LA2 4

KYB

Thrn, it's a Fact

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Overview

Ex6.6.13,14, the minimum-norm least-squares solution to $Ax = y$

Prove that $\overline{x}$ has the smallest Euclidean norm of any element of $\hat{x} + \mathcal{N}(A)$.

Proof.

Let $x \in \hat{x} + \mathcal{N}(A)$. Then $\bar{x} = \operatorname{proj}_{\operatorname{col}(A^T)} x$. So $(\bar{x} - x) \cdot \bar{x} = 0$. By Pythagorean theorem,

$$\|x\|_2^2 = \|x - \bar{x} + \bar{x}\|_2^2 = \|x - \bar{x}\|_2^2 + \|\bar{x}\|_2^2 > \|\bar{x}\|_2^2.$$

$\square$

7.2 The spectral theorem for normal matrices

Definition

Let $A \in \mathbb{C}^{n\times n}$. We say that A is normal if and only if $A^*A = AA^*$.

If A is Hermitian, A is normal, but the converse need not to be true.

Lemma (346)

Let $A \in \mathbb{C}^{n\times n}$ be normal. Then

$$\|Ax\|_2 = \|A^*x\|_2.$$

Theorem (347)

Let $A \in \mathbb{C}^{n\times n}$ be normal. If (λ, x) is e.pair of A, then $(\bar{\lambda}, x)$ is e.pair of A^*.

Theorem (348)

Let $A \in \mathbb{C}^{n\times n}$ be normal. Then e.vecs of A corr to distinct e.vals are orthogonal.

Lemma (349)

Let $A \in \mathbb{C}^{n\times n}$ be normal. Then

$$\mathrm{col}(A^*) = \mathrm{col}(A) \text{ and } \mathcal{N}(A^*) = \mathcal{N}(A).$$

Recall

$$\mathbb{C}^n = \mathrm{col}(A^*) \oplus \mathcal{N}(A)$$

Theorem (350)

Let $A \in \mathbb{C}^{n\times n}$ *be normal and* $\lambda \in \mathbb{C}$ *be e.val of* A. *Then* $\mathrm{m.\,geo}(\lambda) = \mathrm{m.\,alg}(\lambda)$.

Recall

TFAE

- $\mathrm{m.\,geo}(\lambda) = \mathrm{m.\,alg}(\lambda)$;
- $\mathcal{N}((A-\lambda I)^2) = \mathcal{N}(A-\lambda I)$;
- $\mathrm{col}(A-\lambda I) \cap \mathcal{N}(A-\lambda I) = \{0\}$.

Theorem (351)

Let $A \in \mathbb{C}^{n\times n}$ *be normal. Then there exists a unitary matrix* $X \in \mathbb{C}^{n\times n}$ *and a diagonal matrix* $D \in \mathbb{C}^{n\times n}$ *such that* $A = XDX^*$.

Observation

Let $x, y \in \mathbb{R}^n$. Then the dot product of x and y is

$$x \cdot y = \sum x_i y_i = \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = y^T x.$$

On the other hand

$$xy^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & \ddots & \vdots \\ x_n y_1 & \cdots & x_n y_n \end{bmatrix}$$

Moreover, for $z \in \mathbb{R}^n$

$$(xy^T)z = x(y^T z) = x(z \cdot y) = (z \cdot y)x$$

Using this property, we can define new concept of product, say the <u>outer product</u>.

Definition

Let U and V be inner product spaces over $\mathbb{R}$ or $\mathbb{C}$. If $u \in U$ and $v \in V$, then the outer product of u and v is the opreator $u \otimes v : V \to U$ defined by

$$(u \otimes v)(w) = \langle w, v \rangle u$$

Suppose $A \in \mathbb{C}^{n \times n}$ has the spectral decomposition $A = XDX^*$, where

$$X = [x_1 | \cdots | x_n], D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n),$$

then for any $v \in \mathbb{C}^n$

$$Av = XDX^*v = XD \begin{bmatrix} \langle v, x_1 \rangle \\ \vdots \\ \langle v, x_n \rangle \end{bmatrix} = X \begin{bmatrix} \lambda_1 \langle v, x_1 \rangle \\ \vdots \\ \lambda_n \langle v, x_n \rangle \end{bmatrix} = \sum_{i=1}^{n} \lambda_i \langle v, x_i \rangle x_i$$

Hence $A = \sum_i \lambda_i x_i \otimes x_i$.

