Chapter 3

STEADY-STATE SYSTEM ANALYSIS. PERFORMANCE INDICES

The behavior of control systems (CSs) in several regimes is determined by the system structure, and by the type and parameter values of the controller. Both the permanent and the transient regimes are of interest. The control system performance can be assessed on the basis of *criteria to appreciate the performance (the quality*) and, as part of these criteria, on the basis of *indices to assess the performance (the qualit*y) also called **performance indices** [1]. The local criteria to assess the quality / the performance of CSs concern the system behavior in a certain operating regime or a certain CS mathematical characterization, and they make use of specific performance indices: overshoot σ_1, settling time t_s, steady-state error ε_∞ (or e_∞), constant speed error $\varepsilon_{r\infty}$ (or $e_{r\infty}$), static coefficient γ, etc. The main information concerning the ways to assess the quality of control systems will be treated in Sub-chapter 3.5, and the readers are invited to address [1] for additional information.

3.1. Operating regimes of control systems

Two categories of regimes characterize the operation of control systems:

- permanent regimes,
- transient regimes, usually viewed as transitions from a permanent regime to another one.

The permanent regime*s* are particular operating regimes where system's variables or their variations with respect to time take constant values. The permanent regimes can be installed only in *stable systems*. The following permanent regimes are of interest in the analysis and design of CSs:

- **The steady-state regime** (SSR), which is installed in a system when the input variables take constant values (in time):

$$w(t) = w_\infty \sigma(t), \; v(t) = v_\infty \sigma(t) , \tag{3.1-1}$$

where $w(t)$ is the reference input, $v(t)$ is the disturbance input, and $\sigma(t)$ is the unit step signal. The effect of (3.1-1) is the annulations of the transient regime in the system. Accordingly, all variables of the system take constant values called **steady-state values** (SSVs). This happens theoretically for $t \to \infty$ and practically after a time interval of approximately five-ten times the sum of large time constants of the system. In this contest w_∞ in (3.1-1) is the steady-state value of $w(t)$, and v_∞ is the steady-state value of $v(t)$.

- **The constant speed regime** (CSR), which is installed in a system (at $t \to \infty$) when one of the inputs has a linear variation with respect to time and the other one (ones) being considered constant of even zero:

$$w(t) = t\, w_\infty \sigma(t), \; v(t) = v_\infty \sigma(t)\,. \qquad (3.1\text{-}2)$$

The CSR is specific to the operation and testing of tracking systems. The example indicated by (3.1-2) concerns the reference tracking systems.

- **The constant acceleration regime** (CAR), which is installed in a system (at $t \to \infty$) when one of the inputs has a parabolic type variation (the other input (inputs) are constant of even zero). This input is usually the reference input:

$$w(t) = \frac{1}{2} t^2 \; w_\infty \sigma(t), \; v(t) = v_\infty \sigma(t)\,. \qquad (3.1\text{-}3)$$

The operating point (o.p.) of a system is the set of values of all characteristic variables of that system. If a permanent regime is installed in that system, the o.p. will be called **permanent operating point**. If a steady-state regime is installed in that system, the o.p. will be called **steady-state operating point** (s.s.o.p.). The notations of the coordinates of the s.s.o.p. are indicated are $A_0(u_0, v_0, y_0, \mathbf{x}_0)$ or $A_\infty(u_\infty, v_\infty, y_\infty, \mathbf{x}_\infty)$.

The graphical characterizations of the dependencies between the SSVs of different variables are called **static characteristic**s (SCs). In this context, the static characteristics can be defined only for stable systems. A particular case concerns the systems with transfer functions (t.f.s) that do not contain pure integral (I) components or pure derivative (D) components.

The CS performance with respect to different variations of the inputs is determined and it can be influenced by the CS design by means of:

- the proper choice of the CS structure,

- the choice of the controller type and the values of the tuning parameters (for a certain CS structure and a certain controller).

The following permanent regimes are of interest in control systems applications:

- With respect to the reference input $w(t)$:
  - the steady-state regime,
  - the constant speed regime,
  - the constant acceleration regime.
- With respect to the disturbance input $v(t)$:
  - the steady-state regime.

Remark: In case of linear systems the particular operating regimes with respect to these two inputs can be studied separately. The superposition principle can be next applied.

3.2. Computation of steady-state values of control systems

The computation of the s.s.o.p.s of a CS and, next, of the SSVs can be solved in several ways. The following *frequent situations* are practically used:

- The system is known by SCs concerning steady-state dependencies between several variables. It is desired to obtain the equivalent SCs (e-SCs) of the system and, using these characteristics, to compute the s.s.o.p.s.
- The system is known by its (parametric) mathematical models (MMs) that characterize several steady-state dependencies between system's variables. It is desired to compute the s.s.o.p.s of the overall system.
- The system operates and it is desired to obtain experimentally the coordinates of the s.s.o.p.s and, using these coordinates, to get the SCs.

The first case can be solved as follows:

- The system structure is known and the SCs of several of CS's blocks are known. These blocks are supposed to be linear or nonlinear but with continuous CSs.
- It is required to obtain the e-SCs and, using these characteristics, the significant s.s.o.p.s.

This is treated with sufficient details and presented in different forms in [2], [3], [4]. Special situations are related to one or more of CS's blocks that contain integral component(s).

A. Conditions to install the SSR and to compute the SSVs of a system. The steady-state regime can be installed in a physical / dynamical system:

- Only if the system is stable.
- If the inputs (i.e., the two conventional inputs applied to the CS) are constant with respect to time, namely:

$$w_\infty = \text{const} \;\text{ and }\; v_\infty = \text{const} .$$

- After the transient regimes are annulated, theoretically for $t \to \infty$.

Therefore, installing an SSR assumes **the annulations of derivative and integral effects in the system**. This formulation can be transposed in the following mathematical expressions:

(a) The case of continuous-time systems. The following SSR mathematical conditions characterize the system operation:

- For the state variables:

$$\mathbf{x}' = \mathbf{0} \Leftrightarrow \mathbf{x}'_\infty = \mathbf{0} . \tag{3.2-1}$$

- **For the integral (I) blocks**, Fig. 3.2.1 (a), or with separate I component:

$$u_\infty = 0 \to y_\infty = \text{const} . \tag{3.2-2}$$

The blocks with I behavior do not have SCs. For the zero input, the SSR output y_∞ can take any constant value in the domain of possible values. This value is determined by the past evolution (the history) of the system.

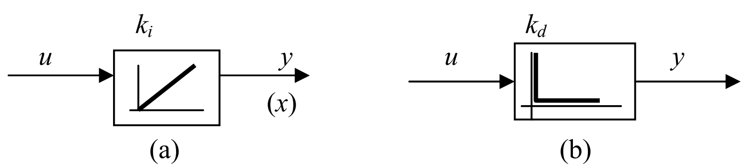


Fig. 3.2-1. Notations for the SSR characterization of I (a) and D (b) blocks.

- **For the derivative (D) blocks**, Fig. 3.2-1 (b), or with separate D component:

$$\forall u_\infty = \text{const} \to y_\infty = 0 . \tag{3.2-3}$$

The blocks with D behavior do not have SCs.

- **For the proportional blocks** (P-, including PT1, PT2,…PTn):

$$y_\infty = k\, u_\infty . \tag{3.2-4}$$

The blocks with P behavior have SCs.

Generally speaking, if the continuous-time system is stable and characterized by a rational t.f. (the presence of the time delay does not affect the steady-state behavior):

$$H(s)=\frac{B(s)}{A(s)}=\frac{b_m s^m+...+b_1 s+b_0}{a_n s^n+...+a_1 s+a_0}, m \le n, b_0 \ne 0, a_0 \ne 0\,, \tag{3.2-5}$$

then

$$k=\lim_{s\to 0} H(s)=H(0)=\frac{b_0}{a_0} \Rightarrow y_\infty = k\, u_\infty\,. \tag{3.2-6}$$

Equation (3.2-6) shows that a given value $u_\infty = \text{const} \ne 0$ results in the value $y_\infty = \text{const} \ne 0$. Equation (3.2-2) is obtained from (3.2-5) and (3.2-6) for $a_0 = 0$ that characterizes the I blocks. Equation (3.2-3) results from (3.2-5) and (3.2-6) for $b_0 = 0$ that characterizes the D blocks.

The graphical representation of the dependence $y_\infty = f(u_\infty)$ is **the static characteristic** (SC) of the system (of the block, of the component, of the subsystem, etc.). The previous formulation can be continued as follows in relation with the model (3.2-5):

- If $a_0 = 0$ but $b_0 \ne 0$, this characterizes a system with an I component (module / block) and the condition (3.2.-2) is applied.
- If $b_0 = 0$ but $a_0 \ne 0$, this characterizes a system with a D component (module / block) and the condition (3.2.-3) is applied.

(b) The case of discrete-time systems. The principles presented for continuous-time systems are valid for discrete-systems as well but the mathematical formulations are modified accordingly:

- For the state variables:

$$\mathbf{x}_{k+1} = \mathbf{x}_k \Leftrightarrow \Delta\mathbf{x}_k = \mathbf{x}_{k+1} - \mathbf{x}_k = \mathbf{0}\,, \tag{3.2-7-a}$$

with the general notation $\mathbf{x}_k = \mathbf{x}(k)$.

- For the I blocks:

$$u_\infty = 0,\ y_\infty = \text{const}\,, \tag{3.2-7-b}$$

- For the D blocks:

$$u_\infty = \text{const},\ y_\infty = 0, \tag{3.2-7-c}$$

- For the P blocks:

$$y_\infty = k\, u_\infty . \tag{3.2-7-d}$$

Generally speaking, if the system is characterized by the stable rational t.f.

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_m z^m + \ldots + b_1 z + b_0}{a_n z^n + \ldots + a_1 z + a_0}, m \le n, \tag{3.2-8}$$

and the system does not contain:

- pure D components, i.e., $B(1) \neq 0$,
- pure I components, i.e., $A(1) \neq 0$,

then the final value theorem applied to the system with a constant input $u_\infty = \text{const}$ leads to

$$k = \lim_{z \to 1} H(z) = H(1) \Rightarrow y_\infty = k\, u_\infty . \tag{3.2-9}$$

B. The computation of the steady-state values of a system and of the particular case of a control system. Knowing the SSVs of a system and of a CS is of interest because of the following reasons which allow:

- the computation of all s.s.o.p.s,
- the construction of the SCs of several blocks of the CS and of the overall CS; therefore, it is next possible to know the possibilities of CS operation in several transient regimes (for example, characterized by large variations of the inputs accounting for limitations specific to CS blocks),
- the correct design of the components of the CS (actuators, measuring elements, etc.).

