Algebraic Topology
- Dunkin's Torus 5 -

KYB

Thrn, it's a Fact

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The Fundamental Group

- Retractions and Fixed Points
- The Fundamental Theorem of Algebra

Retractions and Fixed Points

Definition

If $A \subset X$, a *retraction* of X onto A is a continuous map $r : X \to A$ such that $r|A$ is the identity map of A. If such a map r exists, we say that A is a *retract* of X.

Lemma (55.1)

If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion $j : A \to X$ is injective.

Retractions and Fixed Points

Theorem (55.2, No-retraction theorem)

There is no retraction of B^2 *onto* S^1.

Lemma (55.3)

Let $h : S^1 \to X$ *be a continuous map. Then the following conditions are equivalent:*

(1) h *is nulhomotopic.*

(2) h *extends to a continuous map* $k : B^2 \to X$.

(3) h_* *is the trivial homomorphism of fundamental groups.*

Corollary (55.4)

The inclusion map $\mathrm{j} : S^1 \to \mathbb{R}^2 - \mathbf{0}$ is not nulhomotopic. The identity map $\mathrm{i} : S^1 \to S^1$ is not nulhomotopic.

Theorem (55.5)

Given a nonvanishing vector field on B^2, there exists a point of S^1 where the vector field points directly inward and a point of S^1 where it points directly outward.

Note

- A *vector field* on B^2 is an ordered pair $(x, v(x))$, where $x \in B^2$ and v is a continuous map of B^2 into $\mathbb{R}^2$.

$$\mathbf{v}(x) = v_1(x)\mathbf{i} + v_2(x)\mathbf{j}$$

  where $v = (v_1, v_2)$.
- A vector field is said to be *nonvanishing* if $v(x) \neq \mathbf{0}$ for every x.

Retractions and Fixed Points

Theorem (55.6, Brouwer Fixed-point Theorem for the Disc)

If $f : B^2 \to B^2$ *is continuous, then there exists a point* $x \in B^2$ *such that* $f(x) = x$*.*

Corollary (55.7)

Let A be a 3 by 3 matrix of positive real numbers. Then A has a positive real eigenvalue.

Theorem (55.8)

There is an $\epsilon > 0$ such that for every open covering $\mathcal{A}$ of T by sets of diameter less than ϵ, some point of T belongs to at least three elements of $\mathcal{A}$.

Ex 55.1

Show that if A is a retract of B^2, then every continuous map $f : A \to A$ has a fixed point.

Ex 55.2

Show that if $h : S^1 \to S^1$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

Ex 55.4

Suppose that you are given the fact that for each n, there is no retraction $r : B^{n+1} \to S^n$. Prove the following:

(a) The identity map $i : S^n \to S^n$ is not nulhomotopic.

(b) The inclusion map $j : S^n \to \mathbb{R}^{n+1} - \mathbf{0}$ is not nulhomotopic.

(c) Every nonvanishing vector field on B^{n+1} points directly outward at some point of S^n, and directly inward at some point of S^n.

(d) Every continuous map $f : B^{n+1} \to B^{n+1}$ has a fixed point.

(e) Every $n + 1$ by $n + 1$ matrix with positive real entries has a positive eigen value.

(f) If $h : S^n \to S^n$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

The Fundamental Theorem of Algebra

Theorem (56.1,The Fundamental Theorem of Algebra)

A polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree $n > 0$ *with real or complex coefficients has at least one (real or complex) root.*

Proof, Step1

Consider the map $f : S^1 \to S^1$ given by $f(z) = z^n$, where z is a complex number.

> Claim : f_* is injective.

Let $p_0 : I \to S^1$ be the standard loop in S^1,

$$p_0(s) = e^{2\pi i s} = (\cos 2\pi s, \sin 2\pi s).$$

Then

$$f(p_0(s)) = (e^{2\pi i s})^n = e^{2\pi i n s} = (\cos 2\pi n s, \sin 2\pi n s).$$

$f \circ p_0$ corresponds to the integer n under the standard isomorphism of $\pi_1(S^1, b_0)$ with the integers, whereas p_0 correspond to the number 1.

Proof, Step2

Claim : If $g : S^1 \to \mathbb{R}^2 - \mathbf{0}$ is the map $g(z) = z^n$, then g is not nulhomotopic.

Let $j : S^1 \to \mathbb{R}^2 - \mathbf{0}$ be the inclusion map. Then $g = j \circ f$. f_* and j_* are injective because S^1 is a retract of $\mathbb{R}^2 - \mathbf{0}$. Therefore, g_* is injective, and hence, g cannot be nulhomotopic.

Proof, Step3

Claim : If a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

with $|a_{n-1}| + \cdots + |a_1| + |a_0| < 1$, then the equation has a root lying in the unit ball B^2.

Assume it has no such root. Then we can define a map $k : B^2 \to \mathbb{R}^2 - \mathbf{0}$ by the equation

$$k(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_z z + a_0.$$

Let $h = k|S^1$. Because h extends to a map of the unit ball into $\mathbb{R}^2 - \mathbf{0}$, the map h is nulhomotopic.
On the other hand, define a homotopy F between h and g by

$$F(z, t) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0).$$

But it is a contradiction because g is not nulhomotpic.

Proof, Step4

Now let a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

be given, and let us choose a real number $c > 0$ and substitute $x = cy$. Then

$$y^n + \frac{a_{n-1}}{c}y^{n-1} + \cdots + \frac{a_1}{c^{n-1}}y + \frac{a_0}{c^n} = 0.$$

Choose c large enough so that

$$\left|\frac{a_{n-1}}{c}\right| + \cdots \left|\frac{a_1}{c^{n-1}}\right| + \left|\frac{a_0}{c^n}\right| < 1.$$

By the Step 3, it has a root $y = y_0$, and then the original equation has the root $x_0 = cy_0$.