NAVIGATIONAL ALGORITHMS
Sight Reduction with Matrices

For each **body**
Date UT1

Observation
Hs, Z

Nautical Almanac

GHA, Dec

Observed Altitude
Ho

Equations

DR
Be, Le

$$\begin{bmatrix} \cos Dec_1 \cdot \cos GHA_1 & \cos Dec_1 \cdot \sin GHA_1 & \sin Dec_1 \\ \cos Dec_2 \cdot \cos GHA_2 & \cos Dec_2 \cdot \sin GHA_2 & \sin Dec_2 \\ \cos Be \cos Le & \cos Be \sin Le & \sin Be \end{bmatrix} \begin{bmatrix} x_F \\ y_F \\ z_F \end{bmatrix} = \begin{bmatrix} \sin H_1 \\ \sin H_2 \\ 1 \end{bmatrix}$$

Cartesian to Spherical

$$B_F = ATAN2(z_F, \sqrt{x_F^2 + y_F^2})$$
$$L_F = ATAN2(y_F, x_F)$$

True Position
B_F, L_F

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*http://sites.google.com/site/navigationalalgorithms/*

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Abstract

An analytical method for obtain the position by celestial sights using matrix algebra is described. The simplicity of the process allows the use of a hand held calculator or PDA.

Finding position by stars

If we take some sight with a sextant, we shall be able to obtain our position by intersecting the associated circles of position, COP.
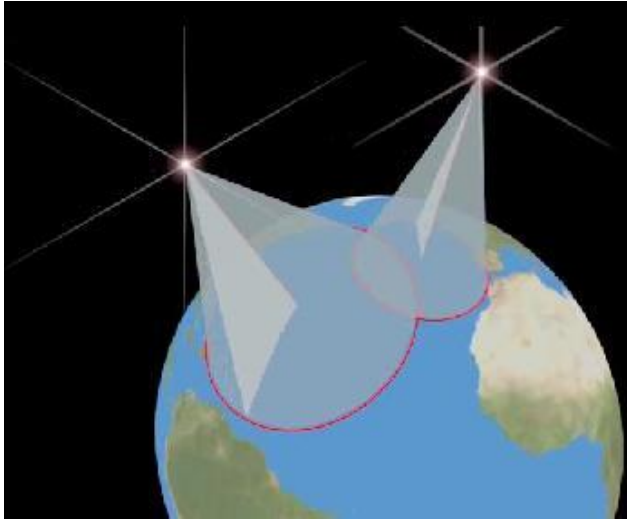
Circles of Equal Altitude

For one sight, the vectorial equation of the circle of equal altitude is:

$$\vec{OP} \bullet \vec{GP} = \cos(90º - Ho)$$

And in Cartesian coordinates we have:

$$\{OP\} = \begin{bmatrix} \cos B \cdot \cos L \\ \cos B \cdot \sin L \\ \sin B \end{bmatrix}$$

$$\{GP\} = \begin{bmatrix} \cos Dec \cdot \cos GHA \\ \cos Dec \cdot \sin GHA \\ \sin Dec \end{bmatrix}$$

Where OP is the observer's position, GP the geographical position of the celestial body, GHA the Greenwich Hour Angle, Dec the declination, and Ho the celestial body's observed altitude.

The position is obtained by solving the system of linear equations of each COP for the Cartesian coordinates, and transforming them to spherical ones.

Take care with the signs of the variables; the mathematical functions use a different criterion to the nautical one:

- $-90 \leq B \leq 90º$ [+N/-S]
- $0 \leq L \leq 360º$ [+ W ⇨ E]
- $-90 \leq Dec \leq 90º$ [+N/-S]
- $0 \leq GHA \leq 360º$ [+ W ⇨ E]

Here is supposed that all observations are obtained simultaneously, if not the motion of the vessel and the motion of the bodies may be incorporated into the algorithm.

Two Observations

If two bodies are shot there are two solutions or one.

- Two solutions: The two COPs have two points of intersection. Our DR position determines what the real one is.

- One solution: if the two COPs are tangents. This theorist case is very improbable in navigation

There are three unknown parameters, the Cartesian coordinates of the true position, hence three equations are needed in order to calculate them: the two COP equations and the fact that from the observer's position the zenith altitude is 90º.

