# Seminarul 10

P. rezolvate

1) $(X, Y)$ v. aleatoare.

Densitatea de probabilitate

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & x \in [0,1], \ y \in [0,1] \\ 0, & \text{în rest} \end{cases}$$

a) Să se arate că $f$ e densitate de probabilitate

$f$ densitate de probabilitate dacă:

- $f_{X,Y}(x,y) \geq 0, (\forall)(x,y) \in \mathbb{R}^2$
- $\iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = 1$

$f \geq 0 \ (\forall)(x,y) \in \mathbb{R}^2$

$$\iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = \int_0^1 \left( \int_0^1 4xy\,dy \right) dx = \int_0^1 \left( 2xy^2 \Big|_0^1 \right) dx =$$

$$= \int_0^1 (2x - 0)\,dx = \int_0^1 2x\,dx = x^2 \Big|_0^1 = 1$$

$\Rightarrow f$ densitate de probabilitate

b) Să se vizualizeze ev. $A: (X < 0.5$ și $Y > 0.5)$ și să se calculeze probabilitatea $P(A)$

Not $S = [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$

$$P(A) = \iint_S f_{X,Y}(x,y)\, dx\, dy = \int_0^{\frac{1}{2}} \left( \int_{\frac{1}{2}}^{1} 4xy\, dy \right) dx = \int_0^{\frac{1}{2}} \left( 2xy^2 \Big|_{\frac{1}{2}}^{1} \right) dx =$$

$$= \int_0^{\frac{1}{2}} \left(2x - x \cdot \frac{1}{2}\right) dx = \frac{3}{2} \int_0^{\frac{1}{2}} x\, dx = \frac{3}{2} \cdot \frac{x^2}{2} \Big|_0^{\frac{1}{2}} = \frac{3}{2} \cdot \frac{1}{8} = \frac{3}{1}$$

c) Să se det. f.ct. de repartiție a vectorului $(X, Y)$

Fct. de repartiție

$$F(x,y) = P(X \le x, Y \le y)$$

$$F(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, dt\, ds =$$

$$= \int_{-\infty}^{x} \left( \int_{-\infty}^{y} 4ts\, dt \right) ds = \int_{-\infty}^{x} \left( 2t^2 s \Big|_{-\infty}^{y} \right) ds = \int_{-\infty}^{x} 2y^2 s\, ds = y^2 x^2$$

d) Să se det. densitățile marginale $f_X(x)$ și $f_Y(y)$

Densitățile marginale

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \quad , \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_0^1 4xy\, dy = 2xy^2 \Big|_0^1 = 2x \quad , (\forall) x \in [0,1]$$

$$f_Y(y) = 2y \quad (\forall) y \in [0,1]$$

$$\Rightarrow f_X(x) = \begin{cases} 2x, & x \in [0,1] \\ 0, & \text{în rest} \end{cases}$$

$$f_Y(y) = \begin{cases} 2y, & y \in [0,1] \\ 0, & \text{în rest} \end{cases}$$

e) Să se det. fct. de repartiție marginale $F_X(x)$ și $F_Y(y)$

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} 2t\,dt = x^2$$

$$F_X(y) = y^2$$

f) Să se studieze dacă cele 2 variabile X și Y sunt independente

$$\left.\begin{array}{l} f_{X,Y}(x,y) = 4xy \\ f_X(x) \cdot f_Y(y) = 2x \cdot 2y = 4xy \end{array}\right| \Rightarrow \text{independente}$$

Independente dacă $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

g) Să se calculeze $P(X+Y<1)$

$$P(X+Y<1) = \int_0^1 \left( \int_0^{1-x} 4xy\,dy \right) dx = \int_0^1 \left( 2xy^2 \Big|_0^{1-x} \right) dx =$$

$$= \int_0^1 \left( 2x(1-x)^2 \right) dx = 2\left( \int_0^1 x(1-2x+x^2)\,dx \right) = 2\left( \int_0^1 x\,dx - 2\int_0^1 x^2\,dx + \int_0^1 x^3\,dx \right) =$$

$$= 2\left(\frac{1}{2}^{(6} - \frac{2}{3}^{(4} + \frac{1}{4}^{(3}\right) = \cancel{2}^{6}\left(\frac{6-8+3}{\cancel{12}}\right) = \frac{1}{6}$$

2) $(X, Y)$ v.a.

