Analysis
- PMA 19 -

KYB

Thrn, it's a Fact

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Overview

Integration of Differential Forms

- Integration
- Primitive Mappings
- Partitions of Unity
- Change of Variables
- Exercises

Integration

Definition

Suppose I^k is a k-cell in $\mathbb{R}^k$, consisting of all

$$\mathbf{x} = (x_1, \cdots, x_k)$$

such that

$$a_i \leq x_i \leq b_i \quad i = 1, \cdots, k.$$

I^j is the j-cell in $\mathbb{R}^j$ defined by the first j inequalities, $a_i \leq x_i \leq b_i$, and f is a real continuous function on I^k. Put $f = f_k$, and define f_{k-1} on I^{k-1} by

$$f_{k-1}(x_1, \cdots, x_{k-1}) = \int_{a_k}^{b_k} f_k(x_1, \cdots, x_k)\, dx_k.$$

The uniform continuity of f_k on I^k shows that f_{k-1} is continuous on I^{k-1}. Hence we can repeat this process and obtain functions f_j, continuous on I^j, such that f_{j-1} is the integral of f_j with respect to x_j, over $[a_j, b_j]$. After k steps we arrive at a number f_0, which we call the *integral of f over I^k*, we write it in the form

$$\int_{I^k} f(\mathbf{x})\, d\mathbf{x} \quad \text{or} \quad \int_{I^k} f.$$

Integration

Write $L(f)$ for the integral in the previous definition, and $L'(f)$ for the result obtained by carrying out the k integrations in some other order.

Theorem (10.2)

For every $f \in \mathscr{C}(I^k)$, $L(f) = L'(f)$.

Proof.

If $h(\mathbf{x}) = h_1(x_1) \cdots h_k(x_k)$, where $h_j \in \mathscr{C}([a_j, b_j])$, then

$$L(h) = \prod_{i=1}^{k} \int_{a_i}^{b_i} h_i(x_i)\, dx_i = L'(h).$$

If $\mathscr{A}$ is the set of all finite sums of such functions h, it follows that $L(g) = L'(g)$ for all $g \in \mathscr{A}$. Also, $\mathscr{A}$ is an algebra of functions on I^k to which the Stone-Weierstrass theorem applies.
Put $V = \prod_1^k (b_i - a_i)$. If $f \in \mathscr{C}(I^k)$ and $\epsilon > 0$, there exists $g \in \mathscr{A}$ such that $\|f - g\| < \epsilon/V$, where $\|f\|$ is defined as $\max |f(\mathbf{x})|$ ($\mathbf{x} \in I^k$). Then $|L(f-g)| < \epsilon$, $|L'(f-g)| < \epsilon$, and since

$$L(f) - L'(f) = L(f-g) + L'(g-f)$$

we conclude that $|L(f) - L'(f)| < 2\epsilon$. $\square$

Integration

Definition

- The *support* of a (real or complex) function f on $\mathbb{R}^k$ is the closure of the set of all points $\mathbf{x} \in \mathbb{R}^k$ at which $f(\mathbf{x}) \neq 0$.
- If f is a continuous function with compact support, let I^k be any k-cell which contains the support of f, and define

$$\int_{\mathbb{R}^k} f = \int_{I^k} f.$$

Integration

Example

Let Q^k be the k-simplex which consists of all points $\mathbf{x} = (x_1, \cdots, x_k)$ in $\mathbb{R}^k$ for which $x_1 + \cdots + x_k \leq 1$ and $x_i \geq 0$ for $i = 1, \cdots, k$. If $f \in \mathscr{C}(Q^k)$, extend f to a function on I^k by setting $f(\mathbf{x}) = 0$ off Q^k, and define

$$\int_{Q^k} f = \int_{I^k} f.$$

Here I^k is the "unit cube" defined by $0 \leq x_i \leq 1$ $(1 \leq i \leq k)$. Since f may by discontinuous on I^k, the existence of the integral $\int_{I^k} f$ needs proof.

Primitive Mappings

Our next goal is the change of variables.

