# Analysis
# - PMA 5 -

KYB

Thrn, it's a Fact

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# Overview

Basic Topology
Exercises

# Exercises

## Definition

A metric space is called separable if it contains a countable dense subset.

## Ex 2.22

Show that $\mathbb{R}^k$ is separable.

# Exercises

## Definition

A collection $\{V_\alpha\}$ of open subsets of $X$ is said to be a base for $X$ if the following is true:

- For every $x \in X$ and every open set $G$ in $X$ such that $x \in G$, there is $V_\alpha$ such that

$$x \in V_\alpha \subset G.$$

In other words, every nonempty open subset $G$ is of the form $G = \cup_\alpha V_\alpha$ for some $V_\alpha$'s.

## Ex 2.23

Prove that every separable metric space has a countable base.

# Exercises

## Definition

Let $X$ be a topological space.

(1) $X$ is compact if every open cover has a finite subcover.

(2) $X$ is limit point compact if every infinite subset of $X$ has a limit point.

(3) $X$ is sequentially compact if every sequence in $X$ has a convergent subsequence.

If $X$ is a metric space, the the above statements are all equivalent.

# Exercises

## Ex 2.24

Let $X$ be a metric space in which every infinite subset has a limit point. Prove that $X$ is separable.

# Exercises

## Ex 2.25

Prove that every compact metric space $K$ has a countable base, and that $K$ is therefore separable.

# Exercises

## Ex 2.26

Let $X$ be a metric space in which every infinite subset has a limit point. Prove that $X$ is compact.

# Exercises

## Definition

Let $p$ be a point of a metric space $X$ and let $E$ be a subset of $X$. $p$ is called a condensation point of $E$ if every neighborhood of $p$ contains uncountably many points of $E$.

## Ex 2.27

Suppose $E \subset \mathbb{R}^k$, $E$ is uncountable, and let $P$ be the set of all condensation points of $E$. Prove that $P$ is perfect and that at most countably many points of $E$ are not in $P$. In other words, show that $P^c \cap E$ is at most countable.

# Exercises

## Ex 2.28

Prove that every closed set in a separable metric space is the union of a perfect set and a set which is at most countable. As Corollary, every countable closed set in $\mathbb{R}^k$ has isolated points.

# Exercises

## Ex 2.29

Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments.

# Exercises

## Ex 2.30

- If $\mathbb{R}^k = \cup_1^\infty F_n$, where each $F_n$ is a closed subset of $\mathbb{R}^k$, then at least one $F_n$ has a nonempty interior.
- Equivalently, if $G_n$ is a dense open subset of $\mathbb{R}^k$, then $\cap_1^\infty G_n$ is not empty, in fact, it is dense in $\mathbb{R}^k$.

# The End