COMP 1201 Algorithmics

Shortest Path

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Single Source Shortest Path

- Given a graph $G = (V, E)$ with a real-valued weight function $w : E \rightarrow \mathbb{R}$ we often ask the question:
- What is the minimal cost (shortest) path from $s \in V$ to all other vertices of the graph.
- We will look at two algorithms that perform that task
    1. Bellman-Ford.
    2. Dijkstra.
- First we need some definitions and theorems.

- Let $G = (V, E)$ with the associated weight function $w : E \to \mathbb{R}$.
- The total weight of a path $p = (v_0, \ldots, v_k)$ is the sum of weights of individual edges:

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

- The shortest path cost δ

$$\delta(u, v) = \begin{cases} \min_p w(p) & \text{if there at least one path from u to v} \\ \infty & \text{otherwise} \end{cases}$$

Negative weights and cycles

- Even if a path contains edges with negative weight a shortest path can still be defined.
- It is undefined if the path contains a negative weight **cycle**.
- This is because we can "cross" the cycle as many times as we want, every time lower the cost.
- Therefore in the case when there is a negative cycle on a path from u to v then we set $\delta(u, v) = -\infty$ where $\delta(a, b)$ is the shortest path cost from a to b.

Example of Negative Cycles

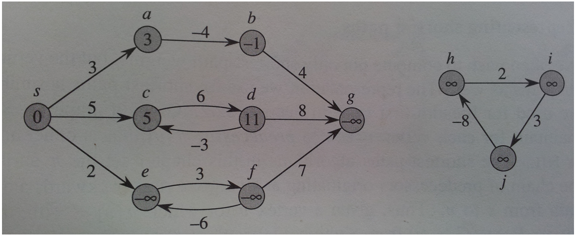


Figure: from CLRS

- $\delta(s,a)=3$,$\delta(s,b)=-1$,$\delta(s,c)=5$,$\delta(s,d)=11$.
- (e,f) form a negative cycle therefore any node reachable from s through this cycle has $\delta=-\infty$
  $\delta(s,e)=\delta(s,f)=\delta(s,g)=-\infty$
- h,i,j are not reachable from s thus
  $\delta(s,h)=\delta(s,i)=\delta(s,j)=\infty$

Representation of Shortest Paths

- In all the algorithms that we will deal with, we maintain for every vertex v its predecessor $v.\pi$ (which could be NULL)
- At **termination** $v.\pi$ will be the predecessor of v on a shortest path from source s to v.
- We also maintain a value $v.\delta$ which at termination will be the value of the shortest path cost from source s to v.
- During the execution of the algorithm $v.\delta$ will be **an upper bound** on the value of the shortest path cost.

Relaxation

- **Relaxing** an edge (u, v) means testing if we can improve the shortest path cost of v by using the edge (u, v).
- If we can then we update $v.\delta$ and $v.\pi$.

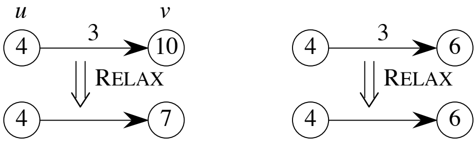


Figure: from CLRS

- In the figure to the left the cost of v was changed to the new cost (7) whereas to the right it was not changed since the new cost (7) is bigger than the current (6).
- What is NOT shown is the change to $v.p$ in the first case.

Initialization and Relaxation

- Initially all vertices (except the source) have cost ∞ and no predecessors (including the source).

---

function `INITIALIZE(G,s)`
  foreach $v \in V$ **do**
    $v.\delta \leftarrow \infty$
    $v.\pi \leftarrow NULL$
  $s.d \leftarrow 0$

---

function `RELAX(u,v)`
  if $v.\delta > u.\delta + w(u,v)$ **then**
    $v.\delta \leftarrow u.\delta + w(u,v)$
    $v.\pi \leftarrow u$

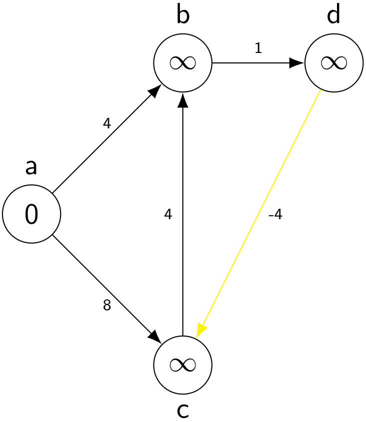
Bellman-Ford Algorithm

- The Bellman-Ford algorithm computes the shortest path from a given source to all other nodes in the graph.
- It works with negative weights and can detect negative cycles.
- It uses the function RELAX to compute the shortest path.
- RELAX is applied to all edges in the graph $|V| - 1$ times.

Example

First iteration

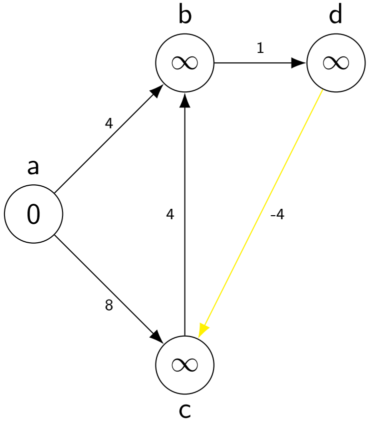
Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



Node	δ	π
a	0	NIL
b	∞	NIL
c	∞	NIL
d	∞	NIL

Example

Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



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Example

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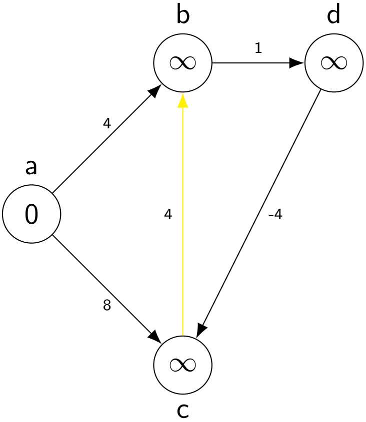

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Example

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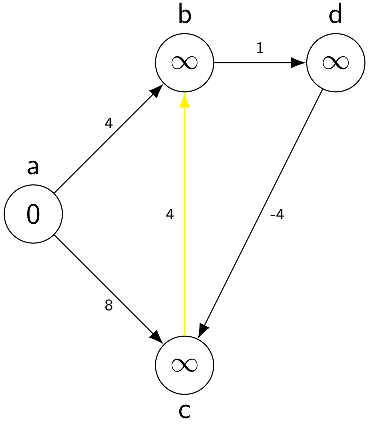

Node	δ	π
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Example

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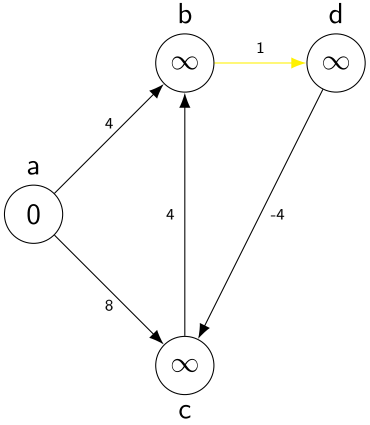

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Example

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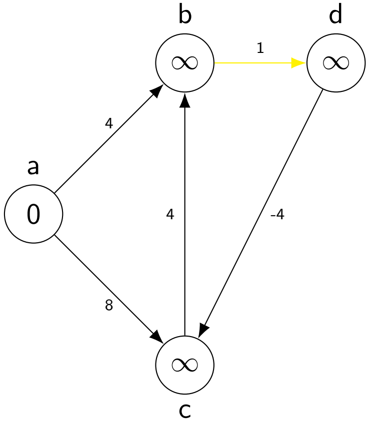

Node	δ	π
a	0	NIL
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d	∞	NIL

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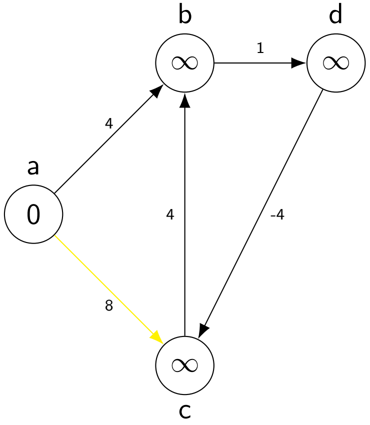
Example

Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



Node	δ	π
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Example

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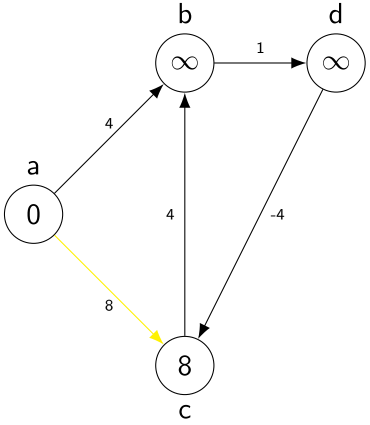
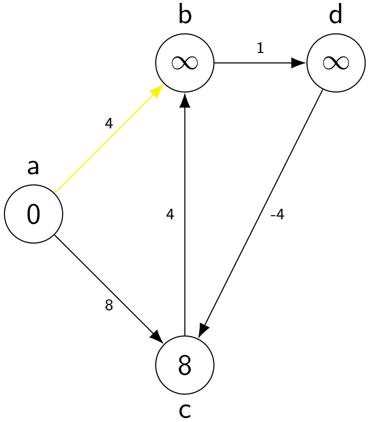


Node	δ	π
a	0	NIL
b	∞	NIL
c	8	a
d	∞	NIL

Example

Consider edges in order $\{$(d,c),(c,b),(b,d),(a,c),(a,b)$\}$



Node	δ	π
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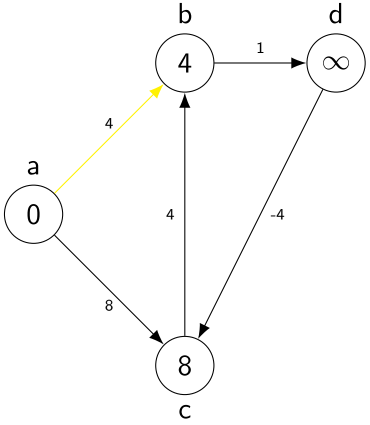
Example

Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}

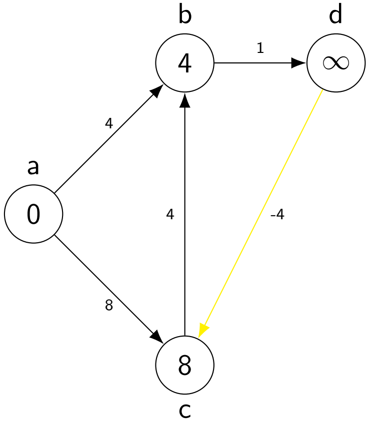


Node	δ	π
a	0	NIL
b	4	a
c	8	a
d	∞	NIL

Example

Second iteration

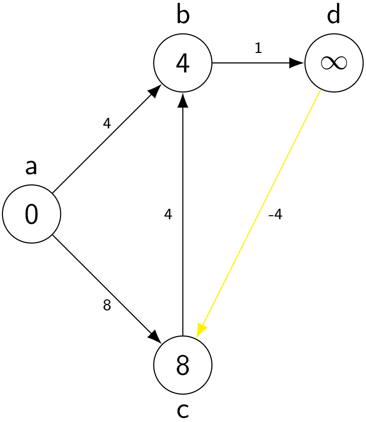
Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



Node	δ	π
a	0	NIL
b	4	a
c	8	a
d	∞	NIL

Example

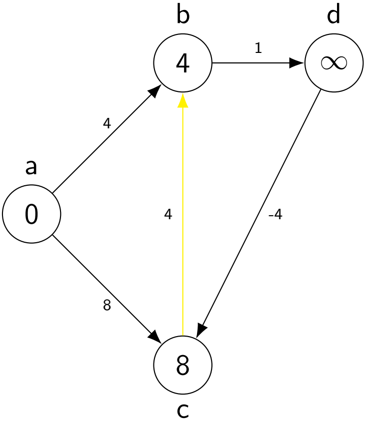
Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



Node	δ	π
a	0	NIL
b	4	a
c	8	a
d	∞	NIL

Example

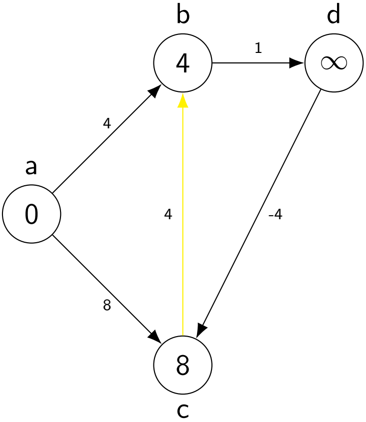
Consider edges in order $\{(d,c),(c,b),(b,d),(a,c),(a,b)\}$



Node	δ	π
a	0	NIL
b	4	a
c	8	a
d	∞	NIL

Example

Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



Node	δ	π
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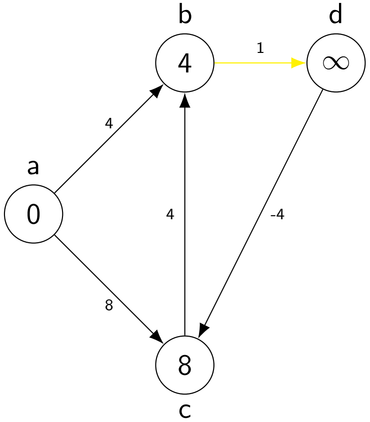
Example

Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



Node	δ	π
a	0	NIL
b	4	a
c	8	a
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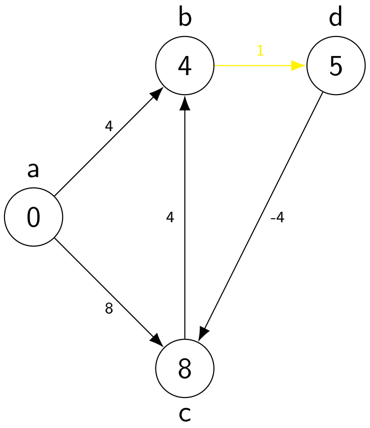
Example

Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



Node	δ	π
a	0	NIL
b	4	a
c	8	a
d	5	b

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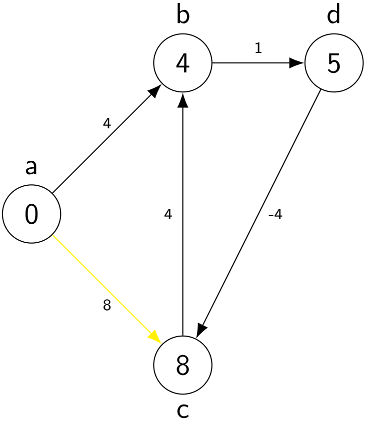
Example

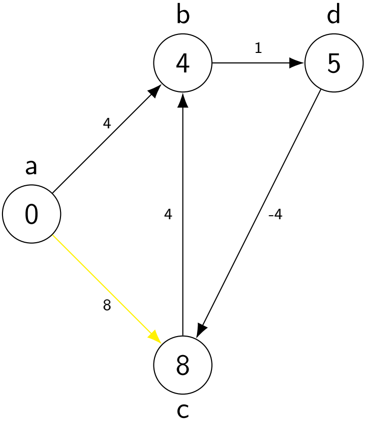
Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



Node	δ	π
a	0	NIL
b	4	a
c	8	a
d	5	b

Example

Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



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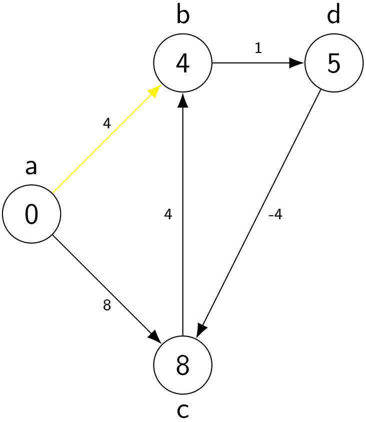

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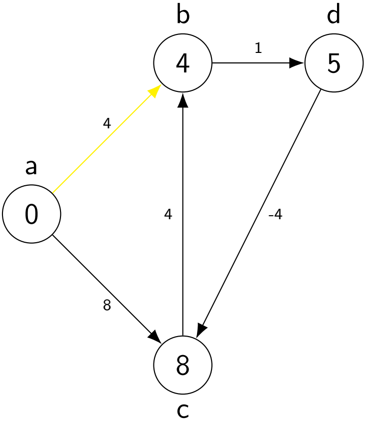
Example

Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}

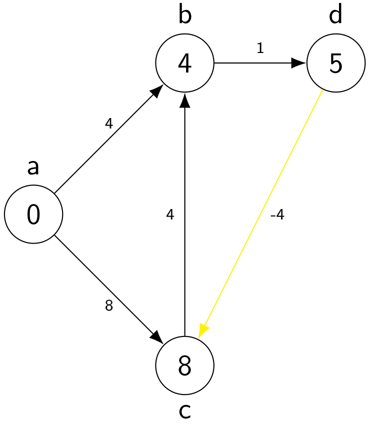


Node	δ	π
a	0	NIL
b	4	a
c	8	a
d	5	b

Example

Third iteration

Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



Node	δ	π
a	0	NIL
b	4	a
c	8	a
d	5	b

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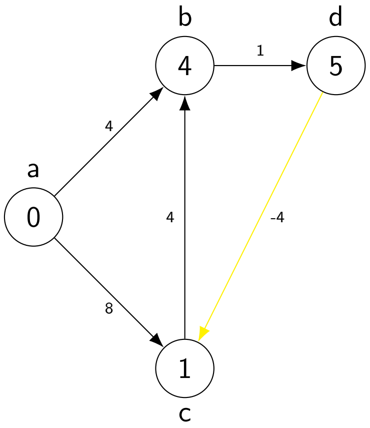
Example

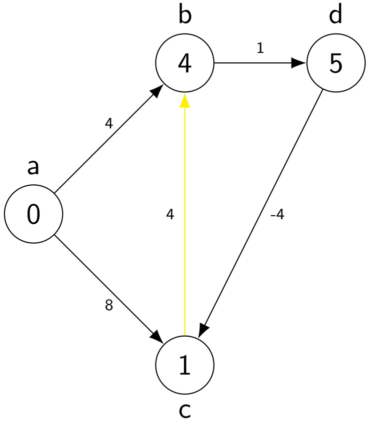
Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



Node	δ	π
a	0	NIL
b	4	a
c	1	d
d	5	b

Example

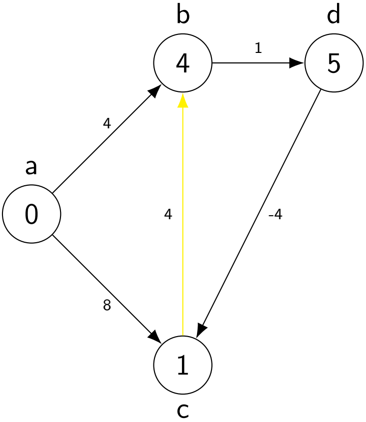
Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



Node	δ	π
a	0	NIL
b	4	a
c	1	d
d	5	b

Example

Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



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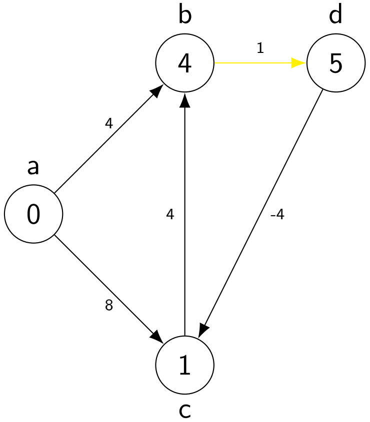

Node	δ	π
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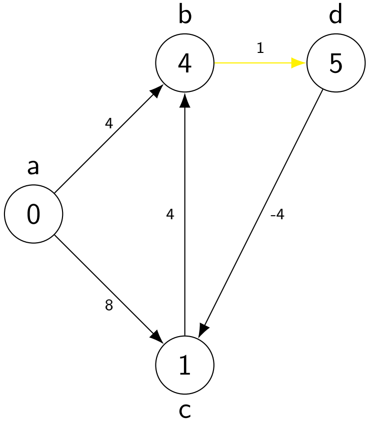
b	4	a
c	1	d
d	5	b

Example

Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



Node	δ	π
a	0	NIL
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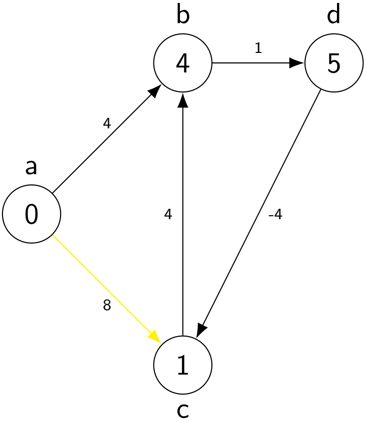
Example

Consider edges in order {(d,c),(c,b),(b,d),(a,c),(a,b)}



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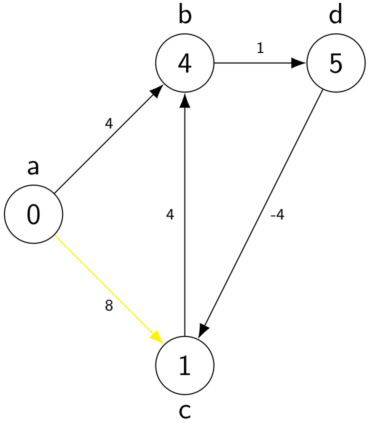

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Example

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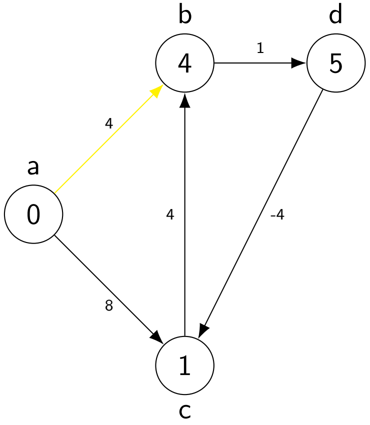

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Example

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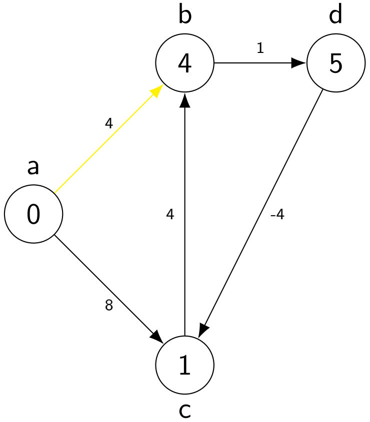


Node	δ	π
a	0	NIL
b	4	a
c	1	d
d	5	b

Pseudo Code

Algorithm 1: Bellman-Ford algorithm

```
INITIALIZE(G,s)
for i ← 1 to V − 1 do
    foreach (u, v) ∈ E do
        RELAX(u,v)
```

- The number of iterations depends on the order in which edges are "relaxed" but at it is at most (n-1)
- In the previous example 3 iterations were needed (which is the maximum since there are 4 vertices)

Complexity of Bellman-Ford

- The initialization is $O(|V|)$.
- the double loop is $O(|V| \cdot |E|)$.
- Therefore the total cost of the Bellman-Ford is $O(|V| \cdot |E|)$.
- Bellman-Ford is slower than Dijkstra's algorithm but it can handle negative weights.
- There is another formulation of Bellman-Ford using dynamic programming

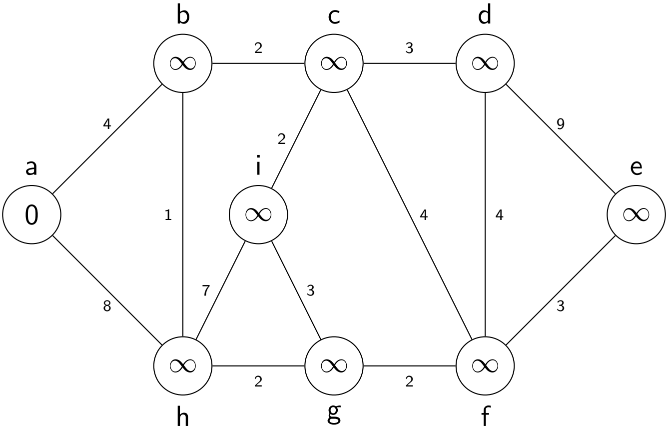
Dijkstra's Algorithm

- Dijkstra's algorithm is another single source shortest path.
- It works when all weights are **positive**.
- We will see that it is faster than the Bellman-Ford algorithm.
- It maintains a set S of nodes whose shortest paths have been determined
- All other nodes are kept in a min-priority queue to keep track of the next node to process.

