LA2 1

KYB

Thrn, it's a Fact

mathrnfact@gmail.com

September 17, 2020

Overview

Tutoring - Linear Algebra 2

- New tools - ‘norm’ and ‘inner product’ (Ch6)
- Factorication of matrices (Ch6,7,8,9) $\rightarrow Ax = b$
- Vector analysis (Ch10)

6.1 Norms and inner products

In calculus, we define the size of points (vectors) by

$$\|(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}.$$

And we define two kinds of multiplication of two points, one is the dot product and the other is the cross product.

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = x_1x_2 + y_1y_2 + z_1z_2$$
$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$$

Definition

Let V be a vectir space over $\mathbb{R}$. A norm on V is a function $\| \cdot \| : V \to \mathbb{R}$ that satisfies the following propeties:

1. $\|u\| \geq 0$ for all $u \in V$, and $\|u\| = 0$ if and only if $u = 0$;
2. $\|\alpha u\| = |\alpha|\|u\|$ for all $\alpha \in \mathbb{R}$ and all $u \in V$;
3. $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.

Example

For $V = \mathbb{R}^n$,

$$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$$

is called Euclidean norm. In general,

$$\|x\|_p = (x_1^p + \cdots + x_n^p)^{1/p}$$

is called p-norm(or l_p-norm) for $p \geq 1$.

Definition

Let V be a vector space over $\mathbb{R}$. An inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the following properties:

$$\langle u, v \rangle = \langle v, u \rangle$$

$$\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle u, w \rangle$$

$$\langle u, u \rangle \geq 0 \text{ and } \langle u, u \rangle = 0 \Leftrightarrow u = 0$$

Operators

- $(F, +, \cdot)$: field
- $(V, +, \cdot)/F$: vector space

$$\begin{aligned} +, \cdot &: F \times F \to F \\ + &: V \times V \to V \\ \cdot &: F \times V \to V \\ \langle \cdot, \cdot \rangle &: V \times V \to F \end{aligned}$$

Lemma (267)

Let V be a vector space over $\mathbb{R}$, and let $\langle \cdot, \cdot \rangle$ be an inner product on V. If either u or v is the zero vector, then $\langle u, v \rangle = 0$.

Theorem (268, The Cauchy-Schwarz inequality)

Let V be a vector space over $\mathbb{R}$, and let $\langle\cdot,\cdot\rangle$ be an inner product on V. Then

$$|\langle u, v\rangle| \leq \langle u, u\rangle^{1/2}\langle v, v\rangle^{1/2}.$$

Theorem (269)

Let V be a vector space over $\mathbb{R}$, and let $\langle \cdot, \cdot \rangle$ be an inner product on V. Then

$$\|u\| = \sqrt{\langle u, u \rangle}$$

defines a norm on V.

Example (l^p norms on $\mathbb{R}^n$)

For any $p \in [1, \infty]$, one can define a norm on $\mathbb{R}^n$ by

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p} \quad \text{for } p < \infty$$

$$\|x\|_\infty = \max\{|x_i| : i = 1, \cdots, n\}$$

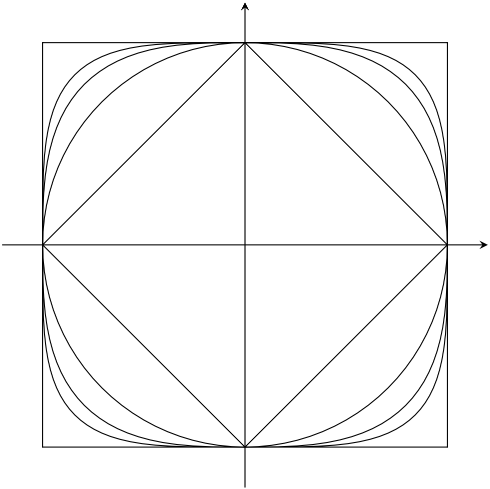


Figure: the unit balls when $n = 2$, $p = 1, 2, 3, 4, \infty$

Example (L^2 inner product and norm for functions)

On $C[a,b]$,

$$\langle f, g\rangle_2 = \int_a^b f(x)g(x)dx$$

defines an inner product and it induces a norm

$$\|f\|_2 = \sqrt{\langle f, f\rangle_2} = \left(\int_a^b f(x)^2 dx\right)^{1/2}.$$

Example (L^p norm)

In general for $p \geq 1$,

$$\|f\|_p = \left(\int_a^b f(x)^p dx \right)^{1/p}$$
$$\|f\|_\infty = \max\{|f(x)| : a \leq x \leq b\}.$$

is a norm.