Ex7.2.3

Let $A \in F^{n\times n}$ be given, define $T : F^n \to F^n$ by $T(x) = Ax$, and let $\mathcal{X} = \{x_1, \cdots, x_n\}$ be a basis for F^n. Prove that $[T]_{\mathcal{X},\mathcal{X}} = X^{-1}AX$.

Proof.

$$\begin{tikzcd} F^n_{\mathcal{S}} \arrow[r, "A"] & F^n_{\mathcal{S}} \\ F^n_{\mathcal{X}} \arrow[u, "{[I]_{\mathcal{X},\mathcal{S}}}"] \arrow[r, "{[T]_{\mathcal{X},\mathcal{X}}}"] & F^n_{\mathcal{X}} \arrow[u, "{[I]_{\mathcal{X},\mathcal{S}}}"'] \end{tikzcd}$$

Denote $(\beta_1, \cdots, \beta_n) \in F^n_{\mathcal{X}}$ by $x = \sum \beta_i x_i \in F^n$. Note that $[x]_{\mathcal{S}} = x$ for all $x \in F^n$ and $e_i = [x_i]_{\mathcal{X}}$. Then

$$[I]_{\mathcal{X},\mathcal{S}} e_i = [I]_{\mathcal{X},\mathcal{S}}[x_i]_{\mathcal{X}} = [Ix_i]_{\mathcal{S}} = [x_i]_{\mathcal{S}} = x_i.$$

So $[I]_{\mathcal{X},\mathcal{S}} = X$, and hence $[T]_{\mathcal{X},\mathcal{X}} = X^{-1}AX$. $\square$

Determinant of linear operator

By the result of Ex7.2.3, we can define the determinant of $L : V \to V$ where V is finite dimensional vector space over F. Fix a basis $\mathcal{X}$. Define

$$\det(L) = \det[L]_{\mathcal{X},\mathcal{X}}.$$

$\det(L)$ is invariant under a choice of a basis.

Observation

Recall that $\det : (F^n) \times \cdots \times (F^n) \to F$ is a function such that

- $\det(e_1, \cdots, e_n) = 1$.
- $\det$ is multilinear.
- $\det(\cdots, v, \cdots, v, \cdots) = 0$.

The first condition is called the normalizing condition.
We can apply $A \in F^{n \times n}$ to $\det$ by

$$\det(A) := \det(Ae_1, \cdots, Ae_n).$$

Step 1

Let $\{x_1, \cdots, x_n\}$ be any basis for F^n. Define $D : (F^n) \times \cdots \times (F^n) \to F$ so that

- $D(x_1, \cdots, x_n) = 1.$
- D is multilinear.
- $D(\cdots, v, \cdots, v, \cdots) = 0.$

D is different from $\det$.
Now for $A \in F^{n \times n}$, define $D(A)$ by

$$D(A) := D(Ax_1, \cdots, Ax_n).$$

Step 2

Let $L : F^n \to F^n$ by $L(x) = Ax$ and $B = [L]_{\mathcal{X},\mathcal{X}}$. Notw that $B_i = [Ax_i]_{\mathcal{X}}$.

$$\begin{aligned}
D(A) &= D(Ax_1, \cdots, Ax_n) \\
&= D(\sum B_{1i_1} x_{i_1}, \cdots, \sum B_{ni_n} x_{i_n}) \\
&= \sum_{i_1} \cdots \sum_{i_n} B_{1i_1} \cdots B_{ni_n} D(x_{i_1}, \cdots, x_{i_n}) \\
&= \sum_{i_1} \cdots \sum_{i_n} B_{1i_1} \cdots B_{ni_n} \det(e_{i_1}, \cdots, e_{i_n}) \\
&= \det(\sum B_{1i_1} e_{i_1}, \cdots, \sum B_{ni_n} e_{i_n}) \\
&= \det(Be_1, \cdots, Be_n) = \det(B) = \det([L]_{\mathcal{X},\mathcal{X}}) \\
&= \det(A).
\end{aligned}$$

This is another reason why $\det(L)$ is well-defined.

Remark

We can directly construct $\det(L)$ for arbitrary $L : V \to V$. Fix a basis $\{x_1, \cdots, x_n\}$. Define $\det : V \times \cdots \times V \to F$ so that

- $\det(x_1, \cdots, x_n) = 1$.
- $\det$ is multilinear.
- $\det(\cdots, v, \cdots, v, \cdots) = 0$.

Now for $L \in \mathcal{L}(V, V)$, define $\det(L)$ by

$$\det(L) := \det(L(x_1), \cdots, L(x_n)).$$

Ex7.2.4

Let $A \in \mathbb{C}^{n\times n}$ be normal. Prove that $A - \lambda I$ is normal for any $\lambda \in \mathbb{C}$.