The SSVs and, accordingly, the SCs of the CS blocks (of the overall CS) can be computed only for limited domains of variations of the CS (determined by CS's functionality) in the vicinity of significant s.s.o.p.s (idle regime, load regime, etc.). In addition, the computation of SSVs can be carried out a priori or a posteriori with respect to system's construction. The following **specific situations** concern the computation of SSVs:

(a) *The experimental computation of SSVs from SSR measurements* conducted on system's variables. This computation is a posteriori with respect to system's construction.

(b) *The analytical computation of SSVs*. This computation – based on system's MMs – can be considered as a priori or a posteriori with respect to system's construction.

(a) The experimental computation of SSVs from measurements conducted on system's variables. This situation is frequently used when the controlled process (CP) and/or the CS are placed in operation. Conducting the measurements is conditioned by the existence of dedicated measurement equipment which is often expensive (transducers, signal converters, process interfaces, measurement and information processing equipment, specialized processing equipment, etc.). The diagram of principle that corresponds to the experimental computation of SSVs of a CS is presented in Fig. 3.2-2 (a).

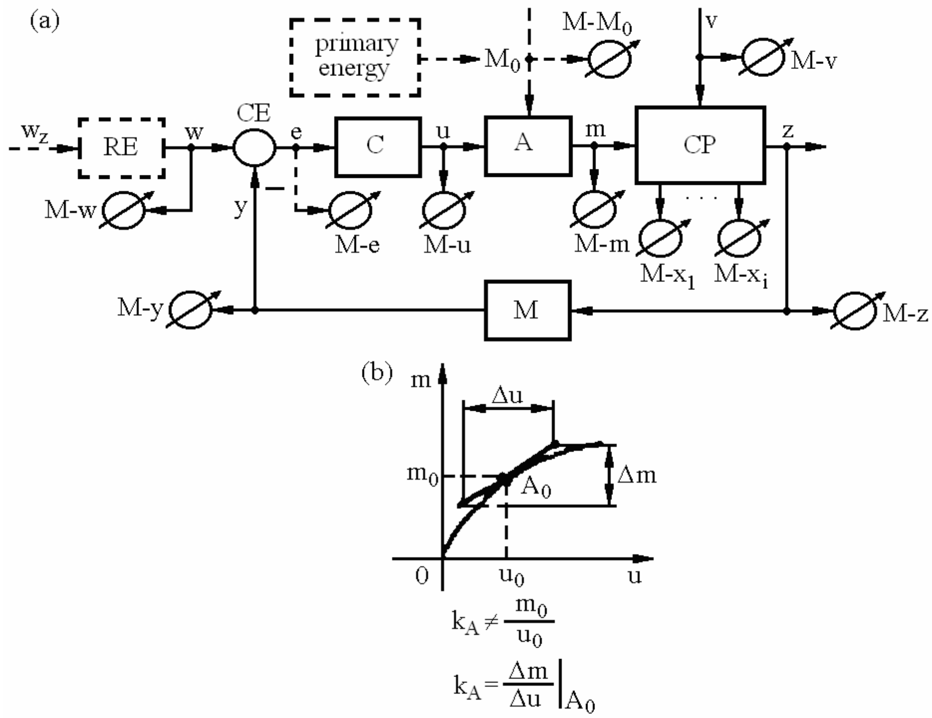


Fig.3.2-2. Diagram of principle for the experimental computation of SSVs.

The following ***details*** should be accounted for:

- *The lack of the access by measurements to some of the internal variables of the CS*, such particular variables concern:
  - the controller (C),
  - the controlled process (CP).

Such situations are solved by measuring only those variables that are accessible to measurements. The values of other variables are next derived on the basis of MMs (if it is possible) by computation or by estimation (by the use of adequate estimators). The SCs are obtained only for blocks for which their input and output SSVs have been obtained.

For installing the SSR in the system is important to have access to the inputs of integral elements (with distinct I component). The access to the control error $e(t)$ is important with this regard. If the controller contains an I component then the SSR relationship is

$$e_{\infty} = 0, \tag{3.2-10}$$

which confirms the installing of an SSR.

Remark: The control error will be indicated in the sequel by $e(t)$ or by $\varepsilon(t)$.

- *The functional block of the CS (the overall CS) can often contain continuous nonlinearities*. These nonlinearities lead to nonlinear SCs of those blocks as the nonlinear SC exemplified in Fig. 3.2-2 (b) for the actuator (A). The use of a linear MM around of an s.s.o.p. will next require the linearization and the expression of linearized MMs by the differences of the input and output variables of the block with respect to the certain s.s.o.p. The differences are

$$\Delta u(t) = u(t) - u_0,\ \Delta y(t) = y(t) - y_0, \tag{3.2-11-a}$$

and the linearized MM attached to the s.s.o.p. A_0 is

$$\Delta y_{\infty} = k\,\Delta u_{\infty}, \tag{3.2-11-b}$$

where

$$k = \left.\frac{dy}{du}\right|_{A_0} \approx \left.\frac{\Delta y}{\Delta u}\right|_{A_0} \tag{3.2-11-c}$$

is the transfer coefficient (gain) of the linear/linearized system and measured in $<k>=<\Delta y>/<\Delta u>$.

- *The SSR dependencies (the SCs) depend on the value of a certain variable of the block (of the system)*. Such situation requires the computation of *families of static characteristics* having as parameter that variable (that functional parameter) of the block.

(b) The analytical computation of SSVs of a system. Several techniques ad used in the analytical computation of SSVs of CSs. These techniques depend on:

- the expression of system's MM (the MM type), the representation of system's structure, the block diagram,
- the linearity / nonlinearity of the MM,
- the information treatment in time (continuous time, discrete time),
- the information concerning some SSVs of the system, etc.

The most frequently used practical situations are synthesized as follows in terms of *case studies*. The following presentation will be focused on the linear (linearized) situation.

(1) The system is characterized by the SS-MM expressed in the time domain. Several remarkable cases are treated as follows.

- ***The case of continuous-time systems***. A linear n-th order stable system with r inputs and q outputs is considered. The SS-MM of this system is

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t). \end{aligned} \qquad (3.2\text{-}12)$$

The SSR relationship is

$$\mathbf{x}^{'}(t) = \mathbf{x}^{'}_{\infty} = \mathbf{0}\,, \qquad (3.2\text{-}13)$$

that results in

$$\mathbf{0} = \mathbf{A}\,\mathbf{x}_{\infty} + \mathbf{B}\,\mathbf{u}_{\infty}\,, \qquad (3.2\text{-}14)$$

$$\mathbf{y}_{\infty} = \mathbf{C}\,\mathbf{x}_{\infty}\,. \qquad (3.2\text{-}15)$$

Equations (3.2-14) and (3.2-15) are a linear algebraic system of (n+q) equations and (n+r+q) unknowns: $n = \dim \mathbf{x}$ – the number of state variables, $r = \dim \mathbf{u}$ – the number of input variables and $q = \dim \mathbf{y}$ – the number of output variables. Solving the algebraic system (3.2-14) and (3.2-15) will lead to (n+q) SSVs accepting that the rest of r SSVs ensure the compatibility of the system.

The SSVs of the output variables can be obtained as the same result if the following equation resulted from (3.2-14) and (3.2-15) is applied:

$$\mathbf{y}_{\infty} = \mathbf{C}\,(-\mathbf{A}^{-1})\,\mathbf{B}\,\mathbf{u}_{\infty}\,. \qquad (3.2\text{-}16)$$

Equation (3.2-16) points out the DC gain.

Remark: If the r known SSVs are the input variables, then the algebraic system (corresponding to a stable CS) will always be compatible.

- ***The case of discrete-time systems***. A linear n-th order stable system with r inputs and q outputs is considered. The SS-MM of this system is

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\,\mathbf{x}_k + \mathbf{B}\,\mathbf{u}_k, \\ \mathbf{y}_k &= \mathbf{C}\,\mathbf{x}_k. \end{aligned} \tag{3.2-17}$$

The SSR relationship is

$$\mathbf{x}_{k+1} = \mathbf{x}_k = \mathbf{x}_\infty, \tag{3.2-18}$$

that results in

$$\mathbf{x}_\infty = \mathbf{A}\,\mathbf{x}_\infty + \mathbf{B}\,\mathbf{u}_\infty,$$

that is equivalent to

$$(\mathbf{I} - \mathbf{A})\,\mathbf{x}_\infty = \mathbf{B}\,\mathbf{u}_\infty, \tag{3.2-19}$$

and

$$\mathbf{y}_\infty = \mathbf{C}\,\mathbf{x}_\infty \tag{3.2-20}$$

or

$$\mathbf{y}_\infty = \mathbf{C}\,(\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{u}_\infty. \tag{3.2-21}$$

Equation (3.2.-21) points out the DC gain.

Equations (3.2-19) and (3.2-21) are a linear algebraic system of (n+q) equations and (n+r+q) unknowns: $n = \dim \mathbf{x}$ – the number of state variables, $r = \dim \mathbf{u}$ – the number of input variables and $q = \dim \mathbf{y}$ – the number of output variables. Solving the algebraic system (3.2-19) and (3.2-21) will lead to (n+q) SSVs accepting that the rest of r SSVs ensure the compatibility of the system.

(2) The system is characterized by the IO-MM or the SS-MM in the operational domain. Several situations involve:

- obtaining the operational expression of the output as function of system's inputs,
- the application of the final value theorem.

For example, considering the general expression of the transfer function matrix with $\lambda = s$ for continuous-time systems and $\lambda = z$ for discrete-time systems, the following relationships hold for Multi Input-Multi Output (MIMO) systems:

$$\mathbf{y}(\lambda) = \mathbf{H}(\lambda)\,\mathbf{u}(\lambda),\ \ \mathbf{H}(\lambda) = \mathbf{C}\,(\lambda\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}, \tag{3.2-22-a}$$

and for Single Input-Single Output (SISO) systems (with the t.f. $H(\lambda)$ instead of the transfer function matrix):

$$y(\lambda) = H(\lambda)\,\mathbf{u}(\lambda),\ H(\lambda) = \mathbf{c}^T\ (\lambda\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}\,. \tag{3.2-22-b}$$

The application of the final value theorem leads to the SSVs for all variables considered as output variables after obtaining the operational dependencies of interest (expressed as follows in the SISO case):

$$\begin{aligned} y_\infty &= \lim_{s\to 0}\{s\frac{u_\infty}{s}H(s)\} = H(0)u_\infty\,, \\ y_\infty &= \lim_{z\to 1}\{\frac{z-1}{z}\frac{zu_\infty}{z-1}H(z)\} = H(1)u_\infty\,. \end{aligned} \tag{3.2-23}$$

(3) The system is characterized by its informational block diagram, where the two basic functional blocks are separately outlined, i.e., the controller (C) and the controlled process (CP). Moreover, for C and CP the D and I blocks must be outlined, and the rest of the blocks have P behavior ($a_0 \neq 0, b_0 \neq 0$). Several situations of interest are pointed out as follows.

