Seminarul 11

Covarianța unui vector (X, Y) se definește ca fiind

$$cov(X, Y) := M[(X - M(X))(Y - M(Y)]$$

Covarianța are următoarele proprietăți:

- $cov(X, Y) = M(XY) - M(X)M(Y)$
- $cov(X, X) = \sigma^2(X)$;
- $cov(X, a) = 0$
- $cov(X, Y) = cov(Y, X)$;
- $cov(aX, Y) = a\,cov(X, Y),\ a \in \mathbb{R}$;
- $cov(X + Y, Z) = cov(X, Z) + cov(Y, Z)$;
- $\sigma^2(X + Y) = \sigma^2(X) + \sigma^2(Y) + 2cov(X, Y)$;
- Dacă în plus X și Y sunt variabile aleatoare independente, atunci
$$\sigma^2(X + Y) = \sigma^2(X) + \sigma^2(Y).$$
- X, Y sunt independente, atunci $cov(X, Y) = 0$ (**Reciproca nu este adevărată!**)

itemize **Coeficientul de corelație** a două variabile aleatoare X și Y, de abateri standard nenule σ_X, σ_Y, este

$$\rho(X, Y) = \frac{cov(X, Y)}{\sigma_X \sigma_Y} = \frac{cov(X, Y)}{\sigma(X)\sigma(Y)} \in [-1, 1]$$

Legătura dintre două variabile X și Y poate fi determinată folosind coeficientul de corelație astfel:

- $\rho(X, Y) = 0$, atunci X și Y sunt **necorelate**;
- $\rho(X, Y)$ este apropiat de zero, atunci X și Y sunt **slab corelate** (intensitatea legăturii dintre ele este redusă);
- $\rho(X, Y) = 1$, atunci $Y = aX + b,\quad a > 0$, X și Y sunt **pozitiv corelate**;
- $\rho(X, Y) = -1$, atunci $Y = aX + b,\quad a < 0$, X și Y sunt **negativ corelate**;
- $|\rho(X, Y)|$ are o valoare apropiată de 1, **relația dintre variabilele aleatoare este "aproape liniară"**, adică valorile (x, y) ale vectorului aleator (X, Y) sunt ușor dispersate în jurul unei drepte de ecuație $y = ax + b$.

Matricea de covarianță a vectorului aleator $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$ este matricea notată cu Σ, ale cărei elemente sunt $\sigma_{ij} = \text{cov}(X_i, X_j),\ i, j \in \{1, \ldots, n\}$. **Observații:**

- $\sigma_{ii} = cov(X_i, X_i) = \sigma^2(X_i)$
- Σ este simetrică și semipozitiv definită
- $\Sigma = M(\mathbf{YY}^T)$, unde $\mathbf{Y} = \mathbf{X} - \mathbf{m} = (X_1 - m_1, X_2 - m_2, \ldots, X_n - m_n)^T$ iar $M(\mathbf{YY^T})$ notează matricea mediilor elementelor matricii $\mathbf{YY^T}$.

P. rezolvate

Distribuție de probabilitate

1)

		Y			Σp_i
		0	1	4	
X	-2	0	0	$\frac{1}{5}$	$\frac{1}{5}$
	-1	0	$\frac{1}{5}$	0	$\frac{1}{5}$
	0	$\frac{1}{5}$	0	0	$\frac{1}{5}$
	1	0	$\frac{1}{5}$	0	$\frac{1}{5}$
	2	0	0	$\frac{1}{5}$	$\frac{1}{5}$
Σq_i		$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	

$p = \frac{1}{5}$

$Y = X^2$

$cov(X, Y) = 0$

X, Y indep.

