UAV Coordinate Frames and Rigid Body Dynamics

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Chapter 2

UAV Coordinate Frames and Rigid Body Dynamics

2.0.1 Rotation Matrices

This section describes the various coordinate systems that are used to describe the position of orientation of aircraft, and the transformation between these coordinate systems. It is necessary to use several different coordinate systems for the following reasons:

- Newton's equations of motion are given the coordinate frame attached to the UAV.
- Some of the on-board sensors take measurements in the body frame, e.g., rate gyros, while some of the sensors take measurements in the inertial frame, e.g., GPS.
- Aerodynamics forces and torques are exerted in the body frame.
- Most system requirements, e.g., flight trajectories, are specified in the inertial frame.

We begin in two dimensions by considering the two coordinate frames shown in Figure 2.1. The vector $\mathbf{p}$ can be expressed in both the $\mathcal{C}_0$ frame (specified by $(\mathbf{i}_0, \mathbf{j}_0)$) and in the $\mathcal{C}_1$ frame (specified by $(\mathbf{i}_1, \mathbf{j}_1)$). In the $\mathcal{C}_0$ frame we have

$$\mathbf{p} = p_x^0 \mathbf{i}_0 + p_y^0 \mathbf{j}_0.$$

Alternatively in the $\mathcal{C}_1$ frame we have

$$\mathbf{p} = p_x^1 \mathbf{i}_1 + p_y^1 \mathbf{j}_1.$$

Setting these two expressions equal to each other gives

$$p_x^0 \mathbf{i}_0 + p_y^0 \mathbf{j}_0 = p_x^1 \mathbf{i}_1 + p_y^1 \mathbf{j}_1.$$

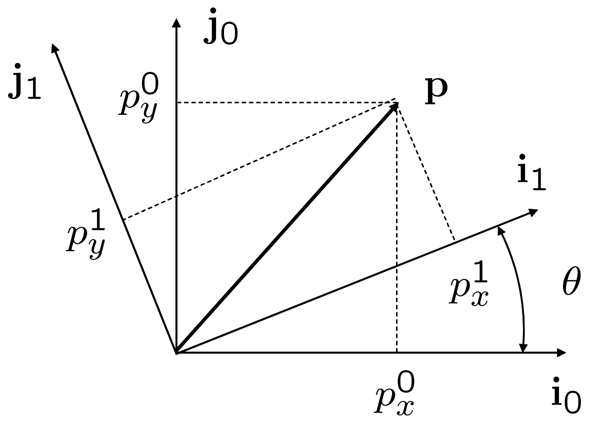

Figure 2.1: Rotation in 2D

Taking the dot product of both sides with $\mathbf{i}_0$ and $\mathbf{j}_0$ respectively, and stacking the result into matrix form gives

$$\mathbf{p}^0 \triangleq \begin{pmatrix} p_x^0 \\ p_y^0 \end{pmatrix} = \begin{pmatrix} \mathbf{i}_0 \cdot \mathbf{i}_1 & \mathbf{i}_0 \cdot \mathbf{j}_1 \\ \mathbf{j}_0 \cdot \mathbf{i}_1 & \mathbf{j}_0 \cdot \mathbf{j}_1 \end{pmatrix} \begin{pmatrix} p_x^1 \\ p_y^1 \end{pmatrix}.$$

Noting that

$$\begin{aligned} \mathbf{i}_0 \cdot \mathbf{i}_1 &= \cos(\theta) \\ \mathbf{i}_0 \cdot \mathbf{j}_1 &= -\sin(\theta) \\ \mathbf{j}_0 \cdot \mathbf{i}_1 &= \sin(\theta) \\ \mathbf{j}_0 \cdot \mathbf{j}_1 &= \cos(\theta) \end{aligned}$$

gives

$$\mathbf{p}^0 = R_{1\to 0}\mathbf{p}^1,$$

where

$$R_{1\to 0} \triangleq \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

One interpretation is that $R_{1\to 0}$ transforms vectors expressed in frame $\mathcal{C}_1$ to an equivalent expression in $\mathcal{C}_0$.