The algorithm is described in the appendix.

Iteration for the assumed position

If the assumed position is far away from the true one, solving the matrix system it obtained a bad solution for the position. Then an iterative process can improve the solution setting new assumed position equal to solution and solving once again until the error is acceptable.
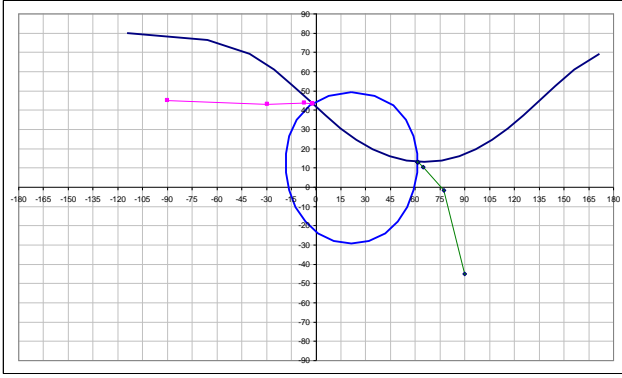
2 COP and solution points in an iterative process for the two intersections

3 Observations

For three sights the system of equations is determined, and is it possible to obtain an explicit solution:

	Body	Ho	GHA	Dec
1	Body1	Ho1	GHA1	Dec1
2	Body2	Ho2	GHA2	Dec2
3	Body3	Ho3	GHA3	Dec3

$$\begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} = \frac{1}{\sin Ho_k} \begin{bmatrix} \cos Dec_k \cdot \cos GHA_k \\ \cos Dec_k \cdot \sin GHA_k \\ \sin Dec_k \end{bmatrix}$$

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \begin{bmatrix} x_F \\ y_F \\ z_F \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$|M_3| = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$x_F = \frac{|Mx|}{|M_3|}$$

$$|M_x| = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}$$

$$y_F = \frac{|My|}{|M_3|}$$

$$|M_y| = \begin{vmatrix} x_1 & 1 & z_1 \\ x_2 & 1 & z_2 \\ x_3 & 1 & z_3 \end{vmatrix}$$

$$z_F = \frac{|Mz|}{|M_3|}$$

$$|M_z| = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

```
M3 = x1*y2*z3+y1*z2*x3+z1*x2*y3-
(z1*y2*x3+x1*z2*y3+y1*x2*z3)

Mx = y2*z3+y1*z2+y3*z1-
(y2*z1+y1*z3+y3*z2)

My = z2*x3+z1*x2+z3*x1-
(z2*x1+z1*x3+z3*x2)

Mz = x2*y3+x1*y2+x3*y1-
(x2*y1+x1*y3+x3*y2)
```

