Top11 The Urysohn Lemma, Tietze Extension Theorem

KYB

Thrn, it's a Fact

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Exercise

Let X be a metric space and let A, B be disjoint closed sets. Show that there exsits a continuous function $f : X \to [0,1]$ such that

$$f(x) = \begin{cases} 0 & \text{for every } x \in A \\ 1 & \text{for every } x \in B \end{cases}$$

Proof.

$$f(x) = \frac{d(x,A)}{d(x,A) + d(x,B)}$$

$\square$

Bump function

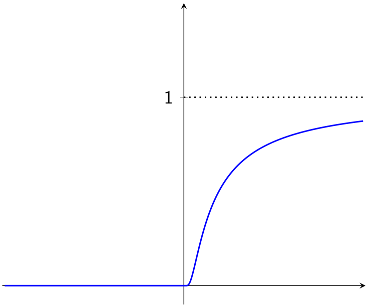
In $\mathbb{R}^n$, there exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{supp}(f) = \overline{\{\boldsymbol{x} \in \mathbb{R}^n : f(x) \neq 0\}}$ is compact.

For $n = 1$.

Step1

Consider $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

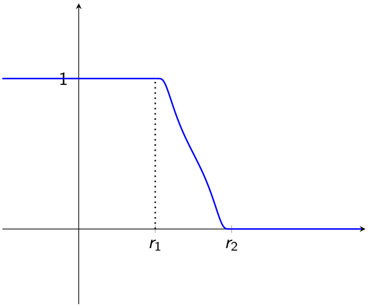


Then f is ∞-differentiable and $0 \leq f < 1$

Step2

Given any real number r_1 and r_2 such that $r_1 < r_2$, define $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) = \frac{f(r_2 - x)}{f(r_2 - x) + f(x - r_1)}$$



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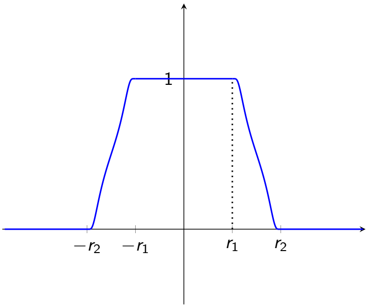
Step3

Given any real number r_1 and r_2 such that $r_1 < r_2$, define $H : \mathbb{R}^n \to \mathbb{R}$ such that

$$H(x) = h(|x|)$$

. Then

$$\begin{array}{rl} H \equiv 1 & \text{on } \overline{B}(0, r_1) \\ 0 < H < 1 & \text{for all } x \in B(0, r_2) - \overline{B}(0, r_1) \\ H \equiv 0 & \text{on } \mathbb{R}^n - B(0, r_2) \end{array}.$$



Theorem

Theorem

Let X be a normal space, let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map $f : X \to [a, b]$ such that

$$f(x) = \begin{cases} a & \textit{for every } x \in A \\ b & \textit{for every } x \in B \end{cases}$$

Step1

Let P be the set of all rational numbers in $[0, 1]$. Then there are open sets U_p for each p in P such that $A \subset U_0$,$U_1 = X - B$,and if $p < q$ then $\overline{U}_p \subset U_q$.

Step2

Extend $\{U_p\}$ to all rational numbers p in $\mathbb{R}$ by

$$U_p = \varnothing \text{ if } p < 0 \qquad\qquad U_p = X \text{ if } p > 1$$

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Step3

Given $x \in X$, define $\mathbb{Q}(x) = \{p : x \in U_p\}$ Then for all $x \in X$,

$$(1, \infty) \subset \mathbb{Q}(x) \subset [0, \infty)$$

Thus $\mathbb{Q}(x)$ is bounded below and $\inf \mathbb{Q}(x) \in [0, 1]$.

Step4

Define $f(x) = \inf \mathbb{Q}(x)$. Then this f is a desired function because

1. For all $x \in B$, $x \notin A \subset U_0 \subset U_p \subset U_1 = X - B$ for all $p \in \mathbb{Q} \cap [0, 1]$. Thus $\mathbb{Q}(x) = (0, \infty)$ and $f(x) = 1$.
2. For all $x \in A$, $x \in U_p$ for all rational $p \geq 0$. Thus $\mathbb{Q}(x) = 0$.

Definition

If A and B are two subsets of the topological space X, and if there is a continuous function $f: X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say A and B **can be separated by a continuous function**.