Inverting $R_{1\to 0}$ gives

$$\mathbf{p}^1 = R_{1\to 0}^{-1}\mathbf{p}^0 \triangleq R_{0\to 1}\mathbf{p}^0,$$

where

$$R_{0\to 1} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}^{-1} = \frac{1}{\cos^2(\theta) + \sin^2(\theta)} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = R_{1\to 0}^T.$$

In general, the inverse of a rotation matrix will be its transpose.

An alternate interpretation of $R_{1\to 0}$ is that the coordinate axis $\mathcal{C}_0$ has been rotated into the coordinate axis $\mathcal{C}_1$ by an angle of θ.

In three dimensions, a rotation of θ about the z-axis is given by

$$R_{1\to 0} \triangleq \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Alternatively, a rotation of θ about the x-axis is given by

$$R_{1\to 0} \triangleq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix},$$

and a rotation about the y-axis is given by

$$R_{1\to 0} \triangleq \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}.$$

2.0.2 UAV Coordinate Frames

For UAVs there are several coordinate systems that are of interest.

The inertial frame $\mathcal{C}_I$. The inertial coordinate system is an earth fixed coordinate system with origin at the defined home location. The x-axis of $\mathcal{C}_I$ points North, the y-axis points East, and the z axis points toward the center of the earth.

The vehicle frame $\mathcal{C}_v$. The origin of the vehicle frame is at the center of mass of the UAV. However, the axes of $\mathcal{C}_v$ are aligned with the axis of the inertial frame $\mathcal{C}_I$. In other words, the x-axis points North, the y-axis points East, and the z-axis points toward the center of the earth.

The body frame $\mathcal{C}_b$. The origin of the body frame is also at the center of mass of the UAV. The x-axis points out the nose of the UAV, the y-axis points out the right wing, and the z-axis point out the belly. As the attitude of the UAV moves, the body frame remains fixed with respect to the airframe.

The wind frame $\mathcal{C}_w$. To maintain lift, the UAV is required to maintain a positive pitch angle with respect to the velocity vector of the UAV. It is often convenient to align the body fixed reference frame with the velocity vector. This is the purpose of the wind frame. The origin of the wind frame is the center of mass of the UAV. The x-axis is aligned with the projection of the velocity vector on the $x-z$ plane of the body axis. The y-axis points out the right wing of the UAV and the z-axis is constructed to complete a right handed orthogonal coordinate systems.

Figure 2.2 shows the body and the wind axes. The velocity vectory of the UAV points in the direction of the x_w axis which makes a angle α with the x_b axis.

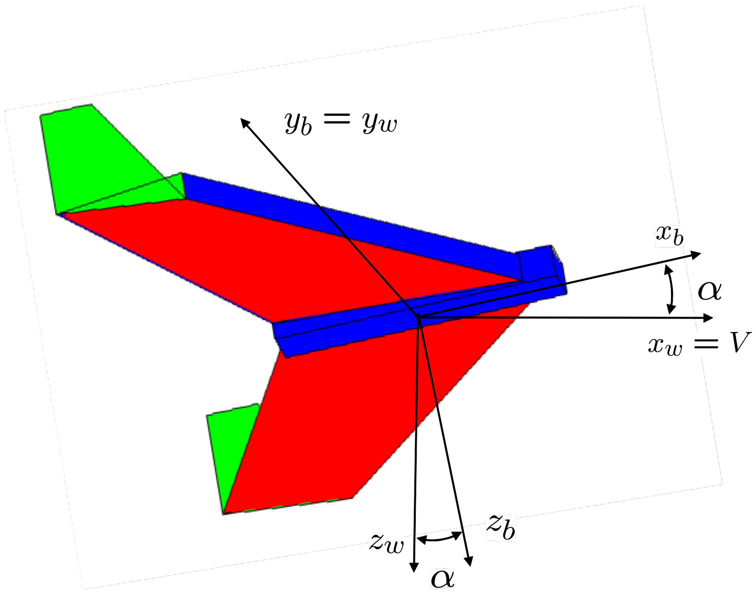

Figure 2.2: The body and the wind axes of the UAV. The x-axis of the body frame $\mathcal{C}_b$ is aligned with the body of the UAV, whereas the x-axis of the wind frame $\mathcal{C}_w$ is aligned with the velocity vector of the UAV. The angle between x_b and x_w is the angle of attack α.