Definition

- If $\mathbf{G}$ maps on open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, and if there is an integer m and a real function g with domain E such that
$$\mathbf{G}(\mathbf{x}) = \sum_{i \neq m} x_i \mathbf{e}_i + g(\mathbf{x})\mathbf{e}_m \quad (\mathbf{x} \in E),$$
then we call $\mathbf{G}$ primitive. In this case, we can also write
$$\mathbf{G}(\mathbf{x}) = \mathbf{x} + [g(\mathbf{x}) - x_m]\mathbf{e}_m.$$
- If g is differentiable at some point $\mathbf{a} \in E$, so is $\mathbf{G}$. The matrix $[\alpha_{ij}]$ of the operator $\mathbf{G}'(\mathbf{a})$ has
$$(D_1 g)(\mathbf{a}), \cdots, (D_m g)(\mathbf{a}), \cdots, (D_n g)(\mathbf{a})$$
at its mth row. For $j \neq m$, we have $\alpha_{jj} = 1$ and $\alpha_{ij} = 0$ if $i \neq j$. The Jacobian of $\mathbf{G}$ at $\mathbf{a}$ is thus given by
$$J_{\mathbf{G}}(\mathbf{a}) = \det[\mathbf{G}'(\mathbf{a})] = (D_m g)(\mathbf{a}),$$
and $\mathbf{G}'(\mathbf{a})$ is invertible if and only if $(D_m g)(\mathbf{a}) \neq 0$.

Primitive Mappings

Definition

A linear operator B on $\mathbb{R}^n$ that interchanges some pair of members of the standard basis and leaves the other fixed will be called a *flip*.

For example,

$$B(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + x_4\mathbf{e}_4) = x_1\mathbf{e}_1 + x_2\mathbf{e}_4 + x_3\mathbf{e}_3 + x_4\mathbf{e}_2$$

interchanges $\mathbf{e}_2$ and $\mathbf{e}_4$, and thus is a flip.

Primitive Mappings

In the proof that follows, we shall use the projections $P_0, \cdots, P_n$ in $\mathbb{R}^n$ defined by $P_0\mathbf{x} = \mathbf{0}$ and

$$P_m\mathbf{x} = x_1\mathbf{e}_1 + \cdots x_m\mathbf{e}_m$$

for $1 \le m \le n$. Thus P_m is the projections whose range an null space are spanned by $\{\mathbf{e}_1, \cdots, \mathbf{e}_m\}$ and $\{\mathbf{e}_{m+1}, \cdots, \mathbf{e}_n\}$, respectively.

Theorem (10.7)

Suppose $\mathbf{F}$ *is a* $\mathscr{C}'$*-mapping of an open set* $E \subset \mathbb{R}^n$ *into* $\mathbb{R}^n$, $\mathbf{0} \in E$, $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, *and* $\mathbf{F}'(\mathbf{0})$ *is invertible. Then there is a neighborhood of* $\mathbf{0}$ *in* $\mathbb{R}^n$ *in which representation*

$$\mathbf{F}(\mathbf{x}) = B_1 \cdots B_{n-1}\mathbf{G}_n \circ \cdots \circ \mathbf{G}_1(\mathbf{x})$$

is valid, where each $\mathbf{G}_i$ *is a primitive* $\mathscr{C}'$*-mapping in some neighborhood of* $\mathbf{0}$; $\mathbf{G}_i(\mathbf{0}) = \mathbf{0}$, $\mathbf{G}_i'(\mathbf{0})$ *is invertible, and each* B_i *is either a flip or the identity operator.*

Primitive Mappings

Proof, Step1

Put $\mathbf{F} = \mathbf{F}_1$. Assume $1 \leq m \leq n-1$, and make the following induction hypothesis;

> V_m is a neighborhood of $\mathbf{0}$, $\mathbf{F}_m \in \mathscr{C}'(V_m)$, $\mathbf{F}_m(\mathbf{0}) = \mathbf{0}$, $\mathbf{F}'_m(\mathbf{0})$ is invertible, and
>
> $$P_{m-1}\mathbf{F}_m(\mathbf{x}) = P_{m-1}\mathbf{x} \quad \mathbf{x} \in V_m.$$

Then,

$$\mathbf{F}_m(\mathbf{x}) = P_{m-1}\mathbf{x} + \sum_{i=m}^{n} \alpha_i(\mathbf{x})\mathbf{e}_i,$$

where $\alpha_m, \cdots, \alpha_n$ are real $\mathscr{C}'$-functions in V_m.
Hence

$$\mathbf{F}'_m(\mathbf{0})\mathbf{e}_m = \sum_{i=m}^{n} (F_m\alpha_i)(\mathbf{0})\mathbf{e}_i.$$

Since $\mathbf{F}'_m(\mathbf{0})$ is invertible, the left side is not $\mathbf{0}$, and therefore there is a k such that $m \leq k \leq n$ and $(D_m\alpha)(\mathbf{0}) \neq 0$.

