Analysis
- PMA 17 -

KYB

Thrn, it's a Fact

mathrnfact@gmail.com

October 18, 2021

Overview

The Contraction Principle

Definition (9.22)

Let X be a metric space, with metric d. If φ maps X into X and if there is a number $c < 1$ such that

$$d(\varphi(x), \varphi(y)) \le cd(x, y)$$

for all $x, y \in X$, then φ is said to be a *contraction* of X into X.

Theorem (9.23)

If X is a complete metric space, and if φ is a contraction of X into X, then there exists one and only one $x \in X$ such that $\varphi(x) = x$.

The Inverse Function Theorem

Theorem (9.24)

Suppose $\mathbf{f}$ *is a* $\mathscr{C}'$*-mapping of an open set* $E \subset \mathbb{R}^n$ *into* $\mathbb{R}^n$, $\mathbf{f}'(\mathbf{a})$ *is invertible for some* $\mathbf{a} \in E$, *and* $\mathbf{b} = \mathbf{f}(\mathbf{a})$. *Then*

(a) *there exist open sets* U *and* V *in* $\mathbb{R}^n$ *such that* $\mathbf{a} \in U$, $\mathbf{b} \in V$, $\mathbf{f}$ *is one-to-one on* U, *and* $\mathbf{f}(U) = V$;

(b) *if* $\mathbf{g}$ *is the inverse of* $\mathbf{f}$, *defined in* V *by*

$$\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x},$$

then $\mathbf{g} \in \mathscr{C}'(V)$.

The Inverse Function Theorem

Theorem (9.25)

If $\mathbf{f}$ is a $\mathscr{C}'$-mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and if $\mathbf{f}'(\mathbf{x})$ is invertible for every $\mathbf{x} \in E$, then $\mathbf{f}(W)$ is an open subset of $\mathbb{R}^n$ for every open set $W \subset E$. In other words, $\mathbf{f}$ is an open mapping of E into $\mathbb{R}^n$.

The Implicit Function Theorem

Notation(9.26)

- If $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \cdots, y_m) \in \mathbb{R}^m$, let us write $(\mathbf{x}, \mathbf{y})$ for the point
$$(x_1, \cdots, x_n, y_1, \cdots, y_m) \in \mathbb{R}^{n+m}.$$
- Every $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ can be split into two linear transformations A_x and A_y by
$$A_x\mathbf{h} = A(\mathbf{h}, \mathbf{0}), \quad A_y\mathbf{k} = A(\mathbf{0}, \mathbf{k})$$
for $\mathbf{h} \in \mathbb{R}^n$, $\mathbf{k} \in \mathbb{R}^m$. Then $A_x \in L(\mathbb{R}^n)$, $A_y \in L(\mathbb{R}^m, \mathbb{R}^n)$, and
$$A(\mathbf{h}, \mathbf{k}) = A_x\mathbf{h} + A_y\mathbf{k}.$$

The Implicit Function Theorem

Theorem (9.27)

If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and if A_x is invertible, then there corresponds to every $\mathbf{k} \in \mathbb{R}^m$ a unique $\mathbf{h} \in \mathbb{R}^n$ such that $A(\mathbf{h}, \mathbf{k}) = \mathbf{0}$.

The Implicit Function Theorem

Theorem (9.28)

Let $\mathbf{f}$ *be a* $\mathscr{C}'$*-mapping of an open set* $E \subset \mathbb{R}^{n+m}$ *into* $\mathbb{R}^n$*, such that* $\mathbf{f}(\mathbf{a},\mathbf{b}) = \mathbf{0}$ *for some point* $(\mathbf{a},\mathbf{b}) \in E$*.*
Put $A = \mathbf{f}'(\mathbf{a},\mathbf{b})$ *and assume that* A_x *is invertible.*
Then there exist open sets $U \subset \mathbb{R}^{n+m}$ *and* $W \subset \mathbb{R}^m$*, with* $(\mathbf{a},\mathbf{b}) \in U$ *and* $\mathbf{b} \in W$*, having the following property: To every* $\mathbf{y} \in W$ *corresponds a unique* $\mathbf{x}$ *such that*

$$(\mathbf{x},\mathbf{y}) \in U \quad \text{and} \quad \mathbf{f}(\mathbf{x},\mathbf{y}) = 0.$$

If this $\mathbf{x}$ *is defined to be* $\mathbf{g}(\mathbf{y})$*, then* $\mathbf{g}$ *is a* $\mathscr{C}'$*-mapping of* W *into* $\mathbb{R}^n$*,* $\mathbf{g}(\mathbf{b}) = \mathbf{a}$*,*

$$\mathbf{f}(\mathbf{g}(\mathbf{y}),\mathbf{y}) = \mathbf{0} \quad \mathbf{y} \in W,$$

and

$$\mathbf{g}'(\mathbf{b}) = -(A_x)^{-1} A_y.$$

The Implicit Function Theorem

Example (9.29)

Take $n = 2$, $m = 3$, and $\mathbf{f} = (f_1, f_2)$ given by

$$\begin{aligned} f_1(x_1, x_2, y_1, y_2, y_3) &= 2e^{x_1} + x_2 y_1 - 4y_2 + 3 \\ f_2(x_1, x_2, y_1, y_2, y_3) &= x_2 \cos x_1 - 6x_1 + 2y_1 - y_3. \end{aligned}$$

If $\mathbf{a} = (0, 1)$ and $\mathbf{b} = (3, 2, 7)$, then $\mathbf{f}(\mathbf{a}, \mathbf{b}) = 0$.

$$[A] = \begin{bmatrix} 2 & 3 & 1 & -4 & 0 \\ -6 & 1 & 2 & 0 & -1 \end{bmatrix} \Longrightarrow [A_x] = \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}, [A_y] = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

and

$$[(A_x)^{-1}] = \frac{1}{20} \begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}$$

gives

$$[\mathbf{g}'(3, 2, 7)] = -\frac{1}{20} \begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{5} & -\frac{3}{20} \\ -\frac{1}{2} & \frac{6}{5} & \frac{1}{10} \end{bmatrix}$$

The Rank Theorem

Definition (9.30)

Suppose X and Y are vector spaces, and $A \in L(X, Y)$.

