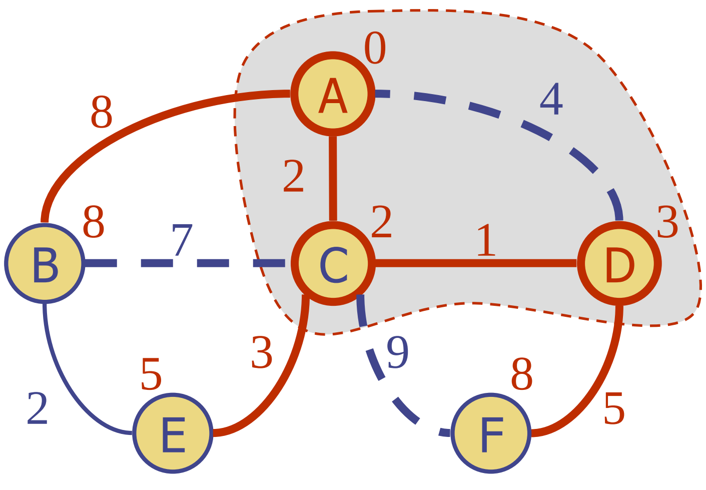
最短通路



weighted graph (加权图、有权图)

- 在某些时候某些场合，并非图的所有边都一样长。出于某些原因和目的，需要给图的每条边加权(某种意义上的值、长度)，也即给每条边赋一个值（权），称为边的长度。

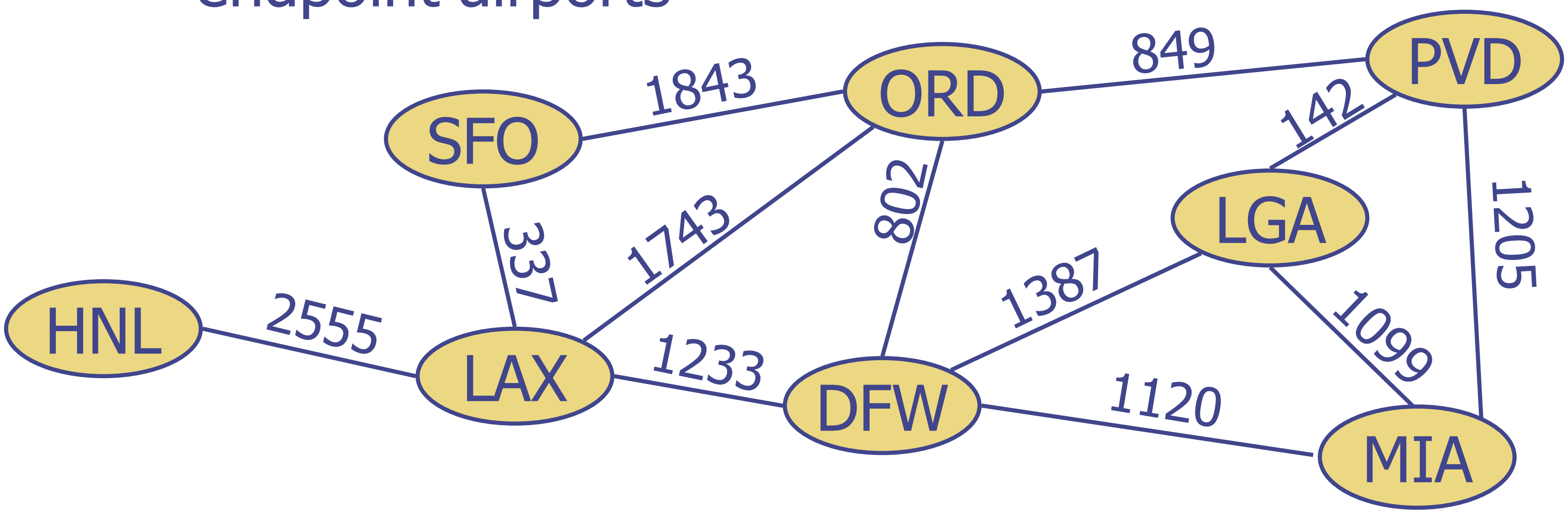
可以将加权图视为一个三元结构：G=(V,E, f),其中f为加权函数；

- 举例说明一下加权的必要性。

- 权可以是距离，时间，成本，费用，带宽，吞吐量，电话呼叫图中的呼叫次数或者说频度等等。

Weighted Graphs

- 加权图中的每条边被赋予一个数值（权）
- Example:
  - In a flight route graph, the weight of an edge represents the distance in miles between the endpoint airports



给结点赋权的图

- Weights can also be attached to the vertices instead of the edges or can be attached to both vertices and edges. The resulting graph is called a *weighted graph.*

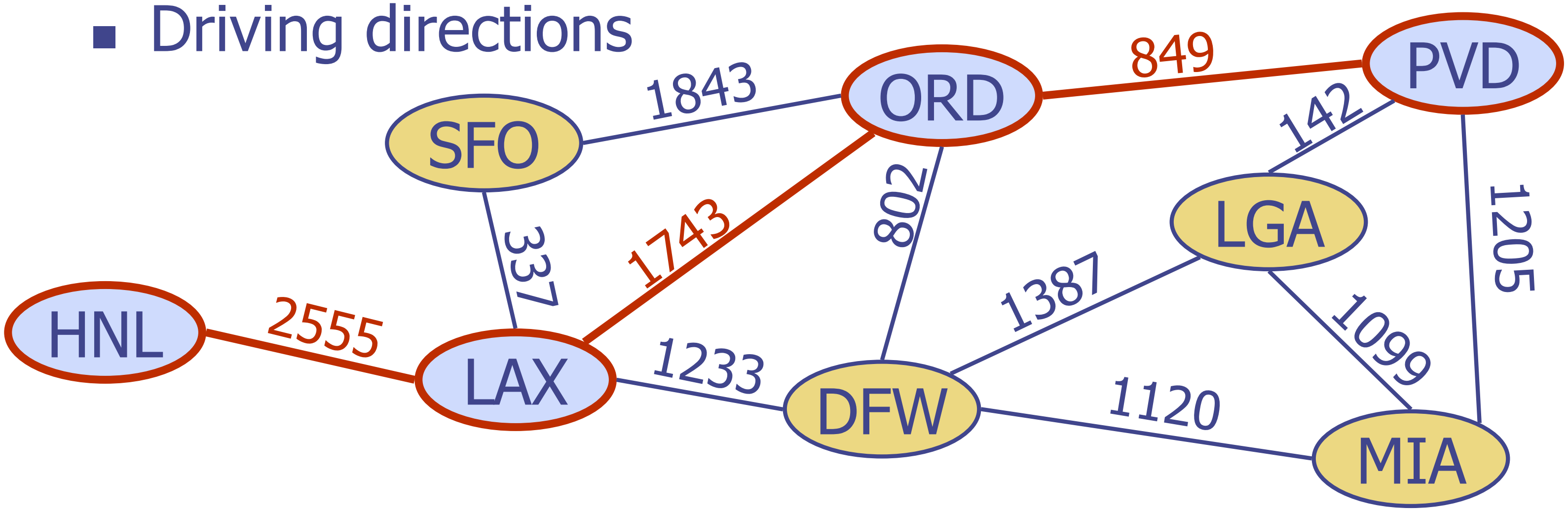
- 给结点赋权的例子…

- 例如：社交网络图，每个结点代表人，可以给结点赋权（年龄，收入等等）

Shortest Path Problem

- **最小通路问题：**给定有权图中的两个不同结点，寻找两个结点之间总权最小的通路

- E.G: Shortest path between MIA and Honolulu
- Applications：
  - Internet packet routing
  - Flight reservations
  - Driving directions



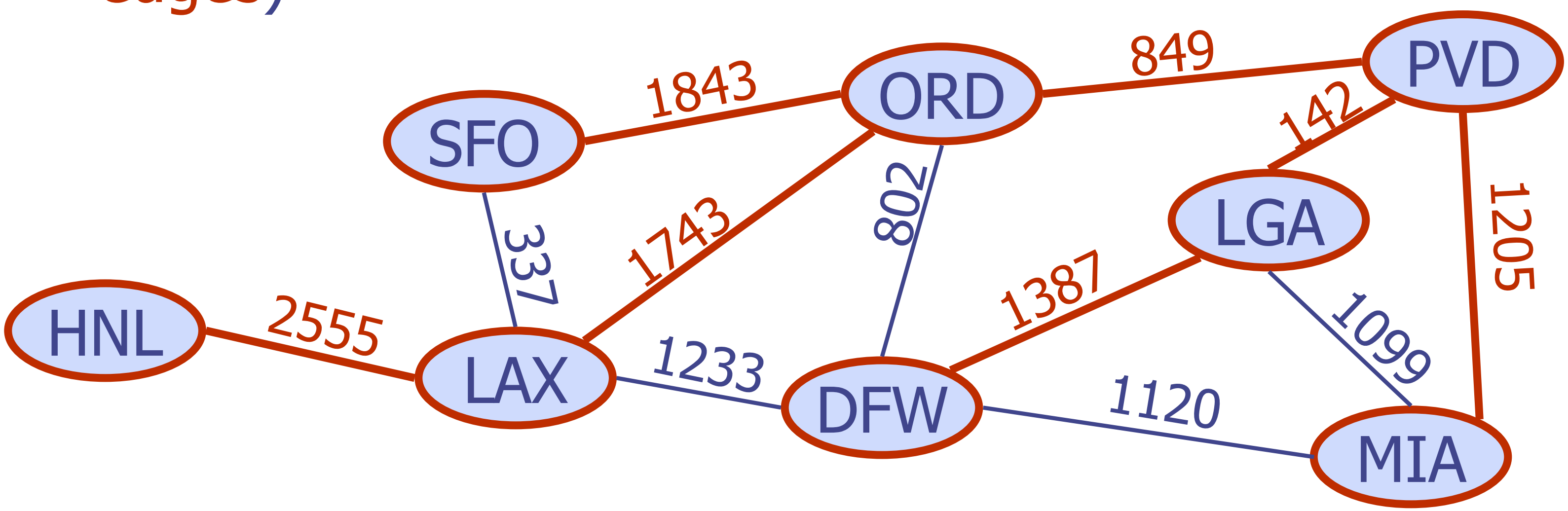
Shortest Path Properties

性质1：最短通路的子路本身也一定是一条最短通路。（Why?）

性质2：连通图中，存在一颗从一个起始结点到其它所有结点的最短路径的树。

Example:

Tree of shortest paths from Providence (those red edges)



Dijkstra‘s 算法 (最经典的、最常用的算法）

single-source shortest path problem in graph theory. 图论最短通路问题的单源算法 （给定一个点到其它所有连接的点的）

Works for both undirected and digraph. 但只适应非负权图。

算法Input: Weighted graph G=(V,E,f) and source vertex $s \in$ V, such that all edge weights are nonnegative

算法Output: Lengths of shortest paths (or the shortest paths themselves) from a given source vertex $s \in$ V to all other vertices

Dijkstra’s Algorithm (迪克斯特拉单源算法)

- **距离**：一个结点到v到另一结点s的距离是v,s间的最短通路的长度，这里的长度是路的所有边的权之和。

- Dijkstra’s algorithm 计算起点s到所有其它所有连接的点的距离。

- Assumptions:
  - 连通
  - 所有边的权值非负
  - 简单图

Dijkstra’s Algorithm (迪克斯特拉单源算法)

- 所谓的“云”：结点集V的子集（从点s开始慢慢扩张，最终包括V的所有点。 贪婪算法）

- “云”算法，或者叫“水淹”算法：给每一个结点v保存一个临时值 ***d***(***v***) ;点在云内时该值表示起点s到v点的距离;v在云外时,表示从s点到v点有一条长度为d(v)的路; 最终将这”云”扩大到整个图

- “云”扩张过程，也即迭代的过程：
  - 开始“云”只包括结点s一个点 (d(s)=0)

  每次迭代做如下的两件事:
  - (1) 把云外的d(u)最小的点u加入到云中（也即离“云”最近的点）
  - (2) 更新“云”外与u邻接的结点的的标记d(v). （关键搞清楚如何更新d(v) ）

  当“云”扩张到了整个图，所有的d(v)都标注完，任务完成。 每个点的标注值d(v)即为点s到v的距离；

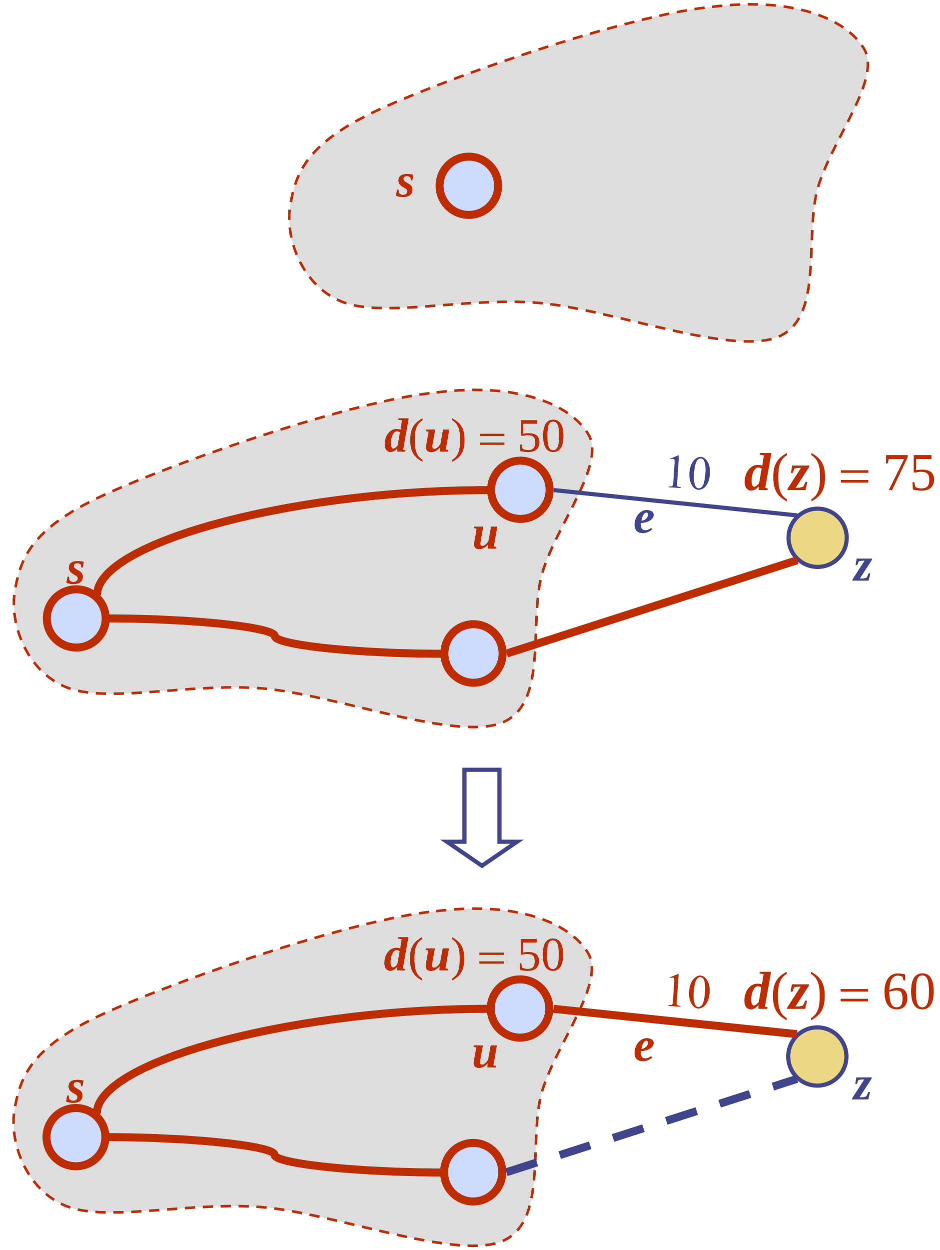
Edge Relaxation

- 第一次给所有与起点s邻接的点中离s最近的点标注一个距离d(v) （也即相应的边长）。每一个与s邻接的结点标注成相应边的长度；其它所有外围的点标为∞

- 寻找云外标注值最小的结点u，将其加入到云内；
  - ***u*** 是最近加入到云中的结点

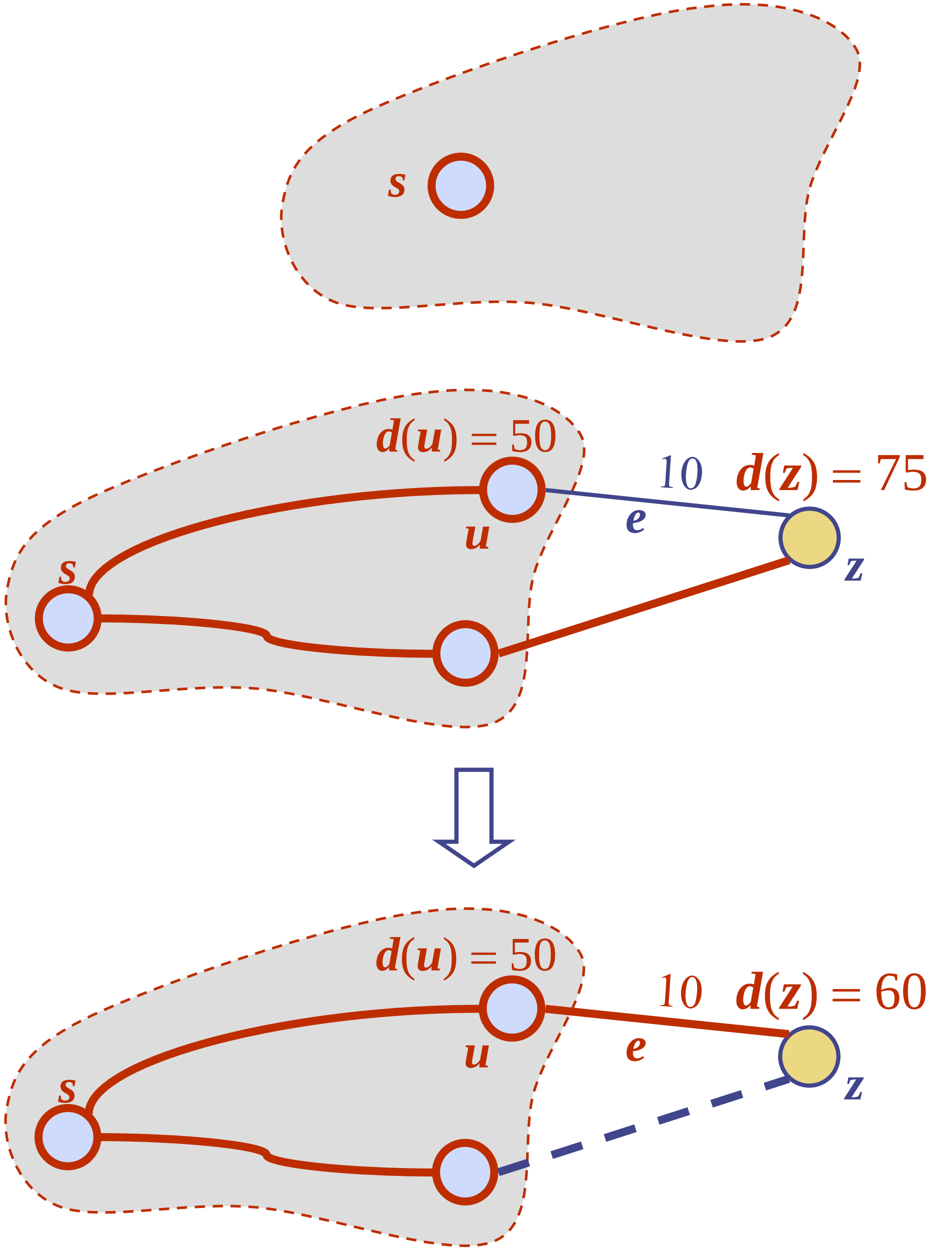
- 逐个更新与云中点u邻接的在云外的结点z的标注值d(z)（“云”周边的）：

$$\boldsymbol{d(z)} = \min\{\boldsymbol{d(z)},\boldsymbol{d(u)} + \boldsymbol{weight(e)}\}$$