Ex7.2.5

Let $A \in \mathbb{C}^{n\times n}$. Prove:

(a) If there exists a unitary matrix $X \in \mathbb{C}^{n\times n}$ and a diagonal matrix $D \in \mathbb{R}^{n\times n}$ such that $A = XDX^*$, then A is Hermitian.

(b) If there exists a unitary matrix $X \in \mathbb{C}^{n\times n}$ and a diagonal matrix $D \in \mathbb{C}^{n\times n}$ such that $A = XDX^*$, then A is normal.

Ex7.2.7

Let $A \in \mathbb{R}^{n\times n}$. We say that A is skew-symmetric if $A^T = -A$.

(a) Prove that any skew symmetric matrix is normal.

(b) Prove that a skew symmetric matrix has only purely imaginary eigenvalues.

Ex7.2.14

Let V be an inner product space over $\mathbb{R}$ or $\mathbb{C}$, and let u, v be nonzero vectors in V.

(a) the rank of $u \otimes v$;

(b) the eigenparis of $u \otimes v$;

(c) the characteristic polynomial, determinant, and trace of $u \otimes v$;

(d) the adjoint of $u \otimes v$.

E7.2.16

Let V be an inner product spave over $\mathbb{R}$ and let $u \in V$ have norm one. Define $T : V \to V$ by

$$T = I - 2u \otimes u.$$

Prove that T is self-adjoint and orthogonal.

Tensor product

Observation

Let U and V be two vector spaces over a field F. Suppose $\dim U = n$ and $\dim V = m$. Then $U \times V = \{(u, v) \mid u \in U, v \in V\}$ is a vector space over F in a natural way;

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2), \alpha(u, v) = (\alpha u, \alpha v).$$

Let $\{u_1, \cdots, u_n\}$ and $\{v_1, \cdots, v_m\}$ be bases for U and V, respectively. Then $\{(u_1, 0), \cdots, (u_n, 0), (0, v_1), \cdots, (0, v_m)\}$ forms a basis for $U \times V$. Thus

$$\dim(U \times V) = \dim U + \dim V.$$

The inner direct sum

Let X be a vector space, and let U and V be two subspace of X such that $U \cap V = \{0\}$. Then every vector of $U + V$ can be written as $u + v$ for some $u \in U$ and $v \in V$ in a unique way. In this case, we write $U + V = U \oplus V$. We call $\oplus$ the inner direct sum.

The outer direct sum

In the previous observation, we can identify U as a subspace of $U \times V$ by $U \cong U \times \{0\}$. Similarly $V \cong \{0\} \times V \subset U \times V$. Since $(U \times \{0\}) \cap (\{0\} \times V) = \{0\}$, $U \times V = (U \times \{0\}) \oplus (\{0\} \times V)$. In this case, we sometimes denote $U \boxplus V \cong U \times V$, which is a vector space containing U and V independently and is isomorphic to $U \times V$. We call $\boxplus$ the outer direct sum.

The direct sum

Since we can identify U and V with subspaces of $X = U \times V$, we don't distinguish between $\oplus$ and $\boxplus$. Thus $U \oplus V$ is a vector space which is generated by U and V independently. Moreover, $\dim(U \oplus V) = \dim U + \dim V$.

Observation

Consider $\mathcal{L}(F^n, F^m) \cong F^{m\times n}$. For all $A \in F^{m\times n}$,

$$A = \sum_{i,j} A_{ij} e_j e_i^T, e_i \in F^n, e_j \in F^m.$$

Let $u \in F^n$ and $v \in F^m$ and $\alpha \in F$.

- $(\alpha v)u^T = v(\alpha u)^T = \alpha(vu^T)$.
- $(v_1 + v_2)u^T = v_1 u^T + v_2 u^T$.
- $v(u_1 + u_2)^T = vu_1^T + vu_2^T$.

So for all $A \in F^{m\times n}$, $A = \sum v_j u_i^T$ for some $u_i \in F^n$ and $v_j \in F^m$, or $F^{m\times n}$ is generated by $\{vu^T \mid u \in F^n, v \in F^m\}$.

