LA11 Ch4

KYB

Thrn, it's a Fact

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Overview

Eigenobjects

For $A \in F^{n \times n}$, $\lambda \in F$ and nonzero vector $x \in F^n$ are called an eigenpair if $Ax = \lambda x$.

If λ is an eigenvalue, $A - \lambda I$ is singular, and thus $\det(\lambda I - A) = 0$.

$p_A(r) := \det(rI - A)$ is called the chracteristic polynomial.

Some results

For eigenvalue λ of A, $E_\lambda(A)$ is a vector space of dimension k. Thus there are eigenvectors $\{x_1, \cdots, x_k\}$ of λ such that they are linearly independent.
k is called the geometric multiplicity.

Let λ_j be distinct eigenvalue of A and let $\mathcal{B}_j = \{x_i^{(j)}\}_{i=1}^{n_j}$ be a basis for $E_{\lambda_j}(A)$. Then $\mathcal{B} = \cup \mathcal{B}_j$ is linearly independent.

Similar

Let $A, B \in F^{n \times n}$. We say A and B are similar if there is invertiable $X \in F^{n \times n}$ such that $B = X^{-1}AX$.
A relation $\sim$ such that $A \sim B$ iff A and B are similar forms an equivalence relation.

Diagonalizable

If A is similar to a diagonal matrix D, we say A is diagonalizable.

Some results

If A and B are similar,

1. $p_A(r) = p_B(r)$
2. $\det(A) = \det(B)$
3. $\mathrm{tr}(A) = \mathrm{tr}(B)$
4. A and B have the same eigen valuses
5. m.geo and m.alg of an eigenvalue λ are the same whether λ is regared as an eigenvalue of A or B.

m.geo $\leq$ m.alg

Some results

Let F be an algebraically closed field. Let $A \in F^{n\times n}$. Then A is Diagonalizable if and only if m.geo$(A) =$ m.alg(A).

Let F be any field and $A \in F^{n\times n}$. If A has n distinct eigenvalues, then A is diagonalizable.

$\{\lambda_1, \dots, \lambda_n\}$(may not distinct) with $\{x_1, \cdots, x_n\}$. Define $X = [x_1|\cdots|x_n]$ and $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. Then

$$
\begin{aligned}
AX &= \left[Ax_1 \,\middle|\, \cdots \,\middle|\, Ax_n\right] = \left[\lambda_1 x_1 \,\middle|\, \cdots \,\middle|\, \lambda_n x_n\right] \\
&= \left[x_1 \,\middle|\, \cdots \,\middle|\, x_n\right] \left[\begin{array}{c|c|c} \lambda_1 & \cdots & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & \cdots & \lambda_n \end{array}\right] = XD
\end{aligned}
$$

Ex4.6.18

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and $r = 1$, $x = (0, 1, 0)$ form an eigenpair of A.

(a) Write $x_1 = x$ and extend $\{x_1\}$ to a basis $\{x_1, x_2, x_3\}$ for $\mathbb{R}^3$.

Proof.

(1) $x_2 = (1, 0, 0)$, $x_3 = (0, 0, 1)$.

(2) $x_2 = (5, 1, 3)$, $x_3 = (4, 3, 2)$. $\square$

Ex4.6.18

(b) Define $X = [x_1|x_2|x_3]$ and compute the matrix $B = X^{-1}AX$. What are the vector v and the matrix C?

$$B = \left[\begin{array}{c|c} \lambda & v \\ \hline 0 & C \end{array}\right]$$

Proof.

(1) $x_2 = (1,0,0)$, $x_3 = (0,0,1)$.

$$\begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

Proof.

(2) $x_2 = (5,1,3)$, $x_3 = (4,3,2)$.

$$\begin{bmatrix} 1 & -22 & -18 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

Ex4.6.18

(c) Verify that $e_1 = (1, 0, 0)$ is an eigenvector of B corresponding to the eigenvalue $r = 1$.

(d) Find another eigenvector z of B, where the first component of z is zero. Let

$$z = \begin{bmatrix} 0 \\ \hline u \end{bmatrix}$$

where $u \in \mathbb{R}^2$. Verify that u is an eigenvector of C corresponding to the eigenvalue $r = 1$.

