Relations

- 关系引入…

- 利用数学的思想、方法对实际生活与应用中各式各样的关系进行分析研究，探讨如何建立数学模型、概念或者结构来表述关系；
- 研究“关系”的规律性的东西，寻找与关系相关的问题的解决工具和方法，最终在实际中加以应用

Focus on Relations

- 1. What is the definition of relation in mathematics? 数学上如何定义关系？
- 2. How to represent real relations using mathematical structure/concept? 如何用数学结构和概念表示关系？
- 3. What kind of properties of relation are there? 关系有哪些个有用的特征？
- 4. How to solve the problems in real applications using relation (or regarding relation)? 如何应用关系理论解决实际问题

Content of Relation 关系主要内容

- Content基本内容:
  - Basic Definitions 基本概念
  - Properties of Relations 关系性质特征
  - Representations of Relations 关系的表示
  - Closure of Relations 关系闭包
  - Compositing Relations关系复合运算
  - Equivalence Relations 等价关系
  - Partial Orders 序关系与格

- 先探讨数学上如何定义关系，如何表示关系

Definition of Relation 关系的定义

- 定义1： (二元关系) 假设A和B是两个集合，A与B的笛卡尔积 的一个子集合，叫做一个从A到B的二元关系。
- 定义2： (多元关系) 假设 有n个非空集合，它们的笛卡尔积 的一个子集合，叫做这n个集合间的一个n元关系
- 如何理解这数学里的为什么要这样定义“关系”？与实际中的“关系”怎么联系起来？

Introduction Examples to Relation

例1： 四支球队a、b、c及d队，他们之间进行了一些比赛，以下一张表格记录了他们之间的比赛结果--胜负关系： a胜b、b胜c、c胜a、d胜a、d胜b、d又胜了c。

为了简单起见，用(a，b)来表示a胜b，于是可以将所有胜负记录表示成{(a， b)，(b， c)，(c， a)，(d， a)，(d， b)，(d， c) }

这就是一张胜负关系表，该表清楚、准确地表现了这四个队 a、b、c、d之间的胜负关系，它就是这四个队之间的一个关系--比赛胜负关系。当我们用集合S表示四个队时，S={a，b，c，d}，那么胜负关系表{(a， b)，(b， c)，(c， a)，(d， a)，(d， b)，(d， c)，(d， b)}就是S与S的笛卡尔积 的一个子集。也就是说用这个子集合表示了这四个队之间的某轮比赛的胜负关系。

Introduction Examples to Relation

- 例2：一个电话号码簿，它里面记录了很多单位或个人的一些电话号码。不难理解，一个号码本就是一个集合。这个号码本，这个集合表示了人、单位跟一些电话号码之间的一种关系，它是一个实实在在的关系
- 如果我们用A表示所有有关的单位和人的集合，用B表示所有相关的电话号码的集合，我们简单地用(a，b)表示a的电话号码是b，其中a,b分别表示A中的一个元素(单位或者人)和B中的一个号码 。那么所有这些有关的序对(a，b)就构成电话号码本，就构成这个号码集合。
- 可以看出这个集合正好是A×B的的一个子集。当有人或有单位的号码发生了变化，这个号码本也相应地发生变化，变成了另外一个号码本，也就是另外一个集合，另外一个子集合，但仍然是A×B的一个子集，另外一个关系。

Introduction Examples to Relation

- 例3： (学生、课程、成绩之间的关系) 假设用集合A表示某大学计算机学院的所有学生，B集合表示计算机学院的所有课程，C集合表示不大于100的非负整数的集合。 那么学生张三的离散数学考试成绩是95分，就可以表示成(张同学，离散数学，95)。
- 将计算机学院所有学生所有课程的这样的记录放在一起，就是一张成绩表，也就是教务管理中的成绩库。 那么这个成绩库就是一个集合，这个集合表示的是计算机学院学生，课程和成绩三者之间的一个关系。而这个集合恰好是集合A、B、C的笛卡尔积 的一个子集。
- 以上三个例子都说明了同一个问题，无论是一个集合内部元素之间的关系，还是不同集合的元素之间的关系，还是多个集合元素之间的关系，都可以表示成相关集合的笛卡尔积的子集。把笛卡尔积的子集当成一个数学模型，那就可以用这个数学模型来表示关系，包括二元关系和多元关系

抽象关系的具体解释

- 例4: 设集合A={a，b，c，d}，S={(a，b)，(c，d)}，根据定义1，S显然是A集合到A集合自身的一个二元关系。这个关系看似是抽象的，但当给a、b、c、d赋予具体的含义，
- 如: abcd分别表示成张三、李四、王五和老六四个人，而(x，y)表示为x与y是朋友，那么二元关系S就表示成四个人之间具有的一个朋友关系。 其中，张三跟李四是朋友，王五跟老六也是朋友，但其他人之间都不是朋友。即便是空集，即空关系，在这里可以理解为集合A的人之间没有人有朋友关系。
- 如果: 根据不同的情况，也可以给出另外的含义和解释。 比如说a=5、b=10、c=3、d=9，那么上面的关系S可以解释为集合A={5，10，3，9}中元素间的整除关系。
- 这个例子说明，一些集合的笛卡尔积的任何一个子集，也即任一个抽象的关系，当给出一些具体的解释后，对应为实际的关系。

Conclusion结论

- Question: 如何用数学的工具或者结构表示实际中的关系?

- We can use the **subset of the Cartesian product of sets** as a mathematical structure to represent real relations in real applications.
- 用笛卡尔集合的子集来表示关系

Relations

Definition. A binary relation R （A到B的二元关系） from set A to set B is a subset of A×B. That is R $\subseteq$ A×B. A binary relation on set A （A上的二元关系） is a subset of A×A.

If *(a,b) $\in$ R we say a is related to b in R, sometime it is noted as aRb.*

用 subset of Cartesian product of sets （笛卡尔集的子集）, which is a kind of mathematical structure, to represent a real relation.

