CSC 561: Neural Networks and Deep Learning

Preliminaries

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Spring 2024

Linear Algebra

Subset of most relevant concepts to Deep Learning

Credit: Deep Learning by Ian Goodfellow, Yoshua Bengio, Aaron Courville

Scalars and vectors

- Scalars
  - integers, real numbers, rational numbers, etc.
  - usually denoted by lowercase letters
- Vectors
  - 1-D array of elements (scalars)

$$\mathbf{x} \in \mathbb{R}^n \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrices and tensors

- Matrices
  - 2-D array of elements

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \quad A \in \mathbb{R}^{m \times n}$$

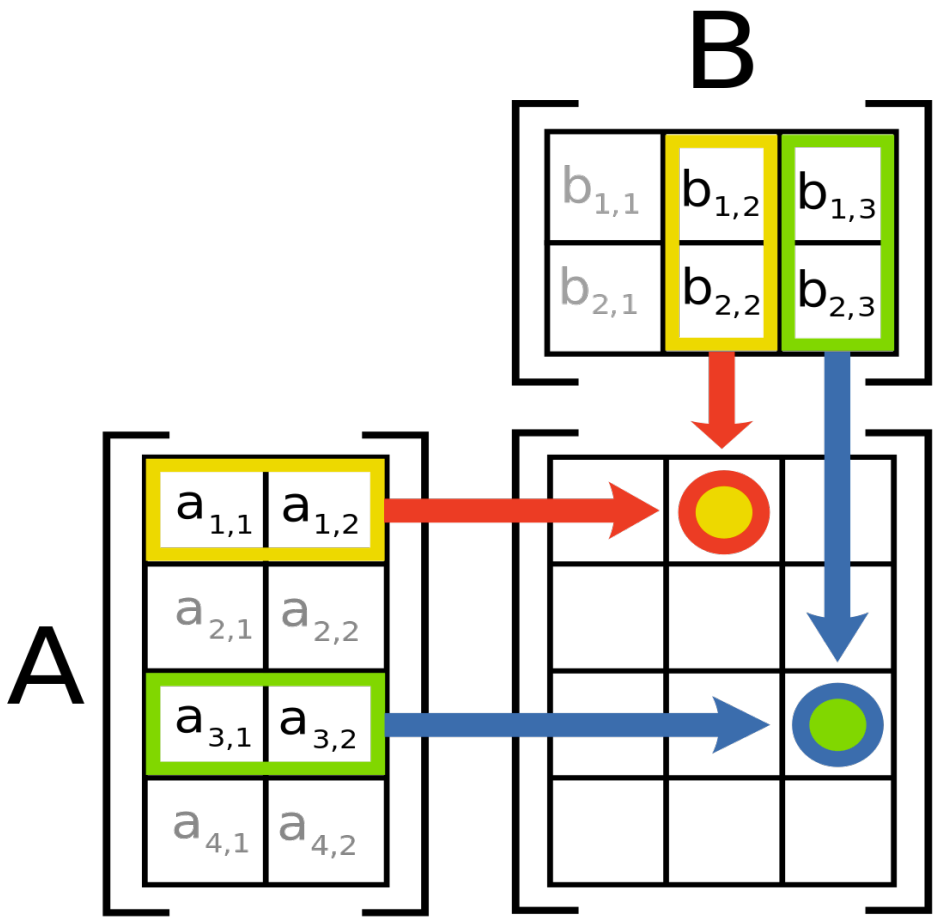
- Tensors
  - homogeneous arrays that may have zero (scalar) or more dimensions

Basics

- Matrix Multiplication

$$C = AB$$

$$C_{i,j} = \sum_{k} A_{i,k} B_{k,j}$$



number of columns in A must be equal to the number of rows in B

https://en.wikipedia.org/wiki/Matrix_multiplication

Dot product

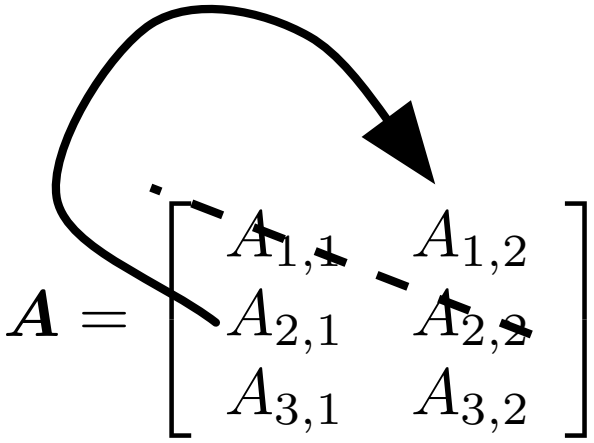
$$[x_1 \quad x_2 \quad \cdots \quad x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

$$\mathbf{x}^T\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$$

$$\mathbf{x}^T\mathbf{y} \in \mathbb{R}$$

Matrix transpose



$$\boldsymbol{A} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ A_{3,1} & A_{3,2} \end{bmatrix} \Rightarrow \boldsymbol{A}^\top = \begin{bmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \end{bmatrix}$$

$$\left(A^T\right)_{i,j} = A_{j,i} \qquad (AB)^T = B^T A^T$$

Identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\forall x \in \mathbb{R}^n, I_n x = x$$

$$A^{-1}A = I_n$$

Norms

- Functions that measure the magnitude ("length") of a vector
  - strictly positive, except for the zero vector
  - think about the distance between zero and the point represented by the vector

$$f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} = \mathbf{0}$$

$$f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y}) \text{ (triangle inequality)}$$

$$\forall \alpha \in \mathbb{R}, f(\alpha \mathbf{x}) = |\alpha| f(\mathbf{x})$$

Norms

ℓ_1-norm: $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$

ℓ_2-norm: $\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$

max norm: $\|\mathbf{x}\|_\infty = \max_i |\mathbf{x}_i|$

Special matrices and vectors

- Unit vector

$$\|\mathbf{x}\|_2 = 1$$

- Symmetric matrix

$$A = A^T$$

- Orthogonal matrix

$$A^T A = A A^T = I$$

$$A^{-1} = A^T$$

Programming with Tensors

Non-negotiable (Python)

- Basic data types
  - booleans, integers, floating point values, strings
- Control flow
- Built-in data structures
  - lists, tuples, dictionaries, sets
  - iterators
- Functions
  - functions can be assigned, passed, returned, and stored
  - lambda functions (inline and anonymous)
- Classes

