Modules

KYB

Thrn, it's a Fact

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Overview

Modules
    Exact Sequences

Recall

Suppose $\varphi : B \to C$ is a surjective homomorphism. Then we have a subobject A of B such that $B/A \cong C$.

Now we consider the reverse situation: given A and C, is there B such that A is a sumobject and $B/A \cong C$? If such B exists, we say B is an extension of C by A.

Definition

Suppose a ring has a 1 and $X, Y, Z, \cdots$ are R-modules.

(1) The pair of homomorphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is said to be *exact* (at Y) if $\mathrm{Im}\,\alpha = \mathrm{Ker}\beta$.

(2) A sequence $\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots$ of homomorphisms is said to be an *exact sequence* if it is exact at every X_n between a pair of homomorphisms.

Proposition

Let A, B and C be R-modules over some ring R. Then

(1) *The sequence $0 \to A \xrightarrow{\psi} B$ is exact (at A) if and only if ψ is injective.*

(2) *The sequence $B \xrightarrow{\varphi} C \to 0$ is exact (at C) if and only if φ is surjective.*

Corollary

The sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ is exact if and only if ψ is injective, φ is surjective, and $\mathrm{Im}\,\psi = \mathrm{Ker}\varphi$.

Definition

The exact sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ is called a *short exact sequence*.

Remark

Suppose $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is exact at Y. Consider the sequence

$$0 \to \operatorname{Im}\alpha \xrightarrow{\iota} Y \xrightarrow{\pi} Y/\mathrm{Ker}\beta \to 0$$

where $\iota : \operatorname{Im}\alpha \to Y$ is an inclusion and $\pi : Y \to Y/\mathrm{Ker}\beta$ is a natural projection. Then this sequence is a short exact sequence. So any exact sequence can be written as a succession of short exact sequences.

Example

(1) $0 \to A \xrightarrow{\iota} A \oplus C \xrightarrow{\pi} C \to 0$

(2) $0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\varphi} \mathbb{Z}/n\mathbb{Z} \to 0$, $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$

(3) If $\varphi : B \to C$ is any homomorphism we may form an exact sequence:

$$0 \to \mathrm{Ker}\varphi \xrightarrow{\iota} B \xrightarrow{\varphi} \mathrm{Im}\,\varphi \to 0.$$

Example

(4) Suppose M is an R-module and S is a set of generators for M. Let $F(S)$ be the free R-module on S. Then the inclusion $S \to M$ induces a homomorphism $F(S) \to M$. Let K be the kernel of this homomorphism. Then

$$0 \to K \xrightarrow{\iota} F(S) \xrightarrow{\varphi} M \to 0$$

is a short exact sequence.

Definition

Let $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ be two short exact sequences of modules.

(1) A *homomorphism of short exact sequences* is a triple α, β, γ of module homomorphisms such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

The homomorphism is an *isomorphism of short exact sequences* if α, β, γ are all isomorphisms, in which case the extensions B and B' are said to be *isomorphic extensions*.

(2) The two exact sequences are called *equivalent* if $A = A'$, $C = C'$, and there is an isomorphism between them as in (1) that is the identity maps on A and C. (i.e., α and γ are the identity). In this case, the corresponding extensions B and B' are said to be *equivalent* extensions.

Example

(1) $m = kn$.

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "n"] \arrow[d, "\alpha"] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "\beta"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathbb{Z}/k\mathbb{Z} \arrow[r, "\iota"] & \mathbb{Z}/m\mathbb{Z} \arrow[r, "\pi'"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0
\end{tikzcd}
$$

Example

(2) Map each module to itself by $x \mapsto -x$.

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "n"] \arrow[d, "-1"] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "-1"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[d, "-1"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "n"] & \mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0
\end{tikzcd}
$$

This is an isomorphism of short exact sequences but is not equivalence of sequences.

Example

(3) Consider the maps

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\psi"] \arrow[d, "\mathrm{id}"] & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r, "\varphi"] \arrow[d, "\beta"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\psi'"] & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r, "\varphi'"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where

- $\psi(a) = (a, 0)$, $\varphi(a, b) = b$;
- $\psi'(b) = (0, b)$, $\varphi'(a, b) = a$.

If $\beta(a, b) = (b, a)$, this diagram commutes, hence giving an equivalence of the two exact sequences that is not identity isomorphism.

