Top13 Paracompactness, Complete Metric Spaces

KYB

Thrn, it's a Fact

mathrnfact@gmail.com

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Overview

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Recall

Definition

A collection $\mathcal{A}$ of subsets of X is said to be locally finite in X if every point of X has a neighborhood that intersects only finitely many elements of $\mathcal{A}$.

Definition

Let $\mathcal{A}$ be a collection of subsets of X. A collection $\mathcal{B}$ of subsets of X is said to be a refinement of $\mathcal{A}$ if for each element B of $\mathcal{B}$, there is an element A of $\mathcal{A}$ containing B.

Definition (Paracompact)

X is paracompact if every open covering $\mathcal{A}$ of X has a locally finite open refinement $\mathcal{B}$ that covers X.

Theorem (41.1)

Every paracompact Hausdorff space X is normal.

Theorem (41.2)

Every closed subspace of a paracompact space is paracompact.

Note

- A paracompact subspace of a Hausdorff space X need not be closed in X.
- A subspace of a paracompact space need not be paracompact.

Lemma (41.3)

Let X be regular. TFAE:
Every open covering of X has a refinement that is:

(1) An open covering of X and countably locally finite.

(2) A covering of X and locally finite.

(3) A closed covering of X and locally finite.

(4) An open covering of X and locally finite.

Theorem (41.4)

Every metrizable space is paracompact.

Theorem (41.4)

Every regular Lindelöf space is paracompact.

Note

- The product of two paracompact spaces need not be paracompact.
- $\mathbb{R}^\omega$ is paracompact in both the product and uniform topologies.
- It is not known whether $\mathbb{R}^\omega$ is paracompact in the box topology.
- The product space $\mathbb{R}^J$ is not paracompact if J is uncountable.

Definition

Let $\{U_\alpha\}$ be an indexed open covering of X. An indexed family of continuous functions $\phi_\alpha : X \to [0,1]$ is said to be a partition of unity on X dominated by $\{U_\alpha\}$ if

1. $\mathrm{supp}(\phi_\alpha) \subset U_\alpha$ for each α
2. $\{\mathrm{supp}(\phi_\alpha)\}$ is locally finite
3. $\sum \phi_\alpha(x) = 1$ for each x.

Lemma (41.6 Shrinking lemma)

Let X be a paracompact Hausdorff space; let $\{U_\alpha\}_{\alpha \in J}$ be an indexed family of open sets covering X. Then there exists a locally finite indexed family $\{V_\alpha\}_{\alpha \in J}$ of open sets covering X such that $\bar{V}_\alpha \subset U_\alpha$ for each α.

Theorem (41.7)

Let X be a paracompact Hausdorff space; let $\{U_\alpha\}_{\alpha \in J}$ be an indexed family of open sets covering X. Then there exists a partition of unity on X dominated by $\{U_\alpha\}$.

Partitions of unity are most often used in mathematics to "patch together" functions that are defined locally so as to obtain a function that is defined globally.

Theorem (41.8)

Let X be a paracompact Hausdorff space; let $\mathcal{C}$ be a collection of subsets of X; for each $C \in \mathcal{C}$, let ϵ_C be a positive number. If $\mathcal{C}$ is locally finite, there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) > 0$ for all x, and $f(x) \leq \epsilon_C$ for $x \in C$.

Ex41.1

Give an example to show that if X is paracompact, it does not follow that for every open covering $\mathcal{A}$ of X, there is a locally finite subcollection of $\mathcal{A}$ that covers X.

Proof.

In $\mathbb{R}$, $\mathcal{A} = \{(-\infty, n) : n \in \mathbb{Z}\}$. $\square$

Ex41.2(a)

Show that the product of a paracompact space and a compact space is paracompact.

Proof

Main idea : Given open cover $\mathcal{A}$, find refinement $\{U_\alpha \times V_{\alpha,i} : i = 1, \cdots, n_\alpha\}$ where $\bigcup V_{\alpha,i} = Y$.

