k objects may be selected from n objects. (Problem 6.24 helps you to derive the value of $\binom{n}{k}$.) For each positive integer n, n factorial (written $n!$) is defined to be $1 \times 2 \times \cdots \times n$. So $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$. $0!$, zero factorial, is defined to be 1. With this notation the binomial coefficient may be written

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{n(n-1)\cdots(k+1)}{(n-k)(n-k-1)\cdots 1} \qquad (1)$$

Example 6.4. This is illustrated with the following two cases:

1. Of 10 residents, three are to be chosen to cover a hospital service on a holiday. In how many ways may the residents be chosen? The answer is

$$\binom{10}{3} = \frac{10!}{7!\,3!} = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}{(1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7)(1 \times 2 \times 3)} = 120$$

2. Of eight consecutive patients, four are to be assigned to drug A and four to drug B. In how many ways may the assignments be made? Think of the eight positions as eight objects; we need to choose four for the drug A patients. The answer is

$$\binom{8}{4} = \frac{8!}{4!\,4!} = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}{(1 \times 2 \times 3 \times 4)(1 \times 2 \times 3 \times 4)} = 70$$

The binomial probability, $b(k; n, \pi)$, may be written

$$b(k; n, \pi) = \binom{n}{k} \pi^k (1-\pi)^{n-k} \qquad (2)$$

Example 6.5. Ten patients are treated surgically. For each person there is a 70% chance of successful surgery (i.e., $\pi = 0.7$). What is the probability of only five or fewer successful surgeries?

$$\begin{aligned}
P[\text{five or fewer successful cases}] &= P[\text{five successful cases}] + P[\text{four successful cases}] \\
&\quad + P[\text{three successful cases}] + P[\text{two successful cases}] \\
&\quad + P[\text{one successful case}] + P[\text{no successful case}] \\
&= b(5; 10, 0.7) + b(4; 10, 0.7) + b(3; 10, 0.7) + b(2; 10, 0.7) \\
&\quad + b(1; 10, 0.7) + b(0; 10, 0.7) \\
&= 0.1029 + 0.0368 + 0.0090 + 0.0014 + 0.0001 + 0.0000 \\
&= 0.1502
\end{aligned}$$

(*Note*: The actual value is 0.1503; the answer 0.1502 is due to round-off error.)

The binomial probabilities may be calculated directly or found by a computer program. The mean and variance of a binomial random variable with parameters π and n are given by

$$\begin{aligned}
E(Y) &= n\pi \\
\text{var}(Y) &= n\pi(1-\pi)
\end{aligned} \qquad (3)$$