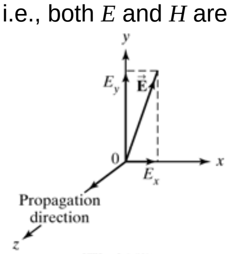
Chapter 14 Matrix Treatment of Polarization

14.1 Mathematical Representation of Polarized Light: Jones Vector

- Light is an EM wave; based on Maxwell' equations, light may be treated as a transverse EM wave, i.e., both E and H are perpendicular to the direction of propagation



- What is polarization of the wave? The *direction* of the electric field *vector* E is known as the *polarization* of the EM wave
  - Linearly polarized: If the vector E does remain in a fixed direction
  - Circular / elliptical polarized: If E can rotate uniformly in the plane x, y
  - Randomly polarized: If the vector E changes randomly with time

http://www.ee.buffalo.edu/faculty/cartwright/java_applets/polarization/index.htm

14.1 Mathematical Representation of Polarized Light: Jones Vector

- Electric field of an EM wave propagating along the z-direction:

$$\boxed{\tilde{E} = E_{0x} e^{i(kz-\omega t+\varphi_x)}\hat{x} + E_{0y} e^{i(kz-\omega t+\varphi_y)}\hat{y} = \tilde{E}_0 e^{i(kz-\omega t)}}$$

- Jones Vector: $\boxed{\tilde{E}_0 = \begin{bmatrix} \tilde{E}_{0x} \\ \tilde{E}_{0y} \end{bmatrix} = \begin{bmatrix} E_{0x} e^{i\varphi_x} \\ E_{0y} e^{i\varphi_y} \end{bmatrix}}$

- Normalized vector $\boxed{\begin{bmatrix} a \\ b \end{bmatrix}}$: $\boxed{|a|^2 + |b|^2 = 1}$

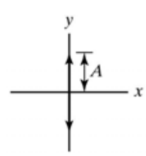
14.1 Mathematical Representation of Polarized Light: Jones Vector

Linear polarization:

The relative phase has to be zero (or $2m\pi$): $\boxed{\Delta\varphi = \varphi_y - \varphi_x = 2m\pi}$

Linear polarization along y:
($E_{0x} = 0$)

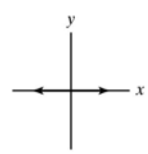
$$\boxed{\tilde{E}_0 = \begin{bmatrix} E_{0x} e^{i\varphi_x} \\ E_{0y} e^{i\varphi_y} \end{bmatrix} = A\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$



(a)

Linear polarization along x:
($E_{0y} = 0$)

$$\boxed{\tilde{E}_0 = \begin{bmatrix} E_{0x} e^{i\varphi_x} \\ E_{0y} e^{i\varphi_y} \end{bmatrix} = A\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

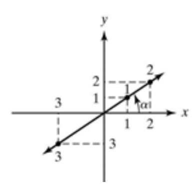


(b)

Linear polarization along angle α:

($E_{0x} = A\cos\alpha$)
($E_{0y} = A\sin\alpha$)

$$\boxed{\tilde{E}_0 = \begin{bmatrix} E_{0x} e^{i\varphi_x} \\ E_{0y} e^{i\varphi_y} \end{bmatrix} = A\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}}$$

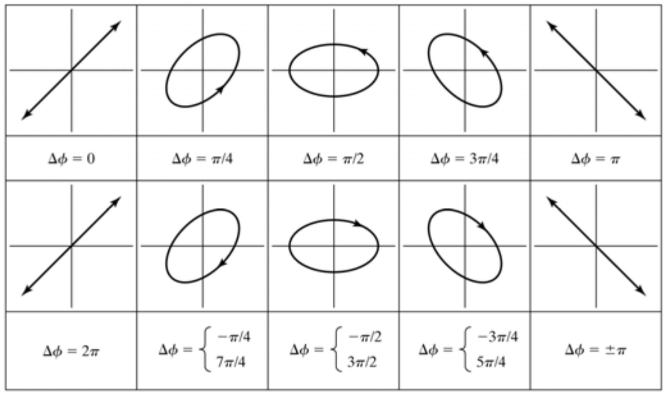


For a given a vector: $\boxed{\tilde{E}_0 = \begin{bmatrix} a \\ b \end{bmatrix}}$ the inclination angle: $\boxed{\alpha = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{E_{0y}}{E_{0x}}\right)}$

