CSC 411

Computer Organization (Fall 2024)
Lecture 7: Floating Point (part 1)

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Fractional binary numbers

- Used to represent fractional numbers
- Binary point
  - separates the integer and fractional parts (similar to the decimal point)
  - bits to the right of binary point represent fractional powers of 2

2^i	2^{i-1}	…	2	1	$1/2$	$1/4$	…	2^{-j+1}	2^{-j}
b_i	b_{i-1}	…	b_1	b_0 •	b_{-1}	b_{-2}	…	b_{-j+1}	b_{-j}

$$\sum_{k=-j}^{i} b_k 2^k$$

1 1 . 0 1 0 =

1 1 0 1 . 1 0 1 =

Practice

- Convert fractional binary numbers to decimal

$$\begin{aligned} 1.10 &= \\ 11.001 &= \\ 100.01 &= \\ 11.111 &= \\ 0.010 &= \end{aligned}$$

Practice

- Convert a decimal to a fractional binary number
  - 349.84375

$$\begin{array}{rl|c|} 349/2 = 174 & R & 1 \\ 174/2 = 87 & R & 0 \\ 87/2 = 43 & R & 1 \\ 43/2 = 21 & R & 1 \\ 21/2 = 10 & R & 1 \\ 10/2 = 5 & R & 0 \\ 5/2 = 2 & R & 1 \\ 2/2 = 1 & R & 0 \\ 1/2 = 0 & R & 1 \end{array} \uparrow$$

$$\begin{array}{r|l} 0.84375 * 2 = & 1.6875 \\ 0.6875 * 2 = & 1.375 \\ 0.375 * 2 = & 0.75 \\ 0.75 * 2 = & 1.5 \\ 0.5 * 2 = & 1.0 \\ 0.0 * 2 = & 0 \end{array} \downarrow$$

101011101.110110

Normalized scientific notation

- Represents binary numbers in the form $1.xxx\ldots \times 2^k$
  - always has a single 1 to the left of the binary point
  - exponent adjusts to maintain this form
  - analogous to decimal scientific notation
- Forms the basis of the IEEE 754 standard
- Examples
  - 101.11 => normalized 1.0111×2^2
  - 0.00101 => normalized 1.01×2^{-3}
  - 1000.101 => normalized 1.000101×2^3

Practice

- Convert to **normalized scientific notation**
  - single non-zero digit to the left of the binary point

$$\begin{aligned} 10.10 &= \underbrace{1.010}_{\text{significand}} \times \underbrace{2}_{\text{base}}^{\overbrace{1}^{\text{exponent}}} \\ 110.001 &= \\ 1111.1010 &= \\ 0.010101 &= \\ 0.0000010 &= \end{aligned}$$

Observations

- Not all decimal fractions have exact binary equivalents
  - can only represent numbers of the form $\frac{x}{2^k}$
  - e.g., $\frac{1}{3} = 0.33333\ldots$ is represented as 0.01010101…
    - requires an infinite series of 0s and 1s
  - large repeating decimals can only be approximated within a certain degree of accuracy
- 0.111111… represents a number just below 1.0

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \frac{1}{2^i} + \ldots = 1.0 - \varepsilon$$

Practice

- Apply the ‘multiply by 2’ method to convert 0.1 from decimal to fractional binary
  - explain what is happening
- Type 0.1 + 0.2 on a python terminal

IEEE standard 754

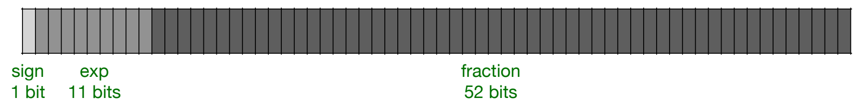
- Defines a common format for representing real numbers in computers
  - developed in response to divergence of representations
  - first standardized in 1985 and revised in 2008
  - supported by major CPUs (almost universally adopted)
  - provides different precision levels (**half, single, double, quadruple**) for various needs
- Standardizes the following layout :
  - more **exp** bits leads to a wider range of numbers
  - more **fraction** bits leads to higher precision

`sign`	`exp`	`fraction`

Precision

- **Single-precision** (32 bits) — `float`
  - good balance of performance and range

- **Double-precision** (64 bits) — `double`
  - higher precision for demanding calculations



Precision

Format	Smallest positive value (*)	Largest positive value (*)	Precision (**)
single	~$1.401 \cdot 10^{-45}$	~$3.403 \cdot 10^{+38}$	6-9 digits
double	~$4.941 \cdot 10^{-324}$	~$1.798 \cdot 10^{+308}$	15-17 digits

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(*) Smallest/largest negative values are the same as their positive
counterparts, but negative.

