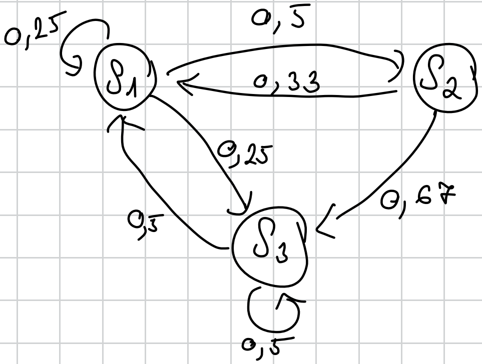
Seminarul 12

P. rezolvate

1) $S = \{1, 2, 3\}$

$$Q = \begin{pmatrix} 0,25 & 0,5 & 0,25 \\ 0,33 & 0 & 0,67 \\ 0,5 & 0 & 0,5 \end{pmatrix}$$

a) Să se deseneze graful asociat acestui lanț Markov



b) $P(X_4 = 3 \mid X_3 = 2) = p_{23} = Q(2,3) = 0,67$

c) $P(X_3 = 1 \mid X_2 = 1) = p_{11} = Q(1,1) = 0,25$

$p_{ij} = P(X_{n+1} = j \mid X_n = i) = Q(i,j)$

d) $P(X_0 = 1) = \frac{1}{3}$

$$P(X_0 = 1, X_1 = 2) = ?$$

$$s_0 = 1 \quad , \quad s_1 = 2$$

$$P(X_0 = s_0, X_1 = s_1, \ldots, X_{n-1} = s_{n-1}, X_n = s_n) =$$

$$= \pi_0(s_0) \cdot Q(s_0, s_1) \cdot Q(s_1, s_2) \cdot \ldots \cdot Q(s_{n-1}, s_n)$$

unde $\pi_0(s_0) = P(X_0 = s_0)$

$$P(X_0 = 1, X_1 = 2) = \pi_0(1) \cdot Q(1,2) = P(X_0 = 1) \cdot Q(1,2) =$$

$$= \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

e) $P(X_0 = 1) = \frac{1}{3}$

$P(X_0 = 1, X_1 = 2, X_2 = 3) = ?$

$$P(X_0 = 1, X_1 = 2, X_2 = 3) = \pi_0(1) \cdot Q(1,2) \cdot Q(2,3) = P(X_0 = 1) \cdot$$

$$\cdot P(X_1 = 2 \mid X_0 = 1) \cdot P(X_2 = 3 \mid X_1 = 2) = P(X_0 = 1) \cdot Q(1,2) \cdot Q(2,3) = \frac{1}{3} \cdot \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} = \frac{1}{9}$$

2) $S = \{0, 1\}$

$$Q = \begin{pmatrix} 0,5 & 0,5 \\ 0,33 & 0,67 \end{pmatrix}$$

$$P(X_0 = 0) = 1$$

a) $P(X_4 = 1 \mid X_0 = 0, X_1 = 1, X_2 = 1, X_3 = 0)$

Proprietatea Markoviană

$$P(X_{n+1}=j \mid X_0=s_0, X_1=s_1, \dots, X_{n-1}=s_{n-1}, X_n=i) =$$
$$= P(X_{n+1}=j \mid X_n=i)$$

$$P(X_4=1 \mid X_0=0, X_1=1, X_2=1, X_3=0) = P(X_4=1 \mid X_3=0) = p_{01} =$$
$$= Q(0,1) = 0,5 = \frac{1}{2}$$

b) probabilitatea ca sistemul să se găsească în starea 1 la momentul $n=3$

$$\tilde{\pi}_0 = [\,P(X_0=0) \quad P(X_0=1)\,]^T$$

din enunț $\Rightarrow$ $\tilde{\pi}_0 = [\,1 \quad 0\,]^T$ (lanțul pleacă din starea 0, cu probabilitate 1)

$$P(X_3=1) = ?$$

Aflăm distribuția de probabilitate a stărilor la momentul $n=3$, folosind formula $\tilde{\pi}_n^T = \tilde{\pi}_0^T \cdot Q^n$

$$\tilde{\pi}_3^T = \tilde{\pi}_0^T \cdot Q^3 = [1 \quad 0] \cdot \begin{pmatrix} 0,5 & 0,5 \\ 0,33 & 0,67 \end{pmatrix}^3 = \left[\frac{29}{72} \quad \frac{43}{72}\right]$$

$$\tilde{\pi}_3 = P(X_3=1)$$

c) Să se det. probabilitatea ca, dacă la mom. $n=2$ lanțul se află în modul 1, în următorii doi pași să treacă în modul

