Chapter 11. Fraunhofer Diffraction

Introduction

- Diffraction is any deviation from geometrical optics that results from obstruction of a wavefront of light.
- For example, an opaque screen with a round hole represents such an obstruction
- Different apertures (such as single slit, multiple slits, circular) will have different diffraction patterns

Slit aperture

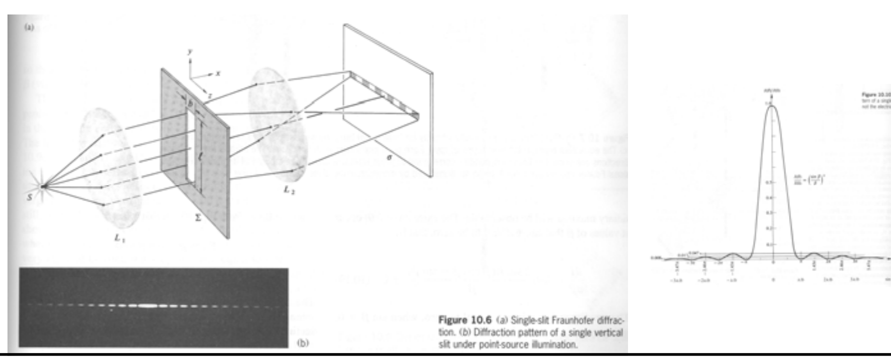
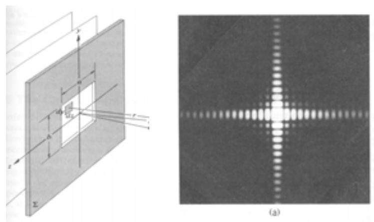


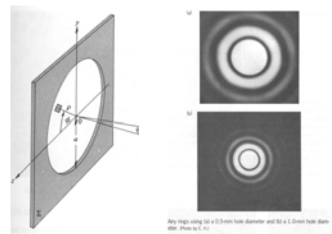
Figure 10.6 (a) Single-slit Fraunhofer diffraction. (b) Diffraction pattern of a single vertical slit under point-source illumination.

Introduction

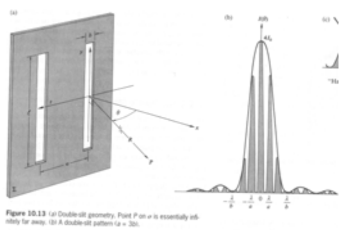
Rectangle aperture

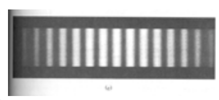


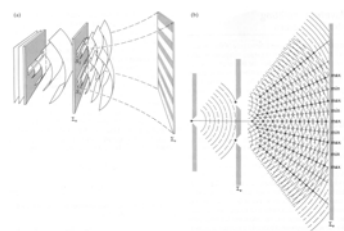
Circular aperture



Double-slit aperture ⟷ *Young' double-slit experiment*





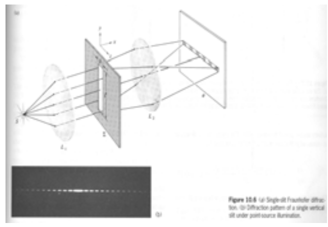


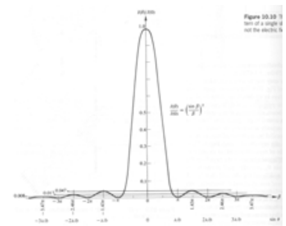
Introduction

- Difference between interference and diffraction
- There is no fundamental physical distinction between diffraction and interference.
- Diffraction is often distinguished from interference on this bases:
  - It is customary, to speak of diffraction when taking account the finite size of the slits as a continuous array of sources rather than point source in interference.
- Two types of diffractions:
  - Fraunhofer diffraction, or far-field diffraction
  - Fresnel diffraction or near-field diffraction
- Fraunhofer diffraction: Both the source and observation screen are effectively far enough from the diffraction aperture, so that wavefronts arriving at the aperture and observation screen may be considered plane

11.1 Diffraction From a Single Slit

- It is customary, to speak of diffraction when taking account the finite size of the slits as a continuous array of sources rather than point source in interference.