14.1 Mathematical Representation of Polarized Light: Jones Vector

- Relative phase determines the type of polarization:

$$\boxed{\Delta\varphi = \varphi_y - \varphi_x}$$



14.1 Mathematical Representation of Polarized Light: Jones Vector

<u>Circular polarization:</u>

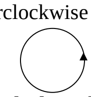
- If relative phase $\boxed{\Delta\varphi = \varphi_y - \varphi_x = \pm\pi/2}$, and $\boxed{E_{0x} = E_{0y}}$, then it is a circular polarized

By setting $\varphi_x = 0$ and $\varphi_y = \pi/2$: $\boxed{\tilde{E}_0 = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = A\begin{bmatrix} 1 \\ i \end{bmatrix}}$ or $\boxed{\begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \end{bmatrix} = \begin{bmatrix} E_{0x}e^{-i\omega t} \\ E_{0y}e^{-i\omega t + i\frac{\pi}{2}} \end{bmatrix}}$

E_x leads E_y by $\pi/2$

$\Rightarrow \boxed{\begin{bmatrix} E_x \\ E_y \end{bmatrix} = A\begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix}} \Rightarrow$

Counterclockwise rotation

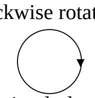


Left-circularly polarized (LCP)

By setting $\varphi_x = 0$ and $\varphi_y = -\pi/2$:

$\boxed{\tilde{E}_0 = \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} = A\begin{bmatrix} 1 \\ -i \end{bmatrix}} \Rightarrow$

Clockwise rotation



Right-circularly polarized (RCP)

- Jones Vectors $\boxed{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}}$ and $\boxed{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}}$ represent *LCP* and *RCP* light, respectively

14.1 Mathematical Representation of Polar $\boxed{\cos(\alpha+\varepsilon)=\cos\alpha\cos\varepsilon-\sin\alpha\sin\varepsilon}$

Elliptical polarization:

- General cases: $\boxed{E_{0x} \neq E_{0y}}$ and relative phase $\boxed{\Delta\varphi = \varphi_y - \varphi_x = \varepsilon}$ (E_x leads E_y by ε)
- Let's find the general elliptical equation by eliminating (kz-ωt):

$$\boxed{\begin{aligned} E_x &= E_{0x}\cos(kz-\omega t) \\ E_y &= E_{0y}\cos(kz-\omega t+\varepsilon) \end{aligned}} \longrightarrow \boxed{E_x / E_{0x} = \cos(kz-\omega t)}$$

$$\boxed{E_y / E_{0y} = \cos(kz-\omega t+\varepsilon) = \cos(kz-\omega t)\cos\varepsilon - \sin(kz-\omega t)\sin\varepsilon}$$

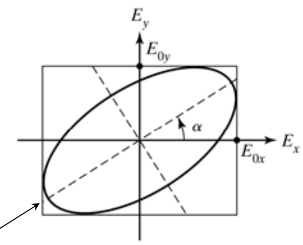
…………..

- Elliptical equation:

$$\boxed{\left(\frac{E_y}{E_{0y}}\right)^2 + \left(\frac{E_x}{E_{0x}}\right)^2 - 2\left(\frac{E_x}{E_{0x}}\right)\left(\frac{E_y}{E_{0y}}\right)\cos\varepsilon = \sin^2\varepsilon}$$

An ellipse making an angle α relative to the x-axis:

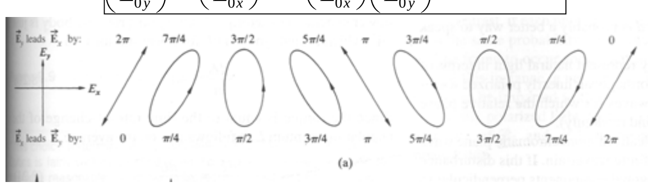
$$\boxed{\tan 2\alpha = \frac{2E_{0x}E_{0y}\cos\varepsilon}{E_{0x}^2 - E_{0y}^2}}$$



Principle axes of the ellipse

14.1 Mathematical Representation of Polarized Light: Jones Vector

- Elliptical equation: $\boxed{\left(\frac{E_y}{E_{0y}}\right)^2 + \left(\frac{E_x}{E_{0x}}\right)^2 - 2\left(\frac{E_x}{E_{0x}}\right)\left(\frac{E_y}{E_{0y}}\right)\cos\varepsilon = \sin^2\varepsilon}$