0, adică $P(X_4 = 0 | X_2 = 1)$

Probabilitatea ca lanțul să treacă din nodul inițial i în nodul j după n pași:

$$P(X_n = j | X_0 = i) = Q^n(i,j).$$

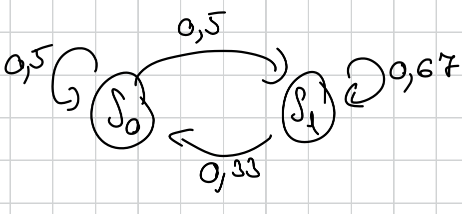
$$\boxed{P(X_{n+k} = j \mid X_k = i) = Q^n(i,j)}$$

$$Q^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} + \frac{1}{6} & \frac{1}{4} + \frac{1}{6} \\ \frac{1}{6} + \frac{2}{9} & \frac{1}{6} + \frac{4}{9} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{10}{24} & \frac{10}{24} \\ \frac{7}{18} & \frac{11}{18} \end{pmatrix} = \begin{pmatrix} \frac{5}{12} & \frac{5}{12} \\ \frac{7}{18} & \frac{11}{18} \end{pmatrix}$$

$$P(X_4 = 0 \mid X_2 = 1) = Q^2(1,0) = \frac{7}{18}$$

d) Să se studieze dacă acest lanț este ireductibil și aperiodic. În caz afirmativ, să se det. distribuția sa de echilibru



Graf tare conex (∃ drum de la orice nod la orice nod) ⇒ este ireductibil

Pt. a spune că e aperiodic e suficient să arătăm că are o singură stare aperiodică, adică de perioadă 1.

$Q(0,0) = 0{,}5 > 0 \Rightarrow S_0$ aperiodică $\Rightarrow$ lanț aperiodic

Lanț ireductibil și aperiodic $\Rightarrow$ pt. orice distribuție inițială π_0 șirul distribuțiilor de probabilitate la momentul n, (π_n) este convergent, iar limita acestui șir este un vector probabilist $\tilde{\pi}$ ce nu depinde de distribuția inițială.

Se det. din relația $Q^T \pi = \tilde{\pi}$

$$\lambda = 1 \Rightarrow (Q^T - I_2)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -\frac{a}{2} + \frac{b}{3} \\ \frac{a}{2} - \frac{b}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -3a + 2b = 0 \\ 3a - 2b = 0 \end{cases}$$

$$\Leftrightarrow 3a = 2b \Rightarrow b = \frac{3a}{2}$$

$$S_{\lambda=1} = \left\{ \left(a, \frac{3a}{2}\right)^T \right\} \quad ; \ ex: (2, 3)^T$$

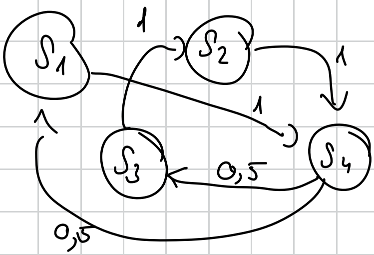
Vector probabilist $\Leftrightarrow$ vector cu suma coord. = 1

$$\Rightarrow \tilde{\pi} = \left(\frac{2}{5}, \frac{3}{5}\right)^T$$

$a + \frac{3a}{2} = 1 \Leftrightarrow 5a = 2 \Leftrightarrow a = \frac{2}{5} \Rightarrow \tilde{\pi} = \left(\frac{2}{5}, \frac{3}{5}\right)^T$

4) $S = \{1, 2, 3, 4\}$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$



este ireductibil

Perioada unui nod

$$T_i = \text{cmmdc}\{n \in \mathbb{N}^* \mid Q^n(i,i) > 0\}$$

Nodul 4 :

$4 \to 1 \to 4$ $\quad l = 2$

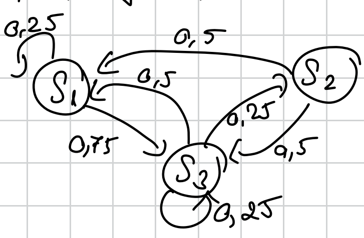
$4 \to 3 \to 2 \to 4$ $\quad l = 3$

$\Rightarrow T_4 = \text{cmmdc}(2, 3) = 1 \Rightarrow T_4 = 1$, lanț ireductibil $\Big| \Rightarrow$ lanț aperiodic

