3RD MONDAY
OCT 9 2023

So far: random variables, their density or mass function, expected value and variance

E.g. $X \sim \text{Binomial}(n, p)$ $\qquad \mathbb{E}(X) = n \times p$

$X = \#$ "successes" in n independent trials, each has prob. p of success $\qquad \text{Var}(X) = np(1-p)$

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, 1, \ldots, n$$

Next: Poisson random variable

N simulations

$x_1, x_2, \ldots, x_N$

$5, 35, 21, \ldots, 32$

$k = 0, 1, \ldots, n$

$$\frac{\# x_i = k}{N} \simeq \mathbb{P}(X = k) \qquad N \to \infty$$

X is called Poisson(λ) if values $\{0, 1, 2, \ldots\}$

model counts

$$\boxed{\mathbb{P}(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}} \qquad k = 0, 1, 2, \ldots$$

Poisson(λ) arises as an approximation to Binomial(n, p)

$n \to \infty \quad p \to 0 \quad$ so that $np \to \lambda$

$X \sim \text{Binomial}(N, p) \qquad \mathbb{E}[X] = \lambda = Np$

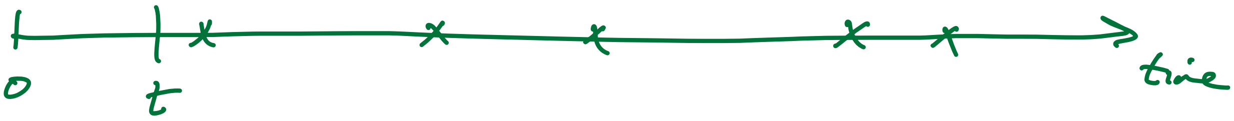
$p = \lambda / N$

$Y \sim \text{Binomial}(N, \lambda/N) \longleftrightarrow \mathbb{E}(X) = N \lambda/N = \lambda$

$\updownarrow$

$X \sim \text{Poisson}(\lambda) \qquad \mathbb{E}(X) = \lambda$

Poisson Process "events"



rate parameter = # "events" / unit time.

cosmic rays/ minute = 1.2 (on average)

In 6 minutes, how many do we expect?

length time × rate = # in interval

$6 \times 1.2 = 7.2$ rays/ 6 minute interval

Poisson process → # occurring in any interval of time is a Poisson r.v.

λ = rate process

$\mathbb{P}[\text{in } [0,t] \text{ no events}] = \mathbb{P}[N = 0] = \dfrac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$

events in $[0,t] \sim \text{Poisson}(\lambda t) \hookrightarrow \text{Poisson}(\lambda t)$