$$X = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

$$M(X) = 0$$

$$Y = \begin{pmatrix} 0 & 1 & 4 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix} \qquad M(Y) = \frac{2}{5} + \frac{8}{5} = 2$$

$$cov(X, Y) = M(XY) - M(X) \cdot M(Y)$$

$$XY = \begin{pmatrix} -8 & -4 & -2 & -1 & 0 & 1 & 2 & 4 & 8 \\ \frac{1}{5} & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & \frac{1}{5} \end{pmatrix}$$

$$M(XY) = -\frac{8}{5} - \frac{1}{5} + \frac{1}{5} + \frac{8}{5} = 0$$

$$cov(X,Y) = M(X,Y) - M(X)\cdot M(Y) = 0 - 0\cdot 2 = 0$$

$$P(X=2, Y=1) = 0$$
$$P(X=2)\cdot P(Y=1) = \frac{1}{5}\cdot\frac{2}{5} = \frac{2}{25}$$

$\Rightarrow$ Fals $\Rightarrow$ nu sunt independente (dependente)

2) X v.a. $M(X)=3$, $\sigma^2(X)=1$

$Y = -2X+5$

$$\rho(X,Y) = \frac{cov(X,Y)}{\sigma(X)\cdot\sigma(Y)}$$

$cov(X,Y) = ?$

$\rho(X,Y) = ?$

matricea de covarianță?

$Y = -2X+5$, $-2 = a < 0$ $\Rightarrow$ negativ corelate $\Rightarrow \rho(X,Y) = -1$

$$\sigma^2(Y) = \sigma^2(-2X+5) = 4\sigma^2(X) = 4 \Rightarrow \sigma(Y) = 2$$

$$\sigma^2(aX+b) = a^2\sigma^2(X)$$

$$\frac{cov(X,Y)}{2\cdot 1} = -1 \Rightarrow cov(X,Y) = -2$$

Matricea de covarianță

$$\Sigma = \begin{pmatrix} \sigma^2(X) & cov(X,Y) \\ cov(X,Y) & \sigma^2(Y) \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

3) X, Y var. aleat. indep.

$X, Y \sim N(0,1)$

$Z = 1 + X + XY^2$, $W = 1 + X$

$cov(Z, W) = ?$

$cov(X, X) = \sigma^2(X)$

X, Y indep $\Rightarrow M(XY) = M(X) \cdot M(Y)$

$cov(X,Y) = cov(Y,X)$

$cov(\alpha, X) = cov(X, \alpha) = 0$

$cov(1 + X + XY^2, 1+X) = cov(1, 1+X) + cov(X + XY^2, 1+X) =$

$= 0 + cov(1+X, X+XY^2) = \underbrace{cov(1, X+XY^2)}_{=0} + cov(X, X+XY^2) =$

$= cov(X+XY^2, X)$

$$cov(Z, W) = \boxed{cov(1 + X + XY^2, 1 + X)} = cov(X + XY^2, X) =$$

$$= cov(X, X) + cov(XY^2, X) = \sigma^2(X) + M(X^2Y^2) - M(XY^2) \cdot M(X)$$

X, Y var. indep $\Rightarrow M(X^2Y^2) = M(X^2) \cdot M(Y^2)$

$$cov(Z, W) = \sigma^2(X) + M(X^2) \cdot M(Y^2) - M(X) \cdot M(X) \cdot M(Y^2) =$$

$$= \sigma^2(X) + M(X^2) \cdot M(Y^2) - M(X)^2 \cdot M(Y^2)$$

$X \sim Norm(m, \sigma^2)$; $M(X) = m$

$\Rightarrow M(X) = 0$, $\sigma^2(X) = 1$

$$\sigma^2(X) = M(X^2) - (\underbrace{M(X)}_{=0})^2$$

$$\sigma^2(X) = M(X^2) \quad \Rightarrow M(X^2) = 1$$

analog $M(Y^2) = 1$

$$\Rightarrow cov(Z, W) = 1 + 1 \cdot 1 - 0 \cdot 1 = 2$$

4) (X, Y) vect. aleat. cont.

$$f(x,y) = \begin{cases} 2 & , 0 \le y \le x \le 1 \\ 0 & , \text{în rest} \end{cases}$$

$\operatorname{cov}(X, Y) = ?$

$$\operatorname{cov}(X, Y) = M(XY) - M(X) \cdot M(Y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^x 2\, dy = 2y \Big|_0^x = 2x$$

