Seminarul 7

P. rezolvate

densitate de probabilitate

1) $f_X(x) = \begin{cases} ce^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \; ; c > 0$

a) $c = ?$ a.î. f_X este densitate de probabilitate

impunem condiția

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} f_X(x)\,dx = c \cdot \int_0^{\infty} e^{-x}\,dx = -ce^{-x}\Big|_0^{\infty} = -c\left(\frac{1}{e^{\infty}} - e^0\right) = \underbrace{-c\cdot(-1)}_{=c}$$

$\Rightarrow f_X(x)$ este distribuție de probab. pt $c = 1$ $\Rightarrow f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

b) fct. de repartiție pt. $F_X(x)$

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{\infty} f_X(t)\,dt$$

pt. $x < 0 \Rightarrow f_X(t) = 0$, $(\forall)\, t < x \Rightarrow F_X(x) = 0$ pt. $x < 0$

pt. $x \geq 0 \Rightarrow F_X(x) = \int_0^x f_X(t)\,dt = \int_0^x e^{-t}\,dt = -e^{-t}\Big|_0^x = -(e^{-x} - 1) = 1 - e^{-x}$

$$F_X(x) = \begin{cases} 1 - e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

c) $P(1 \leq X \leq 3) = ?$, $P(X = 2) = ?$, $P(X > 2) = ?$

$$P(1 \le X \le 3) = F_X(3) - F_X(1) = \cancel{1} - e^{-3} - \cancel{1} + e^{-1} = \frac{1}{e} - \frac{1}{e^3} = \frac{e^2 - 1}{e^3}$$

$P(X = a) = 0 \quad \Rightarrow P(X = 2) = 0$

$$P(X > 2) = 1 - P(X \le 2) = 1 - F_X(2) = 1 - 1 + e^{-2} = e^{-2}$$

d) $M(X) = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$

$$M(X) = \int_{-\infty}^{\infty} x \cdot e^{-x} dx = -xe^{-x} \Big|_0^{\infty} + \int_0^{\infty} e^{-x} dx = -xe^{-x} \Big|_0^{\infty} - e^{-x} \Big|_0^{\infty} =$$

$f'(x) = 1$

$$= -(0 - 0) - \frac{1}{\infty}^{\to 0} + 1 = 1$$

$g(x) = -e^{-x}$

$\sigma^2(X) = M(X^2) - [M(X)]^2$

$\sigma^2(X) = M(X^2) - 1$

$$M(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \underbrace{-2X \cdot e^{-X}}_{0} \Big|_0^{\infty} + \underbrace{2 \int_0^{\infty} x \cdot e^{-x} dx}_{1} = 2$$

$f'(x) = 2x$

$g(x) = -e^{-x}$

$\Rightarrow \sigma^2(X) = 2 - 1 = 1$

2) X v. a. continuă

$$f_X(x) = \begin{cases} x + \frac{1}{2}, & 0 \le x \le 1 \\ 0, & \text{în rest} \end{cases}$$

$M(X^n), (\forall) n \in \mathbb{N} = ?$

$$M(X^n) = \int_{-\infty}^{\infty} x^n \cdot f_X(x)\,dx = \int_0^1 x^{n+1}dx + \frac{1}{2}\int_0^1 x^n dx = \frac{x^{n+2}}{n+2}\Big|_0^1 + \frac{x^{n+1}}{2(n+1)}\Big|_0^1 =$$

$$= \frac{1}{n+2} + \frac{1}{2(n+1)} = \frac{2n+2+n+2}{2(n+1)(n+2)} = \frac{3n+4}{2(n+1)(n+2)}$$

3) $X \sim Unif(-1,1)$

$Y = X^2$

$F_Y(y) = ?$

densitatea de probab. $f_Y(y)$ ale v.a. Y

$y \in [0,1]$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$\boxed{F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & x \in [a,b) \\ 1, & x \ge b \end{cases}}$$

$$F_X(x) = \begin{cases} 0, & x < -1 \\ \frac{x+1}{2}, & x \in [-1,1) \\ 1, & x \ge 1 \end{cases}$$

$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{\sqrt{y}+1}{2} - \frac{1-\sqrt{y}}{2} = \frac{2\sqrt{y}}{2} = \sqrt{y}$$

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \sqrt{y}, & y \in [0,1) \\ 1, & y \ge 1 \end{cases}$$

F_Y continuă $\Rightarrow$ Y v.a. continuă

F_Y derivabilă $\Rightarrow f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}, & y \in (0,1) \\ 0, & \text{în rest} \end{cases}$

4) λ clienți/timp

$Y \sim poiss(\lambda t)$

$X \sim exp\left(\theta = \frac{1}{t}\right)$

$X \sim exp(\theta)$

$F(x) = \begin{cases} 0 & , x < 0 \\ 1 - e^{-\frac{x}{\theta}} & x \geq 0 \end{cases}$

Funcția de repartiție

$f(x) = \begin{cases} 0 & , x < 0 \\ \frac{1}{\theta} e^{-\frac{x}{\theta}} & , x \geq 0 \end{cases}$; $\theta > 0$

