Functions

every time everywhere
函数将无处不在、无时不在

回忆所学的函数

- 哪位同学能描述一下你所知道的函数是什么？
- 高中接触的函数基本上是初等函数。微积分里常用的也是初等函数，或者是初等函数的混合或复合而成的。
- **初等函数**是由幂函数（power function）、指数函数（exponential function）、对数函数（logarithmic function）、三角函数（trigonometric function）、反三角函数（inverse trigonometric function）与常数经过有限次的有理运算（加、减、乘、除、有理数次乘方、有理数次开方）及有限次函数复合所产生，并且能用一个解析式表示的函数。
- **非初等函数**是指不是初等函数的函数。非初等函数的研究与发展是近现代数学的重大成就之一，极大拓展了数学在各个领域的应用，在各个学科中都有十分广泛的应用。是函数论的一个重要的分支。

Formal Definition(函数正式定义)

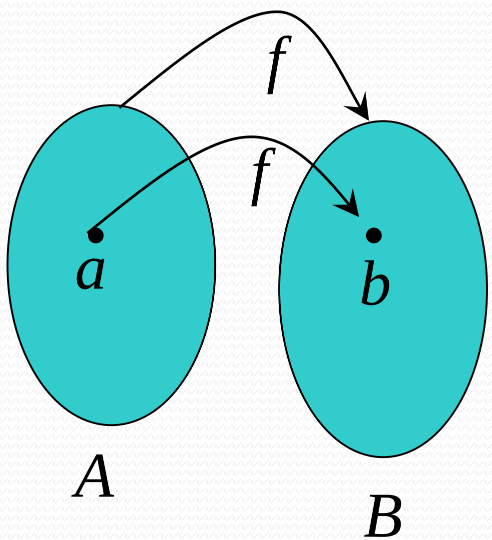
- 函数：For any sets A, B, *function f from A to B* (f: $A \rightarrow B$) is a particular assignment of exactly one element $f(x) \in B$ to each element $x \in A$.
- Also called: mapping or transformation （映射或者变换）
- 问题：线性代数中的线性变换是函数吗？
- 说明：当我们强调是哪个点对于那个点的时候，通常用映射这个概念；当更多的是分析研究函数值的变化的规律性时，更多的是用函数的概念。
- Some further generalizations of this idea:
  - Functions of n arguments （多元函数）: where A is the Cartesian product of some sets.

注：函数不再局限于普通的实数到实数函数；将概念、思维推广应用到更广泛的空间和领域中…

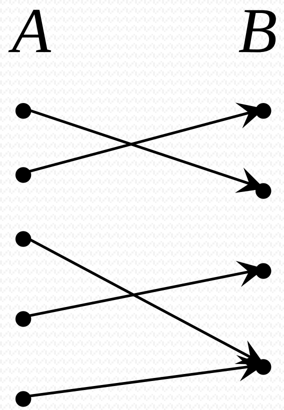
例如：计算机语言中的函数的输入输出就有可能不是一般的数。

Graphical Representations

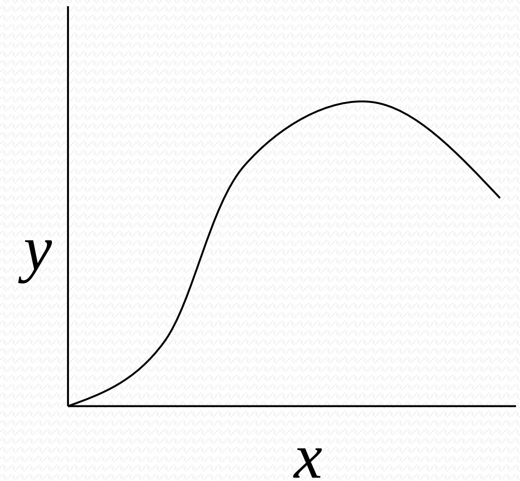
- Functions can be represented graphically in several ways:



Like Venn diagrams

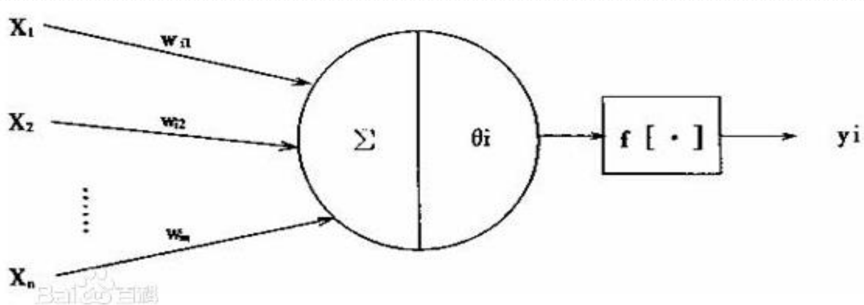


Bipartite Graph



Plot

神经网络—神经元函数及输入输出图模型



$$\begin{cases} Ui = \sum_{j=1}^{n} w_{ij} x_j - \theta_i \\ Yi = f(Ui) \end{cases}$$

