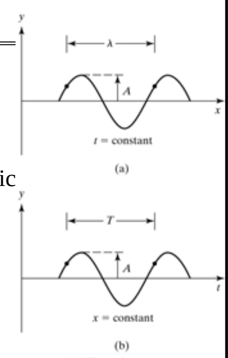
Ch4 Wave Equations & Ch5 Superposition of Waves

<u>Suggested problems</u>:
For Chapter4 & 5: p112 {5, 15, 20} and p129{3}

4.2 Harmonic Waves



- One-dimensional wave equations: $\boxed{\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{\upsilon^2}\dfrac{\partial^2 y}{\partial t^2}}$

- Harmonic wave functions are adapted to represent electromagnetic waves, including light wave

$$\boxed{y(x,t) = A_{\cos}^{\sin}\left[k(x \pm \upsilon t)\right]}$$

- φ : Phase; $\boxed{\varphi = k(x \pm \upsilon t)}$

- Condition of constant phase: When x and t change together in such a way that φ is a constant, the <u>displacement</u> $y = A\sin\varphi$ is also constant

$$\boxed{\varphi = k(x - \upsilon t) = kx - \omega t = \text{constant}} \Rightarrow \boxed{\upsilon = \left|\left(\frac{\partial x}{\partial t}\right)\right|_{\varphi} = \frac{\omega}{k}}$$

- υ : Phase velocity (or wave velocity): speed of wave propagation; this is the velocity at which the phase of any one frequency component of the wave will propagate.

4.2 Harmonic Waves

- Other expressions of harmonic waves:

$$y(x,t) = A_{\cos}^{\sin}\left[k(x \pm \upsilon t)\right]$$
$$y(x,t) = A_{\cos}^{\sin}\left[kx \pm \omega t\right]$$
$$y(x,t) = A_{\cos}^{\sin}\left[2\pi(\frac{x}{\lambda} \pm \frac{t}{T})\right]$$

- Relationship between parameters:

k : propagation constant
k_0 : propagation constant in vacuum
λ : wavelength
λ_0 : wavelength in vacuum
υ : velocity
c : light speed
ω : Angular frequency
ν : frequency
T : period
n : refractive index

$$k = \frac{2\pi}{\lambda} = \frac{2\pi n}{\lambda_0} = nk_0$$

$$\lambda = \frac{\upsilon}{\nu} = \frac{c}{n\nu} = \left(\frac{c}{\nu}\right)\frac{1}{n} = \frac{\lambda_o}{n}$$

$$\upsilon = c/n \qquad \omega = 2\pi\nu$$

$$\upsilon = \frac{\omega}{k} \qquad \nu = \frac{1}{T} \qquad T = \lambda/\upsilon$$

- General expression of harmonic waves:

$$y(x,t) = A_{\cos}^{\sin}\left[k(x \pm \upsilon t) + \varphi_0\right]$$

φ_0 : Initial phase

4.2 Harmonic Waves

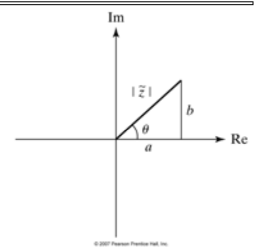
Problems 4-7

For a harmonic wave given by $y = (10\ \text{cm})\sin\left[(628.3/\text{cm})x - (6283/\text{s})t\right]$
Determine (a) wavelength; (b) frequency; (c) propagation constant; (d) angular frequency; (e) period; (f) velocity; (g) amplitude

4.3 Complex Numbers

- Any complex number:

$$\tilde{z} = a + ib = |\tilde{z}|(\cos\theta + i\sin\theta)$$
$$\text{with} \quad |\tilde{z}| = \sqrt{a^2 + b^2}\,, \; \theta = \tan^{-1}\left(\frac{b}{a}\right)$$



Euler formula:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \Longrightarrow \boxed{\tilde{z} = |\tilde{z}|e^{i\theta}}$$

- Complex conjugate: $\boxed{\tilde{z}^* = a - ib = |\tilde{z}|e^{-i\theta}}$

θ	$e^{i\theta}$
0	1
$\pi/2$	i
π	-1
$3\pi/2$	$-i$

- Frequently used values:

4.4 Harmonic Waves as Complex Functions

- Harmonic wave function : $\boxed{y = A^{\sin}_{\cos}\left[kx \pm \omega t\right]}$
- Wave function in complex form : $\boxed{\tilde{y} = Ae^{i(kx-\omega t)}}$

$$\Longrightarrow \quad \boxed{y = \mathrm{Re}(\tilde{y}) = A\cos(kx - \omega t)} \quad \text{or} \quad \boxed{y = \mathrm{Im}(\tilde{y}) = A\sin(kx - \omega t)}$$

