Modules

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Overview

Modules

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Motivation
Construct

Observe

Suppose R is a subring of S with $1_R = 1_S$. Then S is an R-module. If N is an S-module, then N can also be naturally considered as an R-module:

- $(r_1+r_2)n=r_1n+r_2n$ and $r(n_1+n_2)=rn_1+rn_2$;
- $ightharpoonup (r_1r_2)n = r_1(r_2n).$

In general, if $f:R\to S$ is a ring homomorphism with $f(1_R)=1_S$, an S-module N can be considered as an R-module with $r\cdot n=f(r)n$. In this case,

- \triangleright S can be considered as an extension of R;
- ▶ the resulting R-module is said to be obtained from N by restriction of scalars from S to R.

What about the converse?

Example

Consider $\mathbb Z$ as a $\mathbb Q$ -module. Then there is $z\in\mathbb Z$ such that $\frac12\cdot 1=z$. But there is no z in $\mathbb Z$ such that $z+z=\frac12\cdot 1+\frac12\cdot 1=1\cdot 1=1$. So a $\mathbb Z$ -module $\mathbb Z$ cannot be made into a $\mathbb Q$ -module.

Nevertheless, $\mathbb Z$ is contained in a $\mathbb Q$ module, namely $\mathbb Q$ itself. In this sense, we can extend scalar :

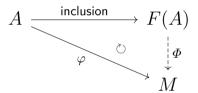
- ▶ for given R-module N, if there is an S-module M such that $N \subset M$, we can identify M as an extension of N.
- More generally, if there is an S-module with an R-module homomorphism $\varphi: N \to M$, by taking $r \cdot n = rf(n)$, we can identify M as an extension of N.

So our first goal is the existence of such S-module M.

Recall, the Universal Property of Free Modules

For any set A there is a free R-module F(A) on the set A and F(A) satisfies the following universal property:

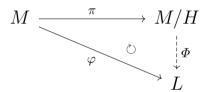
▶ if M is any R-module and $\varphi:A\to M$ is any map of sets, then there is a unique R-module homomorphism $\Phi:F(A)\to M$ such that $\Phi(a)=\varphi(a)$ for all $a\in A$, that is, the following diagram commutes.



Theorem (The Universal Property of Quotient modules)

Let R be a ring with 1 and let M be an R-module and let H be a submodule of M. Then $\pi:M\to M/H$ by $\pi(m)=m+H$ is a surjective R-module homomorphism.

Suppose L is an R-module and $\varphi:M\to L$ is an R-module homomorphism with $H\subset \mathrm{Ker} \varphi$. Then there is a unique R-module homomorphism $\Phi:M/H\to L$ such that φ factors through Φ , i.e. $\varphi=\Phi\circ\pi$ and the diagram commutes.



Proof

Define $\Phi(m+H)=\varphi(m).$ It suffices to show that Φ is well defined.

Tensor Products

Construction

Suppose M is a 'right' R-module and N is a 'left' R-module. Consider $F(M \times N)$ as the free \mathbb{Z} -module. Let H be the subgroup of $F(M \times N)$ generated by all elements of the form

$$(m_1 + m_2, n) - (m_1, n) - (m_2, n),$$

 $(m, n_1 + n_2) - (m, n_1) - (m, n_2),$
 $(mr, n) - (m, rn).$

Then $F(M \times N)/H$ is an abelian group, denoted by $M \otimes_R N$.

- ▶ $M \otimes_R N$ is called the tensor product of M and N over R.
- ▶ The elements of $M \otimes_R N$ is called tensors.
- ▶ The coset $m \otimes n$ of (m, n) in $M \otimes_R N$ is called a simple tensor.

