METR4202

Robotics & Automation

Week 8: [TUT] - Dynamics

Solutions to Week 8 Tutorial

Solutions will be released after the tutorial.

What are we covering for this tutorial?

- Dynamics
  - Determining the Equations of Motion
    - Euler-Lagrange Approach
    - Newton-Euler Approach
  - Calculating the Energy (Lagrangian + Hamiltonian)

Euler-Lagrange

RP Robot Example

Questions

- a) Let the location of the centre of mass of link i be (x_i, y_i). Find (x_i, y_i) for $i = 1, 2$ and their time derivatives in terms of θ and $\dot{\theta}$.
- b) Write the potential energy of each of the two links $\mathcal{P}_1$ and $\mathcal{P}_2$, using the joint variables θ.
- c) Write the kinetic energy of each of the two links $\mathcal{K}_1$ and $\mathcal{K}_2$
- d) What is the Lagrangian $\mathcal{L}$>?
- e) Derive the equations of motion from the Lagrangian and put them in the form: $\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta)$ and identify the coriolis and centripetal terms in $c(\theta, \dot{\theta})$.
- f) Consider the mass matrix $M(\theta)$, and an ellipse in joint-space generated from this matrix. How does this change as θ_2 is increased?
- g) Now consider the end-effector effective mass matrix. How does this change as θ_2 increases. (Consider when $\theta_1 = 0$)

Euler-Lagrange

For Euler-Lagrange, we need to calculate the kinetic ($\mathcal{K}$) energy and potential ($\mathcal{P}$) energy, with respect to the states, θ_1, θ_2, and their time derivatives $\dot{\theta}_1$, $\dot{\theta}_2$

This gives us the Lagrangian ($\mathcal{L}$)

$$\mathcal{L}(\theta, \dot{\theta}) = \mathcal{K}(\theta, \dot{\theta}) - \mathcal{P}(\theta)$$

We can also calculate the total energy, called the Hamiltonian ($\mathcal{H}$)

$$\mathcal{H}(\theta, \dot{\theta}) = \mathcal{K}(\theta, \dot{\theta}) + \mathcal{P}(\theta)$$

Potential Energy

For any rigid-body, we can calculate the potential energy from the centre of mass. We know that the vector field $\mathfrak{g}$, is proportional to the negative gradient of the potiential energy $\mathcal{P}$.

$$m\mathfrak{g} = -\nabla\mathcal{P}$$

It's fair to assume that gravity is constant, which means that we can calculate the potiential as:

$$\begin{aligned}\mathcal{P} &= -\mathfrak{m}\mathbf{x}^\top\mathfrak{g} \\ &= -\mathfrak{m}\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}0 \\ -g\end{bmatrix} \\ &= \mathfrak{m}gy\end{aligned}$$

Kinetic Energy

For any rigid-body, we can calculate the kinetic energy by looking at the velocity and angular velocity about the centre of mass.

$$\mathcal{K} = \frac{1}{2} \begin{bmatrix} \dot{\mathbf{x}}^\top \\ \dot{\boldsymbol{\omega}}^\top \end{bmatrix} \begin{bmatrix} mI & 0 \\ 0 & \mathcal{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}} & \dot{\boldsymbol{\omega}} \end{bmatrix}, \quad \dot{\mathbf{x}}, \boldsymbol{\omega} \in \mathbb{R}^3, \quad \mathcal{I} \in \mathbb{R}^{3\times 3}$$

In the case of a planar mechanism, this reduces to:

$$\mathcal{K} = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) + \frac{1}{2} \mathcal{I} \omega^2$$

$$
\begin{aligned}
\mathcal{K} &= \mathcal{K}_1 + \mathcal{K}_2 \\
&= \frac{1}{2}\left(\mathfrak{m}_1 L_1^2 + \mathcal{I}_1\right)\dot{\theta}_1^2 + \frac{1}{2}\mathfrak{m}_2 \dot{\theta}_2^2 + \frac{1}{2}\left(\mathfrak{m}_2 \theta_2^2 + \mathcal{I}_2\right)\dot{\theta}_1^2 \\
&= \frac{1}{2}\mathfrak{m}_2 \dot{\theta}_2^2 + \frac{1}{2}\left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2 \theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\dot{\theta}_1^2
\end{aligned}
$$