Ex6.1.4

(a) $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$.

(b) $\|x\|_1 \leq \sqrt{n}\|x\|_2$.

(c) $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$.

6.1.6

Define a function $\|\cdot\|$ on $\mathbb{R}^n$ by

$$\|x\| = |x_1| + \cdots + |x_{n-1}|.$$

Prove that $\|\cdot\|$ is not a norm on $\mathbb{R}^n$.

6.1.11

Suppose V is an inner product space and $\|\cdot\|$ is the norm defined by the inner product $\langle\cdot,\cdot\rangle$ on V. Prove that the parallelogram law holds:

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

Using this result, we can prove that neither the l^1 norm nor l^∞ norm on $\mathbb{R}^n$ is defined by an inner product.

Ex6.1.12

Suppose $A \in \mathbb{R}^{n \times n}$ is a nonsingular matrix. Prove that if $\|\cdot\|$ is a norm on $\mathbb{R}^n$, then $\|x\|_A = \|Ax\|$ is again norm on $\mathbb{R}^n$.

Ex6.1.13

Let $\lambda_1, \cdots, \lambda_n$ be positive real numbers. Prove that

$$\langle x, y\rangle = \sum_{i=1}^{n} \lambda_i x_i y_i$$

defines an inner product on $\mathbb{R}^n$.

Ex6.1.15,16

Let U and V be vector spaces over $\mathbb{R}$ with norms $\|\cdot\|_U$ and $\|\cdot\|_V$ respectively. Then each of the following is a norm on $U \times V$:

(a) $\|(u, v)\| = \|u\|_U + \|v\|_V$

(b) $\|(u, v)\| = \sqrt{\|u\|_U^2 + \|v\|_V^2}$

(c) $\|(u, v)\| = \max\{\|u\|_U, \|v\|_V\}$

Moreover if $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_V$ are inner products respectively, then

$$\langle (u, v), (w, z) \rangle = \langle u, w \rangle_U + \langle v, z \rangle_V$$

defines an inner product on $U \times V$.

6.2 The adjoint of a linear operator

Recall

For $A \in F^{m \times n}$ and $B \in F^{n \times p}$,

$$(A^T)_{ij} = A_{ji}$$
$$(AB)^T = B^T A^T$$
$$(A + B)^T = A^T + B^T$$

Theorem (270)

Let $A \in \mathbb{R}^{m \times n}$. Then

$$(Ax) \cdot y = x \cdot (A^T y).$$

Theorem (271)

Let $A \in \mathbb{R}^{m \times n}$. If b is a nonzero vector in $\mathcal{N}(A^T)$, then

$$\mathcal{N}(A^T) \cap col(A) = \{0\}.$$

cf.

In 5.1 Theorem 227, $F^n = \mathcal{N}(A) \oplus col(A)$ if and only if $\mathcal{N}(A) \cap col(A) = \{0\}$.

The adjoint of a linear operator

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear. We can find a linear map $S : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\langle T(x), y \rangle = \langle x, S(y) \rangle.$$

Definition (272)

Let V be an inner product space over $\mathbb{R}$, and $\{u_1, \cdots, u_n\}$ be a basis for V. Then the matrix $G \in \mathbb{R}^{n \times n}$ defined by

$$G_{ij} = \langle u_j, u_i \rangle$$

is called the Gram matrix of the basis $\{u_1, \cdots, u_n\}$.

Theorem (273)

Let V be an inner product space over $\mathbb{R}$, and $\{u_1, \cdots, u_n\}$ be a basis for V, and let G be the Gram matrix for this basis. Then G is nonsingular.

Theorem (274)

Let V be an inner product space over $\mathbb{R}$, and let $x \in V$. Then $\langle x, y \rangle = 0$ for all $y \in V$ if and only if $x = 0$.

Corollary (275)

$\langle x, v \rangle = \langle y, v \rangle$ for all $v \in V$ if and only if $x = y$.

Theorem (276)

Let X and U be finite-dimensional inner product spaces over $\mathbb{R}$, and let $T : X \to U$ be linear. There exists a unique linear operator $S : U \to X$ satisfying

$$\langle T(x), u\rangle_U = \langle x, S(u)\rangle.$$

Denote $S = T^*$.