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0 & , \text{în rest} \end{cases}$$

$$M(X) = \int_0^1 2x^2 dx = \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}$$

$$M(X) = \int_D x \cdot f_X(x)\, dx$$

$$f_Y(y) = \int_y^1 2\, dy = 2(1-y)$$

$$f_Y(y) = \begin{cases} 2(1-y) & , 0 \le y \le 1 \\ 0 & , \text{în rest} \end{cases}$$

$$M(Y) = \int_0^1 2y(1-y)\, dy = 2\left(\int_0^1 y\, dy - \int_0^1 y^2 dy\right) = 2\left(\frac{y^2}{2}\Big|_0^1 - \frac{y^3}{3}\Big|_0^1\right) = 2 \cdot \left(\frac{1}{2} - \frac{1}{3}\right) =$$

$$= \cancel{2} \cdot \frac{1}{\cancel{6}_3} = \frac{1}{3}$$

domeniul mare

domeniul lui y

$$M(XY) = \int_0^1 \int_0^x 2xy\, dx\, dy = \int_0^1 \left(\int_0^x 2xy\, dy\right) dx = \int_0^1 xy^2 \Big|_0^x dx =$$

$$= \int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4} \Rightarrow cov(X,Y) = \frac{1}{4} - \frac{2}{3}\cdot\frac{1}{3} = {}^{9)}\frac{1}{4} - {}^{4)}\frac{2}{9} = \frac{1}{36}$$

P. propuse

5) (X, Y) v. aleat.

$$P(x,y) = \begin{cases} \frac{1}{21} & , x = 0, 1, \dots, 5 ;\ y = 0, 1, \dots, x \\ 0 & , \text{în rest} \end{cases}$$

$cov(X, Y) = ?$

X \ Y	0	1	2	3	4	5	
0	$\frac{1}{21}$	0	0	0	0	0	$\frac{1}{21}$
1	$\frac{1}{21}$	$\frac{1}{21}$	0	0	0	0	$\frac{2}{21}$
2	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	0	0	0	$\frac{3}{21}$
3	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	0	0	$\frac{4}{21}$
4	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	0	$\frac{5}{21}$
5	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{6}{21}$
	$\frac{6}{21}$	$\frac{5}{21}$	$\frac{4}{21}$	$\frac{3}{21}$	$\frac{2}{21}$	$\frac{1}{21}$	

$$X = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ \frac{1}{21} & \frac{2}{21} & \frac{3}{21} & \frac{4}{21} & \frac{5}{21} & \frac{6}{21} \end{pmatrix}$$

$$M(X) = \frac{2+6+12+20+30}{21} = \frac{70}{21} = \frac{10}{3}$$

$$Y = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{2}{21} & \frac{1}{21} \end{pmatrix}$$

$$M(Y) = \frac{5+8+9+8+5}{21} = \frac{35}{21}$$

$$XY = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 10 & 12 & 15 & 16 & 20 & 25 \\ \frac{6}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{2}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} \end{pmatrix}$$

$$M(XY) = (\cancel{1} + \cancel{2} + \cancel{3} + \cancel{8} + \cancel{5} + \cancel{6} + \cancel{8} + \cancel{9} + \cancel{10} + \cancel{12} + \cancel{15} + \cancel{16} + \cancel{20} + \cancel{25}) \cdot \frac{1}{21} =$$

$$= \frac{140}{21}$$

$$cov(X,Y) = M(XY) - M(X) \cdot M(Y) = {}^{7)}\frac{140}{21} - \frac{35}{21} \cdot \frac{10}{7} = \frac{980 - 350}{147}$$

$$= \frac{630}{147} = \frac{210}{49} = \frac{30}{7}$$

b) $$f(x,y) = \begin{cases} \frac{x+y}{3} & , 0 \le x \le 1 \ ; \ 0 \le y \le 2 \\ 0 & , \text{în rest} \end{cases}$$