4.5 Plane Waves

$$\boxed{\psi = A\sin\left[\vec{k}\cdot\vec{r} - \omega t\right]} \quad \text{or} \quad \boxed{\psi = Ae^{i(\vec{k}\cdot\vec{r}-\omega t)}}$$

$$\text{with} \quad \boxed{\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z}$$

4.6 Spherical Waves

$$\boxed{\psi = \left(\frac{A}{r}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)}}$$

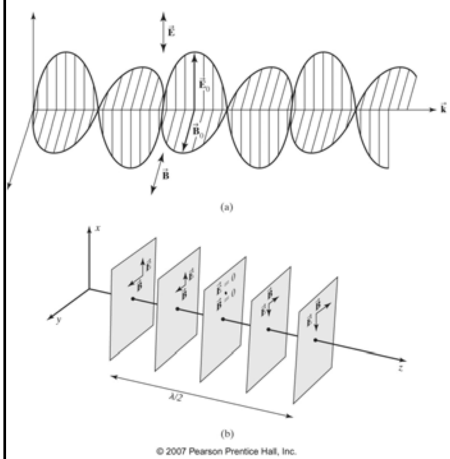
4.8 Electromagnetic Waves

- Harmonic wave functions :

$$\boxed{\vec{E} = \vec{E}_0 \sin(\vec{k}\cdot\vec{r} - \omega t),} \qquad \boxed{E = cB}$$

$$\boxed{\vec{B} = \vec{B}_0 \sin(\vec{k}\cdot\vec{r} - \omega t)} \qquad \boxed{c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}}$$

$$\boxed{\varepsilon_0 = \frac{1}{36\pi}\times 10^{-9},\ \ \mu_0 = 4\pi\times 10^{-7}}$$



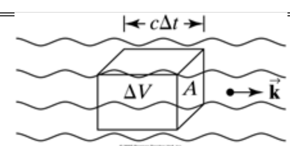
Energy density:

$$\boxed{u_E = \frac{1}{2}\varepsilon_0 E^2 \quad u_B = \frac{1}{2\mu_0}B^2}$$

Total energy density:

$$\boxed{u = u_E + u_B = 2u_E = 2u_B = \varepsilon_0 E^2 = \frac{B^2}{\mu_0}}$$

4.8 Electromagnetic Waves



- Power : $\boxed{\frac{\text{energy}}{\Delta t} = \frac{u\cdot\Delta V}{\Delta t} = \frac{u\cdot Ac\Delta t}{\Delta t} = ucA}$

- Power per unit area S : $\boxed{S = uc}$

- Poynting vector S : $\boxed{\vec{S} = \vec{E}\times\vec{H}}$

$$\boxed{S = EH = \frac{EB}{\mu_0} = \frac{cB^2}{\mu_0} = cu}$$

- Irradiance, E_e: A time average of the power delivered per unit area

$$\boxed{E_e = \left\langle \left|\vec{S}\right| \right\rangle = c\varepsilon_0 E_0^2 \left\langle \sin^2\left(\vec{k}\cdot\vec{r} - \omega t\right)\right\rangle} \qquad \boxed{\left\langle \sin^2\left(\vec{k}\cdot\vec{r} - \omega t\right)\right\rangle = \frac{1}{2}}$$

$$\Rightarrow \boxed{E_e = \frac{1}{2}c\varepsilon_0 E_0^2 = \frac{1}{2\mu_0}cB_0^2}$$

(E_0 and B_0: Amplitudes of E and B fields)

Example 4-2
A laser beam of radius 1 mm carries a power of 6 kW. Determine its average irradiance and the amplitude of its E and B field

4.9 Light Polarization

$$\boxed{\vec{E} = E_0 \sin(kz - \omega t)\hat{x}}$$

$$\boxed{\vec{B} = B_0 \sin(kz - \omega t)\hat{y}}$$

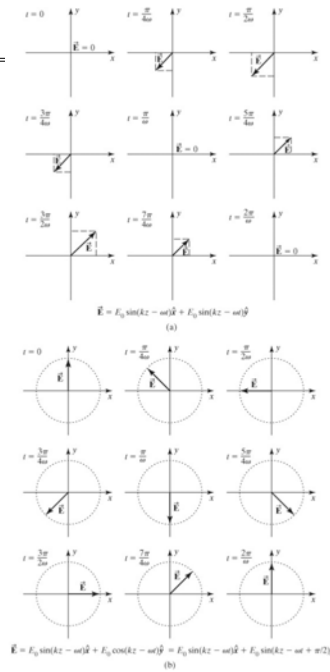
$$\boxed{\vec{S} = \vec{E} \times \vec{H} = E_0 B_0 \sin^2(kz - \omega t)\hat{z}}$$