The orientation of an aircraft is specified by three angles, roll (ϕ), pitch (θ), and yaw (ψ), which are called the Euler angles. The yaw angle is defined as the rotation ψ about the z_v-axis in the vehicle frame $\mathcal{C}_v$. The resulting coordinate frame is denoted $\mathcal{C}_1$ and is shown in Figure 2.3. It can be seen from the figure that the transformation from $\mathcal{C}_v$ to $\mathcal{C}_1$ is given by

$$R_{\text{yaw}}(\psi) = R_{v\to 1} = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

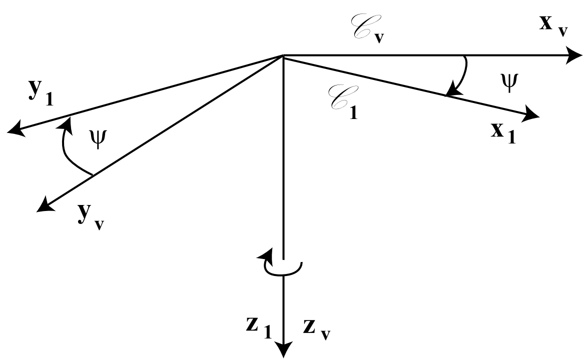

Figure 2.3: Yaw Angle.

The pitch angle is defined as the rotation θ about the y_1 axis in $\mathcal{C}_1$. The resulting coordinate frame is denoted by $\mathcal{C}_2$ and is shown in Figure 2.4. It can be seen from the figure that the transformation

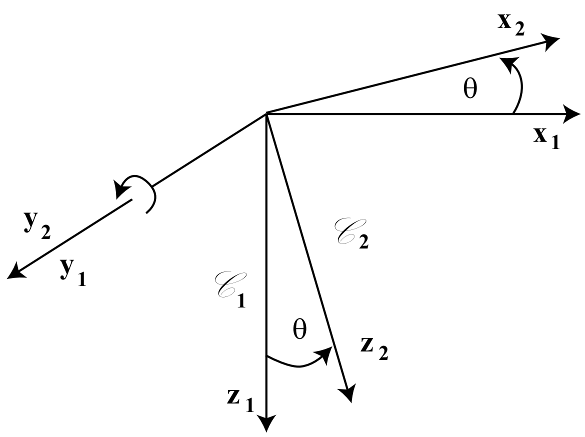

Figure 2.4: Pitch Angle.

from $\mathcal{C}_1$ to $\mathcal{C}_2$ is given by

$$R_{\text{pitch}}(\theta) = R_{1\to 2} = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}.$$

The roll angle is defined as the rotation ϕ about the x_2 axis in $\mathcal{C}_2$. The resulting coordinate frame is the body frame, denoted by $\mathcal{C}_b$, and shown in Figure 2.5. It can be seen from the figure that the

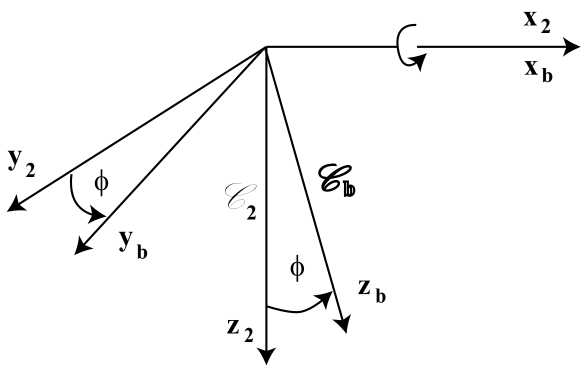

Figure 2.5: Roll Angle.

transformation from $\mathcal{C}_2$ to $\mathcal{C}_1$ is given by

$$R_{\text{roll}}(\phi) = R_{2\to b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix}.$$

Therefore, the complete transformation from the vehicle frame $\mathcal{C}_v$ to the body frame $\mathcal{C}_b$ is given by

$$\mathbf{p}_b = R_{v\to b}\mathbf{p}_v,$$

where

$$\begin{aligned} R_{v\to b} &= R_{\text{roll}}(\phi)R_{\text{pitch}}(\theta)R_{\text{yaw}}(\psi) \\ &= \begin{pmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ s\phi s\theta c\psi - c\phi s\psi & s\phi s\theta s\psi + c\phi c\psi & s\phi c\theta \\ c\phi s\theta c\psi + s\phi s\psi & c\phi s\theta s\psi - s\phi c\psi & c\phi c\theta \end{pmatrix}, \end{aligned}$$

where $c\varphi = \cos(\varphi)$ and $s\varphi = \sin(\varphi)$.