- ***Continuous-time systems which are homogenous with respect to information processing with respect to time.*** The computation of system's SSVs is carried out as follows. The continuous-time and discrete-time cases are treated in similar manner, and the differences appear only in the SSR conditions (the example is given for continuous-time systems):
  - For each typical block (I, D, P, …) the SSR conditions are expressed along with the equation for the computation of SSVs:
    - for I blocks: for $u_\infty = 0 \rightarrow y_\infty = \text{const}$, and this value depends on system's past history,
    - for D blocks: for $u_\infty = \text{const} \rightarrow y_\infty = 0$,
    - for P blocks (PT-n): for $u_\infty = \text{const} \rightarrow y_\infty = k\,u_\infty$.
  - These equations lead to an algebraic system with the dimension depending on system's complexity. The usual dimension is related to (n+q) equations with (n+q+r) SSVs.
  - If a sufficient number of SSVs is known for which the algebraic system is compatible (for example, r SSVs but not any), the rest of SSVs can be computed.

Remark: The presence of an I block in the block diagram of the CS (CP) leads to the zero steady-state input of this block, $u_\infty = 0$. The presence of a D block in the block diagram of the CS (CP) leads to the zero steady-state output of this block, $y_\infty = 0$. Using these conditions, the constant output and input of these blocks are next computed using backward calculations.

Case study. Let us consider the CS with the block diagram given in Fig. 3.2-3, where the reference input is $w(t)$ and the control error is $e(t)$. Two versions of controllers (C) are considered:

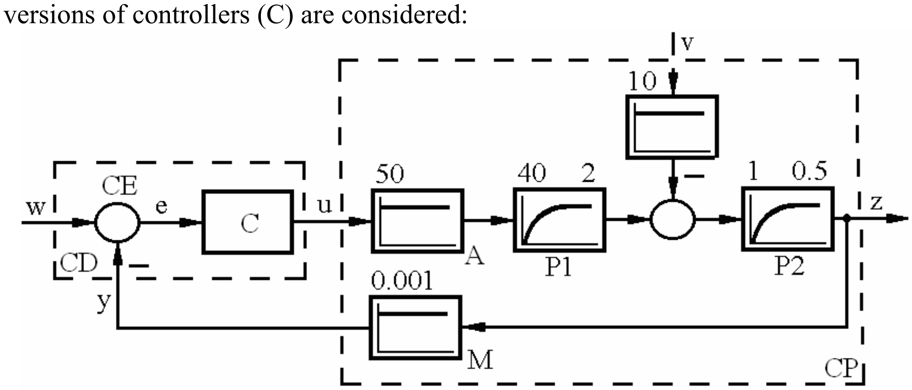


Fig. 3.2-3. Block diagram of the control system in the case study.

- Version 1: PI controller with the t.f. (the index R for the controller also indicates the notation regulator for the controller)

$$H_R(s) = \frac{k_R}{2.5s}(1+2.5\text{s}) . \tag{1}$$

- Version 2: PDT1 controller (lag-lead controller) with the t.f.

$$H_R(s) = \frac{k_R\,(1+2.5\text{s})}{1+25s} . \tag{2}$$

It is required: (1) To determine the transfer characteristics (i.e., the t.f.s with respect to the reference input and with respect to the disturbance input $v(t)$) and the SS-MM for both versions of controllers.

(2) To determine the domain of the values of the controller gain $k_R\,(k_R > 0)$ for which the system is stable. How is the stability of the system influenced by the modification of the time constant $T = 2.5\,\text{sec}$ in the t.f. nominator to $T' = 0.25\,\text{sec}$ and $T'' = 10\,\text{sec}$ if $k_R = 0.5$ for the version 1 and $k_R = 13.5$ for the version 2?

(3) For $w_\infty = 6.5$ and $v_\infty = 500$, to determine the SSVs in the system. How are these SSVs modified by the modifications of k_R and by the modifications of *T* (in the context of the point (2) while fulfilling the stability conditions?

Solution: The computations are given for the controller in the version 1. The reader is invited to give the solution for the controller in the version 2.

(1) The transfer characteristics of the CS are

$$H_w(s) = \frac{z(s)}{w(s)} = \frac{800k_R\ (1+2.5s)}{s^3 + 2.5s^2 + (2k_R + 1)s + 0.8k_R},$$

$$H_v(s) = \frac{z(s)}{v(s)} = -\frac{25s(1+2s)}{s^3 + 2.5s^2 + (2k_R + 1)s + 0.8k_R}.$$

The derivation of the SS-MM of the control system starts with the choice of the state variables: x_1 corresponds to the PI controller (in parallel version of construction), x_2 is the output of the block P1 (of PT1 type), $x_3 = z$ is the output of the block P2 (of PT1 type as well). The SS-MMs of the control system blocks and the connection relationships that characterize the system are

C: $\dot{x}_1 = \dfrac{k_R}{2.5}e$, $u = x_1 + k_R e$,

P1: $\dot{x}_2 = -\dfrac{1}{2}x_2 + \dfrac{40}{2}m$,

P2: $\dot{x}_3 = -\dfrac{1}{0.5}x_3 + \dfrac{1}{0.5}(x_2 - 10v)$, $z = x_3$,

E: $m = 50u$,

M: $y = 0.001x_3$,

CE: $e = w - y$.

These equations are rearranged such that to obtain the SS-MM of the CS:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{b}_w w + \mathbf{b}_v v,$$
$$z = \mathbf{c}^T\mathbf{x},$$

where:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -0.0004k_R \\ 1000 & -0.5 & -k_R \\ 0 & 2 & -2 \end{bmatrix},\ \mathbf{b}_w = \begin{bmatrix} 0.4k_R \\ 1000k_R \\ 0 \end{bmatrix},$$

$$\mathbf{b}_v = \begin{bmatrix} 0 \\ 0 \\ -20 \end{bmatrix},\ \mathbf{c}^T = [0 \quad 0 \quad 1].$$

(2) The characteristic polynomial of the CS

$$\Delta(s) = \det(s\mathbf{I} - \mathbf{A})$$

is computed as the denominator of the transfer characteristics:

$$\Delta(s) = s^3 + 2.5s^2 + (2k_R + 1)s + 0.8k_R.$$

Since $k_R > 0$, the preliminary stability requirements concerning the positive sign of all coefficients in $\Delta(s)$ are fulfilled. The Hurwitz criterion is next applied. The Hurwitz matrix of the system is $\mathbf{H}_H$:

$$\mathbf{H}_H = \begin{bmatrix} 2.5 & 0.8k_R & 0 \\ 1 & 2k_R + 1 & 0 \\ 0 & 2.5 & 0.8k_R \end{bmatrix}.$$

Imposing the stability conditions leads to:

$$H_1 = 2.5 > 0,\ H_2 = 2.5(2k_R + 1) - 0.8k_R > 0,\ H_3 = 0.8k_R H_2 > 0,$$

that result in the domain of values of k_R that guarantee the CS stability:

$$k_R \in (0, \infty).$$

Considering $k_R = 0.5$ and $T^{'} = 0.25\,\text{sec}$, the characteristic polynomial of the CS should be recomputed. The reader is invited do carry out these computations and also the computations for $T^{''} = 10\,\text{sec}$.

(3) The SSR dependencies of the blocks of the CS and the connection relationships are expressed as:

C: $e_\infty = 0$, $u_\infty = x_{1\infty}$

E: $m_\infty = 50u_\infty$,

P1: $x_{2\infty} = 40 m_\infty$,

P2: $z_\infty = x_{3\infty}$, $z_\infty = x_{2\infty} - 10 v_\infty$,

M: $y_\infty = 0.001 x_{3\infty}$,

CE: $e_\infty = w_\infty - y_\infty$.

The above system is solved as follows. The condition $e_\infty = 0$ (for C) leads to $y_\infty = w_\infty = 6.5$, $z_\infty = 1000 y_\infty = 6500$, $x_{2\infty} = 1000 y_\infty + 10 v_\infty = 11500$, $m_\infty = x_{2\infty} / 40 = 287.5$, $u_\infty = m_\infty / 50 = 5.75$, $x_{1\infty} = u_\infty = 5.75$.

These results show that the SSVs are not influenced by the modification of the controller parameters (k_R and T_i) if a PI controller is used.

- ***Systems with non-homogenous information processing with respect to time.*** This case is typical for the digital control of a CP if the CP is characterized by continuous-time MMs and the digital controller is characterized by discrete-time MMs as illustrated in Fig. 3.2-4.

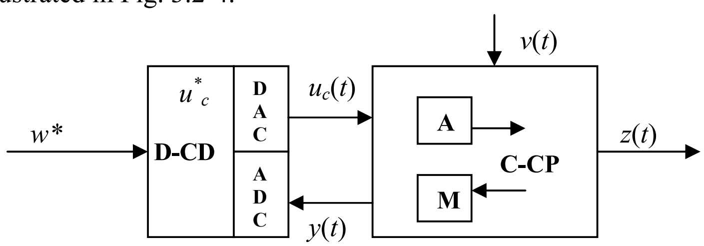


Fig. 3.2-4. Control system with non-homogenous information processing with respect to time.

The two subsystems are connected at the level of the two converters, DAC and ADC. The modern digital control devices (D-DC) employ the information representation on a sufficient large number of bits. This involves that at the level of digital-analog and analog-digital conversions to consider that both representations of the same information (the continuous and the digital ones) are equal, i.e.,

$$\| u(t) \| = \| u^*(t_k) \| = \| u_c(t) \| \text{ for } t = t_k, \quad (3.2\text{-}24\text{-a})$$

$$\| y(t) \| = \| y^*(t_k) \| \text{ for } t = t_k, \quad (3.2\text{-}24\text{-b})$$

$$\| w(t) \| = \| w^*(t_k) \| \text{ for } t = t_k, \quad (3.2\text{-}24\text{-c})$$

The SSR computation will be carried out as follows by methods which are specific to the case study involved. Imposing the additional condition (3.2.24) will lead to solving the specific computation situation.