Step 1

Fix two basis $\mathcal{X} = \{x_1, \cdots, x_n\}$ and $\mathcal{U} = \{u_1, \cdots, u_m\}$.

$$\begin{tikzcd} F^n \arrow[r, "A"] & F^m \\ X \arrow[u] \arrow[r, "T"] & U \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} F^n & F^m \arrow[l, "B"'] \\ X \arrow[u] & U \arrow[u] \arrow[l, "S"'] \end{tikzcd}$$

Thus it suffices to find some matrix B.

Step 2

Write $\alpha = [x]_{\mathcal{X}}$ and $\beta = [u]_{\mathcal{X}}$, or

$$x = \sum_{i=1}^{n} \alpha_i x_i, \quad u = \sum_{j=1}^{m} \beta_j u_j.$$

$$\begin{aligned} \langle T(x), u \rangle_U &= \left\langle T(\sum_i \alpha_i x_i), \sum_j \beta_j u_j \right\rangle_U \\ &= \sum_i \sum_j \langle T(x_i), u_j \rangle_U \alpha_i \beta_j = \alpha \cdot M\beta, \end{aligned}$$

where $M_{ij} = \langle T(x_i), u_j \rangle_U$.

Step 3

Similarly, for given S,

$$\langle x, S(u)\rangle_X = \sum_i \alpha_i \left(\sum_k \langle x_i, x_k\rangle_X (B\beta)_k \right).$$

Since $G = (\langle x_i, x_k\rangle_X)$, $\langle x, S(u)\rangle_X = \alpha \cdot (GB\beta) = \alpha \cdot (GB)\beta$.

Step 4

Finally, $\langle T(x), u\rangle_U = \langle x, S(u)\rangle_X$ iff $\alpha \cdot M\beta = \alpha \cdot (GB)\beta$. Hence $M = GB$, or $B = G^{-1}M$.

Theorem (278)

Let X, U, W be finite-dimensional vector spaces over $\mathbb{R}$, and let $T : X \to U$ and $S : U \to W$ be linear operators.

1. $(T^*)^* = T$;
2. $(ST)^* = T^* S^*$.

Theorem

Let X and U be finite-dimensional inner product spaces over $\mathbb{R}$ and assume that $T : X \to U$ is an invertible linear operator. Then T^* is also invertible and

$$(T^*)^{-1} = (T^{-1})^*.$$

Ex6.2.6

If $A, B \in \mathbb{R}^{m\times n}$ and

$$y \cdot Ax = y \cdot Bx \text{ for } x \in \mathbb{R}^n, y \in \mathbb{R}^m,$$

then $A = B$.

Ex6.2.9

Let $M : \mathcal{P}_2 \to \mathcal{P}_3$ be defined by $M(p) = q$, where $q(x) = xp(x)$. Find M^*, assuming that the $L^2(0,1)$ inner product is imposed on both $\mathcal{P}_2$ and $\mathcal{P}_3$.

Ex6.2.10

Suppose $A \in \mathbb{R}^{n \times n}$ has the following properties: $A^T = A$ and

$$x \cdot Ax > 0 \text{ for all } x \in \mathbb{R}^n, x \neq 0.$$

Prove that $\langle x, y \rangle_A = x \cdot Ay$ for all $x, y \in \mathbb{R}^n$ defines an inner product on $\mathbb{R}^n$.

Ex6.2.11

Let X and U be finite-dimensional inner product spaces over $\mathbb{R}$, and suppose $T : X \to U$ is linear. Defines $S : \mathcal{R}(T^*) \to \mathcal{R}(T)$ by $S(x) = T(x)$.

(a) Prove that S is injective.

(b) The fact that S is injective implies that $\dim(\mathcal{R}(T)) \geq \dim(\mathcal{R}(T^*))$. Prove that $\dim(\mathcal{R}(T)) = \dim(\mathcal{R}(T^*))$.

(c) Then S is surjective, and hence an isomorphism.

Ex6.2.14

Let $f : X \to \mathbb{R}$ be linear, where X is a finite-dimensional inner product space over $\mathbb{R}$. Prove that there exists a unique $u \in X$ such that

$$f(x) = \langle x, u \rangle \text{ for all } x \in X.$$

Dual Spaces

There are two definitions of the dual space of V/F,

1. $V_1^* = \mathcal{L}(V, F)$.
2. $V_2^* = \{f \in \mathcal{L}(V, F) \mid f \text{ is continuous}\}$. (Ch 10.3)

If V is finite dimensional, $V_1^* = V_2^*$, but if V is infinite dimensional, they are different.

