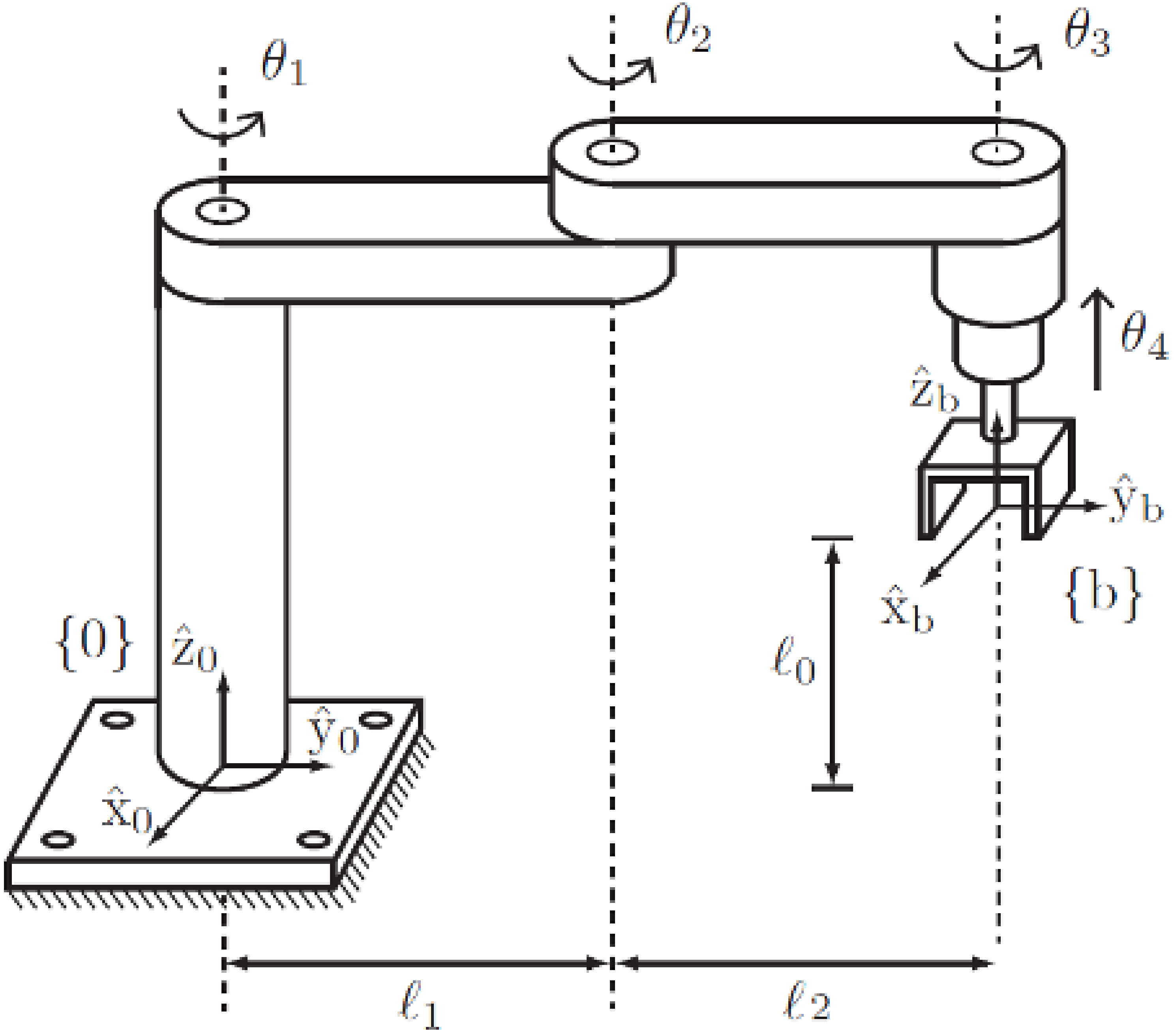
METR4202

Robotics & Automation

Tutorial 3 - Forward Kinematics

Exercise 1: Product of Exponentials


θ_1
θ_2
θ_3
θ_4
$\hat{z}_b$
$\hat{y}_b$
$\hat{x}_b$
$\{b\}$
ℓ_0
$\{0\}$
$\hat{z}_0$
$\hat{y}_0$
$\hat{x}_0$
ℓ_1
ℓ_2

