Analysis
- PMA 3 -

KYB

Thrn, it's a Fact

mathrnfact@gmail.com

January 13, 2021

Overview

Chapter 2 Basic Topology

Contents of Ch2

Sets

$A, B, C, \cdots$ will always denote sets, $a, b, c, \cdots$ will always denote members of sets, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \cdots$ will denote collections of some sets.

Definition

- $\varnothing$ is a set which contains no element.
- $A = B$ iff $x \in A \iff x \in B$.
- $A \subset B$ iff $x \in A \implies x \in B$. In this sense, $A = B$ iff $A \subset B$ and $B \subset A$.
- The power set $\mathcal{P}(A)$ of a set A is the collection of all subset of A, i.e.,

$$X \in \mathcal{P}(A) \iff X \subset A.$$

Sets

Definition

- The union of A and B is a set $A \cup B$ such that

$$x \in A \cup B \iff x \in A \text{ or } x \in B.$$

- The intersection of A and B is a set $A \cap B$ such that

$$x \in A \cap B \iff x \in A \text{ and } x \in B.$$

Sets

Definition

Consider two sets A and B.

- We say $f : A \to B$ is a function if
  1. for all $a \in A$, there is $b \in B$ such that $f(a) = b$;
  2. for $a, a' \in A$ if $f(a) \neq f(a')$, then $a \neq a'$.
- In this case, we call A the domain of f, $f(x)$ is a value of f.
- The set $\{b \in B : f(x) = b \text{ for some } x \in A\}$ is called the range of f.
- If $E \subset A$, $f(E) = \{b : f(x) = b \text{ for some } x \in E\}$. We call $f(E)$ the image of E under f.
- If $E \subset B$, $f^{-1}(E) = \{a \in A : f(a) \in E\}$. We call $f^{-1}(E)$ the inverse image of E under f. If $E = \{y\}$ (a singleton set), write $f^{-1}(y)$ instead of $f^{-1}(E)$.
- If $f : A \to B$ and $g : B \to C$ are functions, a composite of f and g, denoted by $g \circ f$, is a function from A into C such that

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in A.$$

Sets

Remark

Suppose $f : A \to B$ is a function. For $E \subset A$, $f(E) \subset B$ and in general, $f(A) \neq B$.

- If $f(A) = B$, we call f map A onto B. And we say f is a onto map, or surjective.

For $y \in B$, it may $f^{-1}(y) = \varnothing$ (moreover may $f^{-1}(E) = \varnothing$ for some $E \subset B$), and $f^{-1}(y)$ may not be singleton (that is, there may be at least two element $x, y \in f^{-1}(y)$.

- If $f^{-1}(y) = \{x\}$ is singleton for all $y \in f(A)$, we say f is 1-1 (one-to-one), or injective.
  - In this case, $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in A$.
- If f is 1-1 and onto, we say f is a bijective.
  - In this case, we can define the inverse map from B into A, say f^{-1}, by $f^{-1}(y) = x$ where $f(x) = y$.

Sets

Definition

Suppose A and B are sets. We say A and B have the same cardinal number, write $A \sim B$, if there is a 1-1 and onto map $f : A \to B$.

Remark

The relation is an equivalence relation:

- $A \sim A$.
- If $A \sim B$, then $B \sim A$.
- If $A \sim B$ and $B \sim C$, then $A \sim C$.

Sets

Definition

For any positive integer n, let $J_n = \{1, 2, \cdots, n\}$; let J be the set of all positive integers. For any set A, we say:

(a) A is finite if $A \sim J_n$ for some n.

(b) A is infinite if A is not finite.

(c) A is countable if $A \sim J$.

(d) A is uncountable if A is neither finite nor countable.

(e) A is at most countable if A is finite or countable.

Sets

Example

Let A be the set of all integers. Then A is countable:

$$\begin{aligned} A &: 0, 1, -1, 2, -2, 3, -3, \cdots \\ J &: 1, 2, 3, 4, 5, 6, 7, \cdots \end{aligned}$$

or

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ even}, \\ -\frac{n-1}{2} & n \text{ odd} \end{cases}$$

is 1-1 and onto.

Remark

This example shows that even $A \subsetneq B$, it may $A \sim B$.

