Analysis
- PMA 14 -

KYB

Thrn, it's a Fact

mathrnfact@gmail.com

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Overview

Power Series

- If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $|x-a| < R$, f is said to be expanded in a power series about the point $x = a$.
- Without loss of generality, we may assume $a = 0$.

Theorem

Suppose the series

$$\sum_{n=0}^{\infty} c_n x^n$$

converges for $|x| < R$, and define

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad (|x| < R).$$

Then the series converges uniformly on $[-R+\epsilon, R-\epsilon]$, no matter which $\epsilon > 0$ is chosen. The function f is continuous and differentiable in $(-R, R)$, and

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad (|x| < R).$$

Power Series

Corollary

Under the hypotheses of above Theorem, f has derivatives of all orders in $(-R, R)$, which are given by

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n x^{n-k}.$$

In particular,

$$f^{(k)}(0) = k!c_k \quad (k = 0, 1, 2, \cdots).$$

Exercise

Remark

Although a function f may have derivatives of all orders (smooth function), the series $\sum c_n x^n$, where $c_n = f^{(n)}(0)/n!$, need not converge to $f(x)$ for any $x \neq 0$.

Ex 8.1

Define

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \cdots$.

Power Series

Theorem

Suppose $\sum c_n$ *converges. Put*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad (-1 < x < 1).$$

Then

$$\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n.$$

Power Series

Application (another proof of Theorem 3.51)

Theorem 3.51 If $\sum a_n$, $\sum b_n$, $\sum c_n$, converges to A, B, C, and if $c_n = a_0 b_n + \cdots + a_n b_0$, then $C = AB$.

Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n, \quad h(x) = \sum_{n=0}^{\infty} c_n x^n$$

for $0 \leq x \leq 1$. For $0 \leq x < 1$,

$$f(x) \cdot g(x) = h(x).$$

Then

$$f(x) \to A, \quad g(x) \to B, \quad h(x) \to C$$

as $x \to 1$. Hence, $AB = C$.

Power Series

Theorem

Given a double sequence $\{a_{ij}\}$, $i, j = 1, 2, 3, \cdots$, *suppose*

$$\sum_{j=1}^{\infty} |a_{ij}| = b_i \quad i = 1, 2, 3, \cdots ,$$

and $\sum b_i$ *converges. Then*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Exercises

Ex 8.2

Let a_{ij} be the number in the ith row and jth column of the array

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

so that

$$a_{ij} = \begin{cases} 0 & i < j, \\ -1 & i = j, \\ 2^{j-i} & i > j. \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \quad \sum_j \sum_i a_{ij} = 0.$$

Exercises

Ex 8.3

Prove that

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$$

if $a_{ij} \geq 0$ for all i and j (the case $+\infty = +\infty$ may occur).

Power Series

Theorem

Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n x^n,$$

the series converging in $|x| < R$. *If* $-R < a < R$, *then* f *can be expanded in a power series about the point* $x = a$ *which converges in* $|x - a| < R - |a|$, *and*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad (|x-a| < r - |a|).$$

Power Series

Theorem

Suppose the series $\sum a_n x^n$ *and* $\sum b_n x^n$ *converge in the segment* $S = (-R, R)$. *Let* E *be the set of all* $x \in S$ *at which*

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n.$$

If E *has a limit point in* S, *then* $a_n = b_n$ *for* $n = 0, 1, 2, \cdots$. *Hence the equality holds for all* $x \in S$.

The Exponential and Logarithmic Functions

Define

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

for all complex z. Then

- $E(z+w) = E(z)E(w)$;
- $E(0) = 1$;
- $E(z) \neq 0$ for all complex z;
- $E(x) > 0$ for all real x;
- $E(x) \to +\infty$ as $x \to +\infty$ along the real axis;
- $E(x) \to 0$ as $x \to -\infty$ along the real axis;
- if $0 < x < y$, then $E(x) < E(y)$ and $E(-y) < E(-x)$;
- hence E is strictly increasing on the whole real axis;
- $\lim_{h\to 0} \frac{E(z+h)-E(z)}{h} = E(z)$;
- $E(x) = e^x$ for all real x.

The Exponential and Logarithmic Functions

Theorem

Let e^x be defined on $\mathbb{R}^1$. Then

(a) e^x *is continuous and differentiable for all* x;

(b) $(e^x)' = e^x$;

(c) e^x *is a strictly increasing function of* x, *and* $e^x > 0$;

(d) $e^{x+y} = e^x e^y$;

(e) $e^x \to +\infty$ *as* $x \to +\infty$, $e^x \to 0$ *as* $x \to -\infty$;

(f) $\lim_{x \to +\infty} x^n e^{-x} = 0$, *for every* n.