(4) Formulations specific to problems concerning the SSV computations. Several practical situations concerning the system operations are related to different problems of SSV computations. The following situations are exemplified:

- The system operates with an imposed value z_∞ and with an imposed value v_∞. It is required to compute the value of the reference input w_∞ that ensures this s.s.o.p.
- Having a given reference input and a limitation (maximum, minimum) of one / more variables of the system (imposed, for example, by avoiding to exceed a certain maximum admitted values), it is required to compute the SSV v_∞ which ensures the SSR load of the system.
- The system operates with a given reference input and with an imposed output. It is required to compute the load / the disturbance afferent to this case. This problem is not always compatible.

The practical computation of the SSVs, of the SC of the CS is often difficult because the process is nonlinear. The use of a linear measuring element (M) and of a controller with I component, within certain limits, makes the nonlinearities to be invisible in the relation reference input – assessed output, and the SC is equal to the inverse of the SC of M. On the other hand, the usage of a measuring element with adequate nonlinear SC – for example, the inverse of the SC of the feedforward channel – can ensure the linearity of the equivalent SC as well.

3.3. Effects of controller types on steady-state behavior of control systems

Only aspects concerning the properties caused by the type of controller on the steady-state behavior of CS will be discussed as follows. The analysis is related to the classical CS structure given in Fig. 3.3-1, for which the following operational relationships can be expressed:

$$H_v(\lambda) = \frac{H_N(\lambda)}{1 + H_R(\lambda) H_P(\lambda)}, \; H_N(\lambda) = k_N \frac{B_N(\lambda)}{A_N(\lambda)}, \tag{3.3-1}$$

$$H_w(\lambda) = \frac{H_R(\lambda) H_P(\lambda)}{1 + H_R(\lambda) H_P(\lambda)}, \tag{3.3-2}$$

where *w* is the reference input, $\lambda = s$ or *z*, and $H_N(\lambda)$ is the part of $H_P(\lambda)$ passed by the disturbance input *v*.

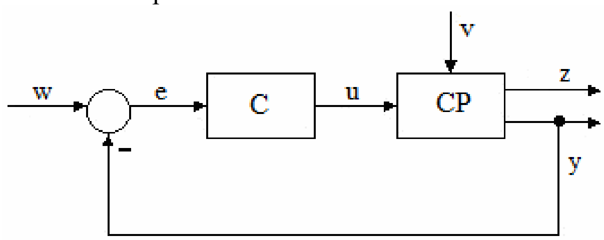


Fig. 3.3-1. Block diagram of the conventional control system (control loop).

Remark: For discrete-time systems, since $H_N(z)$ does not exist usually ($v(t)$ is continuous tin time), $H_N(z)$ is computed as an analytical extension of the discrete-time behavior of the continuous-time system (considering the ZOH).

Using Fig. 3.3-1, the open-loop t.f. is

$$H_0(\lambda) = H_R(\lambda) H_P(\lambda) = \frac{B_0(\lambda)}{A_0(\lambda)}. \tag{3.3-3-a}$$

The presence of the I component in CS structure is illustrated by testing the conditions:

- for continuous-time systems:

$$A_0(0) = 0 \text{, i.e., } s = 0 \text{ is a root,} \tag{3.3-3-b}$$

- for discrete-time systems:

$$A_0(1) = 0 \text{, i.e., } z = 1 \text{ is a root,} \tag{3.3-3-c}$$

and more I components correspond to multiple roots.

The following steady-state behaviors are of interest in practical applications:

A. ***With respect to the reference input*** $w(t)$:

- SSR behavior: $t \to \infty$, $w(t)$ of step type,
- Constant speed regime behavior: $t \to \infty$, $w(t)$ of ramp type.

B. ***With respect to the disturbance input*** $v(t)$:

- SSR behavior: $t \to \infty$, $v(t)$ of step type.

This last case assumes that the system in a SSR determined by a constant reference input w_0 which determines the SSV of the output y_0.

The analysis of the properties caused by the controller in the permanent regime behavior of the conventional will be carried out as follows in the continuous case with λ = *s*. The discrete case, with λ = *z*, can be treated in a similar manner and it leads to the same results and conclusions. The difference concerns the way the final value theorem is applied, which is different in the two cases:

- In the continuous time case, the final value theorem is expressed as

$$\begin{aligned} & y(s) = H_w(s)w(s), \\ & y_\infty = \lim_{t\to\infty} y(t) = \lim_{s\to 0} s\,y(s) = \lim_{s\to 0} sH_w(s)w(s). \end{aligned} \tag{3.3-4-a}$$

- In the discrete time case, the final value theorem is expressed as

$$\begin{aligned} & y(z) = H_w(z)w(z), \\ & y_\infty = \lim_{t\to\infty} y(t) = \lim_{z\to 1}\{\frac{z-1}{z}y(z)\} = \lim_{z\to 1}\{\frac{z-1}{z}H(z)w(z)\}. \end{aligned} \tag{3.3-4-b}$$

The following relationships can be expressed on the basis of Fig. 3.3.-1 and of equations (3.3-1), (3.3-2) and (3.3-3):

$$\begin{aligned} & y(s) = H_w(s)w(s) + H_v(s)v(s) = \frac{H_R(s)H_P(s)}{1+H_R(s)H_P(s)}w(s) \\ & + \frac{H_N(s)}{1+H_R(s)H_P(s)}v(s) \end{aligned} \tag{3.3-5-a}$$

$$e(s) = \frac{1}{1+H_R(s)H_P(s)}w(s) - \frac{H_N(s)}{1+H_R(s)H_P(s)}v(s)\,, \tag{3.3-5-b}$$

where the subscript *R* indicates the controller (also referred to as the regulator), $H_N(s)$ results from (3.31) for λ = s, and the open-loop t.f. is generally expressed as

$$H_0(s) = \frac{k_0}{s^{q_0}}\frac{B_0(s)}{A_0^{'}(s)}e^{-sT_m}\,,\ \ A_0(s) = s^{q_0}A_0^{'}(s)\,. \tag{3.3-6}$$

For the sake of simplicity, the factor that points out the dead time (the transport, delay, the time delay) is omitted in the second equation in (3.3-6), i.e., $T_m = 0$. The generality is not affected because the SSR equation for the dead time element is $e^{-sT_m} \to 1$ for $s \to 0$. $q_0 \in \{0,1,2\}$ in (3.3-6) is the multiplicity order of the pole in origin (the number of I components) of the open-loop system, and $A_0^{'}(0) \neq 0$.

Remark: Several situations of interest concern the discussion on the value of the control signal, $u(s) = f(w(s), v(s))$. This is justified as at the level of the actuator the squared actuating signal, $m^2(t)$, is a measure of the power / energy introduced in the process; therefore, $u^2(t)$ represents a measure *of the (energetic) effort at the level of the control signal*.

The next analysis supposes that the pole in origin modeled by $s^{q_0}, q_0 \in \{0,1,2\}$, is brought by the controller. It is also supposed that the disturbance channel $H_N(\lambda)$ does not contain poles in origin. However, any modification of the conditions (hypotheses) of this analysis can modify its results.

Since the system is linear (linearized), the permanent regime behavior can be analyzed separately:

- with respect to the reference input $w(t)$,
- with respect to the disturbance input $v(t)$.

The final conclusion is next drawn by the principle of superposition.

The open-loop t.f. $H_0(s)$ is substituted from (3.3-6) in (3.3-5) leading to the general input-output relationships:

$$y(s) = \frac{\frac{k_0}{s^{q_0}} \frac{B_0(s)}{A_0^{'}(s)}}{1 + \frac{k_0}{s^{q_0}} \frac{B_0(s)}{A_0^{'}(s)}} w(s) + \frac{k_N \frac{B_N(s)}{A_N(s)}}{1 + \frac{k_0}{s^{q_0}} \frac{B_0(s)}{A_0^{'}(s)}} v(s), \tag{3.3-7-a}$$

$$e(s) = \frac{1}{1 + \frac{k_0}{s^{q_0}} \frac{B_0(s)}{A_0^{'}(s)}} w(s) - \frac{k_N \frac{B_N(s)}{A_N(s)}}{1 + \frac{k_0}{s^{q_0}} \frac{B_0(s)}{A_0^{'}(s)}} v(s) \cdot \tag{3.3-7-b}$$

A. The CS behavior with respect to the reference input. Properties induced by the type of controller. This case concerns mainly the SSR and CSR with respect to the reference input:

$$\begin{aligned} w(s) &= \frac{1}{s} w_\infty, \ w(t) = w_\infty \sigma(t) \ \text{in SSR}, \\ w(s) &= \frac{1}{s^2} w_\infty, \ w(t) = t\, w_\infty \sigma(t) \ \text{in CSR}, \end{aligned} \tag{3.3-8}$$

for $v(t) = 0 \ (v(s) = 0)$. Equations (3.3-7) result in

$$y(s) = \frac{k_0 \frac{B_0(s)}{A_0^{'}(s)}}{s^{q_0} + k_0 \frac{B_0(s)}{A_0^{'}(s)}} w(s), \ e(s) = \frac{s^{q_0}}{s^{q_0} + k_0 \frac{B_0(s)}{A_0^{'}(s)}} w(s). \tag{3.3-9}$$

Remark: The expression (3.3-6) of $H_0(s)$ has been used accepting that $\left.\frac{B_0(s)}{A_0^{'}(s)}\right|_{s \to 0} = 1$. In other words, the steady-state gain (the DC gain) k_0 has been moved out of the rational form.

The application of the final value theorem to the expressions of $y(s)$ and $e(s)$ given in (3.3-9) accounting for (3.3-8), where $q_0 \in \{0,1,2\}$ is a parameter, leads to the results synthesized in Table 3.3-1. Table 3.3-1 shows:

(1) Aspects concerning the SSR:

- Ensuring the condition of zero steady-state control error ($e_\infty = 0$) requires the existence of I component in the controller structure.
- The systems with controllers without I component operate with nonzero control error (they do not ensure the output *exactly equal* to the reference input). This does not indicate that those systems operate wrongly but that they operate in a specific operating mode which can be desired in certain situations.
- The relations given in Table 3.3-1 can be applied in case of linearized systems as well around the s.s.o.p.s $A_0(w_0, y_0, e_0, ...)$, where the differences are used instead of the absolute values of variables: $\Delta y_\infty = y_\infty - y_0$ instead of y_∞, $\Delta w_\infty = w_\infty - w_0$ instead of w_∞,

$\Delta e_\infty = e_\infty - e_0$ instead of e_∞, etc. The absolute values of variables are obtained as $y_\infty = y_0 + \Delta y_\infty$, with Δy_∞ computed using the equations given in Table 3.3-1.

