LA2 11

KYB

Thrn, it's a Fact

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Overview

Ex 10.2.3

Suppose $\{f_k\}$ is a Cauchy sequence in $C[a,b]$ under the L^∞ norm thet converges pointwise to $f:[a,b]\to\mathbb{R}$. Prove that $f_k\to f$ in the L^∞ norm.

Proof

Step1) $\{f_k\}$ is bounded set under L^∞.
Since $\{f_k\}$ is a Cauchy sequence, there is N such that

$$\|f_n - f_m\|_\infty < 1 \text{ for all } n, m \geq N.$$

Let $L = \max\{\|f_1\|_\infty, \cdots, \|f_{N-1}\|_\infty, \|f_N\|_\infty + 1\}$. Then for all n, $\|f_n\|_\infty < L$.
(continued)

Proof

Step2) f is a bounded function.
Let $x \in [a, b]$ and choose M such that

$$|f_k(x) - f(x)| < 1 \text{ for all } k \geq M.$$

Then

$$|f(x)| \leq |f_M(x) - f(x)| + |f_M(x)| < 1 + L.$$

Thus $\|f\|_\infty = \sup\{|f(x)| : a \leq x \leq b\} \leq 1 + L$.
(continued)

Proof

Step3) $f_k \to f$ in L^∞.
Let $x \in [a,b]$ and let $\epsilon > 0$ be given. Choose

- N so that $\|f_n - f_m\|_\infty < \epsilon/3$ for all $n, m \geq N$.
- M so that $|f_n(x) - f(x)| < \epsilon/3$ for all $n \geq M$.

We may assume $M \geq N$. Then for all $n \geq N$,

$$|f_n(x) - f(x)| \leq |f_n(x) - f_M(x)| + |f_M(x) - f(x)| < \frac{2}{3}\epsilon.$$

Thus $\|f_n - f\|_\infty < \epsilon$ for all $n \geq N$.

Completeness of l^2

l^2 is complete space under $\|\cdot\|_2$. Let $\{x_k\}$ be a Cauchy sequence. For given $\epsilon > 0$, choose N such that

$$\|x_n - x_m\|_2 \leq \epsilon \text{ for all } n, m \geq N.$$

Then for each i,

$$(x_n^{(i)} - x_m^{(i)})^2 \leq \sum (x_n^{(i)} - x_m^{(i)})^2 = \|x_n - x_m\|_2^2 < \epsilon$$

So $\{x_k^{(i)}\}$ is a Cauchy sequence in $\mathbb{R}$. Let $x^{(i)} = \lim_{k\to\infty} x_k^{(i)}$.

Claim1) x belongs to l^2.
Note that there is L such that $\|x_k\| \leq L$ for all k. For each n,

$$\sum_{i=1}^{n} (x^{(i)})^2 = \sum_{i=1}^{n} \left(\lim_{k\to\infty} x_k^{(i)} \right)^2 = \lim_{k\to\infty} \sum_{i=1}^{n} (x_k^{(i)})^2 \leq \lim_{k\to\infty} \sum_{i=1}^{\infty} (x_k^{(i)})^2 \leq L^2.$$

(continued)

Completeness of l^2

Claim2) $x_k \to x$ in l^2 norm. Let $\epsilon > 0$ be given. Choose N so that

$$\|x_n - x_m\|_2 \leq \epsilon \text{ for all } m, n \geq N.$$

For any n and $k \geq N$,

$$\sum_{i=1}^{n} \left(x_k^{(i)} - x^{(i)}\right)^2 = \sum_{i=1}^{n} \left(x_k^{(i)} - \lim_{m\to\infty} x_m^{(i)}\right)^2 = \lim_{m\to\infty} \sum_{i=1}^{n} \left(x_k^{(i)} - x_m^{(i)}\right)^2$$
$$\leq \lim_{m\to\infty} \sum_{i=1}^{\infty} \left(x_k^{(i)} - x_m^{(i)}\right)^2 \leq \epsilon^2.$$

So $\|x_k - x\|_2 \leq \epsilon$ for all $k \geq N$.

Proposition

Let X, U be a n-dimensional vector spaces over $\mathbb{R}$. Suppose $\|\cdot\|_U$ is a norm on U. Let $T : X \to U$ be an isomorphism. Then $\|\cdot\|_X : X \to \mathbb{R}$ defined by

$$\|x\|_X = \|T(x)\|_U$$

is a norm.