This method works only for one point of intersection. Otherwise a least squares method must be used to find the most probable celestial position.
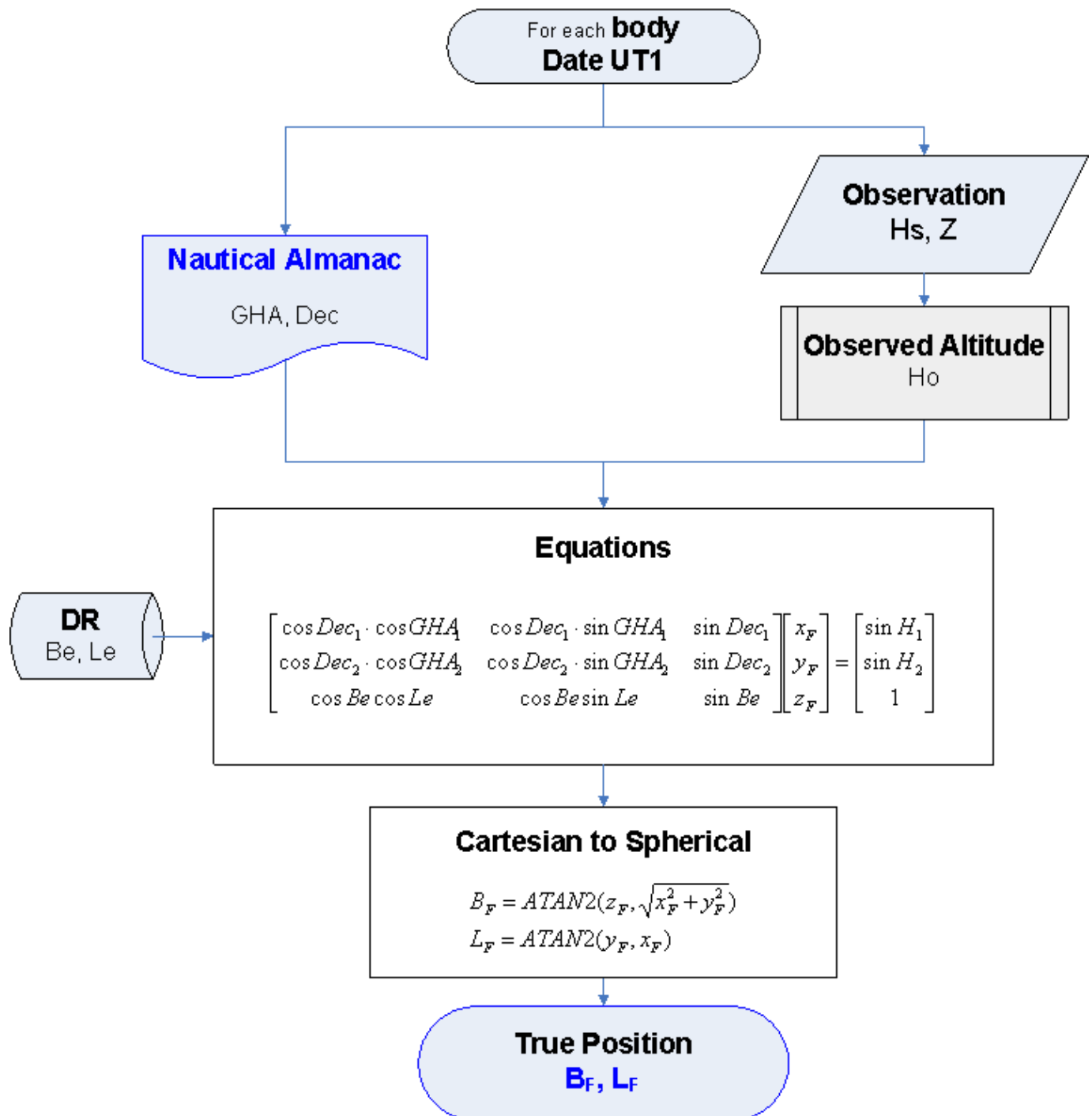
n Observations

The matrix system is overdetermined, and a least squares solution for the fix may be used.

For n ≥ 3 observations it is possible to incorporate the assumed position by doing Dec = B, GHA = L, Ho = 90º, but may be ignored.

The details of the algorithm are described in the appendix.

A1. Algorithms

Sight Reduction with Matrices
2 Observations

For each **body**
Date UT1

Observation
Hs, Z

Nautical Almanac
GHA, Dec

Observed Altitude
Ho

Equations

DR
Be, Le

$$\begin{bmatrix} \cos Dec_1 \cdot \cos GHA_1 & \cos Dec_1 \cdot \sin GHA_1 & \sin Dec_1 \\ \cos Dec_2 \cdot \cos GHA_2 & \cos Dec_2 \cdot \sin GHA_2 & \sin Dec_2 \\ \cos Be \cos Le & \cos Be \sin Le & \sin Be \end{bmatrix} \begin{bmatrix} x_F \\ y_F \\ z_F \end{bmatrix} = \begin{bmatrix} \sin H_1 \\ \sin H_2 \\ 1 \end{bmatrix}$$

