LA2 9

KYB

Thrn, it's a Fact

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Overview

p-norms

Definition (p-norms)

- Let $p \in [1, \infty]$. For $x \in \mathbb{R}^n$,

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \text{ if } p < \infty$$
$$\|x\|_\infty = \max\{|x_i| : i = 1, \cdots, n\}, \text{ if } p = \infty$$

is the l^p-norm on $\mathbb{R}^n$. For $f \in C[a, b]$,

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}, \text{ if } p < \infty$$
$$\|f\|_\infty = \max\{|f(x)| : a \leq x \leq b\}, \text{ if } p = \infty$$

is the L^p-norm on $C[a, b]$.

Remark

We show that $\|\cdot\|_p$ is a norm if $p = 1$, $p = 2$, or $p = \infty$.

For $x \in \mathbb{R}^n$,

(1) $\|x\|_p \geq 0$; and $\|x\|_p = 0$ iff $x = 0$.

(2) $\|\alpha x\|_p = |\alpha| \|x\|_p$.

Similarly, for $f \in C[a, b]$,

(1) $\|f\|_p \geq 0$; and $\|f\|_p = 0$ iff $f = 0$.

(2) $\|\alpha f\|_p = |\alpha| \|f\|_p$.

To show that $\|\cdot\|_p$ satisfies the triangular inequality, we need Hölder's inequality.

Lemma

If $a, b \geq 0$ and $0 < t < 1$, then

$$a^t b^{1-t} \leq ta + (1-t)b,$$

with equality iff $a = b$.

Proof.

If $b = 0$, done. Suppose $b \neq 0$, then dividing both sides by b and setting $x = a/b$, it suffices to show that

$$x^t \leq tx + (1-t)$$

with equality iff $x = 1$. Let $f(x) = x^t - tx$. Then
$f'(x) = tx^{t-1} - t = t(x^{t-1} - 1)$. For $0 \leq x < 1$, $f'(x) < 0$ and for $x > 1$
$f'(x) > 0$. Thus $f(x)$ attains a maximum at $x = 1$, and so

$$x^t - tx \leq 1 - t,$$

with equality iff $x = 1$. $\square$

Hölder's inequality

Suppose $1 < p$ and $1/p + 1/q = 1$. For $x, y \in \mathbb{R}^n$,

$$|x \cdot y| \leq \|x\|_p \|y\|_q.$$

For $f, g \in C[a, b]$,

$$|\langle f, g \rangle| \leq \|x\|_p \|y\|_q.$$

Remark

If $p = q = 2$, this inequality is just Cauchy-Schwarz inequality. So Hölder's inequality is a generalization of Cauchy-Schwarz inequality.

Proof.

Note that

$$|x \cdot y| = |x_1 y_1 + \cdots + x_n y_n| \leq |x_1||y_1| + \cdots + |x_n||y_n|.$$

$$|\langle f, g \rangle| = \left| \int_a^b f(x) g(x) dx \right| \leq \int_a^b |f(x)||g(x)| dx$$

Take $a = |x_i|^p$ and $b = |y_i|^q$. (resp. $a = |f(x)|^p$ and $b = |g(x)|^q$.)

$$a^{1/p} b^{1/q} = |x_i||y|_i \leq \frac{1}{p} a + \frac{1}{q} b = \frac{1}{p} |x_i|^p + \frac{1}{q} |y_i|^q.$$

Suppose $\|x\|_p = \|y\|_p = 1$.

$$|x \cdot y| \leq \frac{1}{p} \left(\sum_{i=1}^n |x_i|^p \right) + \frac{1}{q} \left(\sum_{i=1}^n |y_i|^q \right) = \frac{1}{p} + \frac{1}{q} = 1 = \|x\|_p \|y\|_q.$$

In general, apply $x / \|x\|_p$ and $y / \|y\|_p$. $\square$

Minkowski's inequality

Let $p \in [1, \infty)$.

- For $x, y \in \mathbb{R}^n$,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

- For $f, g \in C[a, b]$,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof.

Note that

$$|x_i + y_i|^p \leq (|x_i| + |y_i|)\, |x_i + y_i|^{p-1}$$

WLOG, assume $p > 1$ and $x + y \neq 0$ and let q be such that $1/p + 1/q = 1$. Note that $(p-1)q = p$. Let $z = (|x_1 + y_1|, \cdots, |x_n + y_n|)$ and apply Hölder's inequality

$$\begin{aligned}
\|x + y\|_p^p = \sum_{i=1}^{n} |x_i + y_i|^p \leq \sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} &+ \sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1} \\
&= |x \cdot z| + |y \cdot z| \leq \|x\|_p \|z\|_q + \|y\|_p \|z\|_q \\
&= \left(\|x\|_p + \|y\|_q\right) \left(\sum_{i=1}^{n} |x_i + y_i|^{q(p-1)}\right)^{1/q}. \\
&= \left(\|x\|_p + \|y\|_q\right) \|x + y\|_p^{p/q}.
\end{aligned}$$