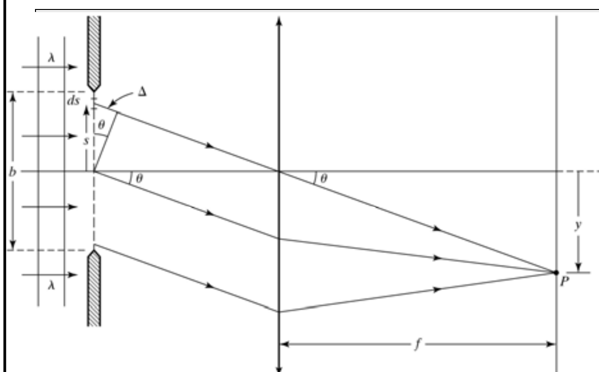




Questions ?

- What are the diffraction patterns (irradiance distributions) for single slit or double slit aperture?
- How to find the irradiance distributions?
- Where are the angular maximum, or minimum positions ?

11.1 Diffraction From a Single Slit



- Parameters: ds, E_L, r, b, θ, f, y

  E_L : Source strength per unit length
  r : Optical-path length from ds to P
  b: Slit width
  θ: angular radius
  f: focal length
  y: vertical displacement on screen

- The electric field of the spherical wavelet contributed by each interval of length ds:

By setting at center $r = r_0$, then any point at slit $r = r_0 + \Delta$

$$\boxed{dE_p = \left(\frac{E_L ds}{r}\right) e^{i(kr-\omega t)}} \quad \Rightarrow \quad \boxed{dE_p = \left(\frac{E_L ds}{r_o + \Delta}\right) e^{i[k(r_o+\Delta)-\omega t]}}$$

$$\boxed{\text{with} \quad \Delta = s \cdot \sin\theta}$$

11.1 Diffraction From a Single Slit

$$\boxed{dE_p = \left(\frac{E_L ds}{r_o + \Delta}\right) e^{i[k(r_o+\Delta)-\omega t]}} \quad \text{with } \Delta = s \cdot \sin\theta \quad \Longrightarrow \quad \boxed{dE_p = \left(\frac{E_L ds}{r_o}\right) e^{i(kr_o - \omega t)} e^{iks\sin\theta}}$$

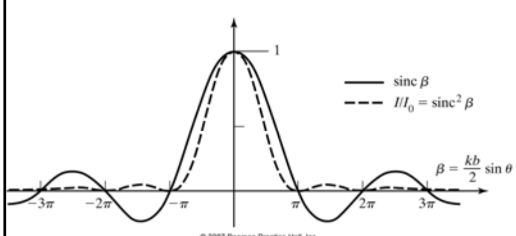
- Total electric field: $E_p = \int_{slit} dE_p = \frac{E_L}{r_o} e^{i(kr_o-\omega t)} \int_{-b/2}^{b/2} e^{iks\sin\theta} ds$

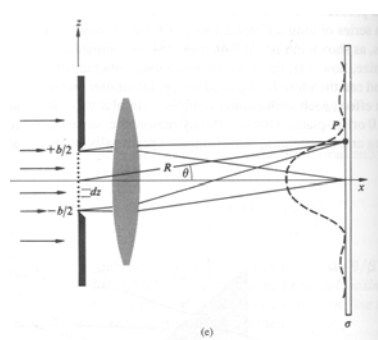
$$\beta = \frac{1}{2} kb\sin\theta \quad \Longrightarrow \quad E_p = \frac{E_L b}{r_o} \frac{\sin\beta}{\beta} e^{i(kr_o-\omega t)}$$

- Amplitude of the electric field: $E_0 = \frac{E_L b}{r_o} \frac{\sin\beta}{\beta}$
- At any angle θ, the irradiance is: $I = \left(\frac{\varepsilon_0 c}{2}\right) E_0^2 = I_0 \frac{\sin^2\beta}{\beta^2}$ with $I_0 = \frac{\varepsilon_0 c}{2}\left(\frac{E_L b}{r_o}\right)^2$
- Consider: $\theta = 0, \quad \sin\beta/\beta = 1$ $\Longrightarrow$ $I = I_0$ **I_0: principle maximum**

11.1 Diffraction From a Single Slit

- At any angle θ, the irradiance is: $I(\theta) = I_0 \frac{\sin^2\beta}{\beta^2} = I_0 \text{sinc}^2\beta$





Question : where are the subsidiary max, min positions ?