Proof

$$\|x\|_X = \|T(x)\|_U \geq 0.$$

If $\|x\|_X = 0$, $\|T(x)\|_U = 0$. So. $T(x) = 0$. Since T is an isomorphism, $x = 0$.

$$\|\alpha x\|_X = \|T(\alpha x)\|_U = \|\alpha T(x)\|_U = |\alpha| \|T(x)\|_U = |\alpha| \|x\|_X.$$

$$\begin{aligned}\|x + y\|_X &= \|T(x + y)\|_U = \|T(x) + T(y)\|_U \\ &\leq \|T(x)\|_U + \|T(y)\|_U = \|x\|_X + \|y\|_X\end{aligned}$$

Proposition

Let V be an n-dimensional vector space over $\mathbb{R}$. Any two norms on V are equivalent.

Proof

Recall that any two norms on $\mathbb{R}^n$ are equivalent. Let $T : \mathbb{R}^n \to V$ be an isomorphism. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ on V be given. Then two induced norms $\|\cdot\|_1^*$ and $\|\cdot\|_2^*$ are equivalent, i.e. there are $c_1, c_2 > 0$ such that

$$c_1\|T(x)\|_1^* \leq \|T(x)\|_2^* \leq c_2\|T(x)\|_1^*$$

Since $\|T(x)\|_i^* = \|x\|_i^*$,

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1.$$

Remark

Let X and U be n-dimensional normed space with $\|\cdot\|_X$ and $\|\cdot\|_U$. Let $T : X \to U$ be an isomorphism. Then T is continuous (moreover, it is homeomorphic). So convergence, continuity, compactness, openness, closedness, Cauchy sequence, etc. hold on any finite dimensional vector spaces.

Functional analysis

Example

Let $V = C[0,1]$ under $L^2(0,1)$ norm, and let $f : V \to \mathbb{R}$ be defined by $f(v) = v(1)$. Then f is linear:

- $f(u+v) = (u+v)(1) = u(1) + v(1) = f(u) + f(v)$
- $f(\alpha v) = (\alpha v)(1) = \alpha v(1) = \alpha f(v)$.

Suppose f is continuous, If $\{v_k\}$ is a sequence in V and $v_k \to v \in V$, then $f(v_k) \to f(v)$ must hold. But for $v_k = x^k$, $\|v_k\|_{L^2(0,1)} \to 0$ as $k \to \infty$, and hence $v_k \to 0$. But $f(v_k) = 1^k = 1 \neq 0 = v(1) = f(v)$. Therefore $f(v_k)$ does not converge to $f(v)$, which shows that f is not continuous.

Remark

Above example shows that there may exists a linear function from V to $\mathbb{R}$ which is not continuous when V is infinite-dimensional.

Definition

Let V be a normed vector space over $\mathbb{R}$. The (continuous) dual space V^* of V is the space of continuous linear functionals defined on V.

Remark

If V is finite dimensional, every linear function $f : V \to \mathbb{R}$ is continuous. Thus $V^* = \mathcal{L}(V, \mathbb{R})$. In this case, $\dim \mathcal{L}(V, \mathbb{R}) = \dim V$. Thus $V \cong V^*$.

Definition

Let V be a normed vector space over $\mathbb{R}$, and let $f : V \to \mathbb{R}$ be linear. We say that f is bounded if and only if there exists a positive number M such that

$$|f(v)| < M \text{ for all } v \in V, \|v\| \leq 1.$$

Theorem (447)

Let V be a normed vector space, and let $f : V \to \mathbb{R}$ be linear. Then f is continuous if and only if it is bounded.

Lemma (448)

Let V be a normed vector space over $\mathbb{R}$ and let $f \in V^*$. Then

$$\begin{aligned}&\sup\{|f(v)| : v \in V, \|v\| \leq 1\}\\ =&\inf\{M > 0 : |f(v)| \leq M \text{ for all } v \in V, \|v\| \leq 1\}.\end{aligned}$$

Theorem (449)

Let V be a normed vector space. For each $f \in V^*$, define

$$\|f\|_{V^*} = \sup\{|f(v)| : v \in V, \|v\|_V \leq 1\}.$$

Then $\|\cdot\|_{V^*}$ defines a norm on V^*.

