2nd Wednesday Oct 4 23

Random variables

- mass functions ✓
- densities ✓
- cumulative distribution functions ✓
- expectation } properties
- variance }

Examples: stochastic optimization

Example toss a p-coin until first time H

X = # tosses required,

possible values 1, 2, - - - - -

Discrete: possible values enumerated

Continuous: values continuous interval, e.g. any possible real value

prob. mass function

$$p_X(k) = \mathbb{P}(X = k) = (1-p)^{k-1} p$$

$$\mathbb{P}(X \le 2) = p(1) + p(2)$$

Continuous random variable

Y = lifetime of a lightbulb

$[0, \infty)$ = possible values

density function f_Y

$$f_Y(y) = \lambda e^{-\lambda y} \qquad y \ge 0 \qquad \lambda \text{ parameter fixed}$$

density function is $f(y) = \lambda e^{-\lambda y}$

Y is an $\exp(\lambda)$ r.v.

$$\mathbb{P}(a \le Y \le b) = \int_a^b f_Y(y)\,dy$$

$$= \int_a^b \lambda e^{-\lambda y}\,dy \qquad \left(\text{If } Y \sim \mathrm{Exp}(\lambda)\right)$$

$$= -e^{-\lambda y}\Big|_{y=a}^{b}$$

$$= -e^{-\lambda b} - -e^{-\lambda a}$$

$$= e^{-\lambda a} - e^{-\lambda b}$$

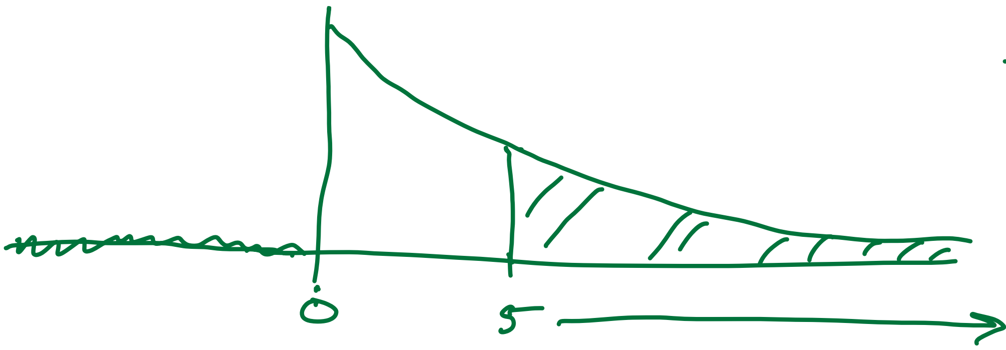
$e^{cx} \Rightarrow \frac{1}{c} e^{cx}$

$x^p \Rightarrow \frac{1}{p+1} x^{p+1}$

$\lambda = 2$

$$\underset{5 \le Y < \infty}{\mathbb{P}(Y \ge 5)} = \int_5^\infty 2e^{-2y}\,dy = -e^{-2y}\Big|_5^\infty$$

$$= 0 + e^{-10}$$



cumulative distribution function

$$F_Y(t) = \mathbb{P}(Y \le t)$$

$$= \int_{-\infty}^{t} f_Y(s)\,ds$$

$h(x) = x^2$

$h(z) = z^2$

$\mathbb{P}(Y = 3) = 0$

$\mathbb{P}(Y = \pi) = 0$

$$= \int_0^t \lambda e^{-\lambda s}\,ds = -e^{-\lambda s}\Big|_0^t$$

$$= 1 - e^{-\lambda t}$$

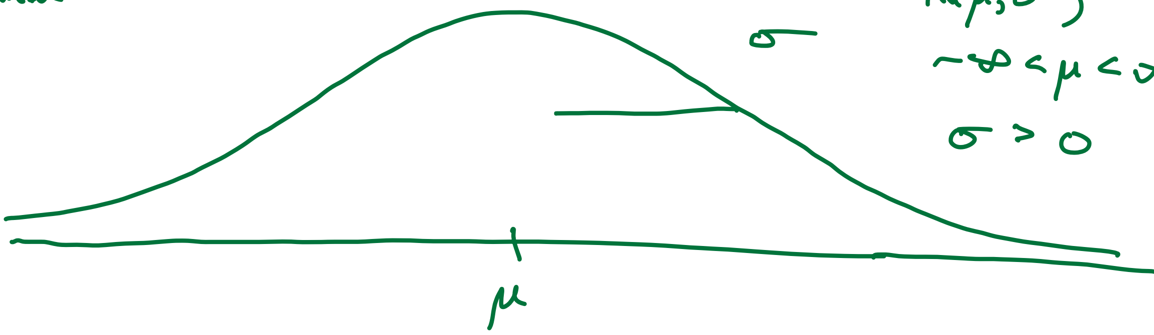
$$\mathbb{P}(3 \le Y \le 11.5) = \mathbb{P}(Y \le 11.5) - \mathbb{P}(Y < 3)$$
$$= F(11.5) - F(3)$$