Numpy

- Library for scientific computing
  - provides a high-performance multidimensional array object and routines for fast operations on these arrays
- The `ndarray` object encapsulates n-dimensional arrays of homogeneous data types
  - many operations performed in compiled code for higher performance
  - have a fixed size at creation
    - changing the size of an ndarray will create a new array and delete the original

```
n = 100000000
array1 = np.random.rand(n)
array2 = np.random.rand(n)
```

```
def dot_python(x, y):
    sum = 0
    for i in range(len(x)):
        sum += x[i] * y[i]
    return sum

def dot_numpy(x, y):
    return np.dot(x, y)
```

```
time_taken = timeit.timeit(lambda: dot_numpy(array1, array2), number=1)
print(f'Numpy Time: {time_taken} seconds')
```

```
Numpy Time: 0.05994947799996453 seconds
```

```
array1 = array1.tolist()
array2 = array2.tolist()
time_taken = timeit.timeit(lambda: dot_python(array1, array2), number=1)
print(f'Python Time: {time_taken} seconds')
```

```
Python Time: 8.325287125999978 seconds
```

The following figures are from:

"*NumPy Illustrated: The Visual Guide to NumPy*" by Lev Maximov

https://betterprogramming.pub/numpy-illustrated-the-visual-guide-to-numpy-3b1d4976de1d

Elegant code

In [3]:
```
a = [1, 2, 3]
[q*2 for q in a]
```

Out[3]: [2, 4, 6]

In [4]:
```
a = np.array([1, 2, 3])
a * 2
```

Out[4]: array([2, 4, 6])

In [1]:
```
a = [1, 2, 3]
b = [4, 5, 6]
[q+r for q, r in zip(a, b)]
```

Out[1]: [5, 7, 9]

In [2]:
```
a = np.array([1, 2, 3])
b = np.array([4, 5, 6])
a + b
```

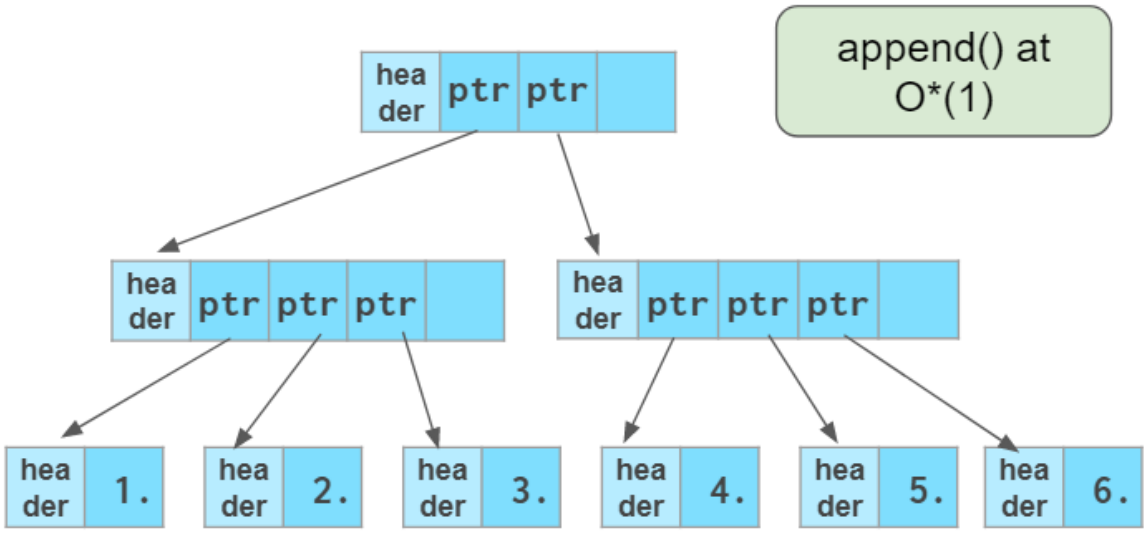
Out[2]: array([5, 7, 9])

Homogeneous and fixed-length

python list **vs** **numpy array**

1.	2.	3.
4.	5.	6.

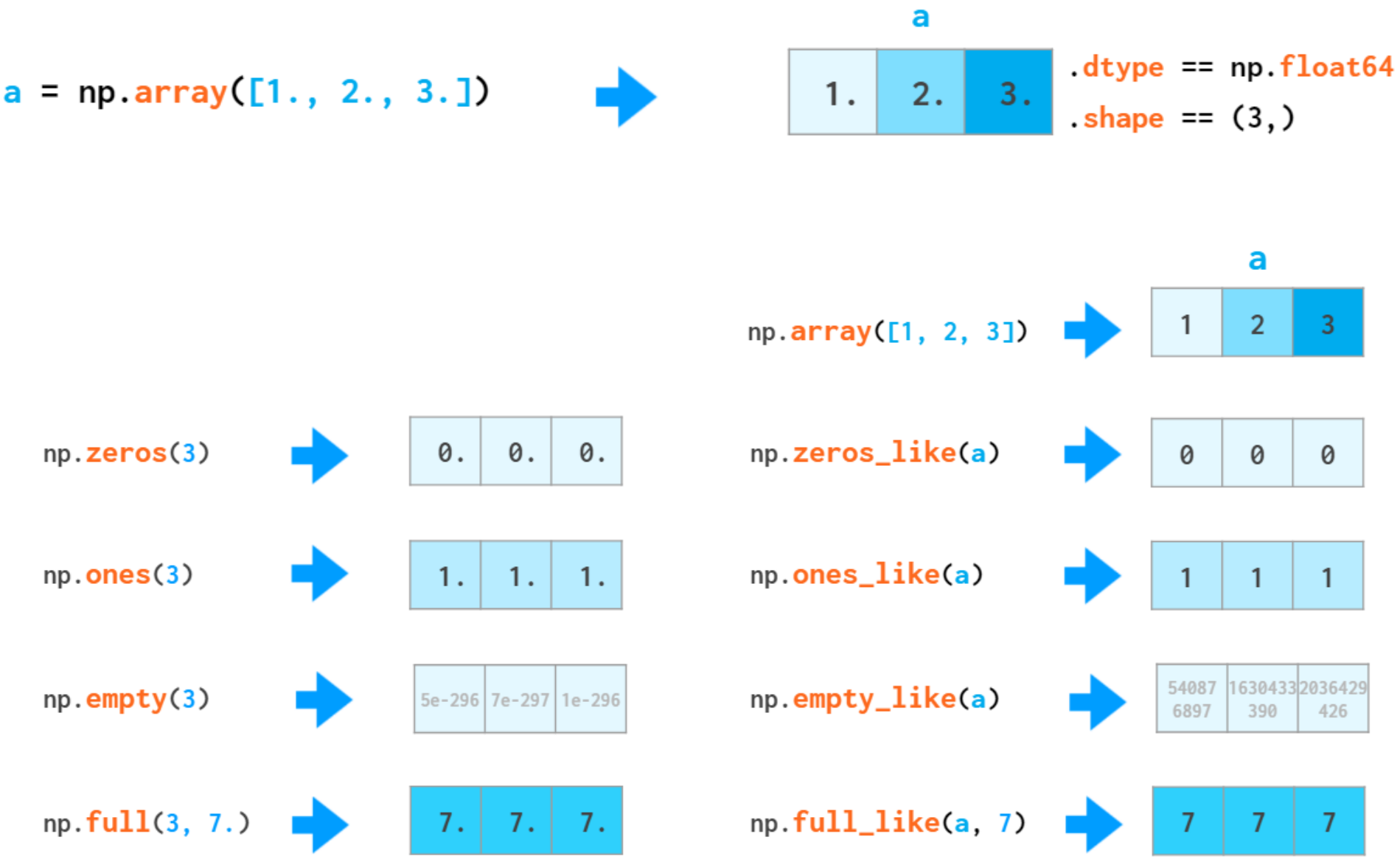
1.	2.	3.
4.	5.	6.



