LA9 Eigen Objects

KYB

Thrn, it's a Fact

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Overview

Determinant

- $\det(A_1, \cdots, A_n) = 0$ iff $\{A_1, \cdots, A_n\}$ is linearly dependent
- $\det(e_1, \cdots, e_n) = 1$
- $\det, \cdots, e_i, \cdots, e_j, \cdots) = -(\det(\cdots, e_j, \cdots, e_i, \cdots)$
- $\det(A) = \sum_{\tau \in S_n} \sigma(\tau) A_{\tau(1)1} \cdots A_{\tau(n)n}$.
- Fix j. Then

$$\begin{vmatrix} \cdots & A_{1j} & \cdots \\ \cdots & A_{2j} & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & A_{nj} & \cdots \end{vmatrix} = \sum_{i=1}^{n} (-1)^n A_{ij} \begin{vmatrix} & \vdots & \vdots & \\ \cdots & A_{i-1,j-1} & A_{i-1,j+1} & \cdots \\ \cdots & A_{i+1,j-1} & A_{i+1,j+1} & \cdots \\ & \vdots & \vdots & \end{vmatrix}$$

- If A is triangular, $\det(A) = \prod_{i=1}^{n} A_{ii}$.

Cramer's Rule

- Notation : For $B \in F^{n\times n}$ and $c \in F^n$,

$$B_j(c) = [B_1|\cdots|\underbrace{c}_{j\text{th}}|\cdots|B_n].$$

If $A \in F^{n\times n}$ is nonsingular, then $Ax = b$ where

$$x_i = \frac{\det(A_i(b))}{\det(A)}$$

Ex 4.3.4

Let $A \in F^{n\times n}$ be nonsingular and $b \in F^n$.

(a) Count the number of arithmetic operations to reduct A to upper triangular matrtix.

Ex 4.3.6

$\det(I_i(x)) = x_i$.

Ex 4.3.7

Suppose $A \in \mathbb{Z}^{n\times n}$ is invertible (in $\mathbb{R}^{n\times n}$). Assume $\det(A) = \pm 1$. Then $A^{-1} \in \mathbb{Z}^{n}$.

Ex 4.3.11, Vandermande Matrix

Suppose $x_0, \cdots, x_n$ are distinct elements in F. Define

$$V(x_0, \cdots, x_n) = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

(a) $\det(V(x_0, \cdots, x_n)) = \prod_{i=1}^{n} (x_i - x_0) \det(V(x_1, \cdots, x_n))$

(b) $\det(V(x_0, \cdots, x_n)) = \prod_{j=0}^{n-1} \prod_{i=1}^{n} (x_i - x_0) = \prod_{0 \le j < i \le n} (x_i - x_j)$

Eigen Values, Vectors, etc

Let $L : V \to V$ be a linear map. $\lambda \in F$ and $x \in V$ are called an eigenpair if

- $x \neq 0$ and
- $L(x) = \lambda x$.

Characteristic Polynomial

Let $A \in F^{n\times n}$. The characteristic polynomial $p_A(r)$ of A is $p_A(r) = \det(rI - A)$.

Remark

There exists a nonzero vector x such that $Ax = \lambda x$ if and only if there is a nonzero vector such that $(\lambda I - A)x = 0$ if and only if $\lambda I - A$ is singular.

- Let $A \in F^{n\times n}$. Define the trace of A, $\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}$.
- Then $p_A(r) = r^n - \operatorname{tr}(A)r^{n-1} + \cdots + (-1)^n \det(A)$.

Algebraically Closed Field

In algebraically closed field F, every polynomial $p(x)$ has a zero, that is, there is $a \in F$ such that $p(a) = 0$.

Example

- $\mathbb{Q}$, $\mathbb{R}$ are not algebraically closed.
- $\mathbb{C}$ is algebraically closed.

In $\mathbb{R}$, the number of roots of $p_A(r)$, say m, $m \leq n$. If $m < n$, we can find all roots in $\mathbb{C}$.

Eigenspaces

An eigenspace $E_\lambda(A)$ of λ is $\mathcal{N}(\lambda I - A)$.

Main Ideal

- If (λ, x) is an eigenpair, then for all $r \neq 0$, (λ, rx) is also eigenpair.
- If (λ, x_1) and (λ, x_2) are eigenpair, then $(\lambda, x_1 + x_2)$ is also eigenpair.

$$\dim E_\lambda(A) = \operatorname{nullity}(\lambda I - A) = \operatorname{m.geo}(\lambda).$$

Ex 4.5.5

Let $A \in \mathbb{R}^{n\times n}$, $\lambda \in \mathbb{R}$, and $z \in \mathbb{C}^n$. Suppose (λ, z) is an eigenpair of A. Show that either $\operatorname{Re} z$ or $\operatorname{Im} z$ is an eigenvector of A.

