Modules

KYB

Thrn, it's a Fact

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Overview

Recall

If R is a subring of a ring S with $1_R = 1_S$, given R-module N, $S \otimes_R N$ is a S-module.
Now we want to obtain an S-module structure on $M \otimes_R N$.

Definition

Let R and S be rings with 1. An abelian group M is called an (S, R)-bimodule if M is a left S-module, a right R-module, and $s(mr) = s(mr)$ for all $s \in S$, $r \in R$, and $m \in M$.

Example

1. Any ring S is an (S, R)-bimodule for any subring R with $1_R = 1_S$.
2. More generally, if $f : R \to S$ is any ring homomorphism with $f(1_R) = 1_S$, then S can be considered as a right R-module with multiplication $s \cdot r = sf(r)$, and becomes an (S, R)-bimodule.
3. Let I be an ideal in the ring R. Then the quotient ring R/I is an $(R/I, R)$-bimodule; $\pi : R \to R/I$ by $\pi(r) = r + I$ is a ring homomorphism with $\pi(1) = 1$.
4. Suppose that R is a commutative ring. Then any left(right) R-module M can always be given the structure of a right(left) R-module by defining $mr = rm$.

Definition

Suppose M is a left (or right) R-module over the commutative ring R. Then the (R, R)-bimodule structure on M defined by letting the left and right R-multiplication coincide, i.e., $mr = rm$ for all $m \in M$ and $r \in R$, will called the standard R-module structure on M.

Remark

Suppose N is a left R-module and M is an (S, R)-bimodule. Give a multiplication by $s(\sum m_i \otimes n_i) = \sum (sm_i) \otimes n_i$. This is a well-defined map from $S \times (M \otimes_R N) \to M \otimes_R N$ and induces an S-module structure on $M \otimes_R N$:

- Given $s \in S$, $(m, n) \mapsto (sm) \otimes n$ is a R-balanced map. Thus
$$(s, m \otimes n) \mapsto (sm) \otimes n$$
is well defined.
- Consider $\iota : M \times N \to M \otimes_R N$ such that $\iota(m, n) = m \otimes n$. ι satisfies
$$\begin{aligned} \iota(m_1 + m_2, n) &= \iota(m_1, n) + \iota(m_2, n), \\ \iota(m, n_1 + n_2) &= \iota(m, n_1) + \iota(m, n_2), \\ r\iota(m, n) &= \iota(rm, n) = \iota(m, rn). \end{aligned}$$

Definition

Let R be a commutative ring with 1 and let M, N, and L be left R-modules. The map $\varphi : M \times N \to L$ is called R-bilinear if it is R-linear in each factor, i.e., if

$$\varphi(r_1m_1 + r_2m_2n) = r_1\varphi(m_1, n) + r_2\varphi(m_2, n),$$
$$\varphi(m, r_1n_1 + r_2n_2) = r_1\varphi(m, n_1) + r_2\varphi(m, n_2)$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, and $r_1, r_2 \in R$.

Corollary

Suppose R is a commutative ring. Let M and N be two left R-modules and let $M \otimes_R N$ be the tensor product of M and N over R, where M is given the standard R-modules structure. Then $M \otimes_R N$ is a left R-module with

$$r(m \otimes n) = (rm) \otimes n = m \otimes (rn),$$

and the map $\iota : M \times N \to M \otimes_R N$ with $\iota(m,n) = m \otimes n$ is an R-bilinear map. If L is any left R-module, then there is a bijection

$$\{R\text{-bilinear maps } \varphi : M \times N \to L\} \leftrightarrow \{R\text{-module homomorphisms } \Phi : M \otimes_R N \to L\}$$

where the correspondence between φ and Φ is given by the commutative diagram.

$$\begin{tikzcd}
M \times N \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & M \otimes_R N \arrow[d, dashed, "\Phi"] \arrow[dl, phantom, "\circlearrowright"] \\
& L
\end{tikzcd}$$

Proof

Example

1. In any tensor product $M \otimes_R N$, we have $m \otimes 0 = 0 \otimes n = 0$.
2. $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$. So there are no nonzero balanced maps from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ to any abelian group.
3. $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.
4. In general, $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ where $d = \gcd(m, n)$.

