Top12 The Tychonoff Theorem and Local Finiteness

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37. The Tychonoff Theorem
Theorems
Exercises
39. Local Finiteness
Theorems
Exercises
41. Paracompact

Overview

37. The Tychonoff Theorem
Theorems
Exercises

39. Local Finiteness
Theorems
Exercises

41. Paracompact

37. The Tychonoff Theorem
Theorems
Exercises
39. Local Finiteness
Theorems
Exercises
41. Paracompact

The Axiom of Choice(Thomas Jech Set Theory)

Every family of nonempty sets has a choice function.

Definition (choice function)

Let $\mathcal{S} = \{U_\alpha\}_{\alpha \in J}$ with $\varnothing \notin \mathcal{S}$. We call f a **choice function** for $\mathcal{S}$ if

$$f\colon J \to \bigcup_{\alpha \in J} U_\alpha \text{ such that}$$

$$f(\alpha) \in U_\alpha \text{ for every } \alpha \in J$$

Note

1. If $U_\alpha = \{x_\alpha\}$ is a singleton for every $\alpha \in J$,
2. If J is finite,
3. If U_α is a finite set of real numbers for every $\alpha \in J$,

a choice function exists.

37. The Tychonoff Theorem

Theorems

Exercises

39. Local Finiteness

Theorems

Exercises

41. Paracompact

The Axiom of Choice(Munkres Topology)

Given a collection $\mathcal{A}$ of disjoin nonempty sets, there exists a set $\mathcal{C}$ consisting of exactly one element from each element of $\mathcal{A}$.
that is,

$$\mathcal{C} \subset \bigcup_{A \in \mathcal{A}} A$$

and $\mathcal{C} \cap A$ is a singleton.

Definition (Partial Order)

A binary relation $<$ on a set P is a **partial ordering** of P if

(i) $p \not< p$ for any $p \in P$

(ii) if $p < q$ and $q < r$, then $p < r$

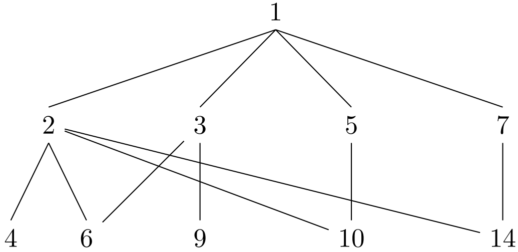
c.f. If $(X, <)$ is a (linear) order set, for any $x, y \in X$ either $x = y$,$x > y$, or $x < y$. However if $(P, <)$ is a partial order, it is possible that $x \neq y$ and $x \not> y$ and $x \not< y$.
Every order set is partial order set.

Example

Give a partial order $\prec$ on $\mathbb{N}$ as follows:
In general, for any $m, n \in \mathbb{N}$ there are $q, r \in \mathbb{N}$ such that

$$m = qn + r \qquad , 0 \leq r < n$$

If $r = 0$, we say n is a divisor of m, and denote $n|m$.
If $n|m$ and $m|n$, $m = n$.
Now define $\preceq$ by $m \preceq n$ if and only if n is a divisor of m. If $m \neq n$ and $m \preceq n$, $m \prec n$.
Then $(\mathbb{N}, \prec)$ is a partial order set.



Zorn's Lemma

Suppose a partially ordered set $(P, \prec)$ has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element.

Well-ordering theorem

Every set can be well-ordered.

Note

Zorn's Lemma $\Leftrightarrow$ Axiom of choice $\Leftrightarrow$ Well-ordering theorem

37. The Tychonoff Theorem
Theorems
Exercises
39. Local Finiteness
Theorems
Exercises
41. Paracompact

Exercise

Zorn's Lemma implies the Axiom of choice(in Thomas Jech Set Theory).

Proof.