- The *null space* of A, $\mathcal{N}(A) = \{\mathbf{x} \in X : A\mathbf{x} = \mathbf{0}\}$.
- The *range* of A, $\mathcal{R}(A) = \{A\mathbf{x} : \mathbf{x} \in X\}$.
- The *rank* of A is defined to be the dimension of $\mathcal{R}(A)$.

The Rank Theorem

Projections(9.31)

Let X be a vector space. An operator $P \in L(X)$ is said to be a *projection* in X if $P^2 = P$.

(a) If P is a projection in X, then every $\mathbf{x} \in X$ has a unique representation of the form

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$$

where $\mathbf{x}_1 \in \mathcal{R}(P)$, $\mathbf{x}_2 \in \mathcal{N}(P)$.

(b) If X is a finite-dimensional vector space and if X_1 is a vector space in X, then there is a projection P in X with $\mathcal{R}(P) = X_1$.

The Rank Theorem

Theorem (9.32)

Suppose m, n, r are nonnegative integers, $m \geq r$, $n \geq r$, $\mathbf{F}$ is a $\mathscr{C}'$-mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and $\mathbf{F}'(\mathbf{x})$ has rank r for every $\mathbf{x} \in E$.
Fix $\mathbf{a} \in E$, put $A = \mathbf{F}'(\mathbf{a})$, let Y_1 be the range of A, and let P be a projection in $\mathbb{R}^m$ whose range is Y_1. Let Y_2 be the null space of P.
Then there are open sets U and V in $\mathbb{R}^n$, with $\mathbf{a} \in U \subset E$, and there is a $1-1$ $\mathscr{C}'$-mapping $\mathbf{H}$ of V onto U such that

$$\mathbf{F}(\mathbf{H}(\mathbf{x})) = A\mathbf{x} + \varphi(A\mathbf{x})$$

where φ is a $\mathscr{C}'$-mapping of the open set $A(V) \subset Y_1$ into Y_2.

Exercises

Ex 9.17

Let $\mathbf{f} = (f_1, f_2)$ be the mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$f_1(x, y) = e^x \cos y, \quad f_2(x, y) = e^x \sin y.$$

(a) What is the range of f?

(b) Show that the Jacobian of f is not zero at any point of $\mathbb{R}^2$. Thus every point of $\mathbb{R}^2$ has a neighborhood in which f is one-to-one. Nevertheless, f is not one-to-one on $\mathbb{R}^2$.

(c) Put $\mathbf{a} = (0, \pi/3)$, $\mathbf{b} = f(\mathbf{a})$, let $\mathbf{g}$ be the continuous inverse of $\mathbf{f}$, defined in a neighborhood of $\mathbf{b}$, such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Find an explicit formula for $\mathbf{g}$, compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verify the $\mathbf{g}'(\mathbf{b}) = \{\mathbf{f}'(\mathbf{g}(\mathbf{b}))\}^{-1}$.

Exercises

Ex 9.19

Show that the system of equations

$$3x + y - z + u^2 = 0$$
$$x - y + 2z + u = 0$$
$$2x + 2y - 3z + 2u = 0$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

Exercises

Ex 9.21

Define f in $\mathbb{R}^2$ by

$$f(x,y) = 2x^3 - 3x^2 + 2y^3 + 3y^2.$$

(a) Find the four points in $\mathbb{R}^2$ at which the gradient of f is zero. Show that f has exactly one local maximum and one local minimum in $\mathbb{R}^2$.

(b) Let S be the set of all $(x,y) \in \mathbb{R}^2$ at which $f(x,y) = 0$. Find those points of S have no neighborhoods in which the equation $f(x,y) = 0$ can be solved for y in terms of x.

Exercises

Ex 9.23

Define f in $\mathbb{R}^3$ by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$

Show that $f(0, 1, -1) = 0$ and $(D_1 f)(0, 1, -1) \neq 0$, and that there exists therefore differential function g in some neighborhood of $(1, -1)$ in $\mathbb{R}^2$, such that $g(1, -1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0.$$

Find $(D_1 g)(1, -1)$ and $(D_2 g)(1, -1)$.

Exercises

Ex 9.24

For $(x, y) \neq (0, 0)$, define $\mathbf{f} = (f_1, f_2)$ by

$$f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad f_2(x, y) = \frac{xy}{x^2 + y^2}.$$

Compute the rank of $\mathbf{f}'(x, y)$, and find the range of $\mathbf{f}$.

Exercises

Ex 9.25

Suppose $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, let r be the rank of A.

(a) Define S as in the proof of Theorem 9.32. Show that SA is a projection in $\mathbb{R}^n$ whose null space is $\mathcal{N}(A)$ and whose range is $\mathcal{R}(S)$.

(b) Use (a) to show that

$$\dim \mathcal{N}(A) + \dim \mathcal{R}(A) = n.$$

The End