Special cases:

- If $\varepsilon = \pm 2m\pi$, $\Rightarrow$ $\boxed{E_y = (E_{0y}/E_{0x})E_x}$ Linear polarized state
- If $\varepsilon = \pm(2m+1)\pi$, $\Rightarrow$ $\boxed{E_y = -(E_{0y}/E_{0x})E_x}$ Linear polarized state
- If $\varepsilon = \pm(2m+1)\pi/2$, $E_{0x} = E_{0y} = E_0$ $\longrightarrow$ $\boxed{E_y^2 + E_x^2 = E_0^2}$ Circular polarized state
  - $\varepsilon = -\pi/2$, E_y leads E_x by $\pi/2$, → RCP
  - $\varepsilon = +\pi/2$, E_x leads E_y by $\pi/2$, → LCP
- If $\varepsilon = \pm(2m+1)\pi/2$, $E_{0x} \neq E_{0y}$ $\longrightarrow$ $\boxed{E_x^2/E_{0x}^2 + E_y^2/E_{0y}^2 = 1}$ Elliptical polarized, with principle axis aligned at coordinate axes
- If $\varepsilon \neq (\pm\pi/2, \pi, \pm 2\pi)$ → Elliptical polarized state, with principle axis rotated by angle α

14.1 Mathematical Representation of Polarized Light: Jones Vector

Any state of polarization:

$$\boxed{\tilde{E}_0 = \begin{bmatrix} E_{0x} e^{i\varphi_x} \\ E_{0y} e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} A \\ b e^{i\varepsilon} \end{bmatrix} = \begin{bmatrix} A \\ B + iC \end{bmatrix}}$$

Normalized Jones Vector: $\boxed{\tilde{E}_0 = \frac{1}{\sqrt{A^2 + B^2 + C^2}} \begin{bmatrix} A \\ B + iC \end{bmatrix}}$

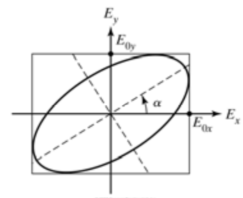
- If the relative phase $\boxed{\Delta\varphi = \varphi_y - \varphi_x = \varepsilon}$ is not special value of mπ, then

1. It represents elliptically polarized light oriented at an *inclination angle* α relative to the *x*-ax

$$\boxed{\tan 2\alpha = \frac{2E_{0x}E_{0y}\cos\varepsilon}{E_{0x}^2 - E_{0y}^2}}$$

with

$$E_{0x} = A$$
$$E_{0y} = \sqrt{B^2 + C^2}$$
$$\varepsilon = \tan^{-1}\left(\frac{C}{B}\right)$$



2. Rotation direction:
   - Adopt the convention that *A* is positive
   - If *C* is positive, then it is counter-clockwise rotation
   - If *C* is negative, then it is clockwise rotation

- If the relative phase is mπ, or $\pi/2$, or $\boxed{E_{0x} = E_{0y}}$, then it could be LP, LCP or RCP

14.1 Mathematical Representation of Polarized Light: Jones Vector

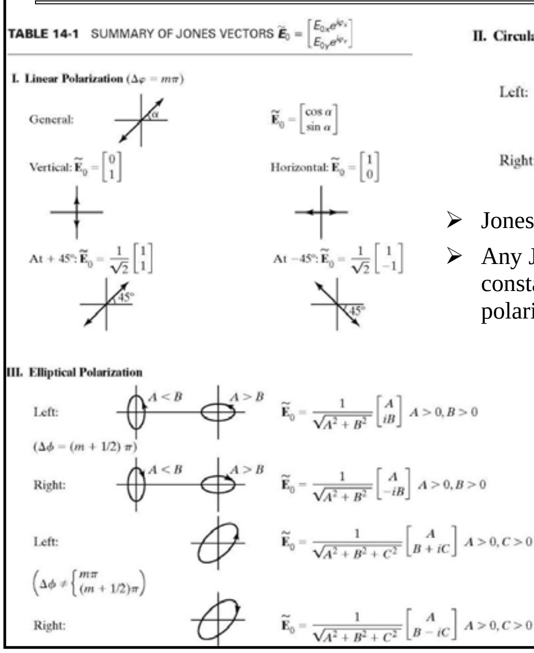


TABLE 14-1 SUMMARY OF JONES VECTORS $\tilde{E}_0 = \begin{bmatrix} E_{0x} e^{i\varphi_x} \\ E_{0y} e^{i\varphi_y} \end{bmatrix}$