RP: The Lagrangian

Question 1d)

Putting it all together we have

$$\begin{aligned}\mathcal{L} &= \mathcal{K} - \mathcal{P} \\ &= \frac{1}{2}\mathfrak{m}_2\dot{\theta}_2^2 + \frac{1}{2}\left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\dot{\theta}_1^2 - \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2\right) g \sin\theta_1\end{aligned}$$

Equations of Motion

From the Lagrangian, we can derive the equations of motion:

$$f_i = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i}$$

f_i and q_i refer to the generalised 'force' and generalised coordinates, which can apply to any system (not just mechanical).
They are defined such that $f_i^\top \dot{q}_i$ is power.
E.g. $P = \tau \cdot \omega = \tau \cdot \dot{\theta}$, or $P = f \cdot v = f \cdot \dot{p}$
For robots defined in this convention, it becomes:

$$\tau_i = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) - \frac{\partial \mathcal{L}}{\partial \theta_i}$$

τ_i are the joint torques/forces and θ_i are the joint angles/distances.

RP: Equations of Motion

Question 1e)

- Let's work out the equations of motion for this system which should be in this form:

$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta)$$

Joint 1:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta_1} &= \frac{\partial}{\partial \theta_1}\left(\frac{1}{2}\mathfrak{m}_2\dot{\theta}_2^2 + \frac{1}{2}\left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\dot{\theta}_1^2 - \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2\right) g \sin\theta_1\right) \\
&= -\left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2\right) g \cos\theta_1 \\
\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} &= \frac{\partial}{\partial \dot{\theta}_1}\left(\frac{1}{2}\mathfrak{m}_2\dot{\theta}_2^2 + \frac{1}{2}\left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\dot{\theta}_1^2 - \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2\right) g \sin\theta_1\right) \\
&= \left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\dot{\theta}_1 \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) &= \left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\ddot{\theta}_1 + 2\mathfrak{m}_2\theta_2\dot{\theta}_1\dot{\theta}_2 \\
\tau_1 &= \left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\ddot{\theta}_1 + 2\mathfrak{m}_2\theta_2\dot{\theta}_1\dot{\theta}_2 + \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2\right) g \cos\theta_1
\end{aligned}
$$

Joint 2:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta_2} &= \frac{\partial}{\partial \theta_2}\left(\frac{1}{2}\mathfrak{m}_2\dot{\theta}_2^2 + \frac{1}{2}\left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\dot{\theta}_1^2 - \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2\right) g \sin\theta_1\right) \\
&= \mathfrak{m}_2\theta_2\dot{\theta}_1^2 - \mathfrak{m}_2 g \sin\theta_1 \\
\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} &= \frac{\partial}{\partial \dot{\theta}_2}\left(\frac{1}{2}\mathfrak{m}_2\dot{\theta}_2^2 + \frac{1}{2}\left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\dot{\theta}_1^2 - \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2\right) g \sin\theta_1\right) \\
&= \mathfrak{m}_2\dot{\theta}_2 \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) &= \mathfrak{m}_2\ddot{\theta}_2 \\
\tau_2 &= \mathfrak{m}_2\ddot{\theta}_2 - \mathfrak{m}_2\theta_2\dot{\theta}_1^2 + \mathfrak{m}_2 g \sin\theta_1
\end{aligned}
$$

$$\begin{cases} \tau_1 &= \left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2 \theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right) \ddot{\theta}_1 + 2\mathfrak{m}_2 \theta_2 \dot{\theta}_1 \dot{\theta}_2 + \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2 \theta_2\right) g \cos \theta_1 \\ \tau_2 &= \mathfrak{m}_2 \ddot{\theta}_2 - \mathfrak{m}_2 \theta_2 \dot{\theta}_1^2 + \mathfrak{m}_2 g \sin \theta_1 \end{cases}$$