Cartesian to Spherical

$$B_F = ATAN2(z_F, \sqrt{x_F^2 + y_F^2})$$
$$L_F = ATAN2(y_F, x_F)$$

True Position
B_F, L_F

Sight Reduction with Matrices
2 Observations
Iteration for assumed position
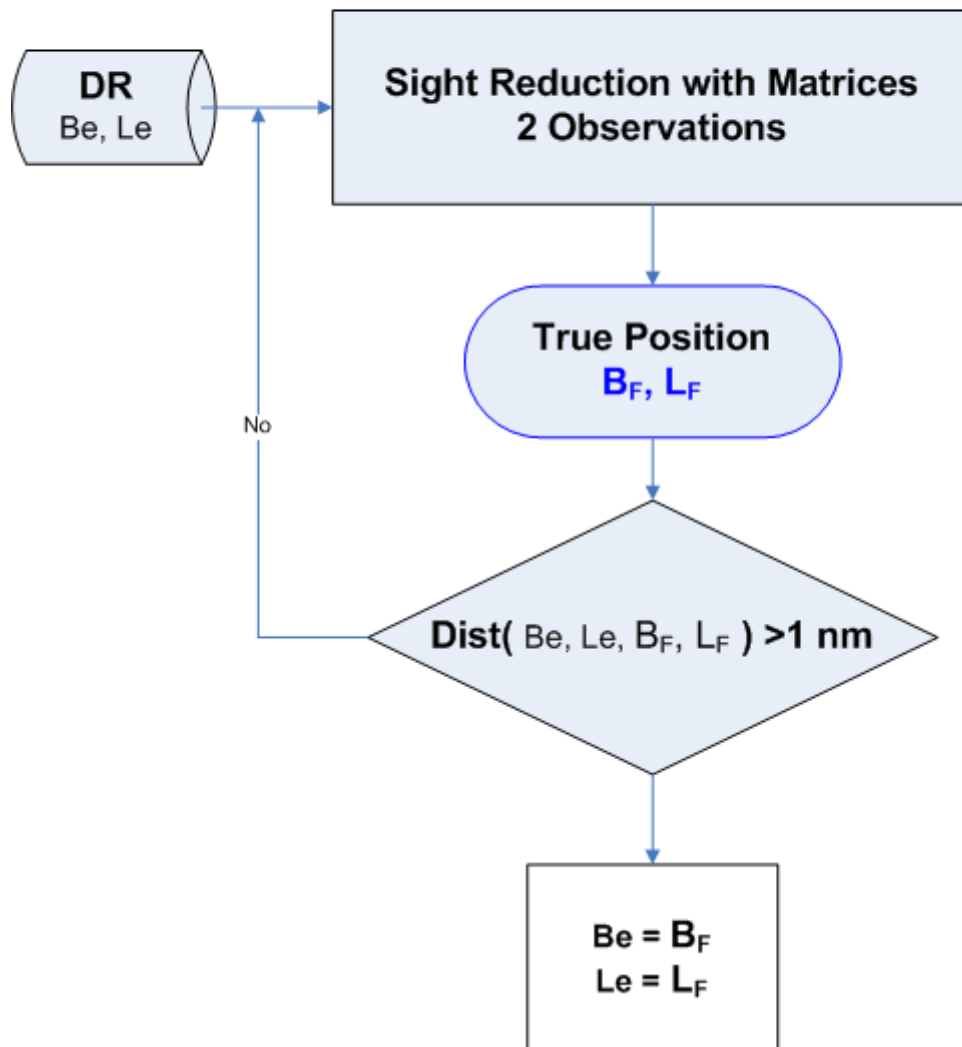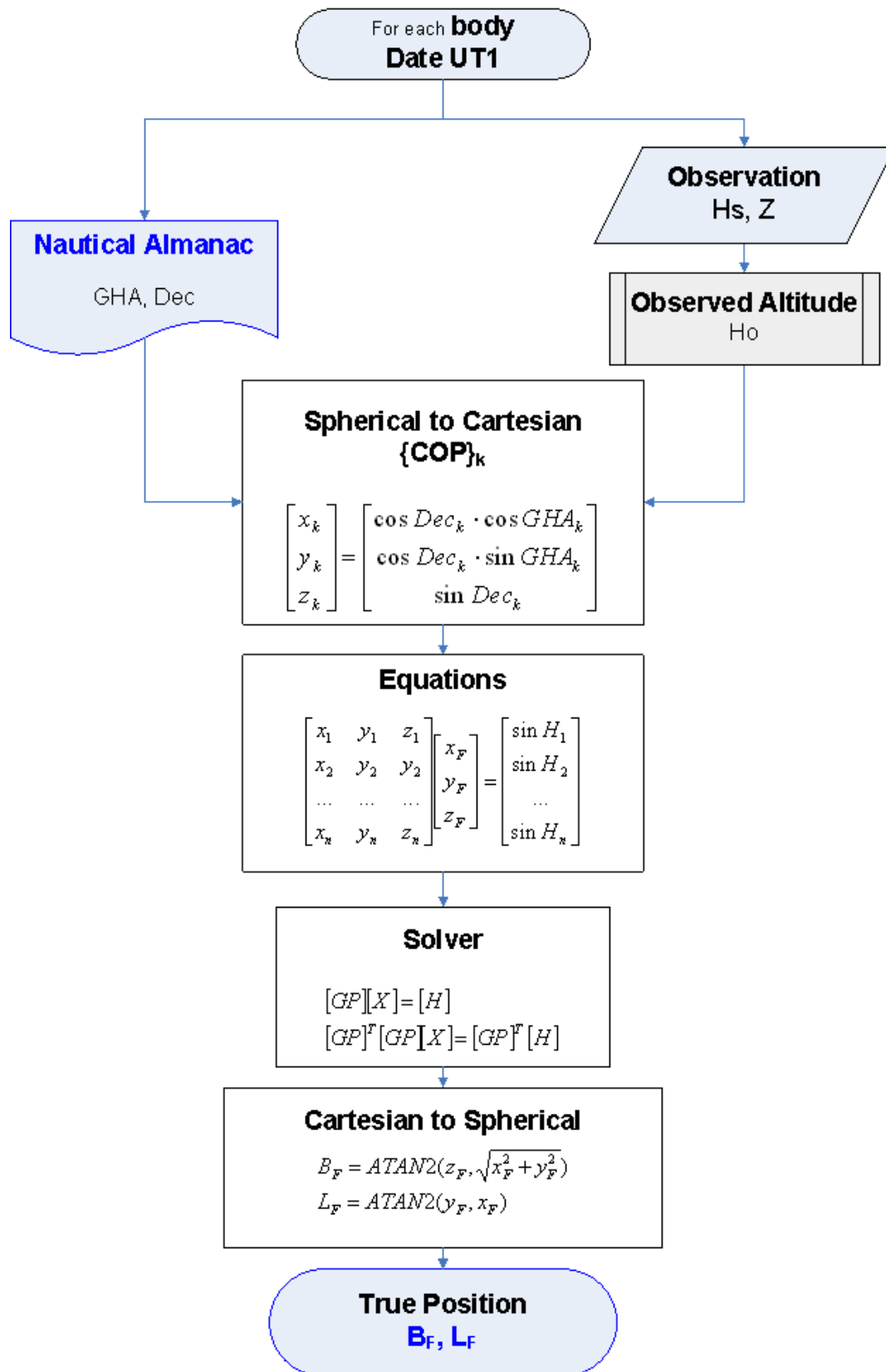
Sight Reduction with Matrices
n Observations

