METR4202

Robotics & Automation

TUT 2: Tutorial - Rigid-body Motions

Representing Rotations as Matrices

Rotations in 2D

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Rotations in 3D

$$R(\theta_x, \theta_y, \theta_z) = R_x(\theta_x) R_y(\theta_y) R_z(\theta_z)$$

$$R_x(\theta_x) = \begin{bmatrix} \cos(\theta_x) & -\sin(\theta_x) & 0 \\ \sin(\theta_x) & \cos(\theta_x) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta_y) = \begin{bmatrix} \cos(\theta_y) & 0 & \sin(\theta_y) \\ 0 & 1 & 0 \\ -\sin(\theta_y) & 0 & \cos(\theta_y) \end{bmatrix}$$

$$R_z(\theta_z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_z) & -\sin(\theta_z) \\ 0 & \sin(\theta_z) & \cos(\theta_z) \end{bmatrix}$$

Rotations in 3D

Other representations? (not assessable)

- Special Orthogonal Group: $\mathrm{SO}(3)$

$$R \in \mathrm{SO}(3) \subset R^{3\times 3}$$
$$\mathrm{SO}(3) : \{R \in R^{3\times 3} | R^\top R = I \wedge \det(R) = 1\}$$

- Unit Quaternions: $\hat{\mathbb{H}}$

$$\hat{\mathbb{H}} : \{q = a + bi + cj + dk \in \mathbb{H} | q^* q = a^2 + b^2 + c^2 + d^2 = 1\}$$

- 2x2 Unitary Matrices: $\mathrm{SU}(2)$

$$\mathrm{SU}(2) : \{U \in \mathbb{C}^{2\times 2} | U^H U = I \wedge |\det(U)| = 1\}$$
$$U = \begin{bmatrix} a + bi & c + di \\ -c + di & a - bi \end{bmatrix},$$

$$\mathrm{SO}(3) \cong \hat{\mathbb{H}} \cong \mathrm{SU}(3)$$

$$\mathrm{SO}(3) \quad \underbrace{\longleftrightarrow}_{\text{isomorphic}} \quad \text{Unit Quaternions} \quad \underbrace{\longleftrightarrow}_{\text{isomorphic}} \quad \mathrm{SU}(2)$$

- Note: all of these are equivalent to each other and represent the same topological spaces.
- In METR4202, you will mostly use the $\mathrm{SO}(3)$ representation.
- Most software implementations of rotations use quaternions for the time and space advantages, since it requires fewer variables and is easy to normalise which $SO(3)$ lacks.
- However, quaternions and $\mathrm{SU}(2)$ are not assessable for the course.

Exercise 1: Basics of Rotation Matrices

Exercise 3.1, Modern Robotics, K. Lynch & F. Park, 2019

In terms of the $\hat{x}_s, \hat{y}_s, \hat{z}_s$ coordinates of a fixed *space* frame $\{s\}$

- The frame $\{a\}$ has
  - The $\hat{x}_a$-axis pointing in the direction $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$
  - The $\hat{y}_a$-axis pointing in the direction $\begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^\top$
- The frame ${b} gas
  - The $\hat{x}_b$-axis pointing in the direction $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$
  - The $\hat{y}_b$-axis pointing in the direction $\begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^\top$

Questions

- a) Draw the three frames by hand (Note: frame origins can be anywhere)
- b) Write down the rotation matrices R_{sa} and R_{sb}
- c) Given R_{sa}, how do you easily calculate R_{sb}^{-1} without calculating the standard matrix inverse?
- d) Given R_{sa} and R_{sb}, how can you calculate R_{ab}?

Exercise 1e)

- Let $R = R_{sb}$ be considered as a transformation operator consisting of a rotation about $\hat{x}$ by -90°.
  Calculate $R_1 = RR_{sa}$ and think of:
  - R_{sa} as a representation of an **orientation**,
  - R as a representation of a **rotation transformation**
    - *We can also use R as a change of basis*
  - R_1 as a **transformed orientation**.
- Does the new orientation R_2 correspond to the rotation of R_{sa} by -90° around
  - the **world**-fixed $\hat{x}_s$-axis, or
  - the **body**-fixed $\hat{x}_a$ axis?

Exercise 1f)

- Use R_{sb} to change the representation of the point $p_b = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top$ from the $\{b\}$ frame to the representation in the $\{s\}$ frame.
  - Note: we are not transforming the point, only changing the representation. ($[p_b]_{\{b\}} \rightarrow [p_s]_{\{s\}}$)
  - This is the same as a change of basis (Recall MATH2000/2001).

Solution 1f)

- We can use the following:

$$p_s = R_{sb} p_b =$$

- We use the rotation from the $\{s\}$ frame to the $\{b\}$ frame, to change the *representation* of the point p from the $\{b\}$ frame to the $\{s\}$ frame.
- For more explanation see:
  - Change of basis, 3Blue1Brown: https://youtu.be/P2LTAUO1TdA

Exercise 1g)

- Choose a point represneted by $p_s = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top$
- Calculate $p' = R_{sb} p_s$
  - How do we interpret the result of this?
  - Is this a change of coordinates or a transformation of a point within a certain frame?
- Calculate $p'' = R_{sb}^\top p_s$
  - How do we interpret the result of this?
  - Is this a change of coordinates or a transformation of a point within a certain frame?

