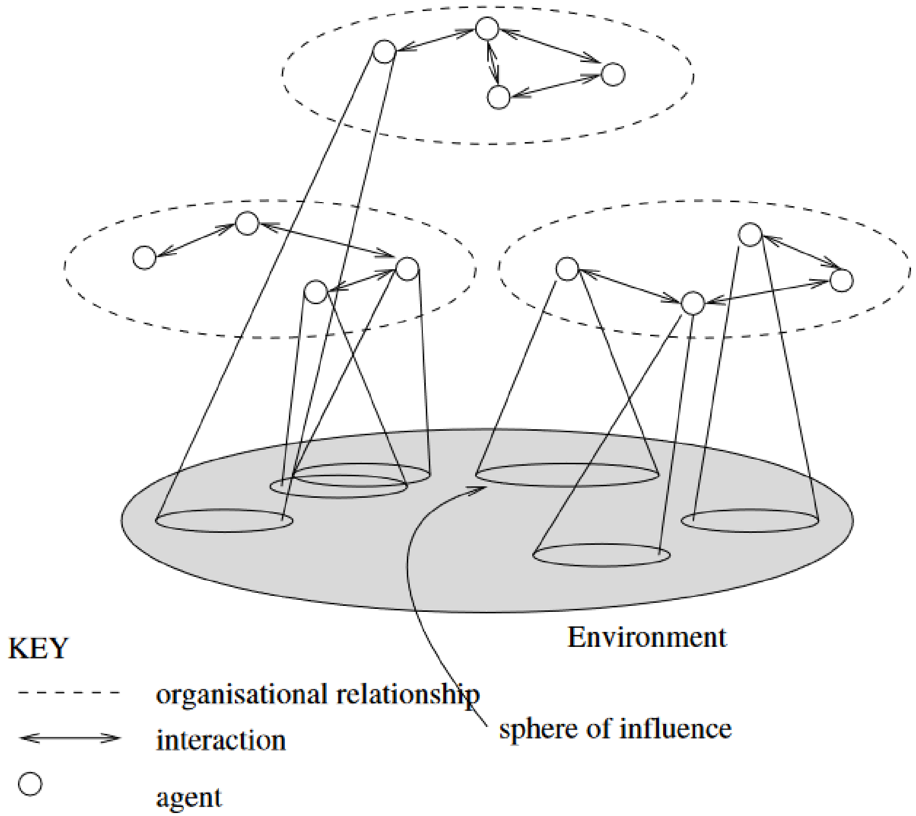
LECTURE 8: Multiagent Decision Making (II)

Introduction to Multi-Agent Systems (MESIIA, MIA)

URV

What are Multi-Agent Systems?

- A multiagent system contains a number of agents that:
  - interact through communication;
  - are able to act in an environment;
  - have different “spheres of influence” (which may coincide); and
  - will be linked by other (organisational) relationships.



Types of Agreement

- **Multiagent encounters (game-like character)**
- **Voting.**
- **Coalition forming.**
- **Auctions**

Cooperative Game Theory

- So far, we have taken a game theoretic view of multi-agent interactions.
  - Non-cooperative games
    - Competitive (Zero-Sum games)
    - **Non- Competitive**
  - Agents/players are self-interested agents.
  - No cooperation due to the following two reasons
    - Binding agreements are not possible (how can we recover the cooperation in this case?)
    - Utility is given to individuals based on individual action.
  - An agent plays strategies that maximize its **utility** and expect each other to play likewise.

- These constraints do not necessarily hold in the real world
  - Contracts, or collective payments can facilitate cooperation, leading to *Coalition Games* and *Cooperative Game Theory*

Example

Three kids, Adam, Bill, and Carmen want to buy ice cream. Adam has \$6, Bill has \$4, and Carmen has \$3. They found that the ice cream shop offers 3 different sizes of ice cream: small (250g) with cost of \$7, medium(375g) with cost of \$9 and large(500g) with cost of \$11.

They can share the ice cream among themselves. How should they cooperate to buy and share the ice cream?

Cooperative Games

- Coalitional games model scenarios where agents can benefit by cooperating.

- Three stages of cooperative action:

Coalitional Structure Generation

Deciding in principle who will work together. It asks the basic question:

Which coalition should I join?

The result: partitions agents into disjoint coalitions. The overall partition is a coalition structure.

Solving the optimization problem of each coalition

Deciding how to work together, and how to solve the “joint problem” of a coalition. It also involves finding how to maximise the utility of the coalition itself, and typically involves joint planning etc.

Dividing the benefits

Deciding “who gets what” in the payoff. Coalition members cannot ignore each other’s preferences,

…if you try to give me a bad payoff, I can always walk away…

We might want to consider issues such as fairness of the distribution.

Formalising Cooperative Games

- Let $Ag = \{1, \ldots, n\}$ to be the set of agents/players
- $C, C`, C_1 \subseteq Ag$ are called coalitions
- $C = Ag$ is the ***grand coalition***
- $\{i\}$ where $i \in Ag$ is a ***singleton coalition***
- Let $v: 2^{Ag} \rightarrow \mathbb{R}$ be the characteristic function of the game. It assigns to every possible coalition a numeric value representing the pay-off that may be distributed between the members of the coalition.
- Then, we can formally define a *cooperative game* (or coalitional game) as a pair $G = \langle Ag, v \rangle$.

A possible function of the Ice-Cream example

Coalition (C)	$v(C)$
ϕ	0
$\{A\}$	0
$\{B\}$	0
$\{C\}$	0
$\{A, B\}$	375
$\{A, C\}$	375
$\{B, C\}$	250
$\{A, B, C\}$	500

Outcomes

- An outcome of a coalitional game $G = \langle Ag, v \rangle$ is a pair $\langle \mathbb{C}, \mathrm{X} \rangle$ where:
  (1) $\mathbb{C} = \{C_1, C_2, \dots, C_k\}$ is the **coalition structure**, i.e., partition of Ag, s.t.
    a. $\bigcup_{i=1}^{k} C_i = Ag$ and
    b. $C_i \cap C_j = \phi, \forall\, i \neq j$
  (2) $\mathrm{X} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is a payoff vector indicating the value of each agent/player, s.t.
    a. $x_i \geq v(\{i\})\ \forall\, i \in Ag$ ***[Individual Rationality]***
    b. $\sum_{i \in C} x_i = v(C)\ \forall\, C \in \mathbb{C}$ ***[Efficiency]***

- Collation and coalition structure
  - If we have 3 agents A, B, C, how many possible coalitions and coalition structures? List them.
- **Outcome examples (Ice-Cream)?**

Outcomes

- An outcome of a coalitional game $G = \langle Ag, v \rangle$ is a pair $\langle \mathbb{C}, \mathrm{X} \rangle$ where:

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  b. $\sum_{i \in C} x_i = v(C)\ \forall\, C \in \mathbb{C}$ ***[Efficiency]***

- **Outcome examples (Ice-Cream)**
  - $\big((\{A, B\}, \{C\}), (200, 175, 0)\big)$, An outcome ✓
  - $\big((\{A, B, C\}), (250, 150, 100)\big)$, An outcome ✓
  - $\big((\{A, B\}, \{C\}), (250, 150, 100)\big)$ not an outcome, why? ✗

Super-Additive Games

A coalitional game $G = \langle Ag, v \rangle$ is super-additive if

- $v(C \cup C`) \geq v(C) + v(C`)$ for all $C, C` \subseteq Ag$ s.t. $C \cap C` = \phi$

- In that case, the coalition that maximises social welfare is the *Grand Coalition*
- So, we can simply reduce the definition of outcome to be only the payoff vector $\mathrm{X} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$
- Is the Ice-Cream example super-additive?

Sub-Additive Games

A coalitional game $G = \langle Ag, v \rangle$ is Sub-additive if

- $v(C \cup C`) < v(C) + v(C`)$ for all $C, C` \subseteq Ag$ s.t. $C \cap C` = \phi$

- In that case, The coalitions that maximis social welfare are *singletons*

- Some games are neither subadditive or superadditive:
  - the characteristic function value calculations need to be determined for each of the possible coalitions!
  - This is exponentially complex.

Which Coalition Should I Join?

- The agent should only join a coalition C which is:
  - *Feasible*: the coalition C really could obtain some payoff than an agent could not object to; and
  - *Efficient*: all of the payoff is allocated
- However, there may be many coalitions
  - Each has a different characteristic function
  - Agents prefer coalitions that are as productive as possible
  - Therefore a coalition will only form if all the members prefer to be in it (I.e. they don't defect to a more preferable coalition)
- Therefore:
  - "*which coalition should I join?*" can be reduced to "*is the coalition stable?*"
    - Is it rational for all members of coalition C to stay with C, or could they benefit by defecting from it?
    - There's no point in me joining a coalition with you, unless you want to form one with me, and vice versa.

Stability

- Let’s consider the ice cream example
  - Players and resources : Adam has \$6, Bill has \$4, and Carmen has \$3
  - Outcomes (ice cream sizes): Small (250), Medium (375), and Large (500)
- As this game is super additive, then the best coalition is the *ground coalition* $\{A, B, C\}$, so:
  - The resources will be in total \$13, and
  - They can buy the large ice cream (500)g
- Which of the following outcomes is stable?
  - (500/3, 500/3, 500/3), (187.5, 187.5, 125), (250, 125, 125), (250, 244, 1)
  - To answer this, we need to define what we mean by *stable outcome*.

Stability

- Let's first introduce the objection term
  - We say that a coalition C *objects* to an outcome for the ground coalition if there is some outcome for C that makes all members of C strictly better off.
  - Formally, $C \subseteq Ag$ objects to an outcome $(x_1, x_2, \ldots, x_n)$ for the grand coalition if there is some outcome $(x'_1, x'_2, \ldots, x'_n)$ for C s.t.:

$$x_i > x_i, i \in C$$

- Hence, we say that an outcome is *stable* if there is no objection, or no agent has incentive to *defect*.
- Stability is a *necessary* but not *sufficient* condition for coalitions to form
  - i.e. Unstable coalitions will never form, but a stable coalition isn't guaranteed to form

Stability

- Which of the following outcomes is stable?
  - (500/3, 500/3, 500/3), (187.5, 187.5, 125), (250, 125, 125), (250, 244, 1)
  - Remember the characteristic function values of the game:

Coalition (C)	$v(C)$
ϕ	0
$\{A\}$	0
$\{B\}$	0
$\{C\}$	0
$\{A, B\}$	375
$\{A, C\}$	375
$\{B, C\}$	250
$\{A, B, C\}$	500

The Core

- Stability can be reduced to the notion of the *core*
- The core of a coalitional game is the set of feasible distributions of payoff to members of a coalition that no sub-coalition can reasonably *object* to.