For each **body**
Date UT1

Observation
Hs, Z

Nautical Almanac

GHA, Dec

Observed Altitude
Ho

Spherical to Cartesian
{COP}$_k$

$$\begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} = \begin{bmatrix} \cos Dec_k \cdot \cos GHA_k \\ \cos Dec_k \cdot \sin GHA_k \\ \sin Dec_k \end{bmatrix}$$

Equations

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & y_2 \\ ... & ... & ... \\ x_n & y_n & z_n \end{bmatrix} \begin{bmatrix} x_F \\ y_F \\ z_F \end{bmatrix} = \begin{bmatrix} \sin H_1 \\ \sin H_2 \\ ... \\ \sin H_n \end{bmatrix}$$

Solver

$$[GP][X] = [H]$$
$$[GP]^T [GP][X] = [GP]^T [H]$$

Cartesian to Spherical

$$B_F = ATAN2(z_F, \sqrt{x_F^2 + y_F^2})$$
$$L_F = ATAN2(y_F, x_F)$$

True Position
B$_F$, L$_F$

A2. Examples

3 Observations

Using the method explained in the *3 Observations* section, not the general one.

Input data:

K	Body	Ho	GHA	Dec
1	Arcturus	22.2	166.829	19.335
2	Vega	70.15	115.277	38.758
3	Dubhe	20.35	226.98	61.909

The equations in matrix form are:
```
| -2.432  0.569  0.876 | |x|   | 1 |
| -0.354  0.750  0.666 | |y| = | 1 |
| -0.924 -0.990  2.537 | |z|   | 1 |
```

Solving the system:
```
|x|  | 0.006|
|y|=| 0.731|
|z|  | 0.682|
```

And the position y latitude and longitude coordinate is:
Fix(42.98208, 89.56648)

Sight Reduction with matrices

http://www.geocities.com/andresruizgonzalez

Assumed Position

Be =   43.3167 °
Le =   2.0000 °

Geographic position & Observed Altitude

n	Body	Dec	GHA	Ho
1	Enif	9.897	338.392	50.775
2	Schedar	56.560	294.303	46.438

Equations in matrix form

$$\begin{vmatrix} 0.91588803 & -0.36277983 & 0.17187179 \\ 0.22679974 & -0.50222783 & 0.83446335 \\ 0.72713001 & 0.02539194 & 0.68603002 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.77466864 \\ 0.72463308 \\ 1 \end{vmatrix}$$

Solution in Cartesian coordinates

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.72712784 \\ 0.02538372 \\ 0.68603263 \end{vmatrix}$$

Solution in Geographic coordinates

B	=	43.3168721 °	46.6831279
L	=	1.99935875 °	88.0006412

Iteration

Set assumed position = solution and solve once again

Sight Reduction with matrices

http://www.geocities.com/andresruizgonzalez

Geographic position & Observed Altitude

n	Body	Dec	GHA	Ho
1	arcturus	19.335	166.829	22.200
2	vega	38.758	115.277	70.150
3	dubhe	61.9090	226.9800	20.350

Equations in matrix form

$$\begin{vmatrix} -0.9187769 & 0.21500662 & 0.33109087 \\ -0.33296938 & 0.70513465 & 0.62603236 \\ -0.32125502 & -0.34426282 & 0.88220084 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.37784079 \\ 0.94058481 \\ 0.34775398 \end{vmatrix}$$

$$\begin{vmatrix} 1.05822439 & -0.3217352 & -0.79605969 \\ -0.3217352 & 0.66195961 & 0.20891489 \\ -0.79605969 & 0.20891489 & 1.279816 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} -0.77205504 \\ 0.62475844 \\ 1.02072501 \end{vmatrix}$$

Solution in Cartesian coordinates

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.00553388 \\ 0.73137457 \\ 0.68160992 \end{vmatrix}$$

Solution in Geographic coordinates

B	=	42.9820851 °
L	=	89.5664846 °

Sight Reduction with matrices
http://www.geocities.com/andresruizgonzalez

Geographic position & Observed Altitude

n	Body	Dec	GHA	Ho
1	Mars	22.393	72.777	29.518
2	Alioth	55.928	357.303	77.542
3	Alkaid	49.2830	343.9330	76.027
4	Arcturus	19.1480	336.8970	57.045

Equations in matrix form

$$\begin{vmatrix} 0.273764 & 0.88313346 & 0.38095742 \\ 0.55961371 & -0.02636135 & 0.82833422 \\ 0.62684273 & -0.18053781 & 0.75794082 \\ 0.86891246 & -0.37067635 & 0.32800942 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.49269697 \\ 0.9764544 \\ 0.97040962 \\ 0.83909807 \end{vmatrix}$$

$$\begin{vmatrix} 1.53605491 & -0.20823613 & 1.32796078 \\ -0.20823613 & 0.95061449 & 0.05617793 \\ 1.32796078 & 0.05617793 & 1.5133306 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 2.01871695 \\ -0.07685291 \\ 2.0072723 \end{vmatrix}$$

Solution in Cartesian coordinates

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.71873376 \\ 0.03556087 \\ 0.69437855 \end{vmatrix}$$

Solution in Geographic coordinates

B = **43.977596** °
L = **2.83251973** °

Sight Reduction with matrices
http://www.geocities.com/andresruizgonzalez

Geographic position & Observed Altitude

n	Body	Dec	GHA	Ho
1	Mars	22.393	72.777	29.518
2	Alioth	55.928	357.303	77.542
3	Alkaid	49.2830	343.9330	76.027
4	Arcturus	19.1480	336.8970	57.045
5	Polaris	89.2900	151.6270	43.370

