Analysis
- PMA 15 -

KYB

Thrn, it's a Fact

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Overview

Fourier Series

Definition

A trigonometric polynomial is a finite sum of the form

$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \quad (x \text{ real})$$

where a_n, b_n are complex numbers. So it can be written in the form

$$f(x) = \sum_{-N}^{N} c_n e^{inx} \quad (x \text{ real})$$

Every trigonometric polynomian is periodic with period 2π.
Since $(e^{inx}/in)' = e^{inx}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\, dx = \begin{cases} 1 & , n = 0 \\ 0 & n = \pm 1, \pm 2, \cdots . \end{cases}$$

Then

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-imx}\, dx$$

for $|m| \le N$. If $|m| > N$, $c_m = 0$.

Remark

The trigonometric polynomial f is real if and only if $c_{-n} = \overline{c}_n$ for $n = 0, 1, 2, \cdots, N$.

Fourier Series

Definition

A trigonometric series is a series of the form

$$\sum_{-\infty}^{\infty} c_n e^{inx} \quad (x \text{ real})$$

If f is an integrable function on $[-\pi, \pi]$, the numbers c_m are called the Fourier coefficients of f, and the series is called the Fourier series of f.

Fourier Series

Definition

Let $\{\phi_n\}$ be a sequence of complex functions on $[a, b]$, such that

$$\int_a^b \phi_n(x)\overline{\phi_m(x)}\,dx = 0 \quad m \neq n.$$

Then $\{\phi_n\}$ is said to be an orthogonal system of functions on $[a, b]$. If, in addition,

$$\int_a^b |\phi_n(x)|^2\,dx = 1$$

for all n, $\{\phi_n\}$ is said to be orthonormal.

Definition

If $\{\phi_n\}$ is orthonormal on $[a, b]$ and if

$$c_n = \int_a^b f(t)\overline{\phi_n(t)}\,dt,$$

we call c_n the nth Fourier coefficient of f relative to $\{\phi_n\}$. We write

$$f(x) \sim \sum_1^\infty c_n\phi_n(x)$$

and call this series the Fourier series of f relative to $\{\phi_n\}$.

Fourier Series

Example

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots$$

Fourier Series

Theorem

Let $\{\phi_n\}$ *be orthonormal on* $[a,b]$. *Let*

$$s_n(x) = \sum_{m=1}^{n} c_m \phi_m(x)$$

be the n*th partial sum of the Fourier series of* f, *and suppose*

$$t_n(x) = \sum_{m=1}^{n} \gamma_m \phi_m(x).$$

Then

$$\int_a^b |f - s_n|^2 \, dx \le \int_a^b |f - t_n| \, dx,$$

and equality holds if and only if

$$\gamma_m = c_m, \quad m = 1, 2, \cdots, n.$$

Fourier Series

Theorem (Bessel Inequality)

If $\{\phi_n\}$ *is orthonormal on* $[a,b]$, *and if*

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_x(x),$$

then

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \int_a^b |f(x)|^2 \, dx.$$

In particular,

$$\lim_{n\to\infty} c_n = 0.$$

Fourier Series

Trigonometric Series

Consider f that have period 2π and that are Riemann-integrable on $[-\pi, \pi]$. Let

$$s_N(x) = s_N(f; x) = \sum_{-N}^{N} c_n e^{inx}$$

where

$$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t)\overline{e^{inx}}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t)e^{-inx}\, dx.$$

Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |s_N(x)|^2\, dx = \sum_{-N}^{N} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx.$$

Fourier Series

Trigonometric Series

$$D_N(x) = \sum_{-N}^{N} e^{inx} = \frac{\sin(N+\frac{1}{2})x}{\sin(x/2)}$$

is called the Dirichlet kernel. Then

$$s_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)\,dt.$$

Fourier Series

Theorem

If, for some x, there are constants $\delta > 0$ and $M < \infty$ such that

$$|f(x+t) - f(x)| \leq M|t|$$

for all $t \in (-\delta, \delta)$, then

$$\lim_{N \to \infty} s_N(f; x) = f(x).$$

Corollary

If $f(x) = 0$ for all x in some segment J, then $\lim s_N(f; x) = 0$ for every $x \in J$.

