# Analysis
# - PMA 8 -

KYB

Thrn, it's a Fact

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# Overview

# Limits of Functions

## Definition

Let $X$ and $Y$ be metric spaces; suppose $E \subset X$, $f$ maps $E$ into $Y$, and $p$ is a limit point of $E$. We write $f(x) \to q$ as $x \to p$, or

$$\lim_{x \to p} f(x) = q$$

if there is a point $q \in Y$ with the following property:

- For every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \epsilon$$

  for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta.$$

# Limits of Functions

### Theorem

*Let $X$, $Y$, $E$, $f$, and $p$ as in the above definition. Then*

$$\lim_{x\to p} f(x) = q \text{ if and only if } \lim_{n\to\infty} f(p_n) = q$$

*for every sequence $\{p_n\}$ in $E$ such that $p_n \neq p$ and $\lim_{n\to\infty} p_n = p$.*

### Corollary

*If $f$ has a limit at $p$, this limit is unique.*

# Limits of Functions

## Definition

- Suppose $f, g : E \to \mathbb{C}$ are functions. Define $f + g$ by
$$(f + g)(x) = f(x) + g(x).$$
Similarly, define $f - g$, $fg$. If $g(x) \neq 0$ on $E$, define $f/g$.
- Let $c \in \mathbb{C}$. If $f(x) = c$ for all $x \in E$, we write $f = c$.
- Suppose $f, g$ are real valued functions. If $f(x) \geq g(x)$ for all $x \in E$, we write $f \geq g$.
- If $\mathbf{f}, \mathbf{g} : E \to \mathbb{R}^k$, we can also define $\mathbf{f} + \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$, $\lambda\mathbf{f}$ for real number $\lambda$.

# Limits of Functions

Suppose $E \subset X$, a metric space, $p$ is a limit point of $E$, $f, g : E \to \mathbb{C}$, and

$$\lim_{x \to p} f(x) = A, \quad \lim_{x \to p} g(x) = B.$$

Then

(a) $\lim_{x \to p} (f + g)(x) = A + B$;

(b) $\lim_{x \to p} (fg)(x) = AB$;

(c) $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$, if $B \neq 0$.

If $\mathbf{f}, \mathbf{g} : E \to \mathbb{R}^k$, then (a) remains true, and (b) becomes

(b') $\lim_{x \to p} (\mathbf{f} \cdot \mathbf{g})(x) = \mathbf{A} \cdot \mathbf{B}$.

# Continuous Functions

## Definition

- Suppose $X$ and $Y$ are metric space, $E \subset X$, $p \in E$, and $f : E \to Y$. Then $f$ is said to be "continuous map at $p$" if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \epsilon$$

  for all points $x \in E$ for which $d_X(x, p) < \delta$.
- If $f$ is continuous at every point of $E$, then $f$ is said to be "continuous on $E$".

## Theorem

*In the situation given in the above definition, assume also $p$ is a limit point of $E$. Then $f$ is continuous at $p$ if and only if $\lim_{x\to p} f(x) = f(p)$.*

# Continuous Functions

### Theorem

*Suppose $X$, $Y$, $Z$ are metric spaces, $E \subset X$, $f$ maps $E$ into $Y$, $g$ maps the $f(E)$ into $Z$, and $h$ is the mapping of $E$ into $Z$ defined by*

$$h(x) = g(f(x)).$$

*If $f$ is continuous at a point $p \in E$ and if $g$ is continuous at the point $f(p)$, then $h$ is continuous at $p$.*

# Continuous Functions

### Theorem

*A mapping $f$ of a metric space $X$ into a metric space $Y$ is continuous on $X$ if and only if $f^{-1}(V)$ is open in $X$ for every open set $V$ in $Y$.*

### Corollary

*A mapping $f$ of a metric space $X$ into a metric space $Y$ is continuous on $X$ if and only if $f^{-1}(C)$ is closed in $X$ for every closed set $C$ in $Y$.*

## Continuous Functions

### Theorem

*Let $f$ and $g$ be complex continuous functions on a metric space $X$. Then $f+g$, $fg$, and $f/g$ are continuous on $X$. (in the case of $f/g$, we must assume that $g(x) \neq 0$, for all $x \in X$.)*