Example

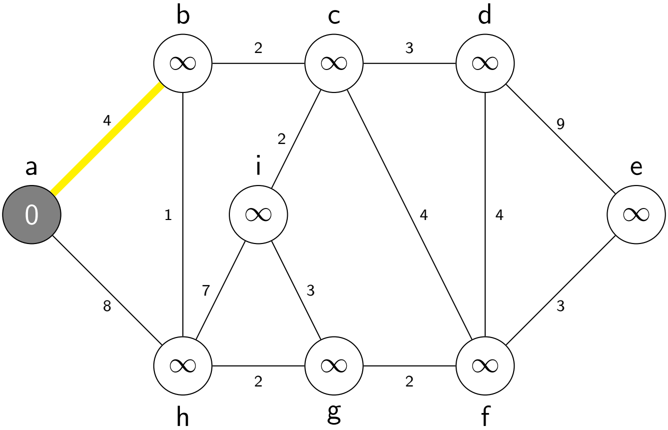
Example

a is source



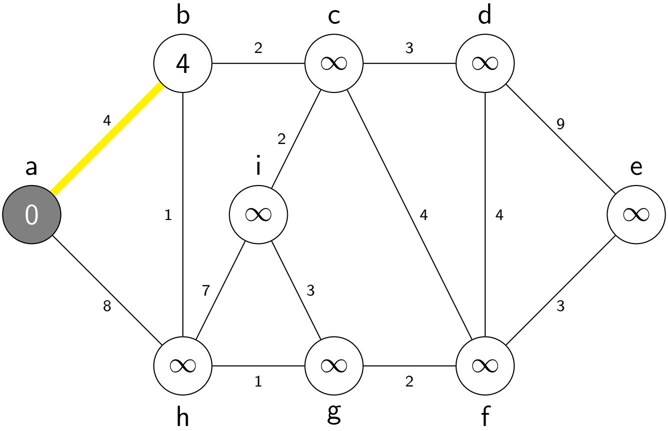
Example

neighbors of a



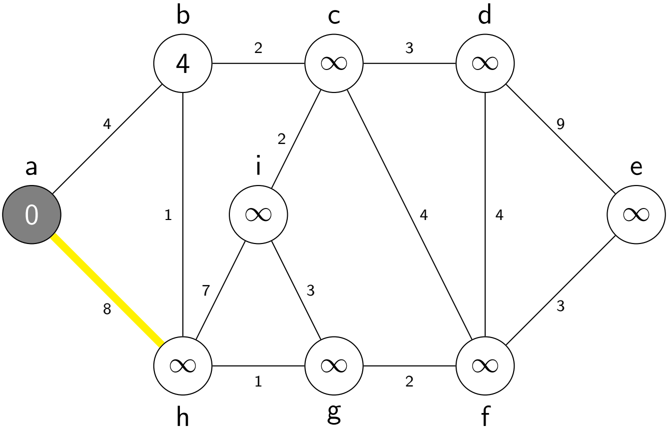
Example

neighbors of a



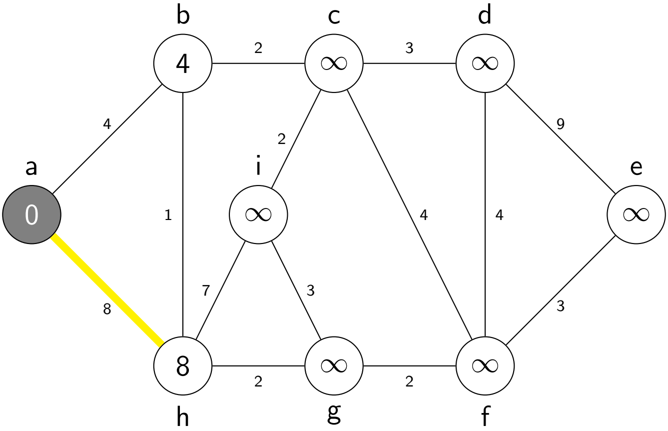
Example

neighbors of a



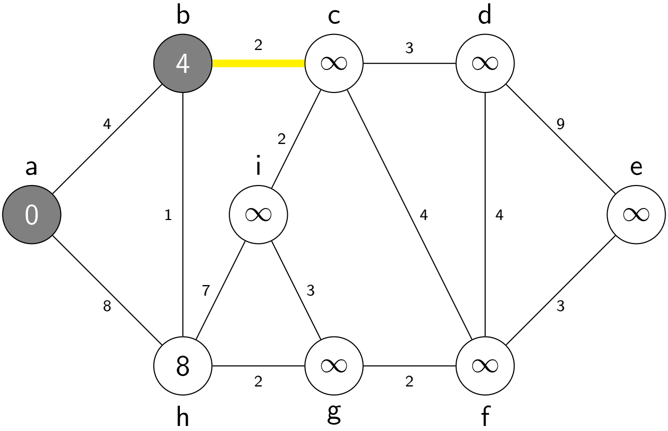
Example

neighbors of a



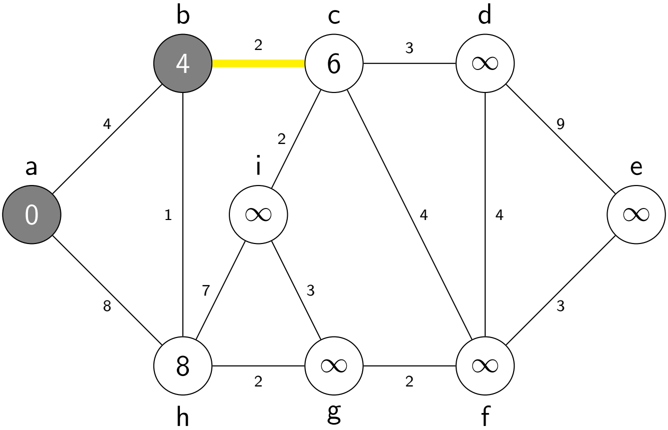
Example

neighbors of b



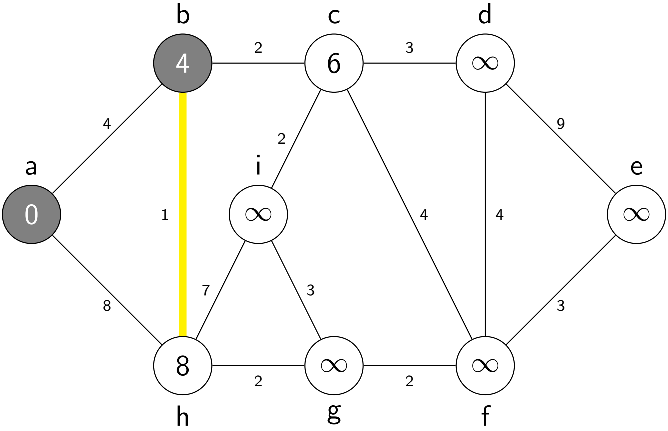
Example

neighbors of b



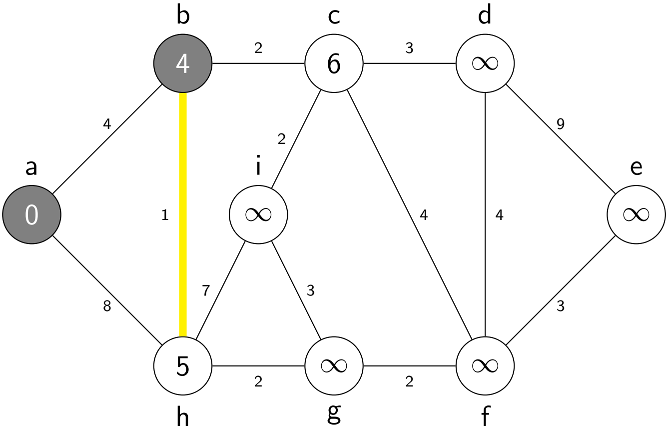
Example

neighbors of b



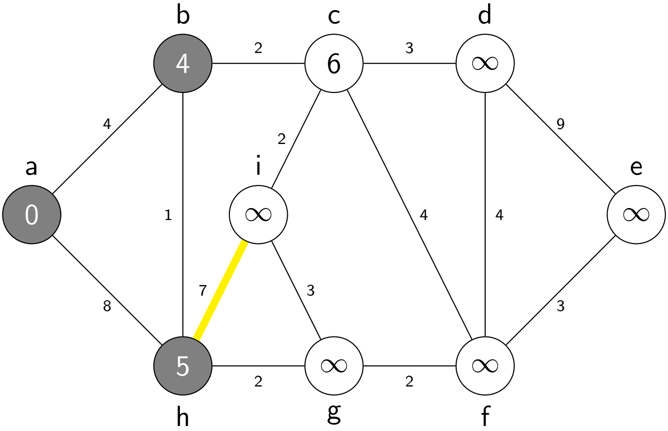
Example

neighbors of b



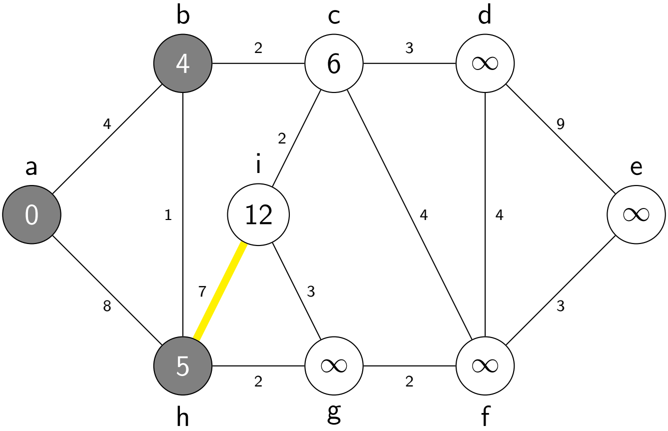
Example

neighbors of h



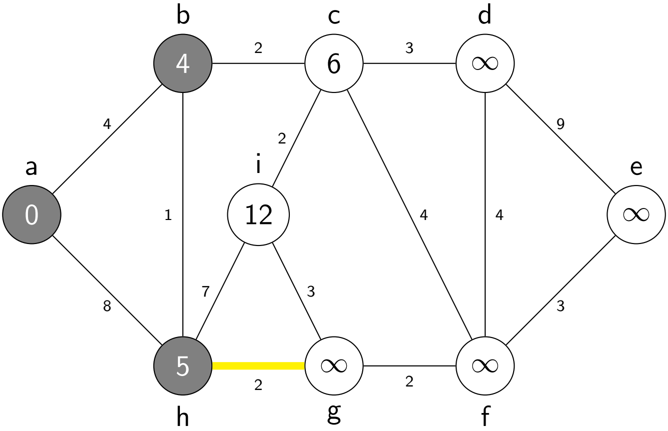
Example

neighbors of h



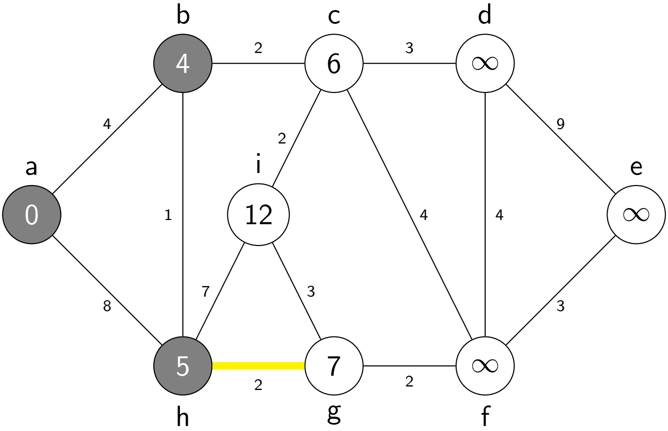
Example

neighbors of h



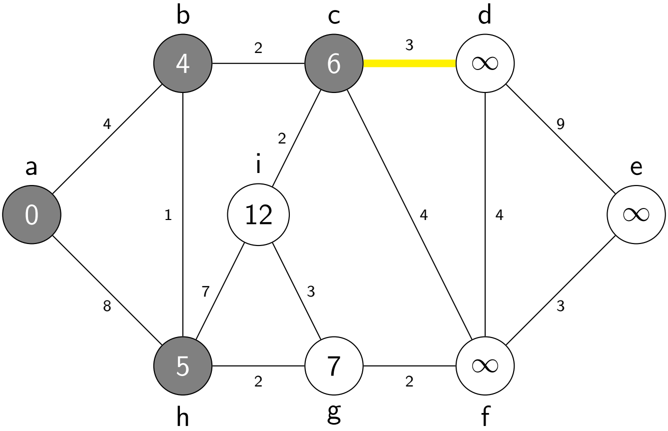
Example

neighbors of h