P. propuse

4) $S = \{1, 2, 3\}$; $P(X_1 = 1) = \frac{1}{3}$, $P(X_1 = 2) = \frac{1}{3}$

a)



$$Q = \begin{pmatrix} 0,25 & 0 & 0,75 \\ 0,5 & 0 & 0,5 \\ 0,5 & 0,25 & 0,25 \end{pmatrix}$$

b) $P(x_1 = 3, x_2 = 1, x_3 = 2, x_4 = 1) = \pi_0(3) \cdot Q(3,1) \cdot$

$\cdot\, Q(1,2) \cdot Q(2,1) = P(x_1 = 3) \cdot \frac{1}{2} \cdot 0 \cdot \frac{1}{2} = \frac{1}{3} \cdot \frac{1}{2} \cdot 0 \cdot \frac{1}{2} = 0$

c) $P(x_1 = 3, x_3 = 2)$

$P(x_0 = s_0, x_1 = s_1, \ldots, x_m = s_m) = P(x_0 = s_0) \cdot$
$\cdot P(x_1 = s_1 \mid x_0 = s_0) \cdot P(x_2 = s_2 \mid x_0 = s_0, x_1 = s_1) \ldots \cdot P(x_m = s_m \mid x_0 = s_0, \ldots, x_{m-1} = s_{m-1})$

$P(x_1 = 3, x_3 = 2) = P(x_1 = 1) \cdot P(x_3 = 2 \mid x_1 = 3) =$

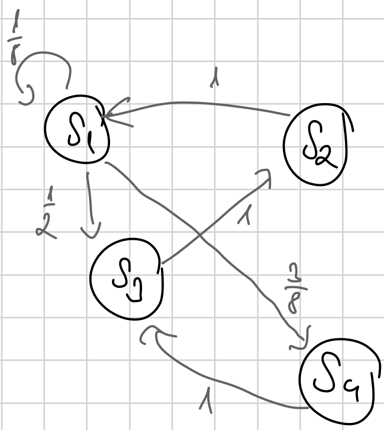
$= \frac{1}{3} \cdot Q^2(3,2)$

$$Q^2 = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{16} + \frac{3}{8} & \frac{3}{16} & \frac{3}{16} + \frac{3}{16} \\ \frac{1}{8} + \frac{1}{4} & \frac{1}{8} & \frac{3}{8} + \frac{1}{8} \\ 3 \cdot \frac{1}{8} & \frac{1}{16} & \frac{3}{8} + \frac{1}{8} + \frac{1}{16} \end{pmatrix} = \begin{pmatrix} \frac{7}{16} & \frac{3}{16} & \frac{6}{16} \\ \frac{6}{16} & \frac{2}{16} & \frac{8}{16} \\ \frac{6}{16} & \frac{1}{16} & \frac{9}{16} \end{pmatrix} = \begin{pmatrix} \frac{7}{16} & \frac{3}{16} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{8} & \frac{1}{2} \\ \frac{3}{8} & \frac{1}{16} & \frac{9}{16} \end{pmatrix}$$

$\Rightarrow P(x_1 = 3, x_3 = 2) = \frac{1}{3} \cdot \frac{1}{16} = \frac{1}{48}$

5) $S = \{1, 2, 3, 4\}$

$$Q = \begin{pmatrix} \frac{1}{8} & 0 & \frac{1}{2} & \frac{3}{8} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



a) este ireductibil

Starea 1 : $1 \to 1$ $\Rightarrow T_1 = 1 \Rightarrow$ aperiodic $\Rightarrow$

$\Rightarrow$ lanț aperiodic

b) $P(X_6 = 2 \mid X_4 = 1) = Q^2(1, 2)$

$$Q^2 = \begin{pmatrix} \frac{1}{8} & 0 & \frac{1}{2} & \frac{3}{8} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{8} & 0 & \frac{1}{2} & \frac{3}{8} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{16} & \frac{1}{2} & \frac{3}{8} + \frac{1}{16} & \frac{3}{64} \\ \frac{1}{8} & 0 & \frac{1}{2} & \frac{3}{8} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$P(X_6 = 2 \mid X_4 = 1) = \frac{1}{2}$