$$f_{X,Y}(x,y) = \begin{cases} 6x, & 0 < x < y < 1 \\ 0, & \text{în rest} \end{cases}$$

a) densitatea marginală $f_X$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_x^1 6x\,dy = 6xy\Big|_x^1 = 6x(1-x)$$

$$f_X(x) = \begin{cases} 6x(1-x) & , x \in (0,1) \\ 0, & \text{în rest} \end{cases}$$

b) Densitatea de prob. a var. aleatoare $(Y \mid X = 0,25)$

$$h(y \mid 0,25) = \frac{f_{XY}(0,25, y)}{f_X(0,25)} = \frac{\cancel{6 \cdot 0,25}}{\cancel{6 \cdot 0,25} \cdot 0,75} = \frac{4}{3}$$

$$h(y \mid 0,25) = \begin{cases} \frac{4}{3} & , y \in (0,25, 1) \\ 0 & , \text{în rest} \end{cases}$$

c) Să se calculeze $P(X > 0,5)$ și $P(Y > 0,5 \mid X = 0,25)$

$$P(X > 0,5) = \int_{0,5}^{1} 6x(1-x)\,dx = 6\int_{0,5}^{1} x - 6\int_{0,5}^{1} x^2 = 3x^2\Big|_{0,5}^{1} - 2x^3\Big|_{0,5}^{1} =$$

$$= 3\left(1-\frac{1}{4}\right) - 2\left(1-\frac{1}{8}\right) = \frac{\cancel{9}^{\,2)}}{4} - \frac{14}{8} = \frac{4}{8} = \frac{1}{2}$$

$$P(Y>0,5 \mid X=0,25) = \int_{0,5}^{1} h(y/0,25)\,dy = \int_{0,5}^{1} \frac{4}{3}\,dy = \frac{4}{3}y\Big|_{0,5}^{1} =$$

$$= \frac{4}{3}\left(1-\frac{1}{2}\right) = \frac{4}{6} = \frac{2}{3}$$

d) Media și dispersia variabilei $(Y \mid X=0,25)$

$$M(Y \mid X=0,25) = \int_{-\infty}^{\infty} y \cdot h(y/0,25)\,dy = \int_{0,25}^{1} y\cdot\frac{4}{3}\,dy = \frac{4}{6}\left(1-\frac{1}{16}\right) = \frac{\cancel{4}}{\cancel{6}_{2}}\cdot\frac{\cancel{15}^{5}}{\cancel{16}_{4}} =$$

$$= \frac{5}{8}$$

$$\sigma^2(X) = M(X^2) - [M(X)]^2$$

$$M(2^2) = \int_{-\infty}^{\infty} y^2 \cdot h(y/0,25)\,dy = \int_{0,25}^{1} y^2\cdot\frac{4}{3}\,dy = \frac{4}{9}\left(1-\frac{1}{64}\right) = \frac{\cancel{4}}{\cancel{9}}\cdot\frac{\cancel{63}^{7}}{\cancel{64}_{16}} = \frac{7}{16}$$

$$\sigma^2(X) = \frac{\cancel{7}^{\,4)}}{16} - \frac{25}{64} = \frac{3}{64}$$

3) $G = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ disc $\Rightarrow x,y \in [-1,1]$

$(X,Y)$ vector aleatoriu

a) Densitatea de probabilitate a vectorului $(X,Y)$

$$f_{XY}(x,y) = \begin{cases} \frac{1}{Aria(G)} = \frac{1}{\pi} & \text{dacă } (x,y) \in G \\ 0 & , \text{ în rest} \end{cases}$$

(pt. că e unif. distribuită ???)

b) Să se det. densitățile marginale $f_X(x)$ și $f_Y(y)$?

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_{-\infty}^{\infty} \frac{1}{\pi}\,dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy =$$

$$= \frac{1}{\pi} \cdot 2\sqrt{1-x^2}$$

$y^2 \le 1-x^2$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$$\Rightarrow f_X(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & , -1 \le x \le 1 \\ 0 & , \text{ în rest} \end{cases}$$

analog $f_Y(y) = \begin{cases} \frac{2}{\pi}\sqrt{1-y^2} , & -1 \le x \le 1 \\ 0 & , \text{ în rest} \end{cases}$

c) Densitatea de probab. a variabilei $(X|Y=y_0)$ pt. $-1 \le y_0 \le 1$

$(X, -1 \le Y \le 1)$

$$g(x|y_0) = \frac{f_{X,Y}(x,y_0)}{f_Y(y_0)} = \begin{cases} \frac{1}{2\sqrt{1-y_0^2}} & , -\sqrt{1-y_0^2} \le x \le \sqrt{1-y_0^2} \\ 0, & \text{ în rest} \end{cases}$$

$$\frac{f_{X,Y}(x,y_0)}{f_Y(y_0)} = \frac{\frac{1}{\cancel{\pi}}}{\frac{2}{\cancel{\pi}}\sqrt{1-y_0^2}} = \frac{1}{2\sqrt{1-y_0^2}}$$

d) Variabilele $X$ și $Y$ sunt independente?