Let $\mathcal{S}$ be a family of nonempty sets. Define $P = \{f : f$ is a choice function on some $\mathcal{Z} \subset \mathcal{S}\}$. Since every finite collection has a choice function, P is not empty.
Give a natural partial order $\subset$ on P. Let $\mathcal{C}$ be a chain in P and define $\overline{f} = \bigcup_{f \in \mathcal{C}} f$. Then for any $f \in \mathcal{C}$, $f \subset \overline{f}$. For each $f \in \mathcal{C}$ with $\mathcal{Z}_f \subset \mathcal{S}$ such that f is a choice function on $\mathcal{Z}_f$, define $\mathcal{Z} = \bigcup_{f \in \mathcal{C}} \mathcal{Z}_f$. Then $\mathcal{Z} \subset \mathcal{S}$ and f is a choice function on $\mathcal{Z}$. Thus $\overline{f} \in P$ and it is an upper bound of $\mathcal{C}$. By Zorn's Lemma, P has a maximal element f_m with $\mathcal{Z} \subset \mathcal{S}$. Suppose $\mathcal{Z} \neq \mathcal{S}$. Then there exsits $Z \in \mathcal{Z}$ such that f_m is not a choice function on $\mathcal{Z} \cup \{Z\}$. Choose $z_0 \in Z$ and define $f_m = f_m \cup \{(Z, z_0)\}$. Then $f_m \subsetneq f_m^*$ and f_m^* is a choice function on $\mathcal{Z} \cup \{Z\}$.(contradiction). Hence f_m is a choice function on $\mathcal{S}$. $\square$

How to use Zorn's Lemma

1. Construct a suitable collection of sets.
2. Define a partial order.
3. For given chain, find an upper bound.

Example

1. (Set theory) Every filter on a set X is contained in an ultrafilter.
2. (Differential geometry) Every smooth atlas $\mathcal{A}$ for a manifold M is contained in a unique maximal smooth chart.
3. (Algebra) In a commutative ring with 1, every proper ideal is contained in a maximal ideal.
4. (Linear Algebra) Every vector space has a basis.
5. (Field Theory) Every filed has an algebraic closure.

Lemma (37.1)

Let X be a set; let $\mathcal{A}$ be a collection of subsets of X having the F.I.P. Then there is a collection $\mathcal{D}$ of subsets of X such that $\mathcal{D}$ contains $\mathcal{A}$, and $\mathcal{D}$ has the F.I.P, and no collection of subsets of X that properly contains $\mathcal{D}$ has this property.

Lemma (37.2)

Let X be a set; let $\mathcal{D}$ be a collection of subsets of X that is maximal with respect to the F.I.P. Then

(a) *Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

(b) *If A is a subset of X that intersects every element of $\mathcal{D}$. then $A \in \mathcal{D}$.*

Theorem (Tychonoff theorem)

An arbitrary product of compact spaces is compact in the product topology.

Step1

Given a collection $\{X_\alpha\}$ of compact spaces, construct $\mathcal{A}$ of $X = \prod X_\alpha$ having the F.I.P.
Goal : $\bigcap_{A \in \mathcal{A}} \bar{A} \neq \varnothing$.

Step2

Extend $\mathcal{A}$ to a maixmal $\mathcal{D}$ with respect ti the F.I.P. It will suffice to show that $\bigcap_{D \in \mathcal{D}} \bar{D} \neq \varnothing$

Step3

Given $\alpha \in J$, consider $\{\overline{\pi_\alpha(D)} : D \in \mathcal{D}\}$. Then this collection has the F.I.P. Let $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$ and $\boldsymbol{x} = (x_\alpha)$.

Step4

$\boldsymbol{x} \in \bar{D}$ for every $D \in \mathcal{D}$.

Ex37.2

A collection $\mathcal{A}$ of subsets of X has the **countable intersection property** if every countable intersection of elements of $\mathcal{A}$ is nonempty. Show that X is a Lindelöf space if and only if for every collection $\mathcal{A}$ of subsets of X having the C.I.P, $\bigcap_{A\in\mathcal{A}} \bar{A}$ is nonempty.

37. The Tychonoff Theorem
Theorems
Exercises
39. Local Finiteness
Theorems
Exercises
41. Paracompact

Definition (Locally Finite)

Let X be a top'l space. A collection of $\mathcal{A}$ of subsets of X is said to be **locally finite** in X if every point of X has a nbd that intersects only finiely many elements of $\mathcal{A}$.