Normal

$N(\mu, \sigma)$

$-\infty < \mu < \infty$

$\sigma > 0$



$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$N \sim \text{Normal}(0,1)$

$$\mathbb{P}(2 \le N \le 10) = \int_2^{10} \boxed{\quad\quad\quad} \, dx$$

must do this integral numerically!
use computer

TYPICAL VALUES

Given a r.v. X, its expected value (expectation)

$$\mathbb{E}(X) = \begin{cases} \sum_x x \underbrace{\mathbb{P}(X=x)}_{p(x)} & X \text{ discrete} \\ \int_{-\infty}^{\infty} x f(x)\, dx & X \text{ continuous} \end{cases}$$

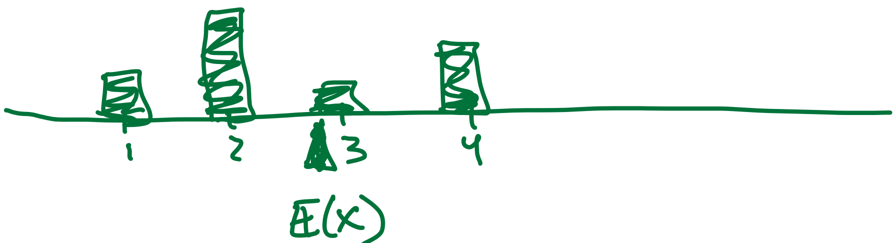
$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$ Bernoulli (p) r.v.

↑ with probability

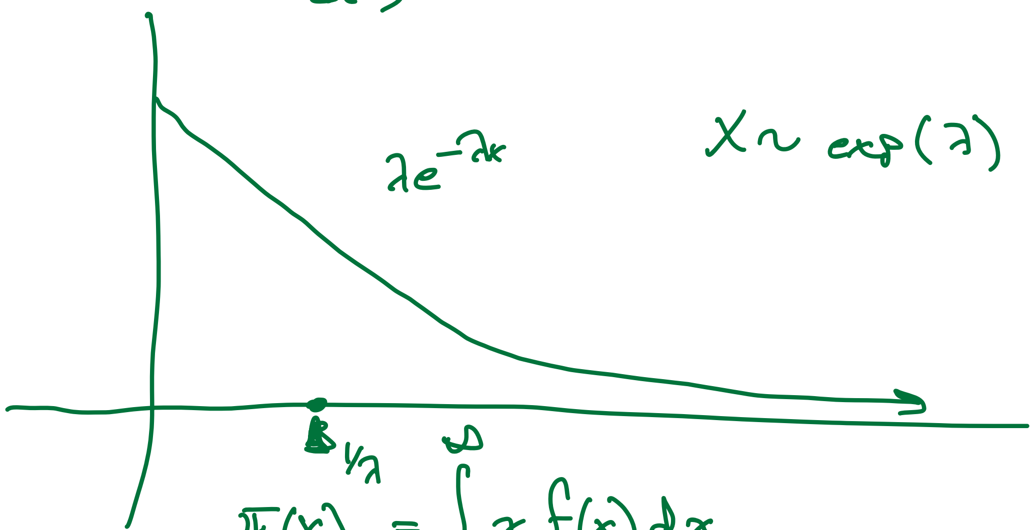
$$\mathbb{E}(X) = 0 \cdot (1-p) + 1 \cdot p = p$$

$X_1, X_2, \ldots$ INDEPENDENT all w/ same DIST'N

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \simeq \mathbb{E}(X)$$



$X \sim \exp(\lambda)$



$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^{\infty} x \lambda e^{-\lambda x}\,dx$$

$u = x \qquad v = -e^{-\lambda x}$

$du = dx \qquad dv = \lambda e^{-\lambda x}\,dx$

$$= -xe^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx$$

$$= 0 + \int_0^{\infty} e^{-\lambda x}\,dx$$

$$= -\frac{1}{\lambda} e^{-\lambda x} \Big|_0^{\infty}$$

$$= \frac{1}{\lambda}$$

$X \sim \exp(\lambda)$

$\mathbb{E}(X) = 1/\lambda$

If $\lambda = \frac{1}{\mathbb{E}(X)}$ is a rate

$X \sim \text{Binomial}(n, p) \leftarrow$ n experiments success failure

p = prob. success on each experiment

X = # successes total

values of $X = \{0, 1, \ldots, n\}$

mass function

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\frac{n!}{k!(n-k)!}$$

$$\mathbb{E}(X) = n \cdot p$$

$$= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$X_j = \begin{cases} 1 & \text{if } j^{\text{th}} \text{ experiment is SUCCESS} \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{X} = \sum_{j=1}^{n} X_j = \#$ successful experiments

