Modules

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Overview

Modules

Exact Sequences

Recall

Suppose $\varphi: B \to C$ is a surjective homomorphism. Then we have a subobject A of B such that $B/A \cong C$.

Now we consider the reverse situation: given A and C, is there B such that A is a sumobject and $B/A \cong C$? If such B exists, we say B is an extension of C by A.

Definition

Suppose a ring has a 1 and X, Y, Z, \cdots are R-modules.

- (1) The pair of homomorphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is said to be exact (at Y) if $\operatorname{Im} \alpha = \operatorname{Ker} \beta$.
- (2) A sequence $\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots$ of homomorphisms is said to be an exact sequence if it is exact at every X_n between a pair of homomorphisms.

Proposition

Let A, B and C be R-modules over some ring R. Then

- (1) The sequence $0 \to A \xrightarrow{\psi} B$ is exact (at A) if and only if ψ is injective.
- (2) The sequence $B \xrightarrow{\varphi} C \to 0$ is exact (at C) if and only if φ is surjective.

Corollary

The sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ is exact if and only if ψ is injective, φ is surjective, and $\operatorname{Im} \psi = \operatorname{Ker} \varphi$.

Definition

The exact sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ is called a *short exact sequence*.

Remark

Suppose $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is exact at Y. Consider the sequence

$$0 \to \operatorname{Im} \alpha \xrightarrow{\iota} Y \xrightarrow{\pi} Y/\operatorname{Ker} \beta \to 0$$

where $\iota:\operatorname{Im} \alpha \to Y$ is an inclusion and $\pi:Y\to Y/\operatorname{Ker}\beta$ is a natural projection. Then this sequence is a short exact sequence. So any exact sequence can be written as a succession of short exact sequences.

- (1) $0 \to A \xrightarrow{\iota} A \oplus C \xrightarrow{\pi} C \to 0$
- (2) $0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\varphi} \mathbb{Z}/n\mathbb{Z} \to 0$, $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$
- (3) If $\varphi: B \to C$ is any homomorphism we may form an exact sequence:

$$0 \to \operatorname{Ker} \varphi \xrightarrow{\iota} B \xrightarrow{\varphi} \operatorname{Im} \varphi \to 0.$$

(4) Suppose M is an R-module and S is a set of generators for M. Let F(S) be the free R-module on S. Then the inclusion $S \to M$ induces a homomorphism $F(S) \to M$. Let K be the kernel of this homomorphism. Then

$$0 \to K \xrightarrow{\iota} F(S) \xrightarrow{\varphi} M \to 0$$

is a short exact sequence.

Definition

Let $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ be two short exact sequences of modules.

(1) A homomorphism of short exact sequences is a triple α, β, γ of module homomorphisms such that the following diagram commutes:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

$$\downarrow^{\alpha} \qquad \downarrow^{\beta} \qquad \downarrow^{\gamma}$$

$$0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0$$

The homomorphism is an isomorphism of short exact sequences if α, β, γ are all isomorphisms, in which case the extensions B and B' are said to be isomorphic extensions.

(2) The two exact sequences are called *equivalent* if A=A', C=C', and there is an isomorphism between them as in (1) that is the identity maps on A and C. (i.e., α and γ are the identity). In this case, the corresponding extensions B and B' are said to be *equivalent* extensions.

(1) m = kn.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

$$\downarrow^{\alpha} \qquad \downarrow^{\beta} \qquad \downarrow^{\gamma}$$

$$0 \longrightarrow \mathbb{Z}/k\mathbb{Z} \xrightarrow{\iota} \mathbb{Z}/m\mathbb{Z} \xrightarrow{\pi'} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

(2) Map each module to itself by $x \mapsto -x$.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

$$\downarrow^{-1} \qquad \downarrow^{-1} \qquad \downarrow^{-1}$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

This is an isomorphism of short exact sequences but is not equivalence of sequences.

(3) Consider the maps

where

- $\psi(a) = (a, 0), \ \varphi(a, b) = b;$
- $\psi'(b) = (0, b), \ \varphi'(a, b) = a.$

If $\beta(a,b)=(b,a)$, this diagram commutes, hence giving an equivalence of the two exact sequences that is not identity isomorphism.

Proposition (The Short Five Lemma)

Let α, β, γ be a homomorphism of short exact sequences

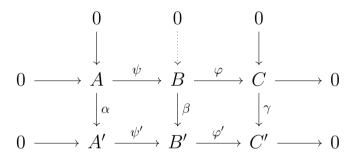
$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\varphi} C \longrightarrow 0$$

$$\downarrow^{\alpha} \qquad \downarrow^{\beta} \qquad \downarrow^{\gamma}$$

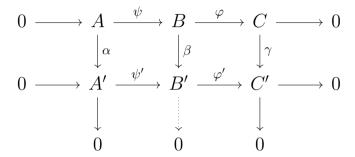
$$0 \longrightarrow A' \xrightarrow{\psi'} B' \xrightarrow{\varphi'} C' \longrightarrow 0$$

- (1) If α and γ are injective, then so is β .
- (2) If α and γ are surjective, then so is β .
- (3) Hence, if α and γ are isomorphisms, then so is β .