Observation

From now on, assume V is a finite dimensional vector space over F. Fix a basis $\mathcal{B} = \{v_1, \cdots, v_n\}$ and define $f_i : V \to F$ by $f_i(v_j) = \delta_{ij}$. Then f_i's are linear maps.

Lemma

$\{f_1, \cdots, f_n\}$ *is linearly indepent.*

Lemma

$\{f_1, \cdots, f_n\}$ *spans* $\mathcal{L}(V, F)$, *and hence it is a basis.*

Observation

Now denote $\mathcal{L}(V, F) = V^*$. Define $\langle \cdot, \cdot \rangle : V \times V^* \to F$ by

$$\langle v, f \rangle = f(v).$$

We call this map $\langle \cdot, \cdot \rangle$ the evaluation map.

Lemma

The evaluation map is a bilinear map.

Fix $v \in V$ and define $f_v = \langle v, \cdot \rangle : V^* \to F$. This map is linear, and thus $f_v \in (V^*)^*$.

$$\begin{aligned} V &\to (V^*)^* \\ v &\mapsto f_v \end{aligned}$$

Hence the dual of the dual space of V is isomorphic to V when V is finite dimensional.

Infinite case

Consider $\mathcal{P}(\mathbb{R})$ the set of all polynomials of $\mathbb{R}$. We know that $\{x^n\}_{n=0}^{\infty}$ is a basis. If we write the dual of x^n by f_n, $\{f_n\}_{n=1}^{\infty}$ is linearly indepent. However, it can not span $\mathcal{P}^*$.
For example, let $f : \mathcal{P} \to \mathbb{R}$ by $f(x^n) = 1$. If $f = \sum_{i=1}^{k} \alpha_i f_{n_i}$, for $n \neq n_i$ for all i, $f(x^n) = \sum \alpha_i f_{n_i}(x^n) = 0$ which contradicts $f(x^n) = 1$.

Infinite case

Moreover $\dim V \lneq \dim V^*$, and V is not isomorphic with $(V^*)^*$. Every linear map $f : \mathcal{P} \to \mathbb{R}$ is completely determined by $f(x^n)$ for $n = 0, 1, 2, \cdots$. For each $r \in \mathbb{R}$, define $f_r : \mathcal{P} \to \mathbb{R}$ by $f_r(x^n) = r^n$. Suppose $\sum_{i=0}^n \alpha_i f_{r_i} = 0$, $r_i \neq 0$ and $r_i \neq r_j$. Then $\sum_{i=0}^n \alpha_i (r_i)^k = 0$. for all k. Then

$$\begin{bmatrix} 1 & r_0 & r_0^2 & \cdots & r_0^n \\ 1 & r_1 & r_1^2 & \cdots & r_1^n \\ \vdots & \vdots & \ddots & & \vdots \\ 1 & r_n & r_n^2 & \cdots & r_n^n \end{bmatrix}$$

is the Vandermonde matrix (see Exercise 4.3.11) whose determinant is $\prod_j \prod_{i>j} (r_i - r_j) \neq 0$. Thus it is invertible and $\alpha_i = 0$ for all i, or $\{f_r \mid r \in \mathbb{R} - \{0\}\}$ is linearly indepent, and this implies $\dim \mathcal{P}$ is uncountable.

6.3 Orthogonal vectors and bases

Pythagorean theorem

If $x \cdot y = 0$, or $\theta = \frac{\pi}{2}$, then

$$\|x \pm y\|_2^2 = \|x\|_2^2 + \|y\|_2^2$$

Theorem (280)

Let V be an inner product space over $\mathbb{R}$, and let x, y be vectors in V. If $\langle \cdot, \cdot \rangle$ is the inner product on V and $\|\cdot\|$ is the corresponding norm, then

$$\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \iff \langle x, y \rangle = 0.$$

Definition

Let V be an inner product space over $\mathbb{R}$.

1. x, y are orthogonal if and only if $\langle x, y \rangle = 0$.
2. $\{u_1, \cdots, u_k\}$ is an orthogonal set if u_i is nonzero vector and $\langle u_i, u_j \rangle = 0$ for all $i \neq j$.

perpendicular, orthogonal, normal

Theorem (282)

Let V be an inner product space over $\mathbb{R}$, and let $\{u_1, \cdots, u_k\}$ be an orthogonal subset of V. Then $\{u_1, \cdots, u_k\}$ is linearly independent.

