LA2 Subspaces, Linear Combination, Spanning Sets, Linearly Independent

KYB

Thrn, it's a Fact

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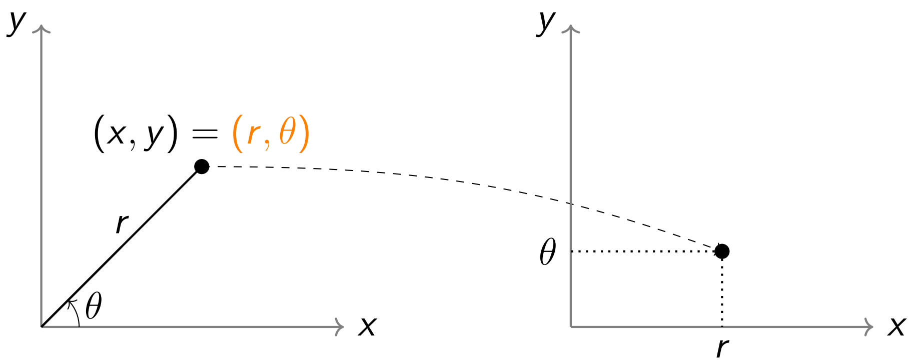
Overview

Polar Coordinate



Definition

Let V be a vector space over a field F, and let S be a subset of V. Then S is a subspace of V if and only if the following are true:

1. $0 \in S$
2. If $\alpha \in F$ and $u \in S$, then $\alpha u \in S$
3. If $u, v \in S$, then $u + v \in S$.

Theorem

Suppose S is a subspace of a vector space V over a field F. Then S is a vector space over F, where the operations on S are the same as the operations on V.

Ex 2.3.1

Let V be a vector space over a field F.

(a) Let $S = \{0\}$ be the subset of V containing only the zero vector. Prove that S is a subspace of V.

(b) Prove that V is a subspace of itself.

Ex 2.3.2, Another Definition of a Subspace

A subspace S of a vector space V is a nonempty subset of V that is closed under addition and scalar multiplication. Prove that this is equivalent to our definition of subspace.

Ex 2.3.3

Let V be a vector space over $\mathbb{R}$ and let $v \in V$ be a nonzero vector. Prove that the subset $\{0, v\}$ is not a subspace of V.

Proof.

1) Since $\mathbb{R}$ is infinite, if W is a subspace, then either $|W|$ or W is infinite. Since $|\{0, v\}| = 2$, $\{0, v\}$ is not a subspace. $\square$

Proof.

2) Suppose $\{0, v\}$ is a subspace. Then $v + v \in \{0, v\}$. So $v + v = 0$ or $v + v = v$. Both cases, $v = 0$ (contradiction). $\square$

Ex 2.3.10

Let V be a vector space over a field F, let $u \in V$, and define $S \subset V$ by $S = \{\alpha v : \alpha \in F\}$. Then S is a subspace of V.

Ex 2.3.19

Let V be a vector space over a field F, and let X, Y be two subspaces of V. Prove or give a counterexample:

(a) $X \cap Y$ is a subspace of V.

(b) $X \cup Y$ is a subspace of V.

Proof.

(a) Yes. $\square$

Ex 2.3.19

Let V be a vector space over a field F, and let X, Y be two subspaces of V. Prove or give a counterexample:

(a) $X \cap Y$ is a subspace of V.

(b) $X \cup Y$ is a subspace of V.

Proof.

(b) No. $F = \mathbb{R}$, $X = \{\alpha(1,0) : \alpha \in \mathbb{R}\}$, $Y = \{\alpha(0,1) : \alpha \in \mathbb{R}\}$. Then $(1,1) \notin X \cup Y$. □

Ex 2.3.20

Let V be a vector space over a field and let S be a nonempty subset of V. Define $\mathcal{T}$ be the collection of all subspace of V containing S, that is,

$$\mathcal{T} = \{T_\alpha | S \subset T_\alpha, T_\alpha \text{ is a subspace of } V\}/$$

Let T be the intersection of all subspaces of V that containing S:

$$T = \bigcap_{T_\alpha \in \mathcal{T}} T_\alpha.$$

Prove:

(a) T is a subspace of V;

(b) T is the smallest subspace of V containing S, in the following sense: If U is a subspace of V and $S \subset U$, then $T \subset U$.

Ex 2.3.20

Proof.
(a) For all $T_\alpha \in \mathcal{T}$, $0 \in T_\alpha \implies 0 \in T$.
$a \in F, u, v \in T$. Then $y, v \in T_\alpha$ for all $T_\alpha \in \mathcal{T}$. So $u + v, \alpha u \in T_\alpha$ for all T_α and $u + v, \alpha u \in T$.

(b) Let $U \in \mathcal{T}$. We want to show $T \subset U$. Let $v \in T$. Then for all $T_\alpha \in \mathcal{T}$, $v \in T_\alpha$. Since $U \in \mathcal{T}$, $v \in U$. $\square$

Ex 2.3.21

Let V be vector space over a field F, and let S and T be subspaces of V. Define

$$S + T = \{s + t : s \in S, t \in T\}.$$

Prove that $S + T$ is the smallest subspace of V containing $S \cup T$.