Definition

A space X is **completely regular** if X satisfies T_1 axiom and for each x_0 and each closed set A not containing x_0, there is a continuous function $f: X \to [0,1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

$T_1 \leftarrow T_2$(Haus) $\leftarrow T_3$(regular) $\leftarrow T_{3\frac{1}{2}}$(completely regular) $\leftarrow T_4$(normal) $\leftarrow T_5$(completely normal)

Theorem

A subspace of a completely regular space is completely regular.
A product of completely sapces if completely regular.

Ex33.1

Examine the proof of the Urysohn lemma, and show that for given r,

$$f^{-1}(r) = \bigcap_{p>r} U_p - \bigcup_{q<r} U_q$$

p, q rational.

Ex33.2

(a) Show that a connected normal space having more than one point is uncountable.

(b) Show that a connected regular space having more than one point is uncountable.

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Ex33.4

Theorem. Let X be normal. There exists a continuous function $f: X \to [0,1]$ such that

$$\begin{aligned} f(x) &= 0 \quad \text{for } x \in A \\ f(x) &> 0 \quad \text{for } x \notin A \end{aligned}$$

iff A is a closed G_δ set in X.

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Ex33.5

Theorem (Strong form of the Urysohn lemma). Let X be normal. There exists a continuous function $f: X \to [0,1]$ such that

$$\begin{array}{ll} f(x) = 0 & \text{for } x \in A \\ f(x) = 1 & \text{for } x \in B \\ 0 < f(x) < 1 & \text{otherwise} \end{array}$$

iff A and B are closed G_δ sets in X.

Ex33.7

Show that every locally compact Hausdorff space is completely regular.

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Ex33.8

Let X be completely regular, let A and B be disjoint closed subsets of X. Show that if A is compact, there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Theorem (Urysohn metrization theorem)

Every regular space X with a countable basis is metrizable.

Recall

- Every regular space with a countable basis is normal.
- If X is homeomorphic to a metrization space Y, then X is metrizable.
- We call an injective continuous $f: X \to Y$ an imbeffing if $f: X \to f(X)$ is a homeomorphism.

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Step1

There exists a countable collection of continuous functions $f_n : X \to [0,1]$ having the property that given any point x_0 of X and any neighborhood U of x_0, there exists an index n such that f_n is positive at x_0 and vanishes ouside U.

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Step2(First version of the proof)

Given the functions f_n, take $\mathbb{R}^\omega$ in the product topology and define a map $F : X \to \mathbb{R}^\omega$ by the rule

$$F(x) = (f_1(x), f_2(x), \cdots)$$

Then

1. F is continuous.(Trivial)
2. F is injective.
3. $F : X \to F(X)$ is homeomorphic.

Thus it is enough to show that for any open U in X, $F(U)$ is open in $F(X)$.

Step3(Second version of the proof)

Replace f_n in Step1 by f_n/n for each n (i.e. we may assume $f_n : X \to [0, 1/n]$.) Take $\mathbb{R}^\omega$ in the uniform topology and define $F : X \to [0,1]^\omega$ by

$$F(x) = (f_1(x), f_2(x), \cdots)$$

Then

1. F is continuous.
2. F is injective.(Step2)
3. $F : X \to F(X)$ is homeomorphic.

By Step2, for every open set U in X, $F(U)$ is open in $F(X)$. Thus it is enough to show that F is continuous.

Theorem (Imbedding theorem)

Let X be a space satisfying T_1 axiom. Suppose that $\{f_\alpha : X \to \mathbb{R}\}_{\alpha \in J}$ is an indexed family of continuous functions f_α satisfying the requirement that for each point x_0 of X and each neighborhood U of x_0, there is an index α such that f_α is positive at x_0 and vanishes outside U. Then the function $F : X \to \mathbb{R}^J$ defined by

$$F(x) = (f_\alpha(x))_{\alpha \in J}$$

is an imbedding of X in $\mathbb{R}^J$. If f_α maps X into $[0,1]$ for each α, then F imbeds X in $[0,1]^J$.

Theorem

A space X is completely regular iff it is homeomorphic to a subspace of $[0,1]^J$ for some J.

Ex34.3

Let X be a compact Hausdorff space. Show that X is metrizable iff X has a countable basis.

Proof.

$(\Rightarrow)$. Ex30.4

$(\Leftarrow)$. Urysohn metrizable theorem. $\square$

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Ex34.4

Let X be a locally compact Hausdorff space. Is it true that if X has a countable basis, then X is metrizable? Is it true that if X is metrizable, then X has a countable basis?

