METR4202

Robotics & Automation

Week 8: [TUT] - Dynamics

(Solutions)

What are we covering for this tutorial?

- Dynamics
  - Determining the Equations of Motion
    - Euler-Lagrange Approach
    - Newton-Euler Approach

Euler-Lagrange

RP Robot Example

For Euler-Lagrange, we need to calculate the kinetic ($\mathcal{K}$) energy and potential ($\mathcal{P}$) energy, with respect to the states, θ_1, θ_2, and their time derivatives $\dot{\theta}_1$, $\dot{\theta}_2$

This gives us the Lagrangian ($\mathcal{L}$)

$$\mathcal{L}(\theta, \dot{\theta}) = \mathcal{K}(\theta, \dot{\theta}) - \mathcal{P}(\theta)$$

We can also calculate the total energy, called the Hamiltonian ($\mathcal{H}$)

$$\mathcal{H}(\theta, \dot{\theta}) = \mathcal{K}(\theta, \dot{\theta}) + \mathcal{P}(\theta)$$

Potential Energy

For any rigid-body, we can calculate the potential energy from the centre of mass. We know that the vector field $\mathfrak{g}$, is proportional to the negative gradient of the potiential energy $\mathcal{P}$.

$$m\mathfrak{g} = -\nabla\mathcal{P}$$

It's fair to assume that gravity is constant, which means that we can calculate the potiential as:

$$\begin{aligned}\mathcal{P} &= -\mathfrak{m}\mathbf{x}^\top\mathfrak{g} \\ &= -\mathfrak{m}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 0 \\ -g \end{bmatrix} \\ &= \mathfrak{m}gy\end{aligned}$$

Kinetic Energy

For any rigid-body, we can calculate the kinetic energy by looking at the velocity and angular velocity about the centre of mass.

$$\mathcal{K} = \frac{1}{2} \begin{bmatrix} \dot{\mathbf{x}}^\top \\ \dot{\boldsymbol{\omega}}^\top \end{bmatrix} \begin{bmatrix} mI & 0 \\ 0 & \mathcal{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}} & \dot{\boldsymbol{\omega}} \end{bmatrix}, \quad \dot{\mathbf{x}}, \boldsymbol{\omega} \in \mathbb{R}^3, \quad \mathcal{I} \in \mathbb{R}^{3\times 3}$$

In the case of a planar mechanism, this reduces to:

$$\mathcal{K} = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}\mathcal{I}\omega^2$$

RP - Position + Velocity

Question 1a)

First we need to calculate the positions of the frames.

$$
\begin{aligned}
x_1 &= L_1 \cos \theta_1 \\
x_2 &= \theta_2 \cos \theta_1 \\
y_1 &= L_1 \sin \theta_1 \\
y_2 &= \theta_2 \sin \theta_1
\end{aligned}
$$

We can take the derivatives of these to get the frame velocities.

$$\dot{x}_1 = -L_1 \dot{\theta}_1 \sin\theta_1$$
$$\dot{x}_2 = \dot{\theta}_2 \cos\theta_1 - \theta_2 \dot{\theta}_1 \sin\theta_1$$
$$\dot{y}_1 = L_1 \dot{\theta}_1 \cos\theta_1$$
$$\dot{y}_2 = \dot{\theta}_2 \sin\theta_1 + \theta_2 \dot{\theta}_1 \cos\theta_1$$

We can also work out the

$$\omega_1 = \dot{\theta}_1$$
$$\omega_2 = \dot{\theta}_1$$

RP - Potential Energy

Question 1b)

For our case, we have

$$
\begin{aligned}
\mathcal{P}_1 &= \mathfrak{m}_1 g y_1 \\
&= \mathfrak{m}_1 g L_1 \sin\theta_1 \\
\mathcal{P}_2 &= \mathfrak{m}_2 g y_2 \\
&= \mathfrak{m}_2 g \theta_2 \sin\theta_1 \\
\mathcal{P} &= \mathcal{P}_1 + \mathcal{P}_2 \\
&= \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2 \theta_2\right) g \sin\theta_1
\end{aligned}
$$

