Analysis
- PMA 6 -

KYB

Thrn, it's a Fact

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Overview

Convergent Sequences

Definition

Let $\{p_n\}$ be a sequence in a metric space X and $p \in X$. We say $\{p_n\}$ converges if there is $p \in X$, and for every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies $d(p_n, p) < \epsilon$. In this sace, we say p_n converges to p and write $p_n \to p$ or

$$\lim_{n\to\infty} p_n = p.$$

If $\{p_n\}$ does not converge, it is said to diverge.

Remark

The "convergent sequence" depends not only on $\{p_n\}$ but also on X (and topology of X).

Convergent Sequences

Example

In $\mathbb{R}$, consider two metric

$$d_1(p,q) = |p-q|$$
$$d_2(p,q) = \begin{cases} 1 & p = q \\ 0 & p \neq q \end{cases}.$$

- Then $\{1/n\}$ converges to 0 in d_1
- but diverges in d_2. In fact, a sequence $\{p_n\}$ converges with d_2 if and only if there is $p \in \mathbb{R}$ and N such that $p_n = p$ for all $n \geq N$ (we say $\{p_n\}$ eventually constant).
- In $\mathbb{R}^+$ with d_1, $\{1/n\}$ diverges.

Convergent Sequences

Definition

- The range of $\{p_n\}$ is the set of all p_n.
- A sequence $\{p_n\}$ is bounded if its range is bounded.

Example

In $\mathbb{C}$ (as $\mathbb{R}^2$),

(a) $s_n = 1/n$.

(b) $s_n = n^2$.

(c) $s_n = 1 + [(-1)^n/n]$.

(d) $s_n = i^n$.

(e) $s_n = 1$.

Convergent Sequences

Theorem

Let $\{p_n\}$ be a sequence in a metric space X.

(a) *$\{p_n\}$ converges to $p \in X$ iff every neighborhood of p contains p_n for all but finitely many n.*

(b) *If $p \in X$, $p' \in X$, and if $\{p_n\}$ converges to p and to p', then $p' = p$.*

(c) *If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.*

(d) *If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$.*

Convergent Sequences

Theorem

Suppose $\{s_n\}$, $\{t_n\}$ *are complex sequences, and* $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$. *Then*

(a) $\displaystyle\lim_{n\to\infty}(s_n + t_n) = s + t.$

(b) $\displaystyle\lim_{n\to\infty} cs_n = cs$, $\displaystyle\lim_{n\to\infty}(c + s_n) = c + s$ *for any number* c.

(c) $\displaystyle\lim_{n\to\infty} s_n t_n = st.$

(d) $\displaystyle\lim_{n\to\infty}\frac{1}{s_n} = \frac{1}{s}$ *if* $s_n \neq 0$ *and* $s \neq 0$.

Convergent Sequences

Theorem

(a) *Suppose* $\mathbf{x}_n \in \mathbb{R}^k$ *and* $\mathbf{x}_n = (\alpha_{1,n}, \cdots, \alpha_{k,n})$. *Then* $\{\mathbf{x}_n\}$ *converges to* $\mathbf{x} = (\alpha_1, \cdots, \alpha_k)$ *if and only if*

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j, \; j = 1, 2, \cdots, k.$$

(b) *Suppose* $\{\mathbf{x}_n\}$, $\{\mathbf{y}_n\}$ *are sequences in* $\mathbb{R}^k$, $\{\beta_n\}$ *is a sequence of real numbers, and* $\mathbf{x}_n \to \mathbf{x}$ *and* $\mathbf{y}_n \to \mathbf{y}$, $\beta_n \to \beta$. *Then*

$$\lim_{n\to\infty} (\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y}, \; \lim_{n\to\infty} \mathbf{x}_n \cdot \mathbf{y}_n = \mathbf{x} \cdot \mathbf{y}, \; \lim_{n\to\infty} \beta_n \mathbf{x}_n = \beta \mathbf{x}$$

Subsequences

Definition

Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < \cdots$.

- Then the sequence $\{p_{n_i}\}$ is called a subsequence of $\{p_n\}$.
- If $\{p_{n_i}\}$ converges, its limit is called a subsequential limit of $\{p_n\}$.

Subsequences

Theorem

(a) *If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X. (i.e., a compact space is sequentially compact.)*

(b) *Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.*

Subsequences

Theorem

The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.

