Analysis
- PMA 1 -

KYB

Thrn, it's a Fact

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Overview

Sets

Definition

If A is any set,

- $x \in A$ means x is a member of A.
- $x \notin A$ means x is not a member of A.
- The *empty set* is a set which contains no element.
- If a set has an element, it is called *nonempty*.
- If B is a set and every element of A is an element of B, we say that A is a subset of B, and write $A \subset B$, or $B \supset A$.
- If, in addition, there is an element of B which is not in A, then A is said to be *proper subset* of B. Note that for every set A, $A \subset A$.
- If $A \subset B$ and $B \subset A$, then $A = B$. Otherwise, $A \neq B$.

Ordered Sets

Definition

Let S be a set. An *order* on S is a relation, denoted by $<$, with the following two properties:

(i) If $x \in S$ and $y \in S$, then one and only one of the statements

$$x < y, x = y, y < x$$

is true.

(ii) If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

$y > x$ means $x < y$ and $x \leq y$ means $x < y$ or $x = y$.

Definition

An *ordered set* is a set S in which an order is defined.

Definition

Suppose S is an ordered set, and $E \subset S$.

- If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is *bounded above*,
- and call β an *upper bound* of E.

Lower bounds are defined in the same way.

Example

- Let $A = \{p \in \mathbb{Q} : p > 0, p^2 < 2\}$. Then we have, for all $p \in S$, $p < 2$. Thus 2 is an upper bound of S. Moreover, A is bounded below by 0.
- Let $B = \{p \in \mathbb{Q} : p > 0, p^2 > 2\}$. For $p \in B$, suppose $p > 0$. Then $p > 1$. If $p < q$, then $p^2 < q^2$. So $p + 1 \in B$. Then you can show that there is no $\beta \in \mathbb{Q}$ which bounds B above. But every element of A is a lower bound of B.

Definition

Suppose S is an ordered set, E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

(i) α is an upper bound of E.

(ii) If $\gamma < \alpha$, then γ is not an upper bound of E.

Then α is called the *least upper bound of* E or the *supremum of* E, and we write

$$\alpha = \sup E.$$

By (ii), such α is unique.

The *greatest lower bound*, or *infimum*, of a set E which is bounded below is defined in the same manner and write $\alpha = \inf E$.

Example

In the previous example, A is bounded above. In fact, the upper bound of A are exactly the members of B. Since B contains no smallest member, A has no least upper bound in $\mathbb{Q}$. Similarly, B is bounded below : the set of all lower bounds of B is consists of A and of all $r \in \mathbb{Q}$ with $r \leq 0$. Since A has no largest member, B has no greatest lower bound in $\mathbb{Q}$.

Example

If $\alpha = \sup E$ exists, then α may or may not be a member of E.

- $E_1 = \{r \in \mathbb{Q} : r < 0\}$
- $E_2 = \{r \in \mathbb{Q} : r \leq 0\}$

Then

$$\sup E_1 = \sup E_2 = 0$$

and $0 \notin E_1$, $0 \in E_2$.

Example

Let $E = \{1/n : n = 1, 2, \cdots\}$. Then

$$\sup E = 1, \inf E = 0$$

and $1 \in E$ but $0 \notin E$.

Definition (The Supremum Axiom)

An ordered set S is said to have the *least-upper-bound property* if the following is ture:

If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

$\mathbb{Q}$ does not have the least-upper-bound property.

Theorem

Suppose S *is an ordered set with the least-upper-bound property,* $B \subset S$, B *is not empty, and* B *is bounded below. Let* L *be the set of all lower bounds of* B. *Then*

$$\alpha = \sup L$$

exists in S, *and* $\alpha = \inf B$. *In particular,* $\inf B$ *exists in* S.

Fields

Definition

A *field* is a set F with two operations, called *addition* and *multiplication*, which satisfy the following so-called "field axioms" (A), (M), and (D):

(A) Axioms for addition

- (A1) If $x, y \in F$, $x + y \in F$.
- (A2) If $x, y \in F$, $x + y = y + x$.
- (A3) If $x, y, z \in F$, $(x + y) + z = x + (y + z)$.
- (A4) $0 \in F$ such that $0 + x = x$ for all $x \in F$.
- (A5) For every $x \in F$, there is $-x \in F$ such that $x + (-1) = 0$.

(M) Axioms for multiplication

- (M1) If $x, y \in F$, $xy \in F$.
- (M2) If $x, y \in F$, $xy = yx$.
- (M3) If $x, y, z \in F$, $(xy)z = x(yz)$.
- (M4) $1 \in F \setminus \{0\}$ such that $1x = x$ for all $x \in F$.
- (M5) For every $x \in F \setminus \{0\}$, there is $1/x \in F$ such that $x(1/x) = 1$.

(D) The distributive law

For all $x, y, z \in F$, $x(y + z) = xy + xz$.

Example

$\mathbb{Q}$ is a field.

See LA1 for details.

Proposition

(A) implies

(a) If $x + y = x + z$, then $y = z$.

(b) If $x + y = x$, then $y = 0$.

(c) If $x + y = 0$, then $y = -x$.

(d) $-(-x) = x$.

(M) implies

(a) If $x \neq 0$ and $xy = xz$, then $y = z$.

(b) If $x \neq 0$ and $xy = x$, then $y = 1$.

(c) If $x \neq 0$ and $xy = 1$, then $y = 1/x$.