11.1 Diffraction From a Single Slit

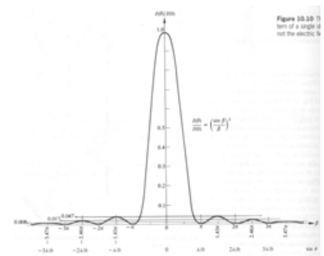
$$I(\theta) = I_0 \frac{\sin^2 \beta}{\beta^2}$$

- Maximum irradiance at center (θ = 0): $\boxed{I(\theta) = I_0}$

- Angular positions of m-th order minimum :

$$\boxed{\sin\beta = 0, \quad \beta = (kb/2)\sin\theta = \mathrm{m}\pi}$$

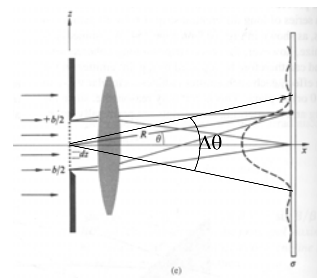
$$\Rightarrow \quad \boxed{b\sin\theta_m = \mathrm{m}\lambda}$$



- Vertical displacement y for zero irradiance :

$$\boxed{y_m \cong f\sin\theta_m} \quad \Rightarrow \quad \boxed{y_m \cong \frac{\mathrm{m}\lambda f}{b}}$$

- Angular spread of the central maximum: or Angular width $\boxed{\Delta\theta = \frac{2\lambda}{b}}$



11.1 Diffraction From a Single Slit

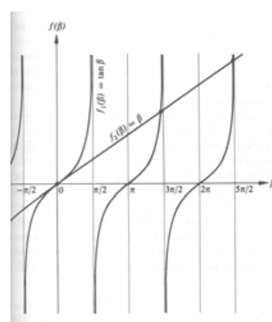
$$I(\theta) = I_0 \frac{\sin^2 \beta}{\beta^2}$$

- The secondary maxima are located at :

$$\boxed{\frac{dI(\theta)}{d\beta} = I_0 \frac{d\left[(\sin\beta/\beta)^2\right]}{d\beta} = I_0\left[2(\sin\beta/\beta)\right]\cdot\left[\frac{\cos\beta}{\beta} - \frac{\sin\beta}{\beta^2}\right] = 0}$$

$$\Rightarrow \quad \boxed{\tan\beta = \beta}$$

$$\Rightarrow \quad \boxed{\beta = \pm 1.4303\pi, \pm 2.459\pi, \pm 3.4707}$$

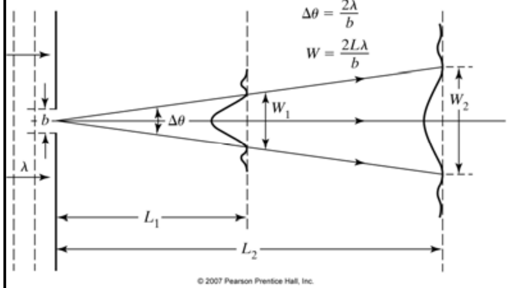


10.8 The points of intersection of the two curves are the solu- Eq. (10.21).

Example 11.1

What is the ratio of irradiances at the central peak maximum to the first of the secondary maxima?

