CSC 461: Machine Learning

Fall 2024

Multinomial logistic regression

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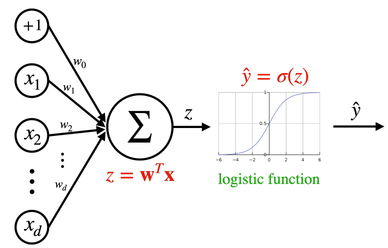
Logistic regression



$$h(\mathbf{x}) = \sigma(\mathbf{w}^T\mathbf{x})$$

probability of class +1

cross-entropy loss

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \boxed{\frac{1}{n}\sum_{i=1}^{n} \log\left(1 + e^{-y^{(i)}\mathbf{w}^T\mathbf{x}^{(i)}}\right)}$$

Gradient (partial derivatives)

$$L(\mathbf{w}) = \frac{1}{n}\sum_{i=1}^{n} \log\left(1 + e^{-y^{(i)}\mathbf{w}^T\mathbf{x}^{(i)}}\right)$$

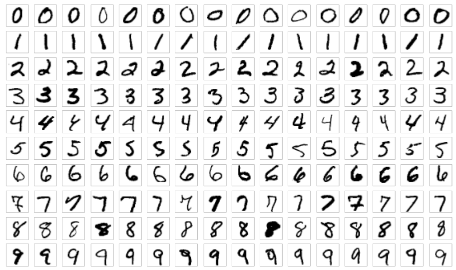
$$\frac{\partial L(\mathbf{w})}{\partial w_j} = -\frac{1}{n}\sum_{i=1}^{n} \sigma\left(-y^{(i)}\mathbf{w}^T\mathbf{x}^{(i)}\right) y^{(i)} x_j^{(i)}$$

$$\nabla_{\mathbf{w}} L(\mathbf{w}) = \left[\frac{\partial L(\mathbf{w})}{\partial w_0}, \ldots, \frac{\partial L(\mathbf{w})}{\partial w_d}\right]$$

Handling multiple classes

MNIST

- The MNIST database is a large database of handwritten digits
  - contains 60,000 training images and 10,000 testing images
  - convolutional neural networks, manages to get an error rate of 0.23%
  - original paper reports an error rate of 0.8% with SVMs



http://yann.lecun.com/exdb/mnist/

Multinomial logistic regression

- Data

$$\mathbf{x} \in \mathbb{R}^d, \quad y \in \{1,2,\ldots,C\}$$

- Binary logistic regression

$$P(y=+1 \mid \mathbf{x}; \mathbf{w}) = \frac{1}{1+e^{-\mathbf{w}^T\mathbf{x}}} = \frac{e^{\mathbf{w}^T\mathbf{x}}}{e^{\mathbf{w}^T\mathbf{x}}+1}$$

- Multiclass logistic regression

$$P(y=c \mid \mathbf{x}; W) = \frac{e^{\mathbf{w}_c{}^T\mathbf{x}}}{\sum_{k=1}^{C} e^{\mathbf{w}_k{}^T\mathbf{x}}}$$

