Seminarul 6

1) $x_i = \begin{pmatrix} 1 & 0 \\ p & 1-p \end{pmatrix}$

Y_i = var. aleatoare ce ia val. 1 dacă în poz i începe un 1 (num de)

$Y_1 = \begin{pmatrix} 1 & 0 \\ p(x_1=1) & p(x_1=0) \end{pmatrix}$ $P(Y_1=1) = P(x_1=1) = p$

$Y_2 = \begin{pmatrix} 1 & 0 \\ pq & 1-pq \end{pmatrix}$ $P(Y_2=1) = P(x_1=0, x_2=1) = q \cdot p$

$Y_3 = \begin{pmatrix} 1 & 0 \\ pq & 1-pq \end{pmatrix}$ $P(Y_3=1) = P(x_2=0, x_3=1) = p \cdot q$

$\vdots$

$N = \sum_{i=1}^{n} Y_i$

$M(N) = \sum_{i=1}^{n} M(Y_i) = 1 \cdot p + (n-1)pq$

$p(x) = \begin{cases} p, & \text{dacă } x=1 \\ 1-p, & \text{dacă } x=0 \\ 0, & \text{în rest} \end{cases}$

Val. medie a unei variabile Bernoulli este $M(x) = 1 \cdot p + 0(1-p) = p$

2) $P(x \le 5) = \sum_{k=0}^{5} C_{20}^{k} (0,2)^k \cdot (0,8)^{20-k}$

$x_i = \begin{pmatrix} 1 & 0 \\ 0,2 & 0,8 \end{pmatrix}$

3) $X = \begin{pmatrix} 1 & 0 \\ 0,75 & 0,25 \end{pmatrix}$

$P(X \leq 6) = \sum_{k=1}^{6} C_{10}^{6} \cdot (0,75)^{k} \cdot (0,25)^{10-k}$ (prob. de esuare)

$P(Y > 4) = \sum_{k=1}^{4} C_{10}^{4} \cdot (0,25)^{k} \cdot (0,75)^{10-k}$ (prob. de succes)

4) $X = Y + 10$

$p = \frac{1}{4} = 0,25$ (p. de succes pt. Y)

$Y \sim Bin(n = 10, p = \frac{1}{4})$

$p_k = P(Y = k) = C_{10}^{k} \cdot p^{k} (1-p)^{10-k}$

$D_X = \{10, 11, \dots, 20\}$

$P(X = 10) = P(Y + 10 = 10) = P(Y = 0) = C_{10}^{0} \cdot p^{0} (1-p)^{10} = \left(\frac{3}{4}\right)^{10}$

$P(X = 11) = P(Y = 1) = 10 \cdot \frac{1}{4} \cdot \left(\frac{3}{4}\right)^{9}$

$\vdots$

$P(X = m) = P(Y = m - 10) = \boxed{C_{10}^{Y} \cdot p^{Y} \cdot (1-p)^{10-Y}}$

$P(X > 15) = P(Y=6) + P(Y=7) + P(Y=8) + P(Y=9) + P(Y=10)$

5) $X \sim Geom\left(p = \frac{1}{3}\right)$

$\boxed{P(X = k) = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{k-1}}$

6) $Y \sim geom(p = 0,95)$

$P(Y = 3) = p \cdot (1-p)^{3-1} = 0,95 \cdot 0,025$

$W = Y - 1$

$M(W) = M(Y-1) = M(Y) - 1 = \frac{1}{p} - 1$

8) a) $X \sim Pois(\lambda)$

$\lambda' = 5 \cdot 0{,}2 = 1$

$P(X=k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$

$P(X=0) = \frac{e^{-1} \cdot 1^0}{0!} = \frac{e^{-1} \cdot 1}{1} = \frac{1}{e}$

b) $Y \sim Pois(\lambda'')$

$\lambda'' = 10 \cdot 0{,}2 = 2$

$P(Y>3) = 1 - P(X \leq 3) = 1 - P(X=0) - P(X=1) - P(X=2) - P(X=3) =$

$= 1 - \frac{e^{-2} \cdot 1}{1} - \frac{e^{-2} \cdot 2}{1} - \frac{e^{-2} \cdot 4}{2} - \frac{e^{-2} \cdot 8}{6}$

9) $Y \sim Bin\left(n = 10, p = \frac{1}{5}\right)$

a) $P(Y=2) = C_{10}^{2} \cdot \left(\frac{1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^8$

b) $P(Y \leq 4) = \sum_{k=1}^{4} C_{10}^{k} \cdot \left(\frac{1}{5}\right)^k \cdot \left(\frac{4}{5}\right)^{10-k}$

10) $X \sim Geom(p)$

partial !