11.2 Beam Spreading



- Angular spread of the central maximum:

$$\boxed{\Delta\theta = \frac{2\lambda}{b}}$$

- Width of the central maximum:

$$\boxed{W = L\Delta\theta = \frac{2L\lambda}{b}}$$

- Criterion for Fraunhofer or far field diffraction:

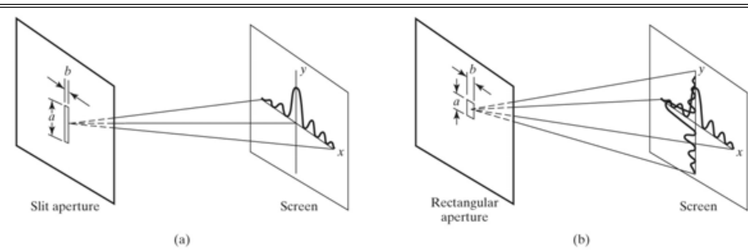
$$\boxed{L_{\min} = \frac{b^2}{2\lambda}} \qquad \boxed{L \gg \frac{b^2}{\lambda}}$$

Example 11.2 (274)

Imagine a parallel beam of 546-nm light of width $b = 0.5\ mm$ propagating a distance of 10 m across the laboratory. Estimate the final width W of the beam due to diffraction spreading

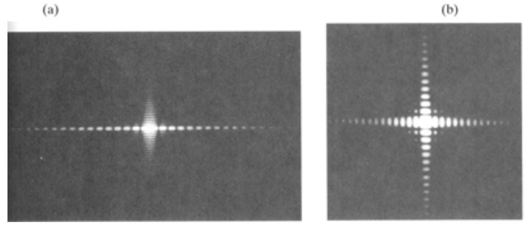
11.3 Rectangular and Circular Apertures

Slit width a, b



$$\boxed{\alpha = \frac{1}{2} ka \sin\theta}$$

$$\boxed{\beta = \frac{1}{2} kb \sin\theta}$$

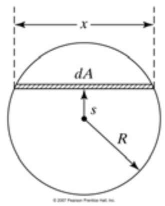


- At any angle θ, the irradiance is: $\boxed{I(\theta) = I_0 \left(\frac{\sin^2\alpha}{\alpha^2}\right)\left(\frac{\sin^2\beta}{\beta^2}\right) = I_0 \mathrm{sinc}^2\alpha\,\mathrm{sinc}^2\beta}$

- Vertical displacement for zero irradiance: $\boxed{y_m \cong \frac{m\lambda f}{b}}$ $\boxed{x_n \cong \frac{n\lambda f}{a}}$

11.3 Rectangular and Circular Apertures

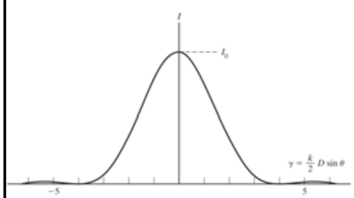
$$\boxed{E_p = \int_{slit} dE_p = \frac{E_L}{r_o} e^{i(kr_o - \omega t)} \int_{-b/2}^{b/2} e^{iks\sin\theta} ds}$$



For circular aperture: $\boxed{E_p = \frac{E_A}{r_o} e^{i(kr_o - \omega t)} \iint_{area} e^{iks\sin\theta} dA}$

$\Rightarrow$ $\boxed{I = I_0 \left(\frac{2J_1(\gamma)}{\gamma} \right)^2}$

with $\gamma = \frac{1}{2} kD\sin\theta$
D: diameter of circular aperture

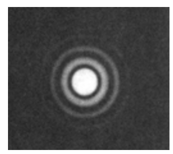


(a)

	γ	$I/I_0 = (2J_1(\gamma)/\gamma)^2$
1st Maximum	0	1
1st Zero	3.832	0
2nd Maximum	5.136	0.0175
2nd Zero	7.016	0
3rd Maximum	8.417	0.00416
3rd Zero	10.173	0
4th Maximum	11.620	0.00160
4th Zero	13.324	0

(b)

- Note: $\boxed{\gamma \to 0, \quad J_1(\gamma)/\gamma = \frac{1}{2}}$
- Maximum irradiance at center (θ = 0): $\boxed{I(0) = I_0}$
- 1st zero irradiance: $\boxed{\gamma = 3.832}$ $\Rightarrow$ $\boxed{D\sin\theta = 1.22\lambda}$
- Far-field angular radius: $\boxed{\Delta\theta_{1/2} = \frac{1.22\lambda}{D}}$