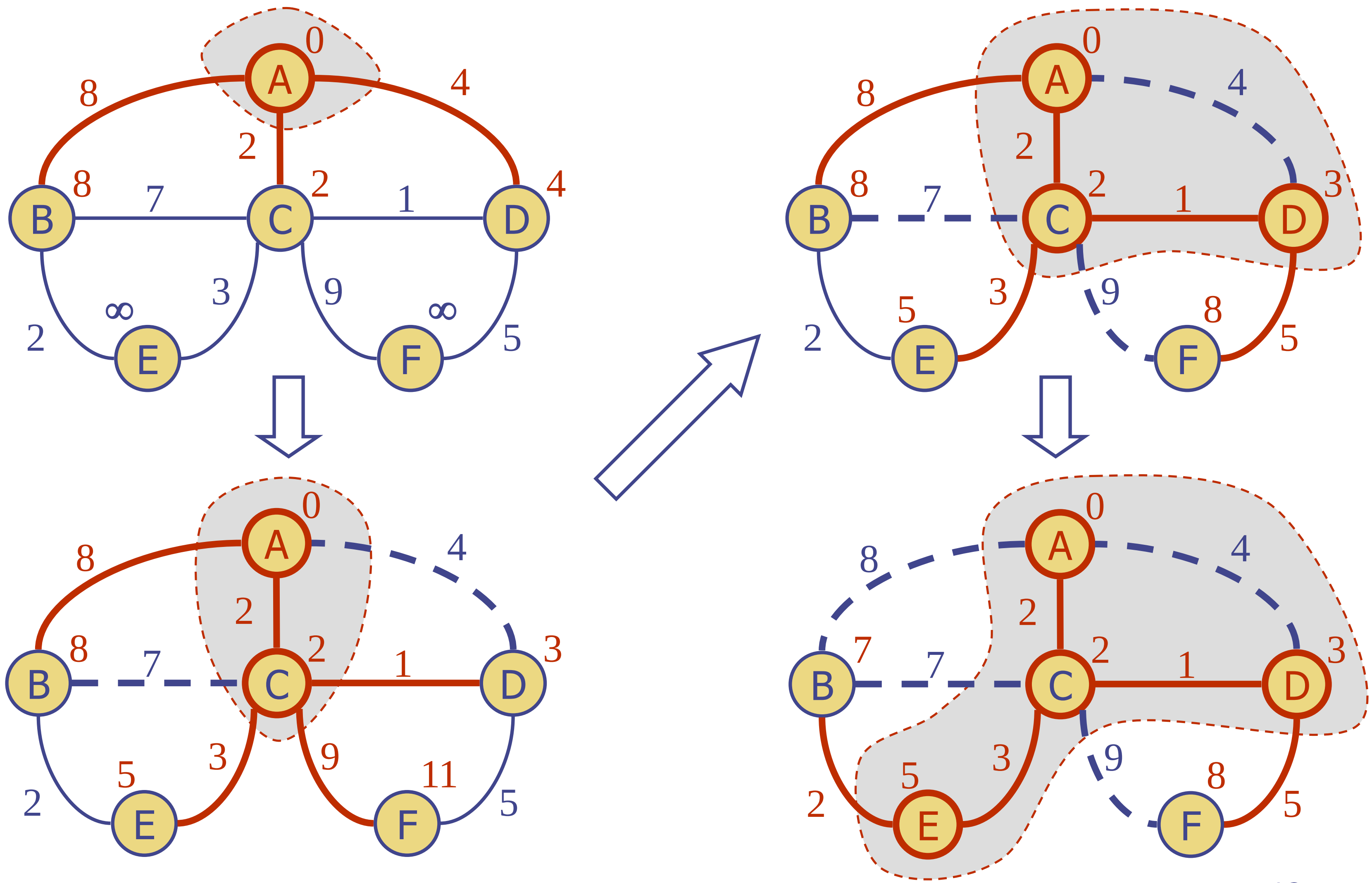


Edge Relaxation

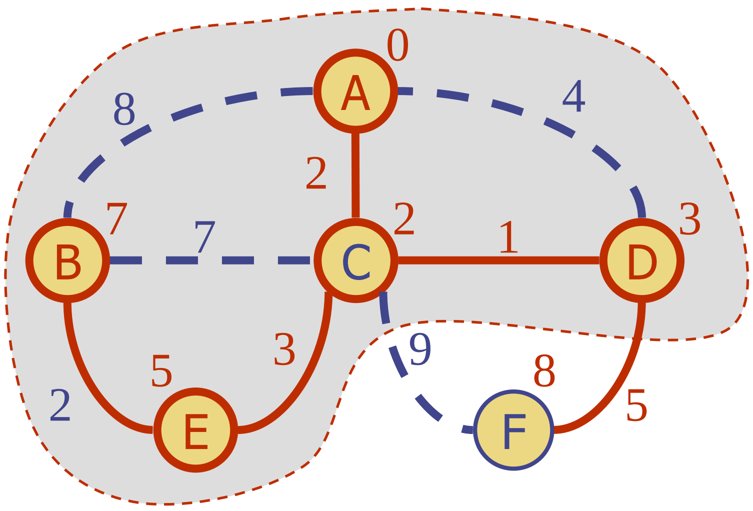
- 将“云”周边邻接的点中标注值(**d(u)**)最小的点加入到“云”中
- 逐个更新与云中最近一次加入到云中的结点邻接的在云外的结点z的标注值d(z) （“云”周边的）
- 直到“云”包含了所有的结点
- *注：这里所谓的“云”其实是结点集V的一个子集。*

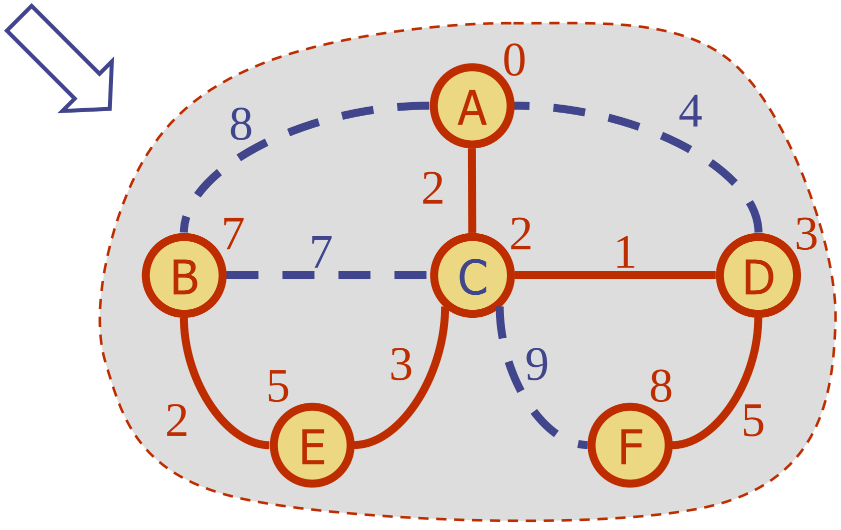


举例：观察云的扩张过程以及点的值d(v)的变化过程



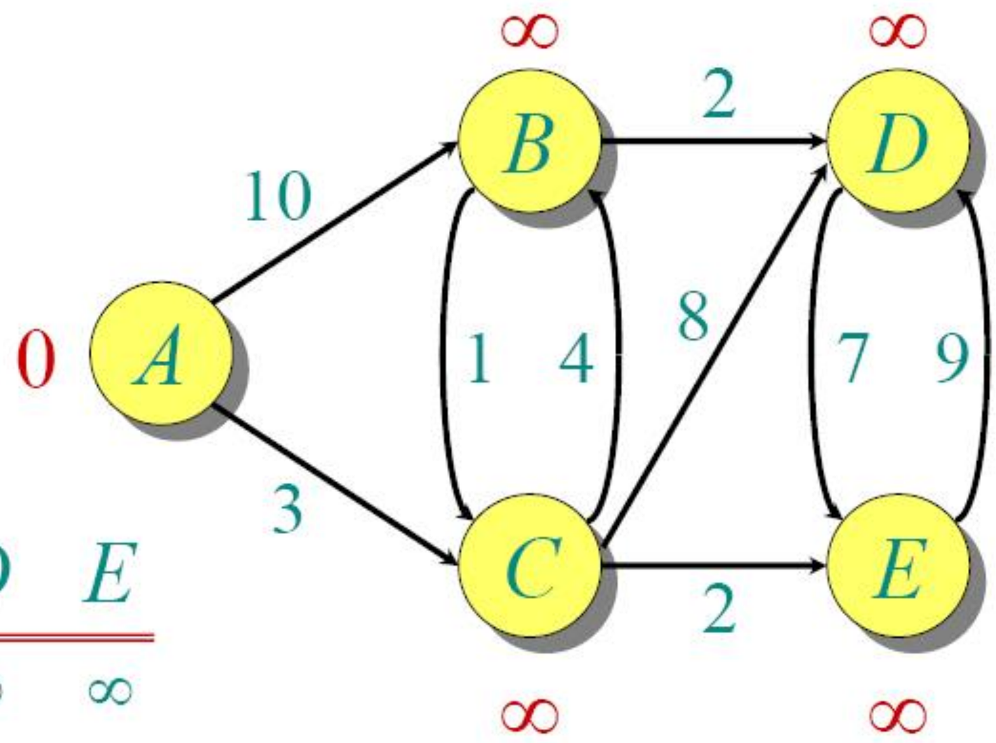
Example (cont.)