Proposition (The Short Five Lemma)

Let α, β, γ be a homomorphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\varphi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\psi'"] & B' \arrow[r, "\varphi'"] & C' \arrow[r] & 0
\end{tikzcd}$$

(1) *If α and γ are injective, then so is β.*

(2) *If α and γ are surjective, then so is β.*

(3) *Hence, if α and γ are isomorphisms, then so is β.*

Proof

(1)

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d, dotted] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\varphi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\psi'"] & B' \arrow[r, "\varphi'"] & C' \arrow[r] & 0
\end{tikzcd}
$$

Proof

(2)

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\varphi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\psi'"] \arrow[d] & B' \arrow[r, "\varphi'"] \arrow[d, dotted] & C' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}
$$

Definition

Let R be a ring and let $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ be a short exact sequence of R-modules. The sequence is said to be *split* if there is an R-module complement to $\psi(A)$ in B. In this case, up to isomorphism, $B = A \oplus C$ (more precisely, $B = \psi(A) \oplus C'$ for some submodule C, and C' is mapped isomorphically onto C by φ: $\varphi(C') \cong C$).

Proposition

The short exact sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ *of* R*-modules is split if and only if there is an* R*-module homomorphism* $\mu : C \to B$ *such that* $\varphi \circ \mu$ *is identity map on* C*.*

Definition

- With notation as in above Proposition, any set map $\mu : C \to B$ such that $\varphi \circ \mu = \mathrm{id}$ is called a *section* of φ.
- If μ is a *homomorphism* as in Proposition, then μ is called a *splitting homomorphism* for the sequence.

Proposition

Let $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ *be a short exact sequence of modules. Then* $B = \psi(A) \oplus C'$ *for some submodule* C' *of* B *with* $\varphi(C') \cong C$ *if and only if there is a homomorphism* $\lambda : B \to A$ *such that* $\lambda \circ \psi$ *is the identity map on* A*.*

Observe

For given exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$, there is a short exact sequence $0 \to \operatorname{Im} f \to B \to B/\mathrm{Ker} g \to 0$. The last term is $B/\mathrm{Ker} g = B/\operatorname{Im} f$. So

$$0 \to \operatorname{Im} f \to B \to B/\operatorname{Im} f \to 0$$

Definition

Let $f : A \to B$ be a R-module homomorphism. Then the *cokernel* of f is the quotient module $B/\operatorname{Im} f$, denoted by $\mathrm{Coker} f$.

Consider a homomorphism of short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A' & \xrightarrow{\psi'} & B' & \xrightarrow{\varphi'} & C' & \longrightarrow & 0
\end{array}
$$

From $B \to B'$, $0 \to \mathrm{Ker}\beta \to B \to B' \to \mathrm{Coker}\beta \to 0$ is a exact sequence.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathrm{Ker}\alpha \arrow[d] & \mathrm{Ker}\beta \arrow[d] & \mathrm{Ker}\gamma \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\varphi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\psi'"] \arrow[d] & B' \arrow[r, "\varphi'"] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
 & \mathrm{Coker}\alpha \arrow[d] & \mathrm{Coker}\beta \arrow[d] & \mathrm{Coker}\gamma \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Lemma (The Snake Lemma)

Suppose

$$\begin{tikzcd} & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\varphi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\psi'"] & B' \arrow[r, "\varphi'"] & C' & \end{tikzcd}$$

is a commutative diagram of R-modules with exact rows. Then there is a homomorphism $\delta : \mathrm{Ker}\gamma \to \mathrm{Coker}\alpha$, called a connecting map such that

$$\mathrm{Ker}\alpha \to \mathrm{Ker}\beta \to \mathrm{Ker}\gamma \xrightarrow{\delta} \mathrm{Coker}\alpha \to \mathrm{Coker}\beta \to \mathrm{Coker}\gamma$$

is an exact sequence. If ψ is injective and φ' is surjective, then

$$0 \to \mathrm{Ker}\alpha \to \mathrm{Ker}\beta \to \mathrm{Ker}\gamma \xrightarrow{\delta} \mathrm{Coker}\alpha \to \mathrm{Coker}\beta \to \mathrm{Coker}\gamma \to 0$$

is exact.