More Examples

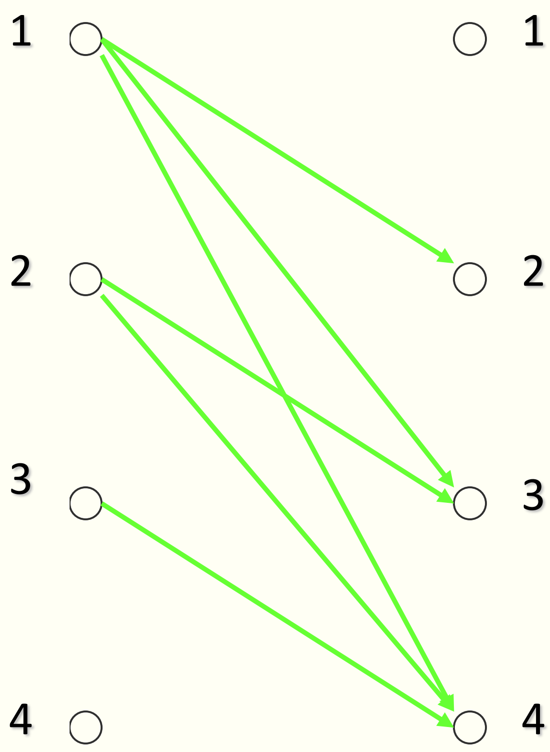
Example: $R = \{(a, b) \mid a, b \in Z^+; a - b \geq 10\}$
$= \{(11, 1), (12, 1), (12, 2), (13, 1), (13, 2), (13, 3), \ldots\}$

Example: $R = \{(a, b) \mid a, b \in Z^+;$ a, b are relatively prime$\}$
$= \{(27, 2), (12, 5), (42, 5), (42, 19), \ldots\}$

Relations on a Set

Example: Let A = {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a < b} ?

Solution: R = { (1,2), (1,3), (1,4), (2,3), (2,4), (3,4)}



R	1	2	3	4
1		X	X	X
2			X	X
3				X
4				

N-ary Relations 多元关系

定义2： (多元关系) Let $A_1, A_2, \ldots, A_n$ be sets. An **n-ary relation** on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$.

The sets $A_1, A_2, \ldots, A_n$ are called the **domains** of the relation, and n is called its **degree**(阶).

If $A = A_1 = A_2 = \ldots, = A_n$, 称之为A上的n元关系。

Example:

Let $R = \{(a, b, c) \mid a = 2b \wedge b = 2c$ with $a, b, c \in Z\}$

What is the degree of R?

The degree of R is 3, so its elements are triples.

Is (2, 4, 8) in R? No.

Is (4, 2, 1) in R? Yes.

Examples of n-ary Relation

Example. Relation as a table.

student	course	time	room
Eve	physic	1CD	122
Frank	physic	2GH	122
Garey	painting	1CD	122
Holms	compiler	1CD	105

- 这个例子也是关系型数据库里的表
- 这几个例子的关系，实际上也是相关集合的笛卡尔积的子集

Note

多元关系是关系型数据库的基础。关系型数据库中每一个表的记录集都一个多元关系

更多地关注二元关系 But we usually confine our study within binary relations unless some extra descriptions provided.

Please remember:

a relation is a set, a subset of the Cartesian product of sets! 一个关系，在数学上就是一个集合，一个笛卡尔集合的子集！

在关系型数据库里就是一个记录集。

Some Special Relations一些特殊关系

- Assume A is a non-empty set 非空集合
- Example 1: R is empty subset of AxA

  Empty Relation空关系

- Example 2: R= { (x,y)|x,y $\in A$ }
- Universal Relation普遍关系

- Example 3: R = {(a,a)|a $\in$ A}
- Identity Relation恒等关系

关系定义域与值域

Let $R \subseteq S \times T$ be a binary relation on S and T.

The **domain 定义域 of R** is the set $Dom(R) = \{s \in S \mid \exists t \in T \text{ where } (s\ R\ t)\}$.
The **image of R 值域** is the set $Im(R) = \{t \in T \mid \exists s \in S \text{ where } (s\ R\ t)\}$.

The **co-domain** (or the range) of R is the set $coDom(R) = T$.

请同学们对比函数的相关概念…

Functions as Relations函数也是关系

Function f:A→B is a relation between A and B, a special relation!

为什么？怎么理解？

$\{(a,f(a))|\ a \in A\}$?

Counting Relations关系计数

How many different relations can we define on a set A with n elements?

A relation on a set A is a subset of $A \times A$.

How many elements are in $A \times A$?

There are n^2 elements in $A \times A$, so how many subsets (relations on A) does $A \times A$ have?

The number of subsets that we can form out of a set with m elements is 2^m. Therefore, 2^{n^2} subsets can be formed out of $A \times A$.

Answer: We can define 2^{n^2} different relations on A.

关系性质

- 二元关系的一些特性：对客观存在的各种各样的关系的一些特性的总结和抽象

- 关注具有这些特性的特殊的二元关系

二元关系的一些特性--自反性

Definition. A relation R on the set A is **reflexive** if and only if (a, a) $\in$ R, for every element a $\in$ A. 自反性

Example. Let A be the set of all integers. Let R = {(a, b)|a≥b}. Then R is *reflexive or not?*

Example. Let A be the set of all positive integers. Let R = {(a, b)|a|b}. Then R is reflexive??.

Example. Let A = 2^N (or P(N)). Let R = {(a, b)| a $\subseteq$ b}. Then R is reflexive??.

Example. Let A be the set of all integers. Let R = {(a, b)|a=-b}. Then R is not reflexive.

More examples please… 请同学们自己举一些例子

Reflexive自反性

Are the following relations on {1, 2, 3, 4} reflexive?

R = {(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)}

No.

R = {(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)}

Yes

R = {(1, 1), (2, 2), (3, 3)}

No.

Special Properties -- irreflexive反自反

Definition: A relation on a set A is called **irreflexive** if $(a, a) \notin R$ for every element $a \in A$.(反自反)

(Equivalently, a relation is *irreflexive if there is no x such that x is* related to itself in the relation. 不存在与自身有关系的元素，这样的关系才是反自反的）

Example：

> and < are irreflexive.

人的集合上的父子关系如何？

Special Properties –symmetric对称性

Definitions:

A relation R on a set A is called **symmetric对称性** if $(b, a)\in R$ whenever $(a, b)\in R$ for all $a, b\in A$.