I. Linear Polarization ($\Delta\varphi = m\pi$)

General: $\tilde{\mathbf{E}}_0 = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$

Vertical: $\tilde{\mathbf{E}}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ Horizontal: $\tilde{\mathbf{E}}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

At $+45°$: $\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ At $-45°$: $\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

III. Elliptical Polarization

Left: $A < B$, $A > B$: $\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{A^2 + B^2}}\begin{bmatrix} A \\ iB \end{bmatrix}$ $A > 0, B > 0$

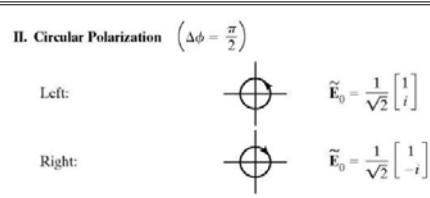
$(\Delta\phi = (m + 1/2)\pi)$

Right: $A < B$, $A > B$: $\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{A^2 + B^2}}\begin{bmatrix} A \\ -iB \end{bmatrix}$ $A > 0, B > 0$

Left: $\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{A^2 + B^2 + C^2}}\begin{bmatrix} A \\ B + iC \end{bmatrix}$ $A > 0, C > 0$

$\left(\Delta\phi \neq \begin{Bmatrix} m\pi \\ (m + 1/2)\pi \end{Bmatrix}\right)$

Right: $\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{A^2 + B^2 + C^2}}\begin{bmatrix} A \\ B - iC \end{bmatrix}$ $A > 0, C > 0$



II. Circular Polarization $\left(\Delta\phi = \frac{\pi}{2}\right)$

Left: $\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$

Right: $\tilde{\mathbf{E}}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

- Jones Vector in the Table is in normalized form
- Any Jones Vector may be multiplied by a real constant, changing the amplitude but not polarization mode

$$\boxed{\begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$ Polarization?

- Any Jones Vector may be multiplied by a factor of $e^{i\varphi}$, i.e., $\varphi_{x,y} \rightarrow \varphi_{x,y} + \varphi$

$$\boxed{i\begin{bmatrix} 1 \\ i \end{bmatrix} \Rightarrow \begin{bmatrix} i \\ -1 \end{bmatrix}}$$ Polarization?

14.1 Mathematical Representation of Polarized Light: Jones Vector

Problems 14-2
Describe completely the state of polarization of each of the following waves:

$$(a)\quad \vec{E} = E_0 \cos(kz - \omega t)\hat{x} - E_0 \cos(kz - \omega t)\hat{y}$$

$$(b)\quad \vec{E} = E_0 \sin 2\pi(z/\lambda - \nu t)\hat{x} - E_0 \sin 2\pi(z/\lambda - \nu t)\hat{y}$$

$$(c)\quad \vec{E} = E_0 \sin(kz - \omega t)\hat{x} - E_0 \sin(kz - \omega t - \pi/4)\hat{y}$$

$$(d)\quad \vec{E} = E_0 \cos(kz - \omega t)\hat{x} + E_0 \cos(kz - \omega t + \pi/2)\hat{y}$$

Problems 14-7
Determine the conditions on the elements A, B, and C of the general Jones vector (Eq.14-9), representing polarized light, that lead to the following special cases: (a) linearly polarized light; (b) elliptically polarized light with major axis aligned along a coordinate axis; (c) circularly polarized light. In each case, from the meanings of A, B, C, deduce the possible values of phase difference between component vibrations.