Sets

Definition

- A sequence in A is a function $f : J \to A$, and denote f by $\{x_n\}$ where $f(n) = x_n$, or $x_1, x_2, x_3, \cdots$.
- x_n's are called the terms of the sequence.

Note that the terms $x_1, x_2, \cdots$ need not be distinct.

Sets

Theorem

Every infinite subset of a countable set A is countable.

Sets

Definition

Let A and Ω be sets, and suppose that with each element α of A, there is associated a subset of Ω which we denote by E_α. Then the set $\{E_\alpha\}$ is called a collection of sets, or a family of sets.

- The union of the sets E_α is the set S such that

  $$x \in S \iff x \in E_\alpha \text{ for some } \alpha \in A.$$

  Write

  $$S = \bigcup_{\alpha \in A} E_\alpha.$$

- Similarly, the intersection of the sets E_α is the set P such that

  $$x \in P \iff x \in E_\alpha \text{ for all } \alpha \in A.$$

  Write

  $$P = \bigcap_{\alpha \in A} E_\alpha.$$

Sets

Remark

- $A \cap \varnothing = \varnothing$.
- $A \cup \varnothing = A$.
- $A \cap B \subset A$.
- $A \subset A \cup B$.
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Sets

Theorem

Let $\{E_n\}_{n=1}^{\infty}$ *be a sequence of countable sets, and put*

$$S = \bigcup_{n=1}^{\infty} E_n.$$

Then S *is countable. Thus a countable union of countable sets is again countable.*

Corollary

Suppose A *is at most countable, and, for every* $\alpha \in A$, B_α *is at most countable. Put*

$$T = \bigcup_{\alpha \in A} B_\alpha.$$

Then T *is at most countable.*

Sets

Theorem

Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \cdots, a_n)$, where $a_k \in A$, and the elements $a_1, \cdots, a_n$ need not to be distinct. Then B_n is countable.

Corollary

The set of all rational numbers is countable.

Sets

Theorem

Let A be the set of all sequences whose elements are the digits 0 and 1. For example, $1, 0, 0, 1, 0, 1, 1, 1, \cdots$ is an element of A. Then A is uncountable.

Exercises

Ex 2.2

A complex number z is said to be algebraic if there are integers $a_0, \cdots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable.

Exercises

Ex 2.3

Prove that there exists real numbers which are not algebraic.

Metric Spaces

Definition

Suppose X is a set.

- A function $d : X \times X \to \mathbb{R}$ is called a metric function, or simply a metric if
  (a) $d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$;
  (b) $d(p,q) = d(q,p)$;
  (c) $d(p,q) \leq d(p,r) + d(r,q)$.
- We call $d(p,q)$ the distance from p to q.
- In this case, we say X is a metric space.
- $x \in X$ is said to be a point of X, instead of an element of X.

Metric Spaces

Example

- For $\mathbb{R}^k$,
$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$$
is a metric.
- For any nonempty set X,
$$d(p, q) = \begin{cases} 1 & p \neq q; \\ 0 & p = q \end{cases}$$
is a metric.

Exercises

Ex 2.11

For $x, y \in \mathbb{R}$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2, \\ d_2(x,y) &= \sqrt{|x-y|}, \\ d_3(x,y) &= \left|x^2-y^2\right|, \\ d_4(x,y) &= |x-2y|, \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Metric Spaces

Definition

On $\mathbb{R}$,

- The segment $(a, b) = \{x : a < x < b\}$.
- The interval $[a, b] = \{x : a \le x \le b\}$.
- Half-open intervals $[a, b)$ and $(a, b]$ can be also defined similarly.

On $\mathbb{R}^k$, suppose $a_i < b_i$ for all $i = 1, \cdots, k$.

- The set of all $\mathbf{x}$ such that $a_i \le x_i \le b_i$ is called a k-cell.
- Let $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$. The set B of all points $\mathbf{y} \in \mathbb{R}^k$ such that

  $$|\mathbf{y} - \mathbf{x}| < r$$

  is called the open ball (resp. closed ball when $\le r$).
- A set $E \subset \mathbb{R}^k$ is convex if

  $$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in E$$

  whenever $\mathbf{x}, \mathbf{y} \in E$, and $0 < \lambda < 1$.

Metric Spaces

Definition

Let X be a metric space with a metric d.