Densitatea de probabilitate

$P(X > t) = P($nu este nici o sosire în $[0,t]) = P(Y=0) = \frac{e^{-\lambda t} \cdot \lambda t^{0}}{0!} = e^{-\lambda t}$

$P(X \leq t) = 1 - P(X > t) = 1 - e^{-\lambda t}$

$x > 0 \Rightarrow F_X(x) = P(X \leq x) = 1 - P(X > x) = 1 - e^{-\lambda x}$

$\theta = \frac{1}{\lambda} \Rightarrow X \sim exp\left(\theta = \frac{1}{\lambda}\right)$

5) $X \sim N(-5, 4)$

$m = -5$

$\sigma^2 = 4 \Rightarrow \sigma = 2$

$Z = \frac{X - m}{\sigma}$, $Z \sim N(0,1)$

a) $P(X < 0) = P\left(\frac{X+5}{2} < \frac{0+5}{2}\right) = P(Z < 2,5) = \Phi(2,5) = 0,99$

b) $P(-7 \le X \le -3) = P\left(\frac{-7+5}{2} < \frac{X+5}{2} < \frac{-3+5}{2}\right) = \Phi(1) - \Phi(-1)$

$\Phi(-x) = 1 - \Phi(x)$

$= \Phi(1) - 1 + \Phi(1)$

$= 2\Phi(1) - 1$

c) $P(X > -3 \mid X > -5) = \frac{P(X > -3)}{P(X > -5)} = \frac{1 - P(X \le -3)}{1 - P(X \le -5)} =$

$= \frac{1 - P\left(X \le \frac{-3+5}{2}\right)}{1 - P\left(X \le \frac{-5+5}{2}\right)} = \frac{1 - P(X \le 1)}{1 - P(X \le 0)} = \frac{1 - \Phi(1)}{1 - \Phi(0)} \approx 0,32$

P. propuse

6) $f_X(x) = \begin{cases} cx^3, & 0 < x \le 1 \\ 0, & \text{în rest} \end{cases}$

a) $c = ?$ a.î. f_X e densitate de probabilitate

$\int_0^1 f_X(x)\,dx = 1$

$\int_0^1 cx^3 dx = c\int_0^1 x^3 dx = c\,\frac{x^4}{4}\Big|_0^1 = c \cdot \frac{1}{4} = \frac{c}{4}$

$\Rightarrow \frac{c}{4} = 1 \Rightarrow c = 4$

b) $F_X(x) = ?$

$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt$

pt. $x < 0$, $f_X(t) = 0 \Rightarrow F_X(x) = 0$

pt. $x \in (0, 1]$, $f_X(t) = 4x^3 \Rightarrow F_X(x) = \int_0^x f_X(t)\,dt = 4 \cdot \frac{t^4}{4}\Big|_0^x = x^4$

pt. $x>1$, $f_X(t)=0$, $(\forall)t>x \Rightarrow F_X(x)=\int_0^1 f_X(t)\,dt+\int_1^x f_X(t)\,dt=1+0$

$$F_X(x)=\begin{cases}0, & x<0\\ x^4, & 0<x\leq 1\\ 1, & x>1\end{cases}$$

c) $P\left(X\leq \frac{2}{3} \mid X>\frac{1}{3}\right)=\dfrac{P\left(X\leq\frac{2}{3}, X>\frac{1}{3}\right)}{P\left(X>\frac{1}{3}\right)}=\dfrac{F_X\left(\frac{2}{3}\right)-F_X\left(\frac{1}{3}\right)}{1-P\left(X\leq\frac{1}{3}\right)}=\dfrac{\frac{2^4-1}{3^4}}{1-F_X\left(\frac{1}{3}\right)}=\dfrac{2^4-1}{3^4-1}$

$P(X=0,4)=0$

$P\left(X>\frac{1}{2}\right)=1-P\left(X\leq\frac{1}{2}\right)=1-F_X\left(\frac{1}{2}\right)=1-\frac{1}{2^4}$

d) $M(x)=\int_{-\infty}^{\infty} x\cdot f_X(x)\,dx=\int_0^1 x\cdot 4x^3\,dx=\frac{4}{5}x^5\Big|_0^1=\frac{4}{5}$

$\sigma^2(X)=M(X^2)-\left(M(X)\right)^2$

$M(x^2)=\int_0^1 x^2 f_X(x)\,dx=\frac{4}{6}x^6\Big|_0^1=\frac{2}{3}$

$\sigma^2(x)=\frac{2}{3}-\frac{16}{25}=\frac{2}{75}$

7) $f_X(x) = \begin{cases} x^2\left(2x+\frac{3}{2}\right) & , 0 < x \le 1 \\ 0 & , \text{în rest} \end{cases}$