Step1

Fix $x \in X$. Then we can find A_α's in $\mathcal{A}$ such that $\{x\} \times Y \subset \bigcup A_\alpha$. The trick is that

> we can find $U_\alpha \subset X$,$V_\alpha \subset Y$ such that $\{x\} \times Y \subset \bigcup U_\alpha \times V_\alpha$ and $U_\alpha \times V_\alpha \subset A_\alpha$ for some $A_\alpha \in \mathcal{A}$.

Then by the compactness of Y, we can fint U_i, V_i for $i = 1, \cdots, n$ such that $\{x\} \times Y \subset \bigcup_1^n U_i \times V_i$. By the tube lemma, we can find $W_x \subset X$ such that

$$\{x\} \times Y \subset W_x \times Y \subset \bigcup_1^n U_i \times V_i$$

Step2

For each $x \in X$, we can find $\{W_x\}$ in Step1 that covers X. By the paracompactness, we can find a locally finite open refinement $\mathcal{B} = \{R_\alpha\}$ that covers X. Then for each x find open O_x that intersects only finitely many elements in $\mathcal{B}$. Now for each α, there is x_α such that $R_\alpha \subset W_{x_\alpha}$. Then define $\mathcal{C} = \{R_\alpha \times V_{x_\alpha,i}\}$

Step3

Claim) $\mathcal{C}$ is a locally finite open refinement of $\mathcal{A}$.

- (refinement) $R_\alpha \times V_{x_\alpha,i} \subset W_{x_\alpha} \times V_{x_\alpha,i} \subset U_{x_\alpha,i} \times V_{x_\alpha,i} \subset A_{x_\alpha,i}$
- (open cover) Trivial.
- (locally finite)

Let $(x, y) \in X \times Y$. Then O_x intersects only finitely many R_α's, say $R_{\alpha_1}, \cdots, R_{\alpha_k}$. Then $O_x \times Y$ is the desired open neighborhood.

Ex.41.5 Expansion lemma

Let $\{B_\alpha\}_{\alpha \in J}$ be a locally finite indexed family of subsets of the paracompact Hausdorff space X. Then there is a locally finite indexed family $\{U_\alpha\}$ of open sets in X such that $B_\alpha \subset U_\alpha$ for each α.

Definition (Cauchy sequence)

Let (X, d) be a metric space. A sequence (x_n) of points of X is said to be a Cauchy sequence in (X, d) if it has the property that given $\epsilon > 0$, there is an integer N such that

$$d(x_n, x_m) < \epsilon \qquad \text{whenever } m, m \geq N$$

The metric space (X, d) is said to be complete if every Cauchy sequence in X converges.

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Lemma (43.1)

A metric space X is complete if every Cauchy sequence in X has a convergent subsequence.

Proof

Given $\epsilon > 0$, choose N large enough that $d(x_n, x_m) < \epsilon/2$ for all $n, m \geq N$. Then choose an i large enough that $n_i \geq N$ and $d(x_{n_i}, x) < \epsilon/2$. Than for $n \geq N$,

$$d(x_n, x) \leq d(x_n, x_{n_i}) + d(x_{n_i}, x) < \epsilon.$$

Lemma (43.3)

Let X be the product space $X = \prod X_\alpha$; let x_n be a sequence of points of X. Then $x_n \to x$ if and only if $\pi_\alpha(x_n) \to \pi_\alpha(x)$.

Definition

Let (Y, d) be a metric space; let $\bar{d}(a, b) = \min\{d(a, b), 1\}$ be the standard bounded metric on Y derived from d. If $x = (x_\alpha)_{\alpha \in J}$ and $x = (y_\alpha)_{\alpha \in J}$ are points of Y^J, let

$$\bar{\rho}(x, y) = \sup\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\}.$$

Theorem (43.5)

If Y is complete d, then the space Y^J is complete in $\bar{\rho}$.

Definition

$$\mathcal{C}(X, Y) \subset Y^X$$

is the set of all continuous functions from X to Y.

$$\mathcal{B}(X, Y) \subset Y^X$$

is the set of all bounded functions from X to Y.

Definition

Let M be a Euclidean space($\mathbb{R}^n$) or smooth n-manifold. Then we define $\mathcal{C}^k$ be the set of all function f such that f has n th derivative $f^{(n)}$ and $f^{(n)}$ is continuous.