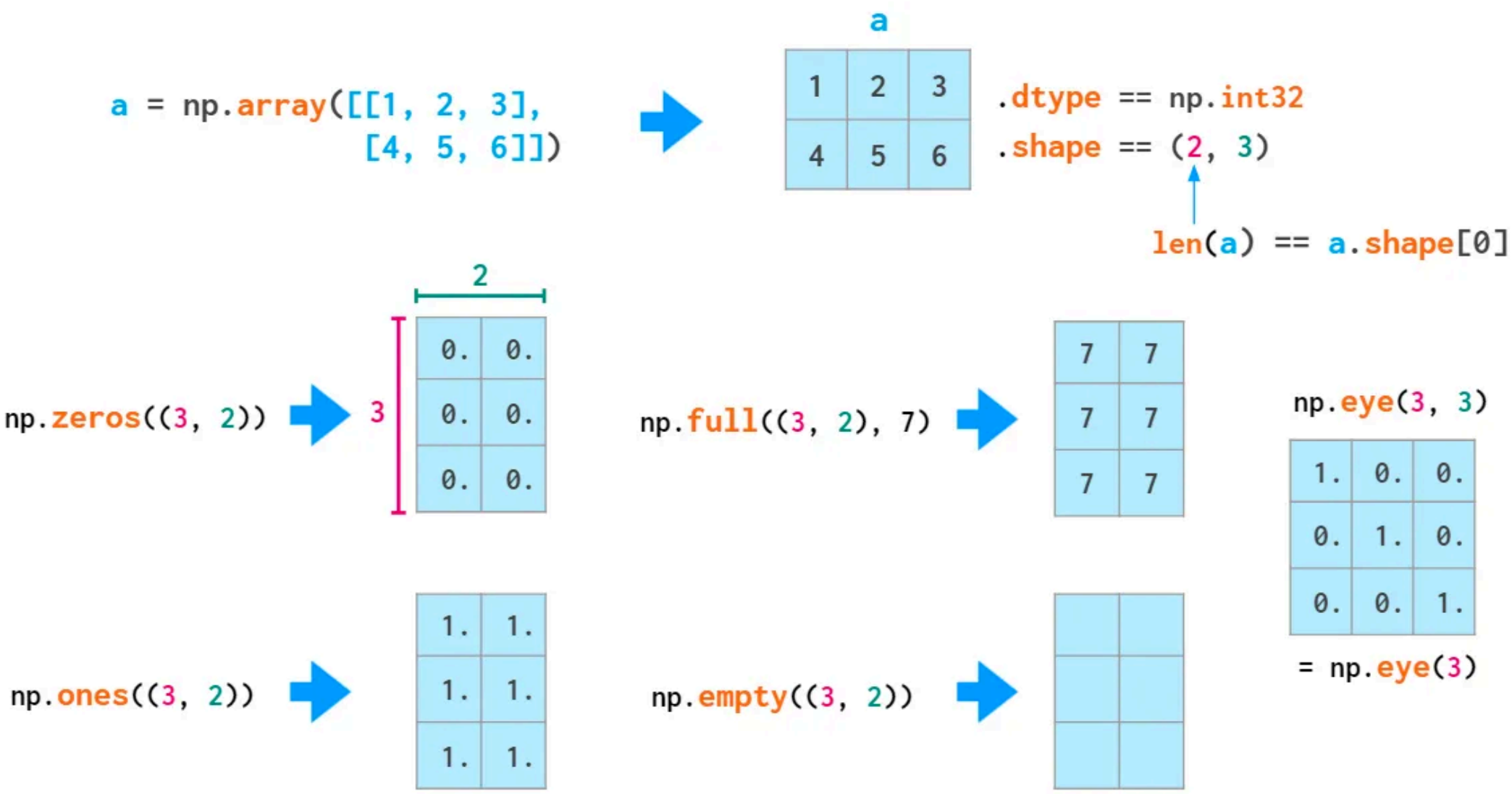
append() at O*(1)

append() at O(N)