Another Dijkstra Animated Example for directed graph （有向图距离）

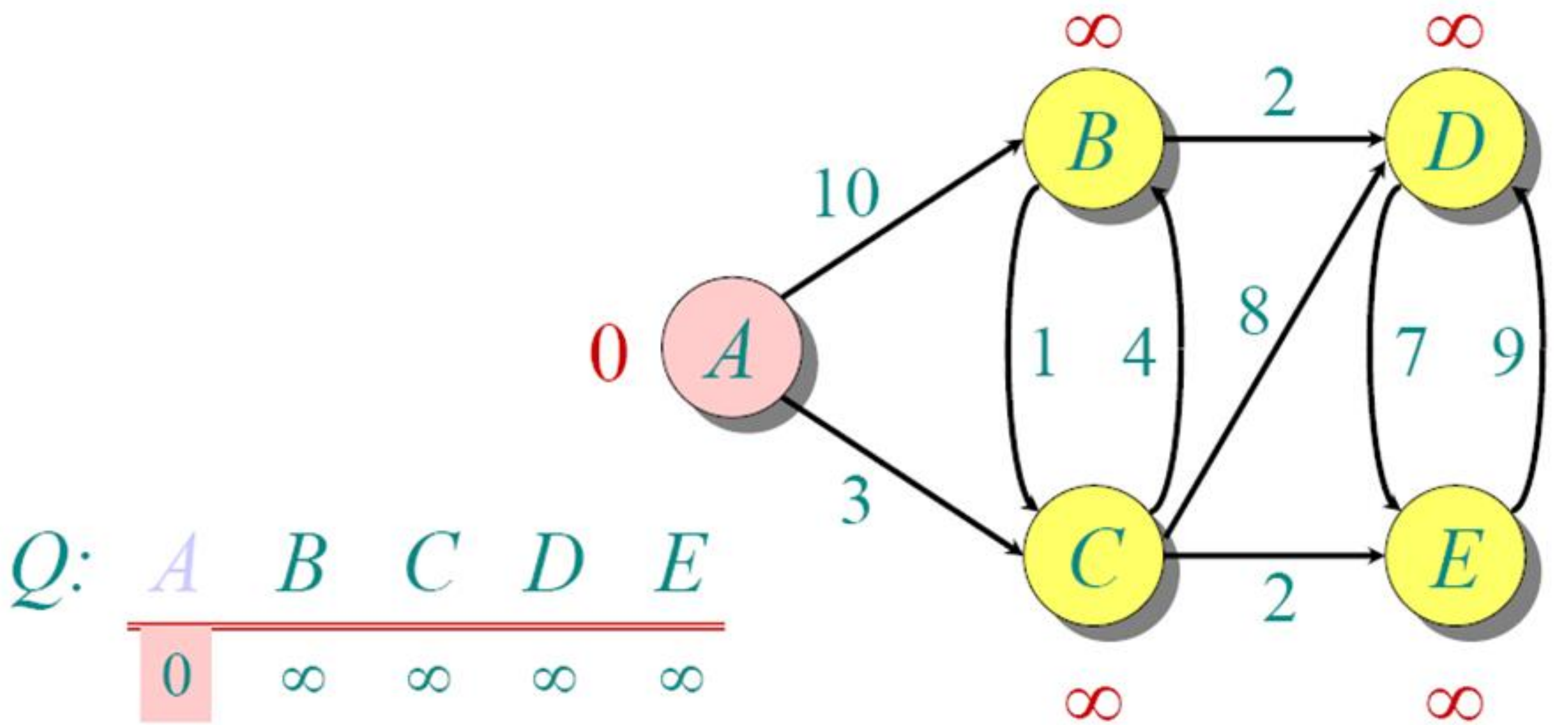
Initialize:



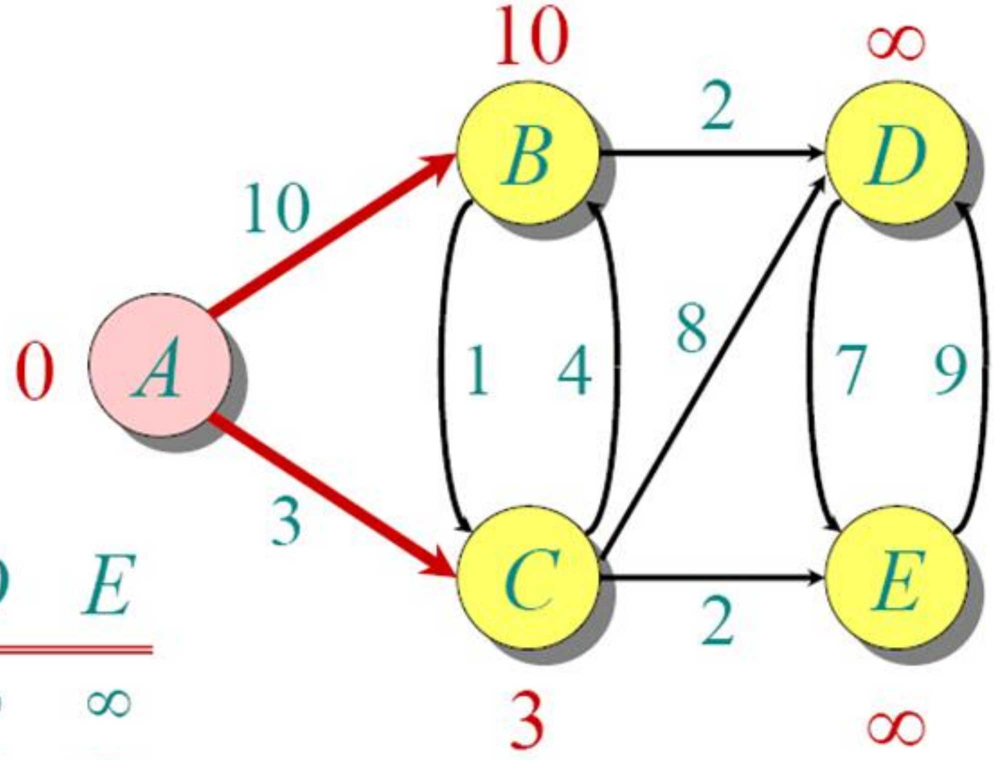
Q:	A	B	C	D	E
	0	∞	∞	∞	∞

S: {}

Dijkstra Animated Example



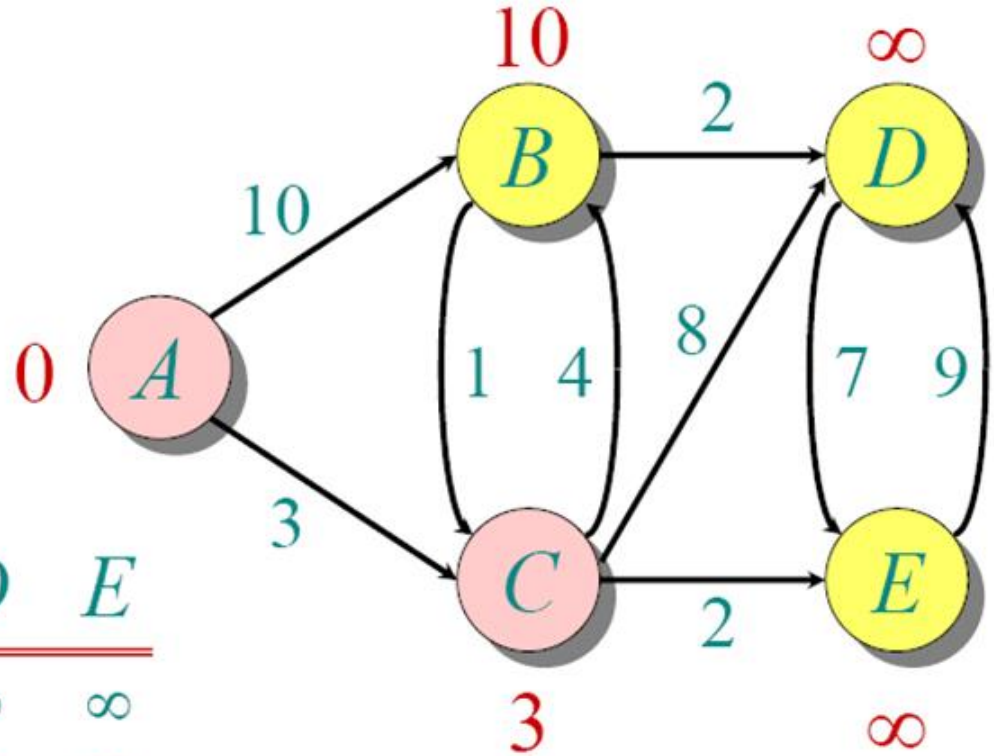
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞

S: { A }

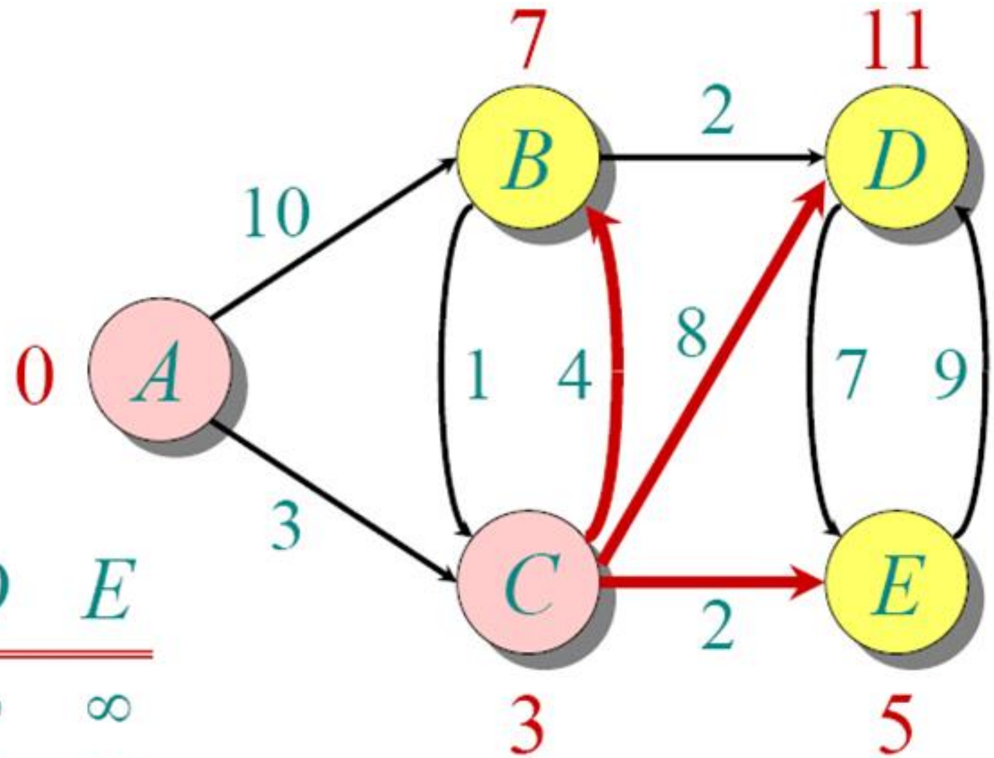
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞

S: { A, C }

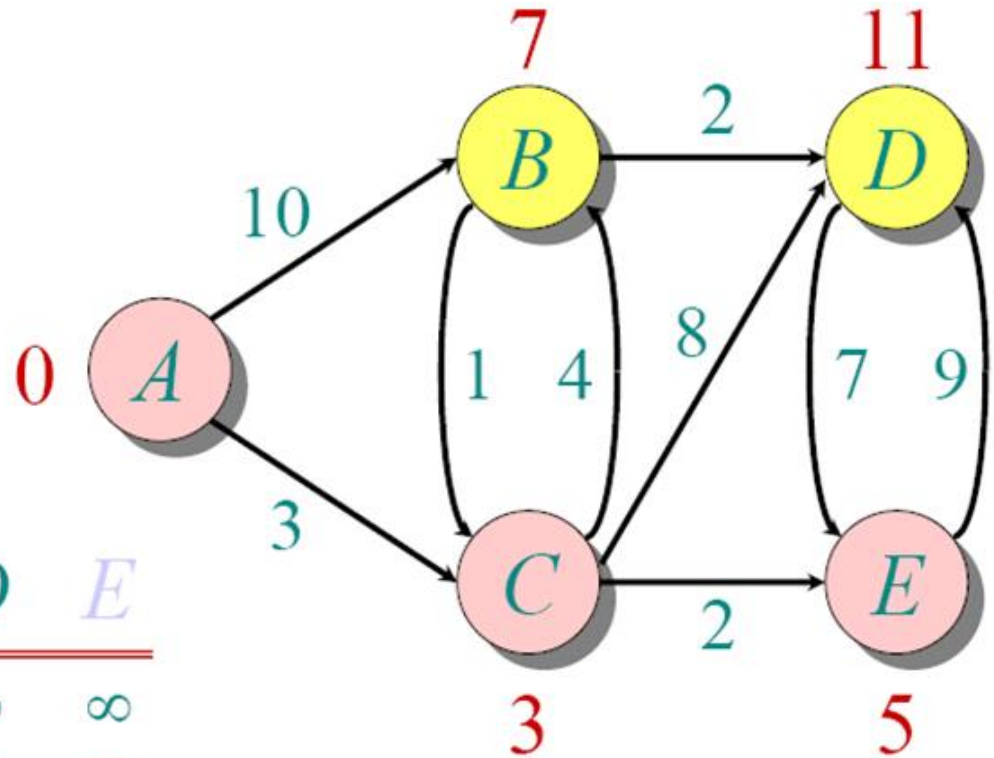
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { *A*, *C* }

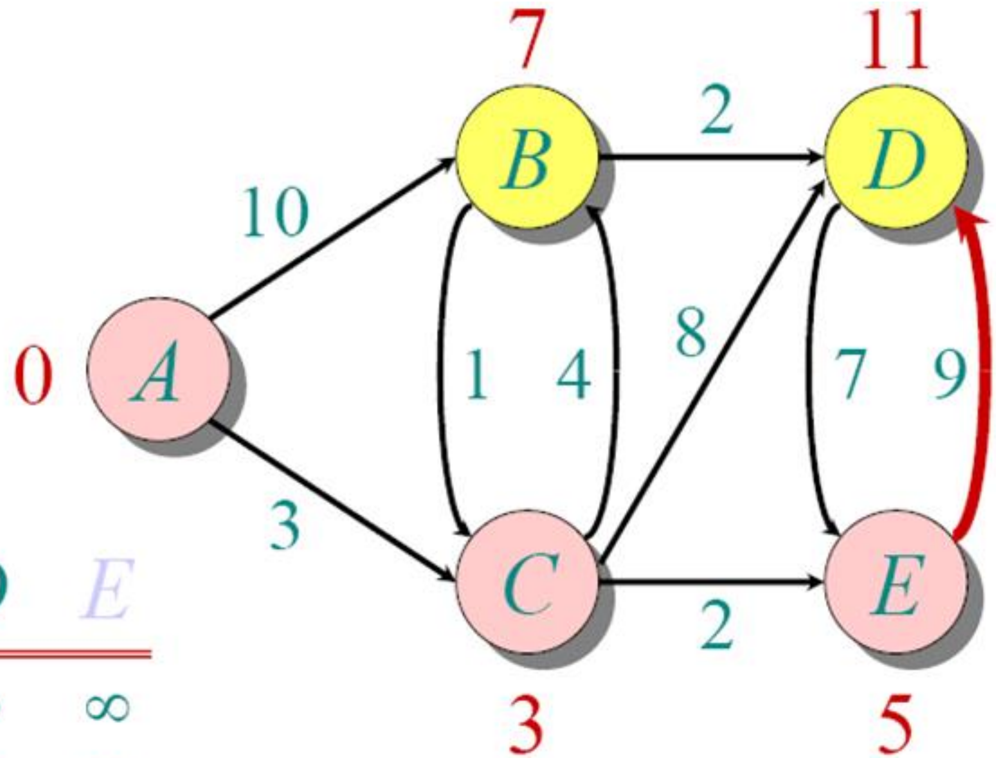
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { A, C, E }

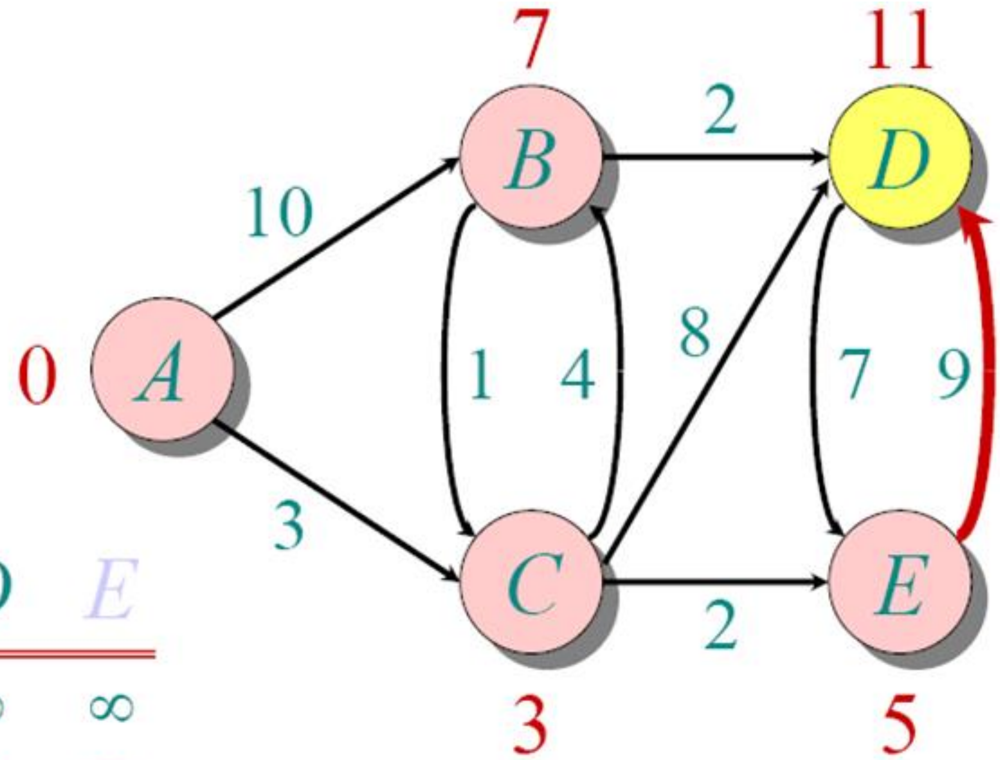
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { A, C, E }

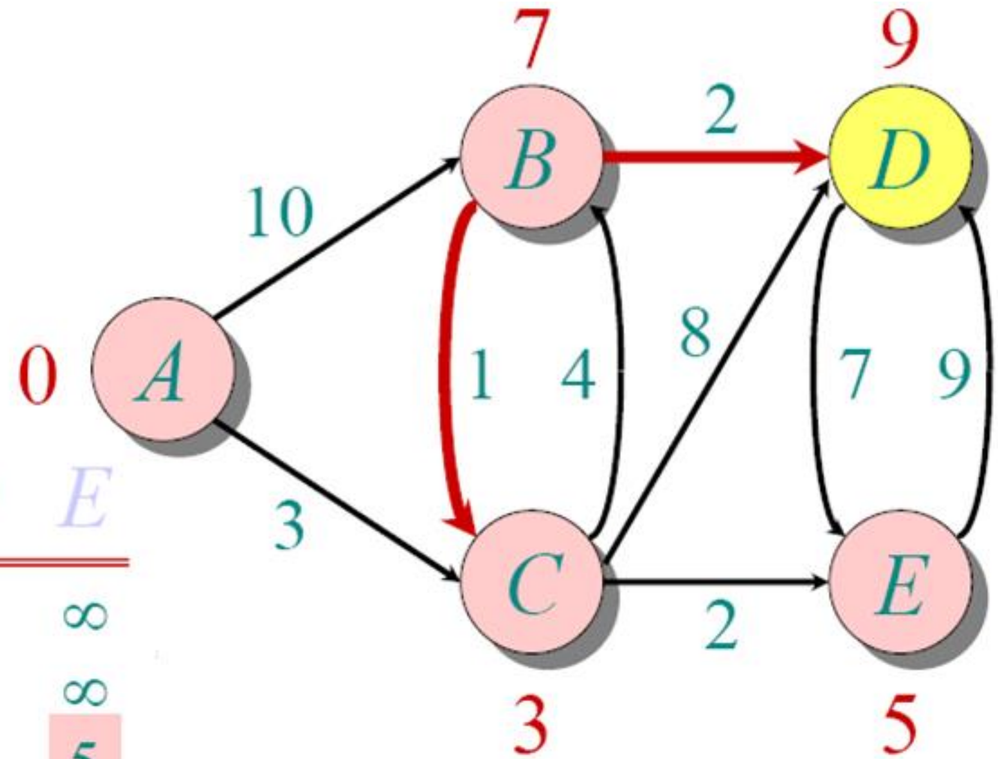
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { A, C, E, B }