Remark

▶ We have the relations

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n,$$

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2,$$

$$(mr) \otimes n = m \otimes (rn).$$

- Every tensor is a finite sum of simple tensors, but a tensor may not be simple.
- ▶ We can define a map $\iota: M \times N \to M \otimes_R N$ by $\iota(m,n) = m \otimes n$. ι satisfies

$$\iota((m_1 + m_2), n) = \iota(m_1, n) + \iota(m_2, n),$$

$$\iota(m, (n_1 + n_2)) = \iota(m, n_1) + \iota(m, n_2),$$

$$\iota((mr), n) = \iota(m, (rn)).$$

Definition

Let M be a right R-module and N be a left R-module and let L be an abelian group. A map $\varphi:M\times N\to L$ is called R-balanced if

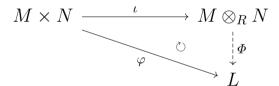
$$\varphi((m_1 + m_2), n) = \varphi(m_1, n) + \varphi(m_2, n),$$

$$\varphi(m, (n_1 + n_2)) = \varphi(m, n_1) + \varphi(m, n_2),$$

$$\varphi((mr), n) = \varphi(m, (rn)).$$

Suppose R is a ring with 1, M is a right R-module, and N is a left R-module. Then $\iota: M \times N \to M \otimes_R N$ is an R-balanced map.

- (1) If $\Phi: M \otimes_R N \to L$ is any group homomorphism from $M \otimes_R N$ to an abelian group L, then the composition map $\varphi = \Phi \circ \iota$ is an R-balanced map from $M \times N \to L$.
- (2) Conversely, suppose L is an abelian group and $\varphi: M \times N \to L$ is any R-balanced map. Then there is a unique group homomorphism $\Phi: M \otimes_R N \to L$ such that φ factors through ι , i.e., $\varphi = \Phi \circ \iota$ and the diagram commutes.



Equivalently, the correspondence $\varphi \leftrightarrow \Phi$ is bijection.

 $\{R\text{-balanced maps}\} \leftrightarrow \{\text{group homomorphisms}\}$

Proof (2) Uniqueness.

Proof (2) Existence.

Remark

This theorem is extremely useful in defining homomorphisms on $M \otimes_R N$. If we define an R-balanced map, then automatically we can define a homomorphism on $M \otimes_R N$.

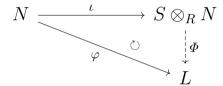
Example

Let R be a subring of a ring S with $1_R=1_S$ and let N be a left R-module. As right R-module S, we can find a map $N\to S\otimes_R N$ by $n\mapsto 1\otimes n$. Moreover, if we define $s(\sum s_i\otimes n_i)=\sum (ss_i)\otimes n_i,\ S\otimes_R N$ is S-module (so R-module). If we write $\iota:N\to S\otimes_R N$ by $\iota(n)=1\otimes n,\ \iota$ is an R-module homomorphism. In general, ι is not injective. In this way, we can consider $S\otimes_R N$ as an extension of scalar of N.

Corollary

Suppose R is a subring of a ring S with $1_R = 1_S$, let N be a left R-module and let $\iota: N \to S \otimes_R N$ be the R-module homomorphism defined by $\iota(n) = 1 \otimes n$.

▶ Suppose L is a left S-module and $\varphi: N \to L$ is an R-module homomorphism. Then there is a unique S-module homomorphism $\Phi: S \otimes_R N \to L$ such that φ factors through Φ , i.e. $\varphi = \Phi \circ \iota$ and the diagram commutes.



Remark

Let $N = \mathbb{Z}/2\mathbb{Z}$. Consider $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. For $q \otimes n \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$,

$$q \otimes n = \frac{q}{2} \otimes 2n = \frac{q}{2} \otimes 0.$$

Since for any $q \in \mathbb{Q}$,

$$q \otimes 0 = q \otimes (0+0) = q \otimes 0 + q \otimes 0.$$

Thus $q \otimes 0 = 0$. Since $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is generated by $q \otimes n = 0$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$.

Example

- (1) For any ring R and any left R-module N, $R \otimes_R N \cong N$.
- (2) If $N \cong \mathbb{R}^n$, $S \otimes_{\mathbb{R}} N \cong \mathbb{S}^n$.

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