Analysis
- PMA 7 -

KYB

Thrn, it's a Fact

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Overview

Power Series

Definition

Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a power series. The numbers c_n are called the coefficients of the series; z is a complex number.

Power Series

Theorem

Given the power seires $\sum c_n z^n$, put

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \ R = \frac{1}{\alpha}.$$

Then

- $\sum c_n$ *converges if* $|z| < R$,
- $\sum c_n$ *diverges if* $|z| > R$.

R *is called the radius of convergence of* $\sum c_n z^n$.

Power Series

Example

(a) $\sum n^n z^n$.

(b) $\sum \frac{z^n}{n!}$.

(c) $\sum z^n$.

(d) $\sum \frac{z^n}{n}$.

(e) $\sum \frac{z^n}{n^2}$.

Summation by Parts

Theorem

Given two sequences $\{a_n\}$, $\{b_n\}$, *put*

$$A_n = \sum_{k=0}^{n} a_k$$

and put $A_{-1} = 0$. *If* $0 \le p \le .q$, *we have*

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

Summation by Parts

Theorem

Suppose

(a) *the partial sums* A_n *of* $\sum a_n$ *form a bounded sequences;*

(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;

(c) $\lim b_n = 0$.

Then $\sum a_n b_n$ *converges.*

Summation by Parts

Theorem (Alternating Series)

Suppose

(a) $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$;

(b) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$;

(c) $\lim c_n = 0$.

Then $\sum c_n$ *converges.*

Absolute Convergence

Definition

The series $\sum a_n$ is said to converges absolutely if the series $\sum |a_n|$ converges.

Theorem

If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Remark

The converse does not hold. $a_n = \frac{(-1)^n}{n}$.

Addition and Multiplication of Series

Theorem

If $\sum a_n = A$ *and* $\sum b_n = B$*, then* $\sum(a_n + b_n) = A + B$ *and* $\sum ca_n = cA$

Addition and Multiplication of Series

Definition

Given $\sum a_n$ and $\sum b_n$, we put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

and call $\sum c_n$ the product of the two given series.

Addition and Multiplication of Series

Theorem

Suppose

(a) $\sum_0^\infty a_n$ *converges absolutely,*

(b) $\sum_0^\infty a_n = A$*,*

(c) $\sum_0^\infty b_n = B$*,*

(d) $c_n = \sum_0^n a_k b_{n-k}$.

Then

$$\sum_{n=0}^{\infty} c_n = AB.$$

Rearrangements

Definition

Let k_n be a sequence in which every positive integers once and only once. Putting

$$a'_n = a_{k_n}$$

we say that $\sum a'_n$ is a rearrangement of $\sum a_n$.

Rearrangements

Example

Consider

$$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots ,$$

$$s' = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots .$$

Rearrangements

Theorem

Let $\sum a_n$ be as series of real numbers which converges, but not absolutely. Suppose

$$-\infty \leq \alpha \leq \beta \leq \infty.$$

Then there exists a rearrangement of $\sum a'_n$ with partial sums s'_n such that

$$\liminf_{n\to\infty} s'_n = \alpha, \limsup_{n\to\infty} s'_n = \beta.$$

Rearrangements

Theorem

If $\sum a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.

Exercises

Ex 3.8

If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, then $\sum a_n b_n$ converges.

Exercises

Ex 3.13

Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Exercises

Ex 3.20

Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

Exercises

Ex 3.21

If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a complete metric space X, if $E_n \supset E_{n+1}$, and if

$$\lim_{n\to\infty} \operatorname{diam} E_n = 0,$$

then $\bigcap_1^\infty E_n$ consists of exactly one point.

Exercises

Ex 3.22

Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove that Baire's theorem,

$\bigcap_1^\infty G_n$ is not empty, in fact it is dense in X.

Exercises

Ex 3.23

Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space. Show that the sequence $\{d(p_n, q_n)\}$ converges.

Exercises

Ex 3.24, Completion

Let X be a metric space.

(a) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X equivalent if

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

Exercises

Ex 3.24, Completion

Let X be a metric space.

(b) Let X^* be the set of all equivalence classes so obtained. If $P, Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n);$$

Show that the number $\Delta(P, Q)$ is an well-defined metric on X^*.

Exercises

Ex 3.24, Completion

Let X be a metric space.

(c) Prove that (X^*, Δ) is complete.

Exercises

Ex 3.24, Completion

Let X be a metric space.

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; Let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, $\varphi(p) = P_p$ is an isometry of X into X^*.

Exercises

Ex 3.24, Completion

Let X be a metric space.

(e) Prove that $\varphi(X)$ is dense in X^*, and $\varphi(X) = X^*$ if X is complete.

The End