$M(X) = ?$

$M\left(\frac{1}{2^X}\right) = ?$

11) $X = \begin{pmatrix} 1 & 0 \\ 0,9 & 0,1 \end{pmatrix}$

a) $Y \sim geom(0,9)$

$P(Y=5) = 0,9 \cdot (0,1)^4$

$M(X) = \frac{1}{p}$

$\sigma^2(X) = \frac{1-p}{p^2}$

12) $X = \begin{pmatrix} 0 & 1 \\ 0,98 & 0,02 \end{pmatrix}$

$Y \sim geom(0,02)$ să eșueze

$P(Y=10) = 0,02 \cdot (0,98)^9$

b) $P(Y \geq 5) = 1 - P(Y=4) - P(Y=3) - P(Y=2) - P(Y=1)$

c) $M(Y) = \frac{1}{p} = \frac{1}{0,02}$

13) $X = \begin{pmatrix} 1 & 0 \\ 0,2 & 0,8 \end{pmatrix}$

a) $Y \sim geom(0,2)$

$P(Y=2) = 0,2 \cdot 0,8$

b) $1 - P(Y=1) - P(Y=2) = 1 - 0,2 - 0,16 = 0,64$

c) $M(\bar{Y}) = M(Y) - 1 = \frac{1}{0,2} - 1$

15) $\lambda = 0{,}01$ erori/pag

$X \sim Pois(0{,}01)$

$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

16) 10 c ... 15 min

x c ..... 1 min

$x = \frac{10}{15} = \frac{2}{3}$ cereri/min

$X \sim Pois(\lambda)$

$\lambda = \frac{2}{3} \cdot 3 = 2$ cereri / 3 min

$P(X=2) = \frac{e^{-2} \cdot 2^2}{2!} = \frac{2}{e^2}$

Ex. rezolvate

2) $p = 0{,}2$

$P(X=k) = C_n^k \cdot p^k \cdot (1-p)^{10-k}$

$Y \sim Bin(20, 0{,}2)$

$P(X \leq 5) = \sum_{k=0}^{5} C_{20}^k \cdot (0{,}2)^k \cdot (0{,}8)^{20-k}$

3) $p = 0{,}75$ (eșuare)
$q = 0{,}25$ (activă)

$Y \sim Bin(10, 0{,}25)$

$P(Y > 4) = \sum_{k=5}^{10} C_{10}^k \cdot (0{,}25)^k \cdot (0{,}75)^{10-k}$

4) $P(X>15) \Rightarrow P(Y>5)$

$Y \sim Bin(10, 0{,}25)$

$$P(Y>5) = \sum_{k=6}^{10} (0{,}25)^{k} \cdot (0{,}75)^{10-k}$$

5) $X \sim geom\left(\frac{1}{3}\right)$

$$P(X=k) = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{k-1}$$

$$P(X=k) = p \cdot (1-p)^{k-1}$$

6) $X \sim geom(0{,}95)$

$$P(X=3) = 0{,}95 \cdot 0{,}05^2 = 0{,}0024$$

$$M(X) = \frac{1}{p}$$

$$M(X-1) = M(X) - 1 = \frac{1}{0{,}95} - 1 = 0{,}05$$

7) $X = \sum_{i=1}^{n} X_i$

$p = \frac{n-(i-1)}{n}$ (probab. de succes)

$$M(X_i) = \frac{1}{p_i} = \frac{n}{n-i+1}$$

$$M(X) = \sum_{i=1}^{n} M(X_i) = \sum_{i=1}^{n} \frac{n}{n-i+1} = n \sum_{i=1}^{n} \frac{1}{i}$$

$H_n = \sum_{i=1}^{n} \frac{1}{i}$ $\qquad H(n) = \ln(n) + O(n)$

?