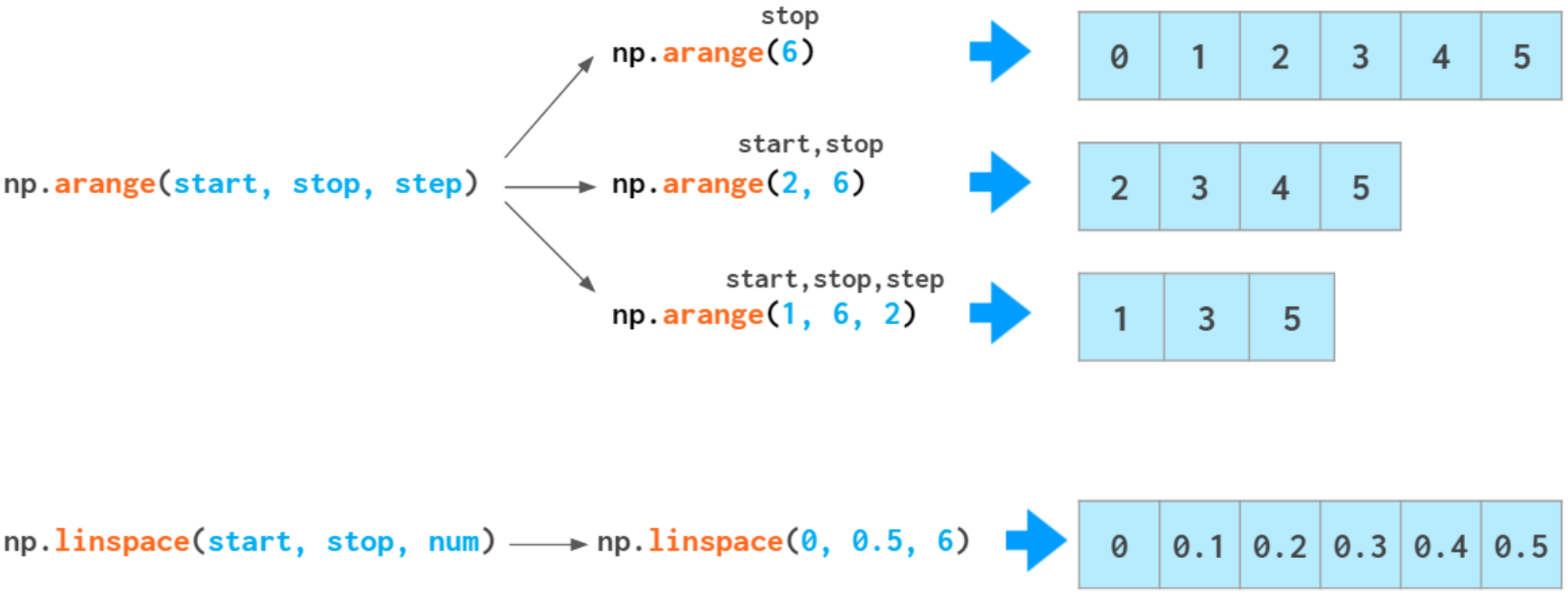
Creating numpy arrays



Creating numpy arrays



Creating numpy arrays



Creating numpy arrays

`np.random.randint(0, 10, 3)`

uniform, x ∈ [0, 10)

Careful!

`np.random.randint(0, 10)` is [0, 10), but

`random.randint(0, 10)` is [0, 10]

`np.random.rand(3)`

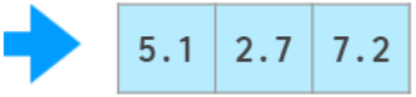
uniform, x ∈ [0, 1)

`np.random.randn(3)`

normal, μ=0, σ=1

`np.random.uniform(1, 10, 3)`

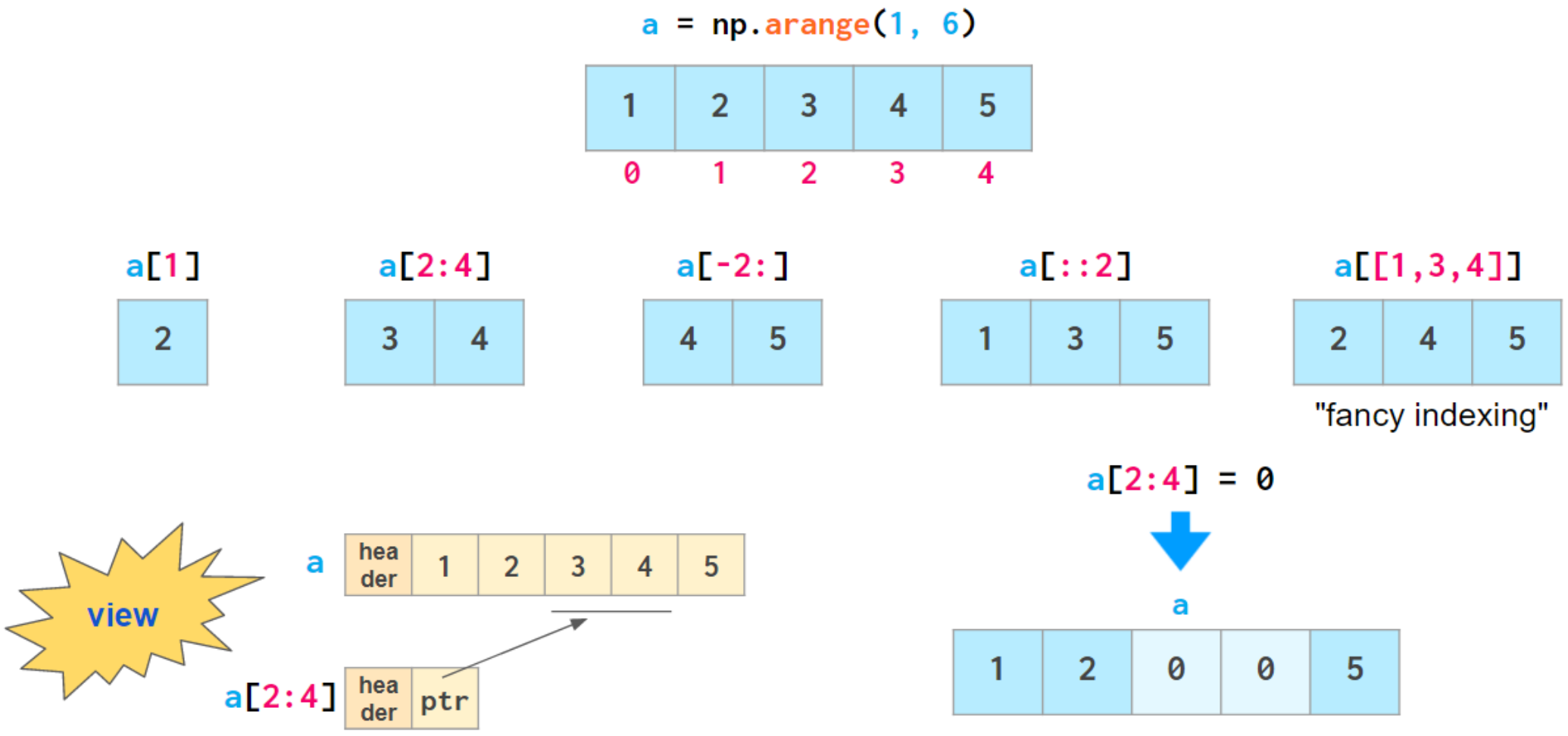
uniform, x ∈ [1, 10)



`np.random.normal(5, 2, 3)`

normal, μ=5, σ=2

Indexing


a = np.arange(1, 6)
1 2 3 4 5
0 1 2 3 4
a[1]
2
a[2:4]
3 4
a[-2:]
4 5
a[::2]
1 3 5
a[[1,3,4]]
2 4 5
"fancy indexing"
view
a
hea der
1 2 3 4 5
a[2:4]
hea der
ptr
a[2:4] = 0
a
1 2 0 0 5