Primitive Mappings

Proof, Step1

Put $\mathbf{F} = \mathbf{F}_1$. Assume $1 \leq m \leq n-1$, and make the following induction hypothesis;

> V_m is a neighborhood of $\mathbf{0}$, $\mathbf{F}_m \in \mathscr{C}'(V_m)$, $\mathbf{F}_m(\mathbf{0}) = \mathbf{0}$, $\mathbf{F}'_m(\mathbf{0})$ is invertible, and
>
> $$P_{m-1}\mathbf{F}_m(\mathbf{x}) = P_{m-1}\mathbf{x} \quad \mathbf{x} \in V_m.$$

Let B_m be the flip that interchanges m and this k (if $k = m$, B_m is the identiy) and define

$$\mathbf{G}_m(\mathbf{x}) = \mathbf{x} + [\alpha_k(\mathbf{x}) - x_k]\mathbf{e}_m \quad \mathbf{x} \in V_m.$$

Then $\mathbf{G}_m \in \mathscr{C}'(V_m)$, $\mathbf{G}_m$ is primitive, and $\mathbf{G}'_m(\mathbf{0})$ is invertible, since $(D_m\alpha_k)(\mathbf{0}) \neq 0$.

Primitive Mappings

Proof, Step1

Put $\mathbf{F} = \mathbf{F}_1$. Assume $1 \le m \le n-1$, and make the following induction hypothesis;

> V_m is a neighborhood of $\mathbf{0}$, $\mathbf{F}_m \in \mathscr{C}'(V_m)$, $\mathbf{F}_m(\mathbf{0}) = \mathbf{0}$, $\mathbf{F}'_m(\mathbf{0})$ is invertible, and
>
> $$P_{m-1}\mathbf{F}_m(\mathbf{x}) = P_{m-1}\mathbf{x} \quad \mathbf{x} \in V_m.$$

The inverse function theorem shows therefore that there is an open set U_m, with $\mathbf{0} \in U_m \subset V_m$, such that $\mathbf{G}_m$ is a 1-1 mapping of U_m onto a neighborhood V_{m+1} of $\mathbf{0}$, in which $\mathbf{G}_m^{-1}$ is continuously differentiable.
Define $\mathbf{F}_{m+1}$ by

$$\mathbf{F}_{m+1}(\mathbf{y}) = B_m \mathbf{F}_m \circ \mathbf{G}_m^{-1}(\mathbf{y}) \quad \mathbf{y} \in V_{m+1}.$$

Then $\mathbf{F}_{m+1} \in \mathscr{C}'(V_{m+1})$, $\mathbf{F}_{m+1}(\mathbf{0}) = \mathbf{0}$, and $\mathbf{F}'_{m+1}(\mathbf{0})$ is invertible. Also, for $\mathbf{x} \in U_m$,

$$P_m \mathbf{F}_{m+1}(\mathbf{G}_m(\mathbf{x})) = P_m B_m \mathbf{F}_m(\mathbf{x}) = \cdots = P_m \mathbf{G}_m(\mathbf{x}),$$

so that

$$P_m \mathbf{F}_{m+1}(\mathbf{y}) = P_m \mathbf{y} \quad \mathbf{y} \in V_{m+1}.$$

Primitive Mappings

Proof, Step2

Since $B_m B_m = I$, with $\mathbf{y} = \mathbf{G}_m(\mathbf{x})$, $\mathbf{F}_{m+1}(\mathbf{y}) = B_m \mathbf{F}_m \circ \mathbf{G}_m^{-1}(\mathbf{y})$ is equivalent to

$$\mathbf{F}_m(\mathbf{x}) = B_m \mathbf{F}_{m+1}(\mathbf{G}_m(\mathbf{x})) \quad \mathbf{x} \in U_m.$$

If we apply this with $m = 1, \cdots, n-1$, we successively obtain

$$\begin{aligned} \mathbf{F} = \mathbf{F}_1 &= B_1 \mathbf{F}_2 \circ \mathbf{G}_1 \\ &= B_1 B_2 \mathbf{F}_3 \circ \mathbf{G}_2 \circ \mathbf{G}_1 = \cdots \\ &= B_1 \cdots B_{n-1} \mathbf{F}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1 \end{aligned}$$

in some neighborhood of $\mathbf{0}$. By Step1, $\mathbf{F}_n$ is primitive.