The Exponential and Logarithmic Functions

Since E is strictly increasing and differentiable on $\mathbb{R}^1$, it has an inverse function L which is also strictly increasing and differentiable and whose domain is $E(\mathbb{R}^1)$. L is defined by

$$E(L(y)) = y \quad (y > 0)$$

or, equivalently, by

$$L(E(x)) = x \quad (x \text{ real}).$$

And

$$(L \circ E)'(x) = L'(E(x)) \cdot E(x) = 1.$$

Writing $y = E(x)$,

$$L'(y) = \frac{1}{y} \quad (y > 0).$$

Taking $x = 0$, then $L(1) = 0$. Hence

$$L(y) = \int_1^y \frac{dx}{x}.$$

The Exponential and Logarithmic Functions

- $L(uv) = L(u) + L(v)$; thus write $L(x) = \log x$.
- $\log x \to +\infty$ as $x \to +\infty$;
- $\log x \to -\infty$ as $x \to 0$;
- $x^\alpha = E(\alpha L(x))$ for $x > 0$, or $x^\alpha = e^{\alpha \log x}$;
- $(x^\alpha)' = \alpha x^{\alpha - 1}$;
- $\lim_{x \to +\infty} x^{-\alpha} \log x = 0$ for every $\alpha > 0$.

Exercises

Ex 8.4

(a) $\lim\limits_{x \to 0} \dfrac{b^x - 1}{x} = \log b$ for $b > 0$;

(b) $\lim\limits_{x \to 0} \dfrac{\log(1+x)}{x} = 1$;

(c) $\lim\limits_{x \to 0} (1+x)^{1/x} = e$;

(d) $\lim\limits_{n \to \infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$.

Ex 8.5

(a) $\lim\limits_{x \to 0} \dfrac{e - (1+x)^{1/x}}{x}$.

(b) $\lim\limits_{n \to \infty} \dfrac{n}{\log n}[n^{1/n} - 1]$.

Exercises

Ex 8.6

Suppose $f(x)f(y) = f(x+y)$ for all real x and y.

(a) Assuming that f is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where c is a constant.

(b) Prove the same thing, assuming only that f is continuous.

Exercises

Ex 8.9

(a) Put $s_N = 1 + (1/2) + \cdots + (1/N)$. Probe that

$$\lim_{N\to\infty} (s_N - \log N)$$

exsits.

(b) Roughly how large must m be so that $N = 10^m$ satisfies $s_N > 100$?

Exercises

Ex 8.10

Prove that $\sum 1/p$ diverges; the sum extends over all primes.

The Trigonometric Functions

Define

$$C(x) = \frac{1}{2}[E(ix) + E(-ix)], \quad S(x) = \frac{1}{2i}[E(ix) - E(-ix)]$$

for real x. Then for real x,

- $C(x)$, $S(x)$ are real
- $E(ix) = C(x) + iS(x)$; $|E(ix)| = 1$;
- $C(0) = 1$, $S(0) = 0$;
- $C'(x) = -S(x)$; $S'(x) = C(x)$;
- There is the smallest positive real x_0 such that $C(x_0) = 0$, and define $\pi = 2x_0$;
- $S(\pi/2) = 1$;
- $E(\pi i/2) = i$; $E(\pi) = -1$; $E(2\pi i) = 1$;
- $E(z + 2\pi i) = E(z)$ for every complex z;

The Trigonometric Functions

Theorem

(a) *The function E is periodic, with $2\pi i$.*

(b) *The functions C and S are periodic, with period 2π.*

(c) *If $0 < t < 2\pi$, then $E(it) \neq 1$.*

(d) *If z is a complex number with $|z| = 1$, there is a unique t in $[0, 2\pi)$ such that $E(it) = z$.*

Exercises

Ex 8.5

(c) $\displaystyle\lim_{x\to 0}\frac{\tan x - x}{x(1-\cos x)}$.

(d) $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{\tan x - x}$.

Exercises

Ex 8.7

If $0 < x < \frac{\pi}{2}$, prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

Exercises

Ex 8.8

For $n = 0, 1, 2, \cdots$, and x real, prove that

$$|\sin nx| \leq n|\sin x|.$$

Exercises

Ex 8.11

Suppose $f \in \mathscr{R}$ on $[0, A]$ for all $A < \infty$, and $f(x) \to 1$ as $x \to +\infty$. Prove that

$$\lim_{t \to 0} t \int_0^\infty e^{-tx} f(x)\, dx = 1 \quad (t > 0).$$

Exercises

Ex 8.23

Let γ be a continuously differentiable closed curve in the complex plane, with parameter interval $[a, b]$, and assume $\gamma(t) \neq 0$ for every $t \in [a, b]$. Define the index of γ to be

$$\mathrm{Ind}(\gamma) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)}\, dt.$$

Prove that $\mathrm{Ind}(\gamma)$ is always an integer.

Exercises

Ex 8.24

Let γ be as in Ex 8.23, and assume in addition that the range of γ does not intersect the negative real axis. Prove that $\mathrm{Ind}(\gamma) = 0$.

Exercises

Ex 8.25

Suppose γ_1 and γ_2 are curves as in Ex 8.23, and

$$|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)| \quad (a \le t \le b).$$

Prove that $\mathrm{Ind}(\gamma_1) = \mathrm{Ind}(\gamma_2)$.

The Algebraic Completeness of the Complex Field

Theorem

Suppose $a_0, \cdots, a_n$ *are complex numbers,* $n \geq 1$, $a_n \neq 0$,

$$P(z) = \sum_0^n a_k z^k.$$

Then $P(z) = 0$ *for some complex number* z.

The End