$\mathbb{E}[\#\text{ events In } [0,t]] = \lambda \cdot t$

W = time until 1st event

$\mathbb{P}(W > t) = e^{-\lambda t}$

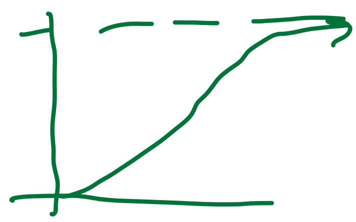
$\Updownarrow$

In $[0,t]$ no events

W = waiting time first event

$F_W(t) = \mathbb{P}(W \le t) = 1 - e^{-\lambda t}$

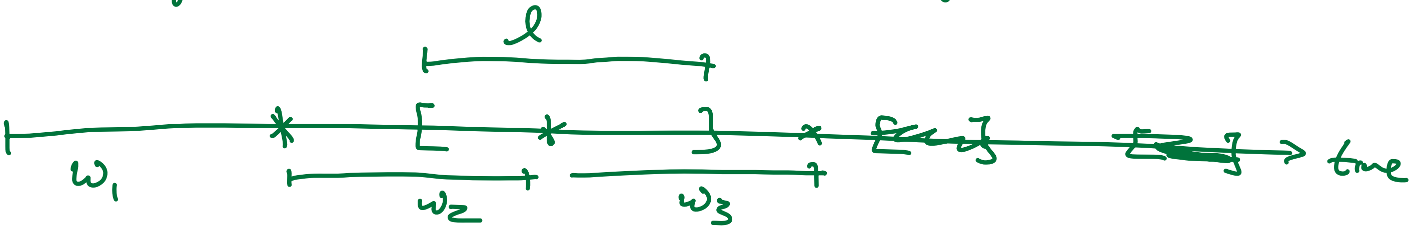
cumulative distribution function



density $\Leftrightarrow f_W(t) = \lambda e^{\lambda t}$ $\quad t \ge 0$

$\exp(\lambda)$ r.v.

Poisson process: let $W_1, W_2, \ldots$ $\exp(\lambda)$ r.v.



events in any interval of time $\sim$ Poisson(λl)

l = length

intervals of time that are disjoint have independent counts of events

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$X \sim$ Poisson (λ)

Var(X)

$$\mathbb{P}(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$\mathbb{E}(X) = \lambda \qquad \mathbb{E}(X) = \sum_k k\,\mathbb{P}(X=k)$$

$\updownarrow$

$$\mathbb{E}(X) = \sum_{k=0}^{\infty} k\frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-1)!}$$

$$k! = k(k-1)\cdots 1$$

$$\frac{k}{k!} = \frac{\cancel{k}}{\cancel{k}(k-1)\cdots}$$

$$= \lambda\sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}$$

$$= \lambda \underbrace{\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}}_{1}$$

$$= \lambda$$

If $X \sim$ Poisson(λ)

$$\text{Var}(X) = \lambda$$

$$SD(X) = \sqrt{\lambda}$$

$$SD(X) = \sqrt{\text{Var}(X)}$$

D = # Defects in panel $\xleftrightarrow{\text{If Poisson}}$ $\mathbb{P}(D=k) =$

If Poisson assumption is correct (ie $D \sim$ Poisson(1) r.v.)

$\mathbb{E}(D) = 1$

$$\mathbb{P}(D=0) = \frac{e^{-1}1^0}{0!} = \frac{1}{e}$$

10000 $D_1, D_2, \ldots D_{1000}$ simulations $\longleftrightarrow$ $\mathbb{E}[\#D_i\text{'s} = 0] = \frac{1}{e}\,10{,}000$

$$\boxed{\mathbb{P}(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}}$$

$$\mathbb{P}(D=1) = \frac{e^{-1}1^1}{1!} = \frac{1}{e}$$

$\mathbb{E}[X|R] = R$ since given R, $X \sim \text{Poisson}(R)$

FACT

$\mathbb{E}[\mathbb{E}[X|R]] = \mathbb{E}[X]$ (proof in slides)

In ~~n~~ this case $\mathbb{E}[X] = \mathbb{E}[R] = 1$

$\quad\quad R \sim \exp(1)$

Comparing two distributions

$X \sim \text{Poisson}(1)$

TWO STEP PROCESS
$R \sim \exp(1)$
$X \sim \text{Poisson}(R)$

First comparison: simulated from, compared to

$\mathbb{E}[X] = 1$ $\quad\quad\quad\quad$ $\mathbb{E}[X] = 1$

$SD(X) = \sqrt{1} = 1$ $\quad\quad\quad\quad$ $\text{Var}(X) > 1$

Hard: Compute via theory the variance of

Easier: Estimate via simulation the variance of

Easier: How to estimate via simulation the ~~mean~~ expectation of a random variable?

$X_1, \ldots, X_N$ simulations from same distribution

how to estimate $\mathbb{E}(X) = ?$

$$\bar{X} = \frac{X_1 + \dots + X_N}{N} \approx \mathbb{E}(X)$$

Variance: $\mathbb{E}\left((X - \mathbb{E}X)^2 \right)$

Sample variance $\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2$

<u>For next time:</u> ~~simulate~~ approx. via simulation the variance of this 2-step distribution