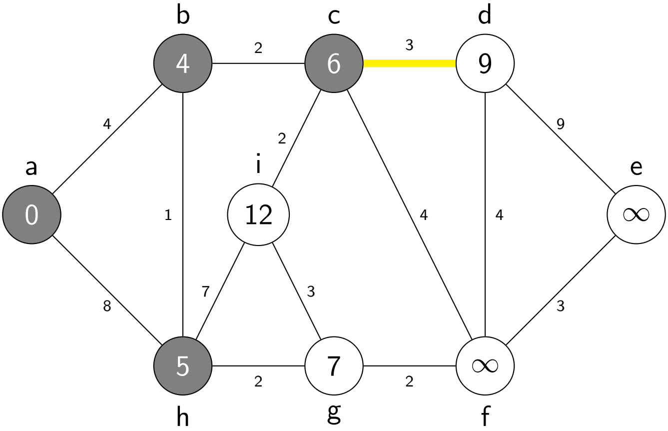
Example

neighbors of c



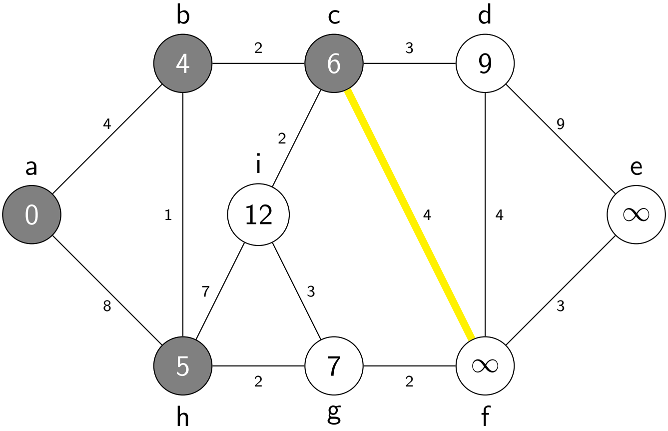
Example

neighbors of c



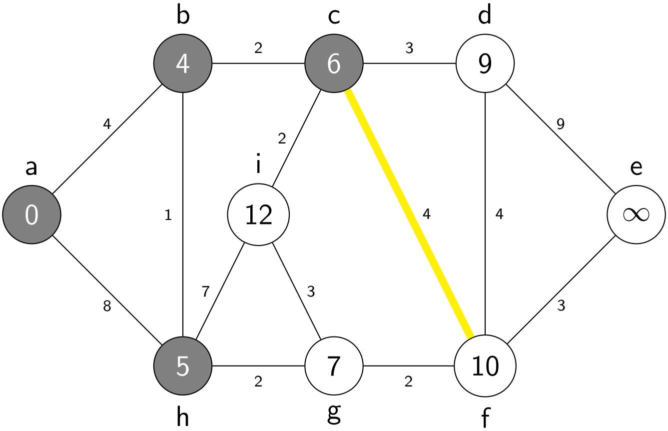
Example

neighbors of c



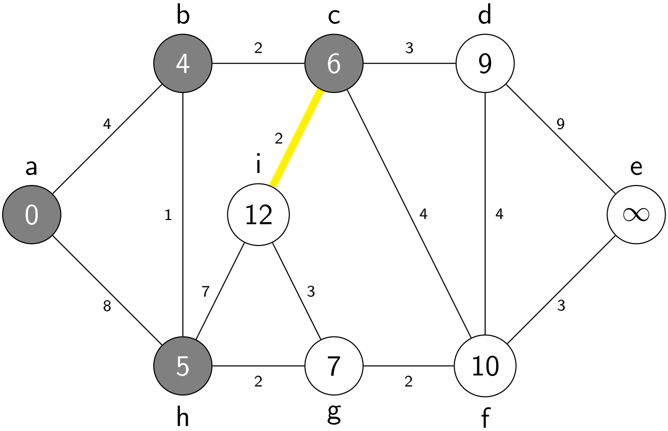
Example

neighbors of c



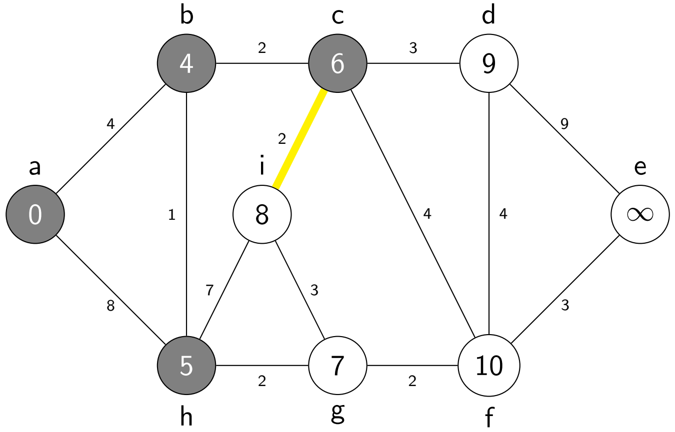
Example

neighbors of c



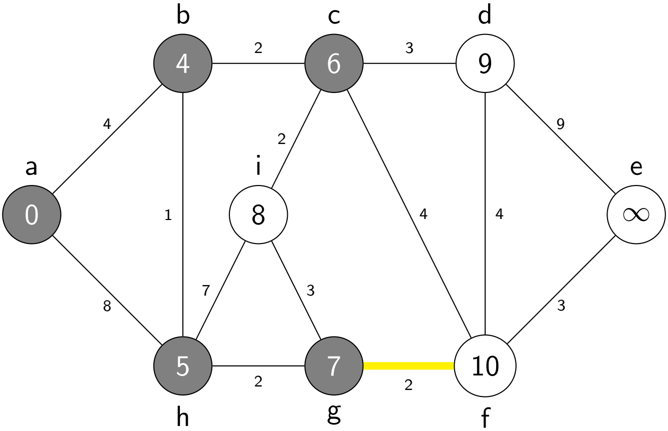
Example

neighbors of c



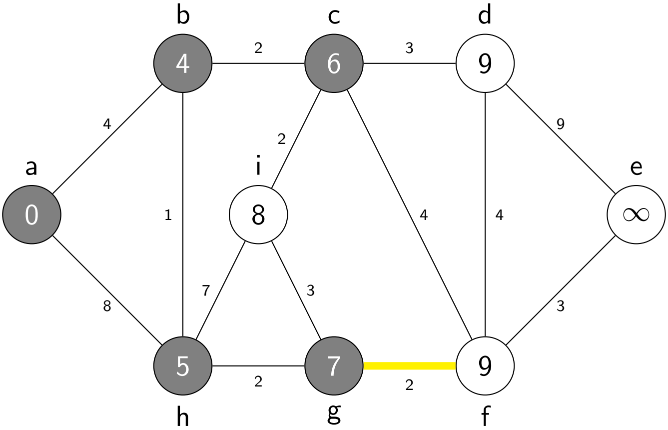
Example

neighbors of g



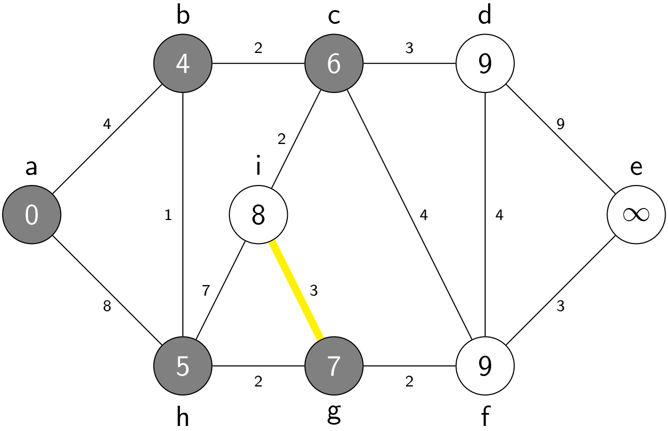
Example

neighbors of g



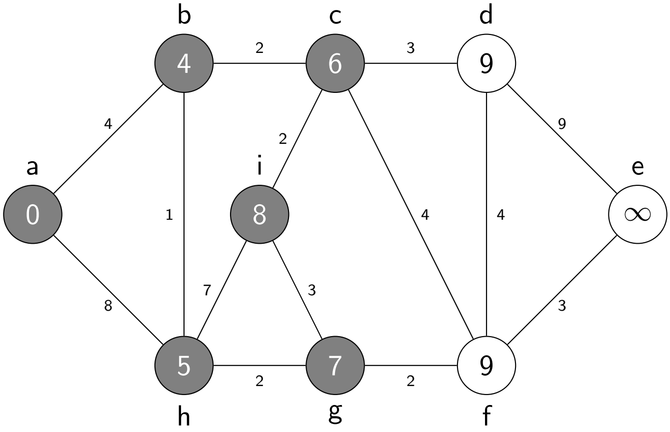
Example

neighbors of g



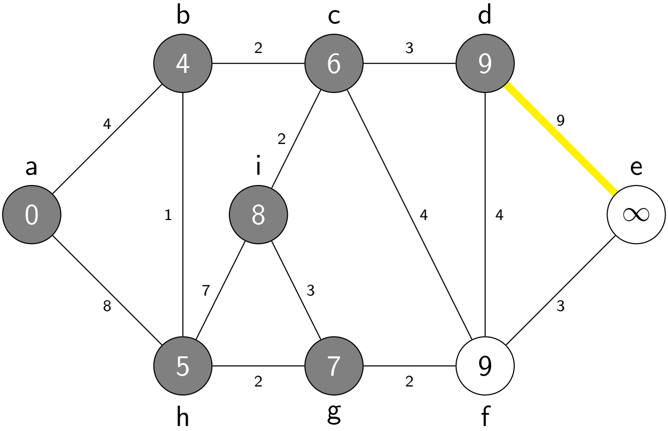
Example

neighbors of i

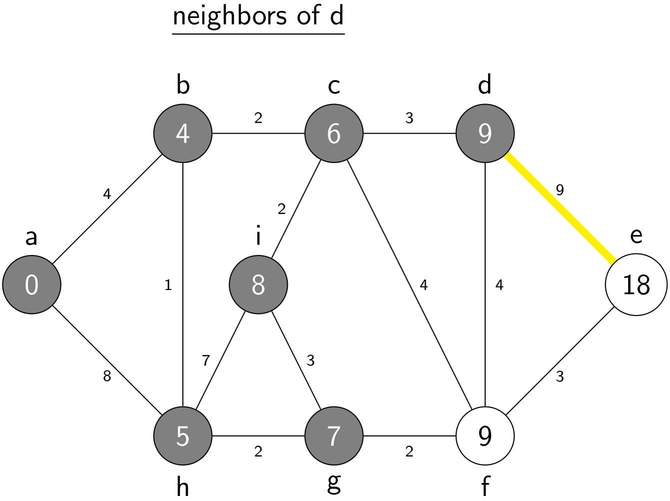


Example

neighbors of d

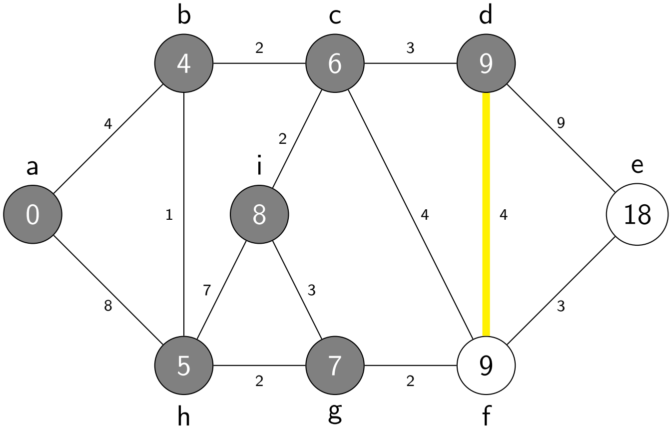


Example



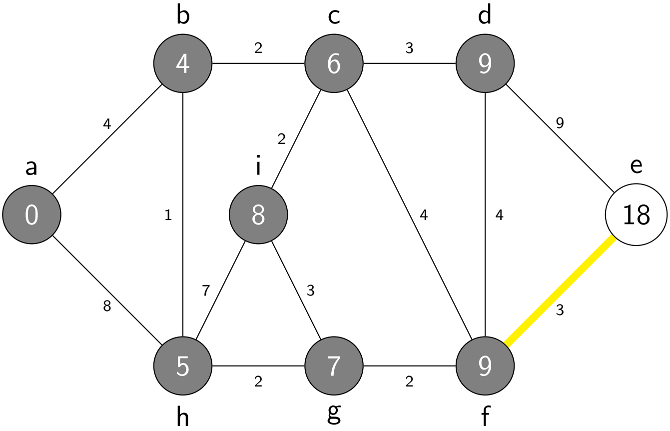
Example

neighbors of d