Hence for all $L \in \mathcal{L}(F^n, F^m)$, $L = \sum u_i \otimes v_j$. If we take bases $\{e_1, \cdots, e_n\}$ and $\{f_1, \cdots, f_m\}$,

$$L = \sum_{i,j} \alpha_{i,j} e_i \otimes f_j.$$

Observation

Let U and V be two vector spaces over a field F. Consider a vector space X which satisfies the followings:

1. X is generated by $\{u \otimes v \mid u \in U, v \in V\}$;
2. $\alpha(u \otimes v) = (\alpha u) \otimes v = u \otimes (\alpha v)$;
3. $(u_1 + u_2) \otimes v = (u_1 \otimes v) + (u_2 \otimes v)$;
4. $u \otimes (v_1 + v_2) = (u \otimes v_1) + (u \otimes v_2)$.

The question is "is there such a vector space?". The answer is "yes".
We write $X = U \otimes V$, a tensor product of U and V.

Suppose $\dim U = n$ and $\dim V = m$ and take bases $\{e_1, \cdots, e_n\}$ and $\{f_1, \cdots, f_m\}$. By the conditions, every vector in $U \otimes V$ is of the form $\sum_i \sum_j \alpha_{ij} e_i \otimes f_j$. Hence $\{e_i \otimes f_j\}_{i,j}$ forms a basis for $U \otimes V$, and

$$\dim(U \otimes V) = \dim U \cdot \dim V.$$

Sequence notation

A sequence on a set S is just a function $a : \mathbb{N} \to S$. Sometimes we write $a = (a_n)$, or $a = \{a_n\}$ where $a_n = a(n)$. In this sense, $x : \mathbb{N} \to S$ can be written as (x_n).
Let $x \in F^n$. Then $x = (x_1, \cdots, x_n)$. So x is a function $x : \{1, \cdots, n\} \to F$ where $x(i) = x_i$.

Set of all functions

Let X and Y be sets and define Y^X a set of all functions $f : X \to Y$. In general, the cadinality of Y^X is $|Y|^{|X|}$.
Now by using the sequence notation, $f \in Y^X$ can be written as $f = (f(x))_{x \in X}$.
If X is finite set, $X = \{x_1, \cdots, x_n\}$, $f = (f(x_i)) = (f_i)$.

Free vector space

Let S be any set and F be a field. We can construct a vector space $\mathcal{F}(S)$ whose basis is S, as follows:

Let $X = \{f : S \to F\}$ be the set of all functions from S to F such that $f(s) = 0$ for all but only finitely many $s \in S$. (We say f has the finite support)
For $x \in X$, write $x_s = x(s)$ for all $s \in S$, and $x = (x_s)_{s \in S}$. We can define $x + y = (x_s + y_s)$ and $\alpha x = (\alpha x_s)$. For each $s \in S$, we can define a fucntion $f^s : S \to F$ by $f^s(t) = \delta_{st}$. Then for all $x \in X$, $x = \sum x_s f^s$. Since $x_s \neq 0$ for only finitely many s, the sum is well defined, and actually it is a linear combination of $\{f^s\}$. Moreover, $\sum \alpha_i f^{s_i} = 0$ implies $\alpha_i = 0$ for all i. Hence $\{f^s\}$ is a basis for X. Now if we identify $s \in S$ with $f^s \in X$, S is a basis for X. Hence $X = \mathcal{F}(S)$ exists, say the free vector space of S over F.

Notation

For $s \in S$, we can write $f^s = 1_s$. This implies $1_s(t) = 1$ for only $t = s$.

Universal property of a free vector space

Let W be a vector space over F. Suppose $f : S \to W$ is a function. Then there is a unique linear map $\tilde{f} : \mathcal{F}(S) \to W$ such that $\tilde{f} \circ \iota(s) = f(s)$.

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathcal{F}(S) \arrow[d, dotted, "\tilde{f}"] \\ & W \end{tikzcd}$$

Proof

Define $\tilde{f}(\sum \alpha_i 1_{s_i}) = \sum \alpha_i s_i$.

Notation

$\mathcal{F}(S) = F^{(S)} = \bigoplus_{s\in S} F$. If $|S| = n < \infty$, $F^{(n)} = F^n = \bigoplus_{i=1}^n F$.
The map $s \mapsto 1_s$ is injective. Define $\iota : S \to \mathcal{F}(S)$ such that $\iota(s) = 1_s$.

Quotient space

Let V be a vector space over F and H be a subspce of V. We can define a relation $\sim$ on V by

$$x \sim y \Longleftrightarrow x - y \in H.$$

This relation satisfies 1)$x \sim x$, 2)$x \sim y$ impleis $y \sim x$, and 3)$x \sim y$ and $y \sim z$ implies $x \sim z$. So $\sim$ is an equivalent relation. Note that $[v] = v + H$.
Consider the set of all equivalece class $V/H = \{[v] \mid v \in V\}$. We can define an addition and a scalar multiplication by

$$[v_1] + [v_2] := [v_1 + v_2], \quad \alpha[v] := [\alpha v].$$

Thus V/H is also vector space. Now define $\pi : V \to V/H$ by $\pi(v) = [v]$. Then π is a serjective linear map whose kernel is H. We call π a canonical projection map.