Theorem (450)

Let V be a normed vector space. Then V^*, under the norm defined in Theorem 449, is complete. (V need not be complete.)

Ex 10.3.3

Prove Theorem 450.

Proof

Let $\{f_k\}$ be a Cauchy sequence in V^*. For each $v \in V$,

$$|f_m(v) - f_n(v)| \leq \|f_m - f_n\|_{V^*} \|v\|_V.$$

Thus $\{f_k(v)\}$ is a Cauchy sequence in $\mathbb{R}$. Define $f : V \to \mathbb{R}$ by

$$f(v) = \lim_{k \to \infty} f_k(v).$$

(continued)

Proof

f is linear) Let $v, w \in V$. Then

$$
\begin{aligned}
f(v+w) &= \lim_{k\to\infty} f_k(v+w) = \lim_{k\to\infty} (f_k(v)+f_k(w)) \\
&= \lim_{k\to\infty} f_k(v) + \lim_{k\to\infty} f_k(w) = f(v)+f(w).
\end{aligned}
$$

Similarly, you can show that $f(\alpha v) = \alpha f(v)$.

Let $\|v\| = 1$ for any $v \in V$. Since f_k is a Cauchy, there is M such that $\|f_k\|_{V^*} \leq M$ for all n. Also for sufficiently large n, $|f(v) - f_n(v)| < 1$.

$$|f(v)| \leq |f(v) - f_n(v)| + |f_n(v)| < 1 + M.$$

Thus f is bounded, that is $f \in V^*$.
(continued)

Proof

Finally, choose N so that $\|f_n - f_m\| < \epsilon/3$ for all $m, n \geq N$. For each $v \in V$ with $\|v\| = 1$, choose $M \geq N$ so that $|f_n(v) - f(v)| < \epsilon/3$ for all $n \geq M$. Then for all $k \geq N$,

$$|f_k(v) - f(v)| \leq |f_k(v) - f_M(v)| + |f_M(v) - f(v)| < 2\epsilon/3.$$

Hence $\|f_k - f\|_{V^*} < \epsilon$ for all $n \geq N$, and so $\|f_k - f\|_{V^*} \to 0$ as $k \to \infty$.

Theorem (451)

Let V be a normed vector space and let f belong to V^*. Then

$$|f(v)| \leq \|f\|_{V^*} \|v\|_V \text{ for all } v \in V.$$

Recall

$\|Ax\| \leq \|A\|\|x\|$ for $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$.

Ex 10.3.4

Prove Theorem 451.

Proof

Let $v \neq 0$.

$$|f(v/\|v\|)| \leq \|f\|.$$

Example

For $p \geq 1$, $L^p(a,b)$ is the set of all functions such that

$$\int_a^b |f(x)|^p dx < \infty$$

in the Lebesgue sense with Lebesgue measure. In this space, two functions are regarded as equal if $\{x : f(x) \neq g(x)\}$ is a measure zero set. Note that in this case, $\int_a^b |f|^p dx = \int_a^b |g|^p dx$ for all $p \geq 1$ (possibly infinite). Now $\|f\|_p$ is well-defined norm on $L^p(a,b)$.

Example

By Hölder inequality, for $p, q \geq 1$ with $1/p + 1/q = 1$

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^p\right)^{1/p} \left(\int_a^b |g(x)|^q dx\right)^{1/q}$$

or

$$\left|\int_a^b f(x)g(x)dx\right| \leq \|f\|_p \|g\|_q.$$

Choose $g \in L^q(a,b)$. We can define a linear functional $l : L^p(a,b) \to \mathbb{R}$ by

$$l(f) = \int_a^b f(x)g(x)dx.$$

Since $|l(f)| \leq \|g\|_q \|f\|_p$, l is bounded. Thus $l \in (L^p(a,b))^*$

Example

Conversely, for any $l \in (L^p(a,b))^*$, there is $g \in L^q(a,b)$ such that

$$l(f) = \int_a^b f(x)g(x)dx.$$

(The proof is not so easy, and so we omit the proof.)
Then we get $L^q(a,b) \cong (L^p(a,b))^*$.

Hilbert Space

- H is a Hilbert space if it is a complete inner product space.
- Hilbert space satisfies the projection theorem.
- Let S be a closed subspace of H. Then $(S^{\perp})^{\perp} = S$.