$$
\begin{tikzcd}
 & \mathrm{Ker}\alpha \arrow[r, "\overline{\psi}"] \arrow[d] & \mathrm{Ker}\beta \arrow[r, "\overline{\varphi}"] \arrow[d] & \mathrm{Ker}\gamma \arrow[d] \arrow[dddll, rounded corners, "\delta"] & \\
 & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] & \\
 & \mathrm{Coker}\alpha \arrow[r, "\overline{\psi'}"] & \mathrm{Coker}\beta \arrow[r, "\overline{\varphi'}"] & \mathrm{Coker}\gamma &
\end{tikzcd}
$$

Proof

Existence of δ.

$$\begin{tikzcd}
 & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\varphi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\psi'"] \arrow[d] & B' \arrow[r, "\varphi'"] & C' & \\
 & \mathrm{Coker}\alpha & & &
\end{tikzcd}$$

Proof

Exactness

- $\mathrm{Ker}\beta \to \mathrm{Ker}\gamma \to \mathrm{Coker}\alpha.$
- $\mathrm{Ker}\gamma \to \mathrm{Coker}\alpha \to \mathrm{Coker}\beta$

Application

Recall the short five lemma. If α and γ are injective, we have

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{Ker}\beta \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\varphi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\psi'"] & B' \arrow[r, "\varphi'"] & C' \arrow[r] & 0
\end{tikzcd}$$

This implies $\mathrm{Ker}\beta = 0$.

Application

If α and γ are surjective, we have

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\varphi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\psi'"] \arrow[d] & B' \arrow[r, "\varphi'"] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathrm{Coker}\beta \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

This implies $B'/\operatorname{Im}\beta = 0$, or $\operatorname{Im}\beta = B'$.

Lemma (The Five Lemma)

Consider a commutative diagram of R-modules and homomorphisms such that each row is exact:

$$\begin{tikzcd}
A_1 \arrow[r, "\psi_1"] \arrow[d, "\alpha_1"] & A_2 \arrow[r, "\psi_2"] \arrow[d, "\alpha_2"] & A_3 \arrow[r, "\psi_3"] \arrow[d, "\alpha_3"] & A_4 \arrow[r, "\psi_4"] \arrow[d, "\alpha_4"] & A_5 \arrow[d, "\alpha_5"] \\
B_1 \arrow[r, "\varphi_1"] & B_2 \arrow[r, "\varphi_2"] & B_3 \arrow[r, "\varphi_3"] & B_4 \arrow[r, "\varphi_4"] & B_5
\end{tikzcd}$$

(1) *If α_1 is surjective and α_2 and α_4 are injective, then α_3 is injective.*

(2) *If α_5 is injective and α_2 and α_4 are surjective, then α_3 is surjective.*

(1) If α_1 is surjective,

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Coker}\psi_1 \arrow[r, "\widetilde{\psi}_2"] \arrow[d, "\widetilde{\alpha}_2"] & A_3 \arrow[r, "\psi_3"] \arrow[d, "\alpha_3"] & \mathrm{Im}\,\psi_3 \arrow[r] \arrow[d, "\widetilde{\alpha}_4"] & 0 \\
0 \arrow[r] & \mathrm{Coker}\varphi_1 \arrow[r, "\widetilde{\varphi}_2"] & B_3 \arrow[r, "\varphi_3"] & \mathrm{Im}\,\varphi_3 \arrow[r] & 0
\end{tikzcd}$$

If α_2 and α_4 are injective, so are $\tilde{\alpha}_2$ and $\tilde{\alpha}_4$.

(2) If α_5 is injective,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\psi_3 & \xrightarrow{\iota_3} & A_3 & \xrightarrow{\psi_3} & \mathrm{Im}\,\psi_3 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \widetilde{\alpha}_2} & & \Big\downarrow{\scriptstyle \alpha_3} & & \Big\downarrow{\scriptstyle \widetilde{\alpha}_4} & & \\
0 & \longrightarrow & \mathrm{Ker}\varphi_3 & \xrightarrow{\jmath_3} & B_3 & \xrightarrow{\varphi_3} & \mathrm{Im}\,\varphi_3 & \longrightarrow & 0
\end{array}
$$

If α_2 and α_4 are surjective, so are $\tilde{\alpha}_2$ and $\tilde{\alpha}_4$.

Exercise, 3×3 lemma

Consider the following commutative diagram in R-modules having exact columns.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

(1) If the bottom two rows are exact, prove that the top row is exact.
(2) If the top two rows are exact, prove that the bottom row is exact.

Proof, (1)

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Proof, (2)

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

The End