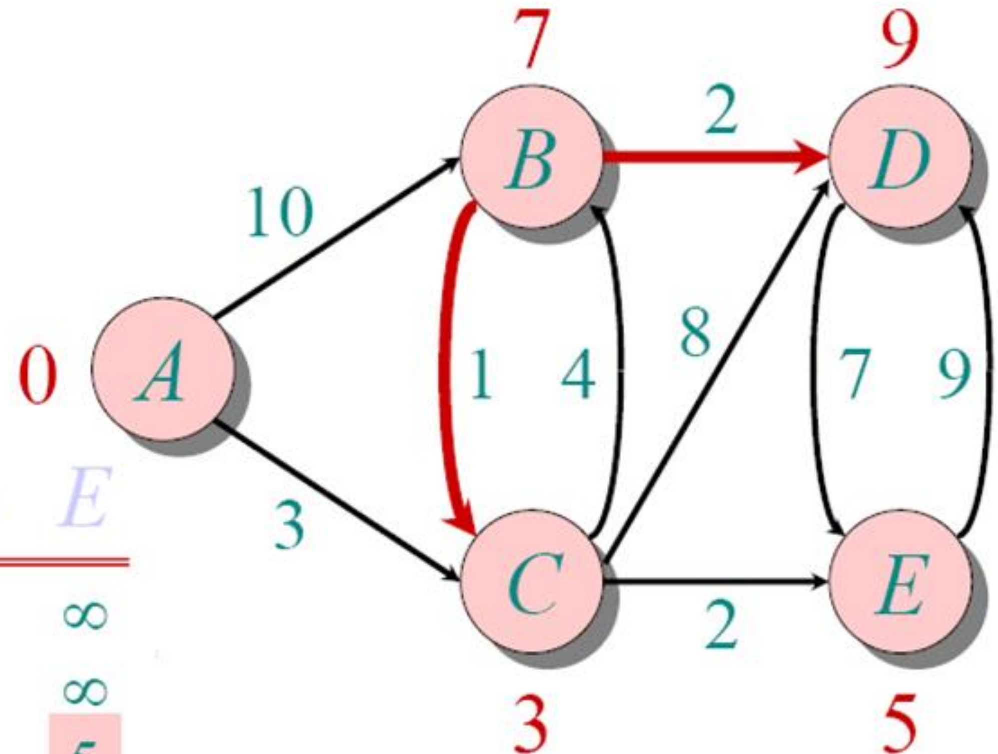
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { *A*, *C*, *E*, *B* }

Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { A, C, E, B, D }

Dijkstra 算法每次迭代要做的两件事

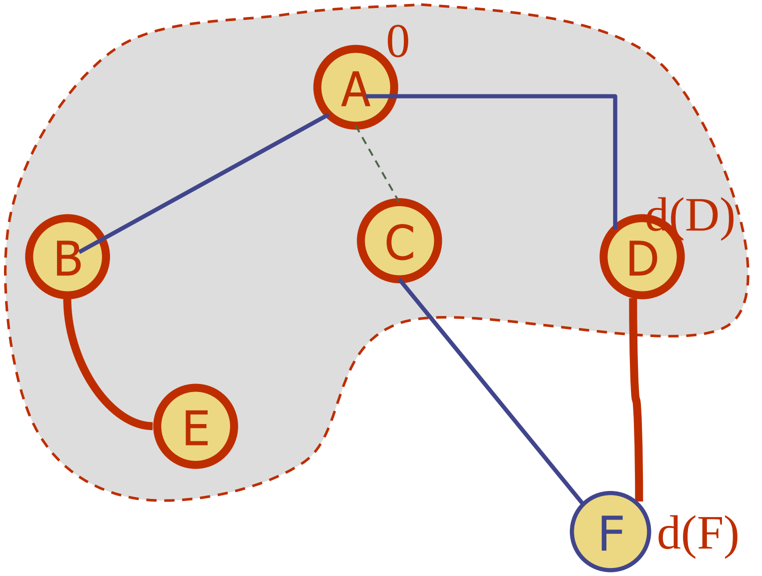
- 1. 从子集S外（“云”外）的所有与S中结点邻接的所有结点中选择标注值d(z)最小的结点**u**，加入到S中；
- 2. 考察对比与**u**结点邻接的在S之外的结点的标注值，做必要的修改。

- 注：一个结点z的标注值d(z)，当它$\neq\infty$时，　所代表的含义是：如果z在子集S中，则它是从起点到点z的最短路径的长度，也即距离；如果在子集S外则说明有某一条从起点到z的路，长度为d(z).这个值的不断修改过程就是寻找更短路的过程，直到找到最短的为止。

Why Dijkstra’s Algorithm Works ?

- Dijkstra’s algorithm is based on the greedy method. It adds vertices by increasing distance.

- Suppose it didn’t find all shortest distances. Let F be the first wrong vertex the algorithm processed.
- When the previous node D, on the true shortest path was considered, its distance was correct.
- But the edge (D,F) was **relaxed** at that time!
- Thus, so long as d(D,F)≥0 (非负边), F’s distance cannot be wrong. That is, there is no wrong vertex.

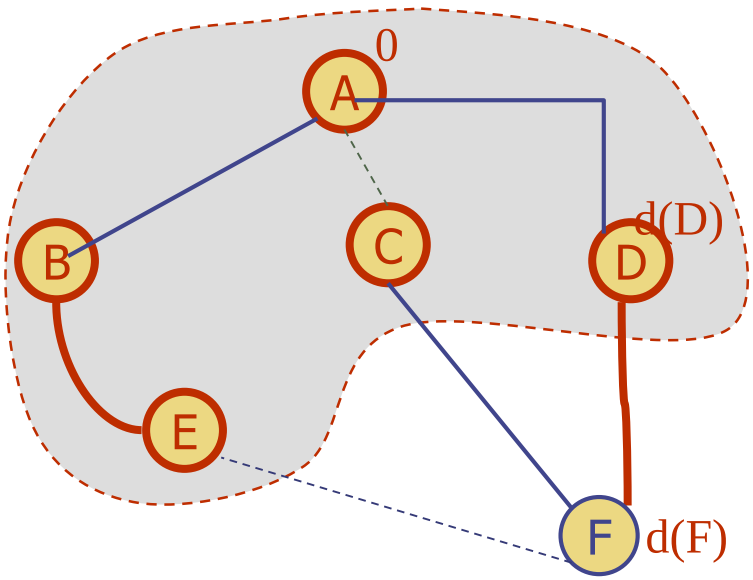


Why Dijkstra’s Algorithm Works**Works--continued**

假设F是第一个出错的结点，也即当F加入到子集的时候，d(F)并非从起点到F的最短通路的长度（并非距离）。

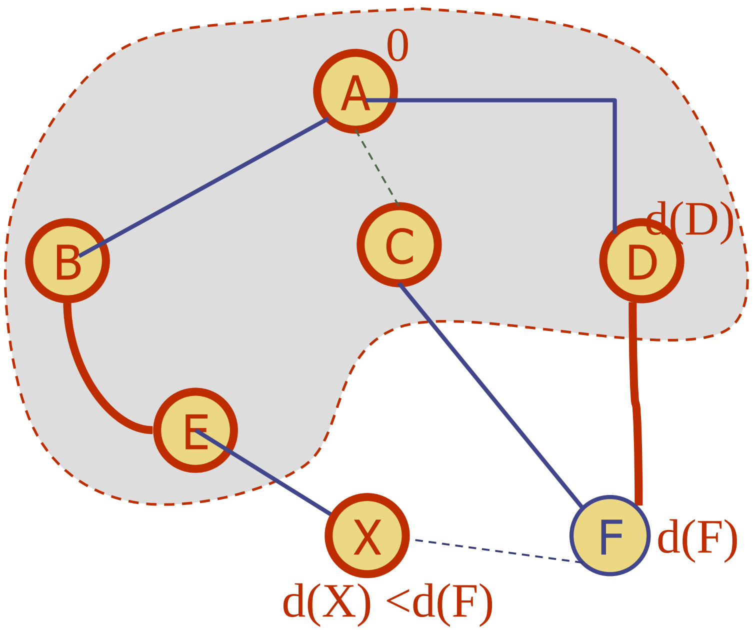
那么F之前已经加入到子集S的结点的标注值都是正确的，也即就是从起点到该结点的距离。假如在F点加入到子集S之前的最后那个结点为**D**，

（1）如果还存在另外一条更短的路径从起点到F点，如右图所示。 由于这条路径是从子集内的起点出发最后到达F点，一定有一段在S内，假设这条路径最后离开S前的那个结点为**E**；如果这条路径从E到F只有一条边，那么在**E**加入到子集S时，F的标注值就应该已经被修改成不会大于这条路径的长度，这样的路径长度不可能小于d(F)，否则与d(F)的选择有矛盾；



Why Dijkstra's Algorithm Works**Works--continued**

2）如果这条较短的路的最后一个S内的结点E到达F不是一条边，那么在S外至少还有一个不同于F的结点，如右边图中的**x**，此时从起点到x的距离必然小于这条路径的长度，当然也小于d(F)，那么x的标注值也一定是小于d(F)的。如果是这样的话，那x点应该比F点先加入到子集S（“云”）中，与F点加入到S的前提条件矛盾。

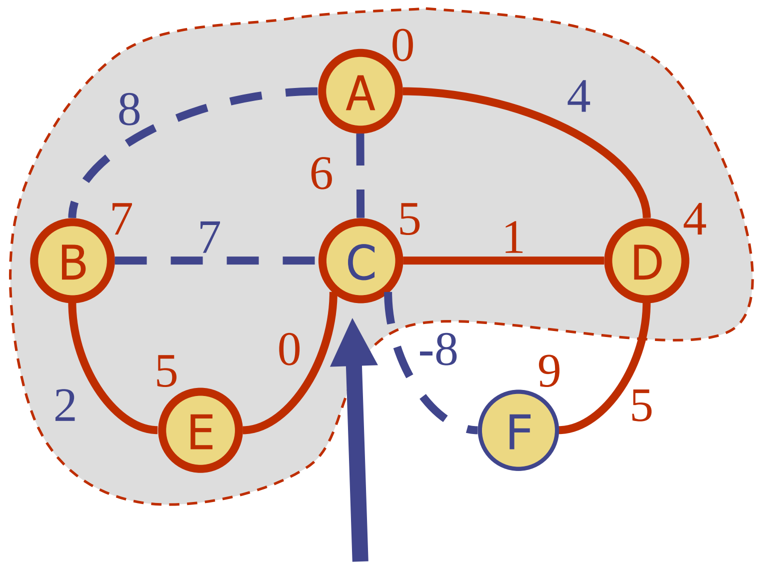


Why It Doesn't Work for Negative-Weight Edges

- Dijkstra's algorithm is based on the greedy method. It adds vertices by increasing distance.