Theorem (453, The projection theorem)

Let H be a Hilbert space over $\mathbb{R}$, and let S be a closed subspace of H.

1. *For any $v \in H$, there is a unique best approximation to v from S, that is, a unique $w \in S$ satisfying*

$$\|v - w\| = \min\{\|v - z\| : z \in S\}.$$

2. *A vector $w \in S$ is the best approximation to v from S if and only if*

$$\langle v - w, z \rangle = 0 \text{ for all } z \in S.$$

Proof

If S is finite-dimensional, we already show that the projection theorem holds. Suppose S is infinite-dimensional. For all $z \in S$, $\|v - z\| \geq 0$. Let $d = \inf\{\|v - z\| : z \in S\}$. Then, we can find a sequence $\{z_k\}$ in S such that

$$\lim_{k \to \infty} \|v - z\| = d.$$

Claim) $\{z_k\}$ is a Cauchy sequence.

$$\begin{aligned}
\|z_m - z_n\|^2 &= \|(z_m - v) - (z_n - v)\|^2 \\
&= 2\|z_m - v\|^2 + 2\|z_n - v\|^2 - \|(z_m - v) + (z_n - v)\|^2. \\
&= 2\|z_m - v\|^2 + 2\|z_n - v\|^2 - 4\left\|\frac{z_m + z_n}{2} - v\right\|^2. \\
&\leq 2\|z_m - v\|^2 + 2\|z_n - v\|^2 - 4d^2
\end{aligned}$$

(continued)

Proof

Since $\|z_k - v\| \to d$, we get $\|z_m - z_n\| \to 0$ as $m, n \to \infty$. Then by take N_1(resp. N_2) so that

$$\|z_m - v\|^2 < d^2 + \epsilon^2/4 \text{ (resp. } \|z_n - v\|^2 < d^2 + \epsilon^2/4)$$

for all $m \geq N_1$(resp.$n \geq N_2$), we get

$$\|z_m - z_n\| < \epsilon$$

for all $m, n \geq \max\{N_1, N_2\}$.
(continued)

Proof

Since H is complete and S is closed, $z_k \to w$ for some $w \in S$. Moreover the continuity of $\|\cdot\|$ implies $\|z_k - v\| \to \|w - v\|$. But we already know that $\|z_k - v\| \to d$. Thus $\|w - v\| = d$ and w is a best approximation to v from S.

The second result can be proved exactly as in Section 6.4 (consider $\|v - (w + tz)\|^2$). And the uniqueness of w is derived from 2.

Remark

If V is an inner product space, every finite dimensional subspace S is closed. Let $x \in V - S$. Then for all $s \in S$, $\|x - s\| > 0$. In particular,

$$0 < \|x - \operatorname{proj}_S x\| \leq \|x - s\|.$$

Let $r = \frac{1}{2}\|x - \operatorname{proj}_S x\|$. Then for $y \in B_r(x)$,

$$\|x - \operatorname{proj}_S x\| \leq \|x - s\| \leq \|x - y\| + \|y - s\|$$

or

$$0 < \frac{1}{2}\|x - \operatorname{proj}_S x\| \leq \|x - \operatorname{proj}_S x\| - \|x - y\| \leq \|y - s\|.$$

Thus for all $s \in S$, $\|y - s\| \neq 0$. Hence

$$B_r(x) \subset V - S,$$

that is, S is closed.

Definition

The orthogonal complement $S^\perp$ of a subspace S is

$$S^\perp = \{v \in H, \langle v, u \rangle = 0 \text{ for all } u \in S\}.$$

Theorem (454)

Let H be a Hilbert space and let S be a closed subspce of H. Then $(S^\perp)^\perp = S$.

Proof

The proof is the same as that of in Section 6.6. The condition that S is closed must be needed because we use the projection theorem.
At first, $S^\perp \cap (S^\perp)^\perp = \{0\}$.
Clearly, $S \subset (S^\perp)^\perp$. Let $x \in (S^\perp)^\perp$ and define $s = \mathrm{proj}_S\, x$. Then $x - s \in S^\perp$ because $\langle x - s, u \rangle = 0$ for all $u \in S$. But $s \in S \subset (S^\perp)^\perp$ and thus $x - s \in (S^\perp)^\perp$. Hence $x - s = 0$, or $x = s$.