We can also write this in matrix-vector form:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2 \theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2 & 0 \\ 0 & \mathfrak{m}_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 2\mathfrak{m}_2 \theta_2 \dot{\theta}_1 \dot{\theta}_2 \\ -\mathfrak{m}_2 \theta_2 \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2 \theta_2\right) g \cos \theta_1 \\ \mathfrak{m}_2 g \sin \theta_1 \end{bmatrix}$$

$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta)$$

We have the mass matrix:

$$M(\theta_1, \theta_2) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2 \theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2 & 0 \\ 0 & \mathfrak{m}_2 \end{bmatrix}$$

The coriolis + centripetal terms:

$$c(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \begin{bmatrix} 2\mathfrak{m}_2 \theta_2 \dot{\theta}_1 \dot{\theta}_2 \\ -\mathfrak{m}_2 \theta_2 \dot{\theta}_2^2 \end{bmatrix}$$

and the gravity terms:

$$g(\theta_1, \theta_2) = \begin{bmatrix} (\mathfrak{m}_1 L_1 + \mathfrak{m}_2 \theta_2) g \cos\theta_1 \\ \mathfrak{m}_2 g \sin\theta_1 \end{bmatrix}$$

Properties of the Dynamics Equations

Mass Matrix

The mass matrix is positive semi-definite and only dependent on the configuration θ.

$$M(\theta) \succeq 0 : \dot{\theta}^\top M(\theta)\dot{\theta} \geq 0, \forall \dot{\theta}$$

Since this is similar to the energy term, this is stating that the kinetic energy is never negative.
Additionally, this matrix must be symmetric.

$$M(\theta) = M^\top(\theta)$$

RP Example:

We can see from this that the terms in this are all positive, which means that it is positive

$$M(\theta_1, \theta_2) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2 \theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2 & 0 \\ 0 & \mathfrak{m}_2 \end{bmatrix}$$

Also, since it is diagonal (in this case), it is also symmetric.

Coriolis + Centripetal Terms

- These can be split into **coriolis** terms which include cross terms between $\dot{\theta}_i$ and $\dot{\theta}_j$, and the **centripetal** terms, which include quadratic velocity terms, e.g. $\dot{\theta}_i^2$.
- We should not get any higher order powers of θ, due to the derivation of the Lagrangian formulation.
  We can also write this out as:

$$c(\theta, \dot{\theta}) = C(\theta, \dot{\theta})\dot{\theta}$$

From this, we can see that if the velocity is zero, this term also goes to zero.

$$\dot{\theta} = 0 \Rightarrow c(\theta, \dot{\theta}) = 0$$

Gravity Terms

- Since the potiential energy is only dependent on the configuration, so are the gravity terms.
- The gravity terms represent the generalised force on the joints due to gravity.

Mass Matrix cont.

What happens when the velocity and gravity is set to zero?

$$\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{q} + g(\theta)$$

$$\dot{\theta} = 0 \Rightarrow \tau = M(\theta)\ddot{\theta}$$

Now, the mass matrix (M) describes how a force/torque on the directly affects the joint acceleration.

I.e. M_{ij} is the inertia of joint i with respect to a torque j, and also the inertia of joint j with respect to torque i, since it is symmetric.

End-effector Mass Matrix

While the mass matrix M describes the relationship between joint torques and joint accelerations, we can also describe the end-effector effective inertia, using the mass matrix $M(\theta)$ and the jacobian inerverse $J^{-1}(\theta)$, both of which are configuration dependent.

$$\Lambda(\theta) = (J^{-1})^\top M (J^{-1})$$

If we have some cartesian coordinates u_i, where

$$u_1 = x, \; u_2 = y,$$

$$J_{ij}(\theta) = \frac{\partial u_i}{\partial \theta_j}(\theta)$$

$$\dot{u}_i = \sum_j J_{ij} \dot{\theta}_j$$

Note that $u_1 = x_n$ and $u_2 = y_n$, for the end-effector.

