq -\ln(e^{y}-1))$$

$$= 1 - F_X(-\ln(e^{y}-1))$$

$$= 1 - e^{-y\theta}$$

$$\Rightarrow \boxed{F_Y(y) = 1 - e^{-y\theta}}$$

$$f_Y(y) = \frac{dF_Y}{dy} = \theta e^{-y\theta} \qquad \Rightarrow \boxed{f_Y(y) = \theta e^{-y\theta}}$$

Example 3: (From random variable wikipedia page)

Suppose $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Calculate $F_Y(y)$ and $f_Y(y)$

for $Y = X^2$

$$\Rightarrow F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$\Rightarrow f_Y(y) = \frac{dF_Y}{dy} = \frac{1}{2\sqrt{y}} \left(f_X(-\sqrt{y}) - f_X(\sqrt{y}) \right) = \frac{e^{-y/2}}{\sqrt{2\pi y}}$$

$$\Rightarrow \boxed{f_Y(y) = \frac{e^{-y/2}}{\sqrt{2\pi y}}}$$