(只要有aRb就一定有bRa)

A relation R on a set A is called **antisymmetric** if $a = b$ whenever $(a, b)\in R$ and $(b, a)\in R$.反对称

(无论是什么元素a,b, 如果aRb与bRa同时成立，那就只能是a=b)

思考：同学、同乡、同龄、臭味相投、相似等关系

Properties –symmetric-examples

Are the following relations on {1, 2, 3, 4} symmetric, antisymmetric (对称，反对称?）

R = {(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)}	symmetric
R = {(1, 1)}	sym. and antisym.
R = {(1, 3), (3, 2), (2, 1)}	antisym
R = {(4, 4), (3, 3), (1, 4)}	antisym.

asymmetric relation 非对称

In mathematics, an asymmetric relation is a binary relation on a set X where for all a and b in X, if a is related to b, then b is not related to a (that is aRb => ¬ （bRa）).

An example of an asymmetric relation is the "less than" relation < between real numbers: if x < y, then necessarily y is not less than x. The "less than or equal" relation ≤, on the other hand, is not asymmetric, because reversing e.g. x ≤ x produces x ≤ x and both are true.
Asymmetry is not the same thing as "not symmetric": the less-than-or-equal relation is an example of a relation that is neither symmetric nor asymmetric. The empty relation is the only relation that is (vacuously) both symmetric and asymmetric.

A relation is asymmetric if and only if it is both antisymmetric and irreflexive. Why?

Properties –transitive 传递性

Definition: A relation R on a set A is called **transitive（可传递的）** if whenever (a, b)∈R and (b, c)∈R, then (a, c)∈R for a, b, c∈A.

Are the following relations on {1, 2, 3, 4} transitive可传递?

R = {(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)} Yes.

R = {(1, 3), (3, 2), (2, 1)} No.

R = {(2, 4), (4, 3), (2, 3), (4, 1)} No.

R= {(1,1), (2,2)}

再思考：同学、同乡、同龄、臭味相投、相似等关系

More examples please…

- 思考问题： 这些特性是实际生活中各种各样的关系的一些什么特征的抽象？

Counting Relations 关系计数

Example: How many different reflexive relations can be defined on a set A containing n elements?A上有多少个自反的二元关系?

Solution: Relations on R are subsets of $A \times A$, which contains n^2 elements.

Therefore, different relations on A can be generated by choosing different subsets out of these n^2 elements, so there are 2^{n^2} relations.

A **reflexive** relation, however, **must** contain the n elements (a, a) for every $a \in A$. **自反的二元关系R={(a,a)|a ∈ A}∪(R- {(a,a)|a ∈ A})**

Consequently, we can only choose among $n^2 - n = n(n-1)$ elements to generate reflexive relations, so there are $2^{n(n-1)}$ of them.

问题：　有多少个反自反的？

Exercises 课外作业

- 5.1节 T3(a),(c), (e)
- T5
- T7 (a), (d)
- T25

关系运算

Combining Relations 构建新的关系

- Relations from A to B are subsets of $A \times B$.

  Thus, they can be combined using set operations （集合运算）.
- Example:
  - Let $A = \{ 1, 2, 3 \}$ and $B = \{ 1, 2, 3, 4 \}$
  - The relations $R_1 = \{ (1,1), (2,2), (3,3) \}$

    and $R_2 = \{ (1,1), (1,2), (1,3), (1,4) \}$ can be combined to obtain:
    - $R_1 \cup R_2 = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (3,3) \}$
    - $R_1 \cap R_2 = \{ (1,1) \}$
    - $R_1 - R_2 = \{ (2,2), (3,3) \}$
    - $R_2 - R_1 = \{ (1,2), (1,3), (1,4) \}$

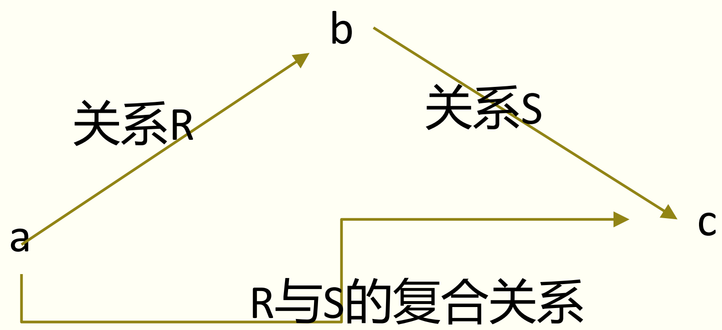
    In each case, the result will be **another relation from A to B**.

逆关系：R^{-1} ： B to A $= \{(b,a) \mid (a,b) \in R, a \in A, b \in B \}$

思考：举例体会上面这些关系运算的实际意义

Composition of Relations
关系复合运算

- Some real examples…



Composition of Relations 关系复合运算

重要的关系运算：

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C.

$$R: A \rightarrow B,\ S: B \rightarrow C$$

The **composite** （复合运算） of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by

S∘R: $A \rightarrow C$, **S∘R={(a,c)|** $a \in A,\ c \in C \wedge \exists b \in B$ 使得$(a, b) \in R \wedge (b, c) \in S$ **}**

In other words, if relation R contains a pair (a, b) and relation S contains a pair (b, c), then S∘R contains a pair (a, c).

Composing Relations--examples

Example: Let D and S be relations on A = {1, 2, 3, 4}.

D = {(a, b) | b = 5 - a}   "b equals (5 – a)"

S = {(a, b) | a < b}   "a is smaller than b"

D = {(1, 4), (2, 3), (3, 2), (4, 1)}

S = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}

S∘D = {   (2, 4),   (3, 3),   (3, 4),   (4, 2),   (4, 3),   (4, 4)}

D maps an element a to the element (5 – a), and afterwards S maps (5 – a) to all elements larger than (5 – a), resulting in **S∘D = {(a,c) | c > 5 – a}** or **S∘D = {(a,c) | a + c > 5}.**

思考问题

从A到B的关系和从B到C的关系的复合关系何时为空?

例:设R是：由一些人的集合A上的有序对(a,b)组成的关系,其中a是b的父母；

S是A上的另一个由有序对(a,b)组成的关系,其中a是b的兄弟姐妹.**那么R·S,S·R分别是什么关系?**

朋友关系复合后是什么样的关系?