- The idea is that an outcome is not going to happen if somebody objects to it!
  - i.e. if the core is *empty*, then no coalition can form

- So, the question "Is the grand coalition stable?" is same as asking:
  "Is the core non-empty?"

The Core and Fair Payoffs

- Sometimes the core is non-empty but is it "*fair*"?
  - Suppose we have $Ag = \{1, 2\}$, with the following characteristic function, v:
    - $v(\{1\}) = 5$
    - $v(\{2\}) = 5$
    - $v(\{1, 2\}) = 20$
  - The outcome $\langle 20, 0 \rangle$ (i.e., agent 1 gets everything) *will not be in the core*, since agent 2 can object; by *working on its own it can do better*, because $v(\{2\}) = 5$
  - However, outcome $\langle 14, 6 \rangle$ is in the core, as *agent 2 gets more than working on its own*, and thus has no objection.

- But is it "*fair*" on agent 2 to get only a payoff of 6, if agent 1 gets 14???

Marginal Contribution

- Let $\delta_i(C)$ to be the amount that agent i adds by joining a coalition $C \subseteq Ag$
  - i.e. the *marginal contribution* of i to C is defined as $\mu_i(C) = v(C \cup \{i\}) - v(C)$
  - Note that if $\mu_i(C) = v\{i\}$ then there is *no added value* from i joining C as the amount i adds is the same as if i earns on its own

Sharing the Benefits of Cooperation

- The Shapley value is best known attempt to define how to divide benefits of cooperation fairly.
  - It does this by taking into account how much an agent contributes.
  - The Shapley value of agent i is the average amount that i is expected to contribute to a coalition.
  - The Shapley value is one that satisfies the axioms opposite!
- The Shapley Value for i, denoted sh_i, is the value that agent $i \in Ag$ is given in the game $\langle Ag, v \rangle$

Symmetry

Agents that make the same contribution should get the same payoff. I.E. the amount an agent gets should only depend on their contribution.

Dummy Player

These are agents that never have any synergy with any coalition, and thus only get what they can earn on their own.

Additivity

If two games are combined, the value an agent gets should be the sum of the values it gets in the individual games.

Shapley Axioms: Symmetry

- Agents that make the same contribution should get the same payoff
  - The amount an agent gets should only depend on their contribution
  - Agents i and j are interchangeable if $\mu_i(C) = \mu_j(C)\ \forall C \subseteq Ag - \{i, j\}$

- The symmetry axiom states:
  - If i and j are interchangeable, then $sh_i = sh_j$

Shapley Axioms: Dummy Player

- Agents that never have any synergy with any coalition, and thus only get what they can earn on their own.
- The amount an agent gets should only depend on their contribution
  - An agent is a dummy player if:
    $$\mu_i(C) = v(\{i\})\ \forall C = Ag - \{i\}$$
  - i.e. an agent only adds to a coalition what it could get on its own
- The dummy player axiom states:
  - If i is a dummy player, then $sh_i = v(\{i\})$

Shapley Axioms: Additivity

- If two games are combined, the value an agent gets should be the sum of the values it gets in the individual games.
  - I.e. an agent doesn't gain or loose by playing more than once
  - Let $G^1 = \langle Ag, v^1 \rangle, G^2 = \langle Ag, v^2 \rangle$ be games with the same agents
  - Let $i \in Ag$ be on of the agents
  - Let sh_i^1, sh_i^2 be the values agent i gets in games G^1 and G^2 respectively
  - Let $G^{1+2} = \langle Ag, v^{1+2} \rangle$ be the game such that $v^{1+2}(C) = v^1(C) + v^2(C)$

- The additivity axiom states:
  - The value sh_i^{1+2} of agent i in game G^{1+2} should be $sh_i^1 + sh_i^2$

Shapley value

- Let $\prod(Ag)$ denote the set of all possible orderings of the agents Ag.
- If $o \in \prod(Ag)$, then let $C_i(o)$ be the agents that appear before i in o, so, we define the Shapley value for agent i, denoted sh_i, as:

$$sh_i = \frac{1}{|Ag|!} \sum_{o \in \prod(Ag)} \mu_i\big(C_i(o)\big)$$

- For example, if $Ag = \{1, 2, 3\}$ then:
  - $\prod(Ag) = \{(1,2,3), (1, 3, 2), (2, 1,3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$
  - For $o = (3, 1, 2)$, find
    - $C_3(o) = ?$
    - $C_1(o) = ?$
    - $C_2(o) = ?$

Shapley Example

- Suppose we have $Ag = \{1, 2\}$, with the opposite characteristic function.
- We first calculate the marginal contribution $\mu_i(C)$ of each agent $i \in C$, for each coalition $C \subseteq Ag$

Coalition (C)	$v(C)$
ϕ	0
$\{\mathbf{1}\}$	5
$\{\mathbf{2}\}$	10
$\{\mathbf{1,2}\}$	20

Coalition (C)	$\boldsymbol{Agent\ i}$	$\boldsymbol{\mu_i(C)}$
ϕ	1	$v(\phi \cup \{1\}) - v(\phi) = 5 - 0 = 5$
	2	$v(\phi \cup \{2\}) - v(\phi) = 10 - 0 = 10$
$\{\mathbf{1}\}$	2	$v(\{1\} \cup \{2\}) - v(\{1\}) = 20 - 5 = 15$
$\{\mathbf{2}\}$	1	$v(\{1\} \cup \{2\}) - v(\{2\}) = 20 - 10 = 10$

- Then, we can calculate the individual Shapley values for each agent i:

  - $sh_1 = \frac{\mu_1(\phi)+\mu_1(\{2\})}{|Ag|!} = \frac{5+10}{2} = 7.5$

  - $sh_2 = \frac{\mu_2(\phi)+\mu_2(\{1\})}{|Ag|!} = \frac{10+15}{2} = 12.5$

Representing Coalitional Games

- It is important for an agent to know if the core of a coalition is non-empty
  - Problem: a naive, obvious representation of a coalitional game is *exponential* in the size of Ag.
  - For n agents we need an input file of $2^n + 1$ lines.
  - e.g. a 100-player game would require 1.2 x 1030 lines
- Now such a representation is:
  - utterly infeasible in practice; and
  - so large that it renders comparisons to this input size meaningless

```
% Representation of a Simple
% Characteristic Function Game

% List of Agents
1,2,3
% Characteristic Function
1 = 5
2 = 5
3 = 5
1,2 = 10
1,3 = 10
2,3 = 10
1,2,3 = 25
```

Representing Characteristic Functions?

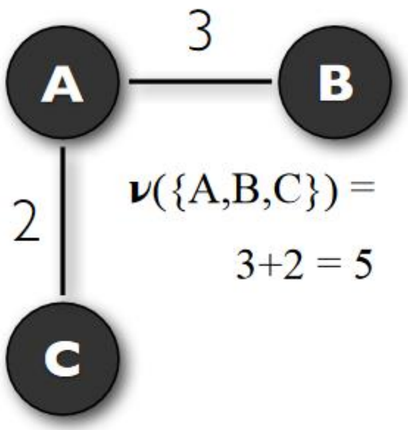
- Two approaches to this problem:
  - try to find a complete representation that is succinct in “most” cases
  - try to find a representation that is not complete but is always succinct

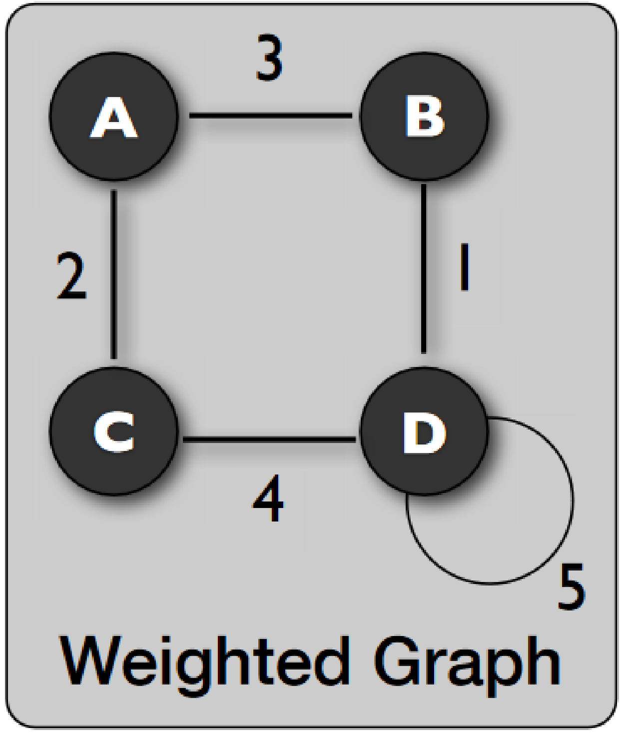
- A common approach:
  - interpret characteristic function over a combinatorial structure.