Table 3.3-1. Stationary dependencies with respect to the reference input.

w	y_∞			e_∞		
	q_0=0	q_0=1	q_0=2	q_0=0	q_0=1	q_0=2
$w(s) = \frac{1}{s} w_\infty$ (SSR)	$\frac{k_0}{1+k_0} w_\infty$	$1 \cdot w_\infty$	$1 \cdot w_\infty$	$\frac{1}{1+k_0} w_\infty$	$0 \cdot w_\infty$	$0 \cdot w_\infty$
$w(s) = \frac{1}{s^2} w_\infty$ (CSR)	∞	∞	∞	∞	$\frac{1}{k_0} \cdot w_\infty$	$0 \cdot w_\infty$

- Table 3.3-1 can be used to obtaining the reference static characteristic of the CS, $y_\infty = f(w_\infty)\big|_{v_\infty = 0}$, shown in Fig. 3.3-2. The biggest angle corresponds to the CS with controllers with I component(s).

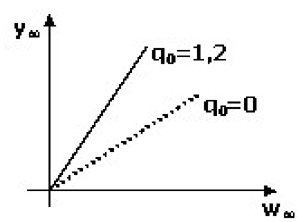


Fig. 3.3-2. Reference static characteristics of CSs.

An immediate consequence is the fact that changing the controller type (from P to I or inversely) leads to the modification of the shape of the reference SC.

(2) Aspects concerning the CSR:

- Ensuring the condition of zero steady-state control error ($e_\infty = 0$) requires the existence of two I components in the controller structure.

- The ∞ values in Table 3.3-1 can be interpreted as at infinite increases of the input (i.e., ramp signals) lead to infinite increased of the output or of the control error.
- Even the systems with controller with I component operate in CSR with nonzero control error (they do not ensure the output *exactly equal* to the reference input). In other words, the tracking systems should be characterized by $q_0 > 1$, with serious difficulties in the stabilization of such systems and in their sensitivity with respect to parametric variations.

B. The CS behavior with respect to the disturbance input. Properties induced by the type of controller. The computations are carried out considering $w(t) = 0 \ (w(s) = 0)$. Equations (3.3-6) and (3.3-7) lead to

$$y(s) = \frac{k_N \dfrac{B_N(s)}{A_N(s)}}{1 + \dfrac{k_0}{s^{q_0}} \dfrac{B_0(s)}{A_0'(s)}} v(s) = \frac{s^{q_0} k_N \dfrac{B_N(s)}{A_N(s)}}{s^{q_0} + k_0 \dfrac{B_0(s)}{A_0'(s)}} v(s) = -e(s), \tag{3.3-10}$$
$$\frac{B_N(0)}{A_N(0)} = 1, \ \frac{B_0(0)}{A_0'(0)} = 1.$$

Remarks. 1. Depending on the place where the disturbance input is applied, the expression of the t.f. $H_N(s)$ is (Fig. 3.3-3):

$H_N(s) = k_P \dfrac{B_P(s)}{A_P(s)}$, if the disturbance is applied to the process input (v_2 in Fig. 3.3-3), and it is called *load disturbance*,

$H_N(s) = 1$, if the disturbance is applied to the process output (v_1 in Fig. 3.3-3), and it is called additive disturbance on the process output.

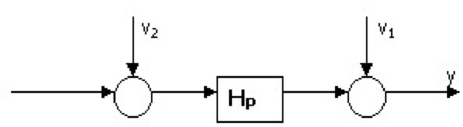


Fig. 3.3-3. Typical (extreme) situation concerning the place where the disturbance input is applied.

2. The real operation of the control system will take place around a s.s.o.p. A_0, with given w_0 which will determine the value of y_0.

The application of the final balue theorem for constant disturbance inputs (other situations are analyzed in similar manner) lead to the SSR dependencies synthesized in Table 3.3-2.

Table 3.3-2. Steady-state dependencies with respect to the disturbance input.

v	$y_\infty = -e_\infty$	$(e_\infty = -y_\infty)$	
	q_0=0	q_0=1	q_0=2
$v(s) = \frac{1}{s} v_\infty$	$\frac{k_N}{1+k_0} v_\infty$	$0 \cdot v_\infty$	$0 \cdot v_\infty$

The information given in Table 3.3-2 leads to the following conclusions:

- The use of controllers with I (2 I) component(s) ensures the rejection of constant disturbances (and the zero steady-control error is also ensured), $y_\infty = 0 \cdot v_\infty$.
- The use of controllers without I component, namely, P-xy controllers (the case of systems with P, PDT1, PD2T2, …, controllers) ensures the following steady-state relationship:

$$y_\infty = \frac{k_N}{1+k_0} v_\infty . \tag{3.3-11}$$

C. The natural static coefficient of control systems. The natural static coefficient γ_n is defined on the basis of (3.3-11) and expressed as follows with respect to absolute values and with respect to differences with certain s.s.o.p.s:

$$\gamma_n = \frac{k_N}{1+k_0} = \frac{y_\infty}{v_\infty}, \quad \gamma_n = \frac{\Delta y_\infty}{\Delta v_\infty} . \tag{3.3-12}$$

Since the two behaviors, with respect to the reference input and with respect to the disturbance input, take place simultaneously and accepting that the CS is linear, the following steady-state relationship can be expressed for the CS:

$$y_\infty = y_0 + \gamma_n \cdot v_\infty , \tag{3.3-13}$$

with y_0 determined by the reference input w_0 by which the s.s.o.p. of the CS is set (according to Table 3.3.2).

The systems are **divided in two categories** that depend on the value of the natural static coefficient γ_n:

- *systems with static coefficient* ($q_0 = 0$) for which:

$$\gamma_n = \frac{k_N}{1+k_0}, \quad k_0 = k_R k_P, \tag{3.3-14}$$

- *systems without static coefficient*, referred to also as *astatic systems* ($q_0 \in \{1,2\}$) for which:

$$\gamma_n = 0. \tag{3.3-15}$$

The static coefficient is a variable with a certain dimension, $<\gamma_n> = <y> / <v>$.

Equations (3.3-14) and (3.3-15) are used in the construction of the load static characteristics of the CS, $y_\infty = f(v_\infty)\big|_{w_\infty=\text{const}}$, shown in Fig. 3.3-4.

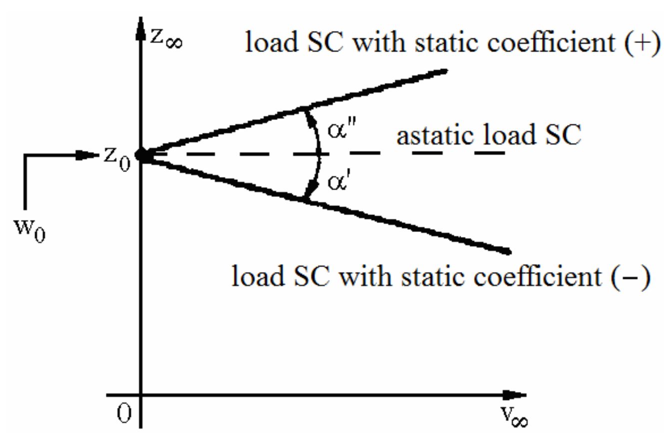


Fig. 3.3-4. Definition of load static characteristics and static coefficient of CS.

Remark: $\gamma_n > 0$ (positive static coefficient) or $\gamma_n < 0$ (negative static coefficient) can be also associated to the sign of disturbance input. This means that γ_n can always be considered as positive but the disturbance input v can be considered with the sign + or –.

It is often preferred to operate practically with the static coefficient *expressed in relative values* when the variables *y* and *v* are expressed in per unit (p.u.) values with respect to their nominal values. This expression of the static coefficient is

$$\hat{\gamma}_n = \frac{\hat{y}_n}{\hat{v}_n} = \frac{y_\infty / y_n}{v_\infty / v_n} = \gamma_n \frac{v_n}{y_n}. \qquad (3.3\text{-}16)$$

The static coefficient can be also expressed in *percent values*:

$$\hat{\gamma}[\%] = \gamma_n \frac{v_n}{y_n} 100\ \%. \qquad (3.3\text{-}17)$$

Accepting that k_P is constant, equation (3.3-14) indicates that the static coefficient can be *modified* by the value of k_R. This is not always possible because any modification of $k_0 = k_R k_P$ modifies the transient regimes in the CS or, in less favorable situation, it affects even the CS stability.

Example 3.3-1: Let us consider the 0 type CS discussed in the previous case study, with the structure given in Fig. 3.2-3 and with the PDT1 controller with $k_{R0} = 13.5$.

It is required: (1) To compute the SSVs of the CS, $\{e_\infty, u_\infty, z_\infty, y_\infty\}$ for $w_\infty = 6.5$ and $v_\infty = 250$.
(2) To compute the value of the natural static coefficient of the CS expressed in natural values, γ_n, and in relative values, $\hat{\gamma}_n$, accepting the nominal values $z_n = 6500$ and $v_n = 500$.
(3) To analyze the implications on the CS behavior of the reduction of the natural static coefficient to the value $\hat{\gamma}_n{}' = -0.01$.

Solution: (1) The SSR dependencies of all blocks and the connection relations are expressed as follows (they are also expressed in the previous case study):

C: $u_\infty = 13.5 e_\infty$,

E: $m_\infty = 50 u_\infty$,

P1: $x_{2\infty} = 40 m_\infty$,

P2: $z_\infty = x_{2\infty} - 10 v_\infty$, M: $y_\infty = 0.001 x_{3\infty}$, $x_{3\infty} = z_\infty$,

CE: $e_\infty = w_\infty - y_\infty$.

For the known values $w_\infty = 6.5$ and $v_\infty = 250$, the following computations are carried out in order to obtain the required SSVs. The expression of u_∞ is substituted from the equation for C in the equation for E, the expression of m_∞ is next substituted form the equation for E in the equation for P1, the expression of $x_{2\infty}$ is next substituted from the equation for P1 in the equation for P2, the expression of z_∞ is next substituted from the equation for P2 in the equation for M, and the expression of y_∞ is substituted from the equation for M in the equation for CE.

E: $m_\infty = 50 \cdot 13.5 e_\infty = 675 e_\infty$,

P1: $x_{2\infty} = 40 \cdot 675 e_\infty = 27000 e_\infty$,

P2: $z_\infty = 27000 e_\infty - 10 v_\infty$,

M: $y_\infty = 0.001(27000 e_\infty - 10 v_\infty) = 27 e_\infty - 0.01 v_\infty$,

CE: $e_\infty = w_\infty - 27 e_\infty + 0.01 v_\infty$.