Corollary (283)

Let V be an n-dimensional inner product space over $\mathbb{R}$. Then any orthogonal set of n vectors in V is a basis for V.

Theorem (284)

Let V be an inner product spave over $\mathbb{R}$ and let $\{u_1, \cdots, u_n\}$ be an orthogonal basis for V. Then any $v \in V$ can be written

$$v = \sum \alpha_j u_j$$

where

$$\alpha_j = \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle}.$$

Since u_j's are not nonzero vector, we may assume $\langle u_j, u_j \rangle = 1$. Then

$$v = \sum \langle v, u_j \rangle u_j.$$

Definition

Let V be an inner product space over $\mathbb{R}$. We say that a subset $\{u_1, \cdots, u_k\}$ of V is an orthonormal set if it is orthogonal and $\|u_j\| = 1$ for each j.

Ex6.3.4

Show that $\{1/2, \sin(\pi nx/L), \cos(\pi nx/L) : n \in \mathbb{Z}_+\}$ is an orthogonal set of $L^2(-L/2, L/2)$.

Ex6.3.12

Let $\{x_1, \cdots, x_n\}$ be an orthonormal set in $\mathbb{R}^n$, and define $X = [x_1|\cdots|x_n]$. Compute $X^T X$ and XX^T.

Ex6.3.13

Let V be an inner product space over $\mathbb{R}$, and let $\{u_1, \cdots, u_K\}$ be an orthogonal subset of V. Prove that, for all $v \in V$,

$$v \in \text{span}\{u_1, \cdots, u_k\} \iff v = \sum_{j=1}^{k} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j .$$

Ex6.3.14

Let $\{u_1, \cdots, u_k\}$ be an orthogonal subset of V, and define $S = \text{span}\{u_1, \cdots, u_k\}$.

(a) Prove that, for all $v \in V - S$, $v - \sum_{j=1}^{k} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j$ is orthogonal to every vector in S.

(b) If $v \in V - S$, then $\|v\| > \left\| \sum_{j=1}^{k} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j \right\|$.

6.4 The projection theorem

Best approximation

Given $v \in V$, and a subspace S of V, we want to find the vector $w \in S$ closest to v, in the sense that

$$w \in S, \|v - w\| \leq \|v - z\| \text{ for all } z \in S.$$

Theorem (289, The projection theorem)

Let V be an inner product space over $\mathbb{R}$, and let S be a finite-dimensional subspace of V.

1. *For any $v \in V$, there is a unique $w \in S$ satisfying*

$$\|v - w\| = \min\{\|v - z\| : z \in S\}.$$

   In this case, we denote $w = \mathrm{proj}_S v$.

2. *$w \in S$ is the best approximation to v from S if and only if $\langle v - w, z\rangle = 0$ for all $z \in S$.*
3. *If $\{u_1, \cdots, u_n\}$ is a basis for S, then*

$$\mathrm{proj}_S v = \sum_{i=1}^{n} x_i u_i,$$

   where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ is the unique solution to the equation $Gx = b$. G is the Gram matrix for the basis and $b_i = \langle v, u_i\rangle$.

Proof of Proj Thm

$w \in S$ is the best approximation to v form S iff $\langle v - w, z \rangle = 0$ for all $z \in S$.
Fix $w \in S$.
1) $y \in S$ iff $y = w + tz$ for some $t \in \mathbb{R}$ and $z \in S$.
2) Consider $\|v - (w + tz)\|^2$.

$$\begin{aligned} \|v - (w + tz)\|^2 &= \langle v - w - tz, v - w - tz \rangle \\ &= \langle v - w, v - w \rangle - 2t\langle v - w, z \rangle + t^2\langle z, z \rangle \\ &= \|v - w\|^2 - 2t\langle v - w, z \rangle + t^2\|z\|^2. \end{aligned}$$

For all $z \in S$ and $t \in \mathbb{R}$,

$$\|v - (w + tz)\|^2 \geq \|v - w\|^2 \iff t^2\|z\|^2 - 2t\langle v - w, z \rangle \geq 0.$$

Fix z and define $\phi(t) = t^2\|z\|^2 - 2t\langle v - w, z \rangle$. $\phi(t) \geq 0$ for all $t \in \mathbb{R}$ iff $\langle v - w, z \rangle = 0$.