Forward Kinematics

We first need to calculate the transformation from the $\{s\}$ frame to the $\{b\}$ frame of the robot, while it is in the zero configuration / home position.

$$M_{sb} = T_{sb}(\theta = 0)$$

$$T_{sb}(\theta) = \left(\prod_{i=1}^{n} e^{[\mathcal{S}_i]\theta_i} \right) M_{sb}$$

$$T_{sb}(\theta) = e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_n]\theta_n} M_{sb}$$

Step 1: Find the screw axes $\mathcal{S}_i$

Joint 1:

$$
\begin{aligned}
\mathcal{S}_2 &= (\hat{\omega}_{s1}, \hat{v}_{s1}) \in \mathbb{R}^6 \\
\hat{\boldsymbol{\omega}}_{s1} &= \hat{\mathbf{z}}_0 && \text{(vector)} \\
\hat{\omega}_{s1} &= (0, 0, 1) && \text{(coordinates in } \{s\} = \{0\}\text{)} \\
q_1 &= (0, 0, \ell_0) \\
\hat{v}_{s1} &= -[\hat{\omega}_{s1}] q_1 && \text{(skew matrix } \leftrightarrow \text{ cross product)} \\
&= -\hat{\omega}_{s1} \times q_1 \\
&= -(0, 0, 1) \times (0, 0, \ell_0) \\
&= (0, 0, 0) && \text{(Note that } \ell_0 \text{ is irrelevant)} \\
\mathcal{S}_1 &= (0, 0, 1, 0, 0, 0)
\end{aligned}
$$

Joint 2:

$$
\begin{aligned}
\mathcal{S}_2 &= (\hat{\omega}_{s2}, \hat{v}_{s2}) \in \mathbb{R}^6 \\
\hat{\omega}_{s2} &= (0, 0, 1) \\
q_2 &= (0, \ell_1, \ell_0) \\
\hat{v}_{s2} &= -[\hat{\omega}_{s2}] q_2 \\
&= -\hat{\omega}_{s2} \times q_2 \\
&= -(0, 0, 1) \times (0, \ell_1, \ell_0) \\
&= (\ell_1, 0, 0) \\
\mathcal{S}_2 &= (0, 0, 1, \ell_1, 0, 0)
\end{aligned}
$$

Joint 3:

$$
\begin{aligned}
\mathcal{S}_3 &= (\hat{\omega}_{s3}, \hat{v}_{s3}) \in \mathbb{R}^6 \\
\hat{\omega}_{s3} &= (0, 0, 1) \\
q_3 &= (0, \ell_1 + \ell_2, \ell_0) \\
\hat{v}_{s3} &= -[\hat{\omega}_{s3}] q_3 \\
&= -\hat{\omega}_{s3} \times q_3 \\
&= -(0, 0, 1) \times (0, \ell_1 + \ell_2, \ell_0) \\
&= (\ell_1 + \ell_2, 0, 0) \\
\mathcal{S}_3 &= (0, 0, 1, \ell_1 + \ell_2, 0, 0)
\end{aligned}
$$

Joint 4:

$$\mathcal{S}_4 = (\hat{\omega}_{s4}, \hat{v}_{s4}) \in \mathbb{R}^6$$
$$\hat{\omega}_{s3} = (0, 0, 0)$$
$$\hat{v}_{s4} = (0, 0, 1)$$
$$\mathcal{S}_4 = (0, 0, 0, 0, 0, 1)$$