Angular Velocities

- An angular velocity (in 3D) is a 'pseudo-vector'
  - For the purposes of robotics, and in this course, you can treat it as a vector.
  - In 2D, we usually treat it as a scalar about the axis perpendicular to the plane, and for an angle theta, we have $\omega = \dot{\theta}$
- It defines the instantaneous rate at which an object rotates about a particular axis.
- Like other vectors such as position and velocity, it needs to be defined with respect to a reference frame.
- We typically use ω (vector) to represent angular velocity, and $\hat{\omega}$ to represent the axis of rotation, such that: $\omega = \hat{\omega}\dot{\theta}$

Exercise 1h)

- An angular velocity ω is represented in $\{s\}$ as $\omega_s = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^\top$
- What is its representation ω_a in $\{a\}$?
- *(from Q1)* In terms of the $\hat{x}_s, \hat{y}_s, \hat{z}_s$ coordinates of a fixed *space* frame $\{s\}$
    - The frame $\{a\}$ has
        - The $\hat{x}_a$-axis pointing in the direction $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$
        - The $\hat{y}_a$-axis pointing in the direction $\begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^\top$

Matrix Logarithm

$$[\hat{\omega}] = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix} = \frac{1}{2\sin(\theta)}\left(R - R^\top\right)$$

$$\theta = \arccos\left(\frac{\mathrm{tr}(R) - 1}{2}\right)$$

Exercise 1i)

- By hand, calculate the matrix logarithm $[\hat{\omega}]\theta$ of R_{sa}.
- Extract the unit angular velocity $\hat{\omega}$ and the rotation angle θ.

Matrix Exponentials

Rodrigues' Formula:

$$R(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin(\theta)[\hat{\omega}] + (1 - \cos(\theta))[\hat{\omega}]^2$$

Exercise 1j)

- Calculate the matrix exponential corresponding to the exponential coordinates of rotation $\hat{\omega}\theta = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^\top$

Transformations

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3), \quad R \in SO(3), \quad p \in \mathbb{R}^3$$

$$T^{-1} = \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix}$$

Exercise 2: Rigid-body motions

- a) Find T_{01} and T_{02} as a function of time t
- b) Find T_{12} as a function of time t

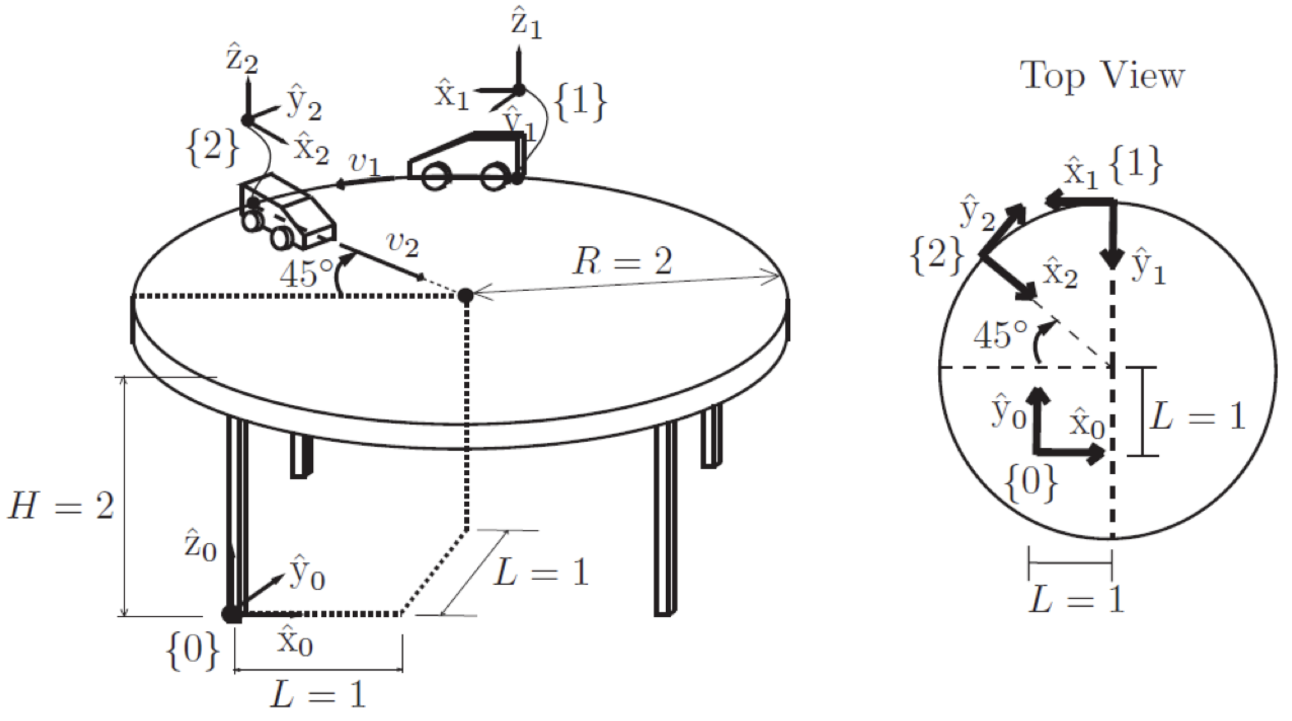

Figure 1 - Positions of the two cars at t = 0 (exercise 2.1)
[Image source: K. Lynch and F. Park, Modern Robotics, 2019]

Bonus!

For some cool python libraries about screw theory check out this GitHub repo by Raghav Mishra (and me)

https://github.com/LaVieEstDure/ScrewRobotics