LA3 Basis, Linear Operator

KYB

Thrn, it's a Fact

mathrnfact@gmail.com

February 13, 2021

Overview

Recall

- Linear Combination:

$$\sum_{i=1}^{k} a_i v_i, \quad a_i \in F, v_i \in V.$$

- Linearly Independent: For all $\{v_1, \cdots, v_k\} \subset X$,

$$\sum_{i=1}^{k} a_i v_i = 0 \implies a_i = 0 \text{ for all } i.$$

- $v_1 \neq 0$

$$\underbrace{\{v_1\} \subset \{v_1, v_2\} \subset \cdots \{v_1, \cdots, v_n\}}_{\text{linearly independent}} \subset \underbrace{\{v_1, \cdots, v_n, v_{n+1}\} \subset \cdots}_{\text{linearly dependent for all } v_{n+1}},$$

Every basis has the same cardinality (even infinite-dimensional).

Definition

A subset X of V is called a basis for V if

1. $\operatorname{span} X = V$
2. X is linearly independent.

i.e., every vector v in V can be written in a unique way as linear combination of elements of X.

Example

- $\{(1,0),(0,1)\} \subset \mathbb{R}^2$.
- $\{1, x, x^2, \cdots, x^n\} \subset \mathcal{P}_n(\mathbb{R}$
- $\{(1,0,0),(1,1,0),(1,1,1)\} \subset \mathbb{R}^3$.

For $n > 0$, let $V = \mathbb{R}^n$ and $e_i = (0, \cdots, \underbrace{1}_{i\text{ th}}, \cdots, 0) \in V$. Then $\{e_1, \cdots, e_n\}$ is a basis for V, and is called the standard basis for V.

Example

Let $\mathcal{P}(\mathbb{R})$ be the set of all polynomials and let $X = \{1, x, x^2, \cdots\}$.

(1) $\operatorname{span} X = \mathcal{P}(\mathbb{R})$

(2) X is linearly independent.

Definition

V is said to be finite-dimensional if $V = \{0\}$ or V has a finite basis.

Example

- $\mathbb{R}^n$ is a finite-dimensional vector space.
- $\mathcal{P}(\mathbb{R})$ is an infinite-dimensional vector space.

Remark

(1) If $\underbrace{\{v_1, \cdots, v_n\}}_{\text{non zero}}$ is linearly dependent, then $\exists k$ such that $u_k \in \text{span}\{v_1, \cdots, \hat{v}_k, \cdots, v_n\}$.

(2) Suppose $\{u_1, \cdots, u_m\}$ is a basis for V. Then for any $n > m$, $\{v_1, \cdots, v_n\}$ is linearly dependent.

(3) Every basis (for finite-dimensional vector space) has the same cardinal. So we can define the dimension of V by $\dim V = |\text{basis}|$.

Example

- $\dim \mathbb{R}^n = n$.
- $\dim \mathcal{P}_n(\mathbb{R}) = n + 1$.

Remark

Basis is not unique!

(1) Span:

$$\begin{aligned}\operatorname{span}\{v_1, \cdots, v_n\} &= \operatorname{span}\{\alpha v_1, v_2, \cdots, v_n\} \quad (\alpha \neq 0)\\ &= \operatorname{span}\{v_1 + v_2, v_2, \cdots, v_n\}\end{aligned}$$

So

$$\operatorname{span}\{v_1, \cdots, v_n\} = \operatorname{span}\{\alpha_1 v_1 + \sum_{i=2}^{n} \alpha_i v_i, v_2, \cdots, v_n\} \quad (\alpha_1 \neq 0).$$

(2) Linearly Independence: Suppose $\{v_1, \cdots, v_n\}$ is linearly independent.

$\rightarrow$ $\{\alpha v_1, \cdots, v_n\}$ is linearly independent for $\alpha \neq 0$.

$\rightarrow$ $\{v_1 + v_2, \cdots, v_n\}$ is linearly independent.

$\rightarrow$ $\{\alpha_1 v_1 + \sum_{i=2}^{n} \alpha_i v_i, , \cdots, v_n\}$ is linearly independent for $\alpha_1 \neq 0$.

Ex 2.6.1

(b) Suppose $\{v_1, \cdots, v_n\}$ is a basis and $u \in V$ but $u \notin \{v_1, \cdots, v_n\}$. Then $\{v_1, \cdots, v_n, u\}$ is linearly dependent.

Ex 2.6.10

$\{1 + x + x^2, 1 - x + x^2, 1 + x + 2x^2\}$ is a basis for $\mathcal{P}_2(\mathbb{Z}_3)$.

Ex 2.6.11

$\mathcal{P}_n(F)$, F is a finite field with $|F| = q$.

(a) If $n \leq q - 1$, $\{1, x, \cdots, x^n\}$ is linearly independent. So $\dim \mathcal{P}_n(F) = n + 1$. Note that $f \in \mathcal{P}_n(F)$ is a polynomial as a function from $F \to F$.