Lemma (455)

Let H be a Hilbert space, and let $f \in H^*$, $f \neq 0$. Then $\ker(f)$ is a closed subspace with co-dimension one. $(\dim(\ker(f))^{\perp} = 1)$

Proof

Suppose $\{v_k\}$ is a sequence in $\ker(f)$ and $v_k \to v \in H$. By continuity of f,

$$f(v) = \lim_{k\to\infty} f(v_k) = \lim_{k\to\infty} 0 = 0.$$

Therefore $v \in \ker(f)$. So $\ker(f)$ is closed.

Suppose u and w are nonzero vectors in $\ker(f)^{\perp}$. Then $f(u)$ and $f(w)$ is nonzero. Then there is $\alpha \in \mathbb{R}$ such that $f(u) - \alpha f(w) = 0$. Since f is linear, $f(u - \alpha w) = 0$, whence $u - \alpha w \in \ker(f)$. But $u - \alpha w \in \ker(f)^{\perp}$, and thus $u - \alpha W = 0$, or $u = \alpha w$. Since f is not the zero functional, $\ker(f)^{\perp}$ contains at least one nonzero vector w and this implies $\ker(f)^{\perp} = \operatorname{span}\{w\}$.

Theorem (456, Riesz representation theorem)

Let H be a Hilbert space over $\mathbb{R}$. If $f \in H^*$, then there exists a unique vector u in H such that

$$f(v) = \langle v, u \rangle_H \text{ for all } v \in H.$$

Moreover, $\|u\|_H = \|f\|_{H^*}$.

Ex 10.3.6

Uniqueness) Suppose $f(v) = \langle v, w \rangle$ for all $v \in H$.

$$0 = f(v) - f(v) = \langle v, w \rangle - \langle v, u \rangle = \langle v, w - u \rangle.$$

Thus $w - u = 0$, or $w = u$.

Proof

Existence) If f is the zero functional, take $v = 0$. Suppose f is nonzero and take any nonzero $w \in \ker(f)^{\perp}$. Define $u \in \ker(f)$ by

$$u = \frac{f(w)}{\|w\|^2} w.$$

Then

$$\langle w, u \rangle = \left\langle w, \frac{f(w)}{\|w\|^2} w \right\rangle = \frac{f(w)}{\|w\|^2} \langle w, w \rangle = f(w).$$

Therefore, $f(w) = \langle w, v \rangle$. Since $\dim \ker(f)^{\perp} = 1$, $\ker(f)^{\perp} = \operatorname{span}\{w\}$. Thus for all $x \in \ker(f)^{\perp}$, $f(x) = \langle x, u \rangle$.
(continued)

Proof

Every vector $v \in H$ can be written as

$$v = x + y, x \in \ker(f)^{\perp}, y \in \ker(f).$$

It follows that

$$f(v) = f(x+y) = f(x) = \langle x, u \rangle = \langle x, u \rangle + \langle y, u \rangle = \langle v, u \rangle.$$

Finally, by the Cauchy-Schwarz inequality,

$$|f(v)| = |\langle v, u \rangle| \leq \|v\| \|u\|,$$

so $\|f\| \leq \|u\|$. Conversely,

$$|f(u)| = |\langle u, u \rangle| = \|u\| \|u\|,$$

so $\|f\| \geq \|u\|$. Hence $\|f\| = \|u\|$.

Ex 10.3.2

Let S be any set and $f, g : S \to \mathbb{R}$ be functions. Prove that

$$\sup\{f(x) + g(x) : x \in S\} \leq \sup\{f(x) : x \in S\} + \sup\{g(x) : x \in S\}.$$

Ex 10.3.5

Suppose H is a Hilbert space and S is a subspace of H that fails to be closed. What is $(S^{\perp})^{\perp}$ in this case?

Proof

Toplogically, $(S^{\perp})^{\perp} = \overline{S}$ (closure of S).

For a subset A of H, a closure $\overline{A}$ of A is the smallest closed subset containing A in the sense:

1. $\overline{A}$ is closed
2. if C is a closed subset containing A, then $\overline{A} \subset C$.