(a) A neighborhood of p is a set $N_r(p) = \{q \in X : d(p,q) < r\}$ for some $r > 0$. r is called the radius of $N_r(p)$.

(b) p is a limit point of E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.

(c) If $p \in E$ and p is not a limit point of E, then p is called an isolated point of E.

(d) E is closed if every limit point of E is a point of E.

(e) p is an interior point of E if there is a neighborhood N of p such that $N \subset E$.

(f) E is open if every point of E is an interior point of E.

(continued)

Metric Spaces

Definition

(g) The complement of E, denoted by E^c, is the set of all points $p \in X$ such that $p \notin E$.

(h) E is perfect if E is closed and if every point of E is a limit point of E.

(i) E is bounded if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.

(j) E is dense in X if every point of X is a limit point of X is a limit point of E, or a point of E or both.

Metric Spaces

Theorem

Every neighborhood is an open set.

Metric Spaces

Theorem

If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Corollary

A finite set has no limit points, and hence closed.

Metric Spaces

Theorem (De Morgan's Laws)

Let $\{E_\alpha\}$ *be a (finite or infinite) collection of sets* E_α*. Then*

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha \left(E_\alpha^c\right).$$

Metric Spaces

Theorem

A set E is open if and only if its complement is closed.

Corollary

A set F is closed if and only if its complement is open.

Metric Spaces

Theorem

(a) *For any collection* $\{G_\alpha\}$ *of open sets,* $\bigcup_\alpha G_\alpha$ *is open.*

(b) *For any collection* $\{F_\alpha\}$ *of closed sets,* $\bigcap_\alpha F_\alpha$ *is closed.*

(c) *For any finite collection* $G_1, \cdots, G_n$ *of open sets,* $\bigcap_{i=1}^n G_i$ *is open.*

(d) *For any finite collection* $F_1, \cdots, F_n$ *of closed sets,* $\bigcup_{i=1}^n F_i$ *is closed.*

Metric Spaces

Example

Let $G_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n = 1, 2, 3, \cdots$. Put $G = \bigcap_{n=1}^{\infty} G_n$. Then $G = \{0\}$ which is not open.

Metric Spaces

Definition

If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X, then the closure of E is the set $\overline{E} = E \cup E'$.

Theorem

(a) $\overline{E}$ *is closed.*

(b) $E = \overline{E}$ *if and only if* E *is closed.*

(c) $\overline{E} \subset F$ *for every closed set* $F \subset X$ *such that* $E \subset F$.

This theorem says the closure $\overline{E}$ *is unique and it is the smallest closed set containing* E.

Metric Spaces

Theorem

Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed.

Metric Spaces

Observe

Suppose X is a metric space with a metic d and Y be a nonempty subset of X. Then d is also a metric on Y.

Suppose $E \subset Y$ be an open subset in X. Then for each $p \in E$, there is $r > 0$ such that $N_r(p) \subset E$. But $E \subset Y$, so every point of $N_r(p)$ belongs to Y. Thus we can define the openness respect to Y.

Say E is open relative to Y if for each $p \in E$ there is $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$. Note that there may be a subset E of Y which is open relative to Y but not open in X.

Metric Spaces

Theorem

Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.

Exercises

Ex 2.6

Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. Do E and E' always have the same limit points?

Exercises

Ex 2.7

Let $A_1, A_2, \cdots$ be subsets of a metric space.

(a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\overline{B}_n = \bigcup_{i=1}^n \overline{A}_i$.

(b) If $B = \bigcup_{i=1}^\infty A_i$, prove that $\overline{B} \supset \bigcup_{i=1}^\infty \overline{A}_i$.

Exercises

Ex 2.9

Let E° denote the set of all interior points of a set E.

(a) Prove that E° is open.

(b) Prove that E is open if and only if $E^\circ = E$.

(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

(d) Prove that the complement of E° is the closure of the complement of E.

(e) Do E and $\overline{E}$ always have the same interiors?

(f) Do E and E° always have the same closures?

Exercises

Ex 2.10

Let X be any nonempty set and define a metric

$$d(p,q) = \begin{cases} 1 & p \neq q; \\ 0 & p = q \end{cases}.$$

Which subsets of the resulting metric space are open? Which are closed?

The End