神经元处理输入输出函数模型

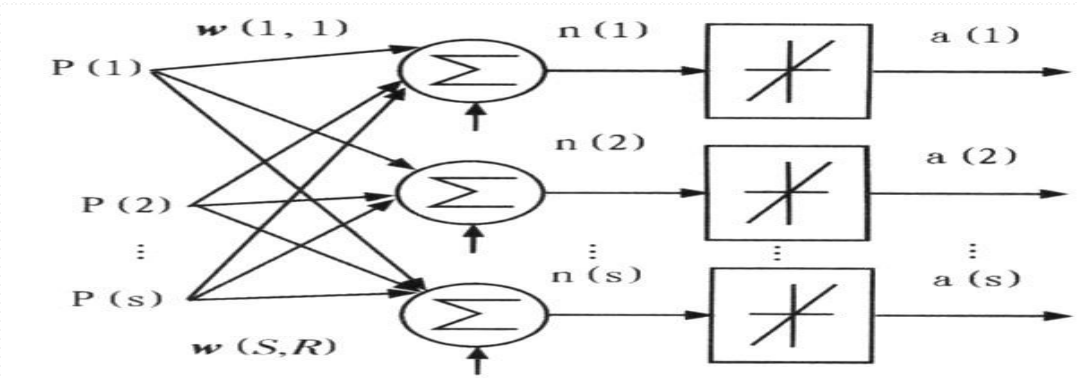
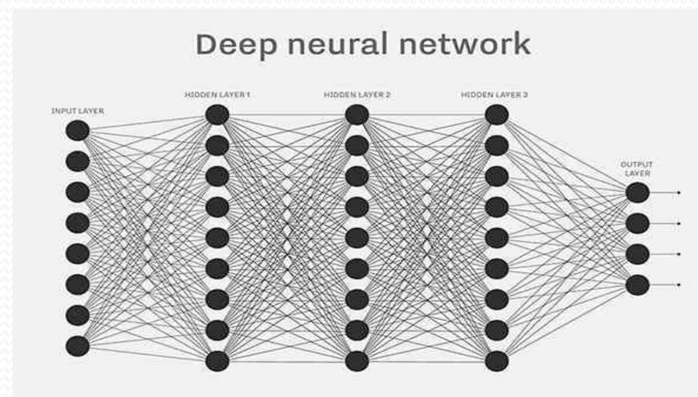
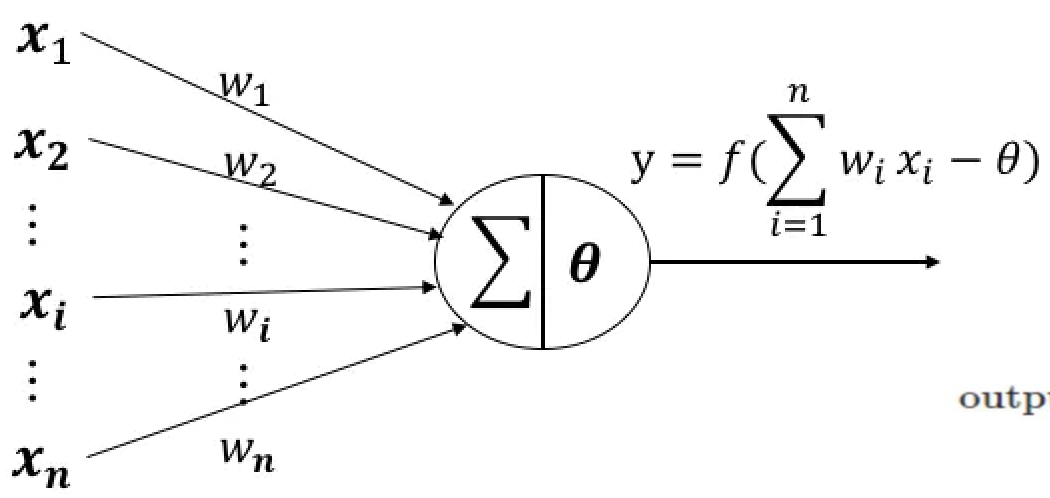


图 1 线性神经网络结构



高级模型（比如深度学习）输入输出

神经网络函数模型

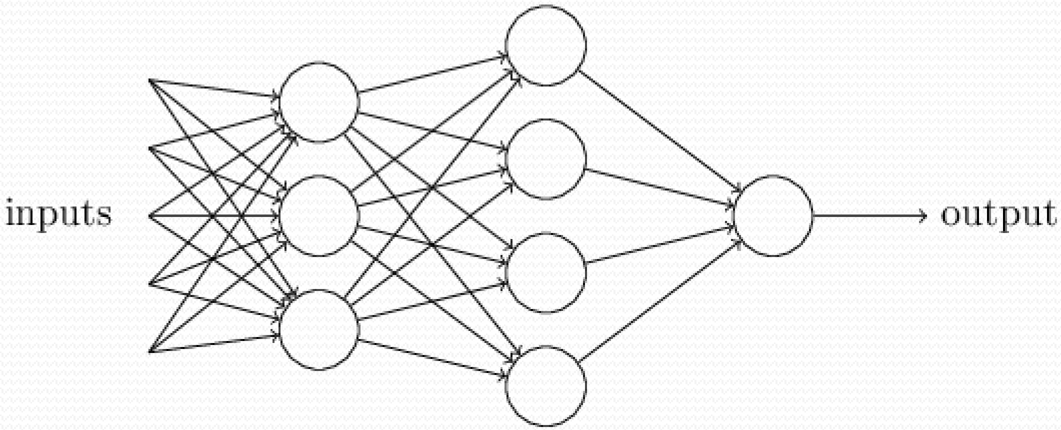


$$\text{output} = \begin{cases} 0 & \text{if } \sum_j w_j x_j \leq \text{ threshold} \\ 1 & \text{if } \sum_j w_j x_j > \text{ threshold} \end{cases}$$

神经网络—神经元模型（权w-阈值Θ)—函数输入输出模型

一个神经元有n个输入，每一个输入对应一个权值w，神经元内会对输入与权重做乘法后求和，求和的结果与偏置做差，最终将结果放入激活函数中，由激活函数给出最后的输出，输出往往是二进制的，0 状态代表抑制，1 状态代表激活。