Proof (1)



Proof (2)



Definition

Let R be a ring and let $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ be a short exact sequence of R-modules. The sequence is said to be *split* if there is an R-module complement to $\psi(A)$ in B. In this case, up to isomorphism, $B = A \oplus C$ (more precisely, $B = \psi(A) \oplus C'$ for some submodule C, and C' is mapped isomorphically onto C by $\varphi \colon \varphi(C') \cong C$).

Proposition

The short exact sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ of R-modules is split if and only if there is an R-module homomorphism $\mu: C \to B$ such that $\varphi \circ \mu$ is identity map on C.

Definition

- ▶ With notation as in above Proposition, any set map $\mu: C \to B$ such that $\varphi \circ \mu = \mathrm{id}$ is called a *section* of φ .
- If μ is a homomorphism as in Proposition, then μ is called a *splitting homomorphism* for the sequence.

Proposition

Let $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ be a short exact sequence of modules. Then $B = \psi(A) \oplus C'$ for some submodule C' of B with $\varphi(C') \cong C$ if and only if there is a homomorphism $\lambda: B \to A$ such that $\lambda \circ \psi$ is the identity map on A.

Observe

For given exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$, there is a short exact sequence $0 \to \operatorname{Im} f \to B \to B/\operatorname{Ker} g \to 0$. The last term is $B/\operatorname{Ker} g = B/\operatorname{Im} f$. So

$$0 \to \operatorname{Im} f \to B \to B/\operatorname{Im} f \to 0$$

Definition

Let $f:A\to B$ be a R-module homomorphism. Then the *cokernel* of f is the quotient module $B/\operatorname{Im} f$, denoted by $\operatorname{Coker} f$.

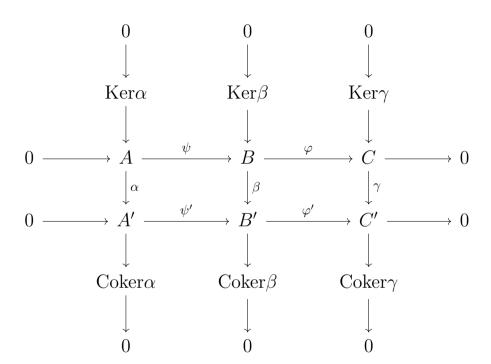
Consider a homomorphism of short exact sequences:

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\varphi} C \longrightarrow 0$$

$$\downarrow^{\alpha} \qquad \downarrow^{\beta} \qquad \downarrow^{\gamma}$$

$$0 \longrightarrow A' \xrightarrow{\psi'} B' \xrightarrow{\varphi'} C' \longrightarrow 0$$

From $B \to B'$, $0 \to \operatorname{Ker}\beta \to B \to B' \to \operatorname{Coker}\beta \to 0$ is a exact sequence.



Lemma (The Snake Lemma)

Suppose

$$\begin{array}{cccc}
A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C & \longrightarrow & 0 \\
\downarrow^{\alpha} & & \downarrow^{\beta} & & \downarrow^{\gamma} \\
0 & \longrightarrow & A' & \xrightarrow{\psi'} & B' & \xrightarrow{\varphi'} & C'
\end{array}$$

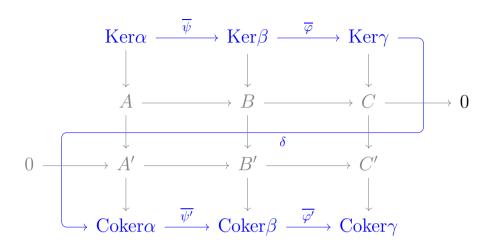
is a commutative diagram of R-modules with exact rows. Then there is a homomorphism $\delta: \operatorname{Ker} \gamma \to \operatorname{Coker} \alpha$, called a connecting map such that

$$\operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \xrightarrow{\delta} \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma$$

is an exact sequence. If ψ is injective and φ' is surjective, then

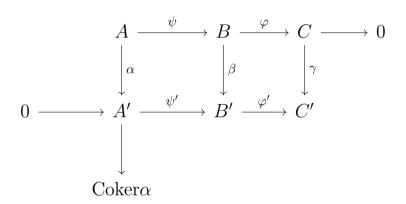
$$0 \to \operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \xrightarrow{\delta} \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma \to 0$$

is exact.



Proof

Existence of δ .



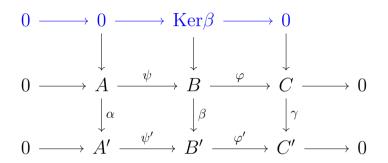
Proof

Exactness

- $ightharpoonup \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha.$
- $\blacktriangleright \operatorname{Ker} \gamma \to \operatorname{Coker} \alpha \to \operatorname{Coker} \beta$

Application

Recall the short five lemma. If α and γ are injective, we have



This implies $Ker\beta = 0$.