RP - Kinetic Energy

Question 1c)

Then we can calculate the kinetic energy of each rigid-body.

$$\mathcal{K}_1 = \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2 \right) + \frac{1}{2} \mathcal{I}_1 \omega_1^2$$

$$\mathcal{K}_2 = \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2 \right) + \frac{1}{2} \mathcal{I}_2 \omega_2^2$$

Link 1:

$$
\begin{aligned}
\mathcal{K}_1 &= \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2 \right) + \frac{1}{2} \mathcal{I}_1 \omega_1^2 \\
&= \frac{1}{2} \mathfrak{m}_1 \left(L_1^2 \dot{\theta}_1^2 \sin^2 \theta_1 + L_1^2 \dot{\theta}_1^2 \cos^2 \theta_1 \right) + \frac{1}{2} \mathcal{I}_1 \dot{\theta}_1^2 \\
&= \frac{1}{2} \left(\mathfrak{m}_1 L_1^2 + \mathcal{I}_1 \right) \dot{\theta}_1^2
\end{aligned}
$$

Link 2:

$$
\begin{aligned}
\mathcal{K}_2 &= \frac{1}{2}\mathfrak{m}_2\left(\dot{x}_2^2+\dot{y}_2^2\right)+\frac{1}{2}\mathcal{I}_2\omega_2^2 \\
\dot{x}_2^2 &= \left(\dot{\theta}_2\cos\theta_1-\theta_2\dot{\theta}_1\sin\theta_1\right)^2 \\
&= \dot{\theta}_2^2\cos^2\theta_1-\dot{\theta}_1\dot{\theta}_2\theta_2\cos\theta_1\sin\theta_1+\theta_2^2\dot{\theta}_1^2\sin^2\theta_1 \\
\dot{y}_2^2 &= \left(\dot{\theta}_2\sin\theta_1+\theta_2\dot{\theta}_1\cos\theta_1\right)^2 \\
&= \dot{\theta}_2^2\sin^2\theta_1+\dot{\theta}_1\dot{\theta}_2\theta_2\cos\theta_1\sin\theta_1+\theta_2^2\dot{\theta}_1^2\cos^2\theta_1 \\
\mathcal{K}_2 &= \frac{1}{2}\mathfrak{m}_2\left(\dot{\theta}_2^2+\theta_2^2\dot{\theta}_1^2\right)+\frac{1}{2}\mathcal{I}_2\dot{\theta}_1^2 \\
&= \frac{1}{2}\mathfrak{m}_2\dot{\theta}_2^2+\frac{1}{2}\left(\mathfrak{m}_2\theta_2^2+\mathcal{I}_2\right)\dot{\theta}_1^2
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{K} &= \mathcal{K}_1 + \mathcal{K}_2 \\
&= \frac{1}{2}\left(\mathfrak{m}_1 L_1^2 + \mathcal{I}_1\right)\dot{\theta}_1^2 + \frac{1}{2}\mathfrak{m}_2\dot{\theta}_2^2 + \frac{1}{2}\left(\mathfrak{m}_2\theta_2^2 + \mathcal{I}_2\right)\dot{\theta}_1^2 \\
&= \frac{1}{2}\mathfrak{m}_2\dot{\theta}_2^2 + \frac{1}{2}\left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\dot{\theta}_1^2
\end{aligned}
$$

RP: The Lagrangian

Question 1d)

Putting it all together we have

$$\begin{aligned}\mathcal{L} &= \mathcal{K} - \mathcal{P} \\ &= \frac{1}{2}\mathfrak{m}_2\dot{\theta}_2^2 + \frac{1}{2}\left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\dot{\theta}_1^2 - \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2\right) g \sin\theta_1\end{aligned}$$

Equations of Motion

From the Lagrangian, we can derive the equations of motion:

$$f_i = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i}$$

f_i and q_i refer to the generalised 'force' and generalised coordinates, which can apply to any system (not just mechanical).