softmax function

$W_{C\times d+1}$ is a matrix where every row is a "class" weight vector

Softmax

$$\sigma(\mathbf{z}) = \frac{e^{z_i}}{\sum_{k=1}^{K} e^{z_k}}$$

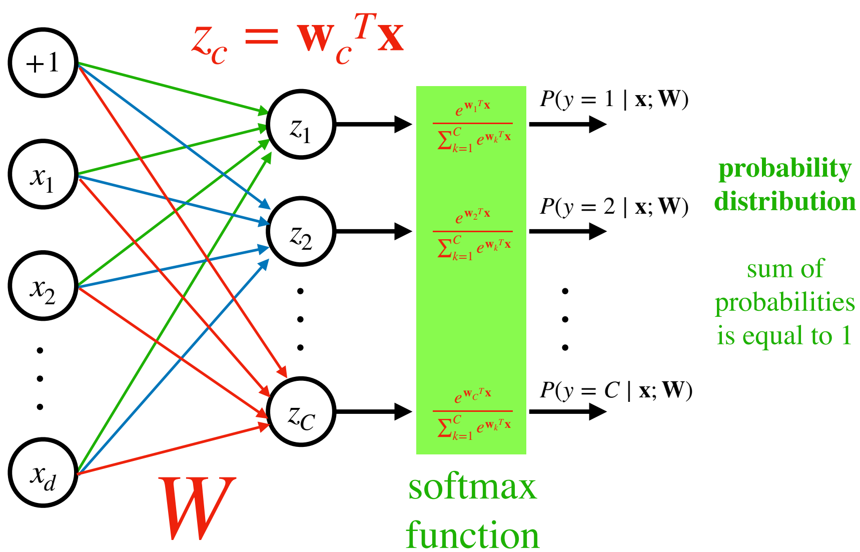
for $i = 1,\ldots,K$ and $\mathbf{z} \in \mathbb{R}^K$

converts a vector of K real numbers into a **probability distribution** of K possible outcomes

Practice

- What is the value of softmax($\mathbf{z}$), given that $\mathbf{z}^T = [-10,10,5,4.3,7]$?

Multinomial logistic regression



Multinomial logistic regression

- Use the **softmax function** for activation
- Predict the label with the highest probability score

$$\hat{y} = \arg\max_{c} P(y = c \mid \mathbf{x}; \mathbf{W})$$

- How to learn the weights?
  - need to define a **Loss Function** … then apply **gradient descent**
  - loss function can be derived using **MLE** (similar to binary logistic regression)

Applying MLE

$$\mathbf{W}^* = \arg\max_{\mathbf{W}} \frac{1}{n} \prod_{i=1}^{n} P(y^{(i)} \mid \mathbf{x}^{(i)}; \mathbf{W})$$

$$= \arg\max_{\mathbf{W}} \frac{1}{n} \prod_{i=1}^{n} \prod_{c=1}^{C} P\left(y^{(i)} = c \mid \mathbf{x}^{(i)}; \mathbf{W}\right)^{t_{i,c}}$$

$$= \arg\max_{\mathbf{W}} \frac{1}{n} \sum_{i=1}^{n} \sum_{c=1}^{C} t_{i,c} \log\left(P(y^{(i)} = c \mid \mathbf{x}^{(i)}; \mathbf{W})\right)$$

$$= \arg\min_{\mathbf{W}} -\frac{1}{n} \sum_{i=1}^{n} \sum_{c=1}^{C} t_{i,c} \log\left(P(y^{(i)} = c \mid \mathbf{x}^{(i)}; \mathbf{W})\right)$$

$$= \arg\min_{\mathbf{W}} -\frac{1}{n} \sum_{i=1}^{n} \sum_{c=1}^{C} t_{i,c} \log\left(\frac{e^{\mathbf{w}_c \mathbf{x}^{(i)}}}{\sum_{k=1}^{C} e^{\mathbf{w}_k \mathbf{x}^{(i)}}}\right)$$

Consider a matrix $t_{n\times C}$ where every row is a **one-hot encoding** of the target variable