- We look at two possible approaches:
  - Induced Subgraph
  - Marginal Contribution Networks

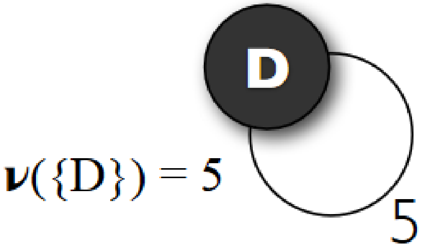
Induced Subgraph

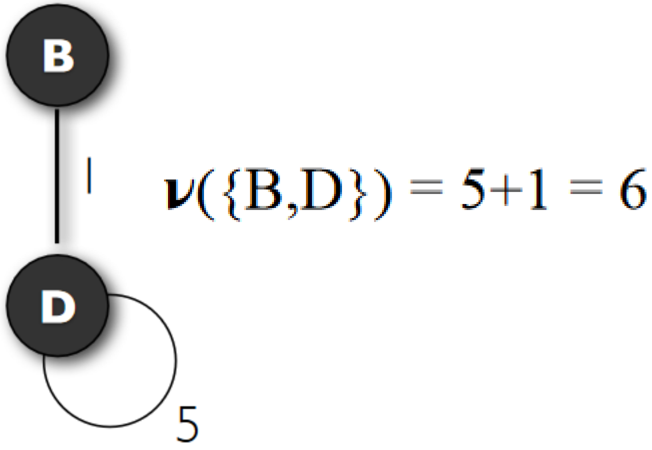
- Represent v as an undirected graph on Ag, with weights $w_{i,j}$ between $i, j \in Ag$.
- Value of coalition C is then:
  - $v(C) = \sum_{i,j \in Ag} w_{i,j}$





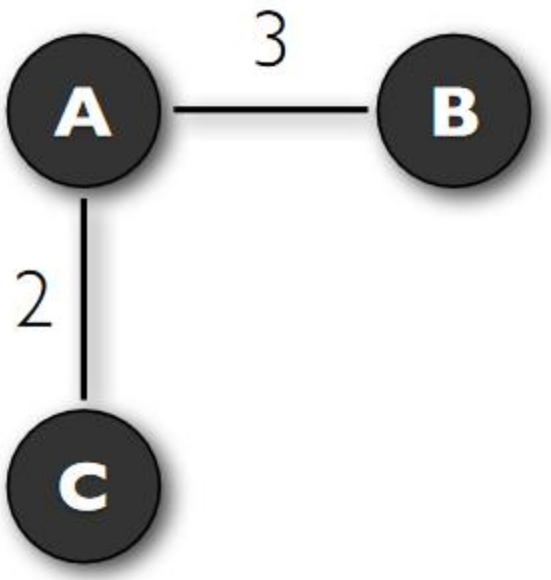
- .e., the value of a coalition $C \subseteq Ag$ is *the weight of the subgraph* induced by C





Induced Subgraph

- Representation is succinct, but not complete
  - there are characteristic functions that cannot be captured using this representation
- Determining emptiness of the core is NP-complete
  - Checking whether a specific distribution is in the core is co-NP-complete
- Shapley value can be calculated in polynomial time
  - $sh_i = \frac{1}{2} \sum_{i \neq j} w_{i,j}$
  - i.e. an agent gets *half the income from the edges in the graph to which it is attached*.



Marginal Contribution Nets

- Characteristic function v represented as rules:

$$\text{pattern} \rightarrow \text{value}$$

  - Pattern is conjunction of agents, a rule applies to a group of agents C if C is a superset of the agents in the pattern.

- Value of a coalition is then sum over the values of all the rules that apply to the coalition.

- We can also allow negations in rules (i.e. for when an agent is not present).

Rule set (rs) 2:

$$\begin{aligned} a \wedge b &\rightarrow 5 \\ b &\rightarrow 2 \\ c &\rightarrow 4 \\ b \wedge \neg c &\rightarrow -2 \end{aligned}$$

$\nu_{rs2}(\{a\}) = 0$	no rules apply
$\nu_{rs2}(\{b\}) = 2 + -2 = 0$	2nd and 4th rules
$\nu_{rs2}(\{c\}) = 4$	3rd rule
$\nu_{rs2}(\{a, b\}) = 5+2+-2 = 5$	1st, 2nd and 4th rules
$\nu_{rs2}(\{a, c\}) = 4$	3rd rule
$\nu_{rs2}(\{b, c\}) = 2+4 = 6$	2nd and 3rd rules
$\nu_{rs2}(\{a, b, c\}) = 5+2+4 = 11$	1st, 2nd and 3rd rules

Marginal Contribution Nets

- Calculating the Shapley value for marginal contribution nets is similar to that for induced subgraphs
  - Again, Shapley's symmetry axiom applies to each agent
    - The contributions from agents in the same rule is equal
  - The additivity property means that:
    - we calculate the Shapley value for each rule
    - sum over the rules to calculate the Shapley value for each agent
  - Handling negative values requires a different method

Calculating the Shapley Value

$$sh_i = \sum_{r \in rs; i \text{ occurs in lhs of } r} sh_i^r$$

where:

$$sh_i^{1 \wedge \cdots \wedge l \to x} = \frac{x}{l}$$

Example:

$$a \wedge b \to 5$$
$$b \to 2$$
$$c \to 4$$

$$sh_a = \frac{5}{2} = 2.5$$
$$sh_b = \frac{5}{2} + \frac{2}{1} = 4.5$$

What is sh_c?

Coalition Structure Generation

- A *coalition structure* is a partition of the overall set of agents Ag into mutually disjoint coalitions.
- Example: if $Ag = \{1,2,3\}$; then there are seven possible coalitions:
  - $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$
- And five possible coalition structures:
  - $\{\{1\}, \{2\}, \{3\}\}, \{\{1\}, \{2,3\}\}, \{\{2\}, \{1, 3\}\}, \{\{3\}, \{1,2\}\}, \{\{1,2,3\}\}$

Given a coalitional game $G = \langle Ag, v \rangle$, we say that the ***socially optimal coalition structure***

$$\mathbb{C}^* = \arg \max_{\mathbb{C} \in \text{partitions of } Ag} V(\mathbb{C})$$

where:

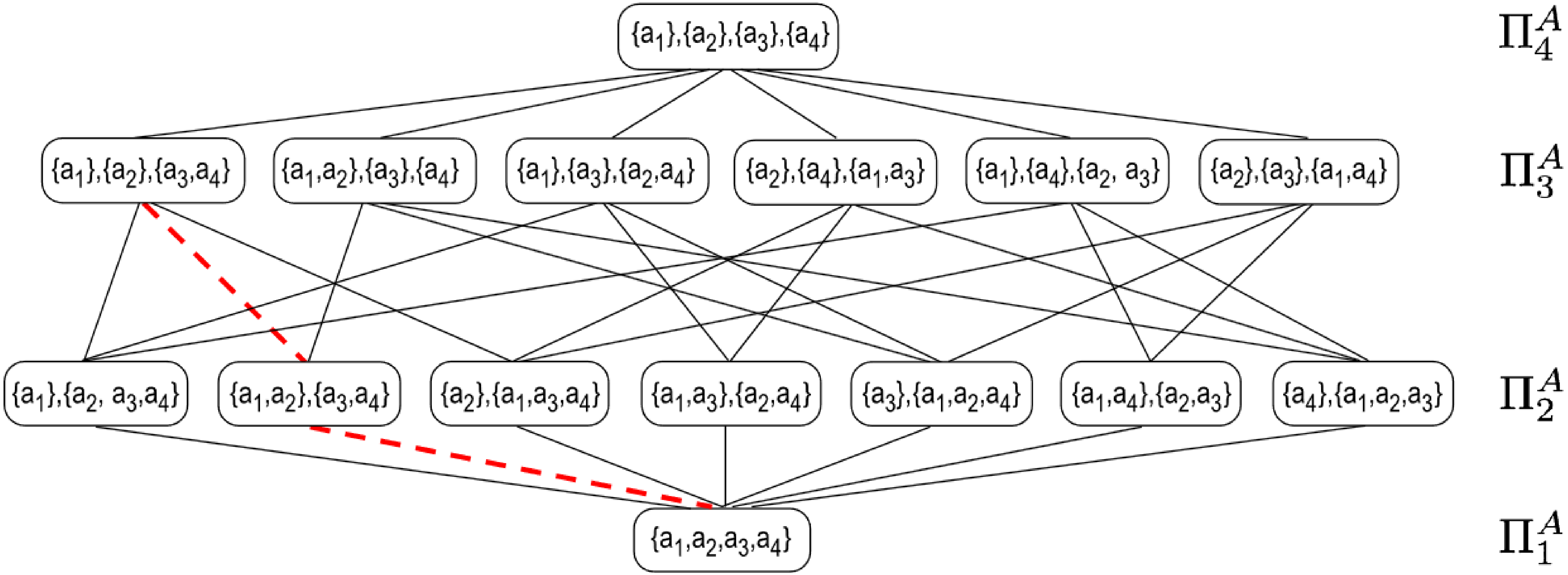
$$V(\mathbb{C}) = \sum_{C \in \mathbb{C}} v(C)$$

- **This problem is computationally hard.**
- **Moreover, it is NP-hard to find an optimal coalition structure given oracle access to the characteristic function**

Coalition Structure Generation

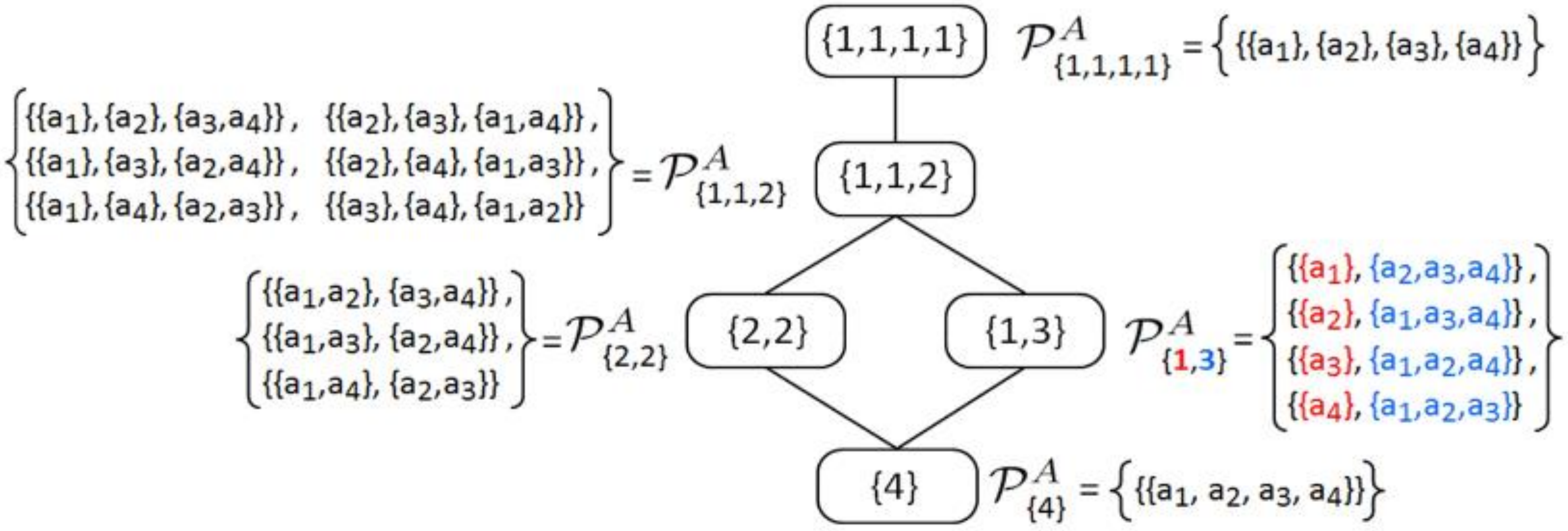
- Space Representation
  - There are two main representations of the space of possible coalition structures:
    - ***Coalition structure graph*** (Sandholm, 1999)
      - Every node represents a coalition structure.
      - Categorized into levels $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_n$, where level P_i contains the nodes that represent all coalition structures containing exactly i coalitions.
      - An edge con-nects two coalition structures if and only if:
        1. they belong to two consecutive levels P_i and P_{i-1}, and
        2. the coalition structure in P_{i-1} can be obtained from the one in P_i by merging two coalitions into one.
    - ***Integer partition-based representation*** (Rahwan et al. 2007)
      - Categorize them based on the sizes of the coalitions they contain.
      - Divide the space of coalition structures into disjoint subspaces that are each represented by an integer partition of n.

