Extension 2 Tips and Tricks

The UNSW Mathematics Society

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1. L'Hopital's Rule

1. L'Hopital's rule

L'Hopital's Rule is a method of evaluating limits involving indeterminate forms. The rule is named after 17th century French mathematician Guillaume de l'Hopital and was published in 1696.

In it's simplest form, l'Hopital's rule states that:

if

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \text{ or } \pm\infty,$$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

1. L'Hopital's rule (continued)

Moreover, l'Hopital's rule may be applied iteratively.

That is:

if

$$\lim_{x \to c} f'(x) = \lim_{x \to c} g'(x) = 0 \text{ or } \pm\infty,$$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} = \lim_{x \to c} \frac{f''(x)}{g''(x)}.$$

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1. L'Hopital's rule: an example

Example 1:

Evaluate $\lim_{x \to 1} \dfrac{2 \log_e x}{x-1}$.

Solution: Setting $f(x) = 2\log_e x, g(x) = x - 1$, we note that

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0.$$

Then by l'Hopital's rule,

$$\lim_{x \to 1} \frac{2 \log_e x}{x-1} = \lim_{x \to 1} \frac{\frac{d}{dx}(2\log_e x)}{\frac{d}{dx}(x-1)} = \lim_{x \to 1} \frac{\frac{2}{x}}{1} = 2. \square$$

1. L'Hopital's rule (continued)

Useful note:
Identities such as

$$\lim_{k\to 0}\frac{\sin x}{x},\ \lim_{k\to 0}\frac{\tan x}{x}$$

may be computed quickly by l'Hopital's rule (meaning you need to remember less!)

$$\lim_{k\to 0}\frac{\sin x}{x} = \lim_{k\to 0}\frac{\cos x}{1} = 1$$

$$\lim_{k\to 0}\frac{\tan x}{x} = \lim_{k\to 0}\frac{\sec^2 x}{1} = 1$$

Only differentiate again if the limit is again indeterminate $\left(\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{-\infty}\right)$.

1. L'Hopital's rule (continued)

Disclaimer! l'Hopital's rule only initially appears in university level calculus and is not an "approved method" of the Board of Studies. As such, in examinations please use HSC methods for working and utilise l'Hopital's rule *only* for checking work and remembering identities.

1. l'Hopital's rule: another example

Example 2:

Evaluate $\lim_{x \to 2} \dfrac{x^3 - 2x^2 + 5x - 10}{x - 2}$.

Solution: HSC method

Step 1. Factorise $p(x) = x^3 - 2x^2 + 5x - 10$. Observe that p(2)=0, then

$$
\begin{array}{r}
x^2 \qquad\qquad + 5 \\
x - 2 \overline{\big)\; x^3 - 2x^2 + 5x - 10} \\
\underline{- x^3 + 2x^2 \qquad\qquad} \\
5x - 10 \\
\underline{- 5x + 10} \\
0
\end{array}
$$

So $p(x) = (x-2)(x^2+5)$.

1. l'Hopital's rule: another example (continued)

Step 2. Thus,

$$\begin{aligned}\lim_{x\to 2}\frac{x^3-2x^2+5x-10}{x-2} &= \lim_{x\to 2}\frac{(x-2)(x^2+5)}{x-2}\\ &= \lim_{x\to 2} x^2+5 = 9. \ \square\end{aligned}$$

1. l'Hopital's rule: another example (continued)

Alternative solution: l'Hopital's rule

Note that

$$\lim_{x \to 2} x^3 - 2x^2 + 5x - 10 = \lim_{x \to 2} x - 2 = 0.$$

Then by L'Hopital's rule,

$$\begin{aligned}
\lim_{x \to 2} \frac{x^3 - 2x^2 + 5x - 10}{x - 2} &= \lim_{x \to 2} \frac{\frac{d}{dx}(x^3 - 2x^2 + 5x - 10)}{\frac{d}{dx}(x - 2)} \\
&= \lim_{x \to 2} \frac{3x^2 - 4x + 5}{1} \\
&= 3(2)^2 - 4(2) + 5 = 9. \ \square
\end{aligned}$$

2. Advanced Combinatorics

General tips:

- For these problems, *always* set out your working clearly. In most questions, this means listing the choices to be made and the number of options in each case; and then combining these by addition or multiplication to get the final answer.
- Remember to count everything!
- Does it involve arrangements? Are they ordered? If it's probability, is it binomial?

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L'Hopital's rule: another example

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Solution: HSC method

Step 1. Factorise $p(x) = x^3 - 2x^2 + 5x - 10$. Observe that p(2)=0, then

$$\begin{array}{r|l} & \quad x^2 \qquad\qquad\quad + 5 \\ \cline{2-2} x - 2 & \quad x^3 - 2x^2 + 5x - 10 \\ & -x^3 + 2x^2 \\ \cline{2-2} & \qquad\qquad\quad 5x - 10 \\ & \qquad\qquad -5x + 10 \\ \cline{2-2} & \qquad\qquad\qquad\quad 0 \end{array}$$

So $p(x) = (x - 2)(x^2 + 5)$.

l'Hopital's rule: another example (continued)

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