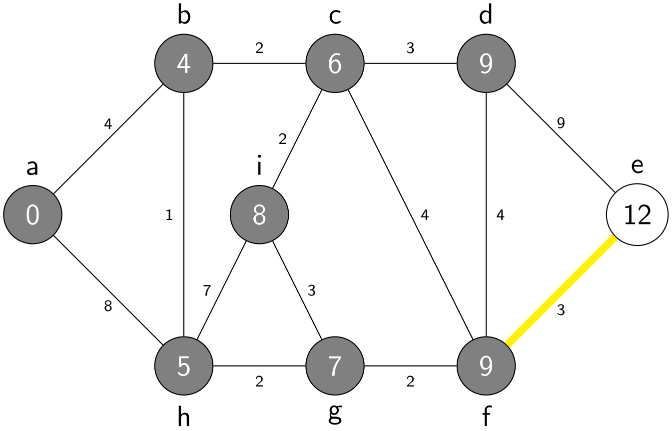
Example

neighbors of f



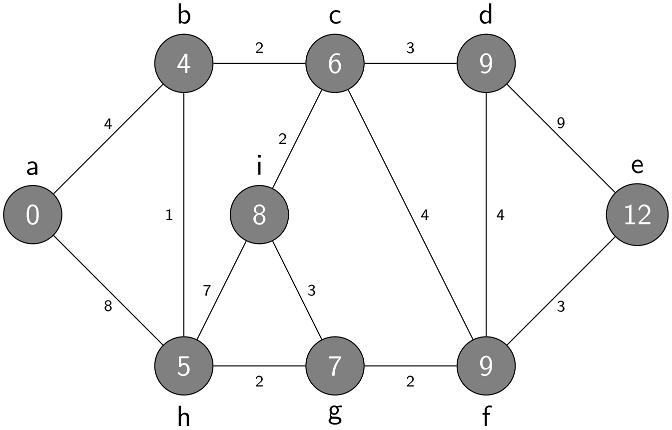
Example

neighbors of f



Example

Done



Complexity

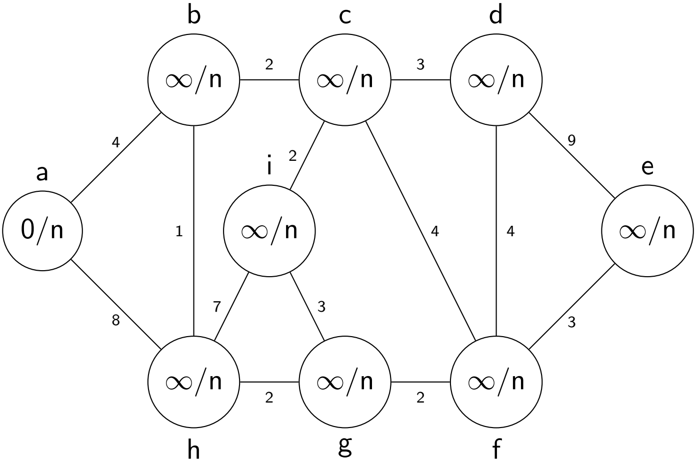
- The running time of Dijkstra's algorithm depends on the implementation of the queue.
- Using a min-heap on a sparse graph gives complexity of $O((V+E)\log V)$.
- This is because the while loop executes V times. The extract-min is $O(\log V)$ for a cost of $V \log V$. The relax includes an key update which means $\log V$. Since each edge is relaxed at most once then the total is E with a cost of $E \log V$.

And the shortest PATH?

- The algorithm computes the shortest distance from the source to all other nodes.
- But in most situations we are interested in actual shortest path from the source to the destinations.
- This can be done by updating and saving the predecessor of each node.