- The direction of energy propagation is the direction of the poynting vector $\boxed{\vec{E} \times \vec{H}}$

- The direction of electric field is known as the polarization of the wave

Linear: $\boxed{\vec{E} = E_{0x} \sin(kz - \omega t)\hat{x} + E_{0y} \sin(kz - \omega t)\hat{y}}$

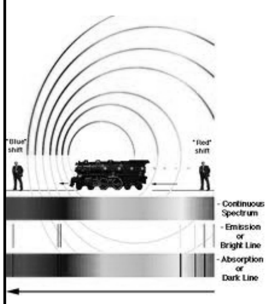
Circular or elliptical: $\boxed{\vec{E} = E_{0x} \sin(kz - \omega t)\hat{x} + E_{0y} \cos(kz - \omega t)\hat{y}}$



4.10 Doppler Effect

- Doppler Effect: A change in the observed frequency of a wave, as of sound or light, occurring when the source and observer are in motion relative to each other, with the frequency increasing when the source and observer approach each other and decreasing when they move apart.

Change of wavelength caused by motion of the source or observer



- sirens
-Astronomy

$$\boxed{\frac{\lambda'}{\lambda} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \simeq 1 - \frac{v}{c}}$$

- Doppler Effect can be used to determine the speed of astronomical sources emitting electromagnetic radiation

- redshift: source is away from us (observer)
- blueshift: source is towards us (observer)

Chapter 5 Superposition of Waves

5-1 Superposition Principle

- Superposition principle: In linear system, when two separate waves arrive at the same place in space wherein they overlap, they will simply add to one another. In other word, the resulting disturbance at each point in the region of overlap is the algebraic sum of the individual constituent waves at that location. $\boxed{\psi = \psi_1 + \psi_2}$

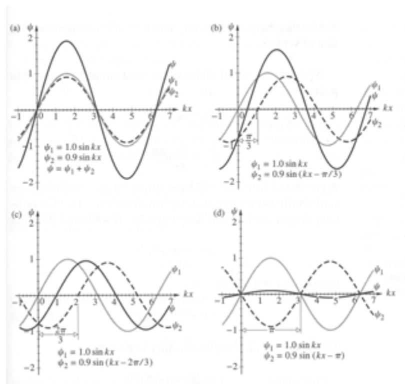


Figure 2.14 The superposition of two sinusoids with amplitudes of $A_1 = 1.0$ and $A_2 = 0.9$. In (a) they are in-phase. In (b) ψ_1 leads ψ_2 by $\pi/3$. In (c) ψ_1 leads ψ_2 by $2\pi/3$. And (d) ψ_1 and ψ_2 are out-of-phase by π and almost cancel each other. To see how the amplitudes can be determined, go to Fig. 2.18.

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$
$$\vec{B} = \vec{B}_1 + \vec{B}_2$$

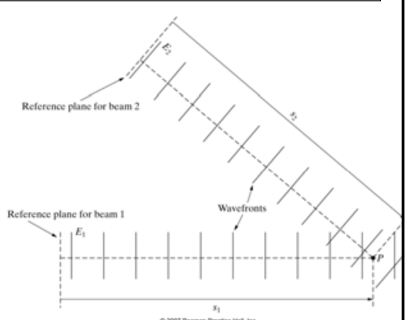
5.2 Superposition of Waves of the Same Frequency

- Harmonic wave functions :

$$E_1 = E_{o1}\cos(ks_1 - \omega t - \varphi_1)$$
$$E_2 = E_{o2}\cos(ks_2 - \omega t - \varphi_2)$$

Set phase:

$$\alpha_1 = ks_1 - \varphi_1, \quad \alpha_2 = ks_2 - \varphi_2 \quad \Rightarrow \quad E_1 = E_{o1}\cos(\alpha_1 - \omega t), \quad E_2 = E_{o2}\cos(\alpha_2 - \omega t)$$



- Phase difference of the two waves, intersecting at fixed point P:

$$\delta = \alpha_2 - \alpha_1 = k(s_2 - s_1) - (\varphi_2 - \varphi_1)$$

- Superposition of the two waves:

$$E_R = E_1 + E_2 = E_{o1}\cos(\alpha_1 - \omega t) + E_{o2}\cos(\alpha_2 - \omega t)$$

5.2 Superposition of Waves of the Same Frequency

- General superposition (by using complex method) :
- Write wave functions in complex form: $\boxed{\tilde{E}_1 = E_{01} e^{i(\alpha_1 - \omega t)}}$, $\boxed{\tilde{E}_2 = E_{02} e^{i(\alpha_2 - \omega t)}}$

$$\Rightarrow \boxed{\tilde{E} = \tilde{E}_1 + \tilde{E}_2 = e^{-i\omega t}(E_{01} e^{i\alpha_1} + E_{02} e^{i\alpha_2})}$$

