Analysis
- PMA 2 -

KYB

Thrn, it's a Fact

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January 13, 2021

Overview

Natural Numbers

Remark

We can construct the set of all natural numbers using empty set $\varnothing$ and basic set theory : put $0 = \varnothing$, $S(0) = \{\varnothing\}$, $SS(0) = S(0) \cup \{S(0)\} = \{\varnothing, \{\varnothing\}\}$, $\cdots$,

$$\underbrace{S \cdots S}_{k+1 \text{ times}}(0) = \underbrace{S \cdots S}_{k \text{ times}}(0) \cup \{\underbrace{S \cdots S}_{k \text{ times}}(0)\}.$$

For convenient, write

$$n = \underbrace{S \cdots S}_{n \text{ times}}(0)$$

and

$$S(n) = n + 1.$$

Then the set $\mathbb{N}$ of all such n satisfies the natural number axioms.

Relations

To construct integers from natural numbers, we need some tools.

Definition

Let X be a nonempty set. A *relation* R is a subset of $X \times X$. If $(x, y) \in R$, write xRy.

Example

An order relation is a relation

Definition

A relation $\sim$ on X is called an *equivalence relation* if

(i) $x \sim x$ for all $x \in X$.

(ii) For $x, y \in X$, if $x \sim y$, then $y \sim x$.

(iii) For $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

Given $x \in X$, the set $\{y \in X : y \sim x\}$ is called the *equivalence class of* x, and denoted by $[x]_\sim$, or simply $[x]$.

Remark

Suppose $\sim$ is an equivalent relation on X. Then

(a) $X = \cup_{x \in X}[x]$.

(b) For $x, y \in X$, if $[x] \cap [y] \neq \varnothing$, $[x] = [y]$.

In this case, we say the set $\{[x] : x \in X\}$ is a partition of X, or the quotient set of X by $\sim$, and denoted by $X/\sim$.

Integers

Remark

Let $X = \mathbb{N} \times \mathbb{N}$ and define $\sim$ by

$$(m, n) \sim (p, q) \iff m + q = n + p.$$

(this relation comes from $m - n = p - q$). Then $\sim$ is an equivalence relation. Let $\mathbb{Z} = X/\sim$. We have a injective function $\iota : \mathbb{N} \to \mathbb{Z}$ by $\iota(n) = [(n, 0)]$. So we can identify $[(n, 0)]$ with n. More general, write $[(m, n)] = m - n$ and $[(0, n)] = -n$.

Remark

Let $m=[(m_1,m_2)], n=[(n_1,n_2)]\in\mathbb{Z}$. Define

$$m+n=[(m_1+n_1,m_1+n_2)],\quad mn=[(m_1n_1+m_2n_2,m_1n_2+m_2n_1)]$$

Then the addition and multiplication are well defined, i.e., if $[(m_1,m_2)]=[(p_1,p_2)]$ and $[(n_1,n_2)]=[(q_1,q_2)]$, then

$$\begin{aligned}[(m_1,m_2)]+[(n_1,n_2)]&=[(p_1,p_2)]+[(q_1,q_2)],\\ [(m_1,m_2)]\cdot[(n_1,n_2)]&=[(p_1,p_2)]\cdot[(q_1,q_2)].\end{aligned}$$

In particular, for $m,n\in\mathbb{N}$, $[(m,0)]+[(n,0)]$ and $[(m,0)]\cdot[(n,0)]$ are usual $m+n$ and mn.

Remark

Let $m = [(m_1, m_2)], n = [(n_1, n_2)] \in \mathbb{Z}$. Define

$$m \leq n \iff m_1 + n_2 \leq n_1 + m_2$$

This relation is well defined and for $m, n \in \mathbb{N}$, $[(m, 0)] \leq [(n, 0)]$ is usual $m \leq n$. Moreover, this relation is still ordered relation.

Rationals

Remark

Similarly, we can construct a rational m/n as follows: Let $X = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ and define $\sim$ by

$$(m,n) \sim (p,q) \iff mq = np$$

(this relation comes from $m/n = p/q$.) Then $\sim$ is an equivalence relation. Let $\mathbb{Q} = X/\sim$. $\mathbb{Q}$ has addition, multiplication and order:

$$\frac{m}{n} + \frac{p}{q} = \frac{mq+np}{nq}, \quad \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$$

$$\frac{m}{n} \leq \frac{p}{q} \iff \begin{cases} mq \leq np & \text{if } nq > 0 \\ mq \geq np & \text{if } nq < 0 \end{cases}$$

Reals

Remark

Let $p, q \in \mathbb{Q}$ with $p < q$. Then we have $p < (p+q)/2 < q$. Thus for given $q \in \mathbb{Q}$, the set

$$A = \{p \in \mathbb{Q} : p < q\}$$

satisfies

(i) A is nonempty and $A \neq \mathbb{Q}$.