关系复合与函数复合

函数是特殊关系

We already know that **functions** are just **special cases** of **relations** (namely those that map each element in the domain onto exactly one element in the codomain).

If we formally convert two functions into relations, that is, write them down as sets of ordered pairs, the composite of these relations will be exactly the same as the composite of the functions (as defined earlier).

函数转换成关系后，函数的复合也就是相应的关系复合

Composing Relations—more examples

- Example: （函数复合与关系复合）
- Suppose R defined on **Z** by: xRy iff $y = x^2$
- S defined on **Z** by: xSy iff $y = x^3$
- What is the composite $S \circ R$?
- 请用集合的形式表示$S \circ R$?

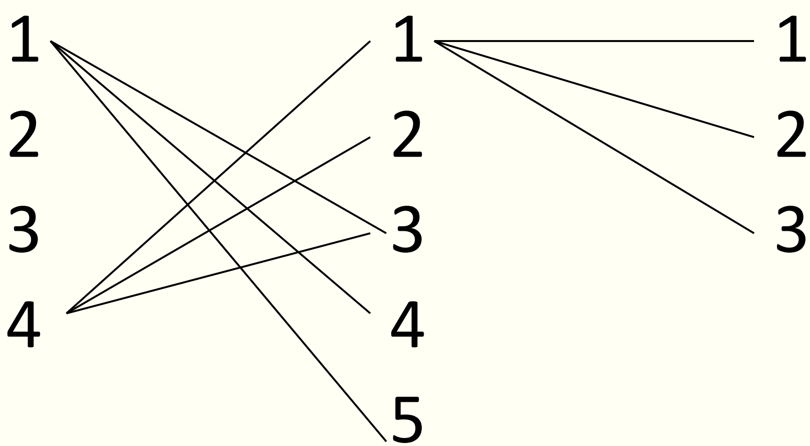
- xRy iff $y = x^2$ xSy iff $y = x^3$

Answer: These are functions (squaring and cubing) so the composite SR is just the function composition (raising to the 6th power). $X(S \circ R)y$ iff $y = x^6$ (in this case $R \circ S = S \circ R$)

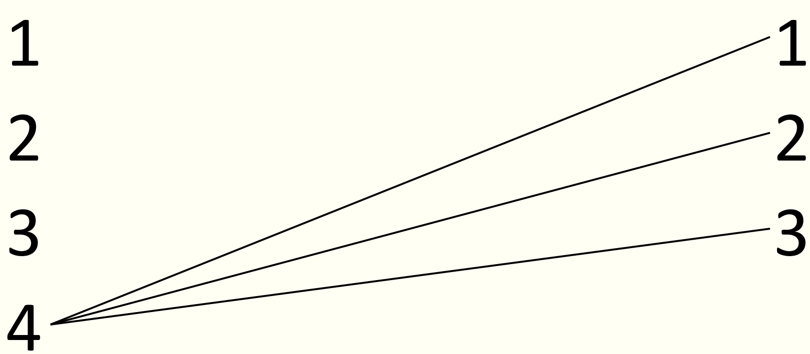
Composing Relations

Compose the following relations:

Composing Relations



A: Draw all possible shortcuts. In our case, all shortcuts went through 1:



Associative Law of Composite
(关系复合运算的结合律）

- If R_1 is a relation from A to B, R_2 from B to C, R_3 from C to D, then

$$(R_3 \circ R_2) \circ R_1 = R_3 \circ (R_2 \circ R_1)$$

- 这里就是要证明对应任意的(a,d) $\in (R_3 \circ R_2) \circ R_1$，必然有(a,d) $\in R_3 \circ (R_2 \circ R_1)$，而且反之也真。

Composition Relations复合运算推广

（多重复合）

Definition: Let R be a relation on the set A. The powers R^n, n = 1, 2, 3, …, are defined inductively by

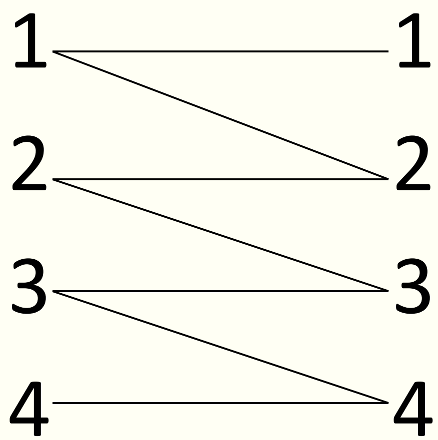
$R^1 = R$

$R^{n+1} = R^n \circ R$

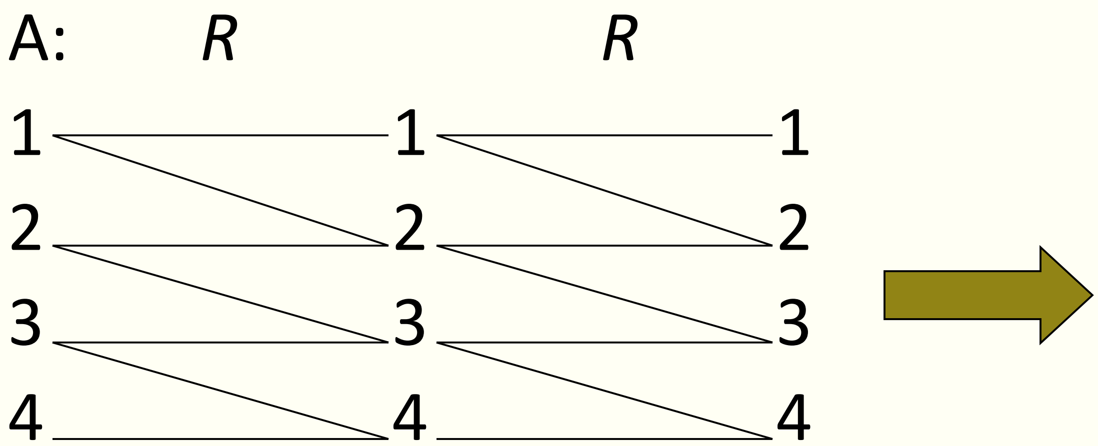
In other words: $R^n = R \circ R \circ \ldots \circ R$ (n times the letter R)