### Theorem

(a) *Let $f_1, \cdots, f_k$ be real functions on a metric space $X$, and let $\mathbf{f}$ be the mapping of $X$ into $\mathbb{R}^k$ defined by*

$$\mathbf{f}(x) = (f_1(x), \cdots, f_k(x)),$$

*then $\mathbf{f}$ is continuous if and only if each of the functions $f_1, \cdots, f_k$ is continuous.*

(b) *If $\mathbf{f}$ and $\mathbf{g}$ are continuous mappings of $X$ into $\mathbb{R}^k$, then $\mathbf{f}+\mathbf{g}$ and $\mathbf{f}\cdot\mathbf{g}$ are continuous on $X$.*

# Continuous Functions

## Example

- If $x_1, \cdots, x_k$ are the coordinates of the point $\mathbf{x} \in \mathbb{R}^k$, the functions $\phi_i$ defined by

$$\phi_i(\mathbf{x}) = x_i$$

  are continuous on $\mathbb{R}^k$.
- Every monomial $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ is continuous on $\mathbb{R}^k$ when $n_1, \cdots, n_k$ are nonnegative integers.
- Every polynomial $P$, given by

$$P(\mathbf{x}) = \sum c_{n_1 \cdots n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

  is continuous on $\mathbb{R}^k$.
- Every rational function in $x_1, \cdots, x_k$, that is, every quotient of two polynomials, is continuous on $\mathbb{R}^k$ whenever the denominator is different from zero.
- Recall that a norm $|\cdot|$ is a function from $\mathbb{R}^k$ into $\mathbb{R}$. $|\cdot|$ is continuous on $\mathbb{R}^k$.
- If $\mathbf{f}$ is a continuous mapping from a metric space $X$ into $\mathbb{R}^k$, and if $\phi$ is defined on $X$ by setting $\phi(p) = |\mathbf{f}(p)|$, then $\phi$ is continuous.

# Exercises

## Ex 4.1

Suppose $f$ is a real function defined on $\mathbb{R}$ which satisfies

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}$. Does this imply that $f$ is continuous?

# Exercises

### Ex 4.2

If $f$ is a continuous mapping of a metric space $X$ into a metric space $Y$, prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. Show that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

# Exercises

## Ex 4.3

Let $f$ be a continuous real function on a metric space $X$. Let $Z(f)$ (the zero set of $f$) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

# Exercises

## Ex 4.4

Let $f$ and $g$ be continuous mappings of a metric space $X$ into a metric space $Y$, and let $E$ be a dense subset of $X$. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$.

# Exercises

## Ex 4.5

If $f$ is a real continuous function defined on a closed set $E \subset \mathbb{R}$, prove that there exist continuous real functions $g$ on $\mathbb{R}$ such that $g(x) = f(x)$ for all $x \in E$. Such $g$ is called continuous extension of $f$ from $E$ to $\mathbb{R}$. Show that the result becomes false if the word "closed" is omitted. Extend the result to vector valued functions.

# Exercises

## Ex 4.7

If $E \subset X$ and if $f$ is a function defined on $X$, the restriction of $f$ to $E$ is the function $g$ whose domain of definition is $E$, such that $g(p) = f(p)$ for $p \in E$.
Define $f$ and $g$ on $\mathbb{R}^2$ by

$$f(0,0) = g(0,0) = 0, \quad f(x,y) = \frac{xy^2}{x^2+y^4}, \quad g(x,y) = \frac{xy^2}{x^2+y^6} \text{ if } (x,y) \neq (0,0).$$

Prove that $f$ is bounded on $\mathbb{R}^2$ and $g$ is unbounded in every neighborhood of $(0,0)$, and $f$ is not continuous at $(0,0)$; nevertheless, the restrictions of both $f$ and $g$ to every straight line in $\mathbb{R}^2$ are continuous.

## Continuity and Compactness

### Definition
A mapping $\mathbf{v}$ of a set $E$ into $\mathbb{R}^k$ is said to be bounded if there is a real number $M$ such that $|\mathbf{f}(x)| \leq M$ for all $x \in E$.