如果一个结点与一条带负权的边关联，被加入到“云”里，就可能导致混乱。

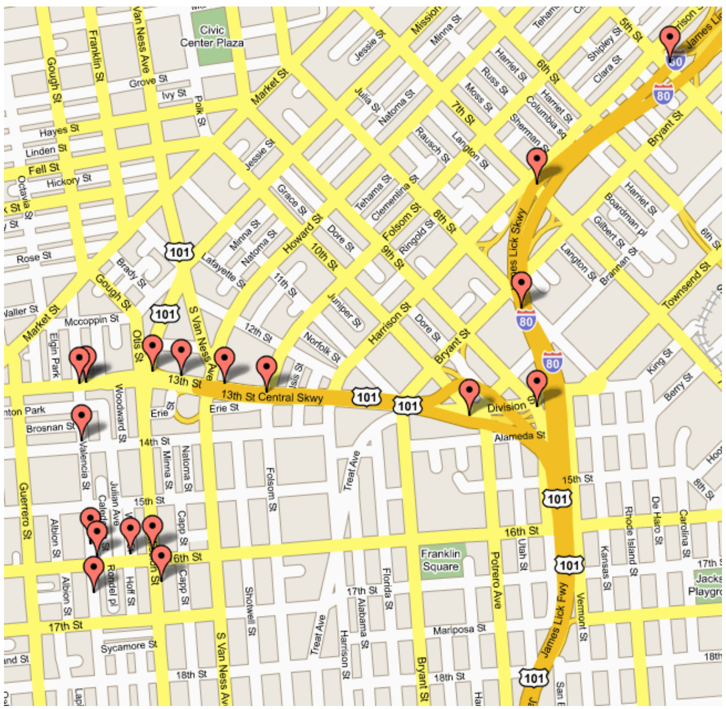
如右图所示：



C's true distance is 1, but it is already in the cloud with d(C)=5!

Applications of Dijkstra's Algorithm

- Traffic Information Systems are most prominent use
- Mapping (Map Quest, Google Maps)
- Routing Systems



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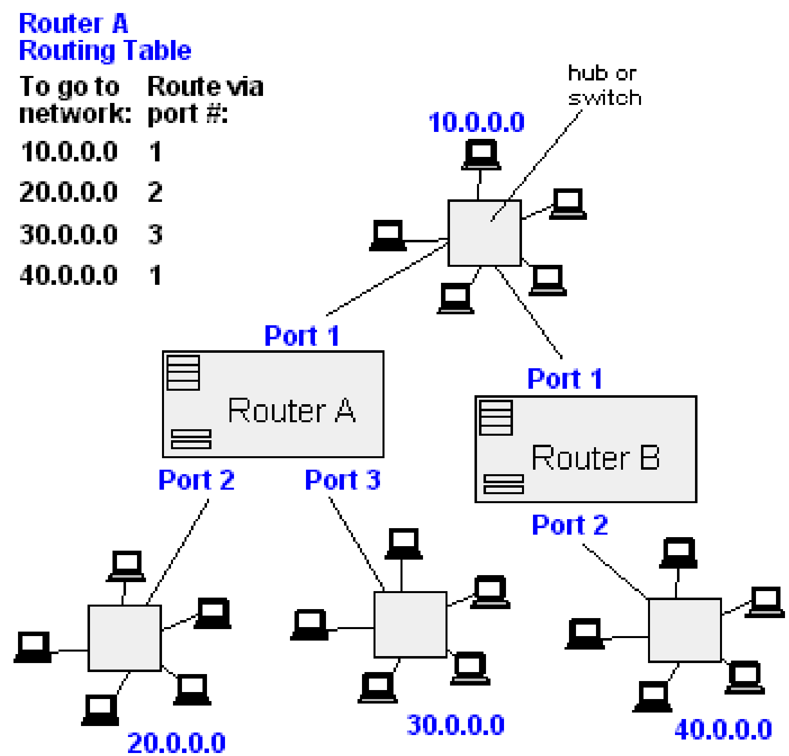


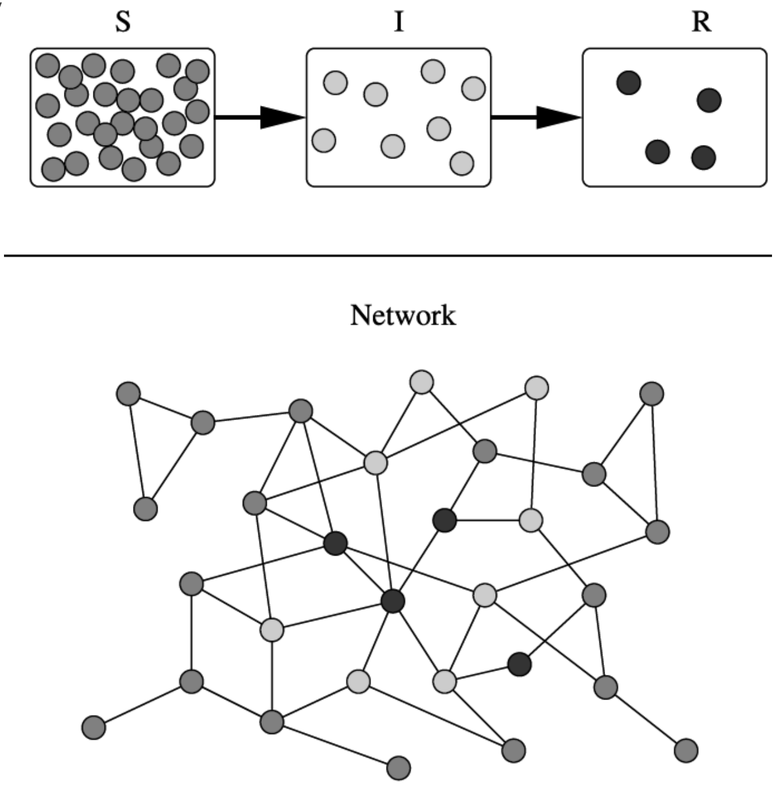
Dijkstra's 算法应用举例

- One particularly relevant: epidemiology

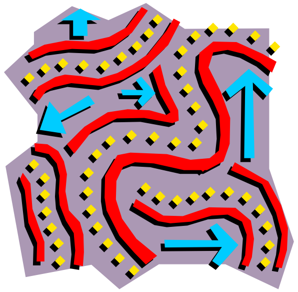
- Prof. Lauren Meyers (Biology Dept.) uses networks to model the spread of infectious diseases and design prevention and response strategies. (传染疾病防控）

- Vertices represent individuals, and edges their possible contacts. It is useful to calculate how a particular individual is connected to others.

- Knowing the shortest path lengths to other individuals can be a relevant indicator of the potential of a particular individual to infect others.

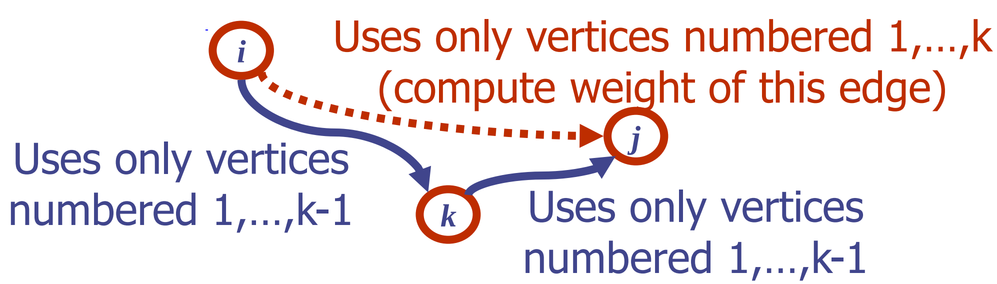


All-Pairs Shortest Paths



- Find the distance between every pair of vertices in a weighted directed graph G.
- We can make n calls to Dijkstra's algorithm (if no negative edges), which takes O(nmlog n) time.
- Likewise, n calls to Bellman-Ford would take O(n^2m) time.
- We can achieve O(n^3) time using dynamic programming (similar to the Floyd-Warshall algorithm).

Algorithm *AllPair*(*G*) {assumes vertices 1,…,***n***}
 for all *vertex pairs (i,j)*
   if $i = j$
      $D_0[i,i] \leftarrow 0$
   else if *(i,j) is an edge in G*
      $D_0[i,j] \leftarrow$ *weight of edge (i,j)*
   else
      $D_0[i,j] \leftarrow +\infty$
 for $k \leftarrow 1$ **to** n **do**
   for $i \leftarrow 1$ **to** n **do**
     for $j \leftarrow 1$ **to** n **do**
       $D_k[i,j] \leftarrow \min\{D_{k-1}[i,j], D_{k-1}[i,k]+D_{k-1}[k,j]\}$
 return D_n



- How to solve the shortest path problem when the graph has negative edges?