$$cov(X,Y) = ?$$

$$f_X(x) = \int_0^2 \frac{x+y}{3}\,dy = \frac{1}{3}\left(\int_0^2 x\,dy + \int_0^2 y\,dy\right) = \frac{1}{3}\left(xy\Big|_0^2 + \frac{y^2}{2}\Big|_0^2\right) =$$

$$= \frac{1}{3}(2x + 2)$$

$$f_X(x) = \begin{cases} \frac{2}{3}(x+1) & , 0 \le x \le 1 \\ 0, & \text{în rest} \end{cases}$$

$$M(X) = \int_0^1 x \cdot f_X(x)\,dx = \frac{2}{3}\int_0^1 x(x+1)\,dx = \frac{2}{3}\left(\int_0^1 x^2\,dx + \int_0^1 x\,dx\right) =$$

$$= \frac{2}{3}\left({}^{2)}\frac{1}{3} + {}^{3)}\frac{1}{2}\right) = \frac{5}{6} \cdot \frac{2}{3} = \frac{10}{18} = \frac{5}{9}$$

$$f_Y(y) = \frac{1}{3}\int_0^1 (x+y)\,dx = \frac{1}{3}\left(\frac{x^2}{2}\Big|_0^1 + yx\Big|_0^1\right) = \frac{1}{3}\cdot\left(\frac{1}{2}+y\right) = \frac{1}{6}+\frac{y}{3}$$

$$f_Y(y) = \begin{cases} \frac{1}{6}+\frac{y}{3} & , y\in[0,2] \\ 0 & , \text{in rest} \end{cases}$$

$$M(Y) = \int_0^2 y\cdot\frac{1}{6}\,dy + \int_0^2 y^2\cdot\frac{1}{3}\,dy = y^2\cdot\frac{1}{12}\Big|_0^2 + y^3\cdot\frac{1}{9}\Big|_0^2 = {}^{3)}\frac{1}{3}+\frac{8}{9} = \frac{11}{9}$$

$$M(XY) = \int_0^1\int_0^2 xy\cdot\frac{x+y}{3}\,dx\,dy = \frac{1}{3}\int_0^1\left(\int_0^2 x^2y\,dy + \int_0^2 xy^2\,dy\right)dx =$$

$$= \frac{1}{3}\cdot\int_0^1\left(x^2\cdot\frac{y^2}{2}\Big|_0^2 + x\cdot\frac{y^3}{3}\Big|_0^2\right)dx = \frac{1}{3}\int_0^1\left(2x^2+\frac{8}{3}x\right)dx = \frac{1}{3}\cdot 2\,\frac{x^3}{3}\Big|_0^1 + \frac{1}{3}\cdot\frac{8}{3}\cdot\frac{x^2}{2}\Big|_0^1 =$$

$$= {}^{2)}\frac{2}{9}+\frac{8}{18} = \frac{12}{18} = \frac{2}{3}$$

$$\operatorname{cov}(X,Y) = M(XY) - M(X)\cdot M(Y) = {}^{27)}\frac{2}{3} - \frac{5}{9}\cdot\frac{11}{9} = \frac{54}{81} - \frac{55}{81} = -\frac{1}{81}$$

7) (X, Y) vector aleat.

$$\Sigma = \begin{pmatrix} 4 & -4 \\ -4 & 25 \end{pmatrix}$$

$\rho(X,Y) = ?$

$\sigma^2(X+2Y) = ?$

$$\Sigma = \begin{pmatrix} \sigma^2(X) & \operatorname{cov}(X,Y) \\ \operatorname{cov}(X,Y) & \sigma^2(Y) \end{pmatrix}$$