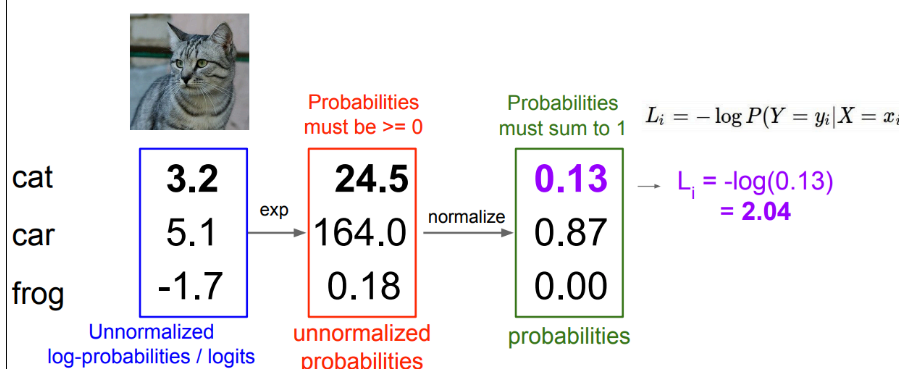
individual loss

$e(h_{\mathbf{w}}(\mathbf{x}), y) = -\log p_c$

predicted probability of the correct class

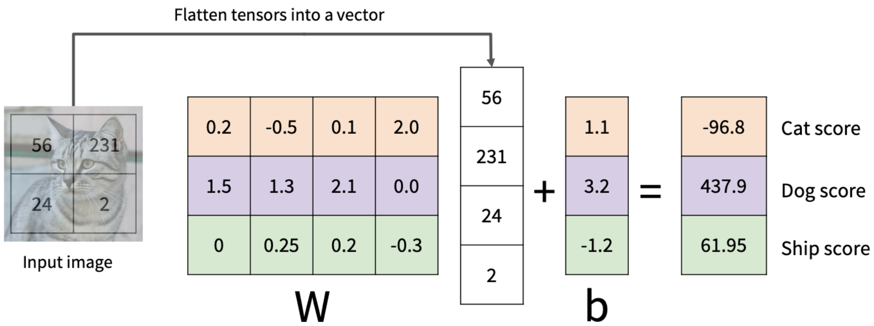
Cross-entropy loss

Softmax and loss

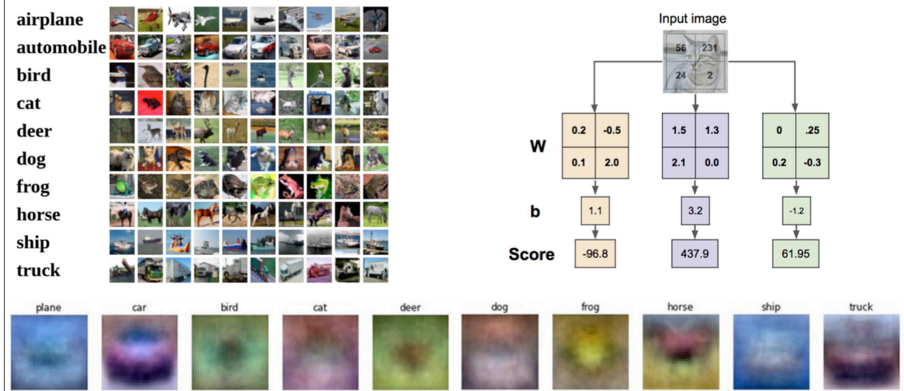


https://cs231n.stanford.edu/schedule.html

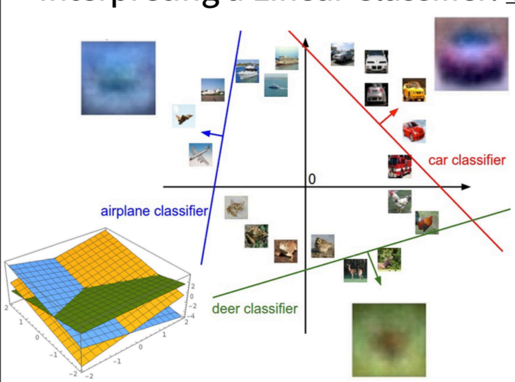
Example with an image with 4 pixels, and 3 classes (cat/dog/ship)
Algebraic Viewpoint



Interpreting a Linear Classifier: Visual Viewpoint



Interpreting a Linear Classifier: Geometric Viewpoint



$$f(x,W) = Wx + b$$

Array of 32x32x3 numbers
(3072 numbers total)

Plot created using Wolfram Cloud

Interactive web demo