Coalition Structure Generation



The coalition structure graph of 4 agents.

Coalition Structure Generation



The integer partition-based representation for 4 agents.

Coalition Structure Generation

- **Methods:**
  - An Anytime Algorithm for Optimal Coalition Structure Generation (Rahwan et al., 2009) – a.k.a ***IP*** algorithm
  - Distributing Coalition Value Calculations to Coalition Members (Riley et al., 2015) - a.k.a (n,s)-sequences

IP Method

- **Let's start by an example**

P_1	value	P_2	value	P_3	value	P_4	value
$\{1\}$	30	$\{1,2\}$	50	$\{1,2,3\}$	90	$\{1,2,3,4\}$	140
$\{2\}$	40	$\{1,3\}$	60	$\{1,2,4\}$	120		
$\{3\}$	25	$\{1,4\}$	80	$\{1,3,4\}$	100		
$\{4\}$	45	$\{2,3\}$	55	$\{2,3,4\}$	115		
		$\{2,4\}$	70				
		$\{3,4\}$	80				

- What is the optimal coalition structure?
- Answer:

IP Method

- **Let's start by an example**

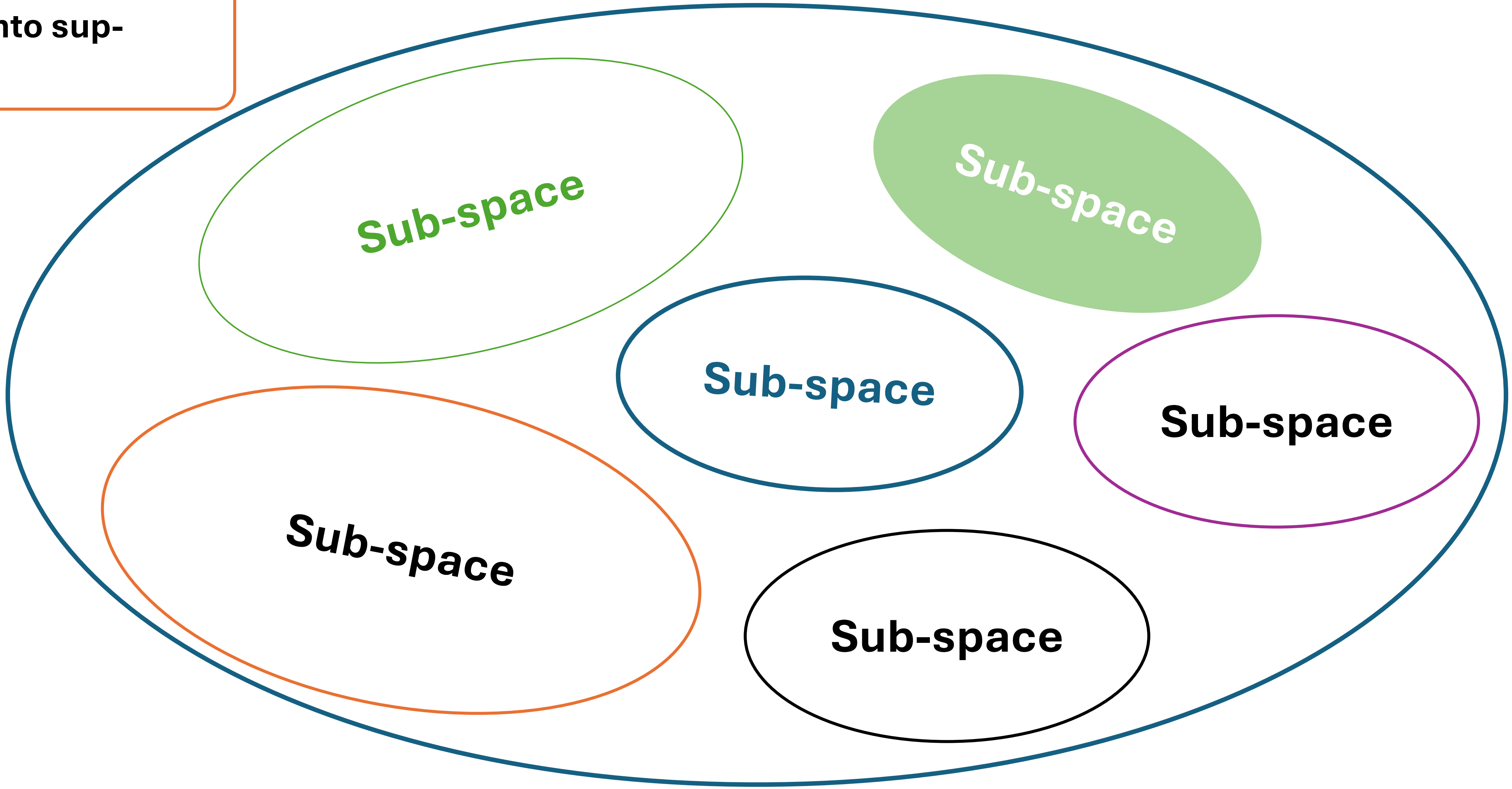
P_1	value	P_2	value	P_3	value	P_4	value
$\{1\}$	30	$\{1,2\}$	50	$\{1,2,3\}$	90	$\{1,2,3,4\}$	140
$\{2\}$	40	$\{1,3\}$	60	$\{1,2,4\}$	120		
$\{3\}$	25	$\{1,4\}$	80	$\{1,3,4\}$	100		
$\{4\}$	45	$\{2,3\}$	55	$\{2,3,4\}$	115		
		$\{2,4\}$	70				
		$\{3,4\}$	80				