Thus $\|x + y\|_p^{p-p/q} = \|x + y\|_p \leq \|x\|_p + \|y\|_p$. $\square$

Review of Calculus

Recall

- Let (x_n) be a seqence in $\mathbb{R}$. We say x_n converges to x if for any $\epsilon > 0$, there is $N \in \mathbb{Z}_+$ such that $|x - x_n| < \epsilon$ for all $n > N$.
- Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For $a \in \mathbb{R}$, we say f is continuous at a if f satisfies the following:
  For any $\epsilon > 0$, there is $\delta > 0$ s.t $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Goal

- Convergence and Continuity
- Compactness
- Completeness

Convergence and Continuity

Definition

Let $\{x_k\}$ be a sequence of vectors in $\mathbb{R}^n$, and let $x \in \mathbb{R}^n$. We say $\{x_k\}$ converges to x if and only if for every $\epsilon > 0$, there exists a positive integer N such that $k \geq N$ implies $\|x_k - x\|_\infty < \epsilon$.

Theorem (418)

Let $\{x^{(k)}\}$ be a sequence in $\mathbb{R}^n$, and $x \in \mathbb{R}^n$. Then $x^{(k)} \to x$ if and only if $x_i^{(k)} \to x_i$ for each i.

Definition

For $p \in [1, \infty]$, let $B_{\epsilon,p}(x) = \{y \in \mathbb{R}^n \mid \|x - y\|_p < \epsilon\}$. We say $B_{\epsilon,p}(x)$ is a ball of radius ϵ centered at x.

Definition

Let $S \subset \mathbb{R}^n$. We say that $y \in \mathbb{R}^n$ is an accumulation point of S if for every $\epsilon > 0$, the open ball $B_{\epsilon,\infty}(y)$ contains infinitely many points of S.

Definition

Let $S \subset \mathbb{R}^n$.
We say S is open if for each $x \in S$, there exists $\epsilon > 0$ such that $B_{\epsilon,\infty}(x) \subset S$.
We say S is closed if $\mathbb{R}^n - S$ is open.

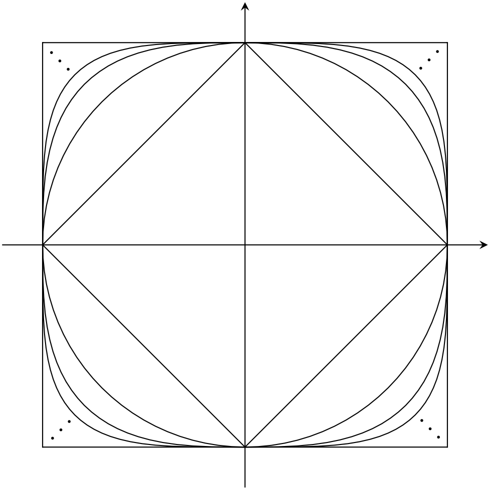

Figure: the unit balls when $n = 2$, $p = 1, 2, 3, 4, \infty$

Definition

Let $S \subset \mathbb{R}^n$, let $f : S \to \mathbb{R}$, and suppose y is an accumulation point of S.

- We say that $f(x)$ converges to $L \in \mathbb{R}$ as $x \to y$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$x \in S, \|x - y\|_\infty < \delta, x \neq y \implies |f(x) - L| < \epsilon.$$

- If $f(x)$ converges to L as $x \to y$, we write

$$\lim_{x \to y} f(x) = L$$

  or $f(x) \to L$ as $x \to y$.
- If there is no real number L such that $f(x) \to L$ as $x \to y$, then we say $f(x)$ diverges as $x \to y$.

Definition

Let $S \subset \mathbb{R}^n$, and let $f : S \to \mathbb{R}$ be a function. We say f is continuous at $x \in S$ if for any $\epsilon > 0$, there is $\delta > 0$ such that

$$y \in S, \|y - x\|_\infty < \delta \implies |f(y) - f(x)| < \epsilon.$$

We say that f is continuous on S if it is continuous at every $x \in S$.

Lemma (423)

Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. Then there exists a constant $M > 0$ such that

$$\|x\| \leq M\|x\|_\infty.$$

Lemma (424, Reverse triangle inequality)

Let V be a vector space over $\mathbb{R}$, and let $\|\cdot\|$ be a norm on V. Then

$$\left|\|x\| - \|y\|\right| \leq \|x - y\|.$$

Theorem (425)

Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. Then $\|\cdot\|$ is a continuous function.

Corollary (426)

Let $y \in \mathbb{R}^n$ be given. Then $\|x\|_\infty \to \|y\|_\infty$ as $x \to y$.

Compact

Definition

Let S be a subset of $\mathbb{R}^n$. We say S is bounded if there exists $R > 0$ such that $\|x\|_\infty \leq R$ for all $x \in S$.