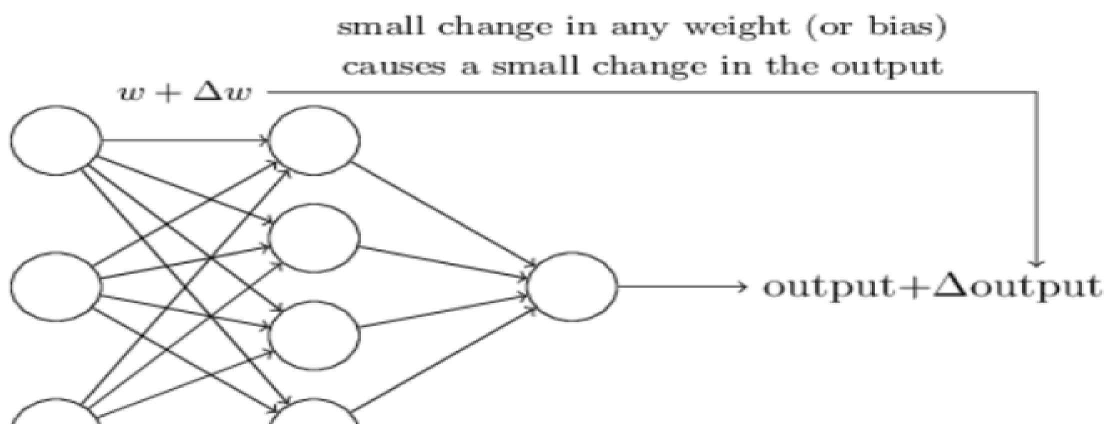
神经元处理输入输出函数模型



神经网络决策模型

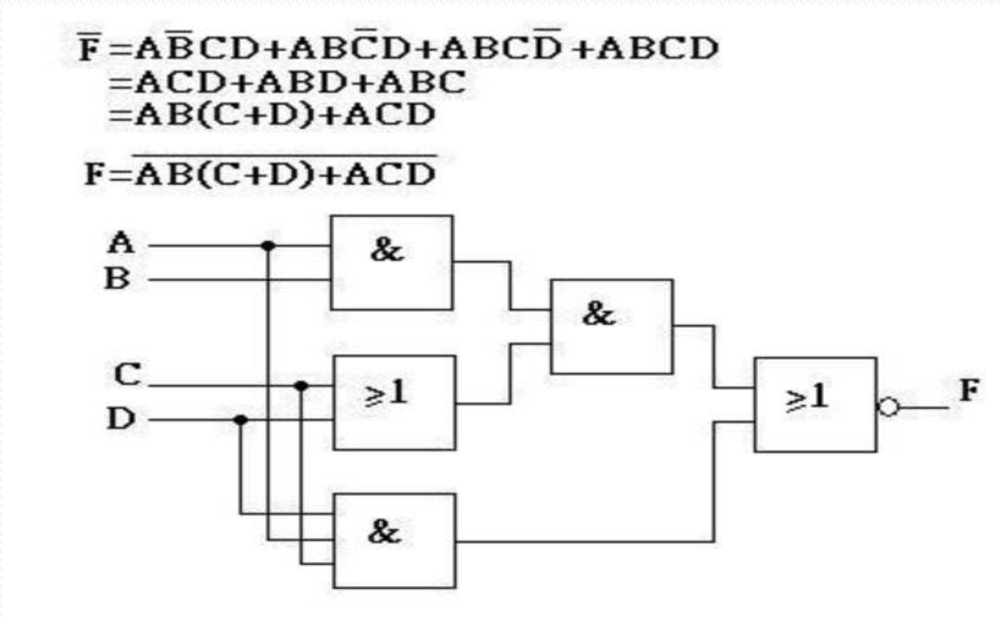
$$
\begin{cases}
U_i = \sum_{j=1}^{n} w_{ij} x_j - \theta_i \\
Y_i = f(U_i)
\end{cases}
$$

神经元处理输入输出函数模型



神经网络微调训练模型

- 逻辑电路：一组输入，得到输出（真值）输出，可以看成一个闭盒，也实际上可以理解为一个函数的实现，一个真值函数的实现

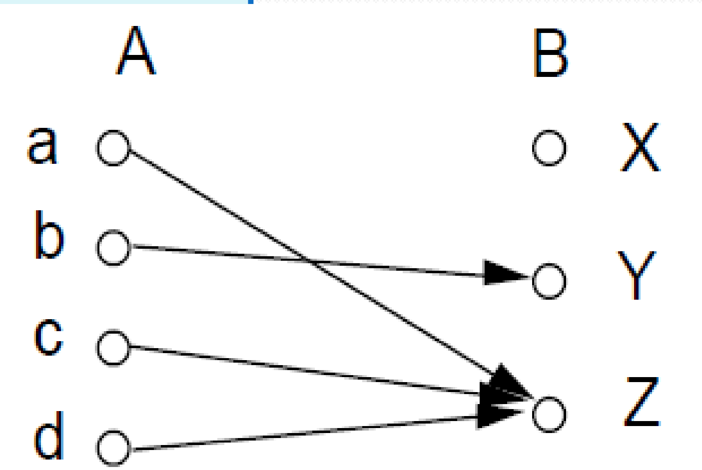


- 再想象一下，校门口的人脸识别系统（车牌识别系统），是不是也是一个函数？
- 输入是一图片，输出是对应的某个人的信息？

Functions （函数）

- If S is a subset of A then f(S) = {f(x) | x in S}.
- The *range* (值域) of f is the set of all images of points in A under f. We denote it by f(A).

Example:
• f(a) = Z
• the image （像） of a,d is Z
• the domain （定义域） of f is A = {a, b, c, d}
• the codomain 伴域 is B = {**X, Y, Z**}
• f(A) = {Y, Z}
• the preimage（原像） of Y is b
• the preimages of Z are {a, c, d}
• f({c,d}) = {**Z**}



Some Function Terminology有关术语

- $f:A \to B$, and $f(a)=b$ (where $a \in A$ & $b \in B$), then:
  - A is the *domain* of f.   定义域
  - B is the *codomain* of f. 伴域
  - b is the *image* of a under f. 像
  - a is a *pre-image* of b under f. *原像*

    In general, b may have more than 1 pre-image. 原像不一定唯一
  - The *range* $R \subseteq B$ of f is $R=\{b \mid \exists a\ f(a)=b\}$. 值域
  - 回忆一下：高中数学里求函数定义域、值域的问题

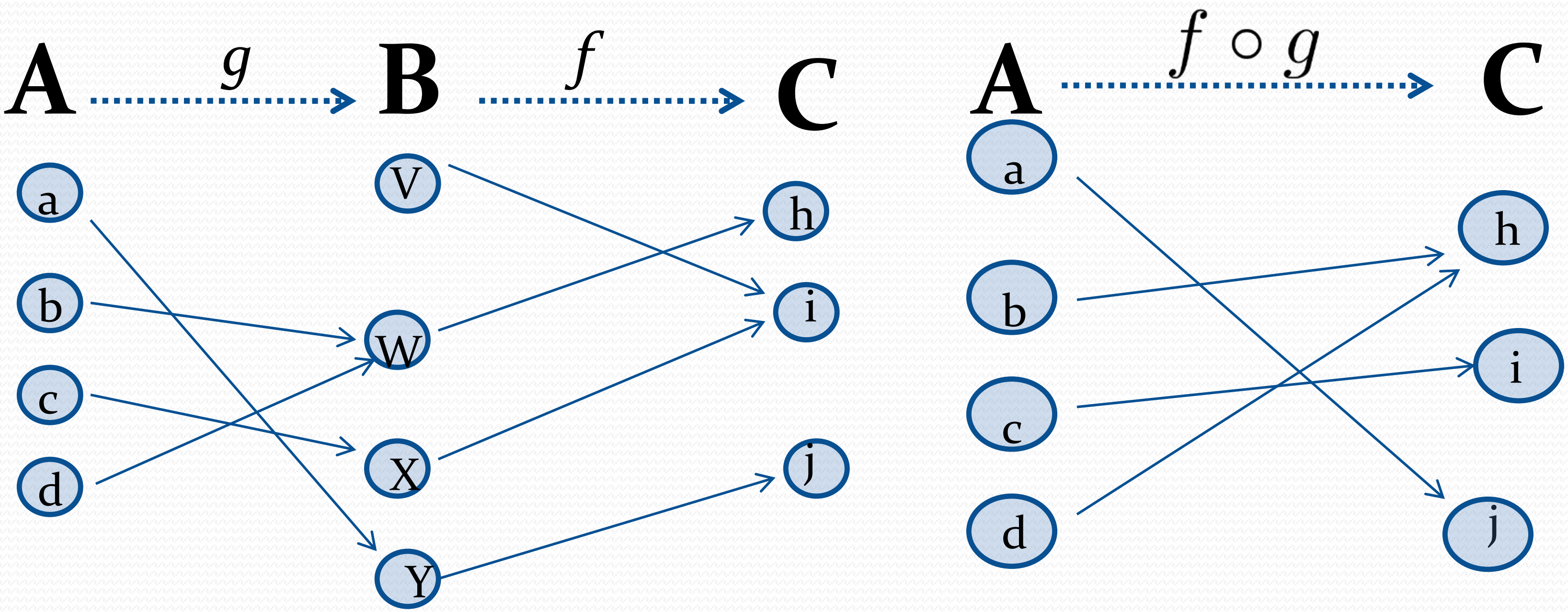
Range versus Codomain 值域与伴域

- The range of a function might *not* be its whole codomain （伴域）.
- The co-domain is the set that the function is *declared* to map all domain values into.
- The range值域 is the *particular* set of values in the codomain that the function *actually* maps elements of the domain onto.