- Assume the resultant wave : $\boxed{\tilde{E} = E_0 e^{i(\alpha - \omega t)} = e^{-i\omega t} \cdot E_0 e^{i\alpha}}$
- By comparing the two terms, we have: $\boxed{E_0 e^{i\alpha} = E_{01} e^{i\alpha_1} + E_{02} e^{i\alpha_2}}$

$$\boxed{E_0^2 = E_0 e^{i\alpha} \cdot (E_0 e^{i\alpha})^* = (E_{01} e^{i\alpha_1} + E_{02} e^{i\alpha_2}) \cdot (E_{01} e^{-i\alpha_1} + E_{02} e^{-i\alpha_2})}$$

$\Rightarrow$ Resultant Amplitude : $\boxed{E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_2 - \alpha_1)}$

Resultant Phase : $\boxed{\tan\alpha = \dfrac{E_{o1}\sin\alpha_1 + E_{o2}\sin\alpha_2}{E_{o1}\cos\alpha_1 + E_{o2}\cos\alpha_2}}$

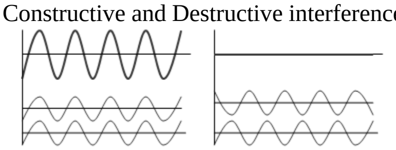
5.2 Superposition of Waves of the Same Frequency

- Resultant Amplitude and Phase :

$$\boxed{E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_2 - \alpha_1),} \qquad \boxed{\tan\alpha = \frac{E_{o1}\sin\alpha_1 + E_{o2}\sin\alpha_2}{E_{o1}\cos\alpha_1 + E_{o2}\cos\alpha_2}}$$

- Assuming : $\varphi_1 = \varphi_2$, phase difference: $\boxed{\delta = \alpha_2 - \alpha_1 = k(s_2 - s_1) = \dfrac{2\pi n}{\lambda_0}(s_2 - s_1)}$
- Interference term : $\boxed{2E_{01}E_{02}\cos\delta}$

Constructive and Destructive interference



Constructive interference: only if $\boxed{\delta = 2m\pi}$

Destructive interference: only if $\boxed{\delta = (2m+1)\pi}$

Constructive Interference:

$$\boxed{\begin{aligned} &\alpha_2 - \alpha_1 = 2m\pi,\ E_0 = E_{01} + E_{02} \\ &E_R = (E_{01} + E_{02})\cos(\alpha_1 - \omega t) \end{aligned}}$$

Destructive Interference:

$$\boxed{\begin{aligned} &\alpha_2 - \alpha_1 = (2m+1)\pi,\ E_0 = E_{01} - E_{02} \\ &E_R = (E_{01} - E_{02})\cos(\alpha_1 - \omega t) \end{aligned}}$$

Young's Double Slit Interference: http://vsg.quasihome.com/interfer.htm

5.2 Superposition of Waves of the Same Frequency

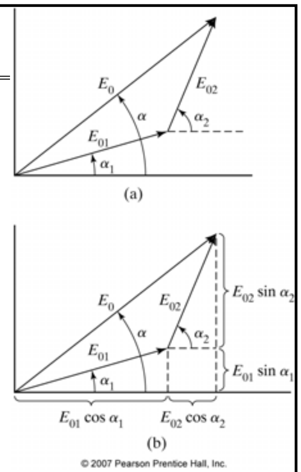
➢ Phase diagram:

1. For 2 harmonic waves (see Fig.a,b):

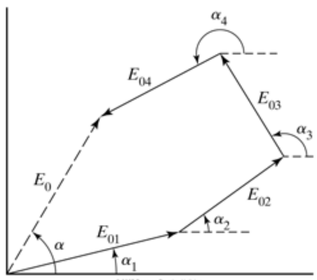
Resultant phase: $$\boxed{\tan\alpha = \frac{E_{o1}\sin\alpha_1 + E_{o2}\sin\alpha_2}{E_{o1}\cos\alpha_1 + E_{o2}\cos\alpha_2}}$$

Resultant amplitude:

$$\boxed{E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_2 - \alpha_1)}$$



2. For N harmonic waves:



$$\boxed{\tan\alpha = \frac{\sum_{i=1}^{N} E_{oi}\sin\alpha_i}{\sum_{i=1}^{N} E_{oi}\cos\alpha_i}}$$

$$\boxed{E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2\sum_{j>i}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}\cos(\alpha_j - \alpha_i)}$$

5.2 Superposition of Waves of the Same Frequency

Example 5-1

Determine the result of the superposition of the following harmonic waves:

$E_1 = 7\cos(\pi/3 - \omega t),\ E_2 = 12\sin(\pi/4 - \omega t),$ and $E_3 = 20\cos(\pi/5 - \omega t)$