# Algebraic Topology
# - Dunkin's Torus 6 -

KYB

Thrn, it's a Fact

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November 5, 2021

# Overview

## The Fundamental Group

- The Borsuk-Ulam Theorem
- Deformation Retracts and Homotopy Type

# The Borsuk-Ulam Theorem

## Definition

- If $x$ is a point of $S^n$, then it *antipode* is the point $-x$.
- A map $h : S^n \to S^m$ is said to be *antipode-preserving* if $h(-x) = -h(x)$ for all $x \in S^n$.

## Theorem (57.1)

*If $h : S^1 \to S^1$ is continuous and antipode-preserving, then $h$ i not nulhomotopic.*

**Theorem (57.2)**

*There is no continuous antipode-preserving map $g : S^2 \to S^1$.*

**Theorem (57.3, Borsuk-Ulam theorem for $S^2$)**

*Given a continuous map* $f : S^2 \to \mathbb{R}^2$*, there is a point* $x$ *of* $S^2$ *such that* $f(x) = f(-x)$.

**Theorem (57.4, The bisection theorem)**

*Given two bounded polygonal regions in $\mathbb{R}^2$, there exists a line in $\mathbb{R}^2$ that bisects each of them.*

## Ex 57.2

Show that if $g : S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all $x$, then $g$ is surjective.

## Ex 57.4

Suppose you are given the fact that for each $n$, no continuous antipode-preserving map $h : S^n \to S^n$ is nulhomotopic. Prove the following:

(a) There is no retraction $r : B^{n+1} \to S^n$.

(b) There is no continuous antipode-preserving map $g : S^{n+1} \to S^n$.

(c) (Bousuk-Ulam theorem) Given a continuous map $f : S^{n+1} \to \mathbb{R}^{n+1}$, there is a point $x$ of $S^{n+1}$ such that $f(x) = f(-x)$.

# Deformation Retracts and Homotopy Type

## Lemma (58.1)

*Let* $h, k : (X, x_0) \to (Y, y_0)$ *be continuous maps. If* $h$ *and* $k$ *are homotopic, and if the image of the base point* $x_0$ *of* $X$ *remains fixed at* $y_0$ *during the homotopy, then the homomorphisms* $h_*$ *and* $k_*$ *are equal.*

## Theorem (58.2)

*The inclusion map* $j : S^n \to \mathbb{R}^{n+1} - \mathbf{0}$ *induces an isomorphism of fundamental groups.*

# Deformation Retracts and Homotopy Type

## Definition

Let $A$ be a subspace of $X$.

- We say $A$ is a *deformation retract* of $X$ if there is a continuous map $H : X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a$ for all $a \in A$.
- The homotopy $H$ is called a *deformation retraction* of $X$ onto $A$. The map $r : X \to A$ defined by $r(x) = H(x, 1)$ is a retraction of $X$ onto $A$, and $H$ is a homotopy between the identity map of $X$ and the map $j \circ r$ where $j : A \to X$ is inclusion.

## Theorem (58.3)

*Let $A$ be a deformation retract of $X$; let $x_0 \in A$. Then the inclusion map*

$$j : (A, x_0) \to (X, x_0)$$

*induces an isomorphism of fundamental groups.*

### Example

Let $B$ denote the $z$-axis in $\mathbb{R}^3$. Consider the space $\mathbb{R}^3 - B$. It has the punctured $xy$-plane $(\mathbb{R}^2 - \mathbf{0}) \times 0$ as a deformation retract.

$$H(x, y, z, t) = (x, y, (1 - t)z)$$

We conclude that the space $\mathbb{R}^3 - B$ has an infinite cyclic fundamental group.

### Example

Consider $\mathbb{R}^2 - p - q$, the *doubly punctured plane*. It has the figure eight space as a deformation retract.

### Example

Another deformation retract of $\mathbb{R}^2 - p - q$ is the theta space

$$\theta = S^1 \cup (0 \times [-1, 1])$$

## Definition

Let $f : X \to Y$ and $g : Y \to X$ be continuous maps. Suppose that the map $g \circ f : X \to X$ is homotopic to the identity map of $X$, and the map $f \circ g : Y \to Y$ is homotopic to the identity map of $Y$.