Proof.

Clearly $S \cup T \subset S + T$.

(1) $0 = 0 + 0 \in S + T$

(2) $\alpha(s + t) = \alpha s + \alpha t \in S + T$

(3) $(s_1 + t_1) + (s_2 + t_2) = (s_1 + s_2) + (t_1 + t_2) \in S + T$

(4) Suppose U is a subspace of V such that $S \cup T \subset U$. Then
$s + t \in S + T \implies s, t \in U \implies s + t \in U.$

$\square$

Definition

Let V be a vector space over a field F, let $u_1, \cdots, u_k \in V$ and let $\alpha_1, \cdots, \alpha_k \in F$. Then

$$\alpha_1 u_1 + \cdots + \alpha_k u_k$$

is called a linear combination of $u_1, \cdots, u_j$.

Definition

Let S be a subset of V. The spanning set span$\{S\}$ is the set of all linear combination of S,

$$\text{span}\{S\} = \{\alpha_1 v_1 + \cdots + \alpha_k v_k : \alpha_i \in F, v_i \in S, k \in \mathbb{N}\}.$$

Theorem

$span\{S\}$ is a subspace of V.

Ex 2.4.1

Let $S = \text{span}\{(-1,-2,4,-2),(0,1,-5,4)\}$ in $\mathbb{R}^4$. Determine if each vector v belongs to S:

(a) $v = (-1, 0, -6, 6)$

Proof.

Suppose $v \in S$. Then $v = \alpha(-1,-2,4,2) + \beta(0,1,-5,4)$ for some $\alpha, \beta \in \mathbb{R}$.

$$\begin{cases} -\alpha = -1 \\ -2\alpha + \beta = 0 \\ 4\alpha - 5\beta = -6 \\ 2\alpha + 4\beta = 6 \end{cases} \implies \begin{cases} \alpha = 1 \\ -2 + \beta = 0 \implies \beta = 2 \\ 4 - 10 = -6 \\ 2 + 4 = 6 \end{cases}$$

$\square$

Ex 2.4.1

Let $S = \text{span}\{(-1,-2,4,-2),(0,1,-5,4)\}$ in $\mathbb{R}^4$. Determine if each vector v belongs to S:

(b) $v = (1,1,1,1)$

Proof.

Suppose $v \in S$. Then $v = \alpha(-1,-2,4,2) + \beta(0,1,-5,4)$ for some $\alpha, \beta \in \mathbb{R}$.

$$\begin{cases} -\alpha = 1 \\ -2\alpha + \beta = 1 \\ 4\alpha - 5\beta = 1 \\ 2\alpha + 4\beta = 1 \end{cases} \implies \begin{cases} \alpha = -1 \\ \beta = -1 \\ -4 + 5 = 1 \\ -2 - 4 = 1 (\text{contradiction}) \end{cases}$$

$\square$

Ex 2.4.10

Show that

$$S_1 = \text{span}\{(1,1,1),(1,-1,1)\} \text{ and } S_2 = \text{span}\{(1,1,1),(1,-1,1),(1,0,1)\}$$

are the same subspace of $\mathbb{R}^3$.

Proof.

Need to show $S_1 \subset S_2$ and $S_1 \supset S_2$. $\square$

Ex 2.4.15

Let V be a vector space over a field F, and let u be a nonzero vector in V. Prove that, for any scalr $\alpha \in F$, $\text{span}\{u\} = \text{span}\{u, \alpha u\}$.

Ex 2.4.16

Let V be a vector space over a field F, and suppose

$$x, u_1, \cdots, u_k, v_1, \cdots, v_l$$

are vectors in V. Assume $x \in \text{span}\{u_1, \cdots, u_k\}$ and $u_j \in \text{span}\{v_1, \cdots, v_l\}$ for $j = 1, 2, \cdots, k$. Prove that $x \in \text{span}\{v_1, \cdots, v_l\}$.

Ex 2.4.17

1. Let V be a vector space over $\mathbb{R}$, and let u, v be any two vectors in V. Prove that $\text{span}\{u, v\} = \text{span}\{u + v, u - v\}$.
2. $F = \mathbb{Z}_2$ and $V = \mathbb{Z}_2^2$, $u = (1, 0)$, $v = (0, 1)$. Then

$$\text{span}\{u, v\} = V, \text{span}\{u + v, u - v\} = \{(0, 0), (1, 1)\}.$$

Definition

Suppose V is a vector space over a field F and X is a subset of V. We say X is linearly independent if for any distinct $x_1, \cdots, x_k \in X$ and $\alpha_1, \cdots, \alpha_k \in F$, $\alpha_1 x_1 + \cdots + \alpha_k x_k = 0$ implies $\alpha_1 = \cdots = \alpha_k = 0$. Otherwise, X is linearly dependent.