Note that for each $x \in H$, $\langle \cdot, x \rangle$ is continuous because for $v \in H$ with $\|v\| = 1$,

$$|\langle v, x \rangle| \leq \|v\| \|x\| = \|x\| < \infty.$$

Thus if $v_k \to v$, then $\langle v_k, x \rangle \to \langle v, x \rangle$ for all $x \in H$.
(continued)

Proof

Cliam1) $S^\perp = \overline{S}^\perp$.

Since $S \subset \overline{S}$, $S^\perp \supset \overline{S}^\perp$. Let $v \in S^\perp$. Then for each $s \in \overline{S}$, there is a sequence $\{s_k\}$ in S which converges to s and $\langle s_k, v\rangle = 0$. Then $\langle s_k, v\rangle \to \langle s, v\rangle = 0$. Thus $v \in \overline{S}^\perp$, and so $S^\perp \subset \overline{S}^\perp$.

Claim2) $\overline{S}$ is a subspace of H.

Let $s, t \in \overline{S}$ with sequences $\{s_k\}$ and $\{t_k\}$ where $s_k \to s$ and $t_k \to t$. Then $s_k + t_k \to s + t$. Similarly, $\alpha s_k \to \alpha s$. Thus $\overline{S}$ is a subspace of H.

Now $(S^\perp)^\perp = (\overline{S}^\perp)^\perp = \overline{S}$

Ex 10.3.7

Let X and U be Hilbert spaces, and let $T : X \to U$ be linear. We say that T is bounded if and only if there exists $M > 0$ such that

$$\|T(x)\|_U \leq M\|x\|_X \text{ for all } x \in X.$$

Prove that T is continuous if and only if T is bounded.

Remark

Let $f \in U^*$. Then $f \circ T : X \to \mathbb{R}$ is continuous and linear. So $f \circ T \in X^*$. Then we have a dual map $T^* : U^* \to X^*$ by $T^*(f) = f \circ T$.

Ex 10.3.8

Let X, U be Hilbert spaces, and let $T : X \to U$ be linear and bounded. Use the Riesz representation theorem to prove that there exists a uique bounded linear operator $T^* : U \to X$ such that

$$\langle T(x), u \rangle_U = \langle x, T^*(u) \rangle_X \text{ for all } x \in X, u \in U.$$

The operator T^* is called the adjoint of T.

Proof

Let $u \in U$ and consider $f(x) = \langle T(x), u \rangle_U$. Since

$$|f(x)| \leq \|T(x)\|_U \|u\|_U \leq M\|x\|_X \|u\|_U,$$

f is bounded. So $f \in X^*$. Now by Riesz representation theorem, there is x^* such that

$$f(x) = \langle T(x), u \rangle_U = \langle x, x^* \rangle_X \text{ for all } x \in X.$$

Define $T^*(u) = x^*$. (continued)

Proof

By the uniqueness of x^*, T^* is well defined function. For $u_1, u_2 \in U$ with x_1^*, x_2^*, let $T^*(u_1 + u_2) = x^*$. Then

$$\begin{aligned}\langle x, x^*\rangle_X &= \langle T(x), u_1 + u_2\rangle_U = \langle T(x), u_1\rangle_U + \langle T(x), u_2\rangle_U \\ &= \langle x, x_1^*\rangle_X + \langle x, x_2^*\rangle_X = \langle x, x_1^* + x_2^*\rangle.\end{aligned}$$

Thus $x^* = x_1^* + x_2^*$. Similarly, you can show that $T^*(\alpha u) = \alpha T^*(u)$. Thus T^* is linear.
(continued)

Proof

Finally, let M be such that

$$\|T(x)\|_U \leq M\|x\|_X \text{ for all } x \in X,$$

and let $u \in U$ with $\|u\|_U = 1$.

$$\|T^*(u)\|_X = \|\langle \cdot, T^*(u)\rangle\|_{X^*} = \|\langle T(\cdot), u\rangle_U\|_{U^*}$$

Since for $x \in X$ with $\|x\|_X = 1$

$$|\langle T(x), u\rangle| \leq \|T(x)\|_U\|u\|_U = \|T(x)\|_U \leq M\|x\|_X = M,$$

$$\|T^*(u)\|_X = \|\langle T(\cdot), u\rangle_U\|_{U^*} \leq M = M\|u\|_U.$$

In general, $\|T^*(u)\|_X \leq M\|u\|_U$. Hence T^* is bounded linear.

The End