Then

$$\mathcal{C} = \mathcal{C}^{(0)} \supset \mathcal{C}^{(1)} \supset \cdots .$$

Define $\mathcal{C}^\infty = \bigcap \mathcal{C}^n$ the set of all smooth functions.

Theorem (43.6)

X : top'l. (Y, d) metric.

- $\mathcal{C}(X, Y)$ is closed in Y^X under $\bar{\rho}$.
- $\mathcal{B}(X, Y)$ is closed in Y^X under $\bar{\rho}$.

Therefore, if Y is complete, these spaces are complete in $\bar{\rho}$.

Definition

(Y, d) metric. We can define another metric on $\mathcal{B}(X, Y)$ by

$$\rho(f, g) = \sup\{d(f(x), g(x)) | x \in X\}$$

For $f, g \in \mathcal{B}(X, Y)$, $\bar{\rho}(f, g) = \min\{\rho(f, g), 1\}$

Ex43.4

Show that the metric space (X, d) is complete if and only if for every nested sequence $A_1 \supset A_2 \supset \cdots$ of nonempty closed sets of X such that $\mathrm{diam} A_n \to 0$, the intersection of the sets A_n is nonempty.

Ex43.5

If (X, d) is a metric space, recall that a map $f : X \to X$ is called a contraction if there is a number $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$. Show that if f is a contraction of a complete metric space, then there is a unique point $x \in X$ such that $f(x) = x$.

Ex43.8

Define $e : X \times \mathcal{C}(X, Y) \to Y$ be the equation $e(x, f) = f(x)$. We call e the evaluation map. Show that if d is a metric for Y and $\mathcal{C}(X, Y)$ has the corresponding uniform topology, then e is continuous.

Ex43.9 Another completion

(X, d) :metric space. Let $\tilde{X}$ be the set of all Cauchy sequences $x = (x_n)_n$. Define $x \sim y$ if $d(x_n, y_n) \to 0$. Let $[x]$ be the equivalence class of x; let $Y = \tilde{X}/\sim$. Define a metric D on Y by

$$D([x], [y]) = \lim d(x_n, y_n)$$

Then Y is complete and there is an isometric imbedding $h : X \to Y$

In the set theory, there are two way to construct real number. One is 'Dedekind cut' and the other is 'completion'.

(a)

Show that $\sim$ is an equivalence relation, and show that D is a well-defined metric

(b)

Define $h : X \to Y$ by letting $h(x)$ be the equivalence class of the constant sequence $(x, x, \cdots)$. Show that h is an isometric imbedding.

(c)

Show that $h(X)$ is dense in Y; indeed, given $x = (x_n)_n \in \tilde{(}X)$, show the $h(x_n)$ of point of Y converges to the point $[x]$.

(d)

Show that if A is a dense subset of a metric space (Z, ρ), and if every Cauchy sequence in A converges in Z, then Z is complete.

(e)

Show that (Y, D) is complete.

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Ex43.10 Uniqueness of the completion

Let $h : X \to Y$ and $h' : X \to Y'$ be isometric imbeddinfs of the metric space (X, d) in the complete metric spaces (Y, D) and (Y', D'), respectively. Then there is an isometry of $(\overline{h(X)}, D)$ with $(\overline{h; (X)}, D')$ that equals $h'h^{-1}$ in the subspace $h(X)$.

Bump Function

Consider $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Prove that f is smooth.

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Bump function

Given $r_1 < r_2$, define $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) = \frac{f(r_2 - x)}{f(r_2 - x) + f(x - r_1)}.$$

Prove that h is smooth.

Bump function

Given any real number r_1 and r_2 such that $r_1 < r_2$, define $H : \mathbb{R}^n \to \mathbb{R}$ such that

$$H(x) = h(|x|).$$

Then

$$\begin{array}{cl} H \equiv 1 & \text{on } \overline{B}(0, r_1) \\ 0 < H < 1 & \text{for all } x \in B(0, r_2) - \overline{B}(0, r_1) \\ H \equiv 0 & \text{on } \mathbb{R}^n - B(0, r_2) \end{array} .$$

and H is smooth.