- Then the maps $f$ and $g$ are called *homotopy equivalences*, and each is said to be a *homotopy inverse* of the other.

If $f : X \to Y$ is a homotopy equivalence of $X$ with $Y$ and $h : Y \to Z$ is a homotopy equivalence of $Y$ with $Z$, then $h \circ f : X \to Z$ is a homotopy equivalence of $X$ with $Z$.

- So we can say that two spaces have the same *homotopy type* if they are homotopy equivalent.

## Lemma (58.4)

*Let $h, k : X \to Y$ be continuous maps; let $h(x_0) = y_0$ and $k(x_0) = y_1$. If $h$ and $k$ are homotopic, there is a path $\alpha$ in $Y$ from $y_0$ to $y_1$ such that $k_* = \hat{\alpha} \circ h_*$. Indeed, if $H : X \times I \to Y$ is the homotopy between $h$ and $k$, then $\alpha$ is the path $\alpha(t) = H(x_0, t)$.*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\alpha}"] \\
& \pi_1(Y, y_1)
\end{tikzcd}$$

## Corollary (58.5)

*Let* $h, k : X \to Y$ *be homotopic continuous maps; let* $h(x_0) = y_0$ *and* $k(x_0) = y_1$*. If* $h_*$ *is injective, or surjective, or trivial, so is* $k_*$*.*

## Corollary (58.6)

*Let* $h : X \to Y$*. If* $h$ *is nulhomotopic, then* $h_*$ *is the trivial homomorphism.*

## Theorem (58.7)

*Let $f : X \to Y$ be continuous; let $f(x_0) = y_0$. If $f$ is a homotopy equivalence, then*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

*is an isomorphism.*

## Ex 58.1

Show that if $A$ is a deformation retract of $X$, and $B$ is a deformation retract of $A$, then $B$ is a deformation retract of $X$.

## Ex 58.5

Recall that a space $X$ is said to be *contractible* if the identity map of $X$ to itself is nulhomotopic. Show that $X$ is contractible if and only if $X$ has the homotopy type of a one-point space.

## Degree of $h : S^1 \to S^1$

We define the *degree* of a continuous map $h : S^1 \to S^1$ as follows:

Let $b_0$ be the point $(1, 0)$ of $S^1$; choose a generator $\gamma$ for the infinite cyclic group $\pi_1(S^1, b_0)$. If $x_0$ is any point of $S^1$, choose a path $\alpha$ in $S^1$ from $b_0$ to $x_0$, and define $\gamma(x_0) = \hat{\alpha}(\gamma)$. The element $\gamma(x_0)$ is independent of the choice of the path $\alpha$, since the fundamental group of $S^1$ is abelian.

## Degree of $h : S^1 \to S^1$

Now given $h : S^1 \to S^1$, choose $x_0 \in S^1$ and let $h(x_0) = x_1$. Consider the homomorphism

$$h_* : \pi_1(S^1, x_0) \to \pi_1(S^1, x_1).$$

Since both groups are infinite cyclic, we have

$$h_*(\gamma(x_0)) = d \cdot \gamma(x_1) \quad \cdots (*)$$

for some integer $d$, if group is written additively. The integer $d$ is called the *degree* of $h$ and is denoted by deg $h$.

The degree of $h$ is independent the choice of the generator $\gamma$; choosing the other generator would merely change the sign of both sides of (*)

## Ex 58.9

(a) Show that $d$ is independent of the choice of $x_0$.

(b) Show that if $h, k : S^1 \to S^1$ are homotopic, they have the same degree.

(c) Show that $\deg(h \circ k) = (\deg h)(\deg k)$.

(d) Compute the degrees of the constant map, the identity map, the reflection map $\rho(x_1, x_2) = (x_1, -x_2)$, and the map $h(z) = z^n$, where $z$ is a complex number.

(e) Show that if $h, k : S^1 \to S^1$ have the same degree, they are homotopic.