Home position / Zero configuration

We need to calculate the transformation from the space-frame (base) to the body-frame (end-effector)

$$\{s\} = \{0\} \rightarrow \{b\} = \{4\}$$

$$M = M_{sb} = M_{0,4} = T(\theta = 0)$$

The frame axes are aligned, so we have:

$$\hat{x}_0 = \hat{x}_4,\, \hat{y}_0 = \hat{y}_4,\, \hat{z}_0 = \hat{z}_4 \Rightarrow R_{0,4} = I$$

In addition, the translation expressed in the $\{s\}$ frame is:

$$\mathbf{p}_{0,4} = 0\hat{\mathbf{x}}_0 + (\ell_1 + \ell_2)\,\hat{\mathbf{y}}_0 + \ell_0\hat{\mathbf{z}}_0$$

$$\Rightarrow p_{0,4} = (0, \ell_1 + \ell_2, \ell_0)$$

$$
\begin{aligned}
M &= \begin{bmatrix} R_{0,4} & p_{0,4} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \ell_1 + \ell_2 \\ 0 & 0 & 1 & \ell_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Summary

We get the following screw axes:

$$\begin{aligned}
\mathcal{S}_1 &= (0, 0, 1, 0, 0, 0) \\
\mathcal{S}_2 &= (0, 0, 1, \ell_1, 0, 0) \\
\mathcal{S}_3 &= (0, 0, 1, \ell_1 + \ell_2, 0, 0) \\
\mathcal{S}_4 &= (0, 0, 0, 0, 0, 1)
\end{aligned}$$

These are expressed in the coordinate system of the frame

$$\{s\} = \{0\} \text{ (space frame)}$$

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} M$$

$$
\begin{aligned}
e^{[\mathcal{S}]\theta} &= \begin{bmatrix} e^{[\hat{\omega}]\theta} & G_{\hat{\omega}}(\theta)\hat{v} \\ 0 & 1 \end{bmatrix} \\
e^{[\omega]\theta} &= I + \sin(\theta)[\hat{\omega}] + (1 - \cos(\theta))\,[\hat{\omega}]^2 \\
G_{\hat{\omega}}(\theta) &= I\theta + (1 - \cos(\theta))[\hat{\omega}] + (\theta - \sin\theta)[\hat{\omega}]^2
\end{aligned}
$$

- We want to be able to implement this in code.
- This can easily be done in Python using numpy

Python Example

ex_01.py

```
import numpy as np

def skew(w):
    w_skew = np.array([
        [0,     -w[2],  w[1]],
        [w[2],      0, -w[0]],
        [-w[1], w[0],      0]
    ])
    return w_skew

def exp3(w_hat, theta):
    w_skew_hat = skew(w_hat)
    return np.eye(3) + np.sin(theta) * w_skew_hat \
            + (1 - np.cos(theta)) * (w_skew_hat @ w_skew_hat)
```

ex_01.py

```
def G(w_hat, theta):
    w_skew_hat = skew(w_hat)
    return np.eye(3) * theta + (1 - np.cos(theta)) * w_skew_hat \
            + (theta - np.sin(theta)) * (w_skew_hat @ w_skew_hat)

def exp6(S, theta):
    w_hat = S[:3]
    v_hat = S[3:]
    R = exp3(w_hat, theta)
    p = G(w_hat, theta) @ v_hat
    T = np.vstack((
        np.hstack((R, np.expand_dims(p, 1))),
        np.array([[0.0, 0.0, 0.0, 1.0]], dtype=np.float64)
    ))
    return T
```

ex_01.py

```
if __name__ == "__main__":
    l0 = 1
    l1 = 1
    l2 = 1
    theta1 = 0
    theta2 = np.pi / 2
    theta3 = -np.pi / 2
    theta4 = 1
    M = np.array([
        [1, 0, 0, 0],
        [0, 1, 0, l1 + l2],
        [0, 0, 1, l0],
        [0, 0, 0, 1]
    ])
    S1 = np.array([0, 0, 1, 0, 0, 0])
    S2 = np.array([0, 0, 1, l1, 0, 0])
    S3 = np.array([0, 0, 1, l1 + l2, 0, 0])
    S4 = np.array([0, 0, 0, 0, 0, 1])
    T1 = exp6(S1, theta1)
    T2 = exp6(S2, theta2)
    T3 = exp6(S3, theta3)
    T4 = exp6(S4, theta4)
    T = T1 @ T2 @ T3 @ T4 @ M
    print(T)
```