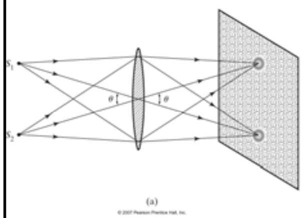
Diffraction image of the circular aperture is called the Airy disc

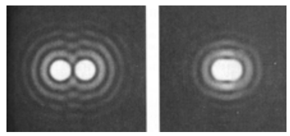
11.3 Rectangular and Circular Apertures

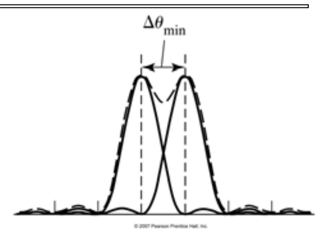
Example 11.3 (278)

Find the diameter of the Airy disc at the center of the diffraction pattern formed on a wall at a distance L = 10 m from a uniformly illuminated circular aperture of diameter D = 0.5 mm. Assume that the illuminating light has wavelength of λ = 546 nm. Compare the beam spread to that from the slit of width b =0.5 mm of Example 11-2.

11.4 Resolution







Rayleigh's criterion for just resolvable diffraction pattern

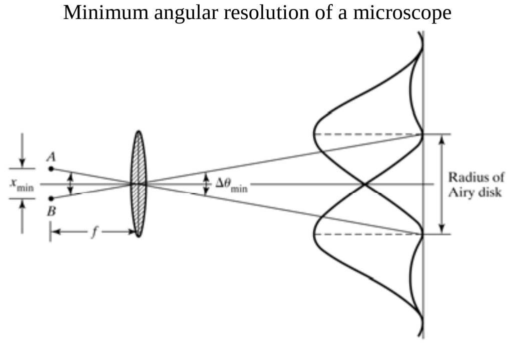
- Rayleigh's criterion: the max. of one pattern falls directly over the 1st min. of the other
- Limit of resolution: $\boxed{(\Delta\theta)_{\min} = \dfrac{1.22\lambda}{D}}$ D: diameter of lens

Example 11.4 (280)

Suppose that each lens on a pair of binoculars has a diameter of 35 mm. How far apart must two stars be before they are theoretically resolvable by either of the lenses in the binoculars?

11.4 Resolution

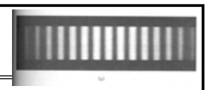
Minimum angular resolution of a microscope

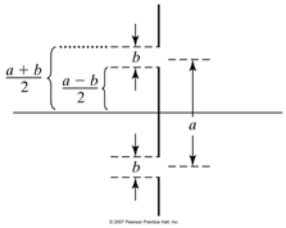


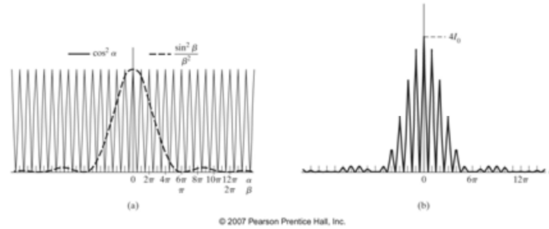
$$\boxed{x_{\min} = f(\Delta\theta)_{\min} = f\left(\frac{1.22\lambda}{D}\right)}$$

- Resolution of a microscope: is roughly equal to the wavelength of light used

11.5 Double-Slit Diffraction







- Parameters: Slit width b and slit separation a

Single Slit: $\boxed{E_p = \int\limits_{slit} dE_p = \frac{E_L}{r_o} e^{i(kr_o - \omega t)} \int_{-b/2}^{b/2} e^{iks\sin\theta} ds}$

Double Slit: $\boxed{E_p = \frac{E_L}{r_o} e^{i(kr_o - \omega t)} \left(\int_{(a-b)/2}^{(a+b)/2} e^{iks\sin\theta} ds + \int_{-(a+b)/2}^{-(a-b)/2} e^{iks\sin\theta} ds \right)}$