Careful when making copies

python list

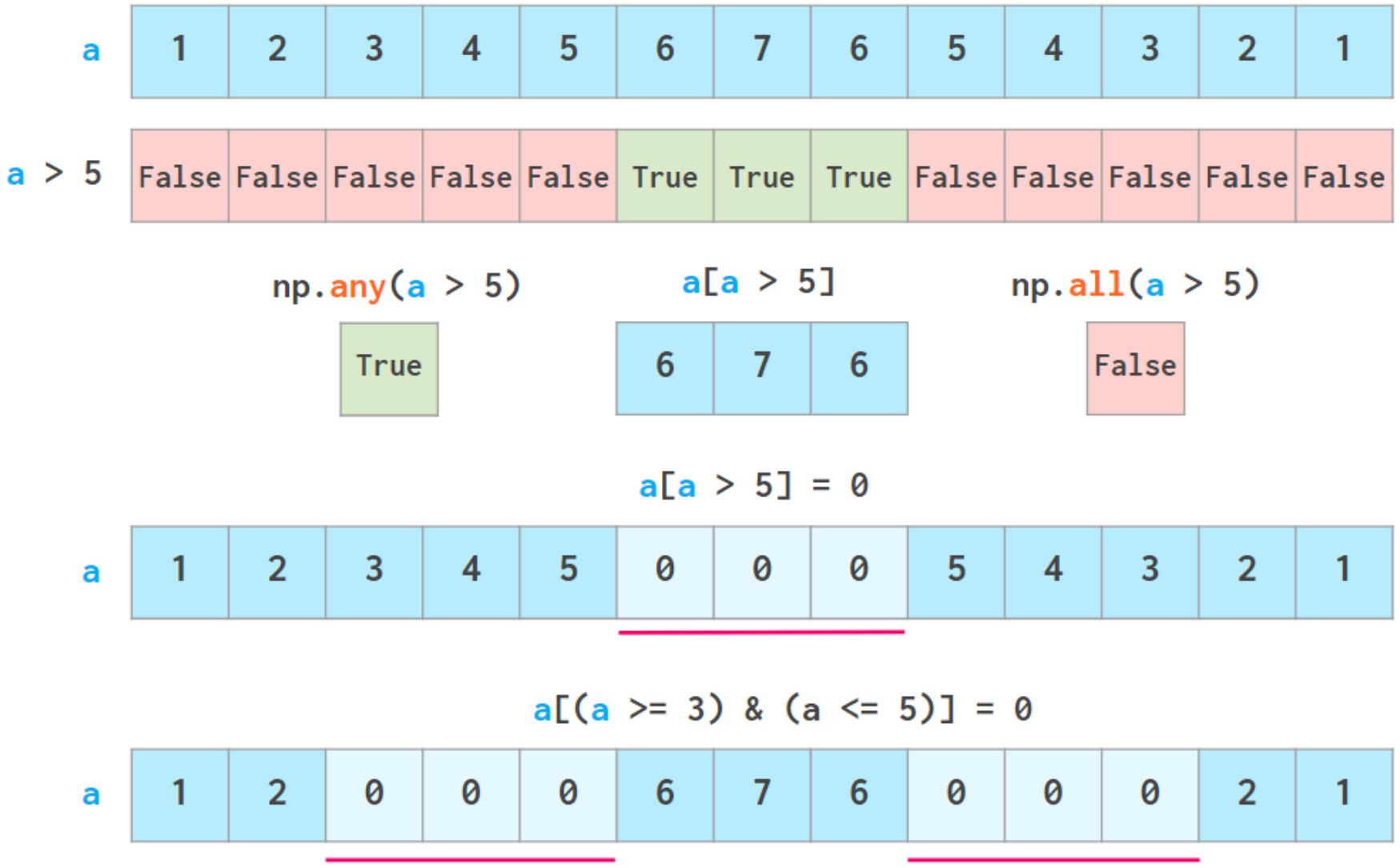
```
a = [1, 2, 3]
b = a          # no copy
c = a[:]       # copy
d = a.copy()   # copy
```

vs

numpy array

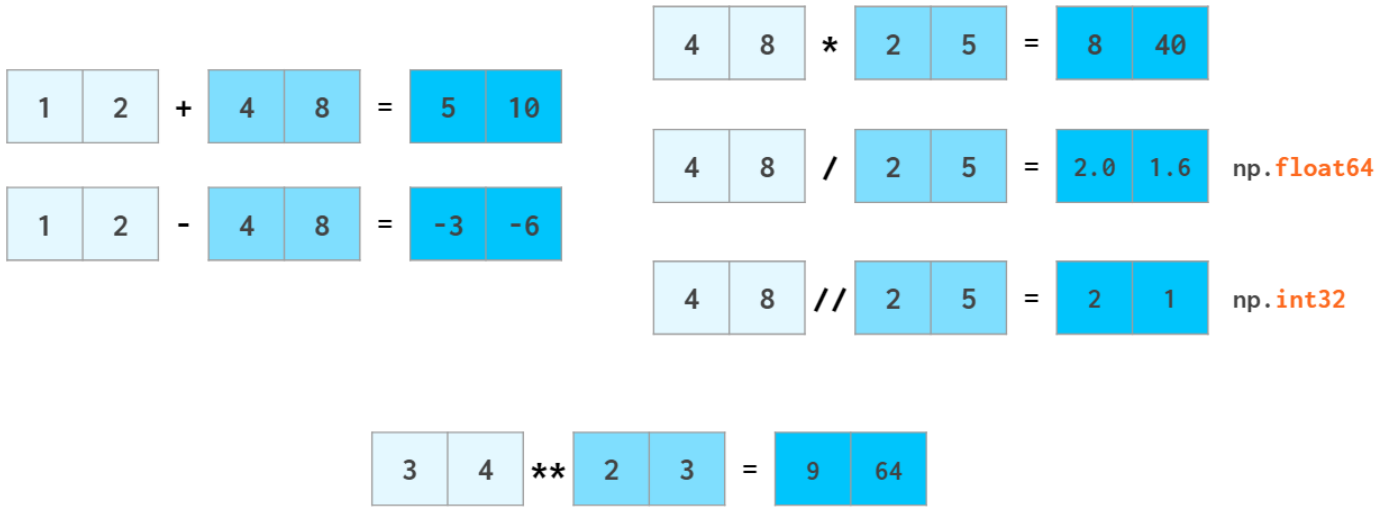
```
a = np.array([1, 2, 3])
b = a          # no copy
c = a[:]       # no copy!!!
d = a.copy()   # copy
```