(ii) If $p \in A$, $r \in \mathbb{Q}$ and $r < p$, then $r \in A$.

(iii) If $p \in A$, then $p < r$ for some $r \in A$.

Reals

Step1

A subset α is called a *cut* if

(i) α is nonempty and $\alpha \neq \mathbb{Q}$.

(ii) If $p \in \alpha$, $q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$.

(iii) If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.

Let $\mathbb{R}$ be the set of all cuts. Due to the above remark, $\mathbb{R}$ is nonempty.

Step 2

Now the letter $p, q, r, \cdots$ will be always rationals and $\alpha, \beta, \gamma, \cdots$ will denote cuts. Define $\alpha < \beta$ if α is a proper subset of β. Then this relation is an order.

Step 3

$\mathbb{R}$ has the least-upper-bound property.

Step 4

Define

$$\alpha + \beta = \{r + s : r \in \alpha, s \in \beta\},$$

and $0^* = \{q \in \mathbb{Q} : q < 0\}$. Then $\mathbb{R}$ satisfies the axiom (A) with zero element 0^*.

Step 5

If $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.

Step 6

Define $\mathbb{R}^+ = \{\alpha > 0^*\}$. If $\alpha, \beta \in \mathbb{R}^+$, define $\alpha\beta$ to be the set of all $p \leq rs$ for some $r \in \alpha, s \in \beta, r > 0, s > 0$. Define $1^* = \{q < 1\}$. Then the axioms (M) and (D) hold with $\mathbb{R}^*$ in place of F, and with 1^* in the role of 1.

If $\alpha > 0^*$ and $\beta > 0^*$, then $\alpha\beta > 0^*$.

Step 7

We complete the definition of multiplication by setting $\alpha 0^* = 0^* \alpha = 0^*$, and by setting

$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0^*, \beta < 0^*, \\ -[(-\alpha)\beta] & \text{if } \alpha < 0^*, \beta > 0^*, \\ -[\alpha(-\beta)] & \text{if } \alpha > 0^*, \beta < 0^*. \end{cases}$$

Step 8

For each $r \in \mathbb{Q}$, let $r^* = \{p \in \mathbb{Q} : p < r\}$. Then

(a) $r^* + s^* = (r+s)^*$,

(b) $r^* s^* = (rs)^*$,

(c) $r^* < s^*$ if and only if $r < s$.

Step 9

By step 8, $\mathbb{Q}^* = \{r^* : r \in \mathbb{Q}\}$ is isomorphic to $\mathbb{Q}$ (preserves sums, product, and order). Thus we can identify r with r^*. In this sense, $\mathbb{Q} \subset \mathbb{R}$.

Complex Numbers

Definition

A complex number is an ordered pair (a, b) of real numbers. Let $x = (a, b)$ and $y = (c, d)$ be two complex numbers. We write $x = y$ if and only if $a = c$ and $b = d$. Define

$$x + y = (a + c, b + d), \quad xy = (ac - bd, ad + bc).$$

Theorem

These definition of addition and multiplication turn the set of all complex numbers into a field, with $(0, 0)$ and $(1, 0)$ in the role of 0 and 1.

Theorem

For any real numbers a and b,

$$(a, 0) + (b, 0) = (a + b, 0) \quad (a, 0)(b, 0) = (ab, 0).$$

Now we can identify $(a, 0)$ with a.

Definition

Define $i = (0, 1)$.

Theorem

$i^2 = -1$.

Theorem

If a and b are real, then $(a, b) = a + bi$.

Definition

If a, b are real and $z = a + bi$, then the complex number $\bar{z} = a - bi$ is called the *conjugate of* z. The number a and b are the *real part* and the *imaginary part* of z, respectively. And write $a = \mathrm{Re}(z)$, $b = \mathrm{Im}(z)$.

Theorem

If z and w are complex, then

(a) $\overline{z + w} = \bar{z} + \bar{w}$.

(b) $\overline{zw} = \bar{z}\bar{w}$.

(c) $z + \overline{z} = 2\,\mathrm{Re}(z)$ *and* $z - \overline{z} = 2i\,\mathrm{Im}(z)$.

(d) $z\overline{z}$ *is real and positive except when* $z = 0$.

Definition

If z is a complex number, its absolute value $|z|$ is the nonnegative square root of $z\overline{z}$.

Remark

If x is real, $\overline{x} = x$. So $|x| = \sqrt{x^2}$.

Theorem

(a) $|z| > 0$ *unless* $z = 0$, *and* $|0| = 0$.

(b) $|\overline{z}| = |z|$.

(c) $|zw| = |z||w|$.

(d) $|\mathrm{Re}(z)| \leq |z|$.

(e) $|z + w| \leq |z| + |w|$.

Theorem

If $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ are complex numbers, then

$$\left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^n.$$

Ex1.8

Prove that no order can be defined in the complex field that turns it into an ordered field.

The End