2.0.3 UAV State Variables

The state variables of the UAV are the following twelve quantities

$$
\begin{aligned}
x &= \text{the inertial position of the UAV along } x_I \text{ in } \mathcal{C}_I, \\
y &= \text{the inertial position of the UAV along } y_I \text{ in } \mathcal{C}_I, \\
h &= \text{the altitude of the aircraft measured along } z_v \text{ in } \mathcal{C}_v, \\
u &= \text{the body frame velocity measured along } x_b \text{ in } \mathcal{C}_b, \\
v &= \text{the body frame velocity measured along } y_b \text{ in } \mathcal{C}_b, \\
w &= \text{the body frame velocity measured along } z_b \text{ in } \mathcal{C}_b, \\
\phi &= \text{the roll angle defined with respect to } \mathcal{C}_2, \\
\theta &= \text{the pitch angle defined with respect to } \mathcal{C}_1, \\
\psi &= \text{the yaw angle defined with respect to } \mathcal{C}_v, \\
p &= \text{the roll rate measured along } x_b \text{ in } \mathcal{C}_b, \\
q &= \text{the pitch rate measured along } y_b \text{ in } \mathcal{C}_b, \\
r &= \text{the yaw rate measured along } z_b \text{ in } \mathcal{C}_b.
\end{aligned}
$$

The state variables are shown schematically in Figure 2.6.

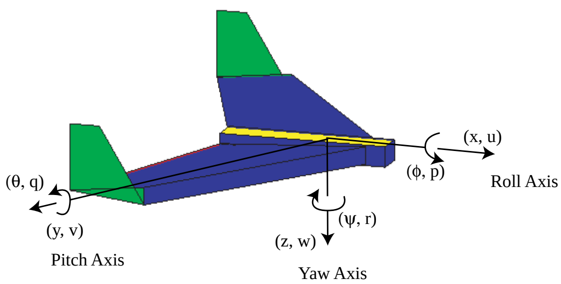

Figure 2.6: Definition of Axes

2.0.4 UAV Kinematics

The state variables x, y, and h are inertial frame quantities, whereas the velocities u, v, and w are body frame quantities. Therefore the relationship between position and velocities is given by

$$
\begin{aligned}
\frac{d}{dt}\begin{pmatrix} x \\ y \\ -h \end{pmatrix} &= R_{b\to v}\begin{pmatrix} u \\ v \\ w \end{pmatrix} \\
&= R_{v\to b}^{T}\begin{pmatrix} u \\ v \\ w \end{pmatrix} \\
&= \begin{pmatrix} c\theta c\psi & s\phi s\theta c\psi - c\phi s\psi & c\phi s\theta c\psi + s\phi s\psi \\ c\theta s\psi & s\phi s\theta s\psi + c\phi c\psi & c\phi s\theta s\psi - s\phi c\psi \\ -s\theta & s\phi c\theta & c\phi c\theta \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}.
\end{aligned}
$$

The relationship between absolute angles ϕ, θ, and ψ, and the angular rates p, q, and r is also complicated by the fact that these quantities are defined in different coordinate frames. The angular rates are defined in the body frame $\mathcal{C}_b$, whereas the roll angle ϕ is defined in $\mathcal{C}_b$ as shown in Figure 2.5, the pitch angle θ is defined in $\mathcal{C}_2$ as shown in Figure 2.4, and the yaw angle ψ is defined in the vehicle frame $\mathcal{C}_1$ as shown in Figure 2.3. Therefore we have

$$
\begin{aligned}
\begin{pmatrix} p \\ q \\ r \end{pmatrix} &= R_{b\to b}\begin{pmatrix} \dot\phi \\ 0 \\ 0 \end{pmatrix} + R_{2\to b}\begin{pmatrix} 0 \\ \dot\theta \\ 0 \end{pmatrix} + R_{1\to b}\begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix} \\
&= I_3\begin{pmatrix} \dot\phi \\ 0 \\ 0 \end{pmatrix} + R_{\text{roll}}(\phi)\begin{pmatrix} 0 \\ \dot\theta \\ 0 \end{pmatrix} + R_{\text{roll}}(\phi)R_{\text{pitch}}(\theta)\begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi c\theta \\ 0 & -s\phi & c\phi c\theta \end{pmatrix}\begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix}.
\end{aligned}
$$