(1) $u = (1, 5)$.

(2) $u = (-18, 22)$.

Ex4.6.18

(e) Find another eigenvector $v \in \mathbb{R}^2$ of C, so that $\{u, v\}$ is linearly independent. Write

$$w = \begin{bmatrix} 0 \\ \hline v \end{bmatrix}.$$

Is w another eigenvector of B?

(1),(2) $v = (1, 1)$.

Ex4.7.3

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be definded by

$$T(x) = (x_1 + x_2 + x_3, ax_2, x_1 + x_3),$$

where $a \in \mathbb{R}$ is a constant. For which values of a does there exist a basis $\mathcal{X}$ for $\mathbb{R}^3$ such that $[T]_{\mathcal{X},\mathcal{X}}$ is diagonal?

Proof.

1) $p_T(r) = r(r-2)(r-a)$.
2) $T(1,0,-1) = (0,0,0)$ and $T(1,0,1) = (2,0,2)$.
3) $(a-1, a^2-2a, 1)$ is an eigenvector for a.($E_a(A)$ has nullity 1.) $\square$

Ex4.7.13

X fin.dim over $\mathbb{C}$ with basis $\mathcal{B} = \{u_1, \cdots, u_k, v_1, \cdots, v_l\}$. Let $U = \text{span}\{u_1, \cdots, u_k\}$ and $V = \text{span}\{v_1, \cdots, v_l\}$. Let $T : X \to X$ be linear s.t. $T(U) \subset U$.

- There exists an eigenvector of T belonging to U.
-
$$[T]_{\mathcal{X},\mathcal{X}} = \left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right]$$
- Show that if V is also invariant under T, then $[T]_{\mathcal{X},\mathcal{X}}$ is block diagonal.

Proof.

$\square$

Goal

$$B_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}$$

$$A = \left[\begin{array}{c|c|c|c} B_1 & & & \\ \hline & B_2 & & \\ \hline & & \ddots & \\ \hline & & & B_k \end{array}\right]$$

Definition (Invariant subspaces)

X v.sp over F. $T : X \to X$ linear and S subspace of X. We say S is invariant under T iff $T(S) \subset S$.

Example

$S = E_\lambda(T)$, then S is invariant under T.

$A \in F^{n\times n}$, S subspace of F^n invariant under A. Choose a basis $\{x_1, \cdots, x_k\}$ for S and extend to $\{x_1, \cdots, x_n\}$ for F^n. Then

$$
\begin{aligned}
AX &= [Ax_1|\cdots|Ax_n] \\
&= |\cdots|Ax_n] = [AX_1|AX_2] \\
AX_1 &= [\sum_1^k a_{1,i}x_i|\cdots|\sum_1^k a_{k,i}x_i] \\
&= [X_1B_1|\cdots|X_1B_k] \\
&= X_1B = X_1B + X_2O = [X_1|X_2]\left[\frac{B}{O}\right]
\end{aligned}
$$

$$
\begin{aligned}
AX_2 &= [\sum_1^n a_{k+1,i}x_i | \cdots | \sum_1^n a_{n,i}x_i] \\
&= [\sum_1^k a_{k+1,i}x_i + \sum_{k+1}^n a_{k+1,i}x_i | \cdots | \sum_1^k a_{n,i}x_i + \sum_{k+1}^n a_{n,i}x_i] \\
&= [X_1 | X_2] \left[\begin{array}{c} C \\ \hline D \end{array}\right] \\
AX &= X \left[\begin{array}{c|c} B & C \\ \hline O & D \end{array}\right]
\end{aligned}
$$

Definition (Direct Sums)

Let V be a v.sp over F and S, T be a subspace of V. We say V is the direct sum of S and T, denoted by $V = S \oplus T$, iff $V = S + T$ and $S \cap T = \{0\}$.
More generally, if $S_1, \ldots, S_t$ are subspaces of V such that $V = \sum S_i$ and $S_i \cap \sum_{i \neq j} S_j = \{0\}$ for any i, $V = \bigoplus S_i$

Counter Example

If $V = \sum S_i$ and $S_i \cap S_j = \{0\}$ for $i \neq j$, may not $V = \bigoplus S_i$.
For example, consider $S_1 = \text{span}\{e_1\}, S_2 = \text{span}\{e_2 + e_3\}, S_3 = \text{span}\{e_1 + e_2 + e_3\}$ in $\mathcal{R}^3$.