Ex 2.6.11

$\mathcal{P}_n(F)$, F is a finite field with $|F| = q$.

(b) If $n \geq q$, $\{1, x, \cdots, x^{q-1}\}$ is linearly independent. So $\dim \mathcal{P}_n(F) \geq q$.

Note

For a field with finite characteristic (char $F < \infty$), we can view a polynomial $f(x) = a_n x^n + \cdots a_0$ in two ways:

- as a function : in this case, x is determined in F, and write the set of all polynomials as $\mathcal{P}(\mathbb{R})$
- as a new object : in this case, we assume x is indeterminant in F, as write the set of all polynomials as $F[x]$ (See 4.4)

For example, for $\mathbb{Z}_2$,

- $\mathcal{P}_2(\mathbb{Z}_2) = \{0, 1, x, 1 + x\}$:

$$x^2 = x : \begin{array}{c} 0 \mapsto 0 \\ 1 \mapsto 1 \end{array}, \text{ and } x^2 + x = 0 : \begin{array}{c} 0 \mapsto 0 \\ 1 \mapsto 0 \end{array}$$

- $\{0, 1, x, x^2, 1 + x, 1 + x^2, x + x^2, 1 + x + x^2\} \subset F[x]$. In this case, $x \neq x^2$ and $x^2 + x \neq 0$.

Ex 2.6.12

Suppose S, T are subspaces of V with $\dim S = \dim T = n$. If $S \subset T$, then $S = T$.

Ex 2.6.13

Suppose S, T are subspaces of V and $S \subset T$. Then $\dim S \leq \dim T$.

Ex 2.6.14

Suppose S and T are finite dimensional vector spaces. Then

$$\dim(S + T) = \dim S + \dim T - \dim(S \cap T).$$

$$
\begin{tikzcd}
 & S+T \arrow[dl, no head] \arrow[dr, no head] & \\
S \arrow[dr, no head] & & T \arrow[dl, no head] \\
 & S \cap T &
\end{tikzcd}
$$

Ex 2.6.16

Let V be a vector space over a field F, and suppose S and T are subspaces of V satisfying $S \cap T = \{0\}$. Suppose $\{s_1, \cdots, s_k\} \subset S$ and $\{t_1, \cdots, t_l\} \subset T$ are bases for S and T, respectively. Prove that

$$\{s_1, \cdots, s_k, t_1, \cdots, t_l\}$$

is a basis for $S + T$.

Ex 2.6.17

Let U and V be vector sapces over a field F, and let $\{u_1, \cdots, u_n\}$ and $\{v_1, \cdots, v_m\}$ be bases for U and V, respectively. Prove that

$$\{(u_1, 0), \cdots, (u_n, 0), (0, v_1), \cdots, (0, v_m)\}$$

is a basis for $U \times V$.

Ex 2.6.18

Suppose F is a finite filed of char $F = p$. Then $|F| = p^n$.

(a) $0, 1, 1+1, \cdots, \underbrace{1+1+\cdots 1}_{p-1 \text{ times}}$ are all distinct.

Ex 2.6.18

Suppose F is a finite filed of char $F = p$. Then $|F| = p^n$.

(b) Identifying the subfield $\{0, 1, 2, \cdots, p-1\} \subset F$ with $\mathbb{Z}_p$, prove that F is a vector space over $\mathbb{Z}_p$.

Ex 2.6.18

Suppose F is a finite filed of char $F = p$. Then $|F| = p^n$.

(c) $|F| = p^n$.

Summary

- $V \neq \{0\}$: A subset of $\{u_1, \cdots, u_m\}$ (lin.indp) is a basis for span$\{u_1, \cdots, u_m\}$
- V fin.dim: Suppose $\{u_1, \cdots, u_k \subset V$ is linearly independent. If span$\{u_1, \cdots, u_k\} \neq V$, then there are $u_{k+1}, \cdots, u_n$ such that

$$\{u_1, \cdots, u_n\} \text{ is a basis for } V.$$

Theorem

Suppose $\dim V = n$.

1. *If* $\{u_1, \cdots, u_n\}$ *is linearly independent, then it is a basis.*
2. *If* $\mathrm{span}\{u_1, \cdots, u_n\} = V$, *then it is a basis.*

Gaussian-Elimination

$$\{v_1, \cdots, v_n\} \Longrightarrow \{\alpha_1 v_1 + \sum_{i=2}^{n} \alpha_i v_i, v_2, \cdots, v_n\}$$

Ex 2.7.10

Let $S = \text{span}\{v_1, v_2, v_3\} \subset \mathbb{Z}_3^3$, where

$$v_1 = (1, 2, 1), v_2 = (2, 1, 2), v_3 = (1, 0, 1).$$

Find a subset of $\{v_1, v_2, v_3\}$ that is a basis.

