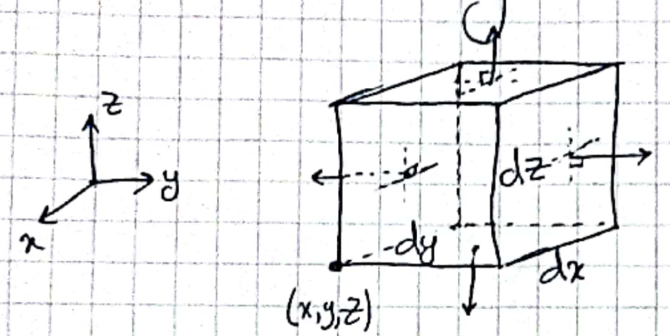
Derivation of Continuity equation and Diffusion PDE

Let's define $\vec{J}$ (Flux) to number of particles per time per area. So:

$$\frac{\Delta N}{\Delta t} = \hat{J} \cdot d\vec{a}$$

Now Consider the following infinitesimal cube



$$\vec{J} = J_x \hat{i} + J_y \hat{j} + J_z \hat{k}$$

Total number of particles exiting through:

x axis $\Rightarrow -J_x(x,y,z)\,dz\,dy + J_x(x+dx,y,z)\,dz\,dy = \Delta N_x$

y axis $\Rightarrow -J_y(x,y,z)\,dx\,dz + J_y(x,y+dy,z)\,dx\,dz = \Delta N_y$

z axis $\Rightarrow -J_z(x,y,z)\,dx\,dy + J_z(x,y,z+dz)\,dx\,dy = \Delta N_z$

$$\Rightarrow \Delta N_T = \Delta N_x + \Delta N_y + \Delta N_z \Rightarrow \Delta C_T = \frac{\Delta N_T}{dx\,dy\,dz}$$

$\Rightarrow$ Total concentration "leaving" the box per time $\Rightarrow \Delta C_T$

$$\Rightarrow \frac{\partial c}{\partial t} = -\Delta C_T = \left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right)$$

$$\Rightarrow \boxed{\frac{\partial c}{\partial t} = -\nabla \cdot \vec{J}}$$

Now to derive the well known PDE equation we need to use

First Fick's law:

$$J_x = -D\frac{\partial c}{\partial x} \quad \text{or} \quad \vec{J} = -D\nabla c$$

This equation Can be derived from statistical mechanics. It gives me the insight that $\vec{J}$ acts like a force and c acts like potential!

Now by applying the Fick's law in the Continuity equation we will have:

$$\frac{\partial c}{\partial t} = -\nabla\cdot(D\nabla c)$$

$$\Rightarrow \boxed{\frac{\partial c}{\partial t} = D\nabla^2 c}$$

In which ∇^2 is the Laplacian operator.