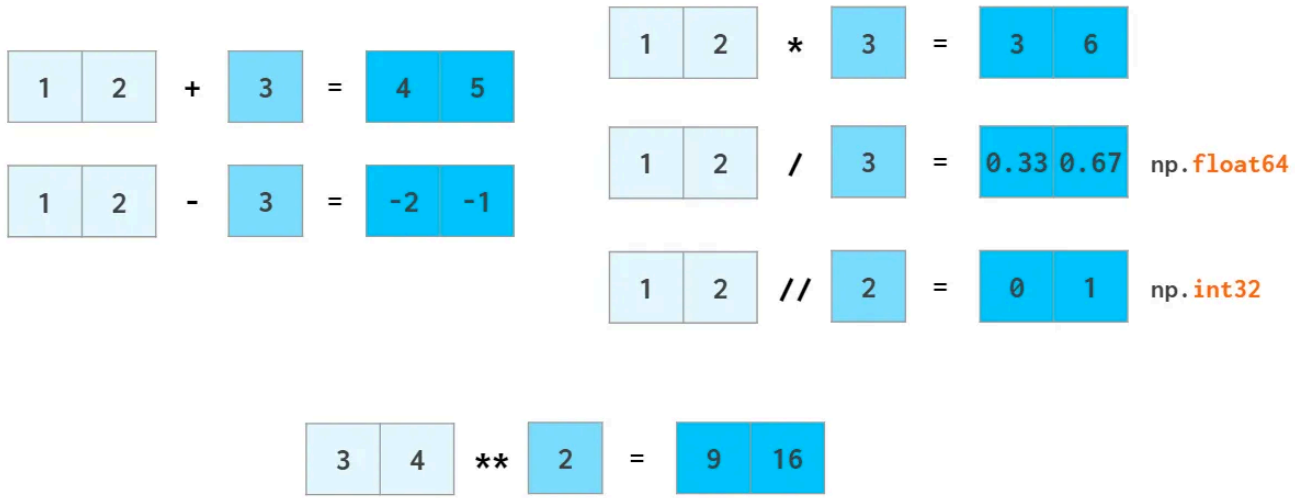
Boolean indexing



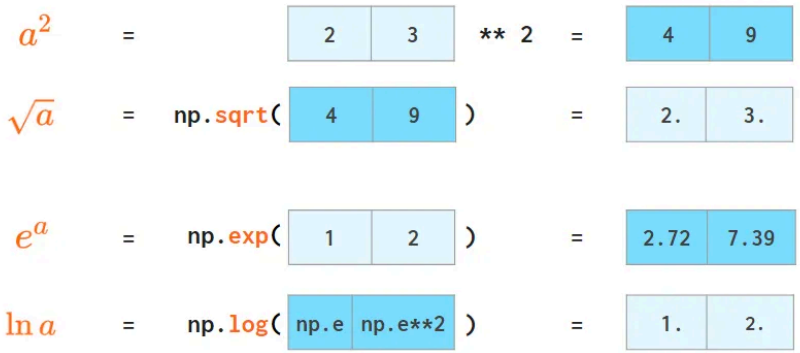
&	and
|	or
^	xor
~	not

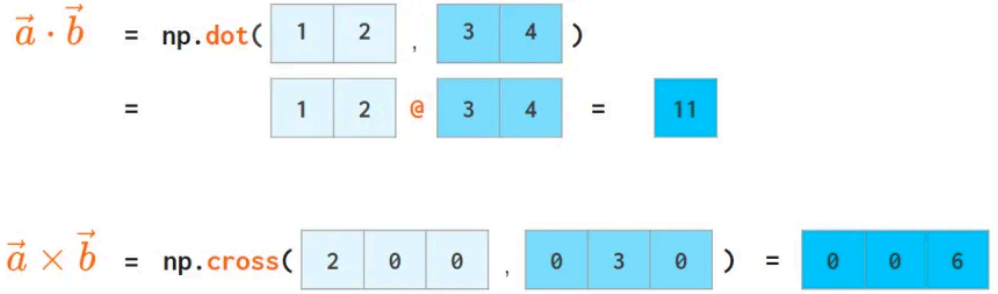
Vector operations

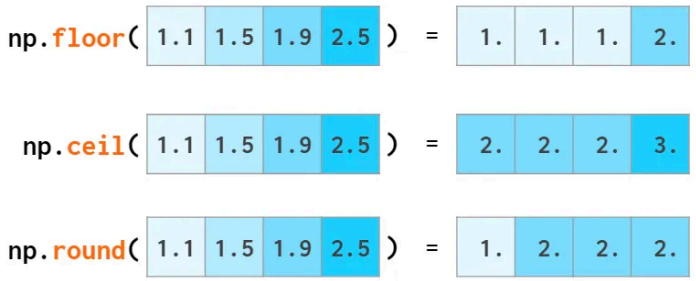


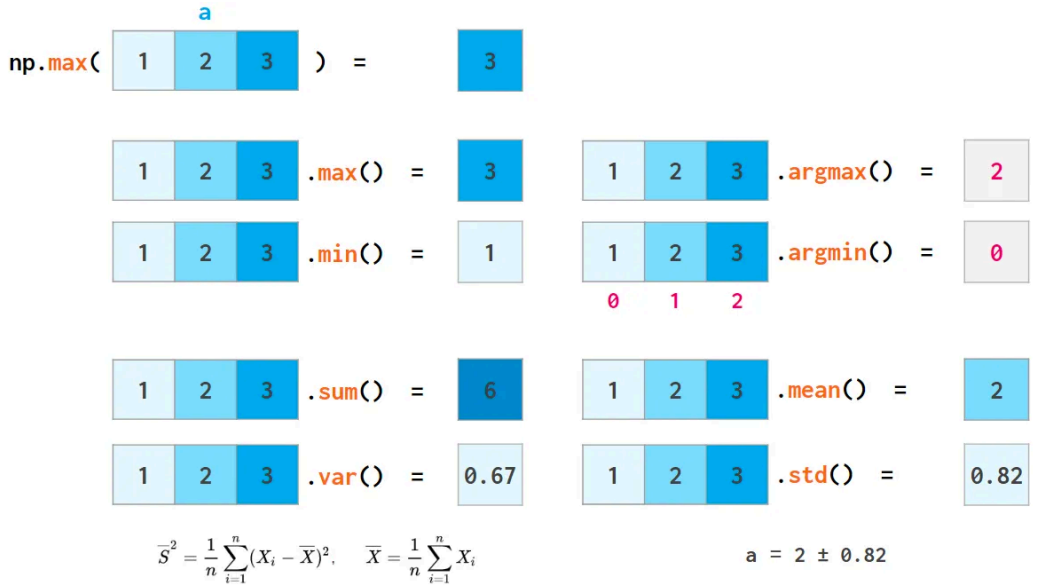


Math functions

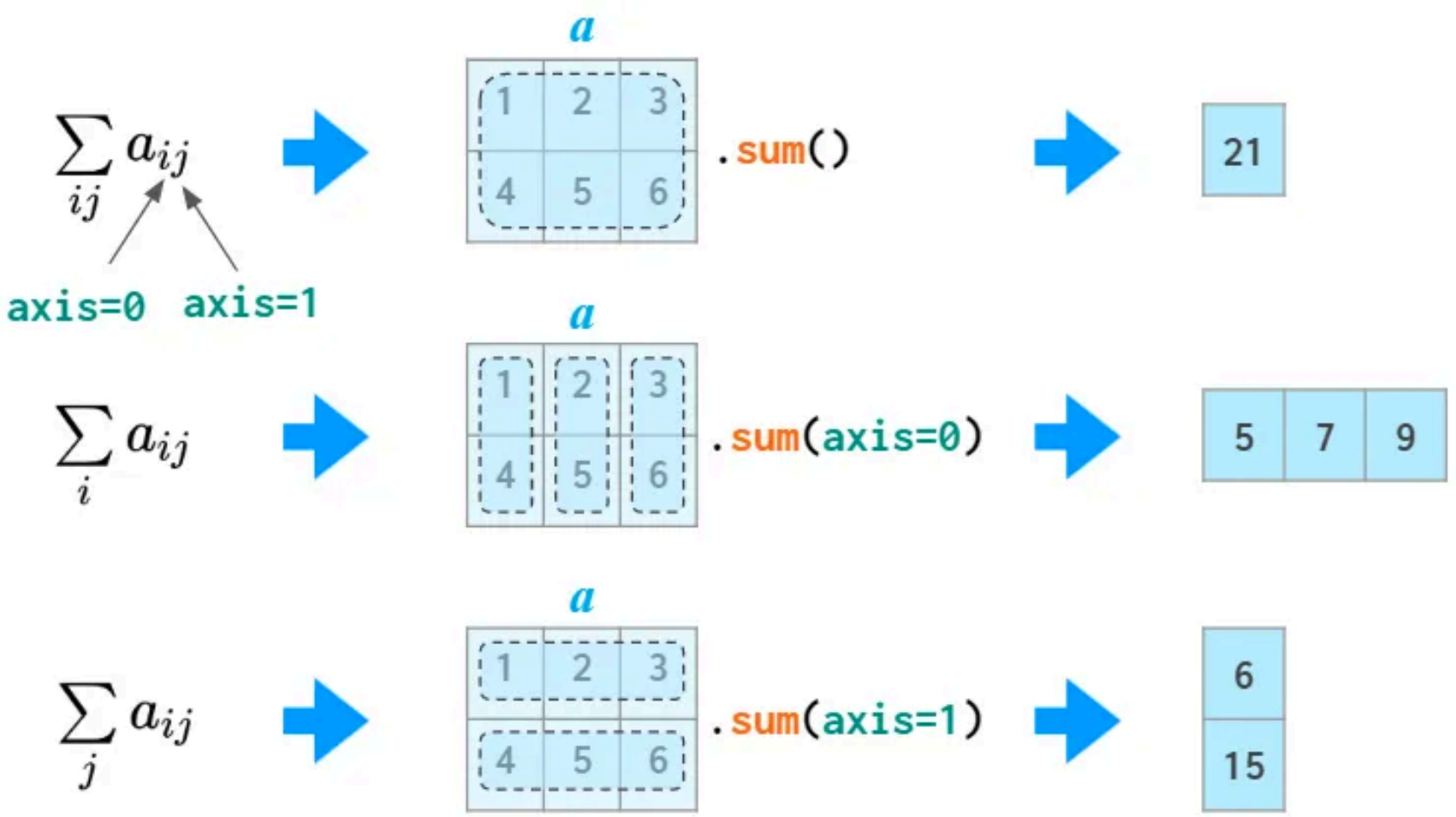




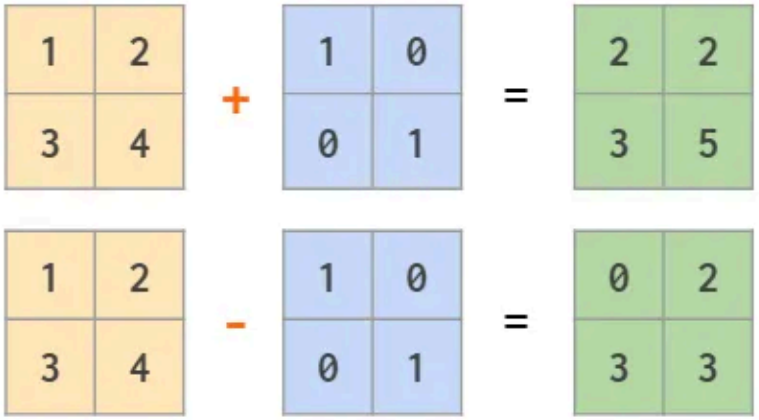


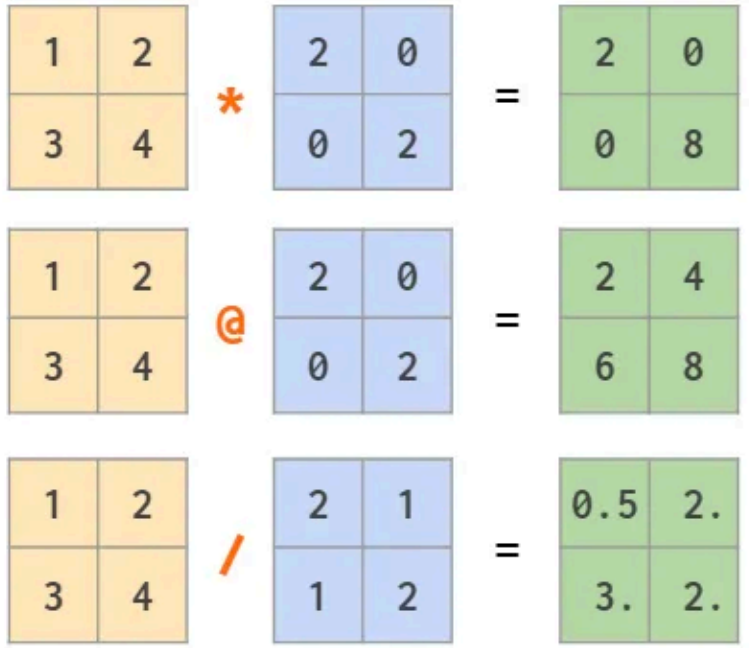


The axis

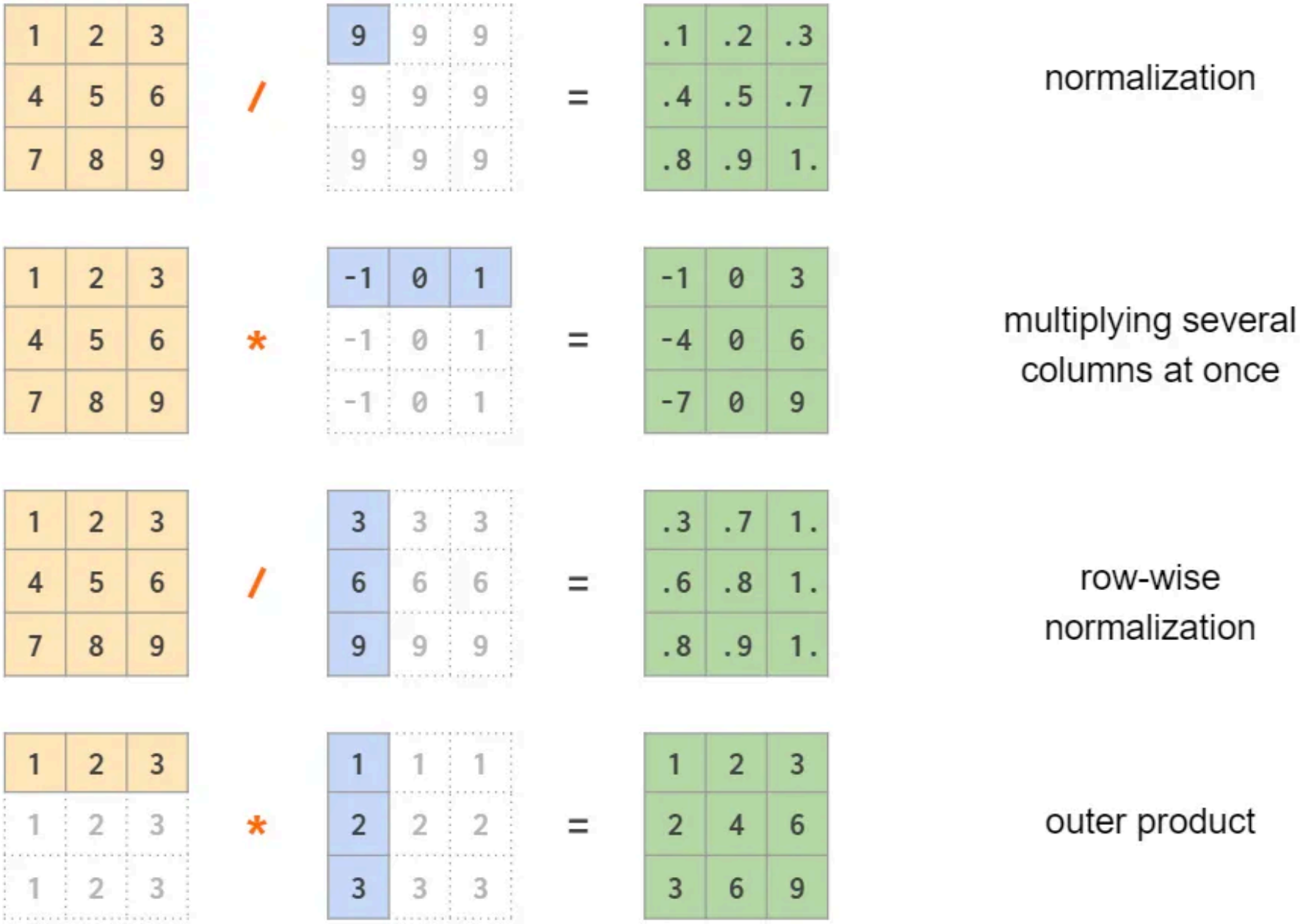

$\sum_{ij} a_{ij}$
axis=0 axis=1
a
1 2 3
4 5 6
.sum()
21
$\sum_{i} a_{ij}$
a
1 2 3
4 5 6
.sum(axis=0)
5 7 9
$\sum_{j} a_{ij}$
a
1 2 3
4 5 6
.sum(axis=1)
6
15

Matrix operations





Broadcasting



General broadcasting rules