$$f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{\pi}\, dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\, dx = \frac{2}{\pi}\cdot\sqrt{1-y^2}$$

$$f_X(x)\cdot f_Y(y) = \frac{2}{\pi}\cdot\sqrt{1-x^2}\cdot\frac{2}{\pi}\cdot\sqrt{1-y^2} = \frac{4}{\pi^2}\cdot\sqrt{\phantom{x}}\cdot\sqrt{\phantom{x}}$$

$$f_{X,Y}(x,y) = \frac{1}{\pi}$$

$\Rightarrow$ nu sunt indep.

e) Pseudocod de simulare a unei valori de obs. a vectorului $(X, Y)$

```
do {
      x = -1 + 2 * urand();
      y = -1 + 2 * urand();
} while (x*x + y*y > 1)
return (x,y);
```

f) $N \sim Geom(p)$

$$P(N=3) = p\cdot(1-p)^2$$

$$p = \iint_G g(x,y)\, dx\, dy = \iint_G \frac{1}{Aria(D)}\, dx\,dy = \frac{1}{Aria(D)} \iint_G 1\, dx\,dy =$$

$$= \frac{Aria\ G}{Aria\ D} = \frac{\pi}{4}$$

$\Rightarrow P(N=3) = \frac{\pi}{4} \cdot \left(1 - \frac{\pi}{4}\right)^2$

P. propuse

4) $(X, Y)$ vector aleator

$$f_{X,Y}(x,y) = \begin{cases} c x^2 y(1+y) & , x \in [0,3] \text{ și } y \in [0,3] \\ 0 & , \text{în rest} \end{cases}$$

a) $c = ?$ a.î. $f$ e densitate de probabilitate

$c x^2 y (1+y) > 0 \Rightarrow c > 0$

$$\int_0^3 \int_0^3 f_{X,Y}(x,y)\, dx\, dy = \int_0^3 \left( \int_0^3 c x^2 y (1+y)\, dy \right) dx =$$

$$\int_0^3 \left( cx^2 \int_0^3 y\, dy + cx^2 \int_0^3 y^2 dy \right) dx = \int_0^3 \left( {}^{3)}\frac{cx^2}{2} \cdot 9 + {}^{2)}\frac{cx^2}{3} \cdot 27 \right) dx =$$

$$= \int_0^3 \frac{27cx^2 + 2 \cdot 27cx^2}{6}\, dx = \frac{27}{2} \cancel{\frac{81}{6}} \int_0^3 cx^2 dx = \frac{27}{2} \cdot c \cdot \frac{x^3}{3} = \frac{27 \cdot \cancel{27}\,9}{2 \cdot \cancel{3}\,1} c$$

$$\Rightarrow \frac{9 \cdot 27}{2} \cdot c = 1 \Rightarrow c = \frac{2}{9 \cdot 27} = \frac{2}{243}$$

b) $A : (1 \le X \le 2$ și $0 \le Y \le 1)$

$P(A) = ?$

$$P(A) = \int_1^2 \int_0^1 \frac{2}{243} x^2 \cdot y(1+y)\, dx\, dy = \frac{2}{243} \int_1^2 \left( \int_0^1 x^2 y (1+y)\, dy \right) dx =$$

$$= \frac{2}{243}\int_1^2 \left(x^2\left(\int_0^1 y\,dy + \int_0^1 y^2 dy\right)\right)dx = \frac{2}{243}\int_1^2 x^2\left(\frac{1}{2}+\frac{1}{3}\right)dx =$$

$$= \frac{\cancel{2}}{243}\cdot\frac{5}{\cancel{6}_3}\cdot\frac{x^3}{3}\Big|_1^2 = \frac{5}{243\cdot 9}\cdot(8-1) = \frac{5\cdot 7}{243\cdot 9} = \frac{35}{2187}$$

c) funcția de repartiție?