Cauchy Sequences

Definition

A sequence $\{p_n\}$ in a metric space X is said to be a Cauchy sequence if for every $\epsilon > 0$ there is an integer N such that $d(p_n, p_m) < \epsilon$ if $n, m \geq N$.

Definition

Let E be a nonempty subset of a metric space X, and let S be the set of all real numbers of the form $d(p, q)$, with $p, q \in E$. The sup of S is called the diameter of E.

Remark

If $\{p_n\}$ is a sequence in X and if $E_N = \{p_N, p_{N+1}, \cdots, \}$,

- $\{p_n\}$ is a Cauchy sequence if and only if $\lim_{N\to\infty} \operatorname{diam} E_N = 0$.

Cauchy Sequences

Theorem

(a) *If $\overline{E}$ is the closure of a set E in a metric space X, then*

$$\operatorname{diam} \overline{E} = \operatorname{diam} E.$$

(b) *If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1}$ and if*

$$\lim_{n\to\infty} \operatorname{diam} K_n = 0,$$

then $\bigcap_1^\infty K_n$ consists of exactly one point.

Cauchy Sequences

Theorem

(a) *In any metric space X, every convergent sequence is a Cauchy sequence.*

(b) *If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X.*

(c) *In $\mathbb{R}^k$, every Cauchy sequence converges.*

Definition

A metric space in which every Cauchy sequence converges is said to be complete.

Cauchy Sequences

Definition

A sequence $\{s_n\}$ of real numbers is said to be

(a) monotonically increasing if $s_n \leq s_{n+1}$ for all n;

(b) monotonically decreasing if $s_n \geq s_{n+1}$ for all n;

Theorem

Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.

Upper and Lower Limits

Definition

Let $\{s_n\}$ be a sequence of real numbers with the following property:

- For every real M, there is an integer N such that $n \geq N$ implies $s_n \geq M$. We then write
$$s_n \to +\infty.$$
- Similarly, if for every real M, there is an integer N such that $n \geq N$ implies $s_n \leq M$. We then write
$$s_n \to -\infty.$$

Upper and Lower Limits

Definition

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers x (in the extended real number system) such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$. Define

$$s^* = \sup E, \;\; s_* = \inf E.$$

s^* is called the upper limit and s_* is called the lower limit of $\{s_n\}$; we use the notation

$$\limsup_{n\to\infty} s_n = s^*, \;\; \liminf_{n\to\infty} s_n = s_*.$$

Remark

$\limsup$ and $\liminf$ always exist.

Upper and Lower Limits

Theorem

Let $\{s_n\}$ be a sequence of real numbers. Let E and s^* have the same meaning as in the above definition. Then s^* has the following two properties:

(a) $s^* \in E$.

(b) *If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$.*

Moreover, s^* is the only number with the properties (a) and (b).

Upper and Lower Limits

Example

(a) Let $\{s_n\}$ be a sequence containing all rationals. Then every real number is a subsequential limit, and

$$\limsup_{n\to\infty} s_n = +\infty, \ \liminf_{n\to\infty} s_n = -\infty$$

(b) Let $s_n = (-1)^n/[1+(1/n)]$. Then

$$\limsup_{n\to\infty} s_n = 1, \ \liminf_{n\to\infty} s_n = -1.$$

(c) For a real-valued sequence $\{s_n\}$, $\lim_{n\to\infty} s_n = s$ if and only if

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s.$$

Upper and Lower Limits

Theorem

If $s_n \le t_n$ for all $n \ge N$, where N is fixed, then

$$\liminf_{n\to\infty} s_n \le \liminf_{n\to\infty} t_n,$$
$$\limsup_{n\to\infty} s_n \le \limsup_{n\to\infty} t_n,$$

Some Special Sequences

Remark

If $0 \le x_n \le s_n$ for $n \ge N$, where N is some fixed number, and if $s_n \to 0$, then $x_n \to 0$.

Theorem

(a) *If* $p > 0$, *then* $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(b) *If* $p > 0$, *then* $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(d) *If* $p > 0$ *and* α *is real, then* $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

(e) *If* $|x| < 1$, *then* $\lim_{n\to\infty} x^n = 0$.