- What is the optimal coalition structure?
- Answer: $\{\{1\},\{2\},\{3,4\}\}$

IP Method: Basic Idea

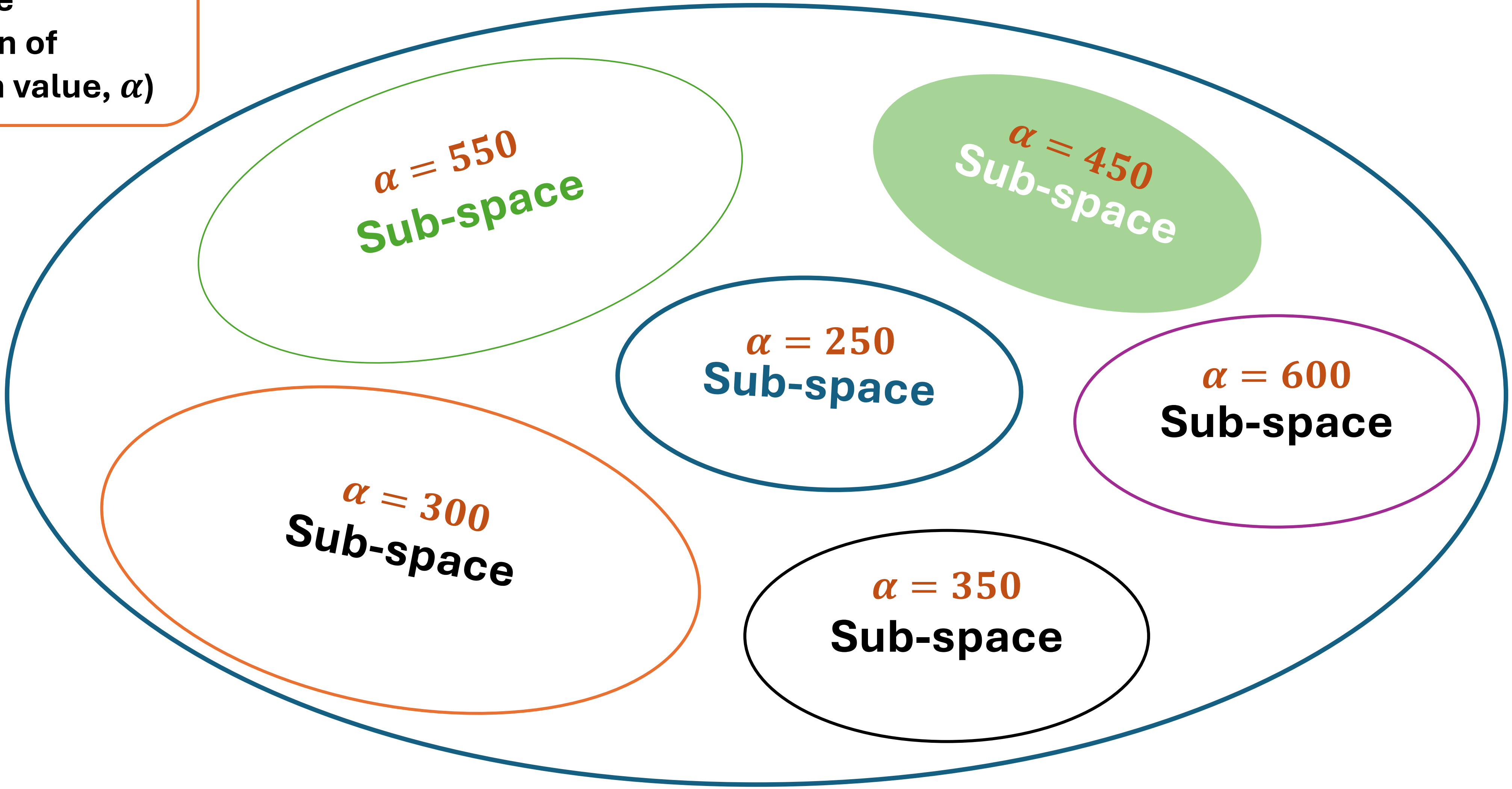
IP Method: Basic Idea

Divide the search space into sup-spaces



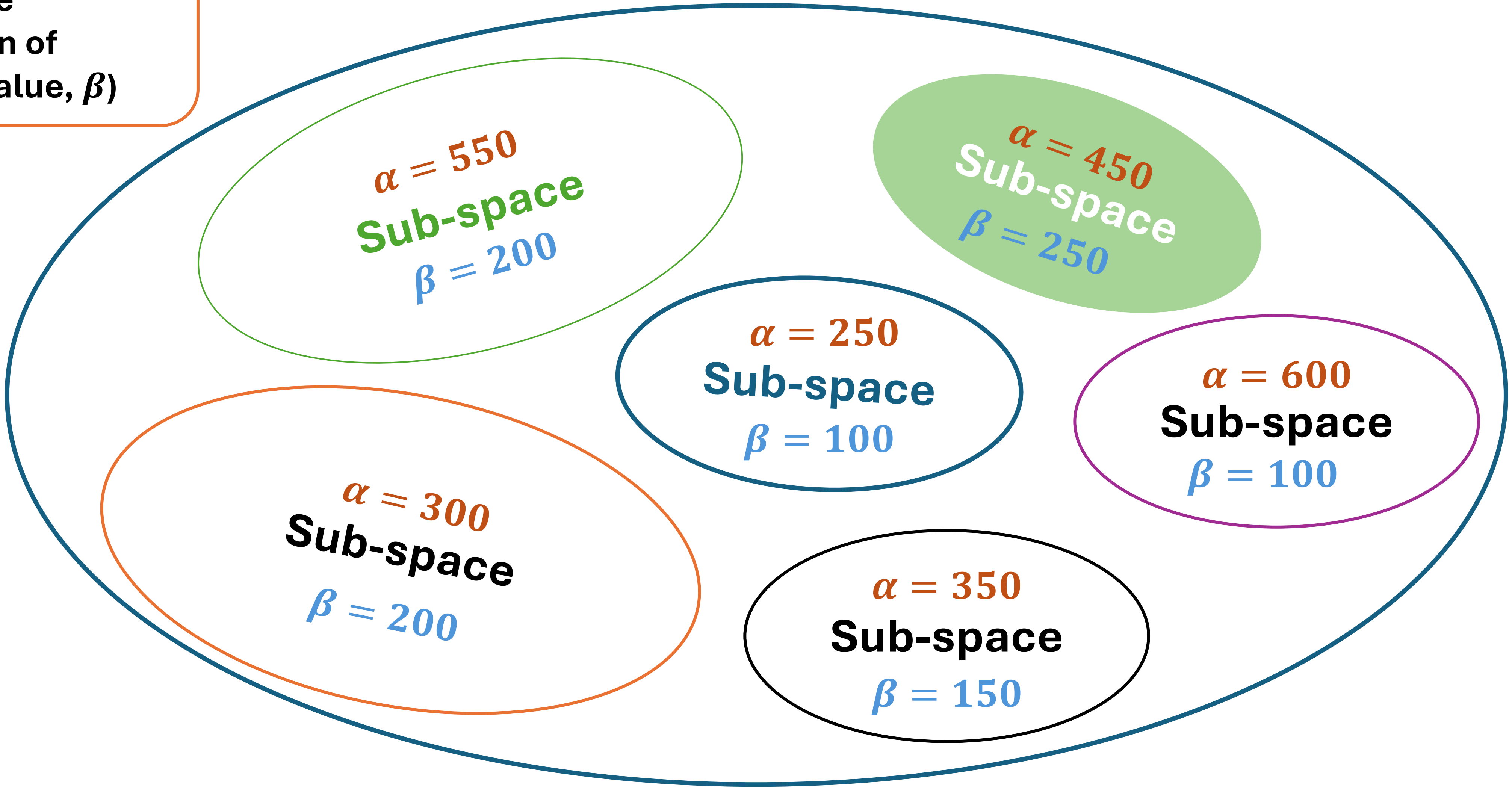
1) Calculate the upper-bound of each sub-space (that is the estimation of maximum value, α)

IP Method: Basic Idea

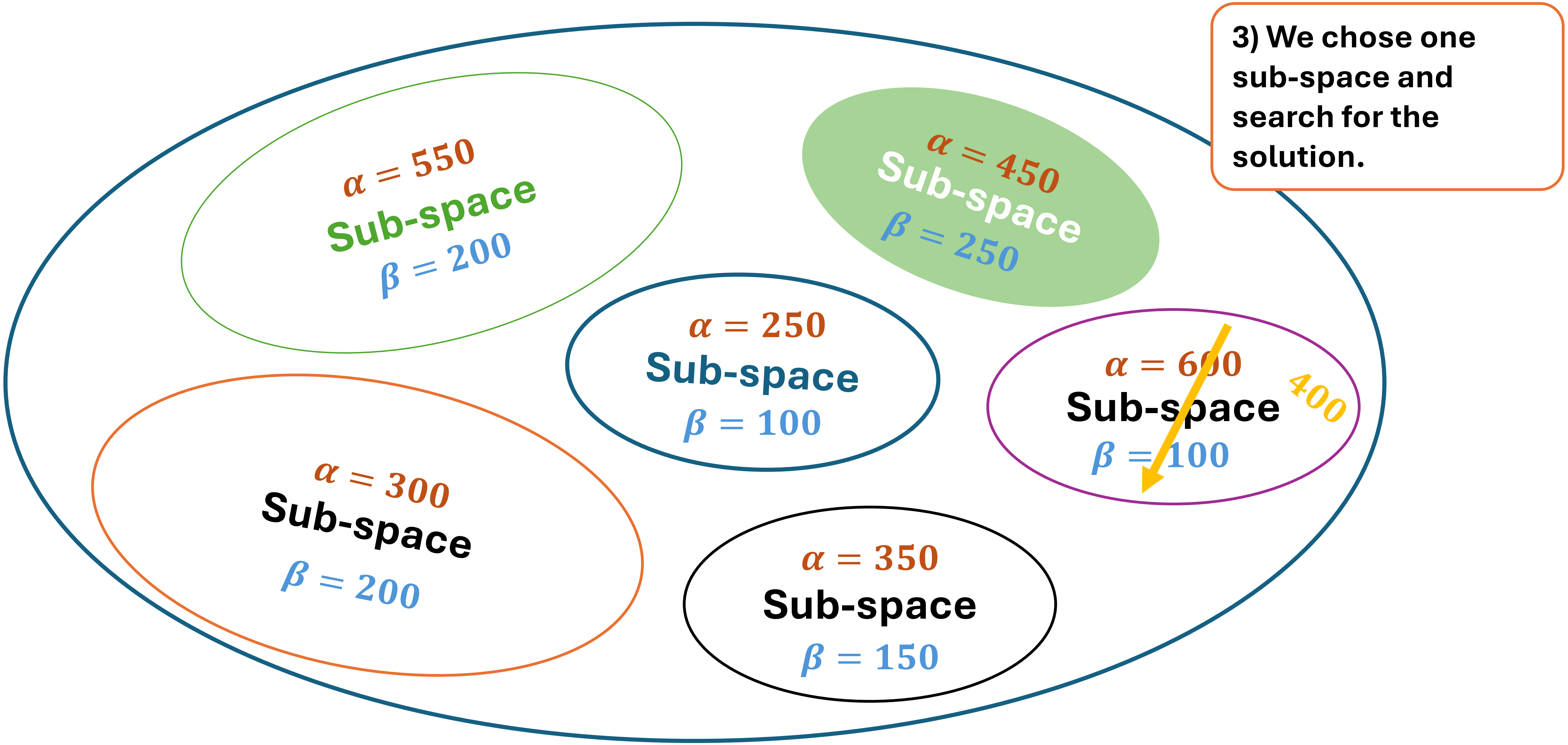


2) Calculate the lower-bound of each sub-space (that is the estimation of average value, β)

IP Method: Basic Idea



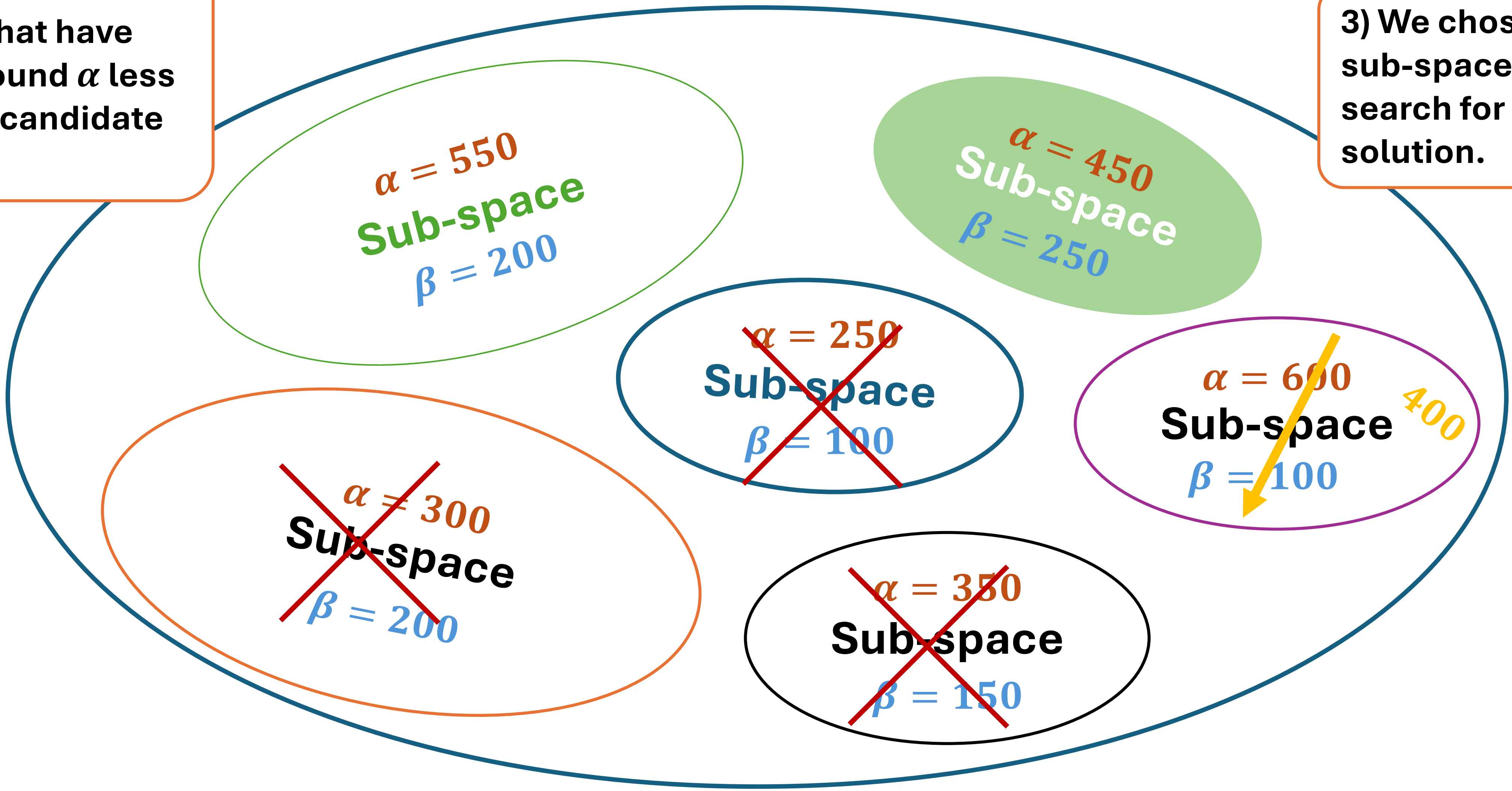
IP Method: Basic Idea



IP Method: Basic Idea

4) Burning: we ignore all the sub-spaces that have upper-bound α less than the candidate solution