Fourier Series

Theorem

If f is continuous (with period 2π) and if $\epsilon > 0$, then there is a trigonometric polynomial P such that

$$|P(x) - f(x)| < \epsilon$$

for all real x.

Fourier Series

Parseval's Theorem

Suppose f and g are Riemann-integrable functions with period 2π, and

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}, \quad g(x) = \sum_{-\infty}^{\infty} \gamma_n e^{inx}.$$

Then

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - s_N(f;x)|^2\,dx &= 0,\\
\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx &= \sum_{-\infty}^{\infty} c_n\overline{\gamma}_n,\\
\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx &= \sum_{-\infty}^{\infty} |c_n|^2.
\end{aligned}$$

Exercises

Ex 8.12

Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \leq \delta$, $f(x) = 0$ if $\delta < |x| \leq \pi$, and $f(x + 2\pi) = f(x)$ for all x.

(a) Compute the Fourier coefficients of f.

(b) Conclude that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \quad 0 < \delta < \pi.$$

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2 \delta} = \frac{\pi - \delta}{2}.$$

(d) Let $\delta \to 0$ and prove that

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}.$$

(e) Put $\delta = \pi/2$.

Exercises

Ex 8.13

Put $f(x) = x$ if $0 \leq x < 2\pi$, and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Ex 8.14

If $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$, prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Exercises

Ex 8.15

With the Dirichlet kernel D_N, put

$$K_n(x) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(x).$$

Prove that

$$K_n(x) = \frac{1}{N+1} \cdot \frac{1-\cos(N+1)x}{1-\cos x}$$

and that

(a) $K_N \geq 0$,

(b) $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} K_N(x)\, dx = 1$,

(c) $K_n(x) \leq \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta}$ if $0 < \delta \leq |x| \leq \pi$.

If $s_N = s_N(f;x)$, consider

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

Prove that

$$\sigma_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\, dt,$$

and hence prove that if f is continuous, with period 2π, then $\sigma_N(f;x) \to f(x)$ uniformly on $[-\pi, \pi]$.

Exercises

Ex 8.16

Prove a pointwise version of Fejér's theorem:
If $f \in \mathscr{R}$ and $f(x+)$, $f(x-)$ exist for some x, then

$$\lim_{N \to \infty} \sigma_N(f; x) = \frac{1}{2}[f(x+) + f(x-)].$$

Exercises

Ex 8.17

Assume f is bounded and monotonic on $[-\pi, \pi)$, with Fourier coefficients c_n.

(a) Use Ex 6.17 to prove that $\{nc_n\}$ is a bounded sequence.

(b) Combine (a) with Ex 8.16 and with Ex 3.14(e), to conclude that

$$\lim_{N\to\infty} s_N(f;x) = \frac{1}{2}[f(x+) + f(x-)]$$

for every x.

(c) Assume only that $f \in \mathscr{R}$ on $[-\pi, \pi]$ and that f is monotonic in some segment $(\alpha, \beta) \subset [-\pi, \pi]$. Prove that the conclusion of (b) holds for every $x \in (\alpha, \beta)$.

The Gamma Function

Definition

For $0 < x < \infty$,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt.$$

The integral converges for these x.

The Gamma Function

Theorem

(a) *The functional equation*

$$\Gamma(x+1) = x\Gamma(x)$$

holds if $0 < x < \infty$.

(b) $\Gamma(n+1) = n!$ *for* $n = 1, 2, 3, \cdots$.

(c) $\log \Gamma$ *is convex on* $(0, \infty)$.

The Gamma Function

Theorem

If f *is a positive function on* $(0, \infty)$ *such that*

(a) $f(x+1) = xf(x)$,

(b) $f(1) = 1$,

(c) $\log f$ *is convex*,

then $f(x) = \Gamma(x)$.

The Gamma Function

Remark

When $0 < x < 1$,

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)},$$

and since $\Gamma(x+1) = x\Gamma(x)$, this equality holds for all $x > 0$.