Range vs. Codomain - Example

- Suppose I declare to you that: "f is a function mapping students in this class to the set of grades {A,B,C,D,E}."
- At this point, you know f's codomain is: {A,B,C,D,E}, and its range is unknown!.
- Suppose the grades turn out all As and Bs, then the range of f is {A,B}, but its codomain is still {A,B,C,D,E}!.

Function Composition Operator
函数复合运算



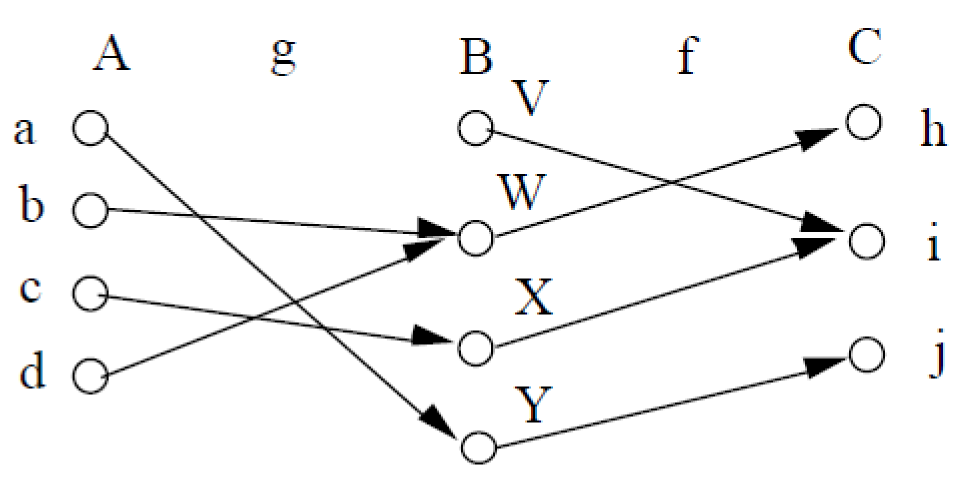
不建议用教材中的**“组合函数、函数组合、函数合成”**这些个名词

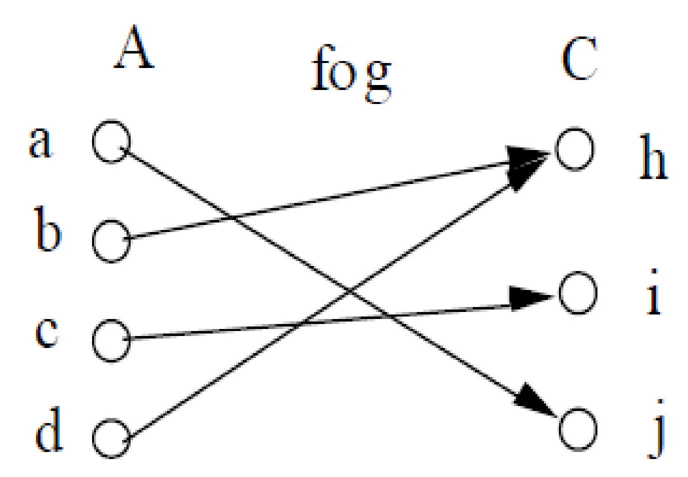
Function Composition Operator 函数复合运算

- For functions g:$A \rightarrow B$ and f:$B \rightarrow C$, there is a special operator called *compose* (“o”).
  - It composes (creates) a new function out of f and g by applying f to the result of applying g.
  - We say $(f \circ g)$:$A \rightarrow C$, where $(f \circ g)(a) :\equiv f(g(a))$.
  - Note $g(a) \in B$, so $f(g(a))$ is defined and $\in C$.
  - Note that “o” is non-commuting. (Generally, $f \circ g \neq g \circ f$. 不可交换的)

Example of Composition Operator
复合运算

Example:





Properties of function
函数性质

注意：这个部分是函数部分的重点

One-to-One Functions

- A function is *one-to-one (1-1)*, or *injective*, or *an injection*, iff **for all x,y∈A, *x≠y implies f(x)* ≠ f(y)**.

- In other words: every element of its range has *only* one pre-image. (一对一函数或者**单射**)
  - Formally: given $f{:}A\rightarrow B$,
    "x is injective" :≡ ($\neg\exists x,y$: $x\neq y$ and $f(x)=f(y)$).

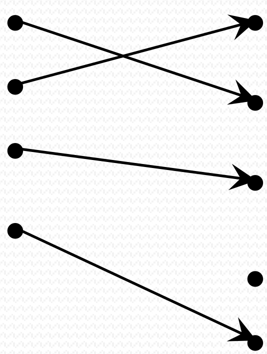
  也即：定义域里不存在两个不同元素对应到同一函数值，或者说不同元素有不同的像

- 值域中的任何元素都有且只有唯一的源像.

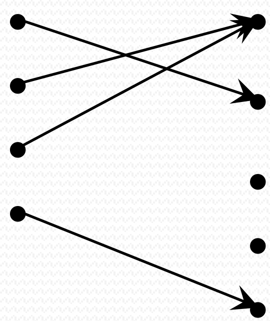
- Domain & range have same cardinality. What about codomain? 定义域与值域有相同的基数

One-to-One Examples

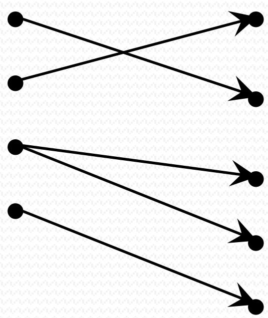
- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one



Not one-to-one



Not even a function!

Sufficient Conditions for 1-1ness
1-1ness的充分条件

- For functions f over numbers, we say:
  - f is *strictly* (or *monotonically*) *increasing* iff $x>y \rightarrow f(x)>f(y)$ for all x,y in domain;
  - f is *strictly* (or *monotonically*) *decreasing* iff $x>y \rightarrow f(x)<f(y)$ for all x,y in domain;
- If f is either strictly increasing or strictly decreasing, then f is one-to-one. *E.g.* x^3
  - *Converse is not necessarily true. Why?*

More Examples

- 1. A is the set of all Chinese citizens. f is a function from a person to her/his ID.
- 2. Any given encryption function from plain text set to cipher text set.
- 3. Records set in a table of database, f is a function from every record to its primary key.

