LA13 Ch5 Exercises

KYB

Thrn, it's a Fact

mathrnfact@gmail.com

February 9, 2021

Overview

Ex5.2.1

$$\det\left(\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}\right) = \det(B)\det(D)$$

Step1

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} B & C \\ 0 & I \end{bmatrix}$$

Step2

$$\det\left(\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}\right) = \det(D)$$

$$\det\left(\begin{bmatrix} B & C \\ 0 & I \end{bmatrix}\right) = \det(B)$$

Ex5.2.2

$$A = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1t} \\ 0 & B_{22} & \cdots & B_{2t} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & B_{tt} \end{bmatrix}$$

Prove that

1. $\det(A) = \det(B_{11}) \cdots \det(B_{tt})$
2. spec$(A) = \bigcup$spec(B_{ii}) and m.geo$_A(\lambda) = \sum$ m.geo$_{B_{ii}}(\lambda)$,m.alg$_A(\lambda) = \sum$ m.alg$_{B_{ii}}(\lambda)$

Ex5.2.4

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(a) $p_A(r)$.
(b) m.alg(1).
(c) Find the smallest k such that $\mathcal{N}((A-I)^{k+1}) = \mathcal{N}((A-I)^k)$.
(d) Show that $\dim(\mathcal{N}(A-I)^k) = m$.

Ex5.2.8

Let λ be an e.val of A and k be any positive integer. Prove that if $x \in \mathcal{N}((A - \lambda I)^k)$ is an e.vec of A, the corr e.val is λ.

Minimal Polynomial

Since $A \in F^{n\times n}$, there is s such that

$$\{I, A, A^2, \cdot, A^{s-1}\}$$

is linearly independent,but

$$\{I, A, A^2, \cdot, A^s\}$$

is linearly dependent. Then there exist unique $c_0, \cdots, c_{s-1} \in F$ such that

$$c_0 I + c_1 A + \cdots c_{s-1} A^{s-1} + A^s = 0$$

Then we say $m_A(r) = c_0 + c_1 r + \cdots c_{s-1} r^{s-1} + r^s$ is the minimal polynomial of A.

Theorem

Let $m_A(r)$ be the minimal polynomial of A. If $p(r) \in F[r]$ satisfies $p(A) = 0$, then there is $q(r) \in F[r]$ such that $p(r) = m_A(r)q(r)$.

Lemma

If (λ, x) is an eigenpair of A and $p(r) \in F[r]$, then $p(A)x = p(\lambda)x$.

Theorem

The roots of $m_A(r)$ are the eigenvalues of A.

The Cayley-Hamilton theorem

Let F be a field, $A \in F^{n\times n}$. Then $p_A(A) = 0$. Thus $m_A(r)$ divides $p_A(r)$.

L

et $A \in \mathbb{C}^{n\times n}$ with distinct eigenvalues $\lambda_1, \cdots, \lambda_t$. Let

$$m_A(r) = (r-\lambda_1)^{k_1}\cdots(r-\lambda_t)^{k_t}$$
$$p_A(r) = (r-\lambda_1)^{m_1}\cdots(r-\lambda_t)^{m_t}$$

Then $k_i \le m_i$ for all $i = 1, \cdots, t$.

Ex5.2.15

Let F be a field and $c_0, \cdots, c_{n-1} \in F$. Define

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdot & -c_0 \\ 1 & 0 & 0 & \cdot & -c_1 \\ 0 & 1 & 0 & \cdot & -c_2 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

Ex4.5.9 shows $p_A(r) = r^n + c_{n-1}r^{n-1} + \cdots + c_1 r + c_0$.

1. Prove that e_1 is a cyclic vector for A.
2. Prove that $m_A(r) = p_A(r)$.

Ex5.3.2

Suppose $T : X \to X$ is a nilpotent linear operator. Prove that 0 is an eigenvalue of T, and it is only eigenvalue.

Ex5.3.3

Let $I : X \to X$ by $I(x) = x$. Then $I + T$ is invertible.

Ex5.3.4

Let k be the index of T. Prove $I + T$ is invertible by proving $S = I - T + T^2 - \cdots + (-1)^{k-1}T^{k-1}$ is the inverse of $I + T$.

Ex5.3.15

$$\det(I + A) = 1$$

where A if nilpotent.

Ex5.3.7

(a) $A \in \mathbb{R}^{n \times n}$ and $A_{ij} > 0$ for all i, j. Prove that A is not nilpotent.

Ex.5.3.13

Suppose X is a fin.v.sp over F and $T : X \to X$ linear.

T/F If T is nilpotent, then T is singular.

T/F if T is singular, then T is nilpotent.

Ex5.4.8

In defining the Jordan canonical form, we list the vectors in an eigenvector/generalized eigenvector chain in the following order:

$$(A-\lambda_i I)^{r_j-1}x_{i,j}, (A-\lambda_i I)^{r_j-2}x_{i,j}, \cdots, x_{i,j}.$$

What is the form of a Jordan block if we list the vectors in the order

$$x_{i,j}, (A-\lambda_i I)x_{i,j} \cdots, (A-\lambda_i I)^{r_j-1}x_{i,j}$$

instead?

The End