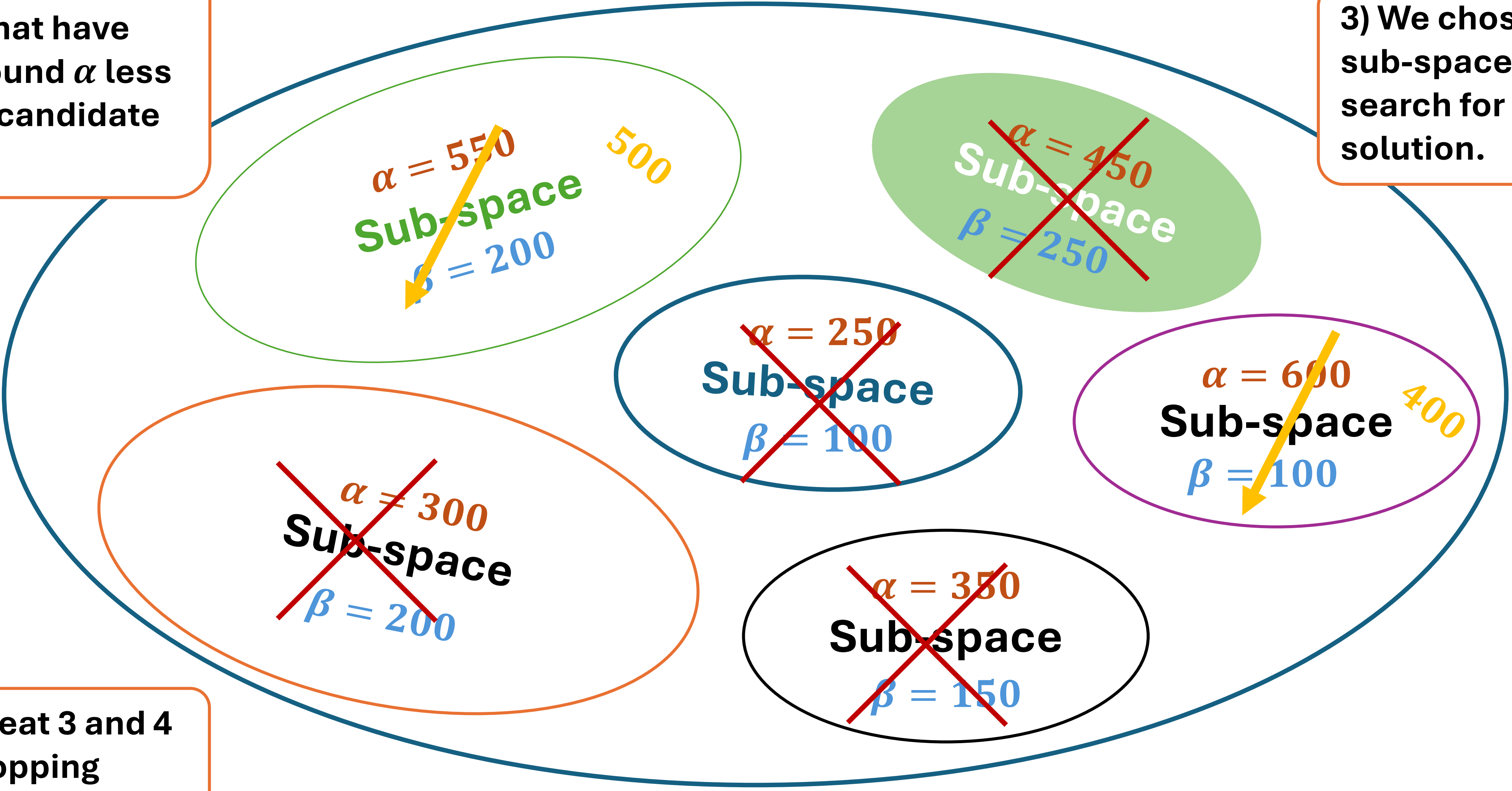
3) We chose one sub-space and search for the solution.



IP Method: Basic Idea

4) Burning: we ignore all the sub-spaces that have upper-bound α less than the candidate solution

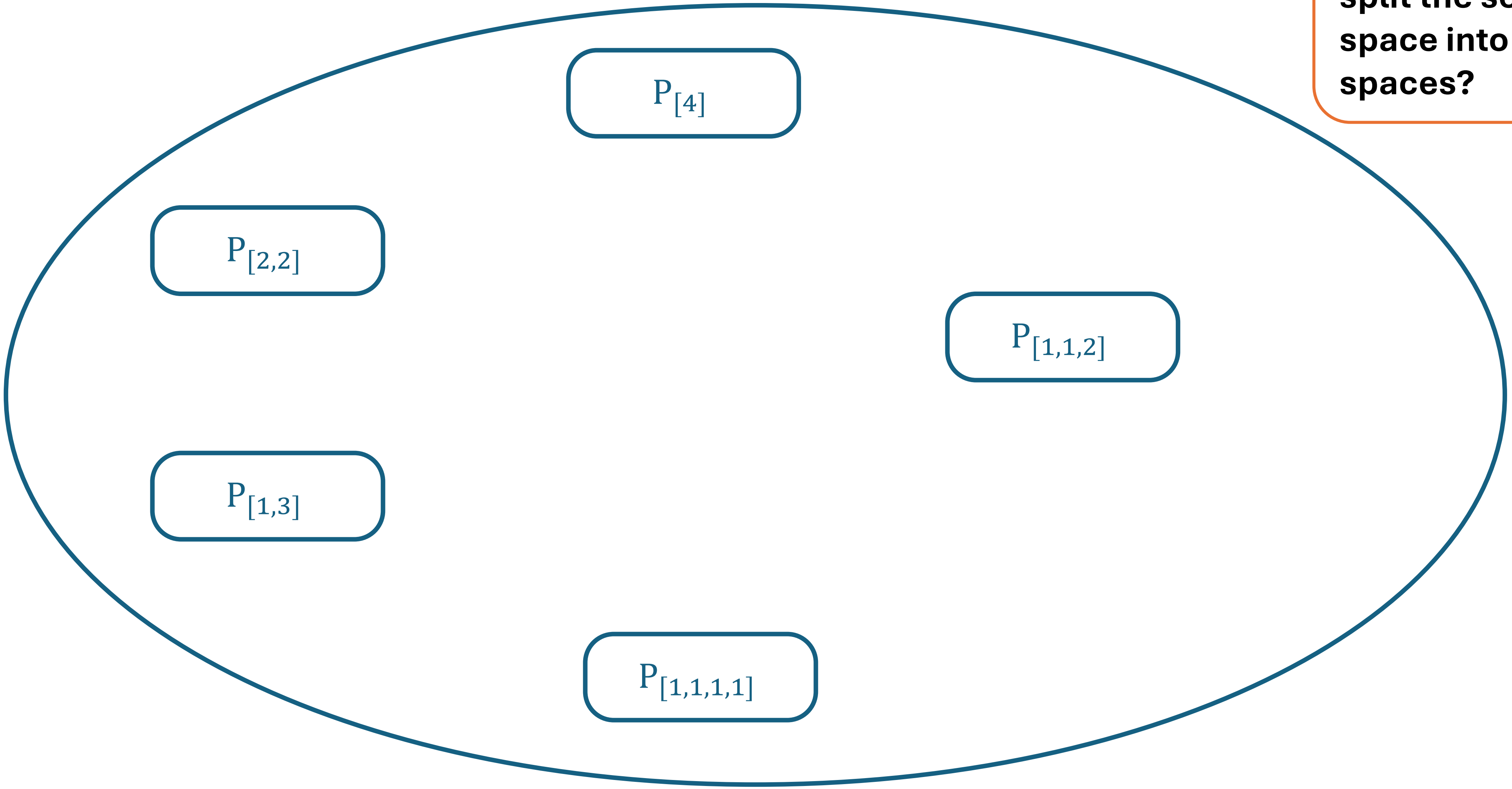
3) We chose one sub-space and search for the solution.



5) We repeat 3 and 4 until a stopping criteria hold

IP Method: Example

Assume we have 4 agents $Ag = \{1, 2, 3, 4\}$, how we split the search space into sub-spaces?



IP Method: Example (1)

Assume we have 4 agents $Ag = \{1, 2, 3, 4\}$, how we split the search space into sub-spaces?