They are defined such that $f_i^\top \dot{q}_i$ is power.

E.g. $P = \tau \cdot \omega = \tau \cdot \dot{\theta}$, or $P = f \cdot v = f \cdot \dot{p}$

For robots defined in this convention, it becomes:

$$\tau_i = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) - \frac{\partial \mathcal{L}}{\partial \theta_i}$$

τ_i are the joint torques/forces and θ_i are the joint angles/distances.

RP: Equations of Motion

Question 1e)

- Let's work out the equations of motion for this system which should be in this form:

$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta)$$

Joint 1:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta_1} &= \frac{\partial}{\partial \theta_1}\left(\frac{1}{2}\mathfrak{m}_2\dot{\theta}_2^2 + \frac{1}{2}\left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\dot{\theta}_1^2 - \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2\right) g \sin\theta_1\right) \\
&= -\left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2\right) g \cos\theta_1 \\
\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} &= \frac{\partial}{\partial \dot{\theta}_1}\left(\frac{1}{2}\mathfrak{m}_2\dot{\theta}_2^2 + \frac{1}{2}\left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\dot{\theta}_1^2 - \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2\right) g \sin\theta_1\right) \\
&= \left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\dot{\theta}_1 \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) &= \left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\ddot{\theta}_1 + 2\mathfrak{m}_2\theta_2\dot{\theta}_1\dot{\theta}_2 \\
\tau_1 &= \left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\ddot{\theta}_1 + 2\mathfrak{m}_2\theta_2\dot{\theta}_1\dot{\theta}_2 + \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2\right) g \cos\theta_1
\end{aligned}$$

Joint 2:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta_2} &= \frac{\partial}{\partial \theta_2}\left(\frac{1}{2}\mathfrak{m}_2\dot{\theta}_2^2 + \frac{1}{2}\left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\dot{\theta}_1^2 - \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2\right) g \sin\theta_1\right) \\
&= \mathfrak{m}_2\theta_2\dot{\theta}_1^2 - \mathfrak{m}_2 g \sin\theta_1 \\
\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} &= \frac{\partial}{\partial \dot{\theta}_2}\left(\frac{1}{2}\mathfrak{m}_2\dot{\theta}_2^2 + \frac{1}{2}\left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2\theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right)\dot{\theta}_1^2 - \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2\theta_2\right) g \sin\theta_1\right) \\
&= \mathfrak{m}_2\dot{\theta}_2 \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) &= \mathfrak{m}_2\ddot{\theta}_2 \\
\tau_2 &= \mathfrak{m}_2\ddot{\theta}_2 - \mathfrak{m}_2\theta_2\dot{\theta}_1^2 + \mathfrak{m}_2 g \sin\theta_1
\end{aligned}
$$

$$\begin{cases} \tau_1 & = \left(\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2 \theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2\right) \ddot{\theta}_1 + 2\mathfrak{m}_2 \theta_2 \dot{\theta}_1 \dot{\theta}_2 + \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2 \theta_2\right) g \cos \theta_1 \\ \tau_2 & = \mathfrak{m}_2 \ddot{\theta}_2 - \mathfrak{m}_2 \theta_2 \dot{\theta}_1^2 + \mathfrak{m}_2 g \sin \theta_1 \end{cases}$$

We can also write this in matrix-vector form:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2 \theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2 & 0 \\ 0 & \mathfrak{m}_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 2\mathfrak{m}_2 \theta_2 \dot{\theta}_1 \dot{\theta}_2 \\ -\mathfrak{m}_2 \theta_2 \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} \left(\mathfrak{m}_1 L_1 + \mathfrak{m}_2 \theta_2\right) g \cos \theta_1 \\ \mathfrak{m}_2 g \sin \theta_1 \end{bmatrix}$$