Lemma (39.1)

Let $\mathcal{A}$ be a locally finite collection of subsets of X. Then

(a) *Any subcollection of $\mathcal{A}$ is locally finite.*

(b) *The collection $\mathcal{B} = \{\bar{A}\}_{A\in\mathcal{A}}$ is locally finite.*

(c) $\overline{\bigcup_{A\in\mathcal{A}} A} = \bigcup_{A\in\mathcal{A}} \bar{A}$.

Definition (Countably locally Finite)

A collection $\mathcal{B}$ of subsets of X is said to be **countably locally finite** if $\mathcal{B}$ can be written as the countalbe union of collections $\mathcal{B}_n$, each of which is locally finite.

Definition

Let $\mathcal{A}$ be a collection of subsets of X. A collection of $\mathcal{B}$ of subsets of X is said to be a **refinement** of $\mathcal{A}$ if for each element B of $\mathcal{B}$, there is an element A of $\mathcal{A}$ containg B. If the elements of $\mathcal{B}$ are open sets, we call $\mathcal{B}$ an **open refinement** of $\mathcal{A}$; if the elements of $\mathcal{B}$ are closed sets, we call $\mathcal{B}$ an **closded refinement** of $\mathcal{A}$;

$\mathcal{A}$에서 불필요한 집합을 버린다는 이미지...

Ex39.1

- $\mathcal{A} = \{(n, n+2) : n \in \mathbb{Z}\}$ is locally fintie in $\mathbb{R}$.
- $\mathcal{B} = \{(0, 1/n) : n \in \mathbb{Z}_+\}$ is locally fintie in $(0, 1)$ but not in $\mathbb{R}$.
- $\mathcal{C} = \{(1/(n+1), 1/n) : n \in \mathbb{Z}\}$ is locally fintie in $\mathbb{R}$.

37. The Tychonoff Theorem
Theorems
Exercises
39. Local Finiteness
Theorems
Exercises
41. Paracompact

Ex39.2

Find a point-finite open covering $\mathcal{A}$ of $\mathbb{R}$ that is not locally finite.
$\mathcal{A}$ is point-finite if each point of $\mathbb{R}$ has in only finitely many element of $\mathcal{A}$.

Proof.

$\mathcal{A} = \{(0, 1/n) : n \in \mathbb{Z}_+\} \cup \{(-\infty, 1), (0, \infty)\}$. $\square$

Ex39.3

Give an example of a collection of sets $\mathcal{A}$ that is not locally finite, such that the collection $\mathcal{B} = \{\bar{A} : A \in \mathcal{A}\}$ is locally finite.

Proof.

Note that $\overline{\mathbb{Q} - \{q\}} = \mathbb{R}$. $\square$

37. The Tychonoff Theorem
Theorems
Exercises
39. Local Finiteness
Theorems
Exercises
41. Paracompact

Ex39.5

Show that if X has a countable basis, a collection $\mathcal{A}$ of subsets of X is countably locally finite if and only if it is countable.

37. The Tychonoff Theorem
Theorems
Exercises
39. Local Finiteness
Theorems
Exercises
41. Paracompact

Compact

1. Every open cover $\mathcal{A}$ of X has a finite subcover $\mathcal{B}$.
2. Every open cover $\mathcal{A}$ of X has a finite open refinement $\mathcal{B}$ that covers X.

Paracompact

Every open cover $\mathcal{A}$ of X has a locally finite refniement $\mathcal{B}$.

Using Paracompactness, we can find a **partition of unity**.

Definition

Let $\{U_\alpha\}$ be an indexed open covering of X. An indexed family of countinuous functions $\phi_\alpha : X \to [0,1]$ is said to be a partition of unity on X dominated by $\{U_\alpha\}$ if

1. $\text{supp}(\phi_\alpha) \subset U_\alpha$ for each α
2. $\{\text{supp}(\phi_\alpha)\}$ is locally finite
3. $\sum \phi_\alpha(x) = 1$ for each x.

37. The Tychonoff Theorem
Theorems
Exercises
39. Local Finiteness
Theorems
Exercises
41. Paracompact