Modules

KYB

Thrn, it's a Fact

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Overview

Observe

Let $f : D \to L$ and $\psi : L \to M$ be two homomorphisms. Then we get a homomorphism $\psi \circ f : D \to L$. That means we have the following commutative diagram:

$$\begin{tikzcd} D \arrow[d, "f"'] \arrow[dr, dashed, "f'"] & \\ L \arrow[r, "\psi"] & M \end{tikzcd}$$

Theorem

Let D,L, and M be R-modules and let $\psi : L \to M$ be an R-module homomorphism. Then the map

$$\psi'(f) = \psi \circ f.$$

is a group homomorphism. If ψ is injective, then ψ' is also injective, i.e.,

$$\textit{if } 0 \to L \xrightarrow{\psi} M \textit{ is exact,}$$

$$\textit{then } 0 \to \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \textit{ is also exact.}$$

Theorem

Let $D, L, M,$ and N be R-modules.

(1) *If*

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0 \text{ is exact,}$$

then the associated sequence

$$0 \to \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(D, N) \text{ is exact.}$$

(2) *$f : D \to N$ lifts to $F : D \to M$ if and only if $f \in \mathrm{Im}\,\varphi'$.*

(3) *φ' is surjective if and only if every homomorphism from D to N lifts to a homomorphism from D to M.*

(4) *$0 \to \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(D, N)$ is exact for all D if and only if $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N$ is exact.*

Proposition

Let P be an R-module. Then the following are equivalent:

(1) *For any R-modules L, M, and N, if*

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$$

is a short exact sequence, then

$$0 \to \mathrm{Hom}_R(P, L) \xrightarrow{\psi'} \mathrm{Hom}_R(P, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(P, N) \to 0$$

is also a short exact sequence.

(2) *For any R-modules M and N, if $M \xrightarrow{\varphi} N \to 0$ is exact, then every R-module homomorphism from P into N lifts to an R-module homomorphism into M, i.e., given $f \in \mathrm{Hom}_R(P, N)$, there is a lift $F \in \mathrm{Hom}_R(P, M)$ making the following diagram commute:*

$$\begin{tikzcd} & P \arrow[dl, dashed, "F"'] \arrow[d, "f"] & \\ M \arrow[r, "\varphi"] & N \arrow[r] & 0 \end{tikzcd}$$

(3) *If P is a quotient of the R-module M, then P is isomorphic to a direct summand of M, i.e., every short exact sequence $0 \to L \to M \to P \to 0$ splits.*

(4) *P is a direct summand of a free R-module.*

Corollary

(1) *Free modules are projective.*

(2) *A finitely generated module is projective if and only if it is a direct summand of a finitely generated free module.*

(3) *Every module is a quotient of a projective module.*

Covariant Functor

Fix D.

- Then given R-module X, $\mathrm{Hom}_R(D, X)$ is an abelian group. So $\mathrm{Hom}_R(D, \_\_)$ behaves like a function.
- Moreover, if $f : X \to Y$ is a R-module homomorphism, then there is an associated group homomorphism $\mathrm{Hom}_R(D, f) : \mathrm{Hom}_R(D, X) \to \mathrm{Hom}_R(D, Y)$.

Roughly speaking, $\mathrm{Hom}_R(D, \_\_)$ maps not only R-modules to abelian groups but also R-module homomorphisms to group homomorphisms. We call this correspondence a covariant functor.

Left Exact Functor

A covariant functor $\mathcal{F}$ is called a *left exact* functor if

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

is an exact sequence, then

$$0 \to \mathcal{F}(X) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(Y) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(Z)$$

is exact. If

$$0 \to \mathcal{F}(X) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(Y) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(Z) \to 0,$$

$\mathcal{F}$ is called an *exact functor*.

Corollary

(1) *For every R-module D, $\mathrm{Hom}_R(D, \_\_)$ is a left exact functor.*

(2) *P is projective module if and only if $\mathrm{Hom}_R(P, \_\_)$ is an exact functor.*

Example

(1) If F is a field, every F-module (F-vector space) is projective.