$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta)$$

We have the mass matrix:

$$M(\theta_1, \theta_2) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2 \theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2 & 0 \\ 0 & \mathfrak{m}_2 \end{bmatrix}$$

The coriolis + centripetal terms:

$$c(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \begin{bmatrix} 2\mathfrak{m}_2 \theta_2 \dot{\theta}_1 \dot{\theta}_2 \\ -\mathfrak{m}_2 \theta_2 \dot{\theta}_2^2 \end{bmatrix}$$

and the gravity terms:

$$g(\theta_1, \theta_2) = \begin{bmatrix} (\mathfrak{m}_1 L_1 + \mathfrak{m}_2 \theta_2)\, g \cos \theta_1 \\ \mathfrak{m}_2 g \sin \theta_1 \end{bmatrix}$$

Properties of the Dynamics Equations

Mass Matrix

The mass matrix is positive semi-definite and only dependent on the configuration θ.

$$M(\theta) \succeq 0 : \dot{\theta}^\top M(\theta)\dot{\theta} \geq 0, \forall \dot{\theta}$$

Since this is similar to the energy term, this is stating that the kinetic energy is never negative.

Additionally, this matrix must be symmetric.

$$M(\theta) = M^\top(\theta)$$

RP Example:

We can see from this that the terms in this are all positive, which means that it is positive

$$M(\theta_1, \theta_2) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2 \theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2 & 0 \\ 0 & \mathfrak{m}_2 \end{bmatrix}$$

Also, since it is diagonal (in this case), it is also symmetric.

Coriolis + Centripetal Terms

These can be split into **coriolis** terms which include cross terms between $\dot{\theta}_i$ and $\dot{\theta}_j$, and the **centripetal** terms, which include quadratic velocity terms, e.g. $\dot{\theta}_i^2$.

We can also write this out as:

$$c(\theta, \dot{\theta}) = C(\theta, \dot{\theta})\dot{\theta}$$

From this, we can see that if the velocity is zero, this term also goes to zero.

$$\dot{\theta} = 0 \Rightarrow c(\theta, \dot{\theta}) = 0$$

RP Example:

$$
\begin{aligned}
c(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2) &= \begin{bmatrix} 2\mathfrak{m}_2\theta_2\dot\theta_1\dot\theta_2 \\ -\mathfrak{m}_2\theta_2\dot\theta_2^2 \end{bmatrix} \\
&= \begin{bmatrix} 2\mathfrak{m}_2\theta_2\dot\theta_2 & 0 \\ 0 & -\mathfrak{m}_2\theta_2\dot\theta_2 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix} \\
&= \mathfrak{m}_2\theta_2\dot\theta_2 \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2\mathfrak{m}_2\theta_2\dot\theta_1 \\ 0 & -\mathfrak{m}_2\theta_2\dot\theta_2 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix} \\
&= \mathfrak{m}_2\theta_2 \begin{bmatrix} 0 & 2\dot\theta_1 \\ 0 & -\dot\theta_2 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix}
\end{aligned}
$$

There are multiple ways you can factorise the terms into the C matrix.

Gravity Terms

Since the potiential energy is only dependent on the configuration, so are the gravity terms.
The gravity terms represent the generalised force on the joints due to gravity.

RP Example

$$g(\theta_1, \theta_2) = \begin{bmatrix} (\mathfrak{m}_1 L_1 + \mathfrak{m}_2 \theta_2)\, g \cos\theta_1 \\ \mathfrak{m}_2 g \sin\theta_1 \end{bmatrix}$$

We can interpret the first entry g_1 as a moment, about the first joint. The force due to gravity on the two masses is $\mathfrak{m}_1 g \cos\theta_1$ and $\mathfrak{m}_2 g \cos\theta_1$, which generates the moments at a distance L_1 and θ_1 respepcptively.

