Modules

KYB

Thrn, it's a Fact

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Overview

Observe

Suppose R is a subring of S with $1_R = 1_S$. Then S is an R-module. If N is an S-module, then N can also be naturally considered as an R-module:

- $(r_1 + r_2)n = r_1 n + r_2 n$ and $r(n_1 + n_2) = rn_1 + rn_2$;
- $(r_1 r_2)n = r_1(r_2 n)$.

In general, if $f : R \to S$ is a ring homomorphism with $f(1_R) = 1_S$, an S-module N can be considered as an R-module with $r \cdot n = f(r)n$.
In this case,

- S can be considered as an extension of R;
- the resulting R-module is said to be obtained from N by restriction of scalars from S to R.

What about the converse?

Example

Consider $\mathbb{Z}$ as a $\mathbb{Q}$-module. Then there is $z \in \mathbb{Z}$ such that $\frac{1}{2} \cdot 1 = z$. But there is no z in $\mathbb{Z}$ such that $z + z = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1 \cdot 1 = 1$. So a $\mathbb{Z}$-module $\mathbb{Z}$ cannot be made into a $\mathbb{Q}$-module.

Nevertheless, $\mathbb{Z}$ is contained in a $\mathbb{Q}$ module, namely $\mathbb{Q}$ itself. In this sense, we can extend scalar :

- for given R-module N, if there is an S-module M such that $N \subset M$, we can identify M as an extension of N.
- More generally, if there is an S-module with an R-module homomorphism $\varphi : N \to M$, by taking $r \cdot n = rf(n)$, we can identify M as an extension of N.

So our first goal is the existence of such S-module M.

Recall, the Universal Property of Free Modules

For any set A there is a free R-module $F(A)$ on the set A and $F(A)$ satisfies the following *universal property*:

- if M is any R-module and $\varphi : A \to M$ is any map of sets, then there is a unique R-module homomorphism $\Phi : F(A) \to M$ such that $\Phi(a) = \varphi(a)$ for all $a \in A$, that is, the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\text{inclusion}"] \arrow[dr, "\varphi"'] & F(A) \arrow[d, dashed, "\Phi"] \arrow[dl, phantom, "\circlearrowright"] \\ & M \end{tikzcd}$$

Theorem (The Universal Property of Quotient modules)

Let R be a ring with 1 and let M be an R-module and let H be a submodule of M. Then $\pi : M \to M/H$ by $\pi(m) = m + H$ is a surjective R-module homomorphism.

- *Suppose L is an R-module and $\varphi : M \to L$ is an R-module homomorphism with $H \subset \mathrm{Ker}\varphi$. Then there is a unique R-module homomorphism $\Phi : M/H \to L$ such that φ factors through Φ, i.e. $\varphi = \Phi \circ \pi$ and the diagram commutes.*

$$\begin{tikzcd} M \arrow[rr, "\pi"] \arrow[drr, "\varphi"'] & & M/H \arrow[d, dashed, "\Phi"] \arrow[dl, phantom, "\circlearrowright"] \\ & & L \end{tikzcd}$$

Proof

Define $\Phi(m + H) = \varphi(m)$. It suffices to show that Φ is well defined.

Tensor Products

Construction

Suppose M is a 'right' R-module and N is a 'left' R-module. Consider $F(M \times N)$ as the free $\mathbb{Z}$-module. Let H be the subgroup of $F(M \times N)$ generated by all elements of the form

$$(m_1 + m_2, n) - (m_1, n) - (m_2, n),$$
$$(m, n_1 + n_2) - (m, n_1) - (m, n_2),$$
$$(mr, n) - (m, rn).$$

Then $F(M \times N)/H$ is an abelian group, denoted by $M \otimes_R N$.

- $M \otimes_R N$ is called the tensor product of M and N over R.
- The elements of $M \otimes_R N$ is called tensors.
- The coset $m \otimes n$ of (m, n) in $M \otimes_R N$ is called a simple tensor.