Theorem (Beta Function)

If $x, y > 0$, *then*

$$\int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

This integral is the so-called beta function $B(x, y)$.

The Gamma Function

Some consequences

$t = \sin^2\theta$ induces

$$2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\, d\theta = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The special case $x = y = \frac{1}{2}$ gives $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
Then

$$\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right).$$

The Gamma Function

Stirling's Formula

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{(x/e)^x\sqrt{2\pi x}} = 1.$$

Proof.

Put $t = x(1+u)$ in

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt,$$

and then

$$\Gamma(x+1) = x^{x+1}e^{-x}\int_{-1}^\infty [(1+u)e^{-u}]^x\,du.$$

Determine $h(u)$ so that $h(0) = 1$ and

$$(1+u)e^{-u} = \exp\left[-\frac{u^2}{2}h(u)\right]$$

if $-1 < u < \infty$, $u \neq 0$. Then

$$h(u) = \frac{2}{u^2}[u - \log(1+u)].$$

The Gamma Function

Proof.

Then h is continuous, and $h(u)$ decreases monotonically from ∞ to 0 as u increases from -1 to ∞.
The substitute $u = s\sqrt{2/x}$

$$\Gamma(x+1) = x^x e^{-x}\sqrt{2x}\int_{-\infty}^{\infty}\psi_x(s)\,ds$$

where

$$\psi_x(s) = \begin{cases}\exp[-s^2 h(s\sqrt{2/x})] & (-\sqrt{x/2} < s < \infty),\\ 0 & s \le -\sqrt{x/2}.\end{cases}$$

$\psi_x(s)$ satisfies

(a) For every s, $\psi_x(s) \to e^{-s^2}$ as $x \to \infty$.

(b) The convergence in (a) is uniform on $[-A, A]$ for every $A < \infty$.

(c) When $x < 0$, then $0 < \psi_x(s) < e^{-s^2}$.

(d) when $s > 0$ and $x > 1$, then $0 < \psi_x(s) < \psi_1(s)$.

(e) $\int_0^\infty \psi_1(s)\,ds < \infty$.

Then the integral converges to $\sqrt{\pi}$ as $x \to \infty$. □

Exercises

Ex 8.22

If α is real and $-1 < x < 1$, prove Newton's binomial theorem

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n.$$

Exercises

Ex 8.30

use Stirling's formula to prove that

$$\lim_{x \to \infty} \frac{\Gamma(x+c)}{x^c \Gamma(x)} = 1$$

for every real constant c.

Exercises

Ex 8.26

Let γ be a closed curve in the complex plane (not necessarily differentiable) with parameter interval $[0, 2\pi]$, such that $\gamma(t) \neq 0$ for every $t \in [0, 2\pi]$.
Choose $\delta > 0$ so that $|\gamma(t)| > \delta$ for all $t \in [0, 2\pi]$. If P_1 and P_2 are trigonometric polynomials such that $|P_j(t) - \gamma(t)| < \delta/4$ for all $t \in [0, 2\pi]$, prove that

$$\mathrm{Ind}(P_1) = \mathrm{Ind}(P_2)$$

by applying Ex 8.25.
Define this common value to be $\mathrm{Ind}(\gamma)$.
Prove that the statements of Ex 8.24 and Ex 8.25 hold without any differentiability assumption.

Exercises

Ex 8.27

Let f be a continuous complex function defined in the complex plane. Suppose there is a positive integer n and a complex number $c \neq 0$ such that

$$\lim_{|z|\to\infty} z^{-n} f(z) = c.$$

Prove that $f(z) = 0$ for at least one complex number z.

Exercises

Ex 8.28

Let $\overline{D}$ be the closed unit disc in the complex plane. Let g be a continuous mapping of $\overline{D}$ into the unit circle T. Prove that $g(z) = -z$ for at least one $z \in T$.

Exercises

Ex 8.29, Brouwer's fixed-point theorem

Prove that every continuous mapping f of $\overline{D}$ into $\overline{D}$ has a fixed point in $\overline{D}$.

The End