- When operating on two arrays, NumPy compares their shapes element-wise
  - starts with the rightmost dimension and works backwards
  - two dimensions are compatible when:
    - they are equal, or
    - one of them is 1
  - otherwise a ValueError: operands could not be broadcast together exception is thrown
- Input arrays do not need to have the same number of dimensions
  - resulting array will have the same number of dimensions as the input array with the greatest number of dimensions
  - the size of each dimension is the largest size of the corresponding dimension among the input arrays
  - missing dimensions are assumed to have size one

```
A      (4d array):  8 x 1 x 6 x 1
B      (3d array):      7 x 1 x 5
Result (4d array):  8 x 7 x 6 x 5

A      (2d array):  5 x 4
B      (1d array):      1
Result (2d array):  5 x 4

A      (2d array):  5 x 4
B      (1d array):      4
Result (2d array):  5 x 4

A      (3d array):  15 x 3 x 5
B      (3d array):  15 x 1 x 5
Result (3d array):  15 x 3 x 5
```

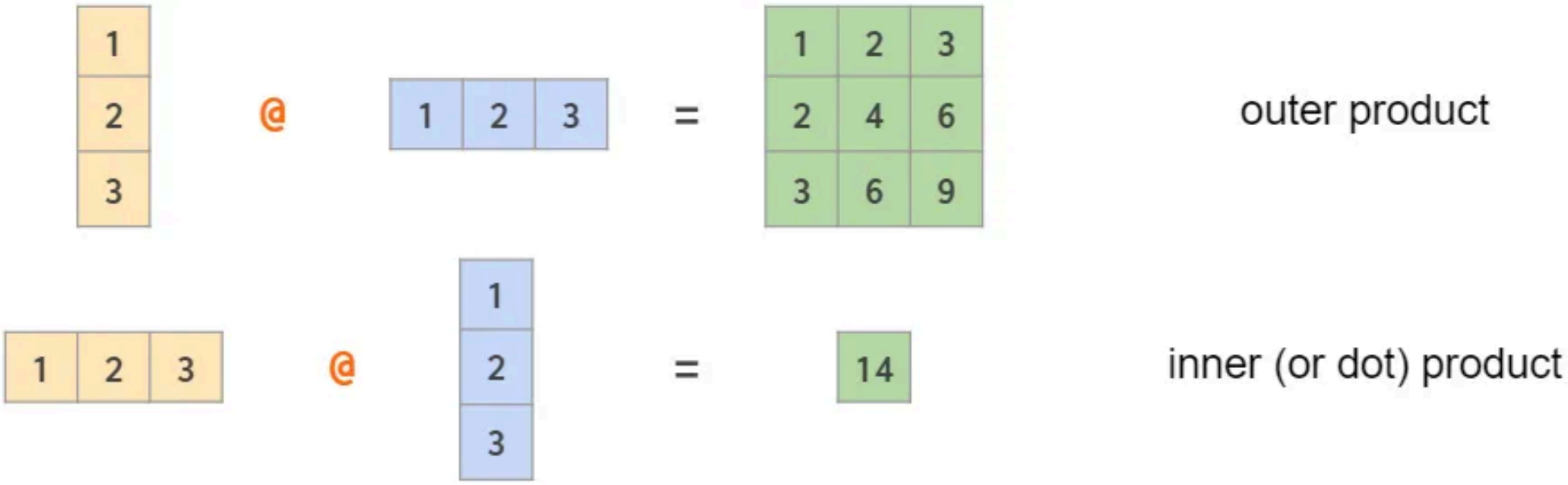
```
A      (3d array):  15 x 3 x 5
B      (2d array):       3 x 5
Result (3d array):  15 x 3 x 5

A      (3d array):  15 x 3 x 5
B      (2d array):       3 x 1
Result (3d array):  15 x 3 x 5

A      (1d array):  3
B      (1d array):  4 # mismatch

A      (2d array):      2 x 1
B      (3d array):  8 x 4 x 3 # mismatch
```

Must know your shapes …



Concatenation