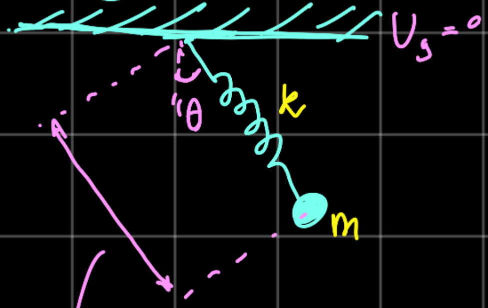
Solving ODE of Spring Pendulum



$L(t) = l + x(t)$

$$R = (L(t)\sin\theta,\ L(t)\cos\theta)$$

$$\Rightarrow \dot{R} = (\dot{x}\sin\theta + \dot{\theta}\cos\theta L(t),\ \dot{x}\cos\theta - \dot{\theta}\sin\theta L(t))$$

$$\Rightarrow |\dot{R}|^2 = \dot{x}^2 + \dot{\theta}^2 (l+x)^2$$

$$\Rightarrow T = \frac{1}{2} m |\dot{R}|^2 = \frac{1}{2} m (\dot{x}^2 + \dot{\theta}^2 (l+x)^2)$$

$$U = \frac{1}{2} k x^2 - mg(l+x)\cos\theta$$

$$\Rightarrow \mathcal{L}_p = T - U = \frac{1}{2} m (\dot{x}^2 + \dot{\theta}^2 (l+x)^2) - \frac{1}{2} k x^2 + mg(l+x)\cos\theta$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$\Rightarrow \boxed{m\ddot{x} = m\dot{\theta}^2 (l+x) - kx + mg\cos\theta}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta}$$

$$\Rightarrow m\ddot{\theta}(l+x)^2 + 2m(l+x)\dot{x}\dot{\theta}^2 = -mg(l+x)\sin\theta$$

$$\Rightarrow \boxed{m\ddot{\theta}(l+x) + 2m\dot{x}\dot{\theta} = -mg\sin\theta}$$

So the equations of motion are:

$$\begin{cases} m\ddot{x} = m\dot{\theta}^2(\ell+x) + mg\cos\theta - kx \\ m(\ell+x)\ddot{\theta} + 2m\dot{x}\dot{\theta} = -mg\sin\theta \end{cases}$$

To solve these equations using Runge-Kutta, we need to write $\ddot{x}, \ddot{\theta}$ explicitly:

⇒ let's assume $m, \ell, g, k = 1$

$$\begin{cases} \ddot{x} = (1+x)\dot{\theta}^2 + \cos\theta - x \\ \ddot{\theta} = \dfrac{-1}{(1+x)}\left[2\dot{x}\dot{\theta} - \sin\theta\right] \end{cases}$$

We now want to put this equation in form of

$$\ddot{\Phi} = F(t, \Phi, \dot{\Phi})$$

So let's define $\Phi = \begin{pmatrix} x \\ \theta \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} (1+x)\dot{\theta}^2 + \cos\theta - x \\ -\dfrac{1}{(1+x)}\left[2\dot{x}\dot{\theta} - \sin\theta\right] \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \vec{F}(t, \Phi, \dot{\Phi})$$

$$\Longrightarrow \ddot{\vec{\Phi}} = \vec{F}(t, \vec{\Phi}, \dot{\vec{\Phi}})$$

This is a second order ODE. to use RK method we need to convert it into two first order ODEs

let's define $\Psi = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} \Phi \\ \dot{\Phi} \end{pmatrix} = \begin{pmatrix} [x, \theta] \\ [\dot{x}, \dot{\theta}] \end{pmatrix}$

$$\Rightarrow \begin{cases} \dot{\psi}_0 = \psi_1 \\ \dot{\psi}_1 = \vec{F}(t, \Psi) \end{cases}$$

$$\Longrightarrow \begin{pmatrix} \dot{\psi}_0 \\ \dot{\psi}_1 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \vec{F}(t, \Psi) \end{pmatrix} = \begin{pmatrix} \dot{\Phi} \\ \vec{F}(t, \Psi) \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \vec{G}$$

$$\Longrightarrow \boxed{\dot{\Psi} = G(t, \Psi)}$$

$$\Psi(t+h) = \Psi(t) + \frac{1}{6}\left(\tilde{g}_0 + 2\tilde{g}_1 + 2\tilde{g}_2 + \tilde{g}_3\right)$$

$g_0 = G(t, \Psi)$; $g_1 = G\left(t + \frac{h}{2}, \Psi + \frac{h}{2}\tilde{g}_0\right)$

$g_2 = G\left(t + \frac{h}{2}, \Psi + \frac{h}{2}\tilde{g}_1\right)$; $g_3 = G(t+h, \Psi + h\tilde{g}_2)$