Series

Definition

Given a sequence $\{a_n\}$, we use the notation

$$\sum_{n=p}^{q} a_n = a_p + a_{p+1} + \cdots + a_q \quad (p \leq q).$$

Consider a sequence $\{s_n\}$ where

$$s_n = \sum_{k=1}^{n} a_k.$$

Write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots .$$

We call $\sum_1^\infty a_n$ a series, and $\sum_1^n a_k$ the partial sums of the series. If $\{s_n\}$ converges to s, we say the series converges and write

$$\sum_{n=1}^{\infty} a_n = s.$$

Series

Theorem

$\sum a_n$ converges if and only if for every $\varepsilon > 0$, there is an integer N such that

$$\left| \sum_{k=n}^{m} a_k \right| \leq \varepsilon$$

if $m \geq n \geq N$.

Theorem

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Theorem

A series of nonnegative terms converges if and only if its partial sums form a bounded sequence.

Series

Theorem (Comparison Test)

(a) *If* $|a_n| \leq c_n$ *for* $n \geq N_0$, *where* N_0 *is some fixed integer, and if* $\sum c_n$ *converges, then* $\sum a_n$ *converges.*

(b) *If* $a_n \geq d_n \geq 0$ *for* $n \geq N_0$, *and if* $\sum d_n$ *diverges, then* $\sum a_n$ *diverges.*

Series of Nonnegative Terms

Theorem

If $0 \leq x < 1$, *then*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

If $x \geq 1$, *then the series diverges.*

Series of Nonnegative Terms

Theorem

Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. *Then the series* $\sum_{n=1}^{\infty} a_n$ *converges if and only if the series*

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converges.

Series of Nonnegative Terms

Theorem

$\sum \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

Theorem

If $p > 1$,

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges; if $p \leq 1$, *the series diverges.*

The Number e

Definition

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

$n! = 1 \cdot 2 \cdot 3 \cdots n$ if $n \geq 1$, and $0! = 1$.

Remark

$$\begin{aligned} s_n &= 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots + \frac{1}{1 \cdot 2 \cdots n} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3. \end{aligned}$$

The Number e

Theorem

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

The Number e

Theorem

e is irrational.

The Number e

Theorem

e is not algebraic.

The Root and Ratio Tests

Theorem (Root Test)

Given $\sum a_n$, *put* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. *Then*

(a) *if* $\alpha < 1$, $\sum a_n$ *converges;*

(b) *if* $\alpha > 1$, $\sum a_n$ *diverges;*

(c) *if* $\alpha = 1$, *the test gives no information.*

The Root and Ratio Tests

Theorem (Ratio Test)

The series $\sum a_n$

(a) *converges if* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(b) *diverges if* $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ *for all* $n \geq n_0$, *where* n_0 *is some fixed integer.*

The Root and Ratio Tests

Theorem

For any sequence $\{c_n\}$ of positive numbers,

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n}$$
$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$$

Exercises

Ex 3.1

If $\{s_n\}$ converges, so does $\{|s_n|\}$. Is the converse true?

Exercises

Ex 3.3

Let $s_1 = \sqrt{2}$ and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}}.$$

Show that s_n converges and $s_n < 2$ for all n.

Exercises

Ex 3.4

Find $\limsup$ and $\liminf$ of the following sequence:

$$s_1 = 0;\ s_{2m} = \frac{s_{2m-1}}{2};\ s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Exercises

Ex 3.5

For any sequences $\{a_n\}$, $\{b_n\}$, show that

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

where the sum on the right is not of the form $\infty - \infty$.

Exercises

Ex 3.7

If $a_n > 0$ and $\sum a_n$ converges, then $\sum \frac{\sqrt{a_n}}{n}$ converges.

Exercises

Ex 3.11

Suppose $a_n > 0$ and $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(a) $\sum \frac{a_n}{1+a_n}$ diverges.

Exercises

Ex 3.11

Suppose $a_n > 0$ and $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(b) Show that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}},$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

Exercises

Ex 3.11

Suppose $a_n > 0$ and $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(c) Show that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n},$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

Exercises

Ex 3.11

Suppose $a_n > 0$ and $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(d) What can we said about

$$\sum \frac{a_n}{1+na_n}, \sum \frac{a_n}{1+n^2a_n}?$$

Exercises

Ex 3.12

Suppose $a_n > 0$ and $\sum a_n$ converges. Put $r_n = \sum_{m=n}^{\infty} a_m$.

(a) Prove that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

Exercises

Ex 3.12

Suppose $a_n > 0$ and $\sum a_n$ converges. Put $r_n = \sum_{m=n}^{\infty} a_m$.

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

The End