14.1 Mathematical Representation of Polarized Light: Jones Vector

Problems 14-9
Specify the polarization mode for each of the following Jones vectors:

$$(a)\ \begin{bmatrix} 3i \\ i \end{bmatrix} \qquad (b)\ \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad (c)\ \begin{bmatrix} 4i \\ 5 \end{bmatrix} \qquad (d)\ \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$(e)\ \begin{bmatrix} 2 \\ 2i \end{bmatrix} \qquad (f)\ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad (g)\ \begin{bmatrix} 2 \\ 6+8i \end{bmatrix}$$

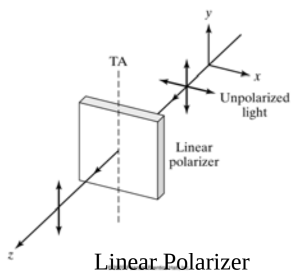
14.2 Mathematical representation of Polarizers: Jones Matrices

- Various optical devices (such as Polarizer, Phase Retarder and Rotator) may modify the state of polarization, which can be described by 2×2 Jones matrices

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- **Linear Polarizer**: is a device that converts an unpolarized or mixed-polarization beam of light into a beam with a single linear polarization state. Polarizers are used in many optical techniques and instruments, and polarization filters find applications in photography and liquid crystal display technology.

  - TA: transmission axis (dashed line, along y axis)



Linear Polarizer

14.2 Mathematical representation of Polarizers: Jones Matrices

- To find Jones matrix for linear polarizer with vertical TA:
  1. For vertical polarized beam (input), must all-pass

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{Polarizer}\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{input} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{output} \Rightarrow \begin{matrix} b = 0 \\ d = 1 \end{matrix}$$

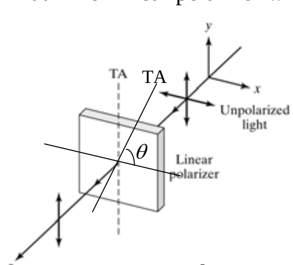
  2. For horizontal polarized beam (input), won't pass

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} a = 0 \\ c = 0 \end{matrix}$$

Jones matrix

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- General matrix for linear polarizer with TA at θ (relative to x-axis):



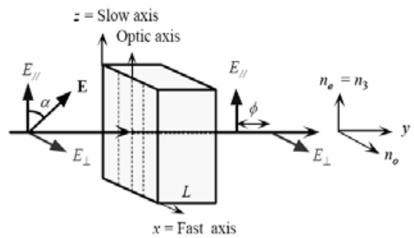
Linear Polarizer

$$M = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

If θ = 90°, then $M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

14.2 Mathematical representation of Polarizers: Jones Matrices

- **Phase Retarder**: A retarder or wave plate is an optical device that alters the polarization state of a light wave travelling through it. A wave plate works by shifting the phase between two perpendicular polarization components of the light wave. A typical wave plate is simply a birefringent crystal with a carefully chosen orientation and thickness.



A retarder plate. The optic axis is parallel to the plate face. The *o*- and *e*-waves travel in the same direction but at different speeds.

Phase difference:

$$\Delta\varphi = \varepsilon_y - \varepsilon_x = \frac{2\pi}{\lambda_o} d\left(|n_o - n_e|\right)$$

Optical path length:

$$OPD = d\left(|n_o - n_e|\right)$$

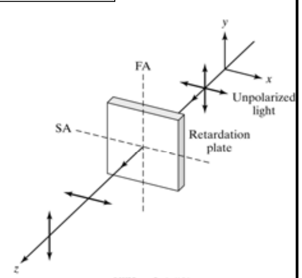
- Quarter-wave plate (QWP): $OPD = (4m+1)\lambda/4 \rightarrow \Delta\varphi = \pm\pi/2$
- Half-wave plate (HWP): $OPD = (4m+1)\lambda/2 \rightarrow \Delta\varphi = \pm\pi$
- Full-wave plate (FWP): $OPD = (4m+1)\lambda \rightarrow \Delta\varphi = \pm 2\pi$

14.2 Mathematical representation of Polarizers: Jones Matrices

- General matrix for phase retarder:

$$M \cdot \begin{bmatrix} E_{0x}e^{i\varphi_x} \\ E_{0y}e^{i\varphi_y} \end{bmatrix} \rightarrow \begin{bmatrix} E_{0x}e^{i(\varphi_x+\varepsilon_x)} \\ E_{0y}e^{i(\varphi_y+\varepsilon_y)} \end{bmatrix} \Longrightarrow M = \begin{pmatrix} e^{i\varepsilon_x} & 0 \\ 0 & e^{i\varepsilon_y} \end{pmatrix}$$

- SA and FA refer to slow axis and fast axis, respectively



1. For QWP (SA vertical): $\varepsilon_y - \varepsilon_x = \pi/2$

$$M = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{i\pi/4}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