Output:

```
% python3 ex_01.py
[[ 1.  0.  0. -1.]
 [ 0.  1.  0.  1.]
 [ 0.  0.  1.  2.]
 [ 0.  0.  0.  1.]]
```

$$T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, repeat the exercise in the $\mathcal{B}$ frame:

$$T_{sb}(\theta) = M_{sb}\left(\prod_{i=1}^{n} e^{[\mathcal{B}_i]\theta_i}\right)$$

$$T_{sb}(\theta) = M_{sb}e^{[\mathcal{B}_1]\theta_1}\dots e^{[\mathcal{B}_n]\theta_n}$$

Answer for screw axes $\mathcal{B}_i$ in $\{b\}$

We get the following screw axes:

$$\mathcal{B}_1 = (0, 0, 1, -\ell_1 - \ell_2, 0, 0)$$
$$\mathcal{B}_2 = (0, 0, 1, -\ell_2, 0, 0)$$
$$\mathcal{B}_3 = (0, 0, 1, 0, 0, 0)$$
$$\mathcal{B}_4 = (0, 0, 0, 0, 0, 1)$$

These are expressed in the coordinate system of the frame

$$\{b\} = \{4\} \text{ (body frame)}$$

How are the two formulations related?

$$T_{sb} = e^{[\mathcal{S}_1]\theta_1} \dots e^{\mathcal{S}_n \theta_n} M_{sb}$$

$$T_{sb} = M_{sb} e^{[\mathcal{B}_1]\theta_1} \dots e^{\mathcal{B}_n \theta_n}$$

$$\begin{aligned}
T_{sb} &= e^{[\mathcal{S}_1]\theta_1} \dots e^{\mathcal{S}_n \theta_n} M_{sb} \\
&= e^{[\mathcal{S}_1]\theta_1} \dots M_{sb} e^{M_{sb}^{-1}[\mathcal{S}_n] M_{sb} \theta_n} \\
&= e^{[\mathcal{S}_1]\theta_1} M_{sb} \cdots e^{M_{sb}^{-1}[\mathcal{S}_n] M_{sb} \theta_n} \\
&= M_{sb} e^{M_{sb}^{-1}[\mathcal{S}_1] M_{sb} \theta_1} \dots e^{M_{sb}^{-1}[\mathcal{S}_n] M_{sb} \theta_n}
\end{aligned}$$