Proof.

1. By Ex32.3(every locally compact Hausdorff space is regular), X is regular. By Urysohn metrizable theorem, X is metrizable.
2. Every discrete topology is metrizable. Take $\mathbb{R}$ in the discrete and it has no countable basis.

$\square$

Ex34.5

Let X be a locally compact Hausdorff space. Let Y be the one-point compactification of X. Is it true that if X has a countable basis, then Y is metrizable? Is it true that if Y is metrizable, then X has a countable basis?

Proof.

Claim) Y is metrizable iff X has a countable basis.
By Ex34.3, it suffices to show that if X has a countable basis, then Y has a countable basis. □

Theorem (Tietze extension theorem)

Let X be a normal space; let A be a closed subspace of X.

(a) *Any continuous map of A into the closed interval $[a, b]$ of $\mathbb{R}$ may be extended to a continuous map of all of X into $[a, b]$.*

(b) *Any continuous map of A into $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.*

Main idea) Construct a sequence of continuous functions s_n defined on X such that s_n converges uniformly, and $|f - s_{n\restriction A}| < \epsilon$ for large n.

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Step1

Consider the case $f\colon A \to [-r, r]$. We want to construct $g : X \to \mathbb{R}$ such that

$$|g| \leq \frac{1}{3}r \qquad\qquad |g(a) - f(a)| \leq \frac{2}{3}r \text{ for all } a \in A.$$

Step2

We may assume $f\colon A \to [-1,1]$. By step1, we can find $g_1 : X \to [-1,1]$ so that

$$|g_1| \leq \frac{1}{3}r \qquad |f(a) - g_1(a)| \leq \frac{2}{3} \text{ for all } a \in A.$$

Consider $f - g_1$. This function maps A into $[-\frac{2}{3}, \frac{2}{3}]$. Thus we can find g_2 such that $|g_2| \leq \frac{1}{3}(\frac{2}{3})^{n-1}$ and $|f(a) - g_1(a) - g_2(a)| \leq (\frac{2}{3})^n$.
Using induction we can find $g_n : X \to \mathbb{R}$ such that

$$|g_n| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} \qquad |f(a) - \sum_{i=1}^{n} g_n(a)| \leq \left(\frac{2}{3}\right)^n \text{ for all } a \in A.$$

Define $g = \sum_{n=1}^{\infty} g_n$. Then g is the desired function.

Step3

Suppose $f\colon A \to \mathbb{R}$. Since $\mathbb{R}$ is homeomorphic to $(-1,1)$, we may assume $f\colon A \to (-1,1)$.
By (a) f can be extended to $g : X \to [-1,1]$. Thus it suffices to show that we can replace $[-1,1]$ to $(-1,1)$.
Define a subset D of X by

$$D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\}).$$

Since g is continuous, D is closed in X and $A \cap D = \varnothing$. By the Urysohn lemma, there is a continuous function $\phi : X \to [0,1]$ such that $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$. Define $h(x) = \phi(x)g(x)$. Then h is the desired function.

Ex35.1

Show that the Tietze extension theorem implies the Urysohn lemma.

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Ex35.2

In the proof of the Tietze theorem, how essential was ther clever decision in Step1 to divide the interval $[-r, r]$ into three equal pieces? Suppose instead that one devides this interval into the three invervals

$$I_1 = [-r, -ar], I_2 = [-ar, ar], I_3 = [ar, r]$$

for some a with $0 < a < 1$. For what values of a other than $a = 1/3$ does the proof go through?

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Ex35.3

Let X be metrizable. Show that the following are equivalent:

(i) X is bounded under every metric that gives the topology.

(ii) Every continuous function $\phi : X \to \mathbb{R}$ is bounded.

(iii) X is limit point compact.

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Ex35.4

Let Z be a topological sapce. If Y is a subspace of Z, we say that Y is a **retract** of Z if there is a continuous map $r : Z \to Y$ such that $r(y) = y$ for each $y \in Y$.

(a) Show that if Z is Hausdorff and Y is a retract of Z, then Y is closed in Z.

(b) Let A be a two-point set in $\mathbb{R}^2$. Show that A is not a retract of $\mathbb{R}^2$.

(c) Let S^1 be the unit circle in $\mathbb{R}^2$; show that S^1 is a retract of $\mathbb{R}^2 - \{\mathbf{0}\}$, where $\mathbf{0}$ is the orgin. Can you conjecture whether or not S^1 is a retract of $\mathbb{R}^2$.

The End

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