LA2 10

KYB

Thrn, it's a Fact

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Overview

Infinite-dimensional vector spaces

Recall

- The set of all polynomials $\mathcal{P}$ is infinite-dimensional.

In this chapter, we will construct another infinite-dimensional vector spaces.

Definition

Let l^2 be the set of all infinite sequences $\{x_i\}_{i=1}^{\infty}$ of $\mathbb{R}$ such that

$$\sum_{i=1}^{\infty} x_i^2 < \infty.$$

Wirte $x = \{x_i\}$.

Goal

We want to show l^2 is a vector space. The scalar multiplication is well-defined by $\alpha \cdot x = \{\alpha x_i\}$ because

$$\sum_{i=1}^{\infty} (\alpha x_i)^2 = \alpha^2 \left(\sum_{i=1}^{\infty} x_i^2 \right) < \infty.$$

But the addition is not easy.

Lemma (435)

Let a and b be two real numbers. Then

$$|2ab| \leq a^2 + b^2.$$

Theorem (436)

For $x, y \in l^2$, define $x + y = \{x_i + y_i\}$. Then l^2 is a vector space over $\mathbb{R}$.

Innder product on l^2

We can define an inner prodcut on l^2 by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i,$$

and a norm

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Theorem (437)

Let X be a vector space over $\mathbb{R}$, and let $\|\cdot\|$ be a norm on X. Then $\|\cdot\|$ is a continuous function.

Lemma (438)

For any positive integer n, the subset $\{e_1, e_2, \cdots, e_n\}$ of l^2 is linearly independent.

Corollary (439)

The space l^2 is infinite-dimensional.

Remark

The Bolzano-Weierstrass theorem fails in l^2.
Let $\overline{B} = \overline{B_1(0)}$ be the closure of the unit ball centered at 0. Then sequence $\{e_k\}$ belongs to $\overline{B}$. If $k \neq j$,

$$\|e_k - e_j\| = \sqrt{2}.$$

So e_k is not a Cauchy sequence. Thus it has no convergent subsequences, and hence $\overline{B}$ is not sequentially compact.

Banach and Hilbert spaces

Cauchy sequence

Let X be a normed space. We say a sequence $\{x_k\}$ is a Cauchy sequence if for all $\epsilon > 0$, there is N such that

$$n, m \geq N \implies \|x_n - x_m\| < \epsilon.$$

Definition

- X is complete if every Cauchy sequence in X converge.
- In this case, we say X is a Banach space.
- If X is a complete inner product space, we say X is a Hilbert space.

Example

Every finite-dimensional vector space is complete. So $\mathbb{R}^n$ is a Hilbert space under the Euclidean dot product, or a Banach space under the l^1 or l^∞ norms (or any other norm).

Definition

Let S be any set and $f, f_k : S \to \mathbb{R}$.

(1) $\{f_k\}$ converges to f pointwisely if for each $x \in S$ and for all $\epsilon > 0$, there is N such that $k \geq N$ implies $|f_k(x) - f(x)| < \epsilon$.

(2) $\{f_k\}$ converges to f uniformly if for all $\epsilon > 0$, there is N such that $k \geq N$ implies $|f_k(x) - f(x)| < \epsilon$ for all $x \in S$.

The difference of above two definitions is that

- in definition (1), N depends on x;
- in definition (2), N does not depend on x.

Example

$L^\infty[a,b]$ norm on $C[a,b]$ is defined by

$$\|f\|_\infty = \max\{|f(x)| \ : \ a \le x \le b\}.$$

Let $\{f_k\}$ be a sequence in $C[a,b]$ and let $f : [a,b] \to \mathbb{R}$ be a function. Then

- $f_k \to f$ in $\|\cdot\|_\infty$ if and only if $f_k \to f$ uniformly.
- If $f_k \to f$ uniformly, then f is continuous.

Proof

Suppose $f_k \to f$ uniformly. Assume that f is continuous (by the second property). For given $\epsilon > 0$, there is N such that for all $x \in [a,b]$ and $k \ge N$,

$$|f_k(x) - f(x)| < \epsilon.$$

Then $\max\{|f_k(x) - f(x)| : a \le x \le b\} < \epsilon$, so $\|f_k - f\|_\infty < \epsilon$. Hence $f_k \to f$ in $\|\cdot\|_\infty$. (continued)

Ex 10.2.2

Suppose $f \in C[a,b]$, $\{f_k\}$ is a sequence in $C[a,b]$, and

$$\|f_k - f\|_\infty \to 0 \text{ as } k \to \infty.$$

Prove that $\{f_k\}$ converges uniformly to f on $[a,b]$.