Ex 4.5.6

Let

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Suppose $A = A^T$.

(a) A has only real eigenvalues.

(b) Under what condition on a, b, c, does A has a multiple eigenvlaue?

Ex 4.5.8

Suppose $A \in \mathbb{R}^{n\times n}$ and n is odd. Then A has a real eigenvalue.

Ex 4.5.9

Let $q(r) = r^n + c_{n-1}r^{n-1} + \cdots + c_0 \in F[r]$ and

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & & 0 & -c_1 \\ 0 & 1 & & 0 & -c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

Prove that $p_A(r) = Q(r)$.

Ex 4.5.10

Let $A \in \mathbb{C}^{n\times n}$, and let $\lambda_1, \cdots, \lambda_n$ listed according to multiplicity. Prove that

(a) $\operatorname{tr}(A) = \lambda_1 + \cdots + \lambda_n$;

(b) $\det(A) = \lambda_1 \cdots \lambda_n$.

Ex 4.5.11

Let $A \in F^{n\times n}$. Prove that $p_A(r) = p_{A^T}(r)$.

Ex 4.5.12

Suppose $A \in F^{n\times n}$ is invertible. Show that an eigenvector of A is also eigenvector. What is the relationship between eigenvalues of A and A^{-1}.

Ex 4.5.14

Let $A \in F^{m\times n}$.

1. If $\lambda \neq 0$ is an eigenvalue of A^TA, then λ is an eigenvalue of AA^T.
2.

Definition

$A, B \in F^{n\times n}$ are similar if there exists invertible $X \in F^{n\times n}$ such that $B = XAX^{-1}$. Then the relation $\sim$ such that $A \sim B$ iff A, B are similar is an equivalance relation.

If A and B are similar,

- $p_A(r) = p_B(r)$
- $\det(A) = \det(B)$
- $\operatorname{tr}(A) = \operatorname{tr}(B)$

Let $\lambda_1, \cdots, \lambda_k$ be all distinct eigenvalues of A. Let $\{x_1^{(i)}, \cdots, x_{n_i}^{(i)}$ be a basis of $E_{\lambda_i}(A)$. Then $\{x_1^{(1)}, \cdots, x_{n_1}^{(1)}, \cdots, x_1^{(k)}, \cdots, x_{n_k}^{(k)}\}$ is linearly independent. Hence, if $\sum n_i = n$, it is a basis for F^n.

Definition

Suppose there are $(\lambda_1, x_1) \cdots, (\lambda_n, x_n)$ such that $\{x_1, \cdots, x_n\}$ is a basis for F^n (in general, there is no such pairs). Define $X = [x_1|\cdots|x_n]$ and $D = \mathrm{Diag}(\lambda_1, \cdots, \lambda_n)$. Then

1. X is invertible
2. $A = XDX^{-1}$

Ex 4.6.1

If λ_i λ_j are distinct eigenvalues of A, then $E_{\lambda_i}(A) \cap E_{\lambda_j}(A) = \{0\}$.

Ex 4.6.13

If F is a finite field, then F is not algebraically closed.

Ex 4.6.14

Suppose $A, B \in F^{n\times n}$ are diagonalizable. Suppose A and B have the same eigenvector. Then $AB = BA$.

Ex 4.6.15

Suppose $A = XDX^{-1}$ where D is a diagonal matrix and X is invertible. Then $A^k = XD^kX^{-1}$.

Ex 4.6.16, Cayley-Hamilton Theorem

Suppose A is diagonalizable. Then $p_A(A) = 0$

Ex 4.6.17

Suppose A is diagonalizable and λ is an eigenvalue of A. Prove $\mathcal{N}((A-\lambda I)^2) = \mathcal{N}(A-\lambda I)$ as follows:

1. Show that $\mathcal{N}(A-\lambda I) \subset \mathcal{N}((A-\lambda I)^2)$
2. Let $A = XDX^{-1}$. Assume $X = [X_1|X_2]$ where the columns of X_1 forms a basis for $\mathcal{N}(A-\lambda I)$ and the columns of X_2 are eigenvectors corresponding to eigenvalues of A unequal to λ. Prove that if $\mathcal{N}(((A-\lambda I)^2) \not\subset \mathcal{N}(A-\lambda I)$, then there exists a vector u of the form $u = X_2 v$ such that
$$(A-\lambda I)u \neq 0, (A-\lambda I)^2 u = 0.$$
3. Complete the proof by showing that $u = X_2 v, (A-\lambda I)^2 u = 0$ imply that $u = 0$.

The End