Example

5. $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0.$
6. $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Q}$-modules.
7. $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \ncong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ as $\mathbb{C}$-modules.

Example

8. General extension of scalars or change of base: Let $f : R \to S$ be a ring homomorphism with $f(1_R) = 1_S$. Then $s \cdot r = sf(r)$ gives S the structure of a right R-module with respect to which S is an (S, R)-bimodule. Then for any left R-module N, the resulting tensor product $S \otimes_R N$ is a left S-module obtained by changing the base from R to S.
9. Let $f : R \to S$ be a ring homomorphism as in the preceding example. Then we have $S \otimes_R R \cong S$ as left S-modules.

Example

10. Let R be a ring (not necessarily commutative), let I be a two sided ideal in R, and let N be a left R-module. Then as previously mentioned, R/I is an $(R/I, R)$-bimodule, so the tensor product $R/I \otimes_R N$ is a left R/I-module. Define

$$IN = \left\{ \sum_{\text{finite}} a_i n_i : a_i \in I, n_i \in N \right\},$$

which is easily seen to be a left R-submodule of N. Then

$$(R/I) \otimes_R N \cong N/IN.$$

Theorem (The "Tensor Product" of Two Homomorphisms)

Let M, M' be right R-modules, let N, N' be left R-modules, and suppose $\varphi : M \to M'$ and $\psi : N \to N'$ are R-module homomorphisms.

1. *There is a unique group homomorphism, denoted by $\varphi \otimes \psi$, mapping $M \otimes_R N$ into $M' \otimes_R N'$ such that $(\varphi \otimes \psi)(m \otimes n) = \varphi(m) \otimes \psi(n)$ for all $m \in M$ and $n \in N$.*
2. *If M, M' are also (S, R)-bimodules for some ring S and φ is also an S-module homomorphism, then $\varphi \otimes \psi$ is a homomorphism of left S-modules. In particular, if R is commutative, then $\varphi \otimes \psi$ is always an R-module homomorphism for the standard R-module structures.*
3. *If $\lambda : M' \to M''$ and $\mu : N' \to N''$ are R-module homomorphisms, then*

$$(\lambda \otimes \mu) \circ (\varphi \otimes \psi) = (\lambda \circ \varphi) \otimes (\mu \circ \psi).$$

Theorem (Associativity of the Tensor Products)

Suppose M is a right R-module, N is an (R,T)-bimodule, and L is a left T-module. Then there is a unique isomorphism

$$(M \otimes_R N) \otimes_T L \cong M \otimes_R (N \otimes_T L)$$

of abelian groups such that $(m \otimes n) \otimes l \mapsto m \otimes (n \otimes l)$. If M is an (S,R)-bimodule, then this is an isomorphism of S-modules.

Corollary

Suppose R is commutative and M, N, and L are left R-modules. Then

$$(M \otimes N) \otimes L \cong M \otimes (N \otimes L)$$

as R-modules for the standard R-modules structures on M, N, and L.

Proof

Definition

Let R be a commutative ring with 1, and let $M_1, \cdots, M_n$ and L be R-modules with the standard R-module structures. A map $\varphi : M_1 \times \cdots \times M_n \to L$ is called n-multilinear over R if it is an R-module homomorphism in each component when the other component entries are kept constant, i.e., for each i,

$$\varphi(\cdots, rm_i + r'm_i', \cdots) = r\varphi(\cdots, m_i, \cdots) + r'\varphi(\cdots, m_i', \cdots)$$