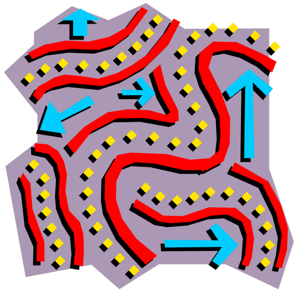
- Bellman-Ford Algorithm （求含负权图的单源最短路径算法，效率很低，但代码很容易写）

(http://blog.csdn.net/xu3737284/article/details/8973615)

- DAG-based Algorithm (http://blog.csdn.net/wall_f/article/details/8204747)

注：这两个算法自己有兴趣的话，上网去搜索学习

Bellman-Ford Algorithm

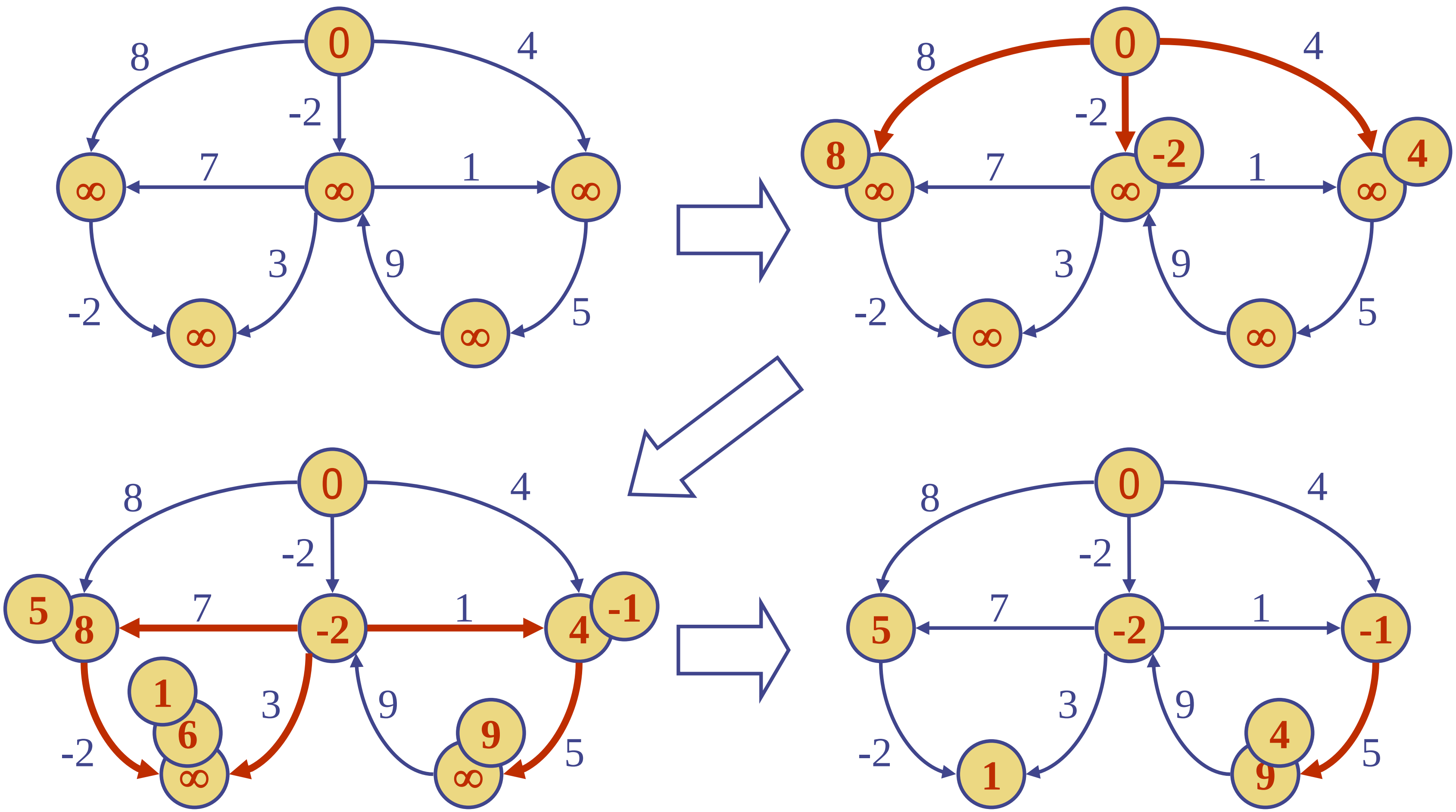


- Works even with negative-weight edges
- Must assume directed edges (有向无环) (for otherwise we would have negative-weight cycles)
- Iteration i finds all shortest paths that use i edges.
- Running time: O(nm).
- Can be extended to detect a negative-weight cycle if it exists
  - How?

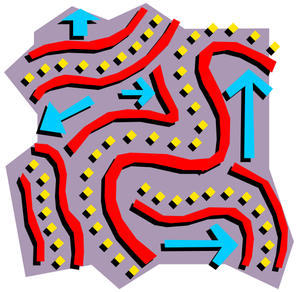
```
Algorithm BellmanFord(G, s)
  for all v ∈ G.vertices()
    if v = s
      setDistance(v, 0)
    else
      setDistance(v, ∞)
  for i ← 1 to n-1 do
    for each e ∈ G.edges()
      { relax edge e }
      u ← G.origin(e)
      z ← G.opposite(u,e)
      r ← getDistance(u) + weight(e)
      if r < getDistance(z)
        setDistance(z,r)
```

Bellman-Ford Example

Nodes are labeled with their d(v) values



DAG-based Algorithm

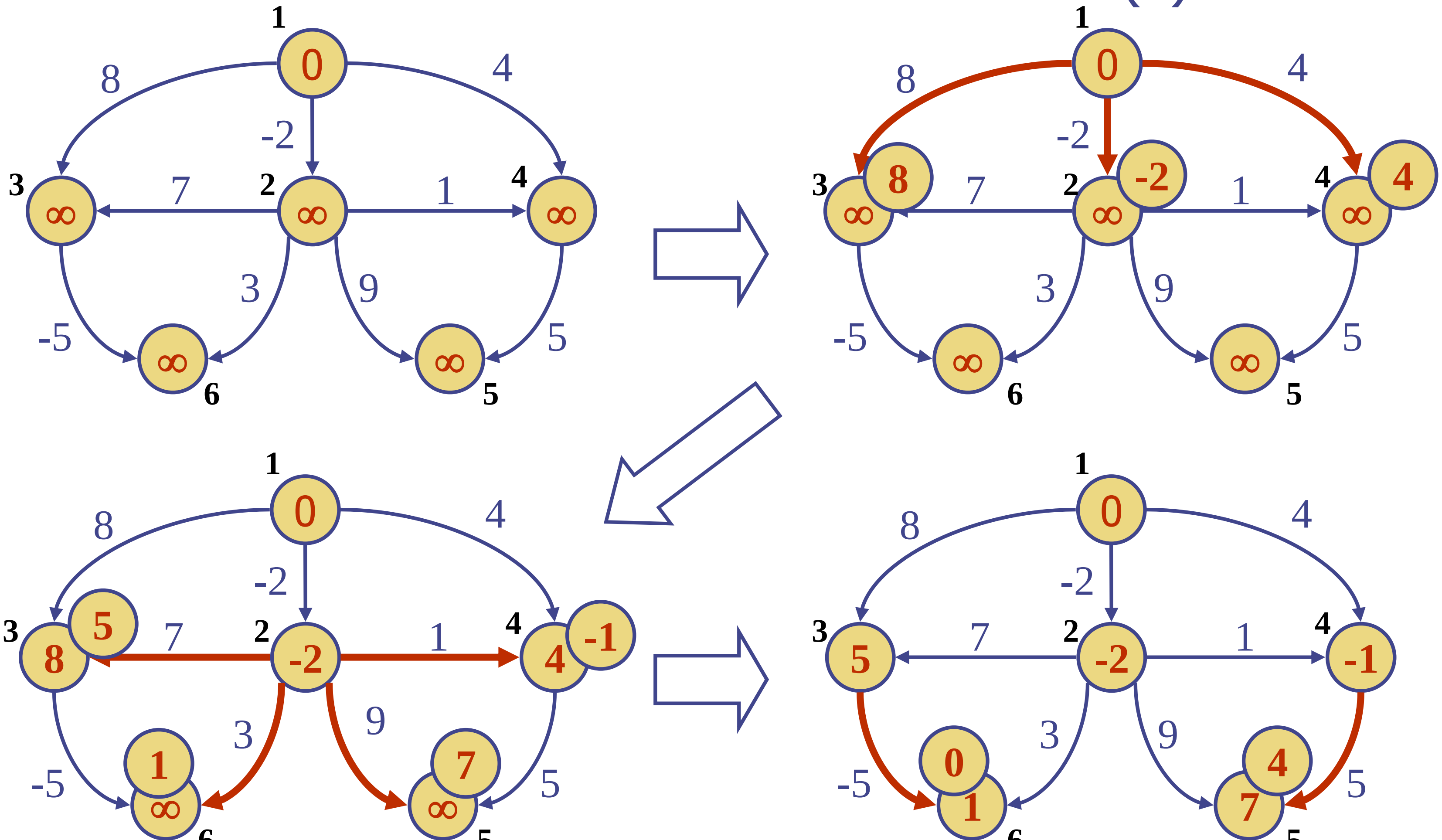


- Works even with negative-weight edges
- Uses topological order
- Doesn't use any fancy data structures
- Is much faster than Dijkstra's algorithm
- Running time: O(n+m).

```
Algorithm DagDistances(G, s)
   for all  v ∈ G.vertices()
      if  v = s
         setDistance(v, 0)
      else
         setDistance(v, ∞)
   Perform a topological sort of the vertices
   for u ← 1 to n do     {in topological order}
      for each  e ∈ G.outEdges(u)
         { relax edge e }
         z ← G.opposite(u,e)
         r ← getDistance(u) + weight(e)
         if  r < getDistance(z)
            setDistance(z,r)
```

DAG Example

Nodes are labeled with their d(v) values



(two steps)

思考Question

- 对一个普通无向连通图，For a connected simple undirected graph (non-weighted), can you figure out an algorithm to calculate the distance between a start vertex v_0 to any other vertex v?

- *Solution: you can set weight 1 to all edges of the graph*

Traveling Salesman Problem 旅行商问题

- Introduction to Traveling Salesman Problem
- 某售货员要到若干城市去推销商品，已知各城市之间的路程(或旅费)。他要选定一条从驻地出发，经过每个城市一次，最后回到驻地的路线，使总的路程(或总旅费)最小。
- 数学化的问题：在带权完全无向图里，求访问每个顶点一次只一次，且最后返回出发点，总权最小的路。这实质是求完全图里总权最小的哈密尔顿回路。
- TSP是一个NP-COMPLETE问题!

最大长度简单回路问题

- 如果把最短通路问题稍微改一下，变成寻找两个结点间的最长的简单回路，就是一个很难的问题。
- 已经证明“最大长度简单回路问题”是NP-COPMLETE问题

练习

- 6.6节 T1， T3