Analysis
- PMA 9 -

KYB

Thrn, it's a Fact

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Overview

The Derivative of Real Function

Definition

- Let f be defined and real-valued on $[a, b]$. For $x \in [a, b]$, form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b, t \neq x),$$

  and define

$$f'(x) = \lim_{t \to x} \phi(t),$$

  provided this limit exists.
- f' whose domain is the set of points of x at which the limit $\lim_{t \to x} \phi(t)$ exists is called the derivative of f.
- If f' is defined at a point x, we say f is differentiable at x, and if f' is defined on at every point of a set $E \subset [a, b]$, we say f is differentiable on E.
- It is possible to consider right-hand and left-hand derivatives.

The Derivative of Real Function

Theorem

Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then f is continuous at x.

Theorem

Suppose f and g are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$. Then $f + g$, fg, and f/g are differentiable at x (for f/g, assume $g(x) \neq 0$), and

(a) $(f + g)'(x) = f'(x) + g'(x)$;

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$;

(c) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$.

The Derivative of Real Function

Theorem (The Chain Rule)

Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If

$$h(t) = g(f(t)) \quad (a \leq t \leq b),$$

then h is differentiable at x, and

$$h'(x) = g'(f(x))f'(x).$$

The Derivative of Real Function

Example

Let f be defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then

$$f'(x) = \begin{cases} \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} & x \neq 0 \\ \text{does not exist} & x = 0. \end{cases}$$

The Derivative of Real Function

Example

Let f be defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Mean Value Theorems

Definition

Let f be a real function defined on a metric space X. We say that f has a local maximum at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$. Similarly, define a local minimum.

Mean Value Theorems

Theorem

Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a, b)$, and $f'(x)$ exists, then $f'(x) = 0$.

Mean Value Theorems

Theorem

If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Theorem

If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which

$$f(b) - f(a) = (b - a)f'(x).$$

Mean Value Theorems

Theorem

Suppose f is differentiable in (a, b).

(a) *If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.*

(b) *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.*

(c) *If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.*

Exercises

Ex 5.1

Let f be defined fro all real x, and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y. Prove that f is constant.

Exercises

Ex 5.2

Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

Exercises

Ex 5.3

Suppose g is a real function on $\mathbb{R}$ with bounded derivative, sat $|g'| \leq M$. Fix $\epsilon > 0$, and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough.

Exercises

Ex 5.5

Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$

Exercises

Ex 5.6

Suppose

(a) f is continuous for $x \geq 0$,

(b) $f'(x)$ exists for $x > 0$,

(c) $f(0) = 0$,

(d) f' is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that g is monotonically increasing.

Exercises

Ex 5.7

Suppose $f'(x)$, $g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

Exercises

Ex 5.8

Suppose f' is continuous on $[a, b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x, t \leq b$.

The Continuity of Derivatives

Theorem

Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

Corollary

If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities on $[a, b]$.

L'Hospital's Rule

Theorem

Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a.$$

If

$$f(x) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to a,$$

of if

$$g(x) \to +\infty \text{ as } x \to a,$$

then

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a.$$

Exercises

Ex 5.11

Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even if $f''(x)$ does not.

Derivatives of Higher Order

Definition

- $f'' = (f')'$.
- $f^{(n)} = (f^{(n-1)})'$.

Taylor's Theorem

Theorem

Suppose f *is a real function on* $[a,b]$, n *is a positive integer,* $f^{(n-1)}$ *is continuous on* $[a,b]$, $f^{(n)}(t)$ *exists for every* $t \in (a,b)$. *Let* α, β *be distinct points of* $[a,b]$, *and define*

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

Then there exists a point x *between* α *and* β *such that*

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n.$$

Exercises

Ex 5.15

Suppose $a \in \mathbb{R}$, f is a twice-differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, on (a, ∞). Prove that

$$M_1^2 \leq 4M_0M_2.$$

Exercises

Ex 5.16

Suppose f is twice-differentiable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

Exercises

Ex 5.17

Suppose f is real, three times differentiable function on $[-1, 1]$, such that

$$f(-1) = 0, f(0) = 0, f(1) = 1, f'(0) = 0.$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$.