$P_{[4]}$: $\{\{1,2,3,4\}\}$

$P_{[2,2]}$:
- $\{\{1,2\},\{3,4\}\}$
- $\{\{1,3\},\{2,4\}\}$
- $\{\{1,4\},\{2,3\}\}$

$P_{[1,1,2]}$:
- $\{\{1\},\{2\},\{3,4\}\}$
- $\{\{1\},\{3\},\{2, 4\}\}$
- $\{\{1\},\{4\},\{2,3\}\}$
- $\{\{2\},\{3\},\{1,4\}\}$
- $\{\{2\},\{4\},\{1,3\}\}$
- $\{\{3\},\{4\},\{1,2\}\}$

$P_{[1,3]}$:
- $\{\{1\},\{2,3,4\}\}$
- $\{\{2\},\{1,3,4\}\}$
- $\{\{3\},\{1,2,4\}\}$
- $\{\{4\},\{1,2,3\}\}$

$P_{[1,1,1,1]}$: $\{\{1\},\{2\},\{3\},\{4\}\}$

IP Method: Example (1)

P_1	value	P_2	value	P_3	value	P_4	value
$\{1\}$	125	$\{1,2\}$	175	$\{1,2,3\}$	200	$\{1,2,3,4\}$	425
$\{2\}$	50	$\{1,3\}$	150	$\{1,2,4\}$	150		
$\{3\}$	75	$\{1,4\}$	100	$\{1,3,4\}$	300		
$\{4\}$	150	$\{2,3\}$	150	$\{2,3,4\}$	150		
		$\{2,4\}$	200				
		$\{3,4\}$	125				

Calculate the upper bound $\alpha^* = 600$

$P_{[2,2]}$
- $\{\{1,2\},\{3,4\}\}$
- $\{\{1,3\},\{2,4\}\}$
- $\{\{1,4\},\{2,3\}\}$

$P_{[1,1,2]}$
- $\{\{1\},\{2\},\{3,4\}\}$
- $\{\{1\},\{3\},\{2,4\}\}$
- $\{\{1\},\{4\},\{2,3\}\}$
- $\{\{2\},\{3\},\{1,4\}\}$
- $\{\{2\},\{4\},\{1,3\}\}$
- $\{\{3\},\{4\},\{1,2\}\}$

$P_{[4]}$
- $\{\{1,2,3,4\}\}$

$P_{[1,3]}$
- $\{\{1\},\{2,3,4\}\}$
- $\{\{2\},\{1,3,4\}\}$
- $\{\{3\},\{1,2,4\}\}$
- $\{\{4\},\{1,2,3\}\}$

$P_{[1,1,1,1]}$
- $\{\{1\},\{2\},\{3\},\{4\}\}$

IP Method: Example (1)

P_1	value	P_2	value	P_3	value	P_4	value
$\{1\}$	125	$\{1,2\}$	175	$\{1,2,3\}$	200	$\{1,2,3,4\}$	**425**
$\{2\}$	50	$\{1,3\}$	150	$\{1,2,4\}$	150		
$\{3\}$	75	$\{1,4\}$	100	$\{1,3,4\}$	**300**		
$\{4\}$	**150**	$\{2,3\}$	150	$\{2,3,4\}$	150		
		$\{2,4\}$	**200**				
		$\{3,4\}$	125				

Calculate the upper bound $\alpha^* = 600$

$\alpha = 400$

$P_{[2,2]}$: $\{\{1,2\},\{3,4\}\}$, $\{\{1,3\},\{2,4\}\}$, $\{\{1,4\},\{2,3\}\}$

$\alpha = 450$

$P_{[1,3]}$: $\{\{1\},\{2,3,4\}\}$, $\{\{2\},\{1,3,4\}\}$, $\{\{3\},\{1,2,4\}\}$, $\{\{4\},\{1,2,3\}\}$

$\alpha = 500$

$P_{[1,1,2]}$: $\{\{1\},\{2\},\{3,4\}\}$, $\{\{1\},\{3\},\{2,4\}\}$, $\{\{1\},\{4\},\{2,3\}\}$, $\{\{2\},\{3\},\{1,4\}\}$, $\{\{2\},\{4\},\{1,3\}\}$, $\{\{3\},\{4\},\{1,2\}\}$

$\alpha = 600$

$P_{[1,1,1,1]}$: $\{\{1\},\{2\},\{3\},\{4\}\}$

$\alpha = 425$

$P_{[4]}$: $\{\{1,2,3,4\}\}$

IP Method: Example (1)

P_1	value	P_2	value	P_3	value	P_4	value
$\{1\}$	125	$\{1,2\}$	175	$\{1,2,3\}$	200	$\{1,2,3,4\}$	**425**
$\{2\}$	50	$\{1,3\}$	150	$\{1,2,4\}$	150		
$\{3\}$	75	$\{1,4\}$	100	$\{1,3,4\}$	**300**		
$\{4\}$	**150**	$\{2,3\}$	150	$\{2,3,4\}$	150		
		$\{2,4\}$	**200**				
		$\{3,4\}$	125				
Avg	100		150		200		425

Calculate the upper bound $\alpha^* = 600$

Calculate the lower bound $\beta^* = 425$

$\alpha = 400$

$P_{[2,2]}$

$\beta = 300$

$\{\{1,2\},\{3,4\}\}$
$\{\{1,3\},\{2,4\}\}$
$\{\{1,4\},\{2,3\}\}$

$\alpha = 500$

$P_{[1,1,2]}$

$\beta = 350$

$\{\{1\},\{2\},\{3,4\}\}$
$\{\{1\},\{3\},\{2,4\}\}$
$\{\{1\},\{4\},\{2,3\}\}$
$\{\{2\},\{3\},\{1,4\}\}$
$\{\{2\},\{4\},\{1,3\}\}$
$\{\{3\},\{4\},\{1,2\}\}$

$\alpha = 425$

$P_{[4]}$

$\beta = 425$

$\{\{1,2,3,4\}\}$

$\alpha = 450$

$P_{[1,3]}$

$\beta = 300$

$\{\{1\},\{2,3,4\}\}$
$\{\{2\},\{1,3,4\}\}$
$\{\{3\},\{1,2,4\}\}$
$\{\{4\},\{1,2,3\}\}$

$\alpha = 600$

$P_{[1,1,1,1]}$

$\beta = 400$

$\{\{1\},\{2\},\{3\},\{4\}\}$

IP Method: Example (2)

P_1	value	P_2	value	P_3	value	P_4	value	P_5	value
{1}	20	{1, 2}	40	{1,2,3}	65	{1,2,3,4}	85	{1,2,3,4,5}	**165**
{2}	15	{1, 3}	30	{1,2,4}	55	{1,2,3,5}	**140**		
{3}	25	{1, 4}	20	{1,2,5}	70	{1,2,4,5}	90		
{4}	**30**	{1,5}	45	{1,3,4}	60	{1,3,4,5}	75		
{5}	10	{2, 3}	40	{1,3,5}	75	{2,3,4,5}	110		
		{2,4}	**65**	{1,4,5}	55				
		{2,5}	20	{2,3,4}	70				
		{3,4}	30	{2,3,5}	**75**				
		{3,5}	45	{2,4,5}	65				
		{4,5}	15	{3,4,5}	60				
Avg	20		35		65		100		165

We have 5 Agents

$P_{[5]}$
$\beta = 165, \alpha = 165$

$P_{[1,4]}$
$\beta = 120, \alpha = 170$

$P_{[2,3]}$
$\beta = 100, \alpha = 140$

$P_{[1,1,3]}$
$\beta = 105, \alpha = 135$

$P_{[1,2,2]}$
$\beta = 90, \alpha = 160$

$P_{[1,1,1,2]}$
$\beta = 95, \alpha = 155$

$P_{[1,1,1,1,1]}$
$\beta = 100, \alpha = 150$

The super upper bound $\boldsymbol{\alpha^*} = 170$

The super lower bound $\boldsymbol{\beta^*} = 165$

IP Method: Example (2)

P_1	value	P_2	value	P_3	value	P_4	value	P_5	value
{1}	20	{1, 2}	40	{1,2,3}	65	{1,2,3,4}	85	{1,2,3,4,5}	**165**
{2}	15	{1, 3}	30	{1,2,4}	55	$\boxed{\{1,2,3,5\}}$	$\boxed{\mathbf{140}}$		
{3}	25	{1, 4}	20	{1,2,5}	70	{1,2,4,5}	90		
$\boxed{\{4\}}$	$\boxed{\mathbf{30}}$	{1,5}	45	{1,3,4}	60	{1,3,4,5}	75		
{5}	10	{2, 3}	40	{1,3,5}	75	{2,3,4,5}	110		
		{2,4}	**65**	{1,4,5}	55				
		{2,5}	20	{2,3,4}	70				
		{3,4}	30	{2,3,5}	**75**				
		{3,5}	45	{2,4,5}	65				
		{4,5}	15	{3,4,5}	60				
Avg	20		35		65		100		165

We have 5 Agents

$P_{[5]}$
$\beta = 165, \alpha = 165$

$P_{[1,4]}$
$\beta = 120, \alpha = 170$

$\cancel{P_{[1,1,3]}}$
$\cancel{\beta = 105, \alpha = 135}$

$\cancel{P_{[1,1,1,2]}}$
$\cancel{\beta = 95, \alpha = 155}$

Answer is
$\{\{4\}, \{1,2,3,5\}\}$

$\cancel{P_{[2,3]}}$
$\cancel{\beta = 100, \alpha = 140}$

$\cancel{P_{[1,2,2]}}$
$\cancel{\beta = 90, \alpha = 160}$

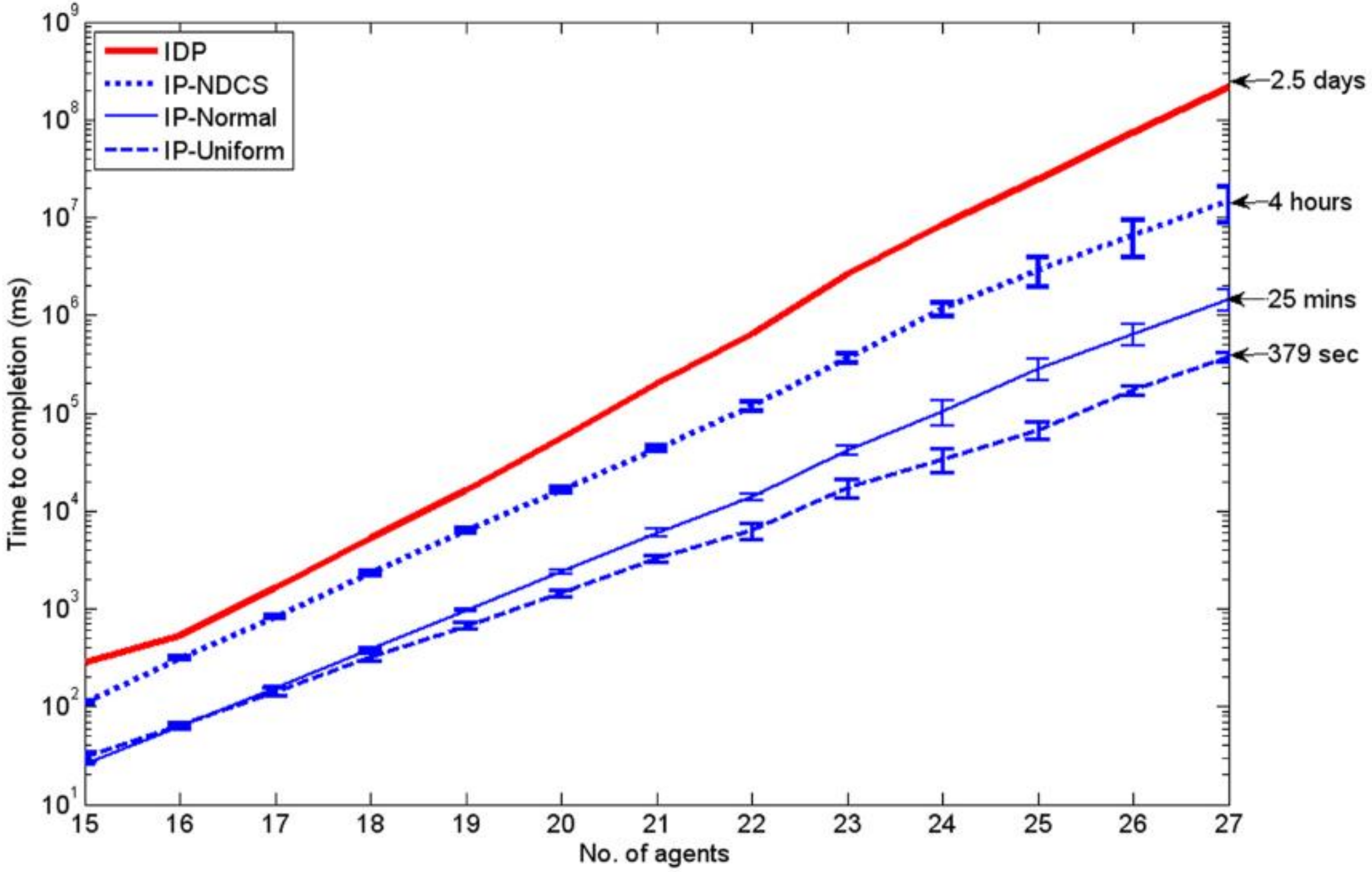
$\cancel{P_{[1,1,1,1,1]}}$
$\cancel{\beta = 100, \alpha = 150}$