(d) If $x \neq 0$, then $1/(1/x) = x$.

The field axioms implies

(a) $0x = 0$.

(b) If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.

(c) $(-x)y = -(xy) = x(-y)$.

(d) $(-x)(-y) = xy$.

Definition

An *ordered field* is a field F which is also an *ordered set*, such that

(i) $x + y < x + z$ if $x, y, z \in F$ and $y < z$,

(ii) $xy > 0$ if $x, y \in F$, and $x > 0$ and $y > 0$.

If $x > 0$, we call x *is positive*; if $x < 0$, x *is negative*.

Proposition

The following statements are true in every ordered field.

(a) If $x > 0$, then $-x < 0$, and vice versa.

(b) If $x > 0$ and $y < z$, then $xy < xz$.

(c) If $x < 0$ and $y < z$, then $xy > xz$.

(d) If $x \neq 0$, then $x^2 > 0$. In particular, $1 > 0$.

(e) If $0 < x < y$, then $0 < 1/y < 1/x$.

The Real Field

Theorem

There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

Theorem

(a) *If* $x, y \in \mathbb{R}$ *and* $x > 0$, *then there is a positive integer* n *such that*

$$nx > y.$$

(b) *If* $x, y \in \mathbb{R}$ *and* $x < y$, *then there exists* $p \in \mathbb{Q}$ *such that*

$$x < p < y.$$

Theorem

For every real $x > 0$ and every integer $n > 0$, there is one and only one positive real y such that $y^n = x$. This number y is written $\sqrt[n]{x}$ or $x^{1/n}$ and called nth roots of x.

Corollary

If a and b are positive real numbers and n is a positive integer, then

$$(ab)^{1/n} = a^{1/n}b^{1/n}.$$

Decimals

Let $x > 0$ be real. Let n_0 be the largest integer such that $n_0 \leq x$. Having chosen $n_0, \cdots, n_{k-1}$, let n_k be the largest integer such that

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \leq x.$$

Let E be the set of these numbers

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \quad (k = 0, 1, 2, \cdots).$$

Then $x = \sup E$. The decimal expansion of x is

$$n_0.n_1n_2n_3 \cdots .$$

The Extended Real Number System

Definition

The extended real number system consists of the real field $\mathbb{R}$ and two symbols, $+\infty$ and $-\infty$. We preserve the original order in $\mathbb{R}$, and define

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$.

Definition

For convention, define

(a) If x is real, $x + \infty = +\infty, x - \infty = -\infty, \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$

(b) If $x > 0$, then $x \cdot (\pm\infty) = \pm\infty.$

(c) If $x < 0$, then $x \cdot (\pm\infty) = \mp\infty.$

Remark

For every subset of the extended real number system, $+\infty$ is an upper bound, and that every nonempty subset has a least upper bound. For example, if $E \subset \mathbb{R}$ is a nonempty unbounded subset, then $\sup E = +\infty$ in the extended real number system.

Euclidean Spaces

Definition

For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered k-tuples

$$\mathbf{x} = (x_1, \cdots, x_k),$$

where $x_1, \cdots, x_k$ are real numbers, called the *coordinates* of $\mathbf{x}$. The elements of $\mathbb{R}^k$ are called points, or vectors, especially when $k > 1$. If $\mathbf{y} = (y_1, \cdots, y_k)$ and α is a real number, put

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \cdots, x_k + y_k),$$

$$\alpha\mathbf{x} = (\alpha x_1, \cdots, \alpha x_k)$$

so that $\mathbf{x} + \mathbf{y}, \alpha\mathbf{x} \in \mathbb{R}^k$. These two operations make $\mathbb{R}^k$ into a *vector space over the real field*. The zero element of $\mathbb{R}^k$ is the point $\mathbf{0} = (0, 0, \cdots, 0)$.

Definition

Define the *inner product* of $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{k} x_i y_i$$

and the *norm* of $\mathbf{x}$ by

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(\sum_{i=1}^{k} x_i^2 \right)^{1/2}.$$

The structure is called euclidean k-space.

Theorem

Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$ *and* α *is real. Then*

(a) $|\mathbf{x}| \geq 0$;

(b) $|\mathbf{x}| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$;

(c) $|\alpha\mathbf{x}| = |\alpha||\mathbf{x}|$;

(d) $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$;

(e) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$;

(f) $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$.

Ex1.1

If r is rational $(r \neq 0)$ and x is irrational, prove that $r + x$ and rx are irrational.

Ex1.4

Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Ex1.5

Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Ex1.6

Fix $b > 1$.

(a) IF m, n, p, q are integers, $n > 0, q > 0$ and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Ex1.6

Fix $b > 1$.

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rationals.

Ex1.6

Fix $b > 1$.

(c) If x is real, define $B(x)$ to be the set of all numbers b^t where t is rational and $t \leq x$. Prove that $b^r = \sup B(r)$ where r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.

Ex1.6

Fix $b > 1$.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Ex1.7

Fix $b > 1$, $y > 0$.

(a) For any positive integer n, $b^n - 1 \geq n(b-1)$.

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

Ex1.7

Fix $b > 1$, $y > 0$.

(c) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

Ex1.7

Fix $b > 1$, $y > 0$.

(d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n.

(e) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

Ex1.7

Fix $b > 1$, $y > 0$.

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

Ex1.7

Fix $b > 1$, $y > 0$.

(g) Prove that this x is unique.

This x is called the *logarithm of* y *to the base* b.

The End