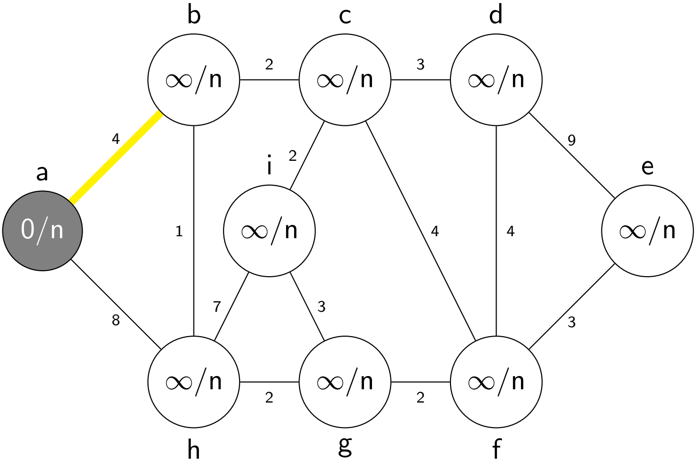
Example

a is source

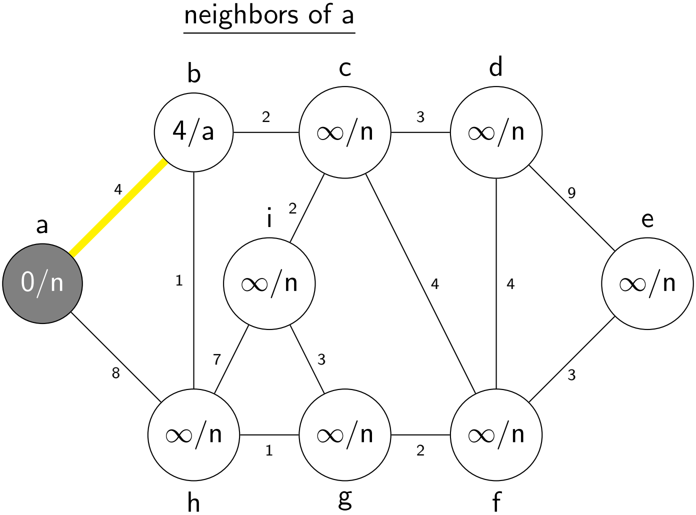


Example

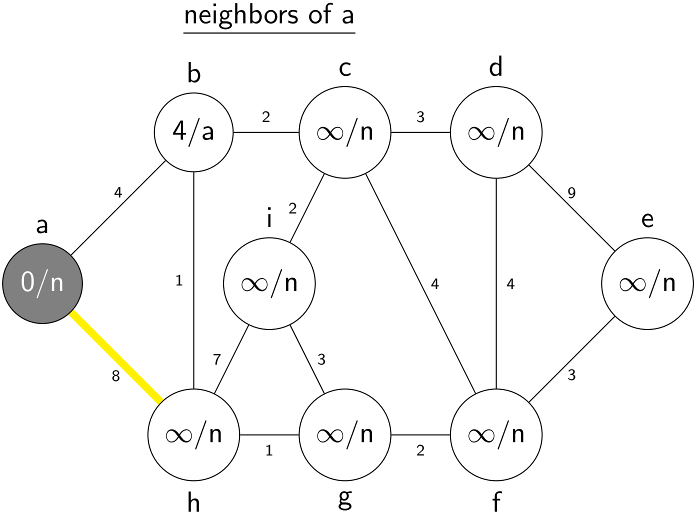
neighbors of a



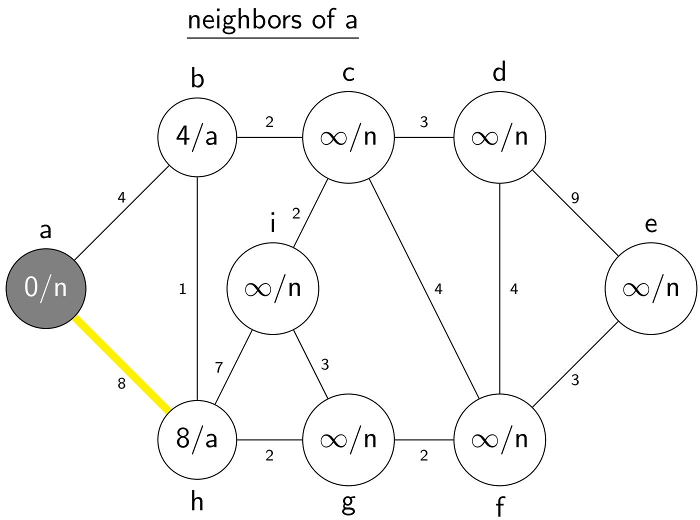
Example



Example



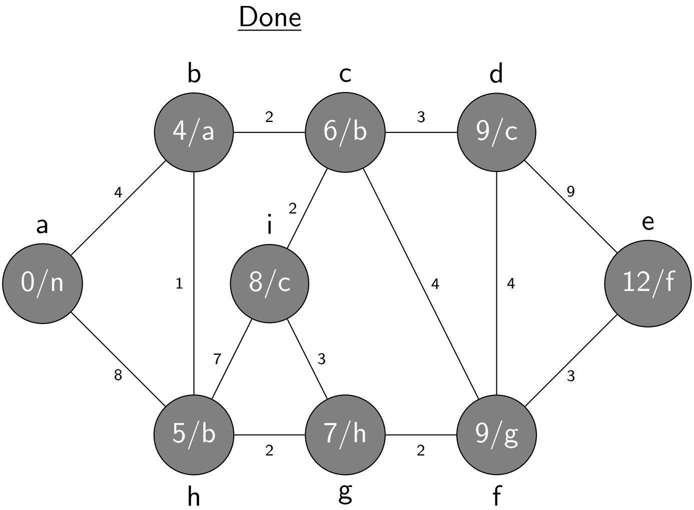
Example



Example

ETC...

Example



Recovering the shortest path

- By iterating backwards over the predecessors we can recover the shortest path.
- For example, consider node e
- $e.p = f, f.p = g, g.p = h, h.p = b, b.p = a$
- So the shortest path from a to e is a, b, h, g, f, e

Dijkstra Pseudo Code

```
DIJKSTRA(G,s);
INITIALIZE(G,s)
S ← ∅
Q ← V
while Q ≠ ∅ do
    u ← EXTRACT-MIN(Q)
    S ← S ∪ {u}
    foreach v ∈ adj[u] do
        RELAX(u,v)
```

Complexity

Let $n = |V|$ and $m = |E|$ and Q is a min-heap.

	DIJKSTRA(G,s);	
	INITIALIZE(G,s)	$O(n)$
1	$S \leftarrow \emptyset$	$O(1)$
2	$Q \leftarrow V$	$O(n)$
3	**while** $Q \neq \emptyset$ **do**	$O(n)$
4	$u \leftarrow$ EXTRACT-MIN(Q)	$O(\log n)$
5	$S \leftarrow S \cup \{u\}$	$O(1)$