The last equation for CE is solved with respect to the unknown e_∞ resulting in

$$e_\infty = \frac{1}{28} w_\infty + \frac{0.01}{28} v_\infty = \frac{6.5}{28} + \frac{0.01 \cdot 250}{28} = \frac{9}{28} = 0.3214.$$

This value is next substituted in the equations for C, P2 and M, and the SSVs are

$$u_\infty = 13.5 \cdot 0.3214 = 4.3389,\ z_\infty = 27000 \cdot 0.3214 - 10 \cdot 250 = 6177.8.$$

(2) Since a 0 type CS is involved, the natural static coefficient is computed using (3.3-14), where $k_N = -10$ and $k_0 = 13.5 \cdot 50 \cdot 40 \cdot 0.01 = 27$. Therefore,

$$\gamma_n = \frac{-10}{1+27} = -0.3571.$$

Equation (3.3-16) is next applied to obtain the static coefficient expressed in relative values:

$$\hat{\gamma}_n = -0.3571\frac{500}{6500} = -0.0275\,.$$

(3) The modified value of the static coefficient $\hat{\gamma}_n^{'} = -0.01$ is expressed in normal values using (3.3.16) modified as:

$$\gamma_n^{'} = \hat{\gamma}_n^{'}\frac{y_n}{v_n} = -0.01\frac{6500}{500} = -0.13\,.$$

The modification of the static coefficient to the value $\gamma_n^{'}$ by means of the controller gain k_R is carried out by the computation of the necessary value of the controller gain, with the notation k_{Rnec}, that ensures the desired natural static coefficient $\gamma_n^{'}$. Equation (3.3-14) is reorganized as follows with this regard:

$$k_{Rnec} = \frac{k_N - \gamma_n^{'}}{\gamma_n^{'} k_P} = \frac{-10 + 0.13}{-0.13 \cdot 2} = 37.9615\,,$$

where the process gain is $k_P = 50 \cdot 40 \cdot 1 \cdot 0.001 = 2$. This value of the controller gain, $k_{Rnec} > k_{R0}$, will affect the CS stability (to be assessed by the reader on the basis of the Hurwitz or Routh-Hurwitz criteria).

3.4. Artificial static coefficients and output coupled systems

The automatic systems that operate with a coupled output, namely with the same output and also called systems that operate in parallel, are frequently used in many industrial applications. For example, the control systems of the synchronous generators coupled to the power systems operate with the same outputs, the power system frequency or voltage.

These systems often require the well stated *distribution of the loa*d (the same disturbance applied to these systems) on each of the subsystems that operate with a coupled output. This requirement can be ensured only if the systems that operate with a coupled output are *with static coefficient*. Moreover, each of the subsystems has its own well computed value of the static coefficient. The requirement can be fulfilled in a convenient manner if the control systems are extended with additional feedforward channels which create artificial static coefficients.

A. The structure of principle of a CS with artificial static coefficient. This structure is presented in Fig. 3.4-1 (a), that points out the “natural” CS and the disturbance compensation block, BC-v, also called the block that creates the artificial static coefficient. The block Σ1 ensures the access to the reference channel of the component that depends on the disturbance input, $w_v(t)$. From a constructive point of view, the channel contains a disturbance measuring element M-v characterized by the gain k_{Mv}, and the block B-v to adjust the static coefficient. B-v can be of proportional (P) type or seldom of PDT1 type. The following equation can be expressed in this context:

$$k_{cv} = k_{bv} k_{Mv} . \tag{3.4-1-a}$$

Generally, the relation can be expressed by the t.f.s of each block:

$$H_{cv}(s) = H_{bv}(s) H_{Mv}(s) . \tag{3.4-1-b}$$

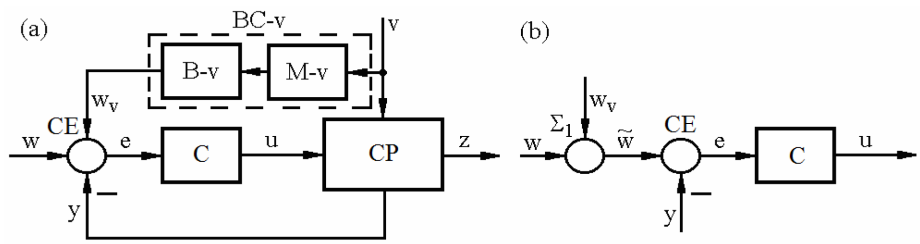


Fig. 3.4-1. Structure of principle of a CS with artificial static coefficient.

Within certain limits, the behavior of the basic CS (the natural one) is affected by the presence of the block BC-v. This justifies the need to filter the disturbance measurement channel. Due to the feedforward character, the presence of the compensation channel does not affect the system stability. Besides, the behavior with respect to the strictly speaking reference input $\tilde{w}(t)$ will be not affected. The design of a CS with artificial static coefficient is conditioned by the fulfillment of two **requirements**:

- the disturbance input $v(t)$ should be easily accessible to measurements,
- the disturbance input $v(t)$ should not affect directly the assessed output $z(t)$.

The design of a CS with artificial static coefficient is based on the creation in the reference input of a permanent and persistent component w_v that depends on the disturbance input. This component is characterized by the following SSR relationship:

$$w_{v\infty} = k_{cv} v_\infty \,. \tag{3.4-2}$$

The sign of application of the component $w_v(t)$ depends on:

- the sign of application of $v(t)$ to CP,
- the value of the natural static coefficient γ_n (if it is nonzero),
- the final desired value for the static coefficient γ_{ad} (artificially created),

and it results after the computations concerning the design of BC-v.

B. The design of CSs with artificial static coefficient. The input variable of the comparing element (CE) is expressed as follows (Fig. 3.4.-1 (b)):

$$\widetilde{w}_\infty = w_\infty + w_{v\infty} = w_\infty + k_{cv} v_\infty \,. \tag{3.4-3}$$

The desired value of the artificial static coefficient is γ_{ad}. The computation relations for the design of the block BC-v depend on the type of the basic CS, and they are presented as follows.

(a) The computation of BC-v for CSs of type 0. The basic structure is characterized by

$$z_\infty = \frac{k_d}{1+k_0} \widetilde{w}_\infty + \frac{k_N}{1+k_0} v_\infty \,, \; \gamma_n = \frac{k_N}{1+k_0} \,. \tag{3.4-4}$$

Equations (3.4-2) and (3.4-3) are substituted in (3.4-4) leading to:

$$z_\infty = \frac{k_d}{1+k_0} w_\infty + \frac{k_d k_{cv} + k_N}{1+k_0} v_\infty \,. \tag{3.4-5}$$

The first term in (3.4-5) characterizes the behavior with respect to the basic reference input $w(t)$ and the second term characterizes the behavior with respect to the disturbance input, i.e., the second term defines *the artificial static coefficient* γ_a:

$$\gamma_a = \frac{z_\infty}{v_\infty} = \frac{k_d k_{cv} + k_N}{1+k_0} \,. \tag{3.4-6}$$

For a given value of the artificial static coefficient $\gamma_a = \gamma_{ad}$ introduced in (3.4-6) the necessary k_{cv} is obtained as $k_{cv\mathrm{nec}}$:

$$k_{cv\mathrm{nec}} = \frac{\gamma_{ad}(1+k_0) - k_N}{k_d} \,. \tag{3.4-7}$$

The resulted sign of $k_{cv\mathrm{nec}}$ is the sign of application of $w_v(t)$ to the block Σ1 in Fig. 3.4-1 (b) and to the block CE in Fig. 3.4-1 (a).

(b) The computation of BC-v for CSs of type 1 and 2. The basic structure is characterized by

$$z_\infty = \frac{1}{k_M}\widetilde{w}_\infty + 0 \cdot v_\infty . \tag{3.4-8}$$

Using (3.4-2) and (3.4-3), the result is

$$z_\infty = \frac{1}{k_M} w_\infty + \frac{k_{cv}}{k_M} v_\infty . \tag{3.4-9}$$

The second term in (3.4-9) defines *the artificial static coefficient* γ_a:

$$\gamma_a = \frac{z_\infty}{v_\infty} = \frac{k_{cv}}{k_M} . \tag{3.4-10}$$

For a given value of the artificial static coefficient $\gamma_a = \gamma_{ad}$ introduced in (3.4-10) the necessary k_{cv} is obtained as $k_{cv\mathrm{nec}}$:

$$k_{cv\mathrm{nec}} = k_M \gamma_{ad} . \tag{3.4-11}$$

Remarks. 1. In the computed value of $k_{cv\mathrm{nec}}$ it is possible to separate the contribution of the disturbance measuring element k_{Mv} and the adjustable contribution of the compensation block $k_{bv\mathrm{nec}}$ from (3.4-1-a):

$$k_{bv\mathrm{nec}} = k_{cv\mathrm{nec}} / k_{Mv} . \tag{3.4-12}$$

2. The artificial static coefficient can be computed in a similar manner for the measured output $y(t)$ as well.

3. The component w_v applied to the CS input ensures not only the modification of the effect of the disturbance input but also a translation of the natural SC of the CS. The new operating points create virtual CSs as illustrated in Fig. 3.4-2.

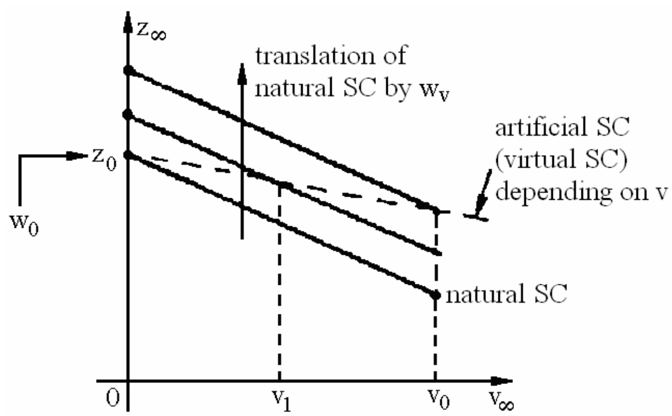


Fig. 3.4-2. Natural SC of CS and its translation by w_v.

Example 3.4-1: The CS structure given in the example 3.3-1 is given, with (1) PI controller and with (2) PDT1 controller with the gain $k_R = 13.5$.