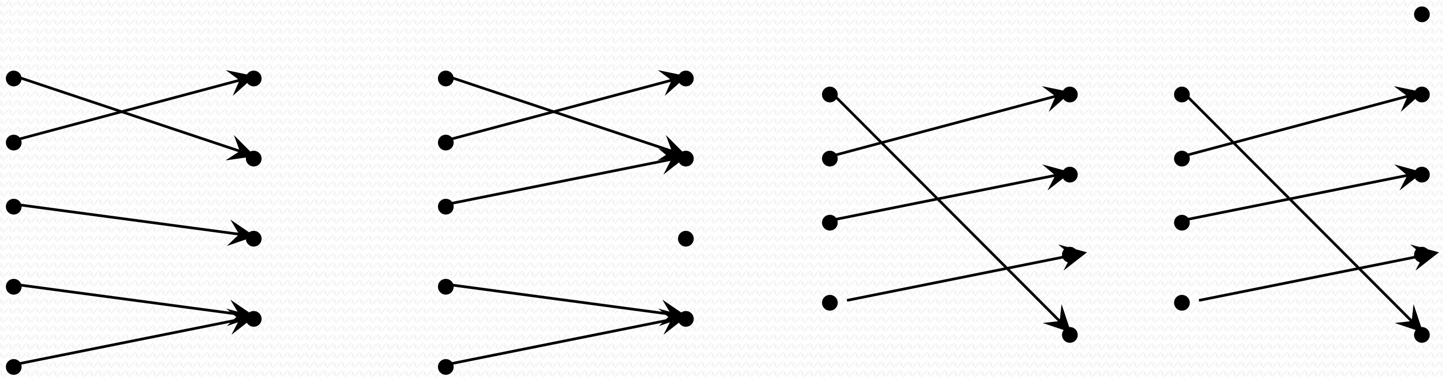
Onto (Surjective 满射) Functions

- A function f:$A \rightarrow B$ is ***onto* or *surjective*** or *a surjection* iff its range is equal to its codomain ($\forall b \in B, \exists a \in A: f(a)=b$).
- f(A) = B? Why?
- 思考: An *onto* function maps the set *A* onto (over, covering) the *entirety* of the set *B*, not just over a piece of it.（体会这里的理解）
- *E.g.*, for domain & codomain **R**, x^3 is onto, whereas x^2 isn't. (Why not?)

- *注：教材翻译成"映上的"*

Examples of Onto 满射

- Some functions that are, or are not, *onto* their codomains:



Onto (but not 1-1)

Not Onto (or 1-1)

Both 1-1 and onto

1-1 but not onto

More Examples

- 1. A: all people in China, B = {0,1,2,3,…, , 100, …, 150}, f is a function from A to B, f(x) is x's age.
- 2. Assume $g(x) = f^{-1}(x)$ for any x in f(A) above, is g a function from f(A) to P(A) (power set of A)? 1-1? Onto?

Showing that *f* is one-to-one or onto

单射和满射的证明

- Suppose that $f: A \to B$.
- To show that f is injective, we have to prove that if $f(x)=f(y)$ for arbitrary $x, y \in A$, then $x = y$ for sure.
- To show that f is not injective, just find paticular elements $x, y \in A$, such that $x \neq y$ but $f(x) = f(y)$
- To show that f is surjective, consider an arbitrary element $y \in B$, and find an element s in A such that $f(x) = y$.
- To show that f is not surjective, to find a particular element $y \in B$, there is no pre-image in A, or for all x in A, $f(x) \neq y$

Showing that f is one-to-one or onto

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

Example 2: Is the function $f(x) = x^2$ from the set of integers to Z onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

Bijections (双射)

- A function *f* is said to be *a one-to-one correspondence*, or *a* ***bijection***, or ***reversible***, or ***invertible***(*可逆的*), iff it is both one-to-one and onto.

  (既是单射又是满射)
- For bijection $f:A \to B$, there exists an *inverse*（*逆*）*of* **f**, written $f^{-1}:B \to A$, which is the unique function such that
  - (where I_A is the identity function on A 恒等函数)

$$f^{-1} \circ f = I_A$$

  - 线性代数里的线性变换是双射吗？

Examples and Questions

- More examples please…
- Encrypt functions and Decrypt functions
- What about the compressing functions?

- Some questions:
- 1. If f, g are 1-1 or (onto) functions, how about the composition of f and g: f o g? 反之如何？
- 2. If f o g is 1-1 （onto）function, how about f and g? 反之如何？