Exponentiation 关系的幂

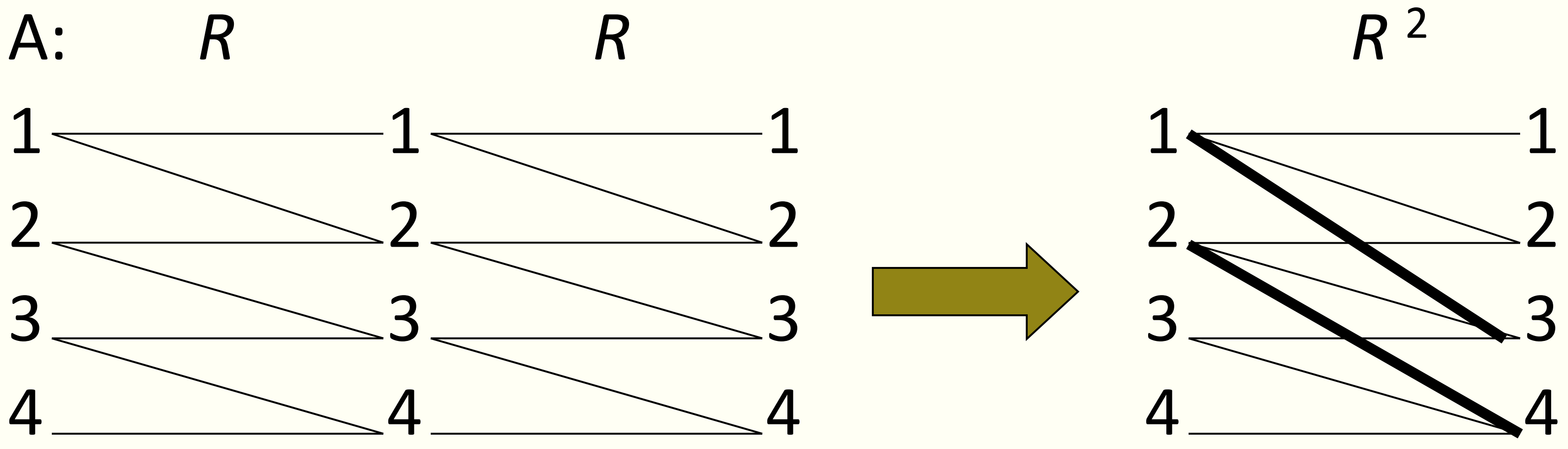
Q: Find R^3 if R is given by:



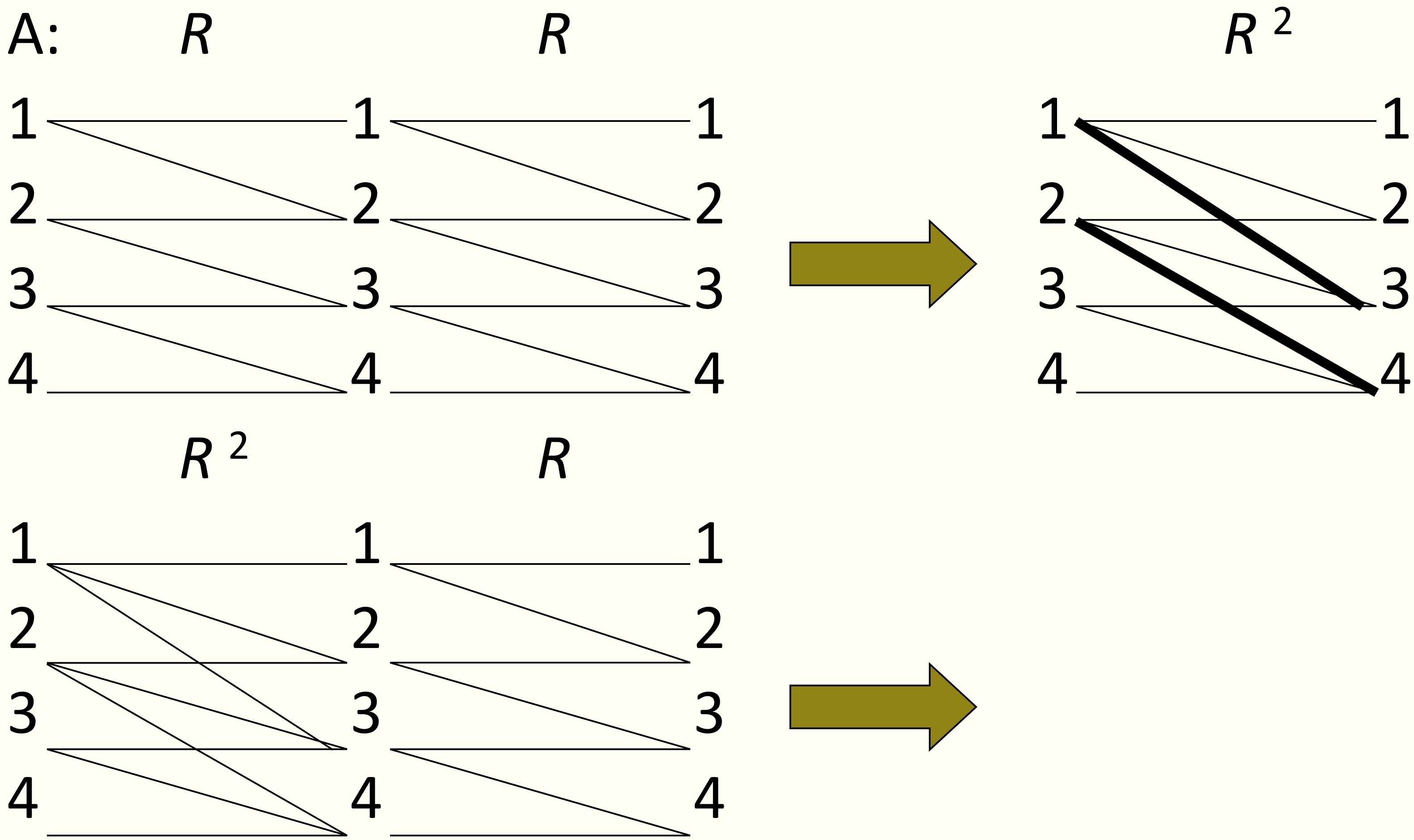
Exponentiation 关系的幂



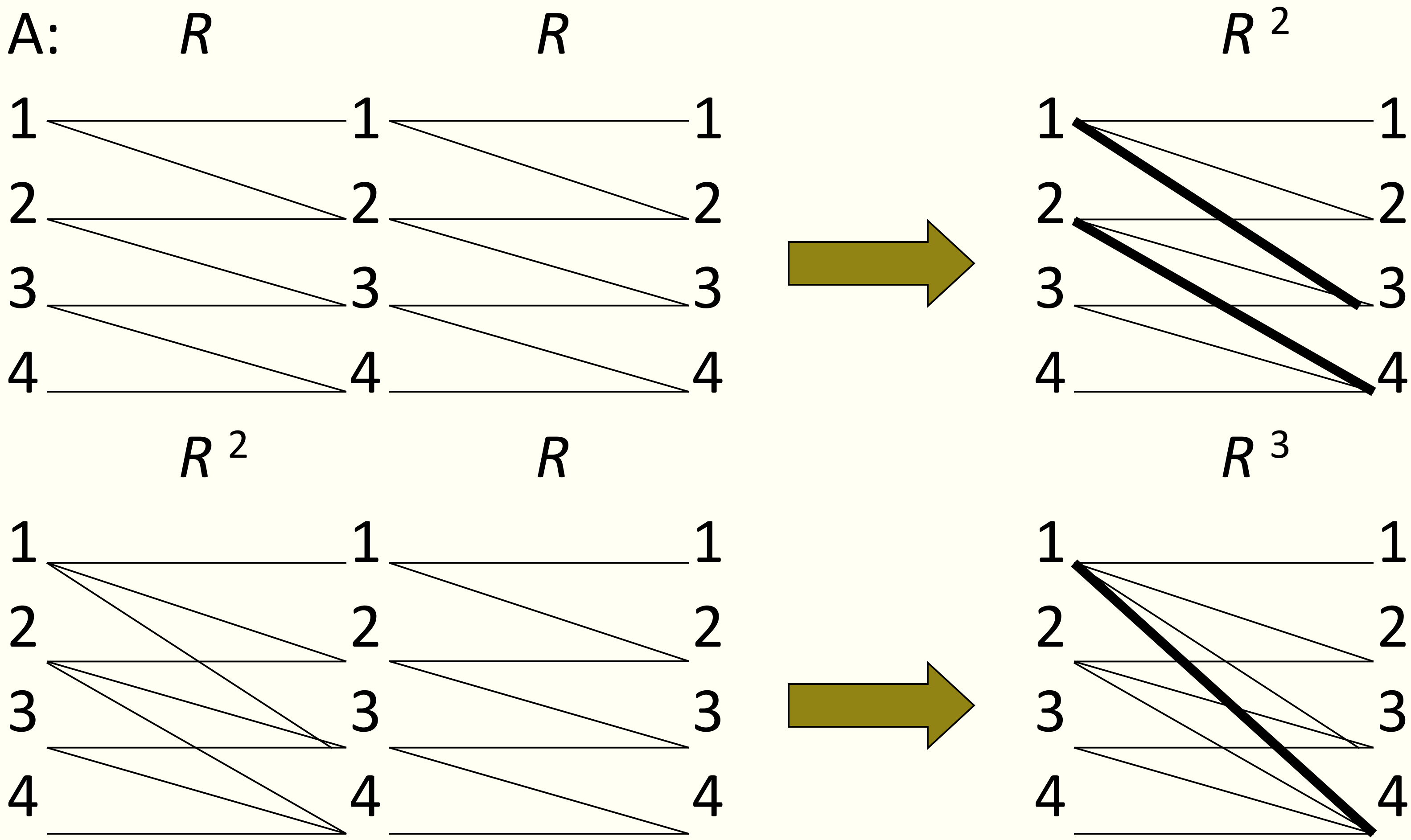
Exponentiation 关系的幂



Exponentiation 关系的幂



Exponentiation



Think about多重关系复合

- If R_1 is a relation from A_1 to A_2, R_2: A_2 to A_3, …, R_n is A_n to A_{n+1}, how to define the composite relation $R_n \circ R_{n-1} \circ \ldots \circ R_2 \circ R_1$?
- 思考：觉得该如何定义这种多重复合？

- How do you know whether $(a_1, a_{n+1}) \in R_n \circ R_{n-1} \circ \ldots \circ R_2 \circ R_1$?

- 当且仅当，存在$a_2 \in A_2$, $a_3 \in A_3, \ldots, a_n \in A_n$满足：$(a_1,a_2) \in R_1$, $(a_2,a_3) \in R_2, \ldots, (a_{n-1}, a_n) \in R_{n-1}$，$(a_n, a_{n+1}) \in R_n$ 为什么？

重要结论

Theorem: The relation R on a set A is transitive if and only if $R^2 \subseteq R$

为什么？

注意：这个定理的应用，能给我们一条判断关系是否可传递的新思路。

想想这个新的思路应该是怎样的？

Composition Relations

重要Theorem: The relation R on a set A is transitive可传递的 if and only if $\mathbf{R^n \subseteq R}$ for all positive integers n.

Why?

Therefore, for a transitive relation R, $R \circ R$ does not contain any pairs that are not in R, so $R \circ R \subseteq R$.

Since $R \circ R$ does not introduce any pairs that are not already in R, it must also be true that $(R \circ R) \circ R \subseteq R \circ R \subseteq R$, and so on, so that $R^n \subseteq R$.

Exercises课外作业

- 5.1节
- T37 (a), (c) ,T41, T43(b),T53

Representing Binary Relations 二元关系的表示

两种表示方法: **Zero-one matrices** and **directed graphs**.(0-1矩阵和有向图）

If R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, then R can be represented by the zero-one matrix $M_R = [m_{ij}]$ （关系矩阵）with

$m_{ij} = 1$, if $(a_i, b_j) \in R$, and
$m_{ij} = 0$, if $(a_i, b_j) \notin R$.