Universal property of quotient spaces

Let V be a vector space over F and H be a subspce of V. Suppose $T : V \to W$ is a linear map such that $H \subset \ker T$. Then there is a unique liner map $\tilde{T} : V/H \to W$ such that $T = \tilde{T} \circ \pi$.

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[dr, "T"'] & V/H \arrow[d, dotted, "\tilde{T}"'] \\ & W \end{tikzcd}$$

Proof

Define $\tilde{T}(v + H) = T(v)$.

The existence of the tensor product

Let U and V be vector spaces over F. Let $X = \mathcal{F}(U \times V)$. Let $H \subset X$ spanned by

$$
\begin{aligned}
1_{(\alpha u, v)} &- 1_{(u, \alpha v)}, \\
1_{(u_1+u_2, v)} &- 1_{(u_1, v)} - 1_{(u_2, v)}, \\
1_{(u, v_1+v_2)} &- 1_{(u, v_1)} - 1_{(u, v_2)}.
\end{aligned}
$$

Then X/H is a vector space over F and it satisfies

1. X/H is generated by $\{[1_{(u,v)}] \mid u \in U, v \in V\}$;
2. $\alpha[1_{(u,v)}] = [1_{(\alpha u,v)}] = [1_{(u,\alpha v)}]$;
3. $[1_{(u_1+u_2,v)}] = [1_{(u_1,v)}] + [1_{(u_2,v)}]$;
4. $[1_{(u,v_1+v_2)}] = [1_{(u,v_1)}] + [1_{(u,v_2)}]$.

If we write $[1_{(u,v)}] = u \otimes v$, X/H is a tensor product of U and V, as desired.

Remark

- When we construct $\mathcal{F}(U \times V)$, $U \times V$ is a set (not a vector space). So for $\alpha \neq 1$, $1_{(\alpha u, \alpha v)} \neq \alpha 1_{(u,v)}$, for example,

$$1_{(\alpha u, \alpha v)}(u, v) = 0.$$

- A vector $x \in U \otimes V$ may not be written as $x = u \otimes v$. But every vector can be written as $x = \sum_i \alpha_i u_i \otimes v_i$. For example, $U = V = F^2$, $e_1 \otimes e_2 + e_2 \otimes e_1 \neq u \otimes v$ for all u, v.
- $u \otimes 0 = 0$ for all u. (cf. $(u, 0) \neq 0$ if $u \neq 0$.)

Universal property of tensor product

Suppose $T : U \times V \to W$ is a bilinear map. There is a unique linear map $\tilde{T} : U \otimes V \to W$ such that

$$T(u, v) = \tilde{T}(u \otimes v).$$

$$\begin{tikzcd} U \times V \arrow[r, "p"] \arrow[dr, "T"'] & U \otimes V \arrow[d, dotted, "\tilde{T}"] \\ & W \end{tikzcd}$$

Note

We have two maps $\iota : U \times V \to \mathcal{F}(U \times V)$ and $\pi : \mathcal{F}(U \times V) \to U \otimes V$. Define $p = \pi \circ \iota$. Then $p(u, v) = u \otimes v$ and $T = \tilde{T} \circ p$. You can easily check p is bilinear.

Proof, Step 1

$$
\begin{tikzcd}
U \times V \arrow[r, "\iota"] \arrow[dr, "T"'] & \mathcal{F}(U \times V) \arrow[d, dotted, "L"] \\
& W
\end{tikzcd}
$$

It is easy to show the uniqueness.
By the universal property of a free vector space, there is a unique linear map $L : \mathcal{F} \to W$ such that $L(\iota(u, v)) = T(u, v)$.

Proof, Step 2

$$\begin{tikzcd} \mathcal{F}(U \times V) \arrow[r, "\pi"] \arrow[dr, "L"'] & U \otimes V \arrow[d, dotted, "\tilde{T}"] \\ & W \end{tikzcd}$$

Since $T(\alpha u, v) - T(u, \alpha v) = 0, T(u_1 + u_2, v) - T(u_1, v) - T(u_2, v) = 0, T(u, v_1 + v_2) - T(u, v_1) - T(u, v_2) = 0$, $H \subset \ker(L)$. By the universal property of a quotient space, there is a unique linear map $\tilde{T} : U \otimes V \to W$ such that $L = \tilde{T} \circ \pi$. Hence $T = \tilde{T} \circ \pi \circ \iota = \tilde{T} \circ p$ ad desired.