Inverting we get

$$
\begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix} = \begin{pmatrix} 1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)\sec(\theta) & \cos(\phi)\sec(\theta) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix}.
$$

2.0.5 Rigid Body Dynamics

In general, Newton's equations of motion are given by

$$
\begin{aligned}
m \left.\frac{d\mathbf{v}}{dt}\right|_I &= \mathbf{F} \\
\mathbf{J} \left.\frac{d\boldsymbol{\omega}}{dt}\right|_I &= \mathbf{M},
\end{aligned}
$$

where $\mathbf{v}_{cg}$ is the inertial velocity of the center of mass, $\mathbf{F}$ is the force exerted on the body in the inertial frame, $\boldsymbol{\omega}$ is the angular velocity, $\mathbf{M}$ is the inertial frame torque, m is the mass,

$$
\mathbf{J} = \begin{pmatrix} J_x & J_{xy} & J_{xz} \\ J_{xy} & J_y & J_{yz} \\ J_{xz} & J_{yz} & J_z \end{pmatrix}
$$

is the inertia tensor expressed in the body frame, and $\frac{d\mathbf{a}}{dt}\big|_I$ is the time derivative of $\mathbf{a}$ in the inertial frame. It is straightforward to argue [5] that if the body is rotating at an angular velocity of ω, then

$$
\left.\frac{d}{dt}\mathbf{a}\right|_I = \left.\frac{d}{dt}\mathbf{a}\right|_b + \boldsymbol{\omega} \times \mathbf{a},
$$

where $d/dt|_b$ is the time derivate in the body frame.

Therefore the general six degrees-of-freedom model for a rigid body is given by [2, p. 101]

$$
\begin{aligned}
m \left.\frac{d\mathbf{v}}{dt}\right|_b &= -\boldsymbol{\omega} \times m\mathbf{v} + \mathbf{F} \\
\mathbf{J} \left.\frac{d\boldsymbol{\omega}}{dt}\right|_b &= -\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \mathbf{M}
\end{aligned}
$$

where $\mathbf{v} = (u, v, w)^T$ is the body frame velocity of the center of gravity expressed in the body frame, $\boldsymbol{\omega} = (p, q, r)^T$ is the angular velocity about the center of gravity expressed in the body frame, $\mathbf{F} = (F_x, F_y, F_z)^T$ is the external force placed on the center of gravity, expressed in the body frame, $\mathbf{M} = (L, M, N)^T$ are the torques about the center of gravity, expressed in the body frame.

Summarizing the equations of motion are given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{h} \end{pmatrix} = \begin{pmatrix} c\theta c\psi & s\phi s\theta c\psi - c\phi s\psi & c\phi s\theta c\psi + s\phi s\psi \\ c\theta s\psi & s\phi s\theta s\psi + c\phi c\psi & c\phi s\theta s\psi - s\phi c\psi \\ s\theta & -s\phi c\theta & -c\phi c\theta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{2.1}$$

$$\dot{\mathbf{v}} = -\boldsymbol{\omega} \times \mathbf{v} + \frac{1}{m}\mathbf{F} \tag{2.2}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)\sec(\theta) & \cos(\phi)\sec(\theta) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} \tag{2.3}$$

$$\dot{\boldsymbol{\omega}} = \mathbf{J}^{-1}\left(-\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \mathbf{M}\right) \tag{2.4}$$

2.1 Exercises

2.1 Develop a Simulink simulation of the equations of motion given in Equations (2.1)–(2.4). Place various forces an torques on the UAV and convince yourself that your simulation gives reasonable results.

Unzip the file homework1.zip. Change the name of the file

`uaveom0_empty.m`

to

`uaveom0.m.`

Find the location in the file where it says “UAV dynamics go here.” Delete the zeros and add the appropriate equations.