Application

If α and γ are surjective, we have

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\varphi} C \longrightarrow 0$$

$$\downarrow^{\alpha} \qquad \downarrow^{\beta} \qquad \downarrow^{\gamma}$$

$$0 \longrightarrow A' \xrightarrow{\psi'} B' \xrightarrow{\varphi'} C' \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow \operatorname{Coker}{\beta} \longrightarrow 0 \longrightarrow 0$$

This implies $B'/\operatorname{Im}\beta=0$, or $\operatorname{Im}\beta=B'$.

Lemma (The Five Lemma)

Consider a commutative diagram of R-modules and homomorphisms such that each row is exact:

$$A_{1} \xrightarrow{\psi_{1}} A_{2} \xrightarrow{\psi_{2}} A_{3} \xrightarrow{\psi_{3}} A_{4} \xrightarrow{\psi_{4}} A_{5}$$

$$\downarrow \alpha_{1} \qquad \downarrow \alpha_{2} \qquad \downarrow \alpha_{3} \qquad \downarrow \alpha_{4} \qquad \downarrow \alpha_{5}$$

$$B_{1} \xrightarrow{\varphi_{1}} B_{2} \xrightarrow{\varphi_{2}} B_{3} \xrightarrow{\varphi_{3}} B_{4} \xrightarrow{\varphi_{4}} B_{5}$$

- (1) If α_1 is surjective and α_2 and α_4 are injective, then α_3 is injective.
- (2) If α_5 is injective and α_2 and α_4 are surjective, then α_3 is surjective.

(1) If α_1 is surjective,

$$0 \longrightarrow \operatorname{Coker} \psi_{1} \xrightarrow{\widetilde{\psi}_{2}} A_{3} \xrightarrow{\psi_{3}} \operatorname{Im} \psi_{3} \longrightarrow 0$$

$$\downarrow^{\widetilde{\alpha}_{2}} \qquad \downarrow^{\alpha_{3}} \qquad \downarrow^{\widetilde{\alpha}_{4}}$$

$$0 \longrightarrow \operatorname{Coker} \varphi_{1} \xrightarrow{\widetilde{\varphi}_{2}} B_{3} \xrightarrow{\varphi_{3}} \operatorname{Im} \varphi_{3} \longrightarrow 0$$

If α_2 and α_4 are injective, so are $\tilde{\alpha}_2$ and $\tilde{\alpha}_4$.

(2) If α_5 is injective,

$$0 \longrightarrow \operatorname{Ker} \psi_{3} \xrightarrow{\iota_{3}} A_{3} \xrightarrow{\psi_{3}} \operatorname{Im} \psi_{3} \longrightarrow 0$$

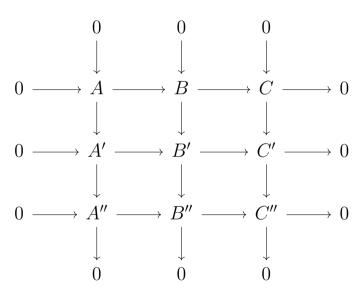
$$\downarrow \tilde{\alpha}_{2} \qquad \qquad \downarrow \tilde{\alpha}_{3} \qquad \qquad \downarrow \tilde{\alpha}_{4}$$

$$0 \longrightarrow \operatorname{Ker} \varphi_{3} \xrightarrow{\jmath_{3}} B_{3} \xrightarrow{\varphi_{3}} \operatorname{Im} \varphi_{3} \longrightarrow 0$$

If α_2 and α_4 are surjective, so are $\tilde{\alpha}_2$ and $\tilde{\alpha}_4$.

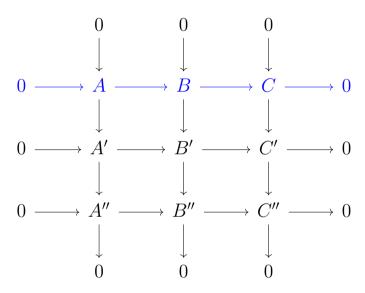
Exercise, 3×3 lemma

Consider the following commutative diagram in R-modules having exact columns.

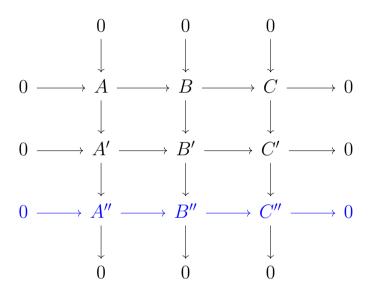


- (1) If the bottom two rows are exact, prove that the top row is exact.
- (2) If the top two rows are exact, prove that the bottom row is exact.

Proof, (1)



Proof, (2)



The End