The super upper bound $\alpha^* = 170$

The super lower bound $\beta^* = 165$

Let's remove the outbound sub-spaces

IP Method



Time to find the optimal solution

(n,s)-sequences

- Riley et.al. proposed a mechanism for calculating *the coalition value calculation share* for an agent, based on the agent's id
  - Given a set of agents in Ag where $n = |Ag|$, the agents are labelled $1 \ldots n$
  - A coalition of size $1 \leq s < n$ can be generated given an (n,s)-sequence t by first calculating the aggregate offset for each position in t and given the agent x, determine its coalition value calculation share (for some $1 \leq x \leq n$):

$$x_i \equiv \begin{cases} x & \text{if } i = 1 \\ \left(x + \sum_{k=0}^{i-2} (t_k + 1) \bmod n \right) & \text{if } 2 \leq i \leq s \end{cases}$$

  - Note that the result is an agent in the range $(1 \leq x \leq n)$
    - i.e. the result is congruent modulo n

Example

- The (n,s)-sequences for coalitions of size s=3, for a set of agents Ag = {1,2,3,4,5,6} are ⟨0,0,3⟩, ⟨0,1,2⟩, ⟨0,2,1⟩, ⟨1,1,1⟩
  - These are used to generate the coalition value calculation shares for each agent x
    - If each agent generates their share…
    - … all of the coalitions of size s will be generated
  - Duplications occur if there is a repeated periodic sub-sequence in the (n-s)-sequence (e.g. ⟨1,1,1⟩)
    - If s=4, then ⟨0,1,0,1⟩ has the repeating sub-sequence ⟨…0,1…⟩, but ⟨0,0,1,1⟩ has no repeating sequence
  - By tracking which agent generates coalitions from the repeated sequence, duplications can be eliminated

Generating Coalition Value Calculation Shares

The following table lists the coalitions generated for the (n,s)-sequences where n=6 and s=3

	⟨0,0,3⟩	⟨0,1,2⟩	⟨0,2,1⟩	⟨1,1,1⟩
CV^3_1	1,2,3	1,2,4	1,2,5	
CV^3_2	2,3,4	2,3,5	2,3,6	
CV^3_3	3,4,5	3,4,6	3,4,1	
CV^3_4	4,5,6	4,5,1	4,5,2	4,6,2
CV^3_5	5,6,1	5,6,2	5,6,3	5,1,3
CV^3_6	6,1,2	6,1,3	6,1,4	

$$x_i \equiv \begin{cases} x & \text{if} \quad i = 1 \\ (x + \sum_{k=0}^{i-2} (t_k + 1)) \bmod n & \text{if} \quad 2 \le i \le s \end{cases}$$

Example

- The (n,s)-sequences for coalitions of size s=3, for a set of agents Ag = {1,2,3,4,5,6} are ⟨0,0,3⟩, ⟨0,1,2⟩, ⟨0,2,1⟩, ⟨1,1,1⟩

- Therefore, if we consider agent 5:
  - C(5, ⟨0,0,3⟩) ≡ {5, 6, 1}
    - i.e. {5, (5+0+1) mod 6, ((5+0+1) + 0 +1) mod 6} ≡ {5, 6, 1}
  - C(5, ⟨0,1,2⟩) ≡ {5, 6, 2}
    - i.e. {5, (5+0+1) mod 6, ((5+0+1) + 1 +1) mod 6} ≡ {5, 6, 2}
  - C(5, ⟨0,2,1⟩) ≡ {5, 6, 3}
    - i.e. {5, (5+0+1) mod 6, ((5+0+1) + 2 +1) mod 6} ≡ {5, 6, 3}
  - C(5, ⟨1,1,1⟩) ≡ {5, 1, 3}
    - i.e. {5, (5+1+1) mod 6, ((5+1+1) + 1 +1) mod 6} ≡ {5, 1, 3}

Generating Coalition Value Calculation Shares

The following table lists the coalitions generated for the (n,s)-sequences where n=6 and s=3

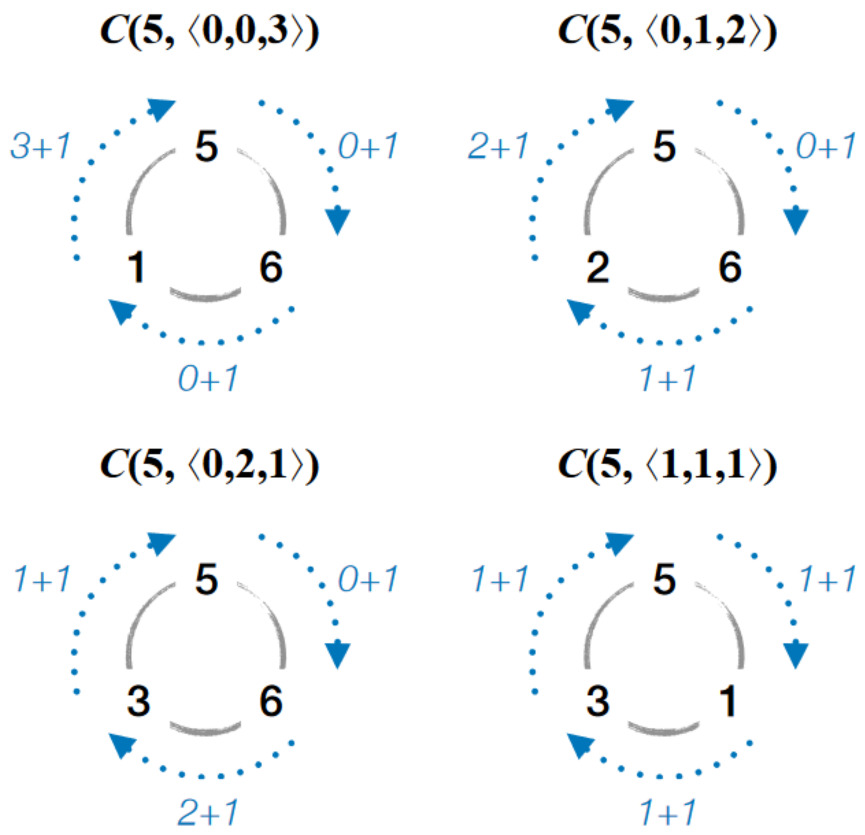
	⟨0,0,3⟩	⟨0,1,2⟩	⟨0,2,1⟩	⟨1,1,1⟩
CV^3_1	1,2,3	1,2,4	1,2,5	
CV^3_2	2,3,4	2,3,5	2,3,6	
CV^3_3	3,4,5	3,4,6	3,4,1	
CV^3_4	4,5,6	4,5,1	4,5,2	4,6,2
CV^3_5	5,6,1	5,6,2	5,6,3	5,1,3
CV^3_6	6,1,2	6,1,3	6,1,4	

$$x_i \equiv \begin{cases} x & \text{if} \quad i = 1 \\ (x + \sum_{k=0}^{i-2} (t_k + 1)) \bmod n & \text{if} \quad 2 \leq i \leq s \end{cases}$$

Example

- The (n,s)-sequences for coalitions of size s=3, for a set of agents Ag = {1,2,3,4,5,6} are ⟨0,0,3⟩, ⟨0,1,2⟩, ⟨0,2,1⟩, ⟨1,1,1⟩

- Therefore, if we consider agent 5:
  - C(5, ⟨0,0,3⟩) ≡ {5, 6, 1}
    - i.e. {5, (5+0+1) mod 6, ((5+0+1) + 0 +1) mod 6} ≡ {5, 6, 1}
  - C(5, ⟨0,1,2⟩) ≡ {5, 6, 2}
    - i.e. {5, (5+0+1) mod 6, ((5+0+1) + 1 +1) mod 6} ≡ {5, 6, 2}
  - C(5, ⟨0,2,1⟩) ≡ {5, 6, 3}
    - i.e. {5, (5+0+1) mod 6, ((5+0+1) + 2 +1) mod 6} ≡ {5, 6, 3}
  - C(5, ⟨1,1,1⟩) ≡ {5, 1, 3}
    - i.e. {5, (5+1+1) mod 6, ((5+1+1) + 1 +1) mod 6} ≡ {5, 1, 3}



Summary

- In this lecture we have looked at mechanisms for identifying coalitions.
  - The notion of a stable coalition game was presented, through the idea of a Core
  - The Shapley Value was then introduced, to determine the contribution that different agents may have on a coalition.
- The problem of representing coalitional games and characteristic functions was then discussed, including:
  - Induced Subgraphs
  - Marginal Contribution Nets.
- We finally talked about Coalition Structure Generation

Readings for this week

- Chapters 13 of the book by M.Wooldridge “An introduction to Multi-Agent Systems” (2nd edition).