6	**foreach** $v \in adj[u]$ **do**	$O(\|adj[u]\|)$
7	RELAX(u,v)	$O(\log n)$

The dominant term comes from the execution of line 7 $O(m)$ times(sum of adjacencies) for a total of $O(m \log n)$.

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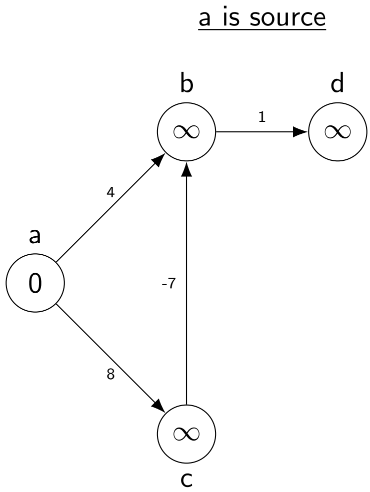
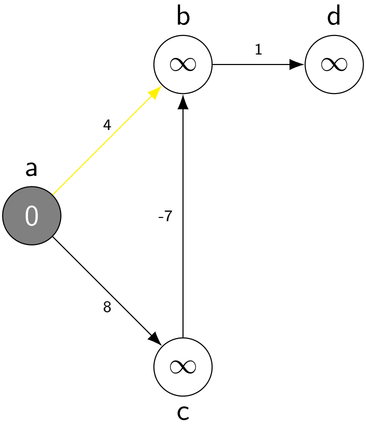
Negative weights counter example

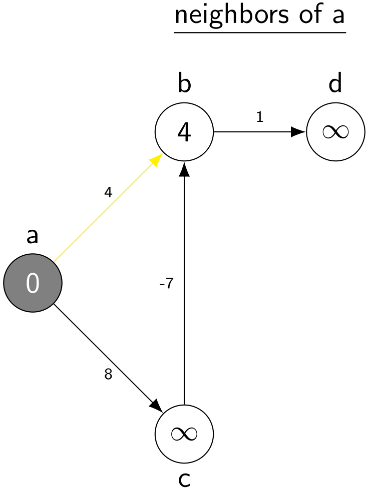


Negative weights counter example

neighbors of a

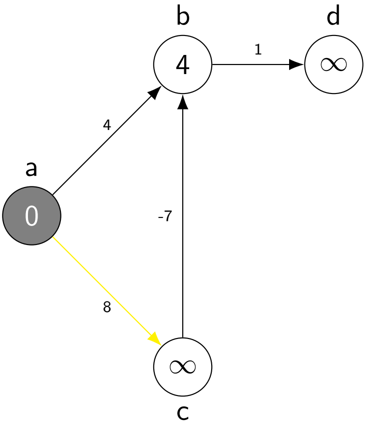


Negative weights counter example

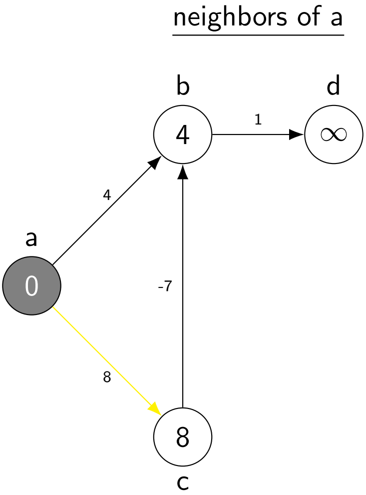


Negative weights counter example

neighbors of a

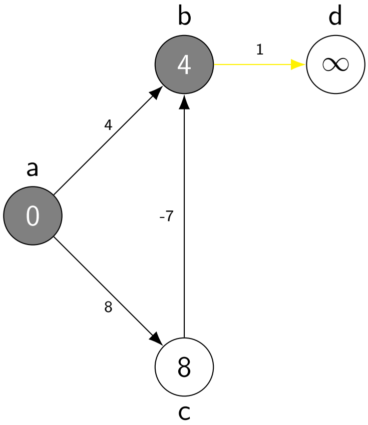


Negative weights counter example



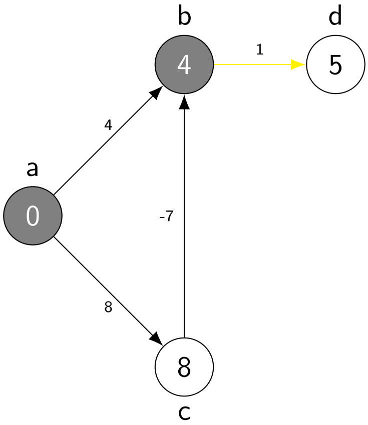
Negative weights counter example

neighbors of b



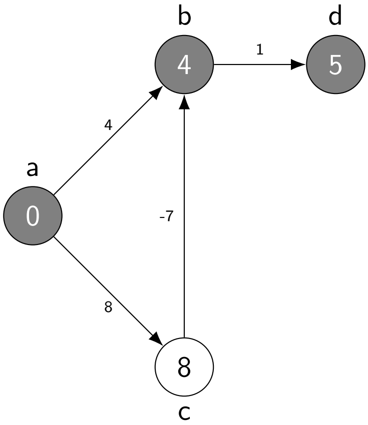
Negative weights counter example

neighbors of b

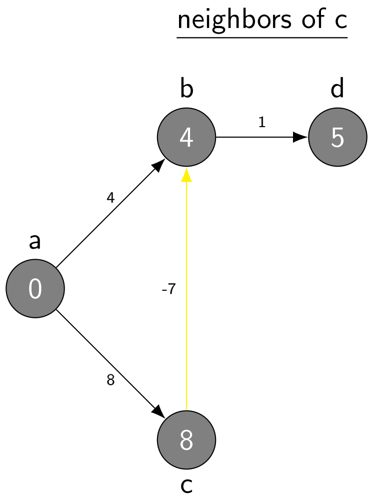


Negative weights counter example

neighbors of d



Negative weights counter example



Negative weights counter example

neighbors of c