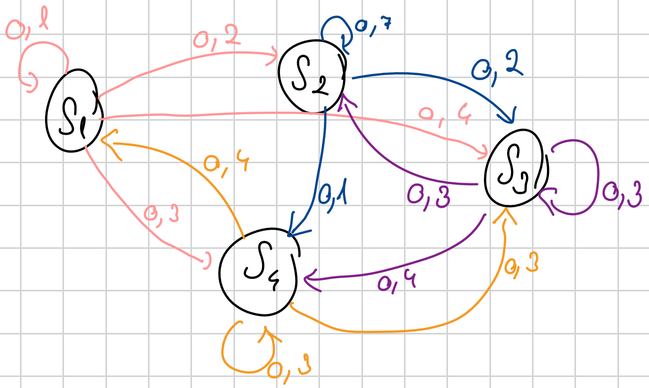
c) $P(X_1 = 2, X_2 = 1, X_3 = 4, X_4 = 1, X_5 = 3) =$

$= \pi_0(2) \cdot Q(2,1) \cdot Q(1,4) \cdot Q(4,1) \cdot Q(3,1) =$

$= P(X_1 = 2) \cdot 1 \cdot \frac{3}{8} \cdot 0 \cdot 0 = \frac{1}{4} \cdot 1 \cdot \frac{3}{8} \cdot 0 = 0$

6) $S = \{1, 2, 3, 4\}$

$$Q = \begin{pmatrix} 0,1 & 0,2 & 0,4 & 0,3 \\ 0 & 0,7 & 0,2 & 0,1 \\ 0 & 0,3 & 0,3 & 0,4 \\ 0,4 & 0 & 0,3 & 0,3 \end{pmatrix}$$



$(\exists)$ drum de la orice nod la orice nod $\Rightarrow$ ired.

Starea 1: $1 \to 1 \Rightarrow d_1 = 1 \Rightarrow$
$\Rightarrow$ nod aperiodic $\Rightarrow$ lanț aperiodic

b) $P(X_4 = 2 \mid X_1 = 1, X_2 = 2, X_3 = 1) = P(X_4 = 2 \mid X_3 = 1) =$
$= Q(1,2) = 0,2$

$$P(X_{n+1} = j \mid X_n = i) = p_{ij} = Q(i,j)$$

$$P(X_4 = 2, X_3 = 2, X_2 = 1 \mid X_1 = 1) = \frac{P(X_4 = 2, X_3 = 2, X_2 = 1, X_1 = 1)}{P(X_1 = 1)} =$$

$$= \frac{\cancel{\pi_0(1)} \cdot Q(1,1) \cdot Q(1,2) \cdot Q(2,2)}{\cancel{\pi_0(1)}} = 0,1 \cdot 0,2 \cdot 0,7 = 0,014$$

c) $\pi_0(1) = \frac{1}{4}$

$$P(x_1 = 1, x_2 = 3, x_3 = 4, x_4 = 1, x_5 = 2) =$$

$$= \pi_0(1) \cdot Q(1,3) \cdot Q(3,4) \cdot Q(4,1) \cdot Q(1,2) =$$

$$= \frac{1}{4} \cdot 0{,}4 \cdot 0{,}4 \cdot 0{,}4 \cdot 0{,}2 = \frac{1}{4} \cdot 0{,}064 \cdot 0{,}2 = \frac{1}{4} \cdot 0{,}0128 = 0{,}25 \cdot$$

$$\cdot 0{,}0128 = 0{,}0032$$

7) $$Q = \begin{array}{c} \\ L \\ C \\ S \end{array} \begin{array}{c} \begin{array}{ccc} L & C & S \end{array} \\ \begin{pmatrix} 0{,}7 & 0{,}1 & 0{,}2 \\ 0{,}5 & 0{,}3 & 0{,}2 \\ 0{,}5 & 0{,}1 & 0{,}4 \end{pmatrix} \end{array}$$

a) $P(x_2 = L \mid x_1 = C) = Q(C, L) = \frac{1}{2}$

b) $$P(x_2 = S, x_1 = S \mid x_0 = C) = \frac{P(x_0 = C, x_1 = S, x_2 = S)}{P(x_0 = C)} =$$

$$= \frac{\cancel{\pi_0(C)} \cdot Q(C,S) \cdot Q(S,S)}{\cancel{\pi_0(C)}} = 0{,}2 \cdot 0{,}4 = 0{,}08$$