复合函数的性质

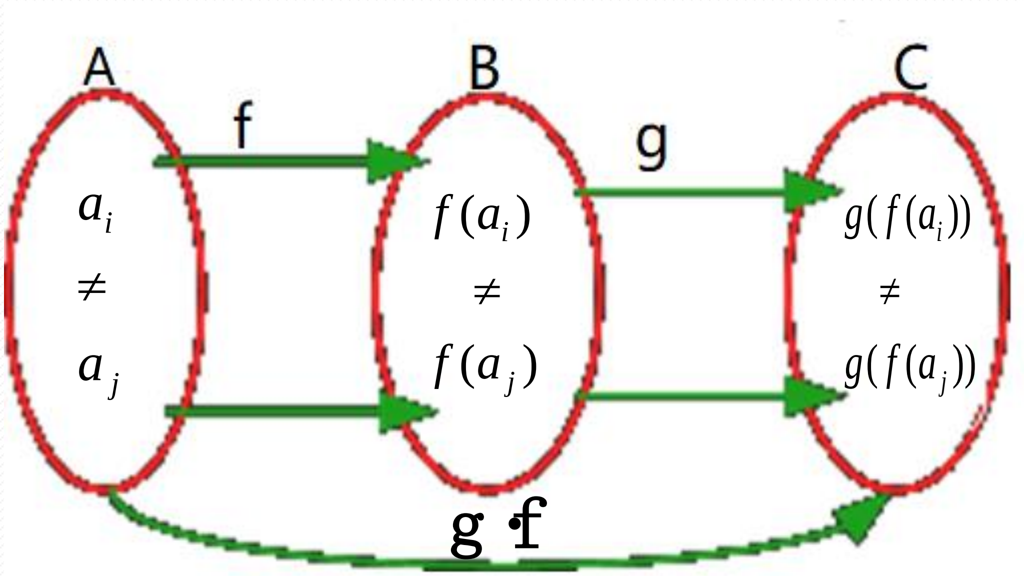
定理　设有函数f：A→B　g：B→C

（1）如果f和g都是单射，则g•f也是单射；

（2）如果f和g都是满射，则g•f也是满射；

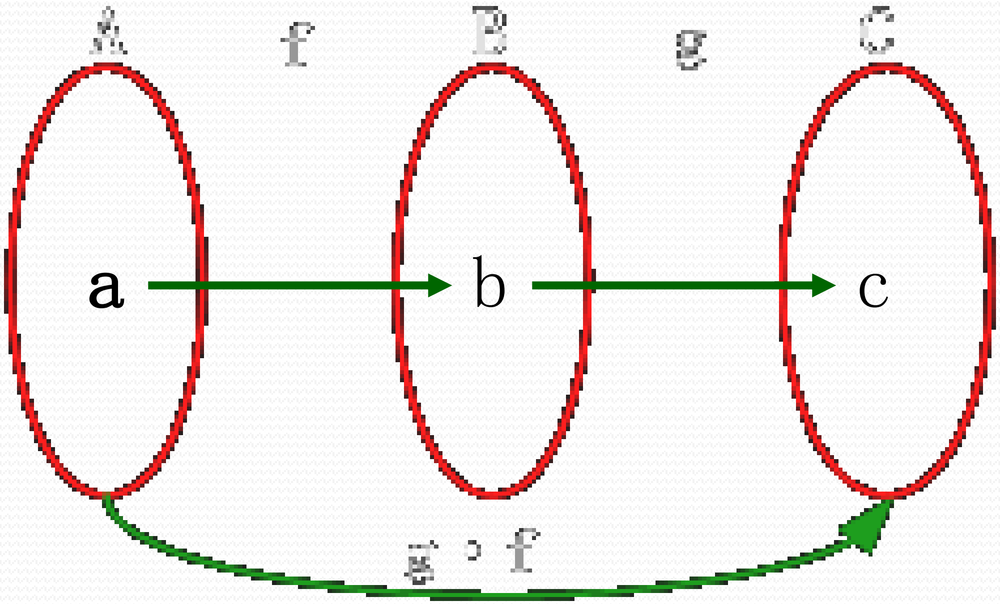
（3）如果f和g都是双射，由g•f也是双射。

证明：（1）



此即 $g \bullet f(a_i) \neq g \bullet f(a_j)$ ，故g•f是单射

(2) 对于集合C中任一元素c，必存在$b\in B$，使得$g(b)=c$。



对于b，又必存在$a\in A$，使得$f(a)=b$，

于是有$g\cdot f(a)=g(f(a))=g(b)=c$，

由c的任意性得$g\cdot f$是满射。

(3) 由(1)和(2)知$g\cdot f$必是双射。

定理 设有函数f：A→B 和g：B→C

（1）如果$g\bullet f$是单射，则f是单射；

（2）如果$g\bullet f$是满射，则g是满射；

（3）如果$g\bullet f$是双射，则f是单射而g是满射。

证明 （1）反证法 假设f不是单射，

则存在元素a_i，$a_j \in A$，$a_i \neq a_j$，但$f(a_i)=f(a_j)$，

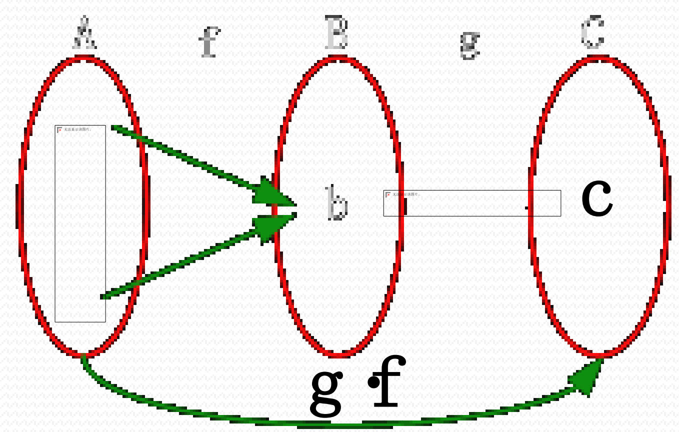
令$f(a_i)=f(a_j)=b$，且令$g(b)=c$，

则$g\bullet f(a_i)=g(f(a_i))=g(b)=c$

$g\bullet f(a_j)=g(f(a_j)=g(b)=c$

$g\bullet f(a_i)=g\bullet f(a_j)$

这与$g\bullet f$ 是单射相矛盾。



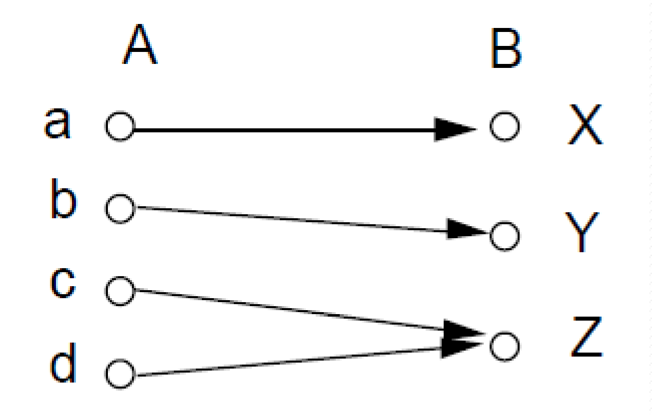
Questions

- 思考：如果f不是单射，　那么g•f如何？
    - 如果g不是满射，那么g•f如何？

Inversion逆 （原像）

- **Definition:** Let S be a subset of B. Then
- $f^{-1}(S) = \{x \mid f(x) \in S\}$
- Note: f doesn't have to be a bi-jection for this definition to hold.

Example:
Let f be the following function:
$f^{-1}(\{Z\}) = \{c, d\}$
$f^{-1}(\{X, Y\}) = \{a, b\}$



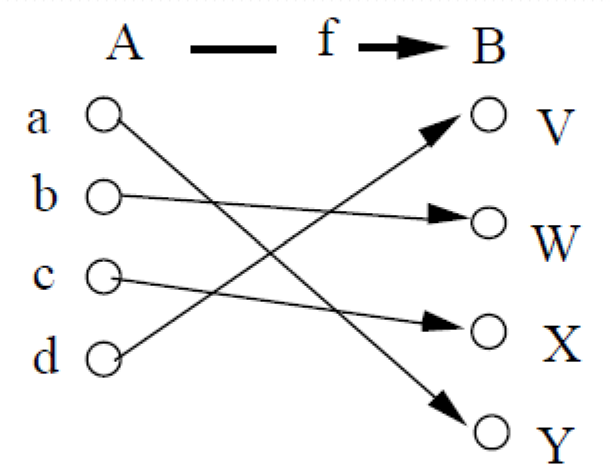
Inverse Function逆函数(反函数）

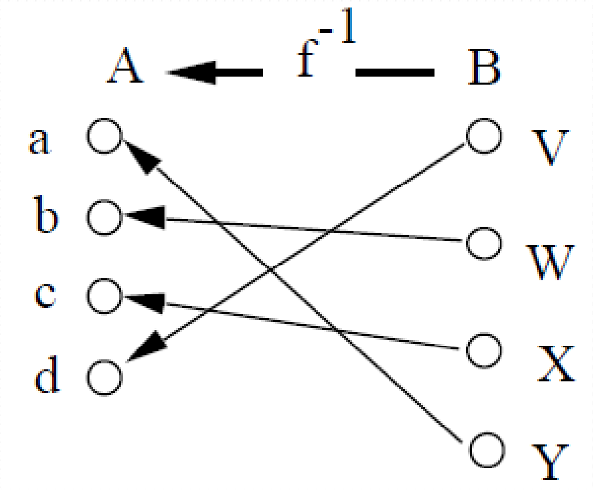
- **Definition:** Let **f** be a bijection from *A* to *B*. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as