8) $X \sim poiss\,(\lambda = 0{,}2)$

$\lambda = 0{,}2$ mes/min

$$P(X=k) = \frac{e^{-\lambda}\cdot \lambda^{k}}{k!}$$

$\lambda' = 5\cdot 0{,}2 = 1$ mes/5min

$Y \sim poiss\,(\lambda')$

$$P(Y=0) = \frac{e^{-\lambda'}\cdot \lambda'^{0}}{0!} = \frac{1}{e}$$

$\lambda'' = 10\cdot 0{,}2 = 2$ mes/10 min

$Z \sim poiss\,(\lambda'')$

$$P(Z>3) = 1 - P(Z\le 3) = 1 - P(Z=0) - P(Z=1) - P(Z=2) - P(Z=3) =$$

$$= 1 - \frac{e^{-2}\cdot 2^{0}}{0!} - \frac{e^{-2}\cdot 2^{1}}{1!} - \frac{e^{-2}\cdot 2^{2}}{2!} - \frac{e^{-2}\cdot e^{3}}{3!}$$

Ex. propuse

9) $p = 0{,}2$

$$X = \begin{pmatrix} 1 & 0 \\ 0{,}2 & 0{,}8 \end{pmatrix}$$

a) $Y \sim Bin\,(10, 0{,}2)$

$$P(Y=2) = C_{10}^{2}\,(0{,}2)^{2}(0{,}8)^{8}$$

b) $P(Y\le 4) = \sum_{k=1}^{4} C_{10}^{k}\,(0{,}2)^{k}\cdot(0{,}8)^{10-k}$

10) $X \sim Geom(p)$

$M(x), M\left(\frac{1}{2^x}\right) = ?$

$$X : \begin{pmatrix} k \\ (1-p)^{k-1} \cdot p \end{pmatrix} \Leftrightarrow \begin{pmatrix} k \\ p \cdot q^{k-1} \end{pmatrix}, k \in \mathbb{N}^\infty$$

$$M(x) = \sum_{k=1}^{\infty} k \cdot p \cdot (1-p)^{k-1} = p \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} = p \cdot \frac{1}{p^2} = \frac{1}{p}$$

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}, \quad |q| < 1 \quad \Big/ ()' \text{ derivăm}$$

$$\sum_{k=1}^{\infty} k \cdot q^{k-1} = \frac{1}{(1-q)^2} = \frac{1}{p^2}$$

$$M\left(\frac{1}{2^x}\right) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot p\,(1-p)^{k-1}$$

11) $p = 0,1 \sim$ eroare
$p' = 0,9 \sim$ corect

a) $X \sim geom\,(0,9)$

$P(X=5) = 0,9 \cdot 0,1^4$

b) $M(x) = \frac{1}{0,9}$

12) $p = 0,02 \sim$ conectare

a) $X = \begin{pmatrix} 1 & 0 \\ 0,02 & 0,98 \end{pmatrix}$

$Y \sim geom\ (0{,}02)$

$P(Y=10) = 0{,}02 \cdot 0{,}98^{9}$

b) $P(Y>5) = 1 - P(Y \leq 5) = 1 - \sum_{i=1}^{5} (0{,}02)(0{,}98)^{i-1}$

c) $M(Y) = \frac{1}{0{,}02} = \frac{100}{2} = 50$

13) $p = 0{,}2 \sim$ colizione

$X = \begin{pmatrix} 1 & 0 \\ 0{,}2 & 0{,}8 \end{pmatrix}$

a) $Y \sim geom\,(0{,}2)$

$P(Y=2) = 0{,}2 \cdot 0{,}8$

b)

$P(Y>2) = 1 - P(Y=1) - P(Y=2) = 1 - 0{,}2(0{,}8)^{0} - 0{,}2 \cdot 0{,}8 = 0{,}64$

c) $M(Y-1) = M(Y) - 1 = \frac{1}{0{,}2} - 1 = 4$

14) $X \sim poiss\,(\lambda)$

$M(X) = ?$

$$M(X) = \sum_{k=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{k}}{k!} = \lambda e^{-\lambda} \cdot \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^{k-1}}{k!}}_{= e^{\lambda}} = \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda$$

$M(X) = \lambda$

15) $\lambda = 0{,}01$ erori/pag

$\lambda' = 1$ erori /100 pag

$Y \sim poiss(\lambda')$

$$P(Y \leq 3) = \frac{e^{-\lambda'} \cdot \lambda'^0}{0!} + \frac{e^{-\lambda'} \cdot \lambda'^1}{1!} + \frac{e^{-\lambda'} \cdot \lambda'^2}{2!} + \frac{e^{-\lambda'} \cdot \lambda'^3}{3!} = \frac{1}{e} + \frac{1}{e} + \frac{1}{2e} + \frac{1}{6e}$$

16) $\lambda = 10$ cer/15 min $\Rightarrow \lambda = \frac{2}{3}$ cereri/min

$Y \sim pois(\lambda')$; $\lambda' = 2$ cer/3 min

$$P(Y=2) = \frac{e^{-\lambda'} \cdot \lambda^2}{2!} = \frac{e^{-2} \cdot 4}{2} = \frac{2}{e^2}$$