(**) Precision refers to to the number of significant digits that
can be represented in a number.
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In C, FLT_MIN and DBL_MIN are defined as the smallest normalized
numbers, and FLT_TRUE_MIN and DBL_TRUE_MIN represent the smallest
positive value that can be represented by the float type, including
denormalized numbers.
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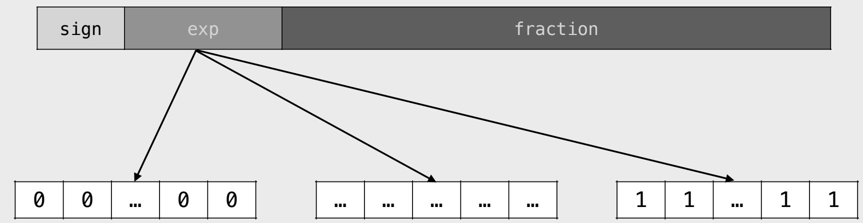
Floating point encoding/decoding

$$(-1)^s M 2^E$$

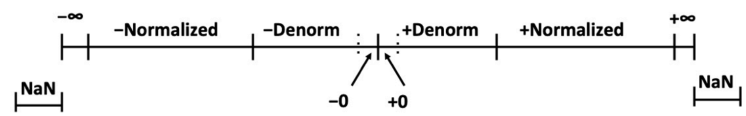
- sign bit S
  - 0 for positive, 1 for negative
- exponent E
  - magnitude of the number, the term 2^E scales the mantissa
- mantissa M
  - a.k.a. **significand**, it captures the significant digits of the number
    - normally in range $[1.0, 2.0)$

Floating point categories

Floating point categories



denormalized values   normalized values   special values



Floating point categories

- Denormalized values
  - used for <u>very small numbers</u>, extending the range of representable values closer to zero
  - lower precision compared to normalized numbers
  - **exp** == 000…000
- Normalized values
  - represent the majority of values, using the full precision of the significand, and a wide range of magnitudes
  - **exp** != 000…000 and **exp** != 111…111
- Special numbers
  - represent special cases like NaN, infinity (positive and negative)
  - **exp** == 111…111

Normalized values

$$(-1)^s M 2^E$$

- **E = exp - bias**
  - exp is the unsigned integer represented by the bits in the exp field
  - bias is $2^{k-1} - 1$, where k is the length in bits of the exp field
    - <u>single precision</u> (bias = $2^{8-1} - 1 = 127$)
    - <u>double precision</u> (bias = $2^{11-1} - 1 = 1023$)
- **M = 1.bb…bb**
  - bb…bb are the bits in the fraction field
  - M is the decimal that corresponds to 1.bb…bb
    - note that <u>M has an implied leading 1</u>

Practice (encoding) $(-1)^s M 2^E$

- Assume a float F = 2024.0
  - convert to fractional binary
    - 11111101000
  - make it form 1.xxx
    - 1.1111101000 x 2^{10}
  - write M and frac
    - M = 1.1111101000
    - frac = 11111010000000000000000
  - write exp using E = exp - bias
    - exp = E + bias = 10 + 127 = 137 = 10001001
  - final binary sequence
    - 0 10001001 11111010000000000000000 0x44FD0000

Practice (encoding) $(-1)^s M 2^E$

- Assume a float F = 0.5
  - convert to fractional binary
    - 0.1
  - make it form 1.xxx
    - 1.0 x 2^{-1}
  - write M and frac
    - M = 1.0
    - frac = 00000000000000000000000
  - write exp using E = exp - bias
    - exp = E + bias = -1 + 127 = 126 = 01111110
  - final binary sequence
    - 0 01111110 00000000000000000000000 0x3F000000

Practice (encoding) $(-1)^s M 2^E$

- Convert the following decimals to hexadecimal representation (single precision)
  - 1.3125
  - 3.53125

Practice (decoding) $(-1)^s M 2^E$

- Assume a float F = 0x43CDA000
  - write the binary
    - divide into **s**, **exp**, **frac**
    - 0 10000111 10011011010000000000000
  - calculate M
    - M = 1.10011011010000000000000 = 1.6064453125
  - calculate E
    - E = exp - bias = 135 - 127 = 8
  - write final number
    - $(-1)^0 * 1.6064453125 * 2^8 = 411.25$

Practice (encoding)

$$(-1)^s M 2^E$$

- Convert the following values in hexadecimal representation (single precision) to decimal
  - 0x41850000
  - 0x43cdb400