Goal

Find $S \subset V$ such that $\mathrm{span}\{S\} = V$ and S is linearly independent.
Suppose $V = \mathrm{span}\{u_1, \cdots, u_n\}$.

(1) if $\{u_1, \cdots, u_n\}$ is linearly independent, we are done.

(2) if $\{u_1, \cdots, u_n\}$ is linearly dependent, there is $u_k \in \mathrm{span}\{u_1, \cdots, \hat{u}_k, \cdots, u_n\}$. (WLOG, assume $k = n$)

(3) repeat for $\{u_1, \cdots, u_n\}$.

Note that if $u \neq 0$, $\{u\}$ is always linearly independent. So these process (1)$\sim$(3) must stop only after finitely many times.

Definition

Let V be a vector space over a field F, and $\mathcal{B}$ be a subset of V. We say $\mathcal{B}$ is a basis if

- $\mathcal{B}$ is linearly independent
- $\operatorname{span}\{\}B = V$.

Remark

- Every vector space V has a basis.
- A basis may not be unique.
- Every basis has the same cardinality.
- We we can define dimension of V over F by $|\mathcal{B}|$ for some basis $\mathcal{B}$.

Example

Suppose V is a nontrivial vector space over a field F.

- $\{u_1, u_2\}$ is linearly dependent if and only if $u_1 = \alpha u_2$ for some $\alpha \in F$.
- $\{v\}$ is linearly dependent if and only if $v = 0$.
- If $0 \in \{u_1, \cdots, u_n\}$, then $\{u_1, \cdots, u_n\}$ is linearly dependent.
- If $\{u_1, \cdots, u_k\}$ is linearly independent and $v \notin \text{span}\{u_1, \cdots, u_k\}$, then $\{u_1, \cdots, u_k, v\}$ is linearly independent.

Ex 2.5.14

Let V be a vector space over a field F and let $\{u_1, \cdots, u_k\}$ be a linearly independent subset of V. Prove or give a counterexample: if $\{v, w\}$ is linearly independent and $v, w \notin \operatorname{span}\{u_1, \cdots, u_k\}$, then $\{u_1, \cdots, u_k, v, w\}$ is linearly independent.

Proof.

In $\mathbb{R}^3$, let $v = (1, 0, 0)$, $w = (1, 1, 0)$, $u = (0, 1, 0)$. Then $\{v, w\}$ and $\{u\}$ are linearly independent and $v, w \notin \operatorname{span}\{u\}$ but $v - w + u = 0$. □

Ex 2.5.15

Let V be a vector space over a field F, and suppose S and T are subspaces of V satisfying $S \cap T = \{0\}$. Suppose $\{s_1, \cdots, s_k\} \subset S$ and $\{t_1, \cdots, t_l\} \subset T$ are both linearly independent sets. Prove that

$$\{s_1, s_2, \cdots, s_k, t_1, t_2, \cdots, t_l\}$$

is a linearly independent subset of V.

Ex 2.5.16

Let V be a vector space over a field F, and let $\{u_1, \cdots, u_k\}$ and $\{v_1, \cdots, v_l\}$ be two linearly independent subsets of V. Find a condition that implies that

$$\{u_1, \cdots, u_k, v_1, \cdots, v_k\}$$

is linearly independent.

Ex 2.5.18

Let U and V be vector spaces over a field F, and define $W = U \times V$. Suppose $\{u_1, \cdots, u_k\} \subset U$ and $\{v_1, \cdots, v_l\} \subset V$ are linearly independent. Prove that

$$\{(u_1, 0), \cdots, (u_k, 0), (0, v_1), \cdots, (0, v_l)\}$$

is a linearly independent subset of W.

Ex 2.5.19

Let V be a vector space over a field F, and let $u_1, \cdots, u_n$ be vectors in V. Suppose a nonempty subset S of $\{u_1, \cdots, u_n\}$ is linearly independent. Prove that $\{u_1, \cdots, u_n\}$ itself is linearly dependent.

Ex 2.5.20

Let V be a vector space over a field F, and suppose $\{u_1, \cdots, u_n\}$ is a linearly independent subset of V. Prove that every nonempty subset of $\{u_1, \cdots, u_n\}$ is also linearly indepedent. Prove that $\{u_1, \cdots, u_n\}$ itself is linearly dependent.

Ex 2.5.21

Let V be a vector space over a field F, and suppose $\{u_1, \cdots, u_n\}$ is linearly dependent. Prove that, given any i, $1 \leq i \leq n$, either u_i is linear combination of $\{u_1, \cdots, \hat{u}_i, \cdots, u_n\}$ or $\{u_1, \cdots, \hat{u}_i, \cdots, u_n\}$ is linearly depedent.

Summary

If

$$\underbrace{\{u_1\} \subset \{u_1, u_2\} \subset \cdots \{u_1, \cdots, u_n\}}_{\text{linearly independent}} \subset \underbrace{\{u_1, \cdots, u_n, u_{n+1}\} \subset \cdots}_{\text{linearly dependent for all } u_{n+1}},$$

then $\{u_1, \cdots, u_n\}$ is a basis for V.

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