Partitions of Unity

Theorem (10.8)

Suppose K is a compact subset of $\mathbb{R}^n$, and $\{V_\alpha\}$ is an open cover of K. Then there exist functions $\psi_1, \cdots, \psi_s \in \mathscr{C}(\mathbb{R}^n)$ such that

(a) $0 \leq \psi_i \leq 1$ *for* $1 \leq i \leq s$;

(b) *each* ψ_i *has its support in some* V_α, *and*

(c) $\psi_1(\mathbf{x}) + \cdots \psi_s(\mathbf{x}) = 1$ *for every* $\mathbf{x} \in K$.

Because of (c), $\{\psi_i\}$ is called a partition of unity, and (b) is sometimes expressed by saying that $\{\psi_i\}$ is subordinate to the cover $\{V_\alpha\}$.

Corollary

If $f \in \mathscr{C}(\mathbb{R}^n)$ and the support of f lies in K, then

$$f = \sum_{i=1}^{s} \psi_i f.$$

Each $\psi_i f$ has its support in some V_α.

Change of Variables

Theorem (10.9)

Suppose T is a 1-1 $\mathscr{C}'$-mapping of an open set $E \subset \mathbb{R}^k$ into $\mathbb{R}^k$ such that $J_T(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in E$. If f is a continuous function on $\mathbb{R}^k$ whose support is compact and lines in $T(E)$, then

$$\int_{\mathbb{R}^k} f(\mathbf{y})\, d\mathbf{y} = \int_{\mathbb{R}^k} f(T(\mathbf{x}))|J_T(\mathbf{x})|\, d\mathbf{x}. \quad \cdots (*)$$

Proof

- If T is a primitive $\mathscr{C}'$-mapping, or if T is a linear mapping which merely interchanges two coordinates, we are done.
- If the theorem is true for transformation P, Q, and if $S(\mathbf{x}) = P(Q(\mathbf{x}))$, then

$$\int f(\mathbf{z})\, d\mathbf{z} = \cdots = \int f(S(\mathbf{x}))|J_S(\mathbf{x})|\, d\mathbf{x}.$$

Thus the theorem is also true for S.

(continued)

Change of Variables

Theorem (10.9)

Suppose T is a 1-1 $\mathscr{C}'$-mapping of an open set $E \subset \mathbb{R}^k$ into $\mathbb{R}^k$ such that $J_T(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in E$. If f is a continuous function on $\mathbb{R}^k$ whose support is compact and lines in $T(E)$, then

$$\int_{\mathbb{R}^k} f(\mathbf{y})\, d\mathbf{y} = \int_{\mathbb{R}^k} f(T(\mathbf{x}))|J_T(\mathbf{x})|\, d\mathbf{x}. \quad \cdots (*)$$

Proof

- Each point $\mathbf{a} \in E$ has a neighborhood $U \subset E$ in which

$$T(\mathbf{x}) = T(\mathbf{a}) + B_1 \cdots B_{k-1}\mathbf{G}_k \circ \mathbf{G}_{k-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x} - \mathbf{a}),$$

  where $\mathbf{G}_i$ and B_i are as in Theorem 10.7. Setting $V = T(U)$, it follows that $(*)$ holds if the support of f lies in V. Thus : Each point $\mathbf{y} \in T(E)$ lies in an open set $V_{\mathbf{y}} \subset T(E)$ such that $(*)$ holds for all continuous functions whose support lies in $V_{\mathbf{y}}$.
- Now let f be a continuous function with compact support $K \subset T(E)$. Since $\{V_{\mathbf{y}}\}$ covers K, $f = \sum \psi_i f$ where each ψ_i is continuous, and each ψ_i has its support in some $V_{\mathbf{y}}$. Thus $(*)$ holds for each $\psi_i f$, and hence also for their sum f.

