A spherical rubber balloon has an initial wall thickness 0.5 mm and diamter 100 mm. It is inflated to a final diameter of 500 mm. Assume that the rubber may be modelled as a neo-Hookean material with a shear modulus of $\mu = 1.0\,\mathrm{MPa}$.

1. Calculate the final wall thickness.

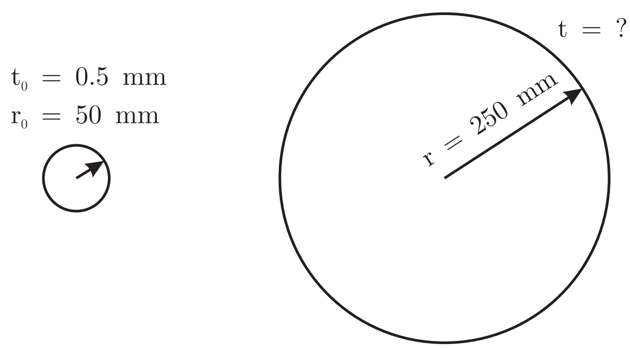
2. Plot a curve of the inflation pressure versus the circumferential stretch curve.

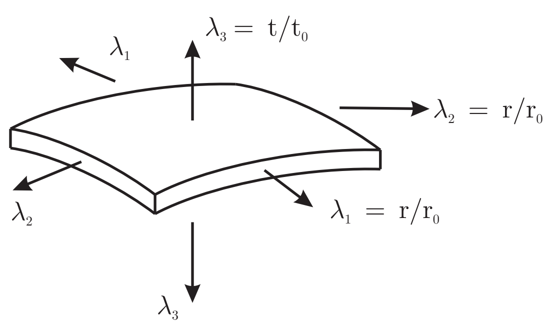
3. Calculate the maximum pressure required to inflate the balloon.

4. Calculate the pressure when the balloon has a final diameter of 500 mm.

Solution:

1. The rubber balloon before and after deformation, as well as an infinitesimal portion of the balloon surface, is diagrammed below.





As illustrated, the three principle stretches are

$$\lambda = \lambda_1 = \lambda_2 = \frac{r}{r_0}$$

$$\lambda_3 = \frac{t}{t_0}$$

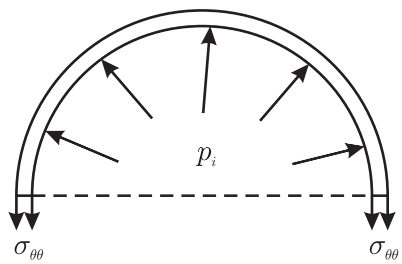
and they are related through the incompressibility condition

$$\lambda_1 \lambda_2 \lambda_3 = 1$$

Combining and rearranging the above relations gives

$$\lambda_3 = \lambda^{-2}$$
$$\frac{t}{t_0} = \left(\frac{r}{r_0}\right)^{-2}$$
$$t = t_0 \left(\frac{r_0}{r}\right)^2 = 0.02\,\mathrm{mm}$$

2. The free body diagram for the thin walled pressure vessel is shown below, and from it we can derive the pressure/stress relation.



$$\sigma_{\theta\theta}(2\pi r t) = \pi r^2 p_i$$
$$\sigma_{\theta\theta}(2t) = r p_i$$
$$\sigma_{\theta\theta} = \frac{p_i r}{2t} \tag{1}$$

The constitutive relation for a neo-Hookean material is

$$\sigma_i = \mu(\lambda_i^2 - p)$$

Using the boundary condition $\sigma_3 = 0$, we can solve for the arbitrary pressure p

$$0 = \mu(\lambda_3^2 - p) \quad \Longrightarrow \quad p = \lambda_3^2 = \lambda^{-4}$$

and hence

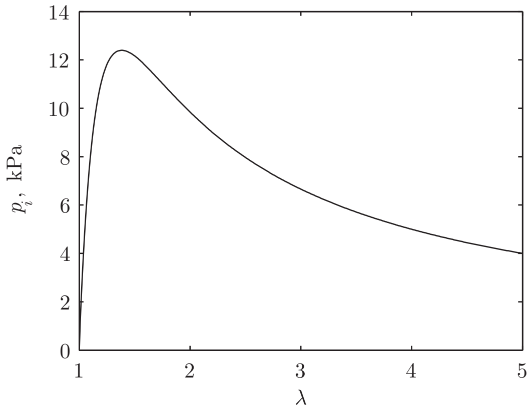
$$\sigma_{\theta\theta} = \mu(\lambda^2 - \lambda^{-4}) \tag{2}$$

Substituting (2) into (1) we get

$$p_i = 2\frac{t}{r}\mu(\lambda^2 - \lambda^{-4}) \tag{3}$$

or

$$p_i = 2\mu\frac{t_0}{r_0}\left(\frac{r_0}{r}\right)\left(\frac{t}{t_0}\right)(\lambda^2 - \lambda^{-4})$$
$$p_i = 2\mu\frac{t_0}{r_0}\lambda^{-1}\lambda^{-2}(\lambda^2 - \lambda^{-4})$$
$$\Longrightarrow \quad \boxed{p_i = 2\mu\frac{t_0}{r_0}(\lambda^{-1} - \lambda^{-7})} \tag{4}$$



3. For a maximum

$$\frac{dp_i}{d\lambda} = 2\mu\frac{t_0}{r_0}\left(-\lambda^{-2} + 7\lambda^{-8}\right) = 0$$
$$\left(-\lambda^{-2} + 7\lambda^{-8}\right) = 0$$
$$7\lambda^{-8} = \lambda^{-2}$$
$$\lambda^{-6} = 1/7$$
$$\lambda = 1.3831$$

Using this stretch in (4)

$$p_{i,\max} = 2\left(1 \times 10^6\right)\left(\frac{0.5}{50}\right)\left(1.3831^{-1} - 1.3831^{-7}\right)$$
$$\boxed{p_{i,\max} = 12.4\,\mathrm{kPa}}$$

4. At the final inflation diameter, we have $\lambda = 5$, and using this in (4)

$$p_{i,\mathrm{final}} = 2\left(1 \times 10^6\right)\left(\frac{0.5}{50}\right)\left(5^{-1} - 5^{-7}\right)$$
$$\boxed{p_{i,\mathrm{final}} = 4\,\mathrm{kPa}}$$