$$F_{X,Y}(x,y) = P(X\le x,\ Y\le y) = \int_{-\infty}^{x}\int_{-\infty}^{y}\frac{2}{243}\cdot x^2 y(1+y)\,dx\,dy =$$

$$= \frac{2}{243}\int_0^x\left(\int_0^y x^2 y(1+y)\,dy\right)dx = \frac{2}{243}\int_0^x x^2\cdot\left(\frac{y^2}{2}+\frac{y^3}{3}\right)dx =$$

$$= \frac{\cancel{2}}{243}\cdot\frac{3y^2+2y^3}{\cancel{6}_3}\cdot\frac{x^3}{3} = \frac{x^3(3y^2+2y^3)}{2187}$$

d) $$f_X(x) = \int_0^3 \frac{2}{243}x^2 y(1+y)\,dy = \frac{2}{243}x^2\left(\int_0^3 y\,dy + \int_0^3 y^2 dy\right) =$$

$$\frac{2}{243}x^2\left(\frac{9}{2}+\frac{^{9}\cancel{27}}{\cancel{3}}\right) = \frac{27x^2}{243} = \frac{x^2}{9}$$

$$f_X(x) = \begin{cases} \dfrac{x^2}{9}, & x\in[0,3] \\ 0, & \text{în rest} \end{cases}$$

$$f_Y(y) = \int_0^3 \frac{2}{243}y(1+y)\,x^2 dx = \frac{2}{243}\cdot y(1+y)\cdot\frac{^{9}\cancel{27}}{\cancel{3}} = \frac{18}{243}y(1+y) = \frac{2}{27}(y+y^2)$$

$$f_Y(y) = \begin{cases} \dfrac{2}{27}y(1+y), & y\in[0,3] \\ 0, & \text{în rest} \end{cases}$$

e) $F_X(x) = P(X \le x) = \int_{-\infty}^{x} \frac{t^2}{9} dt = \frac{x^3}{27}$

$F_Y(y) = P(Y \le y) = \int_{-\infty}^{y} \frac{2}{27}(t+t^2)\, dt = \frac{\cancel{2}}{27}\frac{y^2}{\cancel{2}} + \frac{2}{27 \cdot 3} y^3 = \frac{3y^2 + 2y^3}{81}$

f) $f_X(x) \cdot f_Y(y) = \frac{x^2}{9} \cdot \frac{2}{27} y(1+y) = \frac{2}{243} x^2 y(1+y) = f_{XY}(x,y) \Rightarrow$ indep

5) $f_{X,Y}(x,y) = \begin{cases} x+y, & x \in [0,1], y \in [0,1] \\ 0, & \text{in rest} \end{cases}$

a) $\int_0^1 \int_0^1 (x+y)\, dx\, dy = \int_0^1 \left( \int_0^1 x\, dy + \int_0^1 y\, dy \right) dx = \int_0^1 x\, dx + \int_0^1 \frac{1}{2} dx =$

$= \frac{1}{2} + \frac{1}{2} = 1$

$x + y \ge 0$ pt. $x, y \ge 0$

b) $F_{XY}(x,y) = \int_0^x \left( \int_0^y (s+t)\, dt \right) ds = \int_0^x \left( st \Big|_0^y + \frac{t^2}{2} \Big|_0^y \right) ds =$

$= \int_0^x sy\, ds + \int_0^x \frac{y^2}{2} ds = \frac{x^2 y}{2} + \frac{x y^2}{2}$

$F_{XY}(1,1) = \frac{1}{2} + \frac{1}{2} = 1$

c) $f_X(x) = \int_0^1 (x+y)\, dy = x + \frac{1}{2}$

$f_X(x) = \begin{cases} x + \frac{1}{2}, & \text{pt. } x \in [0,1] \\ 0, & \text{in rest} \end{cases}$

analog $f_Y(y) = \begin{cases} y+\frac{1}{2}, & \text{pt. } y\in[0,1] \\ 0, & \text{în rest} \end{cases}$

d) $f_X(x)\cdot f_Y(y) = \left(x+\frac{1}{2}\right)\cdot\left(y+\frac{1}{2}\right) = xy+\frac{x}{2}+\frac{y}{2}+\frac{1}{4} \neq f_{X,Y}(x,y)$

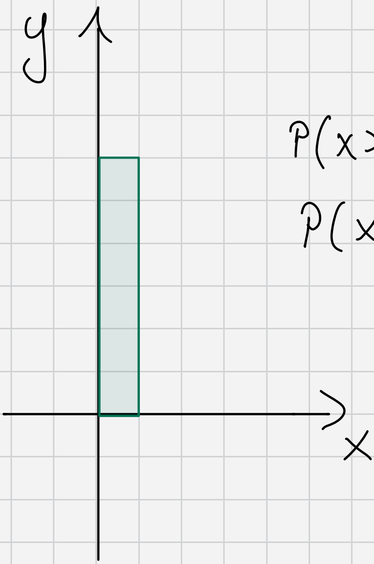
$\Rightarrow$ nu sunt indep.

e) $$M(XY) = \int_0^1\int_0^1 xy(x+y)\,dx\,dy = \int_0^1\left(\int_0^1 x^2y\,dy + \int_0^1 xy^2\,dy\right)dx =$$

$$= \int_0^1\left(x^2\cdot\frac{1}{2} + x\cdot\frac{1}{3}\right)dx = \frac{1}{6}+\frac{1}{6} = \frac{1}{3}$$

