LA2 Extra Exercises

KYB

Thrn, it's a Fact

mathrnfact@gmail.com

October 8, 2020

Overview

Gochenbach

Friedberg

Ex 6.2.14

Let $f : X \to \mathbb{R}$ be linear, where X is a finite-dimensional inner product space over $\mathbb{R}$. Prove that there exists a unique $u \in X$ such that

$$f(x) = \langle x, u \rangle.$$

Exmaple 292 in 6.4

Suppose we believe the variables t and y are related by

$$y = c_0 + c_1 t$$

and the following datas are measured from four trials:

$$(t_i, y_i) = (1, 2.2), (1, 2.3), (2, 4.5), (2, 4.7).$$

Find the most fitting c_0 and c_1.

Ex 6.6.3

Let $A \in \mathbb{R}^{3\times 4}$ be defined by

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 8 & 9 & 12 \end{bmatrix}$$

(a) Find orthogonal bases for $\mathcal{N}(A)$ and $\operatorname{col}(A^T)$.

(b) Write $(1, 1, 1, 1)$ as the combination of the basis elements that you have chosen.

Example 317 in 6.7

Let S be the subspace of $\mathbb{C}^3$ defined by $S = \mathrm{span}\{v_1, v_2\}$, where

$$v_1 = (1+i, 1-i, 1), v_2 = (2i, 1, i)$$

and let $u = (i, 1, 1)$. Find the best approximation to u from S.

Dual Space

Let V be the set of all linear functions from $\mathbb{R}^3$ to $\mathbb{R}$.

(a) Show that V is a vector space over $\mathbb{R}$ with operations

$$(f+g)(x) = f(x) + g(x), (rf)(x) = rf(x)$$

for all $f, g \in V$, $x \in \mathbb{R}^3$ and $r \in \mathbb{R}$.

(b) Find a basis of V.

(c) Show that V is isomorphic to $\mathbb{R}^3$.

(d) Evaluate the Gram matrix for the basis you have chosen in (b).

Friedberg - Linear Algebra-Prentice Hall (2002)

Friedberg Ex6.1.17

Let $T : V \to V$ be a linear operator on an inner product space V. Suppoe $\|T(v)\| = \|v\|$ for all $v \in V$. Show that T is injective.

Friedberg Ex6.1.27

We know that if $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$, then the parallelogram law hold. What if the converse (over $\mathbb{R}$)? (Hint: Define $\langle v, w\rangle = \frac{1}{4}(\|v+w\|^2 - \|v-w\|^2)$.)

The parallelogram law

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

Step 0

We have to prove two things, one is $\langle \cdot, \cdot \rangle$ is an inner product and the other is $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$. The last thing is obvious.

- $\langle v, w\rangle = \langle w, v\rangle$ by the definition.
- $\langle v, v\rangle = \|v\|^2$ is positive definite.

So it suffices to show that $\langle \cdot, \cdot \rangle$ is bilinear.

Step 1

$\langle x, 2y \rangle = 2\langle x, y \rangle.$

Step 2

$\langle x + u, y \rangle = \langle x, y \rangle + \langle u, y \rangle.$

It remains to show that $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$.

Step 3

For $n \in \mathbb{Z}$, $\langle nx, y\rangle = n\langle x, y\rangle$.

Step 4

For $m \in \mathbb{Z}$, $m\langle \frac{1}{m}x, y\rangle = \langle x, y\rangle$.

Step 5

For $r \in \mathbb{Q}$, $\langle rx, y\rangle = r\langle x, y\rangle$.

Step 6

$|\langle x, y\rangle| \leq \|x\|\|y\|$.

Step 7

For $c \in \mathbb{R}$ and for $r \in \mathbb{Q}$,

$$|c\langle x, y\rangle - \langle cx, y\rangle| = |(c-r)\langle x, y\rangle - \langle (c-r)x, y\rangle| \leq 2|c-r|\|x\|\|y\|.$$

Step 8

Using the fact that for given $c \in \mathbb{R}$ and for given $\epsilon > 0$, there is $r \in \mathbb{Q}$ such that

$$|c - r| < \epsilon,$$

conclude that $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

Remark

If V is a normed space over $\mathbb{C}$ and it satisfies the parallelogram law, define

$$\langle v, w \rangle = \frac{1}{4} \left(\|v + w\|^2 + i\|v + iw\|^2 - \|v - w\|^2 - i\|v - iw\|^2 \right).$$

Friedberg Ex6.3.5 (e)

If $T^*T = 0$, then $T = 0$. Similarly, if $TT^* = 0$, then $T^* = 0$.

Friedberg Ex6.4.11

Suppose $T : V \to V$ is linear where V is a complex (not necessarily finite dimensional) inner product space with an adjoint T^*.

(a) If T is self-adjoint, $\langle T(x), x\rangle$ is real for all $x \in V$.

(b) If $\langle T(x), x\rangle = 0$ for all $x \in V$, $T = 0$.

(c) If $\langle T(x), x\rangle$ is real for all $x \in V$, then T is self-adjoint.

Friedberg Ex6.6.8

Let $A, B \in \mathbb{C}^{n\times n}$. If $AB = BA$ and A is normal, then $A^*B = BA^*$. (Hint: $C = 0$ iff $\operatorname{tr}(C^*C) = 0$).

Extra Exercise

Let $A \in \mathbb{C}^{n \times n}$. Suppose A is triangular. Then A is normal if and only if A is diagonal.