Corollary

Let R be a commutative ring and let $M_1, \cdots, M_n, L$ be R-modules. Let $M_1 \otimes \cdots \otimes M_n$ denote any bracketing of the tensor product of these modules and let

$$\iota : M_1 \times \cdots \times M_n \to M_1 \otimes \cdots \otimes M_n$$

be the map defined by $\iota(m_1, \cdots, m_n) = m_1 \otimes \cdots m_n$. Then

1. *for every R-module homomorphism $\Phi : M_1 \otimes \cdots \otimes M_n \to L$, the map $\varphi = \Phi \circ \iota$ is n-multilinear from $M_1 \times \cdots \times M_n \to L$, and*
2. *if $\varphi : M_1 \times \cdots \times M_n \to L$ is an n-multilinear map, then there is a unique R-module homomorphism $\Phi : M_1 \otimes \cdots \otimes M_n \to L$ such that $\varphi = \Phi \circ \iota$.*

$$\begin{tikzcd}
M_1 \times \cdots \times M_n \arrow[r, "\iota"] \arrow[dr, "\varphi"'] \arrow[dr, phantom, shift left=4ex, "\circlearrowright"] & M \otimes \cdots \otimes M_n \arrow[d, dashed, "\Phi"] \\
 & L
\end{tikzcd}$$

$$\{n\text{-multilinear maps}\} \leftrightarrow \{R\text{-module homomorphisms}\}$$

Theorem (Tensor Products of Direct Sums)

Let M, M' be right R-modules and let N, N' be left R-modules. Then there are unique group isomorphisms

$$(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes N)$$
$$M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$$

such that $(m, m') \otimes n \mapsto (m \otimes n, m' \otimes n)$ and $m \otimes (n, n') \mapsto (m \otimes n, m \otimes n')$ respectively. In particular, if R is commutative, these are isomorphisms of R-modules.

In generally, the corresponding result is also true for arbitrary direct sums.

$$M \otimes \left(\bigoplus_{i \in I} N_i \right) \cong \bigoplus_{i \in I} (M \otimes N_i).$$

Corollary (Extension of Scalars for Free Modules)

The module obtained from the free R-module, $N \cong R^n$ by extension of scalars from R to S is the free S-module S^n, i.e.,

$$S \otimes_R R^n \cong S^n$$

as left S-modules.

Corollary

Let R be a commutative ring and let $M \cong R^s$ and $N \cong R^t$ be free R-modules with bases $m_1, \cdots, m_s$ and $n_1, \cdots, n_t$, respectively. Then $M \otimes_R N$ is a free R-module of rank st, with bases $m_i \otimes n_j$, $1 \leq i \leq s$ and $1 \leq j \leq t$, i.e.,

$$R^s \otimes_R R^t \cong R^{st}.$$

More generally, this result still holds for arbitrary rank.

Proposition

Suppose R is a commutative ring and M, N are left R-modules, considered with the standard R-module structures. Then there is a unique R-module isomorphism

$$M \otimes_R N \cong N \otimes_R M$$

mapping $m \otimes n$ to $n \otimes m$.

Proposition

Let R be a commutative ring and let A and B be R-algebras. Then the multiplication $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$ is well defined and make $A \otimes_R B$ into an R-algebra.

Example

The tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is free of rank 4 ad a module over $\mathbb{R}$ with basis given by

$$e_1 = 1 \otimes 1, e_2 = 1 \otimes i, e_3 = i \otimes 1, e_4 = i \otimes i.$$

This tensor product is also ring with $1 = e_1$. Then

$$e_4^2 = (i \otimes i)(i \otimes i) = i^2 \otimes i^2 = (-1) \otimes (-1) = 1.$$

Thus $(e_4 - 1)(e_4 + 1) = 0$, so $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not an integral domain.

Example

As $\mathbb{R}$-algebra, for $r \in \mathbb{R}$ and $x \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, $xr = rx$.
$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ has a structure of a left and right $\mathbb{C}$-modules. For example,

$$\begin{aligned} i \cdot e_1 = i \cdot (1 \otimes 1) = (i \otimes 1) = e_3, \\ e_1 \cdot i = (1 \otimes 1) \cdot i = 1 \otimes i = e_2. \end{aligned}$$

This example shows that even when the rings involved are commutative, there may by natural left and right module structures that are not the same.

Exercise

Let R be a subring of the commutative ring S and let x be an indeterminate over S. Prove that $S[x] \cong S \otimes_R R[x]$ as S-algebras.

The End