$$\Rightarrow M_{sb}^{-1}[\mathcal{S}_i] M_{sb} = [\mathcal{B}_i]$$

The Adjoint

Going from the space frame to the body frame

$$M_{sb}^{-1}[\mathcal{S}_i]M_{sb} = [\mathcal{B}_i]$$

$$\left[\mathrm{Ad}_{M_{sb}^{-1}}\right]\mathcal{S}_i = \mathcal{B}_i$$

$$\mathrm{Ad}_{M_{sb}^{-1}}(\mathcal{S}_i) = \mathcal{B}_i$$

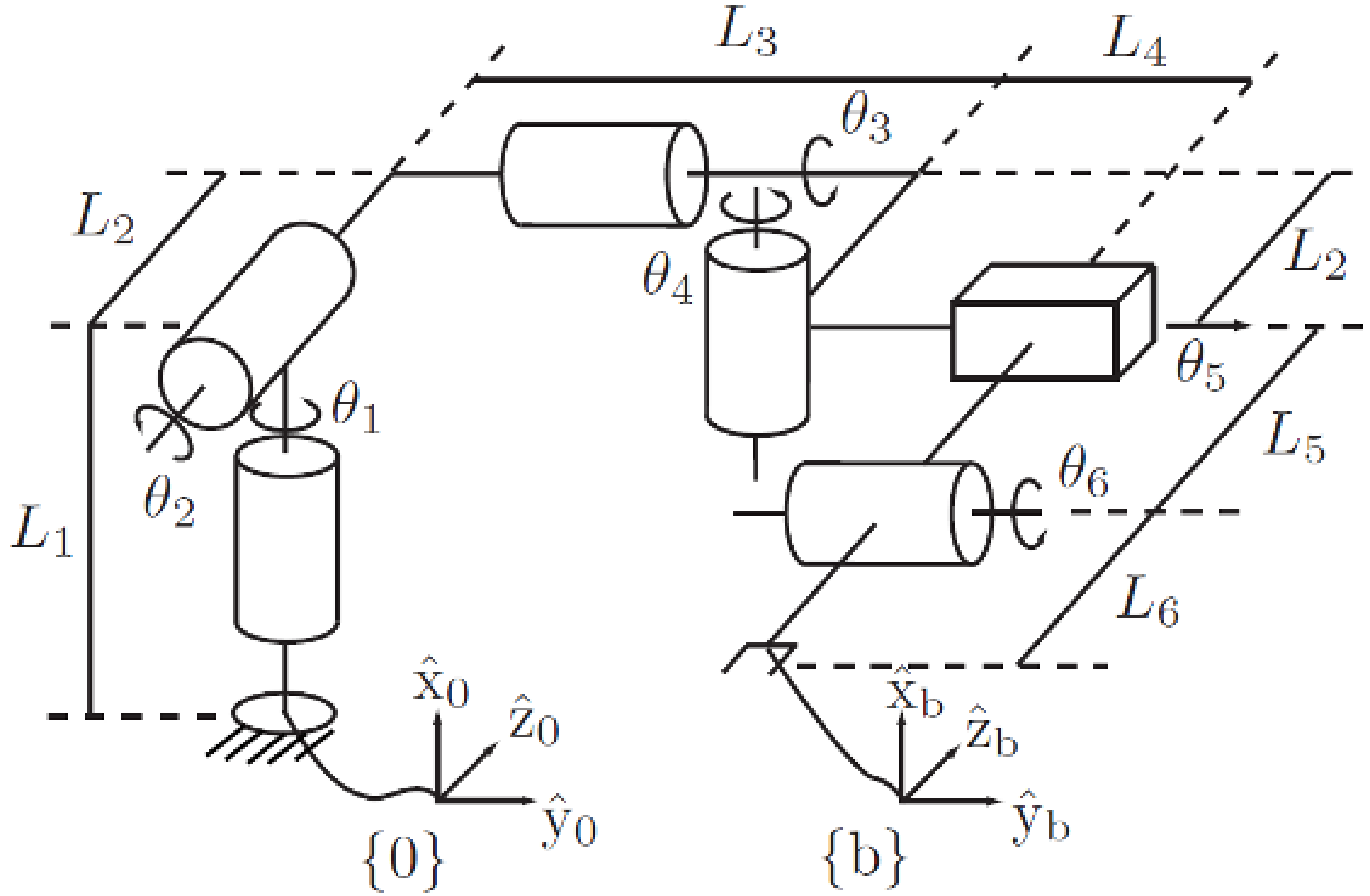
Going from the body frame to the space frame

$$M_{sb}[\mathcal{B}_i]M_{sb}^{-1} = [\mathcal{S}_i]$$

$$[\mathrm{Ad}_{M_{sb}}]\mathcal{B}_i = \mathcal{S}_i$$

$$\mathrm{Ad}_{M_{sb}}(\mathcal{B}_i) = \mathcal{S}_i$$

Exercise 1.2



L_3
L_4
θ_3
L_2
θ_4
L_2
θ_1
θ_5
θ_2
L_5
θ_6
L_1
L_6
$\hat{x}_0$
$\hat{z}_0$
$\hat{x}_b$
$\hat{z}_b$
$\hat{y}_0$
$\hat{y}_b$
{0}
{b}