$\Rightarrow \sigma^2(X) = 4 \Rightarrow \sigma(X) = 2$

$\sigma^2(Y) = 25 \Rightarrow \sigma(Y) = 5$

$\operatorname{cov}(X,Y) = -4$

$$\sigma^2(X+2Y) = \sigma^2(X) + \sigma^2(2Y) + 2\operatorname{cov}(X,2Y) =$$

$$= \sigma^2(X) + 4\sigma^2(Y) + 4\,cov(X,Y) = 4 + 4\cdot 25 + 4\cdot(-4) =$$

$$= 4 + 100 - 16 = 88$$

$$\rho(X,Y) = \frac{cov(X,Y)}{\sigma(X)\cdot\sigma(Y)} = \frac{-4}{2\cdot 5} = -\frac{2}{5}$$

$$\sigma^2(X+Y) = \sigma^2(X) + \sigma^2(Y) + 2\,cov(X,Y)$$

8) X, Y variabile aleat

$Y = X - 2$

$\sigma^2(X) = 0{,}01$

$\rho(X,Y) = ?$

$\Sigma = ?$

$cov(X,\alpha) = 0$

$\sigma(\alpha) = 0$

$cov(X,X) = \sigma^2(X)$

$$\sigma^2(X-2) = \sigma^2(X) + \sigma^2(-2) + 2\,cov(X,-2) =$$

$$= \sigma^2(X) + 0 + 2\cdot 0 = 0{,}01$$

$$\sigma^2(X) = 0{,}01 = \frac{1}{100} \Rightarrow \sigma(X) = \frac{1}{10}$$

$$\sigma(Y) = \frac{1}{10}$$

$$cov(X, X-2) = cov(X,X) + cov(X,-2) = cov(X,X) =$$

$$= \sigma^2(X) = 0{,}01$$

$$\rho(X, X-2) = \frac{cov(X,X-2)}{\sigma(X)\cdot\sigma(X-2)} = \frac{\frac{1}{100}}{\frac{1}{10}\cdot\frac{1}{10}} = 1$$

$$\Sigma = \begin{pmatrix} \sigma^2(X) & cov(X,X-2) \\ cov(X,X-2) & \sigma^2(X-2) \end{pmatrix} = \begin{pmatrix} 0{,}01 & 0{,}01 \\ 0{,}01 & 0{,}01 \end{pmatrix}$$

9) X, Y var. aleatoare

$\rho(X,Y) = 0 \Rightarrow cov(X,Y) = 0$

- X, Y independente $\underset{\not\Leftarrow}{\Rightarrow} cov(X,Y) = 0$

- $\rho(X,Y) = 0 \Rightarrow X, Y$ necorelate
- $\rho(X,Y) = 1 \Rightarrow X, Y$ pozitiv corelate
- $\rho(X,Y) = -1 \Rightarrow X, Y$ negativ corelate
- $\rho(X,Y)$ aproape de 0 $\Rightarrow X, Y$ slab corelate
- $\rho(X,Y)$ aproape de 1 $\Rightarrow X, Y$ relație „aproape liniară"

10) (X,Y) vector aleator cont.

Densitatea de probabilitate

$$f(x,y) = \begin{cases} 4xy, & 0 \le x \le 1 \;;\; 0 \le y \le 1 \\ 0, & \text{în rest} \end{cases}$$

$cov(X,Y) = ?$

$$f_X(x) = \int_0^1 4xy\, dy = 2xy^2 \Big|_0^1 = 2x$$

$$M(X) = \int_0^1 x \cdot 2x\, dx = 2\frac{x^3}{3}\Big|_0^1 = \frac{2}{3}$$

$$f_Y(y) = \int_0^1 4xy\, dx = 2yx^2 \Big|_0^1 = 2y$$

$$M(Y) = \int_0^1 2y^2 dy = \frac{2}{3}$$

$$M(XY) = \int_0^1\int_0^1 4x^2y^2\,dx\,dy = \int_0^1\left(\int_0^1 4x^2y^2\,dy\right)dx = \int_0^1\left(4x^2\cdot\frac{y^3}{3}\Big|_0^1\right)dx =$$

$$= \int_0^1 \frac{4}{3}x^2\,dx = \frac{4x^3}{9}\Big|_0^1 = \frac{4}{9}$$

$$\mathrm{cov}(X,Y) = M(XY) - M(X)\cdot M(Y) = \frac{4}{9} - \frac{2}{3}\cdot\frac{2}{3} = 0$$

$\Rightarrow X, Y$ var. independente