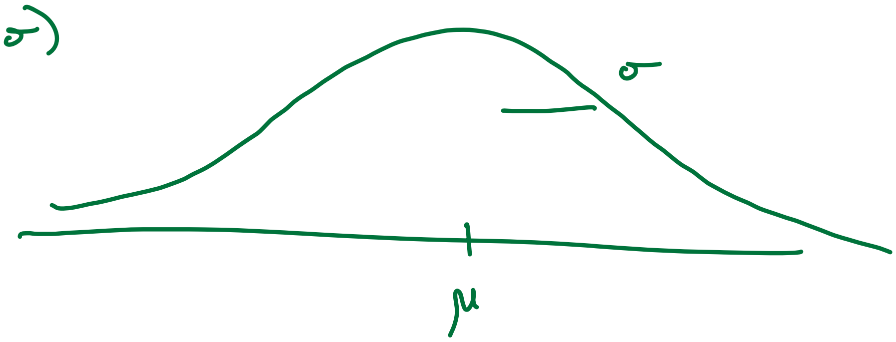
Property of expectation

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$$

$$\mathbb{E}(\mathbb{X}) = \sum_{j=1}^{n} \mathbb{E}(X_j) = \sum_{j=1}^{n} p = np$$

$\mathbb{E}[a\mathbb{X}] = a\,\mathbb{E}(\mathbb{X})$ (another property)

$N(\mu, \sigma)$ Normal density



$$\mathbb{E}(\mathbb{X}) = \mu$$

Variance of a r.v. $\mathbb{X}$

$$Var(\mathbb{X}) = \mathbb{E}\left[(\mathbb{X} - \mathbb{E}X)^2\right]$$

continuous

DISTANCE OF $\mathbb{X}$ to ITS CENTER OF MASS

MEASURES SPREAD

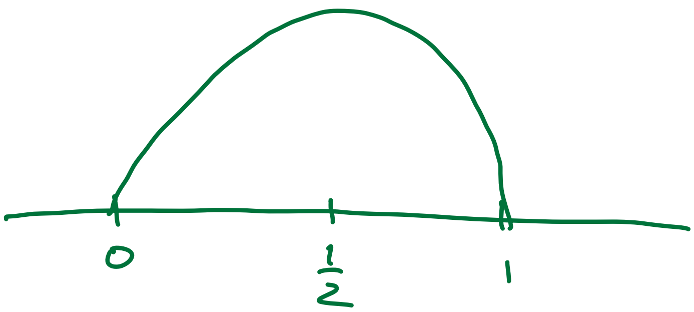
$X \sim$ Bernoulli (p) $\qquad \mathbb{E}(X) = p$

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$(X - \mathbb{E}X)^2 = (X - p)^2 = \begin{cases} (1-p)^2 & \text{w.p. } p \\ (-p)^2 & \text{w.p. } 1-p \end{cases}$$

$$\begin{aligned}
\mathbb{E}[(X - \mathbb{E}X)^2] &= (1-p)^2 p + p^2(1-p) \\
&= (1 - 2p + p^2)p + p^2 - p^3 \\
&= p - 2p^2 + \cancel{p^3} + p^2 - \cancel{p^3} \\
&= p - p^2 = p(1-p)
\end{aligned}$$

$$\mathrm{Var}(X) = p(1-p)$$



$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mathbb{E}X)^2 f(x)\,dx$$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}\left((X - \mathbb{E}X)^2 \right) \\
&= \mathbb{E}[X^2 - 2X\,\mathbb{E}X + (\mathbb{E}X)^2] \\
&= \mathbb{E}(X^2) - \mathbb{E}(2X\,\mathbb{E}X) + \mathbb{E}[(\mathbb{E}X)^2] \\
&= \mathbb{E}(X^2) - 2\,\mathbb{E}(X)\,\mathbb{E}(X) + (\mathbb{E}X)^2 \\
&= \mathbb{E}(X^2) - 2(\mathbb{E}(X))^2 + (\mathbb{E}(X))^2 \\
&= \mathbb{E}(X^2) - (\mathbb{E}(X))^2
\end{aligned}$$

c constant

$\boxed{\mathbb{E}[c] = c}$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx \qquad X \text{ has density } f(x)$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$$

$$\begin{aligned}
\mathrm{Var}(aX) &= E\left[(aX - E(aX))^2 \right] \\
&= E\left[[a(X - EX)]^2 \right] \\
&= E\left[a^2 (X - EX)^2 \right] \\
&= a^2 E\left[(X - EX)^2 \right] \\
&= a^2 \mathrm{Var}(X)
\end{aligned}$$

$$SD(X) = \sqrt{\mathrm{Var}(X)}$$

$$SD(aX) = |a|\, SD(X)$$