Remark

- We have the relations

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n,$$
$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2,$$
$$(mr) \otimes n = m \otimes (rn).$$

- Every tensor is a finite sum of simple tensors, but a tensor may not be simple.
- We can define a map $\iota : M \times N \to M \otimes_R N$ by $\iota(m, n) = m \otimes n$. ι satisfies

$$\iota((m_1 + m_2), n) = \iota(m_1, n) + \iota(m_2, n),$$
$$\iota(m, (n_1 + n_2)) = \iota(m, n_1) + \iota(m, n_2),$$
$$\iota((mr), n) = \iota(m, (rn)).$$

Definition

Let M be a right R-module and N be a left R-module and let L be an abelian group. A map $\varphi : M \times N \to L$ is called R-balanced if

$$\begin{gathered}\varphi((m_1 + m_2), n) = \varphi(m_1, n) + \varphi(m_2, n), \\ \varphi(m, (n_1 + n_2)) = \varphi(m, n_1) + \varphi(m, n_2), \\ \varphi((mr), n) = \varphi(m, (rn)).\end{gathered}$$

Theorem (The Universal Properties of Tensor Product)

Suppose R is a ring with 1, M is a right R-module, and N is a left R-module. Then $\iota : M \times N \to M \otimes_R N$ is an R-balanced map.

(1) *If $\Phi : M \otimes_R N \to L$ is any group homomorphism from $M \otimes_R N$ to an abelian group L, then the composition map $\varphi = \Phi \circ \iota$ is an R-balanced map from $M \times N \to L$.*

(2) *Conversely, suppose L is an abelian group and $\varphi : M \times N \to L$ is any R-balanced map. Then there is a unique group homomorphism $\Phi : M \otimes_R N \to L$ such that φ factors through ι, i.e., $\varphi = \Phi \circ \iota$ and the diagram commutes.*

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & M \otimes_R N \arrow[d, dashed, "\Phi"] \arrow[dl, phantom, "\circlearrowleft"] \\ & L \end{tikzcd}$$

Equivalently, the correspondence $\varphi \leftrightarrow \Phi$ is bijection.

$$\{R\text{-balanced maps}\} \leftrightarrow \{\text{group homomorphisms}\}$$

Proof

(2) Uniqueness.

Proof

(2) Existence.

Remark

This theorem is extremely useful in defining homomorphisms on $M \otimes_R N$. If we define an R-balanced map, then automatically we can define a homomorphism on $M \otimes_R N$.

Example

Let R be a subring of a ring S with $1_R = 1_S$ and let N be a left R-module. As right R-module S, we can find a map $N \to S \otimes_R N$ by $n \mapsto 1 \otimes n$. Moreover, if we define $s(\sum s_i \otimes n_i) = \sum(ss_i) \otimes n_i$, $S \otimes_R N$ is S-module (so R-module). If we write $\iota : N \to S \otimes_R N$ by $\iota(n) = 1 \otimes n$, ι is an R-module homomorphism. In general, ι is not injective. In this way, we can consider $S \otimes_R N$ as an extension of scalar of N.

Corollary

Suppose R is a subring of a ring S with $1_R = 1_S$, let N be a left R-module and let $\iota : N \to S \otimes_R N$ be the R-module homomorphism defined by $\iota(n) = 1 \otimes n$.

- *Suppose L is a left S-module and $\varphi : N \to L$ is an R-module homomorphism. Then there is a unique S-module homomorphism $\Phi : S \otimes_R N \to L$ such that φ factors through Φ, i.e. $\varphi = \Phi \circ \iota$ and the diagram commutes.*

$$
\begin{tikzcd}
N \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & S \otimes_R N \arrow[d, dashed, "\Phi"] \arrow[dl, phantom, "\circlearrowright"] \\
 & L
\end{tikzcd}
$$

Remark

Let $N = \mathbb{Z}/2\mathbb{Z}$. Consider $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. For $q \otimes n \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$,

$$q \otimes n = \frac{q}{2} \otimes 2n = \frac{q}{2} \otimes 0.$$

Since for any $q \in \mathbb{Q}$,

$$q \otimes 0 = q \otimes (0 + 0) = q \otimes 0 + q \otimes 0.$$

Thus $q \otimes 0 = 0$. Since $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is generated by $q \otimes n = 0$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$.

Example

(1) For any ring R and any left R-module N, $R \otimes_R N \cong N$.

(2) If $N \cong R^n$, $S \otimes_R N \cong S^n$.

The End