Uniqueness of tensor products

By using the universal property of the tensor product, there is a unique tensor product up to isomorphism.

Diagram for proof

$$\begin{tikzcd}
U \times V \arrow[r, "\iota"] \arrow[drr, "T"'] & \mathcal{F}(U \times V) \arrow[r, "\pi"] \arrow[dr, dotted, "L"] & U \otimes V \arrow[d, dotted, "\tilde{T}"] \\
& & W
\end{tikzcd}$$

Application of tensor products

Consider a vector space V over $\mathbb{C}$. V is vector space over $\mathbb{R}$. How about the converse?
In general, a vector space V over $\mathbb{R}$ is not a vector space over $\mathbb{C}$. Then we want to find the best approximation of V into a vector space over $\mathbb{C}$.

Scalar extension, or base change

$\mathbb{C}$ is itself a vector space over $\mathbb{R}$. Then $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ is a vector space over $\mathbb{R}$.
Now give a scalar multiplication on $V_{\mathbb{C}}$ by

$$\alpha \cdot (v \otimes c) = v \otimes (\alpha c).$$

In this sense, $V_{\mathbb{C}}$ is a vector space over $\mathbb{C}$.
Moreover $V \otimes_{\mathbb{R}} \mathbb{R}$ is a subspace of $V_{\mathbb{C}}$ and $V \otimes_{\mathbb{R}} \mathbb{R} \cong V$. So we can consider $V_{\mathbb{C}}$ as a scalar extension of V.

The outer product and the tensor product

Suppose U and V are finite dimensional inner product spaces. Define $E : U \times V \to \mathcal{L}(V, U)$ by

$$E(u, v)(w) = \langle w, v \rangle u.$$

We already know that E is a bilinear map. Thus there is a unique linear map $\tilde{E}(\sum u_i \otimes v_i) = \sum \langle \cdot, v_i \rangle u_i$. You can easily chech that $\tilde{E}$ is an isomorphism. So we can identify $u \otimes v = \langle \cdot, v \rangle u$.

In fact, the tensor product is a generalization of the outer product.

Relation of tensor product to linear map

Let U and V be finite diimensional vecsor spaces over F. Consider the map

$$U^* \otimes V \longrightarrow \mathcal{L}(U, V)$$
$$\sum (f_i \otimes v_i) \longmapsto \sum f_i(\cdot) v_i .$$

Clearly it is a linear map. Moreover this map is an isomorphism.

In general there is an isomorphism

$$\mathcal{L}(U \otimes V, W) \cong \mathcal{L}(U, \mathcal{L}(V, W)).$$

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The tensor algebra

Fix a vector space V over F. Then $(V \otimes V) \otimes V \cong V \otimes (V \otimes V)$. Thus we can define $\underbrace{V \otimes \cdots \otimes V}_{n \text{ times}} = \bigotimes_{i=1}^{n} V$.

For a positive integer k, let $T^k(V) = \bigotimes_{i=1}^{k} V$. Then $T^k(V)$ is a vector space over F, and we say an element of $T^k(V)$ is a k-tensor. Define $T^0(V) = F$ and $T^1(V) = V$.

For $v \in T^k(V)$ and $w \in T^l(V)$, we can define $v \otimes w \in T^{k+l}(V)$.

Now define $T(V) = \bigoplus_{k=0}^{\infty} T^k(V)$. Then $T(V)$ is a vector space over F, and there is a multiplication $v \cdot w = v \otimes w$. We call such space F-algebra. Typically, $T(V)$ is a tensor algebra.

Universal property of tensor product

Suppose $T : V_1 \times \cdots \times V_n \to W$ is a multilinear map. There is a unique linear map $\tilde{T} : V_1 \otimes \cdots \otimes V_n \to W$ such that

$$T(v_1, \cdots, v_n) = \tilde{T}(v_1 \otimes \cdots \otimes v_n).$$

$$\begin{tikzcd}
V_1 \times \cdots \times V_n \arrow[r, "p"] \arrow[dr, "T"'] & V_1 \otimes \cdots \otimes V_n \arrow[d, dotted, "\tilde{T}"] \\
& W
\end{tikzcd}$$