$$f^{-1}(y) = x \text{ iff } f(x) = y$$

Note: no inverse function unless f is bijective!

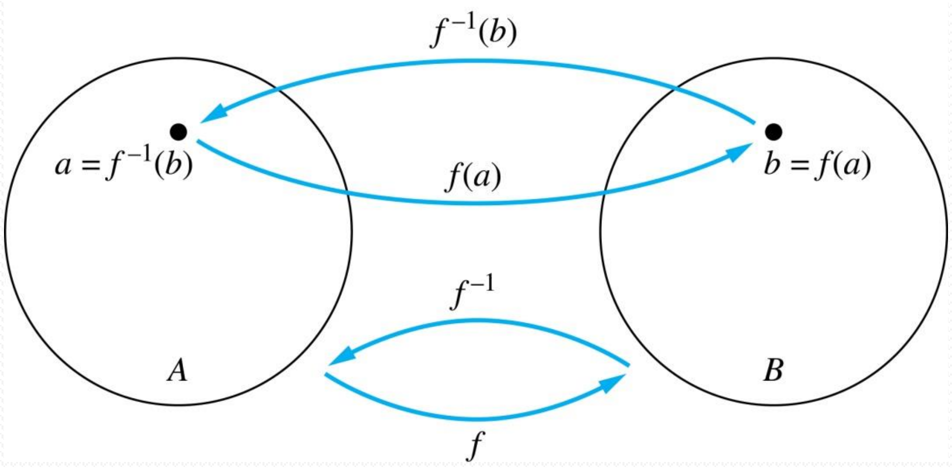
Example:





思考：为什么定义逆函数时，需要有一个前提“双射”？

Inverse Functions逆函数



回忆反三角函数

- Think about why the inverse arcsin(x) of sin(x) defines on [-1, 1] to [-π/2, π/2]?   Why?

逆函数的意义

- 可逆的意义，逆函数的应用…
- 当我们用一个双射（或者一个变换），将一个方面的问题转到另一方面解决后，再通过逆函数（逆过程），回到原来的问题，得到所要的答案或解。
- 例如通讯领域的时域转频域等；线性代数中的线性变换求标准型，通过标准型求解一些问题，再用逆变换，得到原来的解，等等…

A Couple of Key Functions
几个常用的函数介绍（这个部分自己看）

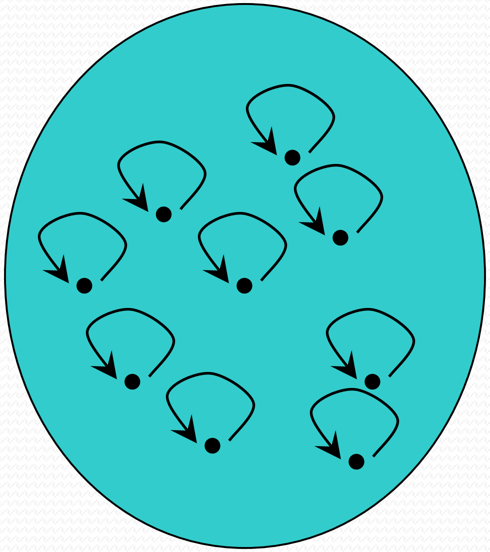
The Identity Function (恒等函数)：

- For any domain A (任意定义域）, the *identity function* $I:A\rightarrow A$ (variously written, I_A, $\mathbf{1}_A$) is the unique function such that $\forall a\in A: I(a)=a$.

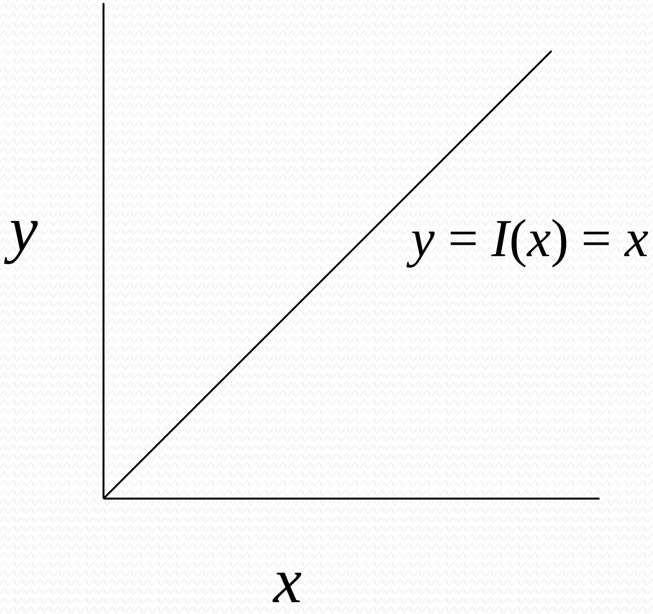
- Note: the identity function is always both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:



Domain and range

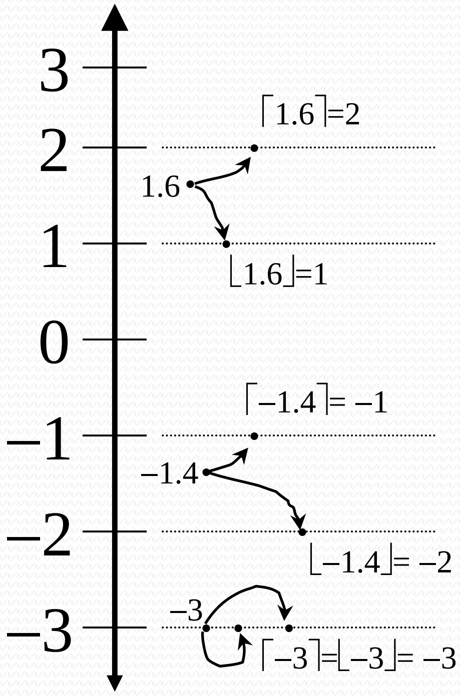


A Couple of Key Functions
几个常用的函数

- In discrete math, we will frequently use the following two functions over real numbers:
  - The ***floor* function** $\lfloor\cdot\rfloor$:**R**→**Z**, where $\lfloor x \rfloor$ ("floor of x") means the largest (most positive) integer $\leq x$. *I.e.*, $\lfloor x \rfloor :\equiv \max(\{i \in \mathbf{Z} \mid i \leq x\})$. 底函数
  - The ***ceiling* function** $\lceil\cdot\rceil$:**R**→**Z**, where $\lceil x \rceil$ ("ceiling of x") means the smallest (most negative) integer $\geq x$. $\lceil x \rceil :\equiv \min(\{i \in \mathbf{Z} \mid i \geq x\})$ 顶函数