This gives a total moment:

$$g_1 = (\mathfrak{m}_1 L_1 + \mathfrak{m}_2 \theta_2)\, g \cos\theta_1$$

The second entry is the force along the prismatic joint, given by the dot product of the force due to gravity, and the joint axis.

$$g_2 = \begin{bmatrix} 0 \\ \mathfrak{m}_2 g \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} = \mathfrak{m}_2 g \sin\theta_1$$

Mass Matrix cont.

What happens when the velocity and gravity is set to zero?

$$\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{q} + g(\theta)$$

$$\dot{\theta} = 0 \Rightarrow \tau = M(\theta)\ddot{\theta}$$

Now, the mass matrix (M) describes how a force/torque on the directly affects the joint acceleration.

I.e. M_{ij} is the inertia of joint i with respect to a torque j, and also the inertia of joint j with respect to torque i, since it is symmetric.

RP - Mass Matrix cont.

Question 1f)

Consider the mass matrix from the example:

$$M(\theta_1, \theta_2) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2 \theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2 & 0 \\ 0 & \mathfrak{m}_2 \end{bmatrix}$$

At rest, the inertia about joint 1 and torque 1, is given by

$$M_{11}(\theta_1, \theta_2) = \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2 \theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2$$

This is the sum of the inertias $\mathcal{I}_1$ and $\mathcal{I}_2$, and the inertias due to the masses $\mathfrak{m}_1$ at a constant distance L_1, and $\mathfrak{m}_2$ at a distance θ_2. When θ_2 increases, the inertia increases, requiring more torque to produce an acceleration.

End-effector Mass Matrix

While the mass matrix M describes the relationship between joint torques and joint accelerations, we can also describe the end-effector effective inertia, using the mass matrix $M(\theta)$ and the jacobian inerverse $J^{-1}(\theta)$, both of which are configuration dependent.

$$\Lambda(\theta) = (J^{-1})^\top M (J^{-1})$$

If we have some cartesian coordinates u_i, where

$$u_1 = x,\ u_2 = y,$$

$$J_{ij}(\theta) = \frac{\partial u_i}{\partial \theta_j}(\theta)$$

$$\dot{u}_i = \sum_j J_{ij} \dot{\theta}_j$$

Note that $u_1 = x_n$ and $u_2 = y_n$, for the end-effector.

RP: End effector Mass Matrix

Question 1g)

We want to get this:

$$\dot{x}_2 = \dot{\theta}_2 \cos\theta_1 - \theta_2 \dot{\theta}_1 \sin\theta_1$$

$$\dot{y}_2 = \dot{\theta}_2 \sin\theta_1 + \theta_2 \dot{\theta}_1 \cos\theta_1$$

to this form:

$$\begin{cases} \dot{x}_2 & = J_{11}\dot{\theta}_1 + J_{12}\dot{\theta}_2 \\ \dot{y}_2 & = J_{21}\dot{\theta}_1 + J_{22}\dot{\theta}_2 \end{cases}$$

With some rearranging we get:

$$\dot{x}_2 = (-\theta_2 \sin\theta_1)\,\dot{\theta}_1 + (\cos\theta_1)\,\dot{\theta}_2$$
$$\dot{y}_2 = (\theta_2 \cos\theta_1)\,\dot{\theta}_1 + (\sin\theta_1)\,\dot{\theta}_2$$

So the Jacobian becomes:

$$J(\theta_1, \theta_2) = \begin{bmatrix} -\theta_2 \sin\theta_1 & \cos\theta_1 \\ \theta_2 \cos\theta_1 & \sin\theta_1 \end{bmatrix}$$