Theorem (428)

Let S be a nonempty, closed, and bounded subset of $\mathbb{R}^n$, and let $\{x^{(k)}\}$ be a sequence in S. Then there exists a subsequence $\{x^{(k_j)}\}$ that converges to a vector $x \in S$.

Theorem (429)

Let S be a nonempty, closed, and bounded subset of $\mathbb{R}^n$, and let $f : S \to \mathbb{R}$ be continuous. Then there exists $m_1, m_2 \in S$ such that

$$f(m_1) \leq f(x) \leq f(m_2) \text{ for all } x \in S.$$

Compactness

Let X be a topological space.

- X is *compact* if every open cover of X has a finite subcover.
- X is *limit point compact* if every infinite subset of X has a accumulation point of it.
- X is *sequntially compact* if every sequence has a convergent subsequence.

In $\mathbb{R}^n$ (with the Euclidean metric), they are all equivalent.

Completeness

Definition

Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$. We say $\{x_k\}$ is a Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{Z}_+$ such that

$$\|x_n - x_m\| < \epsilon \text{ whenever } m, n \geq N.$$

If every Cauchy sequence in X is convergent, we say X is complete.

Theorem (431)

$\mathbb{R}^n$ *is complete.*

Equivalence of norms on $\mathbb{R}^n$

Definition

Let X be a vector space over $\mathbb{R}$, and $\|\cdot\|$ and $\|\cdot\|_*$ be two norms on X. We say that $\|\cdot\|_*$ is equivalent $\|\cdot\|$ if there exists $c_1, c_2 > 0$ such that

$$c_1\|x\| \leq \|x\|_* \leq c_2\|x\|.$$

If we define a relation $\sim$ by $\|\cdot\| \sim \|\cdot\|_*$ if and only if $\|\cdot\|$ is equivalent to $\|x\|_*$, $\sim$ is a equivalece relation.

Theorem (433)

Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. Then $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$.

Corollary (434)

Any two norms on $\mathbb{R}^n$ are equivalent.

Roughly speaking, in analysis in $\mathbb{R}^n$, the choice of norm does not matter.

Ex 10.1.1

Let $\{x^{(k)}\}$ be a sequence in $\mathbb{R}^n$ and suppose $x^{(k)} \to x \in \mathbb{R}^n$. Let $i = 1, \cdots, n$. Prove that the sequence $\{x_i^{(k)}\}$ of real numbers converges to the real number x_i.

From now on, let $\|\cdot\|$ and $\|\cdot\|_*$ be two norms on $\mathbb{R}^n$.

Ex 10.1.3

Prove that if $\{x_k\}$ is a sequence in $\mathbb{R}^n$, then $x_k \to x$ under $\|\cdot\|$ if and only if $x_k \to x$ under $\|\cdot\|_*$.

Ex 10.1.4

Let S be a nonempty subset of $\mathbb{R}^n$. Prove that S is open under $\|\cdot\|$ if and only if S is open under $\|\cdot\|_*$.

Ex 10.1.5

Prove that S is closed under $\|\cdot\|$ if and only if S is closed under $\|\cdot\|_*$.

Ex 10.1.6

Let S be a nonempty subset of $\mathbb{R}^n$. Prove that x is an accumulation point of S under $\|\cdot\|$ if and only if x is an accumulation point of S under $\|\cdot\|_*$.

Ex 10.1.7

Let S be a nonempty subset of $\mathbb{R}^n$, and let $f : S \to \mathbb{R}$ be a function, and let y be an accumulation point of S. Prove that $\lim_{x\to y} f(x) = L$ under $\|\cdot\|$ if and only if $\lim_{x\to y} f(x) = L$ under $\|\cdot\|_*$.

Ex 10.1.8

Let S be a nonempty subset of $\mathbb{R}^n$, and let $f : S \to \mathbb{R}$ be a function, and let y be a point in S. Prove that f is continuous at y $\|\cdot\|$ if and only if f is continuous at y under $\|\cdot\|_*$.

Ex 10.1.9

Let S be a nonempty subset of $\mathbb{R}^n$. Prove that S is bounded under $\|\cdot\|$ if and only if S is bounded under $\|\cdot\|_*$.

Ex 10.1.10

Let S be a nonempty subset of $\mathbb{R}^n$. Prove that S is sequntially compact under $\|\cdot\|$ if and only if S is sequntially compact under $\|\cdot\|_*$.

Ex 10.1.11

Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$. Prove that $\{x_k\}$ is Cauchy under $\|\cdot\|$ if and only if $\{x_k\}$ is Cauchy under $\|\cdot\|_*$.

Ex 10.1.12

Prove that $\mathbb{R}^n$ is complete under $\|\cdot\|$ if and only if $\mathbb{R}^n$ is complete under $\|\cdot\|_*$.

Ex 10.1.13

Let X be a vector space with norm $\|\cdot\|$, and suppose $\{x_k\}$ is a sequence in X converging to $x \in X$ under $\|\cdot\|$. Then $\{x_k\}$ is a Cauchy sequence.

The End