2. For QWP (SA horizontal): $\varepsilon_y - \varepsilon_x = -\pi/2$

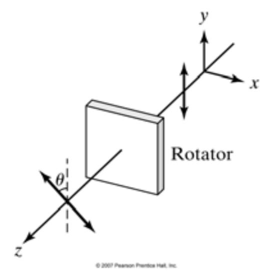
$$M = e^{i\pi/4}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

3. For HWP (SA vertical): $\varepsilon_y - \varepsilon_x = \pi$ $\quad M = \begin{pmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} = e^{-i\pi/2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

4. For HWP (SA horizontal): $\varepsilon_y - \varepsilon_x = -\pi$ $\quad M = e^{i\pi/2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

14.2 Mathematical representation of Polarizers: Jones Matrices

- **Rotator**: The rotator has the effect of rotating the direction of linearly polarized ligth incident on it by some particular angle



$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos(\theta+\beta) \\ \sin(\theta+\beta) \end{pmatrix}}$$

Rotator Input Output

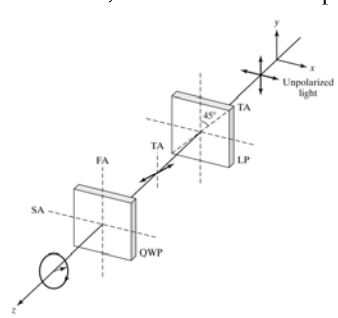
$$\Rightarrow \quad \boxed{M = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}}$$

- If light represented by Jones vector V passes sequentially through a series of polarizing elements (M_1, M_2, M_3….M_m), then **system matrix** is given by:

$$\boxed{\left(M_m ... M_3 M_2 M_1\right) V = M_s V \quad \Rightarrow \quad M_s = M_m ... M_3 M_2 M_1}$$

14.2 Mathematical representation of Polarizers: Jones Matrices

Assume unpolarized light passing a linear polarizer at an angle of 45° and a QWP with SA horizontal, what is the nature of product light?



$$\boxed{e^{i\pi/4}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{e^{i\pi/4}}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}}$$

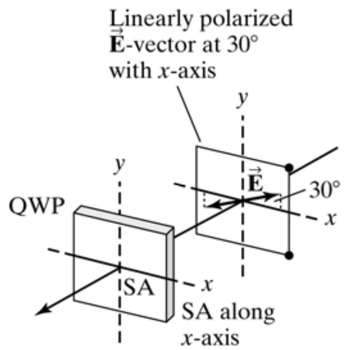
The result is right-circularly polarized

> Example 14-2
>
> Consider the result of allowing left-circularly polarized light to pass through an eighth-wave plate.

14.2 Mathematical representation of Polarizers: Jones Matrices

Problems 14-10

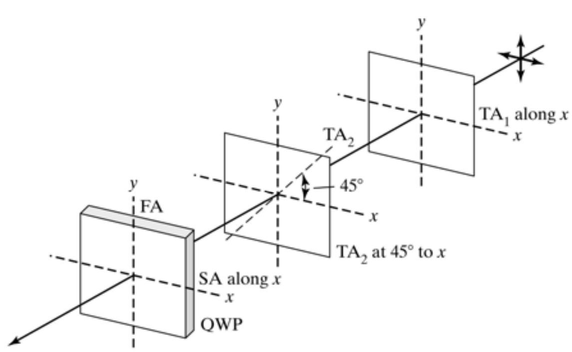
Linearly polarized light with an electric field E is inclined at $+30°$ relative to the x-axis and is transmitted by a QWP with SA horizontal. Describe the polarization mode of the produced light.



14.2 Mathematical representation of Polarizers: Jones Matrices

Problems 14-13

Light linearly polarized with a horizontal transmission axis is sent through another linear polarizer with TA at $+45°$ and then through a QWP with SA horizontal. Use the Jones matrix technique to determine and describe the product light.



14.2 Mathematical representation of Polarizers: Jones Matrices

Problems 14-15

Unpolarized light passes through a linear polarizer with TA at 60° from the vertical, then through a QWP with SA horizontal, and finally through another linear polarizer with TA vertical. Determine, using Jones matrices, the character of the light after passing through (a) the QWP and (b) the final linear polarizer.