$Y = \frac{2}{X} + 3$

$\sigma^2(Y) = ?$

$\sigma^2(aX+b) = a^2\sigma^2(X)$

$\sigma^2\left(\frac{2}{X}+3\right) = 4\sigma^2\left(\frac{1}{X}\right)$

$M\left(\frac{1}{X}\right) = \int_{-\infty}^{\infty} \frac{1}{x} \cdot f_X(x)\,dx = \int_0^1 \frac{1}{\cancel{x}} \cdot x^{\cancel{2}}\left(2x+\frac{3}{2}\right)dx = \int_0^1 \left(2x^2 + \frac{3x}{2}\right)dx = 2\frac{x^3}{3}\Big|_0^1 + \frac{3}{2}\cdot\frac{x^2}{2}\Big|_0^1 =$

$= \frac{2}{3} + \frac{3}{4} = \frac{17}{12}$

$M\left(\left(\frac{1}{X}\right)^2\right) = \int_{-\infty}^{\infty} \frac{1}{x^2} f_X(x)\,dx = \int_0^1 \left(2x+\frac{3}{2}\right)dx = \cancel{2}\frac{x^2}{\cancel{2}}\Big|_0^1 + \frac{3x}{2} = \frac{5}{2}$

$\sigma^2(Y) = 4\left(\frac{5}{2} - \frac{17^2}{144}\right) = \frac{71}{36}$

9) $f_X(x) = \begin{cases} 0 & , x < 0 \\ 2xe^{-x^2} & , x \ge 0 \end{cases}$

a) $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} 2xe^{-x^2}dx = \int_0^{\infty} 2x \cdot \frac{1}{e^{-x^2}}dx = \int_0^{\infty} \frac{1}{e^t}dt = \cancel{e^t} \cdot -\frac{1}{e^{t}}\Big|_0^{\infty} = -e^{-t}\Big|_0^{\infty} = -\frac{1}{e^{\infty}} + 1 = 1$

Not $x^2 = t \Rightarrow 2x = dt$ $\quad x=0 \Rightarrow t=0$

$x=\infty \Rightarrow t=\infty$

b) $F_X(x) = ?$

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(x)\,dx$$

pt. $x<0$ $\quad F_X(x) = 0 \quad (\forall)\, x<0$

pt. $x \geq 0$ $\quad F_X(x) = \int_{-\infty}^{0} f_X(x)\,dx + \int_{0}^{x} f_X(t)\,dx = 0 + \int_{0}^{x} 2t\, e^{-t^2} dx =$

$$= -e^{-t^2}\Big|_0^x = -e^{-x^2} + 1$$

$$F_X(x) = \begin{cases} 0, & x<0 \\ 1-e^{-x^2}, & x \geq 0 \end{cases}$$

c) $P(-0{,}5 \le X \le 3) = F_X(3) - F_X(0{,}5) = \cancel{1} - e^{-9} - \cancel{1} + e^{-0{,}25} = e^{-0{,}25} - e^{-9}$

d) Mediana este acea val. x pt. care $P(X \geq x) = 1 - P(X < x) = \frac{1}{2}$

$$\left.\begin{array}{l} P(X<x) = F_X(x) \\ 1 - P(X \le x) = \frac{1}{2} \end{array}\right| \Rightarrow F_X(x) = \frac{1}{2}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = -e^{-t^2}\Big|_0^x = -e^{-x^2} + 1 = 1 - e^{-x^2} \;\Big|\; \Rightarrow e^{-x^2} = \frac{1}{2}$$

$-x^2 = \ln \frac{1}{2}$

$-x^2 = \ln 1 - \ln 2$

$-x^2 = -\ln 2$

$x = \sqrt{\ln 2}$

8) $f_X(x) = \frac{1}{2} e^{-|x|}, (\forall)\, x \in \mathbb{R}$

$Y = X^2, \quad F_Y(y)$

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} \frac{1}{2} e^{-|t|}\,dt$$

$$x < 0 \Rightarrow \frac{1}{2}\int_{-\infty}^{x} e^{t}\,dt = \frac{1}{2}e^{x}$$

$$x \ge 0 \Rightarrow \frac{1}{2}\int_{-\infty}^{x} e^{-|t|}\,dt = \frac{1}{2}\int_{-\infty}^{0} e^{t}\,dt + \frac{1}{2}\int_{0}^{x} e^{-t}\,dt = \frac{1}{2}\left(1 - \frac{1}{e^{\infty}}\right) - \frac{1}{2}\left(e^{-x} - 1\right) =$$

$$= \frac{1}{2} - \frac{1}{2}e^{-x} + \frac{1}{2} = 1 - \frac{1}{2}e^{-x}$$

$$F_X(x) = \begin{cases} \frac{1}{2}e^{x}, & x < 0 \\ 1 - \frac{1}{2}e^{-x}, & x \ge 0 \end{cases}$$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) =$$

$$= 1 - \frac{1}{2}e^{-\sqrt{y}} - \frac{1}{2}e^{-\sqrt{y}} = 1 - e^{-\sqrt{y}}$$

$$F_Y(y) = \begin{cases} 1 - e^{-\sqrt{y}}, & y \ge 0 \\ 0, & y < 0 \end{cases}$$