Equations in matrix form

$$\begin{vmatrix} 0.273764 & 0.88313346 & 0.38095742 \\ 0.55961371 & -0.02636135 & 0.82833422 \\ 0.62684273 & -0.18053781 & 0.75794082 \\ 0.86891246 & -0.37067635 & 0.32800942 \\ -0.01090296 & 0.00588857 & 0.99992322 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.49269697 \\ 0.9764544 \\ 0.97040962 \\ 0.83909807 \\ 0.68670698 \end{vmatrix}$$

$$\begin{vmatrix} 1.53617379 & -0.20830033 & 1.31705866 \\ -0.20830033 & 0.95064917 & 0.06206605 \\ 1.31705866 & 0.06206605 & 2.51317705 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 2.01122981 \\ -0.07280919 \\ 2.69392655 \end{vmatrix}$$

Solution in Cartesian coordinates

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.7187278 \\ 0.03555919 \\ 0.69438517 \end{vmatrix}$$
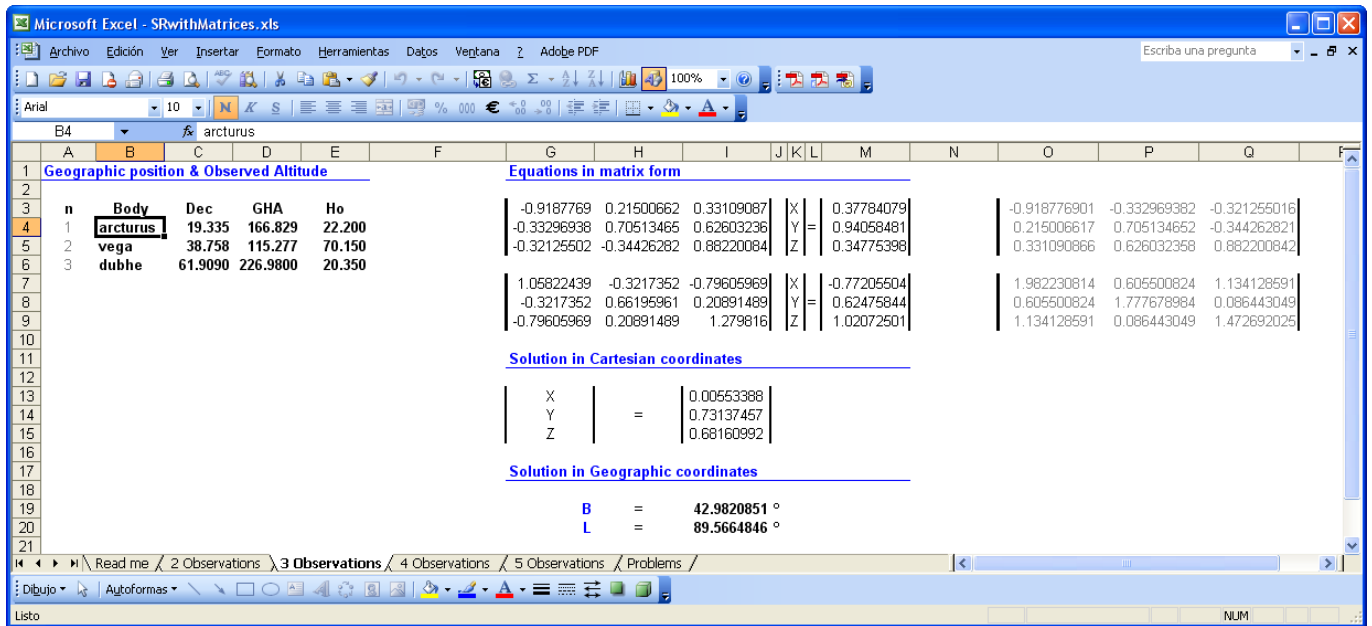
Solution in Geographic coordinates

B = **43.9781085** °
L = **2.83240931** °

A3. Software

An Excel sheet is available for up to 5 observations

A4. Source code

The algorithms are implemented in the Excel sheet

A5. References

- Watkins. R. and Janiczek. P. M., Sight Reduction with Matrices, NAVIGATION, Journal of The Institute of Navigation, Vol. 25, No. 4, Winter 1978-79, pp. 447-48

- Vector equation of the circle of equal altitude. Andrés Ruiz, Navigational Algorithms.