Vector-Valued Functions

Remark

We can define the derivative of complex functions defined on $[a, b]$. If $f_1 = \operatorname{Re} f$ and $f_2 = \operatorname{Im} f$, that is, $f(t) = f_1(t) + if_2(t)$ for $a \leq t \leq b$, then we clearly have

$$f'(x) = f_1'(x) + if_2'(x);$$

also, f is differentiable at x if and only if both f_1 and f_2 are differentiable at x.

Vector-Valued Functions

Definition

Let $\mathbf{f} : [a, b] \to \mathbb{R}^k$ be a function. Let $x \in [a, b]$. If $\mathbf{q} \in \mathbb{R}^k$ exists such that

$$\lim_{t \to x} \left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{q} \right| = 0,$$

define $\mathbf{f}'(x) = \mathbf{q}$. Then $\mathbf{f}'$ is a function with values in $\mathbb{R}^k$.

Remark

If $f_1, \cdots, f_k$ are the components of $\mathbf{f}$, then

$$\mathbf{f}' = (f_1', \cdots, f_k'),$$

and $\mathbf{f}$ is differentiable at a point x if and only if each of the functions $f_1, \cdots, f_k$ is differentiable at x.

Vector-Valued Functions

Remark

Suppose $\mathbf{f}$ and $\mathbf{g}$ are functions from $[a, b]$ to $\mathbb{R}^k$ with $\mathbf{f} = (f_1, \cdots, f_k)$ and $\mathbf{g} = (g_1, \cdots, g_k)$. If $\mathbf{f}$ and $\mathbf{g}$ are differentiable at x, $\mathbf{f} \cdot \mathbf{g}$ is also differentiable at x because

$$\mathbf{f} \cdot \mathbf{g} = f_1 g_1 + \cdots + f_k g_k$$

and

$$(\mathbf{f} \cdot \mathbf{g})'(x) = (\mathbf{f}' \cdot \mathbf{g})(x) + (\mathbf{f} \cdot \mathbf{g}')(x)$$

Vector-Valued Functions

The mean value theorem and L'Hospital's rule fail for complex valued functions.

Example

- Define, for real x,
$$f(x) = e^{ix} = \cos x + i \sin x.$$
- On $(0, 1)$, define $f(x) = x$ and
$$g(x) = x + x^2 e^{i/x^2}.$$

Vector-Valued Functions

Remark

However, there is a consequence of the mean value theorem.

$$|f(b) - f(a)| \leq (b-a) \sup_{a<x<b} |f'(x)|.$$

Theorem

Suppose $\mathbf{f}$ *is a continuous mapping of* $[a,b]$ *into* $\mathbb{R}^k$ *and* $\mathbf{f}$ *is differentiable in* (a,b)*. Then there exists* $x \in (a,b)$ *such that*

$$|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b-a)|\mathbf{f}'(x)|.$$

Exercises

Ex 5.13

Suppose a and c are real numbers, $c > 0$, and f is defined on $[-1, 1]$ by

$$f(x) = \begin{cases} x^a \sin(|x|^{-c}) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(1) f is continuous if and only if $a > 0$.

(2) $f'(0)$ exists if and only if $a > 1$.

(3) f' is bounded if and only if $a \geq 1 + c$.

(4) f' is continuous if and only if $a > 1 + c$.

(5) $f''(0)$ exists if and only if $a > 2 + c$.

(6) f'' is bounded if and only if $a \geq 2 + 2c$

(7) f'' is continuous if and only if $a > 2 + 2c$.

Exercises

Ex 5.14

Let f be a differential real function defined in (a, b). Prove that f is convex if and only if f' is monotonically increasing. Assume next that $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.

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