Exercises

Ex 10.2

For $i = 1, 2, 3, \cdots$, let $\varphi_i \in \mathscr{C}(\mathbb{R})$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y).$$

Then

- f has compact support in $\mathbb{R}^2$
- f is continuous except at $(0, 0)$
- and
$$\int dy \int f(x, y)\, dx = 0 \quad \text{but} \quad \int dx \int f(x, y)\, dy = 1.$$
- Obseve that f is unbounded in every neighborhood of $(0, 0)$.

Exercises

Ex 10.3

(a) If $\mathbf{F}$ is as in Theorem 10.7, put $\mathbf{A} = \mathbf{F}'(\mathbf{0})$, $\mathbf{F}_1(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{F}(\mathbf{x})$. Then $\mathbf{F}_1'(\mathbf{0}) = \mathbf{I}$. Show that

$$\mathbf{F}_1(\mathbf{x}) = \mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x})$$

in some neighborhood of $\mathbf{0}$. This gives another version of Theorem 10.7:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}'(\mathbf{0})\mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x}).$$

(b) Prove that the mapping $(x, y) \to (y, x)$ of $\mathbb{R}^2$ onto $\mathbb{R}^2$ is not the composition of any two primitive mappings, in any neighborhood of the origin.

Exercises

Ex 10.4

For $(x, y) \in \mathbb{R}^2$, define

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y).$$

Prove that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$, where

$$\mathbf{G}_1(x, y) = (e^x \cos y - 1, y)$$
$$\mathbf{G}_2(u, v) = (u, (1 + u) \tan v)$$

are primitive in some neighborhood of $(0, 0)$.

Compute the Jacobians of $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{F}$ at $(0, 0)$. Define

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ is some neighborhood of $(0, 0)$.

Exercises

Ex 10.5

Formulate and prove an analogue of Theorem 10.8, in which K is a compact subset of an arbitrary metric space.

Exercises

Ex 10.6

Strengthen the conclusion of Theorem 10.8 by showing that the functions ψ_i can be made differentiable, and even infinitely differentiable.

Exercises

Ex 10.8

Let H be the parallelogram in $\mathbb{R}^2$ whose vertices are $(1,1)$, $(3,2)$, $(4,5)$, $(2,4)$. Find the affine map T which sends $(0,0)$ to $(1,1)$, $(1,0)$ to $(3,2)$, $(0,1)$ to $(2,4)$. Show that $J_T = 5$. Use T to convert the integral

$$\alpha = \int_H e^{x-y} dx dy$$

to an integral over I^2 and thus compute α.

Exercises

Ex 10.12

- Let I^k be the set of all $\mathbf{u} = (u_1, \cdots, u_k) \in \mathbb{R}^k$ with $0 \le u_k \le 1$ for all i; let Q^k be the set of all $\mathbf{x} = (x_1, \cdots, x_k) \in \mathbb{R}^k$ with $x_i \ge 0$, $\sum x_i \le 1$. Define $\mathbf{x} = T\mathbf{u}$ by

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= (1 - u_1)u_2 \\ \cdots \\ x_k &= (1 - u_1) \cdots (1 - u_{k-1})u_k. \end{aligned}$$

Show that

$$\sum_{i=1}^{k} x_i = 1 - \prod_{i=1}^{k} (1 - u_i).$$

Exercises

Ex 10.12

- Show that T maps I^k onto Q^k, that T is 1-1 in the interior of I^k, and that its inverse S is defined in the interior of Q^k by $u_1 = x_1$ and
$$u_i = \frac{x_i}{1 - x_1 - \cdots - x_{i-1}}$$
for $i = 2, \cdots, k$. Show that
$$J_T(\mathbf{u}) = (1 - u_1)^{k-1}(1 - u_2)^{k-2} \cdots (1 - u_{k-1}),$$
and
$$J_S(\mathbf{x}) = [(1 - x_1)(1 - x_1 - x_2) \cdots (1 - x_1 - \cdots - x_{k-1})]^{-1}.$$

Exercises

Ex 10.13

Let $r_1, \cdots, r_k$ be nonnegative integers, and prove that

$$\int_{Q^k} x_1^{r_1} \cdots x_k^{r_k} \, dx = \frac{r_1! \cdots r_k!}{(k + r_1 + \cdots + r_k)!}$$

The End