Definition

For $\tau \in S_k$ and $v = v_1 \otimes \cdots \otimes v_k \in T^k(V)$, define

$$\tau(v) = v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(k)},$$

and

$$\tau\left(\sum v_1^{(i)} \otimes \cdots \otimes v_k^{(i)}\right) = \sum \tau\left(v_1^{(i)} \otimes \cdots \otimes v_k^{(i)}\right).$$

The symmetric algebra

Consider $T^k(V)$. Let $v = v_1 \otimes \cdots \otimes v_k \in T^k(V)$. For $\tau \in S_k$, $v - v_{\tau(1)} \otimes \cdots \otimes v_{\tau(k)}$ is either 0 or nonzero. We call v is a symmetric if $v - v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(k)} = 0$ for all $\tau \in S_k$. In this case,

$$v = \frac{1}{k!} \sum_{\tau \in S_k} v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(k)}.$$

The symmetric operator

Let $\mathcal{S} : T^k(V) \to T^k(V)$ by

$$\mathcal{S}(v) = \frac{1}{k!} \sum_{\tau \in S_k} v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(k)},$$

for $v = v_1 \otimes \cdots \otimes v_k$.

The alternating algebra

For $\tau \in S_k$, we call v is an alternating if for all $\tau \in S_k$

$$v = \mathrm{sgn}(\tau) v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(k)}.$$

In this case,

$$v = \frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sgn}(\tau) v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(k)}.$$

The alternating operator

Let $\mathcal{A} : T^k(V) \to T^k(V)$ by

$$\mathcal{A}(v) = \frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sgn}(\tau) v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(k)},$$

for $v = v_1 \otimes \cdots \otimes v_k$.

Remark

We only define $\mathcal{S}$ and $\mathcal{A}$ for $v = v_1 \otimes \cdots \otimes v_k$. We can extand these maps in natural way for whole $T^k(V)$, by

$$\mathcal{S}(\sum v_1^{(i)} \otimes \cdots \otimes v_k^{(i)}) = \sum \mathcal{S}(v_1^{(i)} \otimes \cdots \otimes v_k^{(i)}).$$
$$\mathcal{A}(\sum v_1^{(i)} \otimes \cdots \otimes v_k^{(i)}) = \sum \mathcal{A}(v_1^{(i)} \otimes \cdots \otimes v_k^{(i)}).$$

Example

- $m \otimes m$, $m_1 \otimes m_2 + m_2 \otimes m_1$ are symmetric.
- $m_1 \otimes m_2 - m_2 \otimes m_1$ is alternating.

Remark

If v and w are symmetric (or alternating) k-tensros, so is $v + w$. Thus the set of all symmetric (or alternating) k-tensors is subspace of $T^k(V)$. Define $\mathrm{Sym}^k(V)$ the set of all symmetric k-tensors and $\mathrm{Alt}^k(V)$ the set of all alternating k-tensors.

Theorem

$\mathcal{S}(v)$ is symmetric and $\mathcal{A}(v)$ is alternating.

Proof.

$$\tau(\mathcal{S}(v)) = \tau\left(\frac{1}{k!}\sum_{\tau'\in S_k}\tau'(v)\right) = \frac{1}{k!}\sum_{\tau'\in S_k}\tau(\tau'(v)) = \frac{1}{k!}\sum_{\tau'\in S_k}(\tau\circ\tau')(v) = \mathcal{S}(v).$$

$$\tau(\mathcal{A}(v)) = \tau\left(\frac{1}{k!}\sum_{\tau'\in S_k}\operatorname{sgn}(\tau')\tau'(v)\right) = \frac{1}{k!}\sum_{\tau'\in S_k}\operatorname{sgn}(\tau')\tau(\tau'(v))$$
$$= \operatorname{sgn}(\tau)\frac{1}{k!}\sum_{\tau'\in S_k}\operatorname{sgn}(\tau\circ\tau')(\tau\circ\tau')(v) = \operatorname{sgn}(\tau)\mathcal{A}(v).$$

$\square$

The wedge product

Let $v \in T^k(V)$ and $w \in T^l(V)$. We can define the wedge product by

$$v \wedge w = \frac{(k+l)!}{k!l!}\mathcal{A}(v \otimes w).$$

Then $v \wedge w \in \mathsf{Alt}^{k+l}(V)$.

properties

Let $v \in \mathsf{Alt}^k(V)$ and $w \in \mathsf{Alt}^l(V)$.