Theorem

$A \in F^{n\times n}$, define $N = \mathcal{N}(A)$. If there exists a subspace R of F^n such that R is invariant under A and $F^n = N \oplus R$, ther $R = col(A)$.

Corollary

TFAE.

1. *there exists a subspace R of F^n such that R is invariant under A and $F^n = N \oplus R$*
2. *$\mathcal{N}(A) \cap col(A) = \{0\}$, in which case $R = col(A)$.*

Theorem

$\mathcal{N}(A) \cap col(A) = \{0\}$ if and only if $\mathcal{N}(A^2) = \mathcal{N}(A)$.

Lemma

$A \in F^{n\times n}$, $\lambda \in F$. If S is a subspace of F^n, then S is invariant under A iff S is invariant under $A - \lambda I$.

Ex5.1.9

U fin.dim.v.sp over F, $T : U \to U$ linear. Let $\mathcal{U} = \{u_1, \cdots, u_n\}$ be a basis for U and define $A = [T]_{\mathcal{U},\mathcal{U}}$. Suppose $X \in F^{n\times n}$ is an invertiable matrix, and define $J = X^{-1}AX$. For each $j = 1, \cdots n$, define

$$v_j = \sum_1^n X_{ij} u_i.$$

(a) Prove that $\mathcal{V} = \{v_1, \cdots, v_n\}$ is a basis for U.
(b) Prove that $[T]_{\mathcal{V},\mathcal{V}} = J$.

Ex5.1.9

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(b) Prove that $[T]_{\mathcal{V},\mathcal{V}} = J$.

Ex5.1.11

Let V be a finite-dimensional vector space over F, suppose $\{x_1, \cdots, x_n\}$ be a basis for V, and $1 \leq k \leq n-1$. Prove that $V = S \oplus T$ where $S = \text{span}\{x_1, \cdots, x_k\}$ and $T = \text{span}\{x_{k+1}, \cdots, x_n\}$.

Ex5.1.13

$A \in F^{n\times n}$, N is a subspace of F^n that is invariant under A and $F^n = N \oplus \text{col}(A)$. Prove that N must be $\mathcal{N}(A)$.

Ex5.1.14

Suppose A is Diagonalizable. Let $\lambda_1, \cdots, \lambda_t$ be the all distinct eigenvalues. Prove that

$$\mathrm{col}(A - \lambda_1 I) = E_{\lambda_2}(A) + \cdots + E_{\lambda_t}(A)$$

Main Idea

For $\mathbb{C}^{n\times n}$, let λ be an eigenvalue of A. Then

$$\mathcal{N}(A-\lambda I) \subset \mathcal{N}((A-\lambda I)^2) \subset \mathcal{N}((A-\lambda I)^3) \cdots$$

is stable.

$\mathcal{N}(A-\lambda I)$	$\mathcal{N}((A-\lambda I)^2)$	$\mathcal{N}((A-\lambda I)^3)$
$(A-\lambda I)^2 x_1$	$(A-\lambda I)x_1$	x_1
$(A-\lambda I)x_2$	x_2	
$(A-\lambda I)x_3$	x_3	

남은 내용

1. 5.2 Generalized Eigenspaces
   $G_\lambda(A) = \mathcal{N}((A - \lambda I)^k)$.
2. 5.3 Nilpotent Operators
   $T : V \to V$ such that $T^k = 0$.
3. 5.4 The Jordan Canonical Form of a matrix

$$A = \left[\begin{array}{c|c|c|c} B_1 & & & \\ \hline & B_2 & & \\ \hline & & \ddots & \\ \hline & & & B_k \end{array}\right]$$

The End