### Theorem
*Suppose $f$ is a continuous mapping of a compact metric space $X$ into a metric space $Y$. Then $f(X)$ is compact.*

### Theorem
*If $\mathbf{f}$ is a continuous mapping of a compact metric space $X$ into $\mathbb{R}^k$, then $\mathbf{f}(X)$ is closed and bounded. Thus, $\mathbf{f}$ is bounded.*

# Continuity and Compactness

## Theorem

*Suppose $f$ is a continuous real function on a compact metric space $X$, and*

$$M = \sup_{p \in X} f(p), \quad m = \inf_{p \in X} f(p).$$

*Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.*

## Continuity and Compactness

### Theorem

*Suppose $f$ is a continuous 1-1 mapping of a compact metric space $X$ onto a metric space $Y$. Then the inverse mapping $f^{-1}$ defined on $Y$ by*

$$f^{-1}(f(x)) = x$$

*is a continuous mapping of $Y$ onto $Y$.*

## Continuity and Compactness

### Definition

Let $f$ be a mapping of a metric space $X$ into a metric space $Y$. We say that $f$ is uniformly continuous on $X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \epsilon$$

for all $p, q \in X$ for which $d_X(p, q) < \delta$.

### Theorem

*Let $f$ be a continuous mapping of a compact metric space $X$ into a metric space $Y$. Then $f$ is uniformly continuous on $X$.*

## Continuity and Compactness

**Theorem**

*Let $E$ be a nonempty set in $\mathbb{R}$. Then*

(a) *there exists a continuous function on $E$ which is not bounded;*

(b) *there exists a continuous and bounded function on $E$ which has no maximum.*

*If, in addition, $E$ is bounded, then*

(c) *there exists a continuous function on $E$ which is not uniformly continuous.*

# Continuity and Compactness

## Example

For a bijective continuous map $f : X \to Y$, $f^{-1}$ may not be continuous. Let $d_1$ and $d_2$ be metrics on $\mathbb{R}$ by

$$d_1(p,q) = |p-q|, \quad d_2(p,q) = \begin{cases} 1 & p \neq q \\ 0 & p = q \end{cases}.$$

Define $f : (\mathbb{R}, d_2) \to (\mathbb{R}, d_1)$ by $f(x) = x$ and define $g : (\mathbb{R}, d_1) \to (\mathbb{R}, d_2)$ by $g(x) = x$. Clearly, $f^{-1} = g$ and $g^{-1} = f$. $f$ is continuous but $g$ is not continuous.

# Exercises

## Definition

If $f$ is defined on $E$, the graph of $f$ is the set of points $(x, f(x))$, for $x \in E$. In particular, if $E$ is a set of real numbers, and $f$ is real valued, the graph of $f$ is a subset of the plane.

## Ex 4.6

Suppose $E$ is compact, and prove that $f$ is continuous on $E$ if and only if its graph is compact.

# Exercises

## Ex 4.8

Let $f$ be a real uniformly continuous function on the bounded set $E$ in $\mathbb{R}$. Prove that $f$ is bounded on $E$. Show that the conclusion is false if boundedness of $E$ is omitted from the hypothesis.

# Exercises

### Ex 4.12

A uniformly continuous function of a uniformly continuous function is uniformly continuous.

# Exercises

## Ex 4.13

Let $E$ be a dense subset of a metric space $X$, and let $f$ be a uniformly continuous real function defined on $E$. Prove that $f$ has a continuous extension from $E$ to $X$.

# Continuity of Connectedness

### Theorem

*If $f$ is a continuous mapping of a metric space $X$ into a metric space $Y$, and if $E$ is a connected subset of $Y$, then $f(E)$ is connected.*

# Continuity of Connectedness

### Theorem

*Let $f$ be a continuous real functions on the interval $[a, b]$. If $f(a) < f(b)$ and if $c$ is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.*

# Exercises

## Ex 4.14

Let $I = [0, 1]$ be the closed unit interval. Suppose $f$ is continuous mapping of $I$ into $I$. Prove that $f(x) = x$ for at least one $x \in I$.

# Discontinuities

## Definition

Let $E$ be a subset of a metric space $X$ and let $Y$ be a metric space. Suppose $f : E \to Y$ is a function. Let $x \in E$.

- $f$ is discontinuous at $x$ if $f$ is not continuous at $x$.

## Definition

Let $f$ be defined on $(a, b)$. Consider any point $x$ such that $a \le x < b$.

- We write
$$f(x+) = q$$
if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in $(x, b)$ such that $t_n \to x$.
- Similarly, define $f(x-)$ for $a < x \le b$.
- If $f$ is discontinuous at $x$ and if $f(x+)$ and $f(x-)$ exist, then $f$ is said to have a discontinuity of the first kind, or simple discontinuity, at $x$.
- Otherwise the discontinuity is said to be of the second kind.

## Remark

$\lim_{t \to x} f(t)$ exists if and only if $f(x+) = f(x-) = \lim_{t \to x} f(t)$.