1. $(v_1 + v_2) \wedge w = v_1 \wedge w + v_2 \wedge w$.
2. $v \wedge (w_1 + w_2) = v \wedge w_1 + v \wedge w_2$.
3. $(\alpha v) \wedge w = v \wedge (\alpha w)$.
4. $v \wedge w = (-1)^{kl} w \wedge v$

Remark

Then every element of $\mathsf{Alt}^k(V)$ is of the form $\sum \alpha_{i_1\cdots i_k} v_{i_1} \wedge \cdots \wedge v_{i_k}$ where $\{v_1, \cdots, v_n\}$ is a basis for V and $i_1 < i_2 < \cdots < i_k$. Hence $\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid i_1 < i_2 < \cdots < i_k\}$ forms a basis for $Alt^k(V)$, and $\dim \mathsf{Alt}^k(V) = \binom{n}{k}$.

Similarly every element of $\mathsf{Sym}^k(V)$ is of the form $\sum \alpha_{i_1\cdots i_k} v_{i_1} \otimes \cdots \otimes v_{i_k}$ where $i_1 \leq i_2 \leq \cdots \leq i_k$ and

$$\alpha_{i_1\cdots i_k} = \alpha_{i_{\tau(1)}\cdots i_{\tau(k)}}$$

for all $\tau \in S_k$. Clearly, $\dim \mathsf{Sym}^k(V) = \binom{k+n-1}{n-1} = \binom{n+k-1}{k}$.

The determinant

Consider $V = F^n$. Let $A = \mathrm{Alt}^n(V^*)$. We can identify $f_1 \wedge \cdots \wedge f_n \in A$ to a multilinear map $V^n \to F$ by

$$(f_1 \wedge \cdots \wedge f_n)(v_1, \cdots, v_n) = \sum_{\tau \in S_k} f_1(v_{\tau(1)}) \cdots f_n(v_{\tau(n)}).$$

Now let d_i be the covector of e_i, i.e. $d_i(e_j) = \delta_{ij}$. Then

$$\det(v_1, \cdots, v_n) = (d_1 \wedge \cdots \wedge d_n)(v_1, \cdots, v_n)$$

Example

$V = \mathbb{R}^2$.

$$(d_1 \wedge d_2)(u_1e_1 + u_2e_2, v_1e_1 + v_2e_2) = d_1(u_1e_1 + u_2e_2)d_2(v_1e_1 + v_2e_2) \\ - d_1(v_1e_1 + v_2e_2)d_2(u_1e_1 + u_2e_2) = u_1v_2 - u_2v_1.$$

The exterior product

Let $V = \mathbb{R}^3$. Let $u = u_1e_1 + u_2e_2 + u_3e_3$ and $v = v_1e_1 + v_2e_2 + v_3e_3$. Since $V = T^1(V)$, we can compute $u \wedge v$. Note that $S_2 = \{\tau_1 = (1,2), \tau_2 = (2,1)\}$.

$$e_i \wedge e_i = 2\mathcal{A}(e_i \otimes e_i) = e_i \otimes e_i - e_i \otimes e_i = 0.$$

$$\begin{aligned} u \wedge v &= (u_1e_1 + u_2e_2 + u_3e_3) \wedge (v_1e_1 + v_2e_2 + v_3e_3) \\ &= (u_1v_2 - u_2v_1)e_1 \wedge e_2 + (u_2v_3 - u_3v_2)e_2 \wedge e_3 + (u_3v_1 - u_1v_3)e_3 \wedge e_1. \end{aligned}$$

Compare

$$u \times v = (u_1v_2 - u_2v_1)(i \times j) + (u_2v_3 - u_3v_2)(j \times k) + (u_3v_1 - u_1v_3)(k \times i)$$

where $i \times j = k$, $j \times k = i$ and $k \times i = j$. The exterior (wedge) product is a generalization of the cross product.

A relation between determinant and wedge product

$V = \mathbb{R}^3$, d_i is the covector of e_i. Let $u = u_1e_1 + u_2e_2 + u_3e_3$ and $v = v_1e_1 + v_2e_2 + v_3e_3$.

$$(d_1 \wedge d_2)(u, v) = d_1(u)d_2(v) - d_1(v)d_2(u) = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$$

In general,

$$(d_{i_1} \wedge \cdots \wedge d_{i_k})(v^{(1)}, \cdots, v^{(k)}) = \begin{vmatrix} v_{i_1}^{(1)} & \cdots & v_{i_1}^{(k)} \\ v_{i_2}^{(1)} & \cdots & v_{i_2}^{(k)} \\ \vdots & \ddots & \vdots \\ v_{i_k}^{(1)} & \cdots & v_{i_k}^{(k)} \end{vmatrix}$$

Notation

In differential geometry or analysis, $e_i = \partial/\partial x_i$ and $d_i = dx_i$.

The End