Proof

Suppose $f_k \to f$ in $\|\cdot\|_\infty$. For any $\epsilon > 0$, there is N such that

$$\|f_k - f\|_\infty < \epsilon \text{ for all } k \geq N.$$

Then for any $x \in [a,b]$,

$$|f_k(x) - f(x)| \leq \|f_k - f\|_\infty < \epsilon.$$

So $f_k \to f$ uniformly. (continued)

proposition

Let f_k be a sequence in $C[a,b]$ and suppose $f_k \to f$ uniformly. Then f is continuous.

Proof

Let $\epsilon > 0$ be given. Let $x \in [a,b]$. Choose

- N so that $|f_k(y) - f(y)| < \epsilon/3$ for all $y \in [a,b]$ and $k \geq N$.
- δ so that $\|x-y\|_\infty < \delta$ implies $|f_N(x) - f_N(y)| < \epsilon/3$.

Then

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \epsilon$$

for all $\|x-y\|_\infty < \delta$. Thus f is continuous.

Example

Let $f_k(x) = x^k$ on $[0,1]$. Then $f_k \to f$ pointwisely where

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}.$$

This is a counter example that pointwise convergence implies uniform convergence.
Note that $\{x^k\}$ is not a Cauchy in $\|\cdot\|_\infty$. For $m \geq n$, $x^n > x^m$ for $0 < x < 1$.

$$(x^n - x^m)' = nx^{n-1} - mx^{m-1} = x^{n-1}(n - mx^{m-n})$$

Take $m = 2n$. Then $x^n - x^{2n}$ attains a maximum when $x = 1/2^{1/n}$. Thus

$$\left\|x^n - x^{2n}\right\|_\infty = \left(\tfrac{1}{2^{1/n}}\right)^n - \left(\tfrac{1}{2^{1/n}}\right)^{2n} = \frac{1}{2} - \frac{1}{\sqrt{2}}.$$

Theorem (442)

$C[a,b]$ is complete under the L^∞ norm.

Proof

The proof is followed from the completeness of $\mathbb{R}$. At first, for each $x \in [a,b]$, $\{f_k(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Then for each x, $f_k(x)$ converges to some real number, say $f(x)$, i.e.

$$f(x) = \lim_{k\to\infty} f_k(x).$$

(continued)

Ex 10.2.3

Suppose $\{f_k\}$ is a Cauchy sequence in $C[a,b]$ under the L^∞ norm that converges pointwise to $f : [a,b] \to \mathbb{R}$. Prove that $f_k \to f$ in the L^∞ norm.

Proof

Claim) f is continuous.
Let $\epsilon > 0$ be given. Let $x \in [a,b]$. Choose

- N_1 so that $\|f_n - f_m\|_\infty < \epsilon/4$ for all $n, m \geq N_1$
- N_2 so that $|f_k(x) - f(x)| < \epsilon/4$ for all $k \geq N_2$
- $\delta > 0$ so that $\|x - y\|_\infty < \delta$ implies $|f_N(x) - f_N(y)| < \epsilon/4$.

Let $N = \max\{N_1, N_2\}$. For $y \in [a,b]$ such that $\|x - y\|_\infty < \delta$,

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \epsilon/4 + \epsilon/4 + |f_N(y) - f(y)|.\end{aligned}$$

(continued)

Proof

Take N_3 so that N_3 so that $|f_k(y) - f(y)| < \epsilon/4$ for all $k \geq N_3$. Let $M = \max\{N, N_3\}$. Then $M \geq N$, thus

$$|f_N(y) - f(y)| \leq |f_N(y) - f_M(y)| + |f_M(y) - f(y)| < \epsilon/4 + \epsilon/4.$$

So $\|x - y\|_\infty < \delta$ implies $|f(x) - f(y)| < \epsilon$, f is continuous.

It remains to show that $f_k \to f$ in $\|\cdot\|_\infty$. Choose

- N so that $\|f_n - f_m\|_\infty < \epsilon/2$ for all $n, m \geq N$.

Let $x \in [a, b]$. Then there is M such that $|f_k(x) - f(x)| < \epsilon/4$ for all $k \geq M$. Take $L = \max\{N, M\}$. For all $n \geq N$,

$$|f_n(x) - f(x)| \leq |f_n(x) - f_L(x)| + |f_L(x) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $\|f_n - f\|_\infty < \epsilon$ for all $n \geq N$.