(2) $\mathbb{Z}$ is a projective $\mathbb{Z}$-module (because it is free). We can show this directly as follows: suppose $f : \mathbb{Z} \to N$ is a $\mathbb{Z}$-module homomorphism and $\varphi : M \to N$ is a surjective homomorphism. f is uniquely determined by $n = f(1)$. Then f can be lifted to a homomorphism $F : \mathbb{Z} \to M$ by $F(1) = m$ where $\varphi(m) = n$.

Example

(3) $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ is not projective for $n \geq 2$. Consider the following short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0.$$

After taking $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \_\_)$, we get

$$0 \to 0 \xrightarrow{n'} 0 \xrightarrow{\pi'} \mathbb{Z}/n\mathbb{Z} \to 0$$

which is not exact at $\mathbb{Z}/n\mathbb{Z}$.

(4) $\mathbb{Q}/\mathbb{Z}$ is not projective.

$$0 \to \mathbb{Z} \to \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0$$

does not split since $\mathbb{Q}$ contains no submodule isomorphic to $\mathbb{Q}/\mathbb{Z}$.

(5) $\mathbb{Z}$-module $\mathbb{Q}$ is not projective.

(6) The direct sum of two projective modules is again projective.

Theorem

Let D,M, and N be R-modules and let $\varphi : M \to N$ be an R-module homomorphism. Then the map

$$\varphi'(f) = f \circ \varphi.$$

is a group homomorphism. If φ is surjective, then φ' is injective, i.e.,

$$\text{if } M \xrightarrow{\varphi} N \to 0 \text{ is exact,}$$

$$\text{then } 0 \to \mathrm{Hom}_R(N, D) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, D) \text{ is also exact.}$$

Theorem

Let D, L, M, and N be R-modules.

(1) *If*

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0 \text{ is exact,}$$

then,

$$0 \to \operatorname{Hom}_R(N, D) \xrightarrow{\varphi'} \operatorname{Hom}_R(M, D) \xrightarrow{\psi'} \operatorname{Hom}_R(L, D) \text{ is exact.}$$

(2) $f : L \to D$ *lifts to* $F : M \to D$ *if and only if* $f \in \operatorname{Im} \psi'$.

(3) ψ' *is surjective if and only if every homomorphism from* L *to* D *lifts to a homomorphism from* M *to* D.

(4) $0 \to \operatorname{Hom}_R(N, D) \xrightarrow{\varphi'} \operatorname{Hom}_R(M, D) \xrightarrow{\psi'} \operatorname{Hom}_R(L, D)$ *is exact for all* D *if and only if* $L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ *is exact.*

Proposition

Let Q be an R-module. Then the following are equivalent:

(1) *For any R-modules L, M, and N, if*

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$$

is a short exact sequence, then

$$0 \to \mathrm{Hom}_R(N, Q) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, Q) \xrightarrow{\psi'} \mathrm{Hom}_R(L, Q) \to 0$$

is also a short exact sequence.

(2) *For any R-modules L and M, if $0 \to L \xrightarrow{\psi} M$ is exact, then every R-module homomorphism from L into Q lifts to an R-module homomorphism of M into Q, i.e., given $f \in \mathrm{Hom}_R(L, Q)$, there is a lift $F \in \mathrm{Hom}_R(M, Q)$ making the following diagram commute:*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\psi"] \arrow[d, "f"'] & M \arrow[dl, dashed, "F"] \\ & Q & \end{tikzcd}$$

(3) *If Q is a submodule of the R-module M, then Q is a direct summand of M, i.e., every short exact sequence $0 \to Q \to M \to N \to 0$ splits.*

Contravariant Functor

Given D, $\mathrm{Hom}_R(\_\_, D)$ has a name, a *contravariant functor*. ('contravariant' means 'direction revsersing').
A contravariant functor $\mathcal{F}$ is called a *left exact* functor if

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

is an exact sequence, then

$$0 \to \mathcal{F}(Z) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(Y) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(X)$$

is exact. If

$$0 \to \mathcal{F}(Z) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(Y) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(X) \to 0,$$

$\mathcal{F}$ is called an *exact functor*.