Ex 2.7.15

Let V be a vector space over a field F, and let $\{u_1, \cdots, u_n\}$ be a basis for V. Let $v_1, \cdots, v_k \in V$, and suppose

$$v_j = \alpha_{1,j} u_1 + \cdots + \alpha_{n,j} u_n$$

Define the vectors $x_1, \cdots, x_k$ in F^n by

$$x_j = (\alpha_{1,j}, \cdots, \alpha_{n,j}).$$

(a) $\{v_1, \cdots, v_k\}$ is lin.indp $\iff$ $\{x_1, \cdots, x_k\}$ lin.idnp.

(b) $\{v_1, \cdots, v_k\}$ spans V $\iff$ $\{x_1, \cdots, x_k\}$ spans F^n.

Observe

- For $(x_0, y_0), \cdots, (x_n, y_n)$, there is $p(x) = c_0 + c_1 x + \cdots + c_n x^n$ such that $p(x_i) = y_i$.
- The Lagrange Basis

$$\{1, x, \cdots, x^n\} \longleftrightarrow \{L_0(x), L_1(x), \cdots, L_n(x)\}$$

$$L_0(x) = \frac{(x - x_1)\cdots(x - x_n)}{(x_0 - x_1)\cdots(x_0 - x_n)}$$

$$\vdots$$

$$L_i(x) = \frac{(x - x_0)\cdots\widehat{(x - x_i)}\cdots(x - x_n)}{(x_i - x_0)\cdots\widehat{(x_i - x_i)}\cdots(x_i - x_n)}.$$

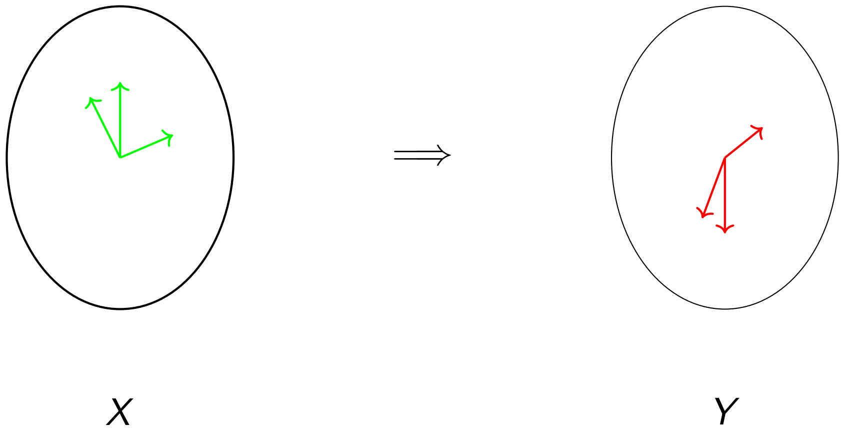
Then

$$L_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and

$$p(x) = y_0 L_0(x) + \cdots + y_n L_n(x).$$

Linear Operators



Linear : $x + y$, αx.

Definition

Let X and U be a vector spaces over a field F, and let $L : X \to U$. We say L is linear if and only if it satisfies the following conditions:

1. $L(\alpha x) = \alpha L(x)$
2. $L(x + y) = L(x) + L(y)$.

Remark

- $L(\alpha_1 x_1 + \cdots + \alpha_k x_k) = \alpha_1 L(x_1) + \cdots + \alpha_k L(x_k)$.
- $L(0) = 0$.
- If $L : X \to U$ and $M : U \to Z$ are linear, then $ML : X \to Z$ is linear.

$$X \xrightarrow{L} U \xrightarrow{M} Z.$$

Matrix

For $A_{ij} \in F$, $A = (A_{ij})$ is called a $m \times n$ matrix:

$$A = (A_{ij}) = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$$

$(A_{1j}, \cdots, A_{mk}$ is the j th column and write

$$A_j = \begin{bmatrix} A_{1j} \\ \vdots \\ A_{mj} \end{bmatrix}$$

$(A_{i1}, \cdots, A_{in})$ is the i th row and write

$$r_i = \begin{bmatrix} A_{i1} & \cdots & A_{in} \end{bmatrix}$$

Matrix

Then we can write A by

$$A = [A_1 | \cdots | A_n] = \begin{bmatrix} r_1 \\ \hline \vdots \\ \hline \\ r_m \end{bmatrix}$$

Multiplication

For $A \in F^{m \times n}$ and $x \in F^n$, we can define Ax by

$$Ax = \sum_{j=1}^{n} A_j x_j.$$

Note that A_j's are vectors and x_j's are scalar. And

$$(Ax)_i = \sum_{j=1}^{n} A_{ij} x_j.$$

Remark

$$\text{Linear Operator} \longleftrightarrow \text{Matrix}$$

Ex 3.1.12

Let $A \in F^{m\times n}$ and $B \in F^{n\times p}$. Find the formula $(AB)_{ij}$.

The End