Example

$L^2(a,b)$ norm on $C[a,b]$ is defined by

$$\|f\|_2 = \left[\int_a^b |f(x)|^2\right]^{1/2}.$$

Then L^2 norm is not equivalent to L^∞ because $C[a,b]$ is not complete under $L^2(a,b)$.
(continued)

Example

Define $f_k \in C[0,1]$ by

$$f_k(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2} - \frac{1}{k+1} \\ \frac{k+1}{2}\left(x - \frac{1}{2} + \frac{1}{k+1}\right), & \frac{1}{2} - \frac{1}{k+1} < x < \frac{1}{2} + \frac{1}{k+1} \\ 1, & \frac{1}{2} + \frac{1}{k+1} \le x \le 1 \end{cases}$$

In $L^2(0,1)$ norm,

$$\|f_m - f_n\|_{L^2(0,1)} \le \sqrt{\frac{2}{n+1}}.$$

So $\{f_k\}$ is a Cauchy sequence. Moreover, $f_k \to f$ under $L^2(0,1)$ where

$$f(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2} \\ 1, & \frac{1}{2} < x \le 1 \end{cases}.$$

Hence $L^2(0,1)$ is not complete.

Remark

Let $f(x)$ be any function such that

$$f(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1 \\ \text{whatever} & x = \frac{1}{2} \end{cases}.$$

Then $f_k \to f$ in $L^2(0,1)$. This implies that in $C[a,b]$ a sequence $\{f_k\}$ may converge to multiple functions.

Remark

$\{f_k\}$ in above example is not a Cauchy in L^∞ norm because if it is a Cauchy, it converges to $f \in C[0,1]$ pointwisely. But $f_k \to f$ pointwisely where

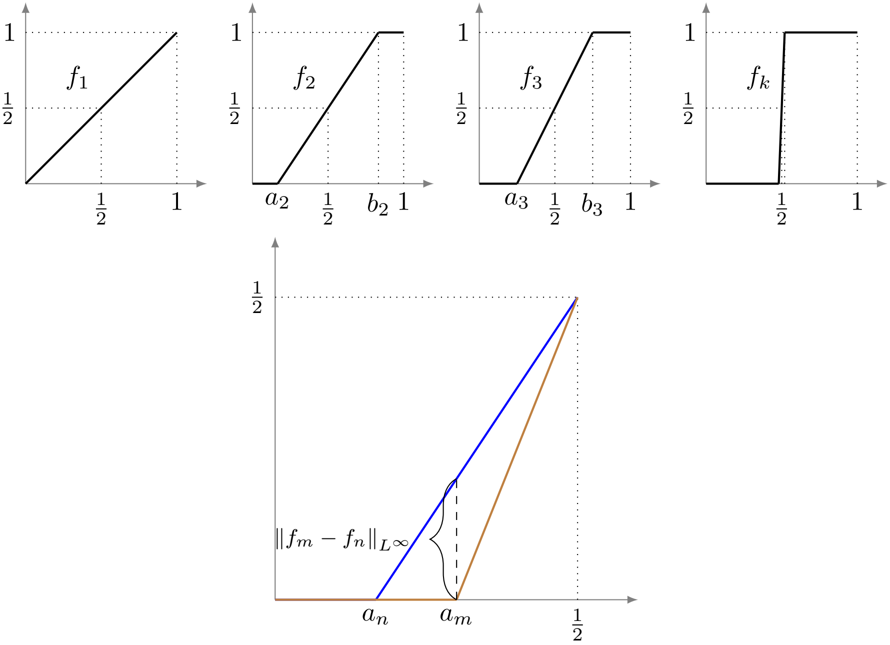
$$f(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2} \\ \frac{1}{2}, & x = \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1 \end{cases},$$

Which is not continuous at $x = \frac{1}{2}$. It happens because if $m > n$,

$$\begin{aligned} f_m(x) - f_n(x) \le f_n\left(\tfrac{1}{2} - \tfrac{1}{m+1}\right) &= \tfrac{n+1}{2}\left(\tfrac{1}{2} - \tfrac{1}{m+1} - \tfrac{1}{2} + \tfrac{1}{n+1}\right) \\ &= \tfrac{1}{2} - \tfrac{1}{2}\tfrac{n+1}{m+1} = \|f_m - f_n\|_{L^\infty} \end{aligned}$$

Take $m = 2n+1$ and then

$$\|f_{2n+1} - f_n\|_{L^\infty} = \frac{1}{2} - \frac{1}{2}\frac{n+1}{2(n+1)} = \frac{1}{4}$$


f_1
f_2
f_3
f_k
1
$\frac{1}{2}$
a_2
b_2
a_3
b_3
$\|f_m - f_n\|_{L^\infty}$
a_n
a_m

Ex 10.2.4

Let $f_k : [0,1] \to \mathbb{R}$ be defined by $f_k(x) = x^k$. Prove that $\{f_k\}$ is Cauchy under the $L^2(0,1)$ norm but not under the $C[0,1]$ norm.

The End