Note that: for creating this matrix we first need to list the elements in A and B in **a particular, but arbitrary order(按照某种约定的元素顺序)**.
按照这种规则，建立起来有限集合之间的二元关系跟0-1矩阵的1-1对于关系。

Representing Relations关系表示

Example: How can we represent the relation $R = \{(2, 1), (3, 1), (3, 2)\}$ from $\{1,2,3\}$ to $\{1,2\}$ as a zero-one matrix?

Solution: The matrix M_R is given by

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Representing Relations 关系表示

A上的2元关系的关系矩阵有何特征？

They are **square** matrices.

What do we know about matrices representing **reflexive** relations?

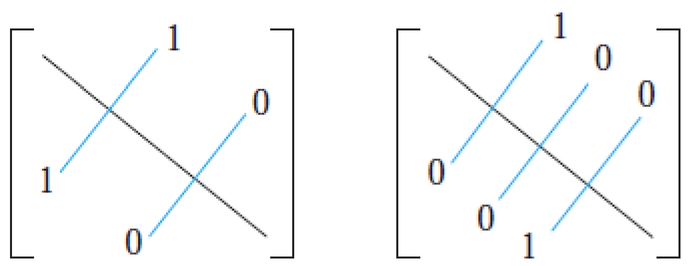
All the elements on the **diagonal** of such matrices M_{ref} must be **1s**.

$$M_{ref} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 1 \end{bmatrix}$$

Representing Relations

对称（反对称）的二元关系的关系矩阵有何特征？

These matrices are symmetric, that is, $M_R = (M_R)^t$.



(a) Symmetric (b) Antisymmetric

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

思考问题：如何利用关系矩阵，求有限集上的对称（反对称）的二元关系的个数？

关系矩阵与关系运算的联系

Example: Let the relations R_1 and R_2 be represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$? 并关系、关系交集的关系矩阵

Solution: These matrices are given by

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- 想象一下：利用矩阵来表示二元关系，会带来什么好处？

- 再思考：关系的差，关系的补，关系的逆关系，如何从关系矩阵求得？

关系矩阵与关系复合运算的联系

the **Boolean product** of two zero-one matrices 0-1
矩阵的布尔积

Let $A = [a_{ij}]$ be an $m \times k$ 0-1 matrix and $B = [b_{ij}]$ be a $k \times n$ 0-1 matrix.

Then the **Boolean product** of A and B, denoted by $A \circ B$, is the $m \times n$ matrix with $(i, j)^{th}$ entry $[c_{ij}]$, where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2i}) \vee \ldots \vee (a_{ik} \wedge b_{kj}).$$

$c_{ij} = 1$ if and only if at least one of the terms $(a_{in} \wedge b_{nj}) = 1$ for some n; otherwise $c_{ij} = 0$.

关系矩阵与关系复合运算的联系

上面分析得到如下重要结论：

定理：$M_{B\circ A} = M_A \odot M_B$ （复合关系的关系矩阵等于关系矩阵之布尔积）

In other words, the matrix representing the **composite** of relations A and B is the **Boolean product** of the matrices representing A and B.

特别地： the **powers of relations**:

$M_{R^n} = M_R^{[n]}$ (n-th **Boolean power**).

关系矩阵与关系复合运算的联系

证明思路：Let us now assume that the zero-one matrices $M_A = [a_{ij}]$, $M_B = [b_{ij}]$ and $M_C = [c_{ij}]$ represent relations A, B, and C $= B \circ A$, respectively.

要证明 $M_C = M_A o M_B$，（分析$M_A o M_B$ 的形成过程）：
$c_{ij} = 1$ if and only if at least one of the terms $(a_{in} \wedge b_{nj}) = 1$ for some n; otherwise $c_{ij} = 0$.
分析$(a_{in} \wedge b_{nj}) = 1$ 代表的实际意义
根据关系矩阵的定义， this means that C contains a pair (x_i, z_j) if and only if there is an element y_n such that (x_i, y_n) is in relation A and (y_n, z_j) is in relation B.

Therefore, $C = B \circ A$ (**composite** of A and B). 说明C恰好是复合关系 $B \circ A$

- 这个定理给了我们求复合关系的新思路，请同学们自己思考一下…

复合关系矩阵计算

Example: Find the matrix representing R^2, where the matrix representing R is given by

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Solution: The matrix for R^2 is given by

$$\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

问题：R 是否可传递？

Solutions 利用关系矩阵

- 1. to calculate the composite of relations from finite set A to finite set B， and from B to C 计算复合关系
- 2. to determine whether a relation on a finite set is transitive or not. 分析判断关系特征
- 3. 更方便用编写程序计算解决问题

Questions 思考问题

- How do you know whether a binary relation R on set A is the subset of another binary relation S on A from their matrices? 如何从关系矩阵出发判断一个关系是否另一个关系的子集（子关系）？

- If you have the matrix of a given binary relation R on A, how can you tell R is transitive or not? 如何利用关系矩阵判断一个二元关系是否是可传递的？

•引导学生思考：从这里能得到什么启发，如何计算判断一个关系是否可传递？

Digraph Representation有向图表示方法

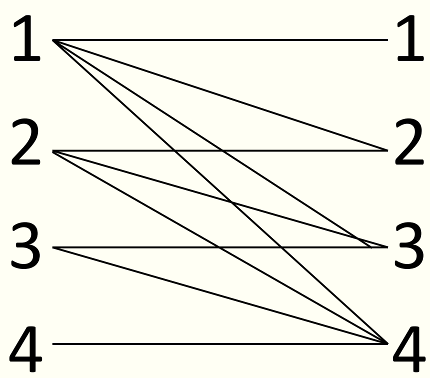
- Relation *R* on a set *A* is represented with a *digraph (有向图)*