Corollary

(1) *For every R-module D, $\mathrm{Hom}_R(\_\_, D)$ is a left exact functor.*

(2) *Q is injective module if and only if $\mathrm{Hom}_R(\_\_, Q)$ is an exact functor.*

Definition

- A $\mathbb{Z}$-module A is called *divisible* if $A = nA$ for all nonzero integers n.
- In general, for an integral domain R, a R-module A is called divisible if $A = rA$ for all nonzero $r \in R$.

Example

$\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible.

Proposition

Let Q be an R-module.

(1) *(Baer's Criterion) The module Q is injective if and only if for every left ideal I of R, any R-module homomorphism $g : I \to Q$ can be extended to an R-module homomorphism $G : R \to Q$.*

(2) *If R is a P.I.D.(that is, every ideal is principal), then Q is injective if and only if $rQ = Q$ for every nonzero $r \in R$.*

(3) *In particular, a $\mathbb{Z}$-module is injective if and only if it is divisible.*

(4) *When R is a P.I.D., quotient modules of injective R-modules are again injective.*

Example

(1) Since $\mathbb{Z}$ is not divisible, $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module.

(2) $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

(3) Since $\mathbb{Z}$ is P.I.D and $\mathbb{Q}$ is injective, $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.

(4) A direct sum of divisible $\mathbb{Z}$-modules is again divisible. Hence a direct sum of injective $\mathbb{Z}$-modules is again injective.

(5) Suppose R is an integral domain (that is, $ab = 0$ implies $a = 0$ or $b = 0$). An R-module A is said to be a *divisible* R-module if $rA = A$ for every nonzero $r \in R$. The proof of Proposition 3 shows that an injective R-module is divisible.

(6) In a field F, every F-module is injective.

Corollary

Every $\mathbb{Z}$-module is a submodule of an injective $\mathbb{Z}$-module.

Theorem

Let R be a ring with 1 and let M be an R-module. Then M is contained in an injective R-module.

Step 1

Let M be a left R-module where R is a ring with 1.

(a) Show that $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ is a left R-module under the multiplication $(r\varphi)(r') = \varphi(r'r)$.

(b) Suppose that $0 \to A \xrightarrow{\psi} B$ is an exact sequence of R-modules. Prove that if every homomorphism $f : A \to M$ lifts to a homomorphism $F : B \to M$ with $f = F \circ \psi$, then every homomorphism $f' : A \to \mathrm{Hom}_{\mathbb{Z}}(R, M)$ lifts to a homomorphism $F' : B \to \mathrm{Hom}_{\mathbb{Z}}(R, M)$ with $f' = F' \circ \psi$.

(c) Prove that if Q is an injective R-module, then $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ is also an injective R-module.

Step 2

This exercise proves that every left R-module M is contained in an injective left R-module.

(a) Show that M is contained in an injective $\mathbb{Z}$-module Q.

(b) Show that $\mathrm{Hom}_R(R, M) \subset \mathrm{Hom}_{\mathbb{Z}}(R, M) \subset \mathrm{Hom}_{\mathbb{Z}}(R, Q)$.

(c) Use the R-module isomorphism $M \cong \mathrm{Hom}_R(R, M)$ and the previous exercise to conclude that M is contained in an injective module.

Exercise

Let P_1 and P_2 be R-modules. Prove that $P_1 \oplus P_2$ is a projective R-module if and only if both P_1 and P_2 are projective.

Exercise

Let Q_1 and Q_2 be R-modules. Prove that $Q_1 \oplus Q_2$ is a injective R-module if and only if both Q_1 and Q_2 are injective.

Exercise

This exercise completes the proof of Proposition 2. Suppose that Q is an R-module with the property that every short exact sequence $0 \to Q \to M_1 \to N \to 0$ splits and suppose that the sequence $0 \to L \xrightarrow{\psi} M$ is exact. Prove that every R-module homomorphism $f : L \to Q$ can be lifted to an R-module homomorphism $F : M \to Q$ with $f = F \circ \psi$.

The End