It is required: To design the disturbance compensation block BC-v such that to ensure the artificial static coefficient $\gamma_{ad} = -1$. The disturbance input is measured by a measuring element with the gain $k_{Mv} = 0.01$. Which is the value of the static coefficient expressed in relative values accepting the nominal values $z_n = 6500$ and $v_n = 500$?

Solution: The solution in the case (1) is given as follows, and the solution in the case (2) is proposed to be given by the reader. Since the CS is of type 0, equation (3.4-11) is applied to obtain the value of k_{cvnec}:

$$k_{cvnec} = k_M \gamma_{ad} = 0.001 \cdot (-1) = -0.001 .$$

The presence of the measuring element M-v with the gain $k_{Mv} = 0.01$ leads to the necessity to apply equation (3.4-12). Therefore, the gain of the block B-v will be

$$k_{bvnec} = k_{cvnec} / k_{Mv} = -0.001 / 0.01 = -0.1 .$$

Remark: The same result can be obtained by solving the system of equations specific to the computation of SSVs:

$$\widetilde{w}_\infty = k_{bv} / k_{Mv} v_\infty , \; z_\infty = \gamma_{ad} v_\infty .$$

The expression of γ_{ad} in relative values, i.e., $\hat{\gamma}_{ad}$, results on the basis of

$$\hat{\gamma}_{ad} = \gamma_{ad} \frac{v_n}{y_n} = -1 \cdot \frac{500}{6500} = -0.0769,$$

$$\hat{\gamma}_{ad}\,[\%] = \gamma_{ad} \cdot 100\ \% = -7.69\ \%.$$

3.4.2. Output coupled control systems

Many particular control systems are coupled by their output variables. It is usual for such systems to impose a well stated distribution of the common load on each of the systems according to certain requirements. Such examples of output coupled systems are the synchronous generators and their control systems which are coupled to a power system. The coupled outputs are:

- the frequency in the power system as a measure of the active power equilibrium (produced ↔ used in the system),
- the terminal voltage of the generators coupled at the level of the power station.

The lack of a rigorous control of the distribution of the common load on the coupled systems leads to the risk to make some of the systems to be over-loaded with respect to other ones. The control of the distribution of the load can be solved by the controllers which control such systems.

If the control loops which are coupled by their output variables would have zero static coefficient, trying to ensure the zero steady-state control error, this will lead to taking over a part of the common load which cannot be computed. Therefore, the output coupling of control systems requires that the CS should have nonzero (natural or artificial) static coefficients in order to ensure a well stated distribution of the common load. The distribution algorithm can be computed in terms of several points of view as, for example, proportional to the nominal load of each system which is coupled. The reader is advised to use [5] for details and additional information concerning the distribution of the load.

3.5. Performance indices for control systems design

The properties (the quality) of a CS can be evaluated on the basis of **quality assessment criteria** and, as part of them, on the basis of **quality indices** also called **performance indices**. The quality assessment criteria are divided in two categories:

- local criteria,
- global criteria.

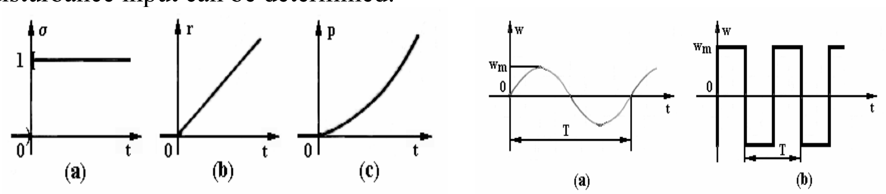
Each type of criterion is associated with the definition of specific performance indices.

3.5.1. Local quality assessment criteria and indices

The local quality assessment criteria make use of the following information to define the performance indices:

- the CS responses in particular operating regimes as, for example, the responses with respect to typical non-periodical or periodical deterministic signals as shown in Fig. 3.5-1,
- several ways to mathematically characterize the CS.

A. Performance indices defined in particular operating regimes of the CS. These indices characterize the system behavior but the conclusions can be extended to the frequency domain as well. The performance indices that can be defined in the CS response with respect to the step signal variation of the reference input or of the disturbance input can be determined:



(A) (B)

Fig. 3.5-1. Signals used in testing and characterizing the properties of CS / CP: non-periodic signals (A) and periodic signals (B).

- in the phase of development of the CS,
- in the phase of CS starting up or exploitation.

The responses with respect to particular input signals (reference input, disturbance input) can be determined relatively easily in both analytical and experimental manner. The results are easily interpretable and they allow for the comparison of difference control solutions.

(a) Performance indices defined in the response with respect to the step signal variation of the reference input. The main indices are defined in the response curve given in Fig. 3.5-2 related to an evolution of the assessed output $z(t)$ or of the measured output $y(t)$.

- *Steady-state indices.* The steady-state error:

$$\delta z_\infty = | z_{\infty d} - z_\infty |, \ \delta z_\infty[\%] = \frac{\delta z_\infty}{\Delta z_\infty} \cdot 100\,\%\,, \tag{3.5-1}$$

i.e., the absolute value of the difference of the steady-state assessed output z_∞ with respect to the desired value $z_{\infty d}$, and δz_∞ characterizes the quality of the physical achievement of the automation equipment.

- *Indices that characterize the dynamic regime behavior*, Fig. 3.5-2:
  - t_s – the settling time,
  - t_1 – the 0% to 100% rise time,
  - t_m – the peak time corresponding to $z_{\max}$,
  - σ_1 – the overshoot also called the maximum overshoot:

$$\sigma_1 = z_{\max} - z_\infty,\ \sigma_1[\%] = \frac{z_{\max} - z_\infty}{z_\infty - z_0} \cdot 100\,\%. \tag{3.5-2}$$

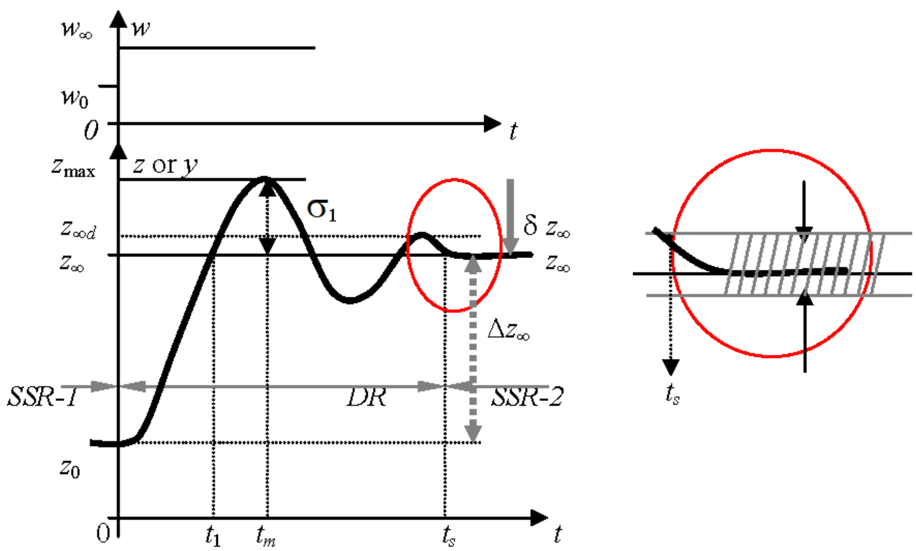


Fig. 3.5-2. Definition of performance indices in the system response with respect to the step signal applied to the reference input.

The CS behavior is appreciated as acceptable if the performance indices meet the following **performance specifications** (requirements):

- t_s, t_1, t_m – of as small as possible values (the imposed values should be correlated to the natural dynamics of the CP).

- σ_1 – with its value imposed within 5% … 15%. The response can be also imposed to be aperiodic, i.e., σ_1 [%]=0.

Imposing too restrictive values for the performance indices $\{t_s, t_1, t_m, \sigma_1\}$ often leads to:

- an increased complexity of the CS,
- the increase of the costs concerning the CS construction.

(b) Performance indices defined in the response with respect to the step signal variation of the disturbance input. A similar way to define these indices is used as in the case of the reference input but this should not be unified. *The natural static coefficient* γ_n is employed in the SSR characterization of the CSs:

$$\gamma_n = \left.\frac{\Delta z_\infty}{\Delta v_\infty}\right|_{w=0} = \frac{k_N}{1+k_0}\,. \tag{3.5-3}$$

Remark: Some applications are characterized by step responses that are different to the typical curve given in Fig. 3.5-2. This can be the effect of the CP properties, of the desired response or of the performance specifications.

B. Performance indices defined in the frequency domain plots. The CS properties are characterized by the closed-loop t.f.s, $H_w(s)$ and $H_v(s)$, or by the open-loop t.f. $H_0(s)$ for a unity feedback CS:

$$H_w(s) = \frac{H_0(s)}{1+H_0(s)}, H_v(s) = \frac{H_N(s)}{1+H_0(s)}, H_0(s) = H_R(s)H_P(s)\,. \tag{3.5-4}$$

The notation $L_0(s)$ is also used for the open-loop t.f. $H_0(s)$. The substitution $s \to j\omega$ leads to the **frequency plots** referred to also as *frequency characteristic*s or *pulsation plots* or *pulsation characteristics*. The CS quality can be assessed on the basis of the following frequency plots for continuous-time systems:

- *the closed-loop frequency plots*:

$$H_w(j\omega) = | H_w(j\omega) | \cdot e^{j\underline{/H_w(j\omega)}}, \tag{3.5-5}$$

- *the open-loop frequency plots*:

$$H_0(j\omega) = | H_0(j\omega) | \cdot e^{j\underline{/H_0(j\omega)}}\,. \tag{3.5-6}$$

(a) Performance indices defined in the closed-loop frequency plots (exemplified in Fig. 3.5-3):

- *the magnitude plot*, $|H_w(j\omega)|$, this plot corresponds to the complementary sensitivity function with the notation $M(\omega)$, and it offers information concerning the system robustness and the filter effect of the system,
- *the phase plot*, $\underline{/H}_w(j\omega)$, with the notation $\alpha(\omega)$.

The following **performance indices** can be defined on the basis of the magnitude plot $M(\omega)$:

- M_m – the peak frequency value (resonant value) which can be used to appreciate the stability degree, the recommended values are $1.1 \le M_m \le 1.5$,
- ω_m – the resonant frequency which characterizes the dynamics of the oscillatory processes,
- Λ_b – the bandwidth.