And the Jacobian inverse is:

$$J^{-1}(\theta_1, \theta_2) = \begin{bmatrix} -\frac{\sin\theta_1}{\theta_2} & \frac{\cos\theta_1}{\theta_2} \\ \cos\theta_1 & \sin\theta_1 \end{bmatrix}$$

$$M(\theta_1, \theta_2) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2 \theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2 & 0 \\ 0 & \mathfrak{m}_2 \end{bmatrix}$$

$$\Lambda = (J^{-1})^\top M J^{-1}$$

$$\Lambda = \begin{bmatrix} -\frac{\sin\theta_1}{\theta_2} & \frac{\cos\theta_1}{\theta_2} \\ \cos\theta_1 & \sin\theta_1 \end{bmatrix}^\top \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} -\frac{\sin\theta_1}{\theta_2} & \frac{\cos\theta_1}{\theta_2} \\ \cos\theta_1 & \sin\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\sin\theta_1}{\theta_2} & \cos\theta_1 \\ \frac{\cos\theta_1}{\theta_2} & \sin\theta_1 \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} -\frac{\sin\theta_1}{\theta_2} & \frac{\cos\theta_1}{\theta_2} \\ \cos\theta_1 & \sin\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{M_1 \sin\theta_1}{\theta_2} & M_2\cos\theta_1 \\ \frac{M_1 \cos\theta_1}{\theta_2} & M_2\sin\theta_1 \end{bmatrix} \begin{bmatrix} -\frac{\sin\theta_1}{\theta_2} & \frac{\cos\theta_1}{\theta_2} \\ \cos\theta_1 & \sin\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} M_1 \frac{\sin^2\theta_1}{\theta_2^2} + M_2\cos^2\theta_1 & \left(M_2 - \frac{M_1}{\theta_2^2}\right)\sin\theta_1\cos\theta_1 \\ \left(M_2 - \frac{M_1}{\theta_2^2}\right)\sin\theta_1\cos\theta_1 & M_1\frac{\cos^2\theta_1}{\theta_2^2} + M_2\sin^2\theta_1 \end{bmatrix}$$

So the end-effector mass matrix becomes:

$$\Lambda(\theta_1, \theta_2) = \begin{bmatrix} M_1 \frac{\sin^2\theta_1}{\theta_2^2} + M_2 \cos^2\theta_1 & \left(M_2 - \frac{M_1}{\theta_2^2}\right)\sin\theta_1\cos\theta_1 \\ \left(M_2 - \frac{M_1}{\theta_2^2}\right)\sin\theta_1\cos\theta_1 & M_1 \frac{\cos^2\theta_1}{\theta_2^2} + M_2 \sin^2\theta_1 \end{bmatrix}$$

We can see that it depends on the angle θ_1, but due to symmetry we should be able to analyse the behaviour from one angle $\theta_1 = 0$.

Now let $\theta_1 = 0$

$$\Lambda(0, \theta_2) = \begin{bmatrix} M_2 & 0 \\ 0 & \frac{M_1}{\theta_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} \mathfrak{m}_2 & 0 \\ 0 & \frac{\mathfrak{m}_1 L_1^2 + \mathfrak{m}_2 \theta_2^2 + \mathcal{I}_1 + \mathcal{I}_2}{\theta_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} \mathfrak{m}_2 & 0 \\ 0 & \frac{\mathcal{I}_{\mathrm{eq}} + \mathfrak{m}_2 \theta_2^2}{\theta_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} \mathfrak{m}_2 & 0 \\ 0 & \frac{\mathcal{I}_{\mathrm{eq}}}{\theta_2^2} + \mathfrak{m}_2 \end{bmatrix}$$

Note the units of these, which should be a mass e.g. kg.

Finally, notice that this approaches a point mass $\mathfrak{m}_2$ when $\theta_2 \to \infty$.