# Discontinuities

## Example

(a) Define

$$f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

(b) Define

$$f(x) = \begin{cases} x & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

(c) Define

$$f(x) = \begin{cases} x+2 & -3 < x < -2 \\ -x-2 & -2 \le x < 0 \\ x+2 & 0 \le x < 1 \end{cases}.$$

(d) Define

$$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

# Exercises

### Ex 4.17

Let $f$ be a real function defined on $(a, b)$. Prove that the set of points at which $f$ has a simple discontinuity is at most countable.

# Exercises

## Ex 4.18

Every rational $x$ can be written in the form $x = m/n$ where $n > 0$ and $m$ and $n$ are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function $f$ defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x \text{ irrational} \\ \frac{1}{n} & x = \frac{m}{n}. \end{cases}$$

Prove that $f$ is continuous at every irrational point, and that $f$ has a simple discontinuity at every point.

# Monotonic Functions

## Definition

Let $f$ be real on $(a, b)$.

- $f$ is said to be monotonically increasing on $(a, b)$ if $a < x < y < b$ implies $f(x) \leq f(y)$.
- $f$ is said to be monotonically decreasing on $(a, b)$ if $a < x < y < b$ implies $f(x) \geq f(y)$.

## Theorem

*Let $f$ be monotonically increasing on $(a, b)$. Then $f(x+)$ and $f(x-)$ exist at every point of $x$ of $(a, b)$. More precisely,*

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t).$$

*Furthermore, if $a < x < y < b$, then $f(x+) \leq f(y-)$.*

## Corollary

*Monotonic functions have no discontinuity of the second kind.*

# Monotonic Functions

### Theorem

*Let $f$ be monotonic on $(a, b)$. Then the set of points of $(a, b)$ at which $f$ is discontinuous is at most countable.*

# Monotonic Functions

## Remark

Given any countable subset $E$ of $(a, b)$, we can construct a function $f$, monotonic on $(a, b)$, discontinuous at every point of $E$, and at no other point of $(a, b)$.

# Exercises

### Definition

A map $f : X \to Y$ is called an open map if $f(V)$ is open in $Y$ whenever $V$ is open in $X$.

### Ex 4.15

Prove that every continuous open mapping of $\mathbb{R}$ into $\mathbb{R}$ is monotonic.

# Infinite Limits and Limits at Infinity

## Definition

For any real $c$, the set of real numbers $x$ such that $x > c$ is called a neighborhood of $+\infty$ and is written $(c, +\infty)$. Similarly, the set $(-\infty, c)$ is a neighborhood of $-\infty$.

## Definition

Let $f$ be a real function defined on $E \subset \mathbb{R}$. We say that

$$f(t) \to A \text{ as } t \to x,$$

where $A$ and $x$ are in the extended real number system, if for every neighborhood $U$ of $A$ there is a neighborhood $V$ of $x$ such that $V \cap E$ is not empty, and such that $f(t) \in U$ for all $t \in V \cap E$, $t \neq x$.

# Infinite Limits and Limits at Infinity

**Theorem**

*Let $f$ and $g$ be defined on $E \subset \mathbb{R}$. Suppose*

$$f(t) \to A, \quad g(t) \to B \quad \text{as } t \to x.$$

*Then*

(a) $f(t) \to A'$ *implies* $A' = A$.

(b) $(f+g)(t) \to A + B$,

(c) $(fg)(t) \to AB$,

(d) $(f/g)(t) \to A/B$.

*where the right members are defined. Note that* $\infty - \infty$, $0 \cdot \infty$, $\infty/\infty$, $A/0$ *are not defined.*

# Exercises

### Ex 4.20

If $E$ is a nonempty subset of a metric space $X$, defined the distance from $x \in X$ to $E$ by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

(a) Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.

(b) Prove that $\rho_E$ is a uniformly continuous function ox $X$, by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

for all $x \in X$, $y \in Y$.

# Exercises

### Ex 4.21

Suppose $K$ and $F$ are disjoint sets in a metric space $X$, $K$ is compact, $F$ is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K$, $q \in F$.

# Exercises

## Ex 4.22

Let $A$ and $B$ be disjoint nonempty closed sets in a metric space $X$, and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)}.$$

Show that $f$ is a continuous function on $X$ whose range lies in $[0, 1]$, that $f(p) = 0$ precisely on $A$ and $f(p) = 1$ precisely on $B$.

# The End