Analysis
- PMA 10 -

KYB

Thrn, it's a Fact

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February 3, 2021

Overview

Differentiation

Exercises

Exercises

Ex 5.18

Suppose f is a real function on $[a,b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a,b]$. Let α, β, and P be as in Taylor's theorem. Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a,b]$, $t \neq \beta$, differentiate $f(t) - f(\beta) = (t-\beta)Q(t)$, $n-1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

Exercises

Ex 5.19

Suppose f is defined in $(-1,1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}.$$

Prove the following statements:

(a) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$.

(b) If $0 < \alpha_n < \beta_n$ and $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, then $\lim D_n = f'(0)$.

(c) If f' is continuous in $(-1,1)$, then $\lim D_n = f'(0)$.

Give an example in which f is differentiable in $(-1,1)$ (but f' is not continuous at 0) and in which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.

Exercises

Ex 5.21

Let E be a closed subset of $\mathbb{R}$. We saw that there is a real continuous function on $\mathbb{R}$ whose zero set is E. Is it possible, for each closed set E, to find such an f which is differentiable on $\mathbb{R}$, or one which is n times differentiable, or even one which has derivatives of all orders on $\mathbb{R}$?

Exercises

Ex 5.22

Suppose f is a real function on $(-\infty, \infty)$. Call x a fixed point of f if $f(x) = x$.

(a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

(b) Show that the function f defined by $f(t) = t + (1 + e^t)^{-1}$ has no fixed point, although $0 < f'(t) < 1$ for all real t.

(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and $x = \lim x_n$, where x_1 is an arbitrary real number and $x_{n+1} = f(x_n)$ for $n = 1, 2, 3, \cdots$.

(d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \cdots .$$

Exercises

Ex 5.23

The function f defined by

$$f(x) = \frac{x^3+1}{3}$$

has three fixed points, say α, β, γ, where

$$-2 < \alpha < -1, \quad 0 < \beta < 1, \quad 1 < \gamma < 2.$$

For arbitrary chosen x_1, define $\{x_n\}$ by setting $x_{n+1} = f(x_n)$.

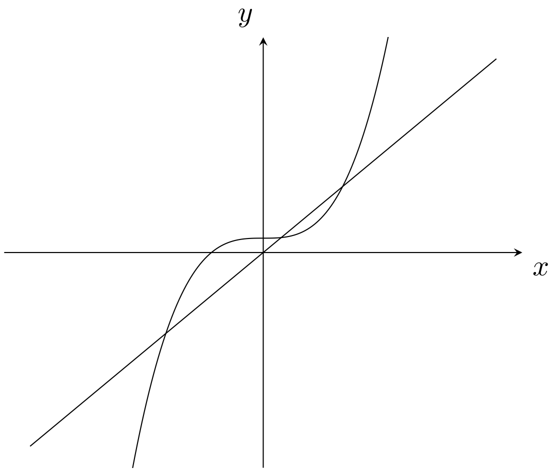
(a) If $x_1 < \alpha$, prove that $x_n \to -\infty$ as $n \to \infty$.

(b) If $\alpha < x_1 < \gamma$, prove that $x_n \to \beta$ as $n \to \infty$.

(c) If $\gamma < x_1$, prove that $x_n \to \infty$ as $n \to \infty$.

Exercises

Ex 5.23



Exercises

Ex 5.25

Suppose f is twice differentiable on $[a,b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a,b]$.
Let ξ be the unique point in (a,b) at which $f(\xi) = 0$.
Complete the details in the following outline of Newton's method for computing ξ.

(a) Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Interpret this geometrically, in terms of a tangent to the graph of f.

(b) Prove that $x_{n+1} < x_n$ and that $\lim x_n = \xi$.

Exercises

Ex 5.25

Suppose f is twice differentiable on $[a,b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a,b]$.
Let ξ be the unique point in (a,b) at which $f(\xi) = 0$.
Complete the details in the following outline of Newton's method for computing ξ.

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

(d) If $A = M/2\delta$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2n}.$$

Exercises

Ex 5.25

Suppose f is twice differentiable on $[a,b]$, $f(a)<0$, $f(b)>0$, $f'(x)\geq\delta>0$, and $0\leq f''(x)\leq M$ for all $x\in[a,b]$.
Let ξ be the unique point in (a,b) at which $f(\xi)=0$.
Complete the details in the following outline of Newton's method for computing ξ.

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x)=x-\frac{f(x)}{f'(x)}.$$

How does $g'(x)$ behave for x near ξ?

(f) Put $f(x)=x^{1/3}$ on $(-\infty,\infty)$ and try Newton's method. What happens?

Exercises

Ex 5.26

Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Exercises

Ex 5.27

Let ϕ be a real function defined on a rectangle R in the plane, given by $a \leq x \leq b$, $\alpha \leq y \leq \beta$. A solution of the initial-value problem

$$y' = \phi(x, y), \quad y(a) = c \quad (\alpha \leq c \leq \beta)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \leq f(x) \leq \beta$, and

$$f'(x) = \phi(x, f(x)) \quad (a \leq x \leq b).$$

Prove that such a problem has at most solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

whenever $(x, y_1), (x, y_2) \in R$.

The End