The Newton-Euler approach

Notation

Some important notation

$$\hat{\mathcal{A}}_i = (\hat{\omega}_i^{\hat{\mathcal{A}}},\ \hat{v}_i^{\hat{\mathcal{A}}})$$

$$\hat{\mathcal{S}}_i = (\hat{\omega}_i^{\hat{\mathcal{S}}},\ \hat{v}_i^{\hat{\mathcal{S}}})$$

$$\mathcal{V}_i = (\omega_i,\ v_i)$$

$$\dot{\mathcal{V}}_i = (\dot{\omega}_i,\ \dot{v}_i)$$

$$T_{i,i-1} = \begin{bmatrix} R_{i,i-1} & p_{i,i-1} \\ 0 & 1 \end{bmatrix}$$

Given Conditions

Given $\mathcal{F}_{n+1} = \mathcal{F}_{\text{ext}}$ and τ

Constants

These can be pre-calculated before the simulation.

$$
\begin{aligned}
M_{ij} &= M_{0,i}^{-1} M_{0,j} \\
M_{i,i-1} &= M_{0,i}^{-1} M_{0,i-1} \\
\hat{\mathcal{A}}_i &= \mathrm{Ad}_{M_{0,i}^{-1}}(\mathcal{S}_i) \\
T_{n+1,n} &= M_{n,n+1}^{-1}
\end{aligned}
$$

NE Algorithm

Forward Propagation of Acceleration/Velocity

For $i = 1$ ton n do

$$
\begin{aligned}
T_{i,i-1} &= e^{-[\hat{\mathcal{A}}_i]\theta_i} M_{i,i-1} \\
\mathcal{V}_i &= \mathrm{Ad}_{T_{i,i-1}}(\mathcal{V}_{i-1}) + \hat{\mathcal{A}}_i \dot{\theta} \\
\dot{\mathcal{V}}_i &= \mathrm{Ad}_{T_{i,i-1}}(\dot{\mathcal{V}}_{i-1}) + \hat{\mathcal{A}}_i \ddot{\theta} + \mathrm{ad}_{\mathcal{V}_i}(\hat{\mathcal{A}}_i)\dot{\theta}_i
\end{aligned}
$$

Backward Propagation of Forces

For $i = n$ to 1 do

$$\mathcal{F}_i = \mathrm{Ad}^\top_{T_{i+1,i}}(\mathcal{F}_{i+1}) + \mathcal{G}_i \dot{\mathcal{V}}_i - \mathrm{ad}^\top_{\mathcal{V}_i}(\mathcal{G}_i \mathcal{V}_i)$$

$$\tau_i = \mathcal{F}_i^\top \hat{\mathcal{A}}_i$$

Decoupled Form for Angular Components

$$
\begin{aligned}
R_{i,i-1} &= e^{-[\hat{\omega}_i^{\hat{\mathcal{A}}}]\theta_i}\,\hat{\omega}_i^{\hat{\mathcal{S}}} \\
\omega_i &= R_{i,i-1}\omega_{i-1} + \hat{\omega}_i^{\hat{\mathcal{A}}}\dot{\theta} \\
\dot{\omega}_i &= R_{i,i-1}\dot{\omega}_{i-1} + \hat{\omega}_i^{\hat{\mathcal{A}}}\ddot{\theta} + (\omega_i \times \hat{\omega}_i^{\hat{\mathcal{A}}})\dot{\theta}_i
\end{aligned}
$$

Using the NE-ID Algorithm

- How can we use the Newton-Euler Inverse Dynamics algorithm to calculate the Mass matrix $M(\theta)$, as well as $c(\theta, \dot{\theta})$ and $g(\theta)$?

Forward Dynamics Simulation

Given a differential equation, derived from forward dynamics, how can we simulate a system's dynamics?

$$\ddot{\theta} = f(\theta, \dot{\theta})$$

Recall that we can approximate the derivatives as:

$$\ddot{\theta} \approx \frac{\Delta \dot{\theta}}{\Delta t} = \frac{\dot{\theta}[k+1] - \dot{\theta}[k]}{t[k+1] - t[k]}$$

$$\dot{\theta} \approx \frac{\Delta \theta}{\Delta t} = \frac{\theta[k+1] - \theta[k]}{t[k+1] - t[k]}$$