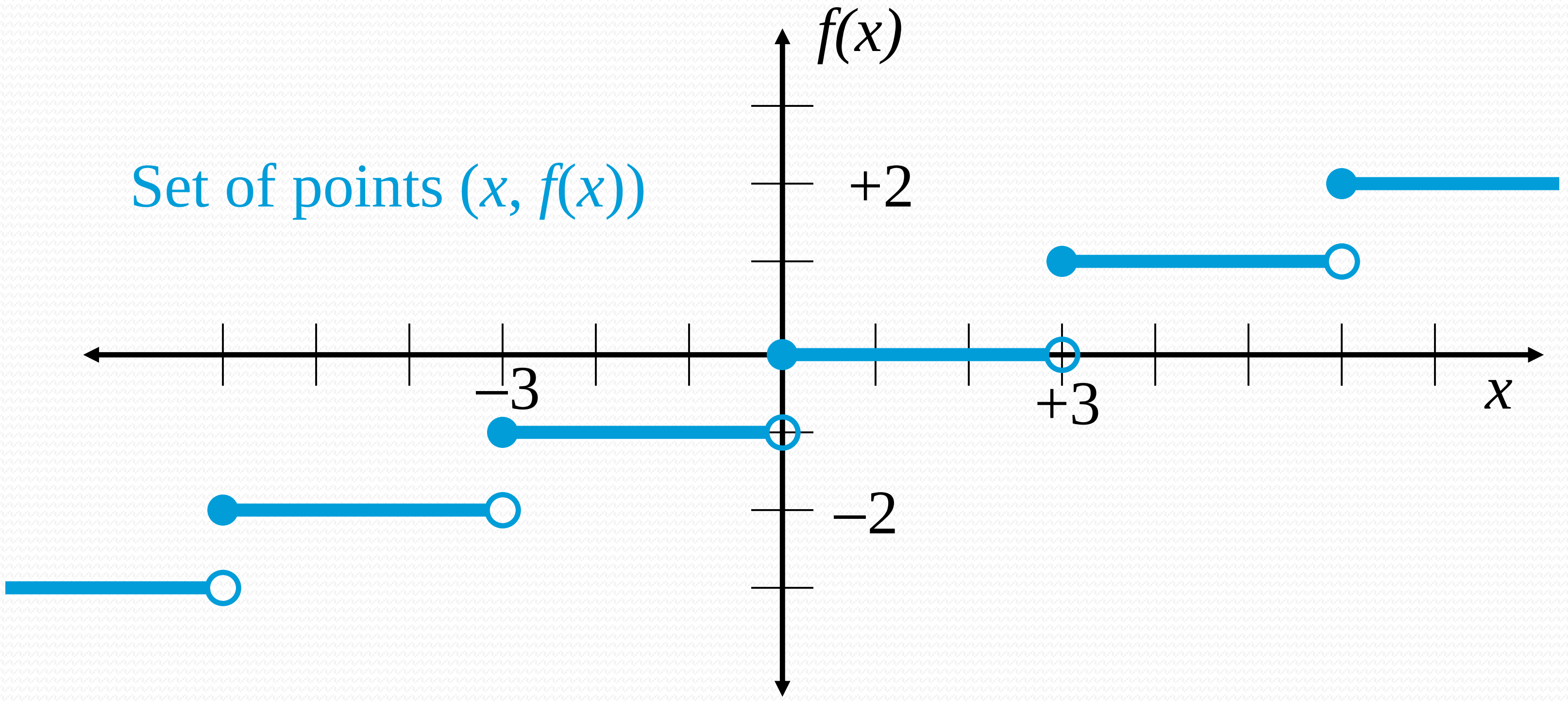
Visualizing Floor & Ceiling
顶函数和底函数的可视化

- Real numbers "fall to their floor" or "rise to their ceiling."
- Note that if $x \notin \mathbf{Z}$, $\lfloor -x \rfloor \neq -\lfloor x \rfloor$ & $\lceil -x \rceil \neq -\lceil x \rceil$
- Note that if $x \in \mathbf{Z}$, $\lfloor x \rfloor = \lceil x \rceil = x$.



Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:



Partial Function 部分函数

- **Definition**： a *partial function* f from a set A to a set B is an assignment to each element a in a subset of A, called the *domain of definition* of f, of a unique element b in B.
- Remark: the sets A and B are called the *domain* and *codomain* of f, respectively. We say that f is *undefined* for elements in A that are not in the domain of definition of f.
- **全函数**（普通意义上的函数）： When the domain of definition of f equals A, we say that f is a ***total function***.
- *Example: $F(x) = 1/x$ on real number set R.*

课外思考题 （曾经的考题）

- 1. 已知A={0，1，2，3，4，..., 100}, B是参加离散数学考试的90个同学. f是A到P(B) (B的幂集)的函数，其中∀k∈A, f(k) = {x|x的离散数学成绩为k，x∈B}. 问：f是否是单射？为什么

- 2. 假设f是A→B的函数，A,B都是非空集合。

∀ b∈B, f^{-1}(b) = {x|f(x)=b, x∈A }说明f^{-1}是B到P(A)的函数

f^{-1}是否是单射，满射和双射？ 为什么？

如果f是满射，所有集合{f^{-1} (b)| b∈B}是否构成A的一个分划？

作业

- 2.3节

10. Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one.

a) $f(a) = b$, $f(b) = a$, $f(c) = c$, $f(d) = d$
b) $f(a) = b$, $f(b) = b$, $f(c) = d$, $f(d) = c$
c) $f(a) = d$, $f(b) = b$, $f(c) = c$, $f(d) = d$

11. Which functions in Exercise 10 are onto?

12. Determine whether each of these functions from $\mathbf{Z}$ to $\mathbf{Z}$ is one-to-one.

a) $f(n) = n - 1$
b) $f(n) = n^2 + 1$
c) $f(n) = n^3$
d) $f(n) = \lceil n/2 \rceil$

13. Which functions in Exercise 12 are onto?

- T27, T35, T37, T45（a)