Algebraic Topology
- Dunkin's Torus 3 -

KYB

Thrn, it's a Fact

mathrnfact@gmail.com

September 14, 2021

Overview

The Fundamental Group

- Covering Spaces

Definition

Let $p : E \to B$ be a continuous surjective map.

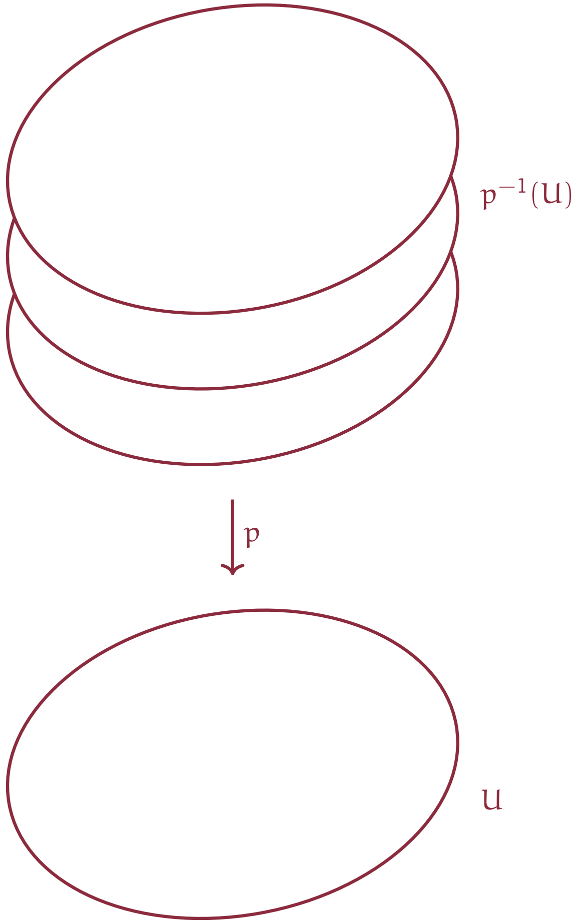
- The open set U of B is said to be *evenly covered* by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphic of V_α onto U.
- The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into *slices*.
- If every point b of B has a neighborhood U that is evenly covered by p, then p is called a *covering map*, and E is said to be a *covering space* of B.

Remark

- If U is evenly covered by p and W is an open set contained in U, then W is also evenly covered by p.

Suppose p is a covering map.

- For each $b \in B$, the subspace $p^{-1}(B)$ of E has the discrete topology.
- For each slice V_α is open in E and intersects the set $p^{-1}(b)$ in a single point; therefore this point is open in $p^{-1}(b)$.
- p is an open map.


$p^{-1}(U)$
p
U

Example

Let X be any space.

- Let $i : X \to X$ be the identity map. Then i is a covering map.
- More generally, let E be the space $X \times \{1, \cdots, n\}$ consisting of n disjoint copies of X. The map $p : E \to X$ given by $p(x, i) = x$ for all i is again a covering map.

Theorem (53.1)

The map $p : \mathbb{R} \to S^1$ given by the equation

$$p(x) = (\cos 2\pi x, \sin 2\pi x)$$

is a covering map.

Remark

If $p : E \to B$ is a covering map, then p is a *local homeomorphism* of E with B.

Example

The map $p : \mathbb{R}_+ \to S^1$ given by the equation

$$p(x) = (\cos 2\pi x, \sin 2\pi x)$$

is surjective, and it is a local homeomorphism. But it is not a covering map, for the point $b_0 = (1, 0)$ has no neighborhood U that is evenly covered by p.

This example shows that the map obtained by restricting a covering map may not be a covering map.

Example

Consider the map $p : S^1 \to S^1$ given in equation by

$$p(z) = z^2.$$

Then p is a covering map.

Theorem (53.2)

Let $p : E \to B$ *be a covering map. If* B_0 *is a subspace of* B*, and if* $E_0 = p^{-1}(B_0)$*, then the map* $p_0 : E_0 \to B_0$ *obtained by restricting* p *is a covering map.*

Theorem (53.3)

If $p : E \to B$ *and* $p' : E' \to B'$ *are covering maps, then*

$$p \times p' : E \times E' \to B \times B'$$

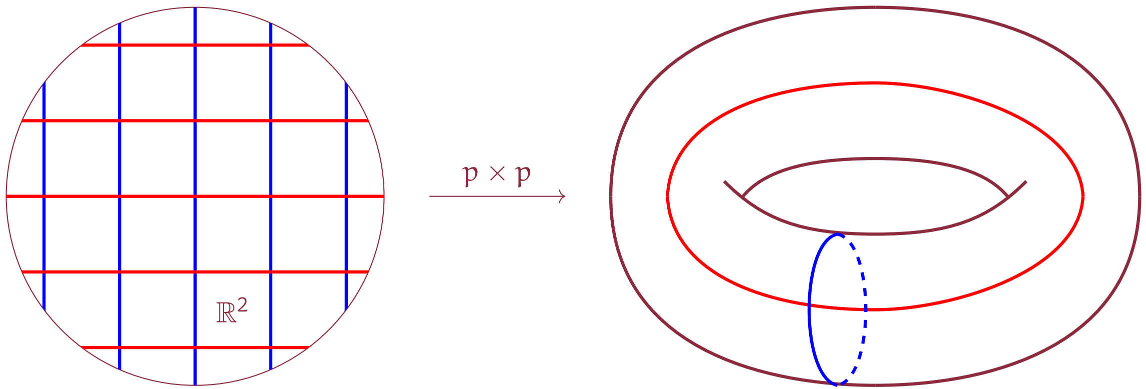
is a covering map.

Example

Consider $T = S^1 \times S^1$, the *torus*. The product map

$$p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$$

is a covering of the torus by the plane $\mathbb{R}^2$.



Example

Let b_0 denoted the point $p(0)$ of S^1; let B_0 denote the subspace

$$B_0 = (S^1 \times b_0) \cup (b_0 \times S^1)$$

of $S^1 \times S^1$. Then B_0 is the union of two circles that have a point in common, we sometimes call it the *figure-eight space*. The space $E_0 = p^{-1}(B_0)$ is the infinite grid

$$E_0 = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}).$$

The map $p_0 : E_0 \to B_0$ obtained by restricting $p \times p$ is thus a covering map.

Example

Consider the covering map
$$p \times i : \mathbb{R} \times \mathbb{R}_+ \to S^1 \times \mathbb{R}_+.$$
If we take the standard homeomorphism of $S^1 \times \mathbb{R}_+$ with $\mathbb{R}^2 - \mathbf{0}$, sending $x \times t$ to tx, the composite gives us a covering
$$\mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2 - \mathbf{0}$$
of the punctured plane by the open upper half-plane.

Ex 53.1

Let Y have the discrete topology. Show that if $p : X \times Y \to X$ is projection on the first coordinate, then p is a covering map.

Ex 53.2

Let $p : E \to B$ be continuous and surjective. Suppose that U is an open set of B that is evenly covered by p. Show that if U is connected, then the partition of $p^{-1}(U)$ into slices is unique.

Ex 53.3

Let $p : E \to B$ be a covering map; let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$. In such a case, E is called a k-*fold covering* of B.

Ex 53.4

Let $q : X \to Y$ and $r : Y \to Z$ be covering maps; let $p = r \circ q$. Show that if $r^{-1}(z)$ is finite for each $z \in Z$, then p is a covering map.

Ex 53.6

Let $p : E \to B$ be a covering map.

(a) If B is Hausdorff, regular, completely regular, or locally compact Hausdorff, then so is E.

(b) If B is compact and $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.