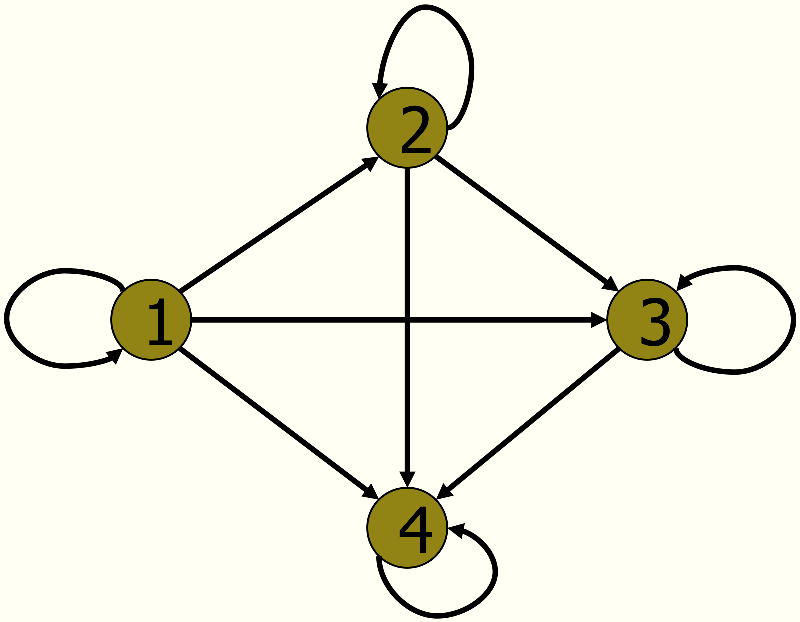
  关系有向图的构成：
  - The set *A* is represented by ***nodes*** (or ***vertices***) 结点集
  - and whenever *aRb* occurs, a ***directed edge*** (or ***arrow***) *a→b* is created. 有向边集
  - Self pointing edges (or ***loops***) are used to represent *aRa*.
    指向自身的小圆弧(单边弧)

Digraph Representation 有向图表示法

Q: Represent previous page's R^3 by a digraph.

R^3





关系的有向图表示

Definition: A **directed graph**, or **digraph**, consists of a set V of **vertices** (or **nodes**) together with a set E of ordered pairs of elements of V called **edges** (or **arcs**).

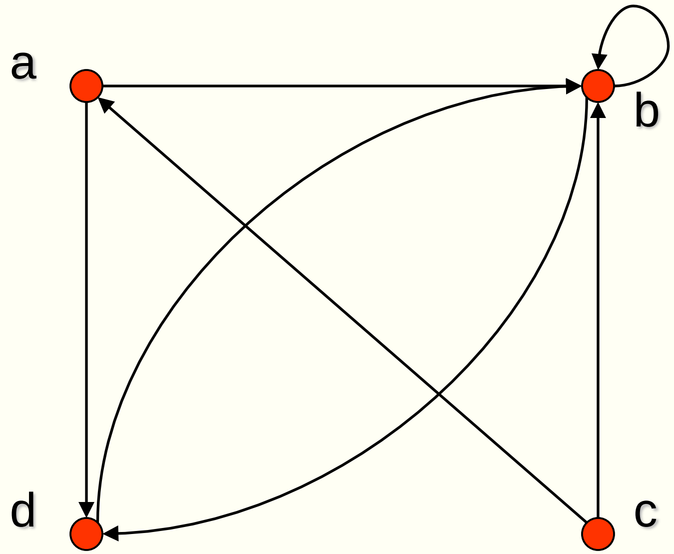
有向图由：结点集，以及结点集之间的元素有序对的集合（有向边集）构成

The vertex a is called the **initial(or source) vertex** of the edge (a, b), and the vertex b is called the **terminal (or target) vertex** of this edge.

（有向边的起点与终点）

关系的有向图表示举例

Example: Display the digraph with V = {a, b, c, d},
E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}.



An edge of the form (b, b) is called a **loop.**

Question: can you tell some properties from the digraph of relation?

关系的有向图表示

显然，任何一个有限集A上的二元关系都可以用一个以A的元素作为顶点集的有向图来表示。

当$(a, b) \in R$ 时，a到b形成一条有向边

反之， 任意顶点集V上的有向图都能对应于(还原出)一个V上的二元关系来表示。

This **one-to-one correspondence** between relations and digraphs means that any statement about relations also applies to digraphs, and vice versa!!

这种1-1对应关系使得二元关系上的任何结论（命题）都可以用到相应的有向图上去，反之也真！

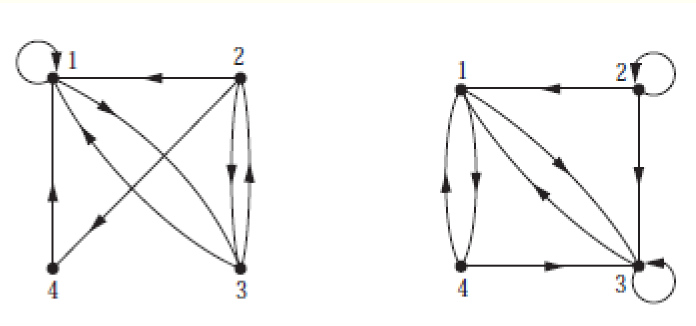
Questions 从关系图分析关系的特征

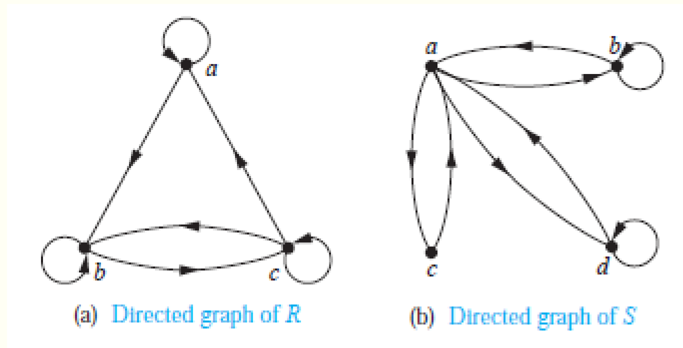
- 1. Can you determine whether a relation is **reflexive** from its digraph? How to? 如何从关系图上分析判断一个集合上的二元关系是否是自反的？

- 2. What about the other properties of relation? 其它性质如何判断？（对称性，反对称性，可传递性）

从关系图分析关系的特征

- Example: Determine whether the relations for the directed graphs shown in the following figures are reflexive, symmetric,
- antisymmetric, and/or transitive.





(a) Directed graph of R (b) Directed graph of S

Summary 关系复合运算总结

- 1. from definition 从定义出发
- 2. from relation graph 利用关系图计算
- 3. from matrix of the relation 利用关系矩阵计算

课外作业

- 5.3 节 T9(a), (d)， T15(a), T27 (这题做在教材上就可以,不用提交), T29;