Proof of Proj Thm

$$\mathrm{proj}_S v = \sum_{i=1}^{n} x_i u_i,$$

$$\begin{aligned}
\langle v - w, u_i \rangle = 0 &\iff \left\langle v - \sum_{j=1}^{n} x_j u_j, u_i \right\rangle = 0 \\
&\iff \langle v, u_i \rangle - \sum_{j=1}^{n} x_j \langle u_j, u_i \rangle = 0 \\
&\iff \sum_{j=1}^{n} x_j \langle u_j, u_i \rangle = \langle v, u_i \rangle
\end{aligned}$$

if and only if x satisfies $Gx = b$ where G is the Gram matrix and $b_i = \langle v, u_i \rangle$.

If $\{u_1, \cdots, u_n\}$ is a orthonormal basis for S,

$$\mathrm{proj}_S v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i.$$

Overdetermined linear systems

Consider a linear system $Ax = y$, where $A \in \mathbb{R}^{m \times n}$, $y \in \mathbb{R}^m$, and $m > n$. By FTLA, col(A) is a proper subspace of $\mathbb{R}^m$. Therefore, if $y \notin$ col(A), the system has no solution. Nevertheless, we need to solve $Ax = y$ in the sense of finding an approximation soultion.

Least-square solution

We wand to find a solution to $Ax = y$ in the sense $\|Ax - y\|_2^2 = \min\{\|Az - y\|_2^2\}$. In this case, we say x is a least-square solution to $Ax = y$.

$$(y - Ax) \cdot w = 0 \text{ for all } w \in \text{col}(A).$$

Since $w \in$ col(A),

$$(y - Ax) \cdot Az = 0 \text{ for all } z \in \mathbb{R}^n.$$

$$A^T(y - Ax) \cdot z = 0 \text{ for all } z \in \mathbb{R}^n.$$

Continue

Thus we get a equation $A^T(y - Ax) = 0$, or

$$A^T A x = A^T y.$$

We call this equation the normal equation of $Ax = y$.

Theorem (291)

Let $A \in \mathbb{R}^{m \times n}$ and $y \in \mathbb{R}^m$ be given. Then $x \in \mathbb{R}^n$ solves

$$\min\{\|Az - y\|_2 : z \in \mathbb{R}^n\} \iff A^T A x = A^T y.$$

Example (Linear regression)

Suppose two variables y and t are thought to be related by the equation $y = c_0 + c_1 t$, whrer c_0, c_1 are unknown constants. Given data $(t_1, y_1), \cdots, (t_m, y_m)$ we can find the equation

$$\begin{aligned} c_0 + c_1 t_1 &= y_1 \\ c_0 + c_1 t_2 &= y_2 \\ \vdots \qquad &\quad \vdots \\ c_0 + c_1 t_m &= y_m \end{aligned}$$

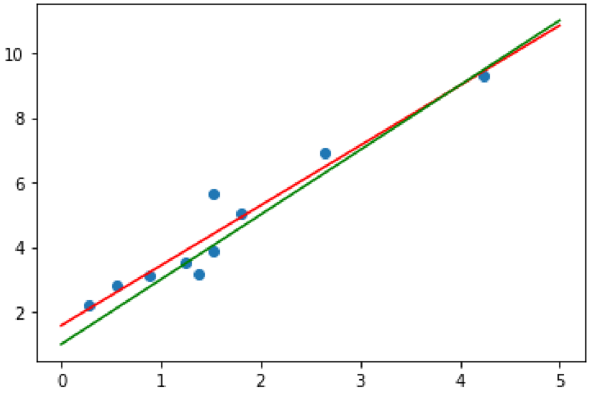
or $Ac = y$ where

$$A = \begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}, c = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}.$$

Code

```
import numpy as np
import matplotlib.pyplot as plt
np.random.seed(15)
# y = 2x+1에대한 linear regression
n = 10
x = np.random.uniform(low = 0.0, high = 5.0, size = n)
error = np.random.normal(size = n) # N(0,1) 정규분포
y = 2*x+1 + error
A = np.array([[1, x[i]]for i in range(n)])
c = np.linalg.solve(A.T@A,A.T@y)
plt.plot(x,y,"o")
X = np.linspace(0.0,5.0,100)
Y = c[0]+c[1]*X
plt.plot(X,Y,"r")
plt.plot(X,2*X+1,"g")
plt.show()
print(c[0],c[1])
```

Result




```
1.5797445996940578 1.8525612078419815
```

The End