The Newton-Euler approach

Notation

Some important notation

$$\hat{\mathcal{A}}_i = (\hat{\omega}_i^{\hat{\mathcal{A}}},\ \hat{v}_i^{\hat{\mathcal{A}}})$$

$$\hat{\mathcal{S}}_i = (\hat{\omega}_i^{\hat{\mathcal{S}}},\ \hat{v}_i^{\hat{\mathcal{S}}})$$

$$\mathcal{V}_i = (\omega_i,\ v_i)$$

$$\dot{\mathcal{V}}_i = (\dot{\omega}_i,\ \dot{v}_i)$$

$$T_{i,i-1} = \begin{bmatrix} R_{i,i-1} & p_{i,i-1} \\ 0 & 1 \end{bmatrix}$$

Given Conditions

Given $\mathcal{F}_{n+1} = \mathcal{F}_{\text{ext}}$ and τ

Constants

These can be pre-calculated before the simulation.

$$\begin{aligned} M_{ij} &= M_{0,i}^{-1} M_{0,j} \\ M_{i,i-1} &= M_{0,i}^{-1} M_{0,i-1} \\ \hat{\mathcal{A}}_i &= \mathrm{Ad}_{M_{0,i}^{-1}}(\mathcal{S}_i) \\ T_{n+1,n} &= M_{n,n+1}^{-1} \end{aligned}$$

NE Algorithm

Forward Propagation of Acceleration/Velocity

For $i = 1$ ton n do

$$T_{i,i-1} = e^{-[\hat{\mathcal{A}}_i]\theta_i} M_{i,i-1}$$

$$\mathcal{V}_i = \mathrm{Ad}_{T_{i,i-1}}(\mathcal{V}_{i-1}) + \hat{\mathcal{A}}_i \dot{\theta}$$

$$\dot{\mathcal{V}}_i = \mathrm{Ad}_{T_{i,i-1}}(\dot{\mathcal{V}}_{i-1}) + \hat{\mathcal{A}}_i \ddot{\theta} + \mathrm{ad}_{\mathcal{V}_i}(\hat{\mathcal{A}}_i)\dot{\theta}_i$$

Backward Propagation of Forces

For $i = n$ to 1 do

$$\mathcal{F}_i = \mathrm{Ad}^\top_{T_{i+1,i}}(\mathcal{F}_{i+1}) + \mathcal{G}_i \dot{\mathcal{V}}_i - \mathrm{ad}^\top_{\mathcal{V}_i}(\mathcal{G}_i \mathcal{V}_i)$$

$$\tau_i = \mathcal{F}_i^\top \hat{\mathcal{A}}_i$$

Decoupled Form for Angular Components

$$
\begin{aligned}
R_{i,i-1} &= e^{-[\hat{\omega}_i^{\hat{\mathcal{A}}}]\theta_i \, \hat{\omega}_i^{\hat{\mathcal{S}}}} \\
\omega_i &= R_{i,i-1}\omega_{i-1} + \hat{\omega}_i^{\hat{\mathcal{A}}}\dot{\theta} \\
\dot{\omega}_i &= R_{i,i-1}\dot{\omega}_{i-1} + \hat{\omega}_i^{\hat{\mathcal{A}}}\ddot{\theta} + (\omega_i \times \hat{\omega}_i^{\hat{\mathcal{A}}})\dot{\theta}_i
\end{aligned}
$$

Using the NE-ID Algorithm

- How can we use the Newton-Euler Inverse Dynamics algorithm to calculate the Mass matrix $M(\theta)$, as well as $c(\theta, \dot{\theta})$ and $g(\theta)$?

Forward Dynamics Simulation

Given a differential equation, derived from forward dynamics, how can we simulate a system's dynamics?

$$\ddot{\theta} = f(\theta, \dot{\theta})$$

Recall that we can approximate the derivatives as:

$$\ddot{\theta} \approx \frac{\Delta\dot{\theta}}{\Delta t} = \frac{\dot{\theta}[k+1] - \dot{\theta}[k]}{t[k+1] - t[k]}$$

$$\dot{\theta} \approx \frac{\Delta\theta}{\Delta t} = \frac{\theta[k+1] - \theta[k]}{t[k+1] - t[k]}$$