Modules

KYB

Thrn, it's a Fact

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February 9, 2021

Overview

Observe

Suppose D is a right R-module. For any homomorphism $f : X \to Y$, $1 \otimes f : D \otimes_R X \to D \otimes_R Y$ is a group homomorphism of abelian groups.

$$D \otimes_R \_\_ : \text{ left } R\text{-modules} \to \text{ abelian groups}$$

If $\psi : L \to M$ is injective, $1 \otimes \psi$ may not be injective. If $\varphi : M \to N$ is surjective, for any $d \otimes n \in D \otimes_R N$, $(1 \otimes \varphi)(d \otimes m) = d \otimes n$ for some $m \in M$ with $\varphi(m) = n$. So $1 \otimes \varphi$ is surjective.

Theorem

Suppose D is a right R-module and L, M, and N are left R-modules.

(1) *If $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is exact, then*

$$D \otimes_R L \xrightarrow{1 \otimes \psi} D \otimes_R M \xrightarrow{1 \otimes \varphi} D \otimes_R N \to 0 \text{ is exact.}$$

(2) *If D is an (S, R)-bimodule, then the associated sequence is exact as left S-modules.*

(3) *The sequence*

$$D \otimes_R L \xrightarrow{1 \otimes \psi} D \otimes_R M \xrightarrow{1 \otimes \varphi} D \otimes_R N \to 0 \text{ is exact}$$

for all right R-module (resp. (S, R)-bimodule) if and only if

$$L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0 \text{ is exact}$$

as abelian sequence (resp. S-modules sequences).

Proposition

Let A be a right R-module. Then the following are equivalent:

(1) *For any left R-modules L, M, and N, if $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is a short exact sequence, then*

$$0 \to D \otimes_R L \xrightarrow{1 \otimes \psi} D \otimes_R M \xrightarrow{1 \otimes \varphi} D \otimes_R N \to 0 \text{ is also a exact sequence.}$$

(2) *For any left R-modules L and M, if $0 \to L \xrightarrow{\psi} M$ an exact sequence of left R-modules, then $0 \to D \otimes_R L \xrightarrow{1 \otimes \psi} D \otimes_R M$ is an exact sequence of abelian groups.*

Corollary

(1) *For any right R-module D, $D \otimes \_\_$ is a covariant right exact functor.*

(2) *$D \otimes \_\_$ is exact functor if and only if D is flat.*

Corollary

Free modules are flat; more generally, projective modules are flat.

Example

(1) Since $\mathbb{Z}$ is a projective $\mathbb{Z}$-module, it is flat. $\mathbb{Z}/2\mathbb{Z}$ is not a flat $\mathbb{Z}$-module.

(2) $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective, but is not flat.

(3) The direct sum of flat modules is flat. In particular, $\mathbb{Q} \oplus \mathbb{Z}$ is flat which is neither projective nor injective.

Theorem (Hom-Tensor Adjoint)

Let M be a right R-module, N be an (S,R)-bimodule, and L be a right S-module. Then there is a group isomorphism

$$\mathrm{Hom}_S(M \otimes_R N, L) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(N, L)),$$

such that $f \mapsto \tilde{f}$ where $\tilde{f}(m)(n) = f(m \otimes n)$.

Theorem

Let M be an (S,R)-module, N be a left R-module, and L be a left S-module. Then there is a group isomorphism

$$\mathrm{Hom}_S(M \otimes_R N, L) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_S(M, L)),$$

such that $f \mapsto \tilde{f}$ where $\tilde{f}(n)(m) = f(m \otimes n)$.

Application

We can prove that $D \otimes_R \_\_$ is a right exact functor using the fundamental theorem of tensor product.

Corollary

If R is commutative, then the tensor product of two projective R-modules is projective.

Summary

(1) Let A be a left R-module. The functor $\mathrm{Hom}_R(A, \_\_)$ is covariant and left exact; A is projective if and only if $\mathrm{Hom}_R(A, \_\_)$ is exact.

(2) Let A be a left R-module. The functor $\mathrm{Hom}_R(\_\_, A)$ is contravariant and left exact; A is injective if and only if $\mathrm{Hom}_R(\_\_, A)$ is exact.

(3) Let A be a right R-module. The functor $A \otimes_R \_\_$ is covariant and right exact; A is flat if and only if $A \otimes_R \_\_$ is exact.

(4) Let A be a left R-module. The functor $\_\_ \otimes_R A$ is covariant and right exact; A is flat if and only if $\_\_ \otimes_R A$ is exact.

(5) Projective modules are flat. The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective but not flat. The $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Q}$ is flat but neither projective nor injective.

Exercise

Let A_1 and A_2 be R-moduiles. Prove that $A_1 \oplus A_2$ is a flat R-module if and only if both A_1 and A_2 are flat. More generally, an arbitrary direct sum $\bigoplus A_i$ of R-modules is flat if and only if each A_i is flat.

Exercise

Prove that the module $M \otimes_R S$ obtained by changing the base from the ring R to the ring S of the flat R-module M is a flat S-module.

Exercise

Prove that A is a flat R-module if and only if for any left R-modules L and M where L is finitely generated, then $\psi : L \to M$ injective implies that also $1 \otimes \psi : A \otimes_R L \to A \otimes_R M$ is injective.

Theorem

A is a flat R-module if and only if for every finitely generated ideal of R, the map from $A \otimes_R I \to A \otimes_R R \cong A$ induced by the inclusion $I \subset R$ is again injective (equivalently, $A \otimes_R I \cong AI \subset A$).

Step 1

If A is flat, then $A \otimes_R I \cong AI$.

Theorem

A is a flat R-module if and only if for every finitely generated ideal of R, the map from $A \otimes_R I \to A \otimes_R R \cong R$ induced by the inclusion $I \subset R$ is again injective (equivalently, $A \otimes_R I \cong AI \subset A$).

Step 2

If $A \otimes_R I \to A \otimes_R R$ is injective for every finitely generated ideal I, $A \otimes_R I \to A \otimes_R R$ is injective for every ideal I. Moreover, if K is any submodule of a finitely generated free module F, then $A \otimes_R K \to A \otimes_R F$ is injective. The same is true for any free module F.

Theorem

A is a flat R-module if and only if for every finitely generated ideal of R, the map from $A \otimes_R I \to A \otimes_R R \cong R$ induced by the inclusion $I \subset R$ is again injective (equivalently, $A \otimes_R I \cong AI \subset A$).

Step 3

Under the assumption of Step 2, Suppose L and M are R-modules and $\psi : L \to M$ is injective. Then $1 \otimes \psi : A \otimes_R L \to A \otimes_R M$ is injective and hence A is flat.

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