- The irradiance can be written as: $\boxed{I = 4I_0 \left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha}$ with $\boxed{\alpha = (ka/2)\sin\theta}$ $\boxed{\beta = (kb/2)\sin\theta}$

I_0: flux-density contribution from either slit

Diffraction **Interference**

11.5 Double-Slit Diffraction

- The irradiance can be written as: $\boxed{I = 4I_0 \left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha}$

I_0: flux-density contribution from either slit

Interference **Diffraction**

- Angular position of <u>Interference maximum</u>: $\boxed{\alpha = (ka/2)\sin\theta = p\pi}$ **or** $\boxed{a\sin\theta = p\lambda}$

- Angular position for <u>Diffraction minima</u>: $\boxed{\beta = (kb/2)\sin\theta = m\pi}$ **or** $\boxed{b\sin\theta = m\lambda}$

- Condition for missing orders: $\boxed{\frac{a}{b} = \frac{p}{m}}$

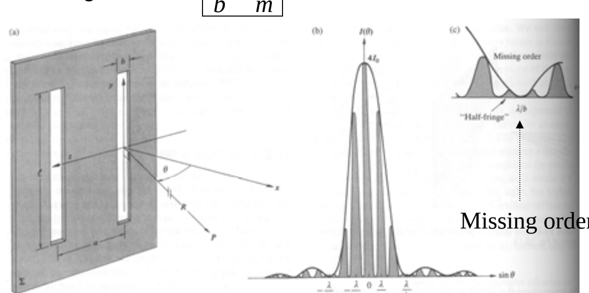


Figure 10.13 (a) Double-slit geometry. Point P on σ is essentially infinitely far away. (b) A double-slit pattern (a = 3b).

11.6 Diffraction from Many Slits

- Irradiance for N-slit diffraction:

$$I(\theta) = I_0 \underbrace{\left(\frac{\sin\beta}{\beta}\right)^2}_{\text{Diffraction}} \underbrace{\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2}_{\text{Interference}}$$

N: number of slit

With $\beta = (kb/2)\sin\theta \qquad \alpha = (ka/2)\sin\theta$

If $N = 2$, $\boxed{I(\theta) = 4I_0\left(\frac{\sin\beta}{\beta}\right)^2 \cos\alpha^2}$

- **Conditions for principle max and secondary min**:

$$\alpha = \frac{p\pi}{N} \quad (p = 0, \pm1, \pm2 \ldots \pm N \ldots \pm 2N)$$

Principle maxima occur for: $p = 0, \pm N, \pm 2N$
Secondary minima occur for: p = all other integer values

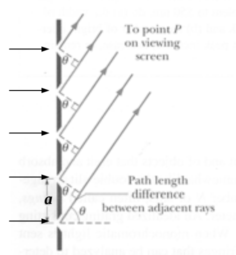
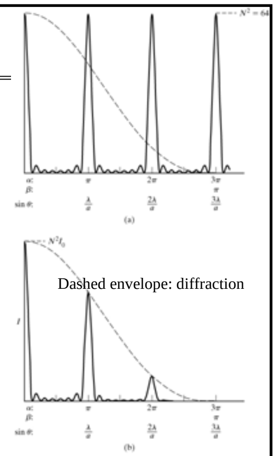


Fig. 37-18 The rays from the rulings in a diffraction grating to a distant point P are approximately parallel. The path length difference between each two adjacent rays is $d \sin\theta$, where θ is measured as shown. (The rulings extend into and out of the page.)



- Applications: A practical device that makes use of multiple-slit diffraction is the diffraction grating

Problems

11.15 (290)

A double-slit diffraction pattern is formed using mercury green light at 546.1nm. Each slit has a width of 0.1mm. The pattern reveals that the 4th-order interference maxima are missing from the pattern.

(a) What is the slit separation?
(b) What is the irradiance of the first three orders of interference fringes, relative to the zeroth-order maximum?

11.19

Make a rough sketch for the irradiance pattern from seven equally spaced slits having a separation-to-width ratio of 4. Label points on the x-axis with corresponding values of α and β.