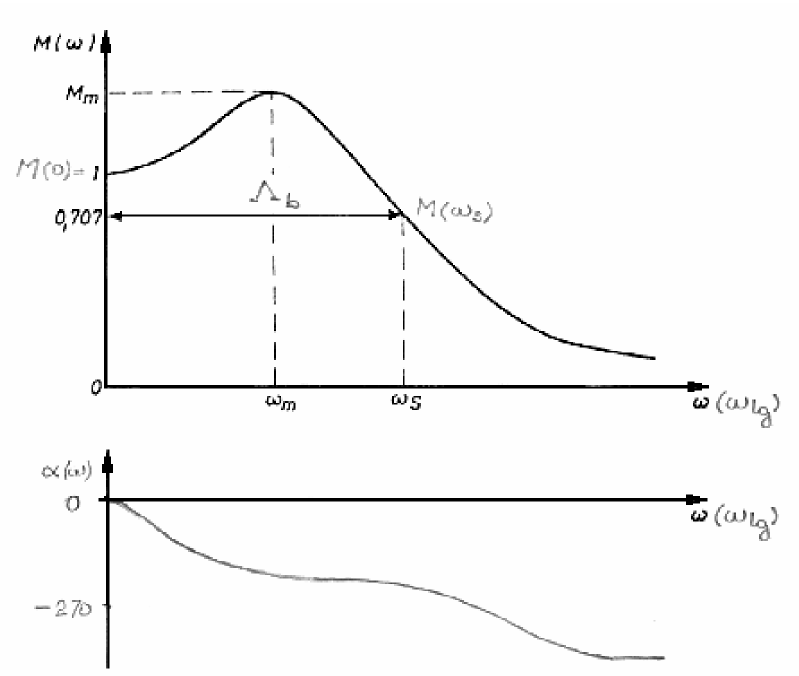
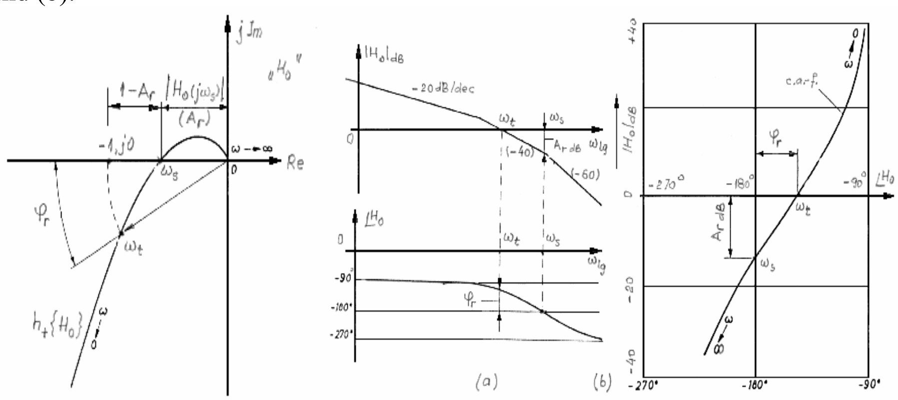


Fig. 3.5-3. Specific forms for the plots $M(\omega)$ and $\alpha(\omega)$ [5].

(b) Performance indices defined in the open-loop frequency plots. The performance indices can be connected to the Nyquist criterion applied to the stability analysis and assessment of control loops. The most important **performance indices** defined in the open-loop frequency plots are:

- $A_{r\,dB}$ – the modulus margin also called the modulus reserve,
- φ_r – the phase margin also called the phase reserve.

Both indices can be defined in the Nyquist hodograph (the Gauss plane) and in the logarithmic frequency plots also called the Bode plots as shown in Fig. 3.5-4 (a) and (b).



(a) (b)

Fig. 3.5-4. Performance indices defined in the Gauss plane (the Nyquist hodograph) (a), and in the open-loop Bode plots and in the modulus-phase plot (b).

- *The modulus margin*, $A_{r\,dB}$, is defined as

$$A_{r\,dB} = 20\lg\frac{1}{|H_0(j\omega_s)|} = -|H_0(j\omega_s)|_{dB}. \tag{3.5-7}$$

- *The phase margin*, φ_r, is defined as

$$\varphi_r = \underline{/H}_0(j\omega_t) + \pi. \tag{3.5-8}$$

ω_s and ω_t are *characteristic pulsations (frequencies) of the open-loop system*. The open-loop cutoff frequency ω_t, correlated with the phase margin, offers information on the system settling time (i.e., on the speed of response of the closed-loop system).

Complementary to the Nyquist hodograph, use is also made of the system sensitivity function (the loop sensitivity function) defined by the following relationship which indicates the t.f. from the reference input to the control error:

$$S_0(s) = \frac{1}{1 + H_0(s)}\,. \tag{3.5-9}$$

The maximum value of the sensitivity function (in the frequency domain) M_s and its inverse M_s^{-1}

$$M_S = \max_{\omega \geq 0} |S(j\omega)|,\ M_s^{-1} = 1/M_S \tag{3.5-10}$$

are employed in the assessment of the robustness of the control system. The circle of radius M_s^{-1}, referred to also as the sensitivity circle, is representative in the assessment of system robustness.

3.5.2. Global quality assessment criteria and indices

A. Optimization in the time domain. The global criteria highlight the quality of CSs in *synthetic formulations* by the use of **integral indices**. The optimal design of a CS is focused on getting *the best controller from a certain point of view* set by means of an integral performance index with the general formulation

$$\mathbf{p}_{opt} = \arg\min_{\mathbf{p}} I(\mathbf{p}), I(\mathbf{p}) = \int_{t_0}^{t_f} F(\boldsymbol{\zeta}(t,\mathbf{p}),t)dt\,. \tag{3.5-11}$$

$F(\boldsymbol{\zeta}(t,\mathbf{p}),t)$ is a vector function of the variables $\boldsymbol{\zeta}$ and $\mathbf{p}$. The vector variable $\boldsymbol{\zeta}$ contains a set of variables (or just a variable) whose evolution characterizes the system quality. The variable $\mathbf{p}$ consists of a set of adjustable parameters (of the controller) with respect to which the index $I(\mathbf{p})$ can be optimized (minimized in case of the example (3.5-11)). $[t_0, t_f]$ in (3.5-11) is the time horizon (the time interval) to evaluate the index (to observe the system); the limits of this interval can take the particular values $t_0 = 0$ and $t_f = \infty$.

The process of setting a certain integral index (with a certain structure of the integral) should be related an (indirect) connection between the expression of the integral in $I(\mathbf{p})$, its minimum and the quality of the CS. This connection is usually reflected by the empirical performance indices {σ_1, t_s, t_1, …} or by some energetic indices (for example, the minimum power consumption).

The use of the integral indices in control applications requires the following two steps:

- *The analysis* of the efficiency of the index in the characterization of the quality of the CS. This involves:
  - the choice of the particular form of the function $F(\boldsymbol{\zeta}(t,\mathbf{p}),t)$, the evaluation of the expression of this function takes place easily only for certain integral indices,
  - the choice of the variables $\boldsymbol{\zeta}$ in the expression of this function,
  - the choice of the particular form of variation of the input variable (of the CS) which determines the trajectory $\boldsymbol{\zeta}(t)$ with respect to which the analysis and design are carried out,
  - the specification of the correspondence between the values of the integral index and the CS quality.
- *The design of the CS*. This step concerns the algorithmic design of the controller and it involves:
  - the minimization of the index with respect to one or more parameters of the controller,
  - the computation of the minimum value of the index, this minimum value is generally not of interest but it should actually be the minimum.

For example, in case of PI controllers, the optimum parameter values $\{k_{c\,\mathrm{opt}}, T_{c\,\mathrm{opt}}\}$ that ensure the minimization of the integral are obtained by solving the parameter optimization equations:

$$\frac{\partial I_*}{\partial k_c}=0,\ \frac{\partial I_*}{\partial T_c}=0\,. \tag{3.5-12}$$

The choice of a certain criterion and of a certain integral index is carried out in accordance with the system particular features, with the types of external signals that are used to assess the index and with the method of assessment of the index.

B. Optimization in the frequency domain. The main requirements of the optimization can be formulated in the frequency domain as well in terms of the following conditions:

$$M_r(\omega)=|H_r(j\omega)|\approx 1 \text{ for as large as possible } \omega\geq 0\,, \tag{3.5-13}$$

$$M_{d1,d2}(j\omega)=|H_{d1,d2}(j\omega)|\approx 0 \text{ for as large as possible } \omega\geq 0\,, \tag{3.5-14}$$

The optimum conditions can be formulated with respect to all components of the frequency domain representation of the system, with respect to the components of the logarithmic frequency plots.

3.5.3. Final remarks

Since the performance requirements imposed to a CS can be very restrictive and often hardly achievable, the design will often accept reasonable tradeoffs to certain indices. Besides, the main relations that characterize a CS illustrate that the design with respect to the reference input and the design with respect to the disturbance input lead to different results.

The design based on performance indices defined in the system response with respect to particular variations of the reference input employs the possibility for simple setting the correspondences between *the time behavior of a system* and *the pole-zero representation of the t.f.* Such correspondences can be expressed as relatively simple relations between the system performance and the pole-zero representation of the system.

The low order systems are approximators for the real behavior of control systems, with results often obtained because of the following reasons:

- neglecting the dominated poles / zeros (i.e., the very small time constants) in the process t.f. or the possible application of the small time constants theorem, this approximation also leads to benchmark-type models [5],
- accepting the time invariance of the parameters and of the process structure,
- the application of the pole-zero cancellation principle.

Knowing the performance of such systems and the effects that result by the extension of the basic configuration with additional poles and/or zeros leads to a suggestive image on the controller design.

References

[1] O. Föllinger, Regelungstechnik, Elitera Verlag, Berlin, 1978.

[2] St. Preitl and R.-E. Precup, Eds., Controllers for Servo Systems: Design Methods (in Romanian: Regulatoare pentru servosisteme: metode de proiectare), Editura Orizonturi Universitare, Timisoara, 2007.

[3] T.-L. Dragomir and St. Preitl, Elemente de teoria sistemelor si reglaj automat, vol I, II, lecture notes, Centrul de multiplicare al Institutului Politehnic “Traian Vuia” din Timisoara, Timisoara, 1979.

[4] T.-L. Dragomir, St. Preitl and A. R. Trica, Regulatoare automate, vol. II, lecture notes, Centrul de multiplicare al Institutului Politehnic “Traian Vuia” din Timisoara, Timisoara, 1989.

[5] St. Preitl, R.-E. Precup and Zs. Preitl, Process Control Structures and Algorithms, vols. 1 and 2 (in Romanian: Structuri si algoritmi pentru conducerea automata a proceselor), Editura Orizonturi Universitare Publishers, Timisoara, 2009.