6) $D = (0,1)\times(0,6)$

$$f_{XY}(x,y) = \begin{cases} \frac{1}{Aria(G)} = \frac{1}{6\cdot 1} = \frac{1}{6}, & (0,1)\times(0,6) \\ 0, & \text{în rest} \end{cases}$$



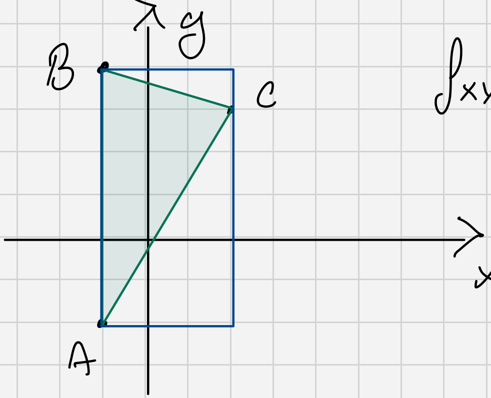

$P(X>Y) = P((X,Y)\in G),\ G=\{(x,y)\in\mathbb{R}^2 : x>y\}$

$$P(X>Y) = 1-P(X\leq Y) = 1-\int_0^1\int_x^6 \frac{1}{6}\,dx\,dy =$$

$$= 1-\int_0^1\left(\frac{1}{6}x\Big|_x^6\right)dx = 1-\int_0^1\left(1-\frac{x}{6}\right)dx =$$

$$= 1-x\Big|_0^1+\frac{x^2}{12}\Big|_0^1 = 1-1+0+\frac{1}{12} = \frac{1}{12}$$

7) $(X,Y)$ vector aleator $[-1,2]\times[-2,4]$ uniform distribuit




$$f_{XY}(x,y) = \begin{cases} \frac{1}{Aria(G)} = \frac{1}{18}, & [-1,2]\times[-2,4] \\ 0, & \text{în rest} \end{cases}$$

$$P((X,Y)\in G) = \frac{A_{\triangle ABC}}{Aria(D)} = \frac{\frac{6\cdot 3}{2}}{3\cdot 6} = \frac{1}{2}$$

```
do
        x = -1 + 3 * urand();
        y = -2 + 6 * urand();
} while ((5*x - 3*y - 1) > 0  && (x + 3*y - 11) < 0)
return (x,y);
```

Ecuațiile dreptelor AC și BC

AC: $y+2 = \frac{3+2}{2+1}\cdot(x+1)$

$y+2 = \frac{5}{3}(x+1) \quad |\cdot 3 \Rightarrow 3y+2 = 5x+5 \Rightarrow 5x-3y-1=0$

BC: $y-3 = \frac{-1}{3}(x-2) \quad |\cdot 3$

$3y-9 = -x+2 \Rightarrow x+3y-11=0$

8) $G = \{ (x,y) \in \mathbb{R}^2 \mid \frac{x^2}{16} + y^2 \le 1 \}$

```
do {
        x = -4 + 8 * urand();
        y = -1 + 2 * urand();
} while (x * x / 16 + y * y > 1);
return (x,y)
```

$a = -4$
$b - a = 8$
$b + 4 = 8 \Rightarrow b = 4$

$a = -1$
$b + 1 = 2 \Rightarrow b = 1$

a) $(x,y) \in [-4, 4] \times [-1, 1]$

b) $\longrightarrow$ vectorul care se află pe conturul elipsei sau chiar pe elipsă
$\longrightarrow$ e ușor de încadrat în dreptunghi

$$P\big((x,y) \in G\big) = \frac{\text{Aria}(G)}{\text{Aria}(D)} =$$

mare

$\Downarrow$

aria (D) minimă

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} \le r^2 \Rightarrow \begin{matrix} a = 4 \\ b = 1 \end{matrix}$$

Aria discului $= \pi \cdot a \cdot b = 4\pi$

$$\Rightarrow P\big((x,y) \in G\big) = \frac{4\pi}{8 \cdot 2} = \frac{\pi}{4}$$

d) Distribuția Geometrică

$N \sim Geom(p)$ $\qquad M(N) = \frac{1}{p}$

$$P(N = k) = p \cdot (1-p)^{k-1}$$

$$P(x = 4) = p \cdot (1-p)^{n-1} = \frac{\pi}{4} \cdot \left(1 - \frac{\pi}{4}\right)^3$$

e) $M(p) = \frac{1}{p} = \frac{4}{\pi}$