Analysis
- PMA 16 -

KYB

Thrn, it's a Fact

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Overview

Linear Transformations

Definition

(a) A nonempty set $X \subset \mathbb{R}^n$ is a *vector space* if $\mathbf{x} + \mathbf{y} \in X$ and $c\mathbf{x} \in X$ for all $\mathbf{x}, \mathbf{y} \in X$ and for all scalars c.

(b) If $\mathbf{x_1}, \cdots, \mathbf{x}_k \in \mathbb{R}^n$ and $c_1, \cdots, c_k$ are scalars, the vector

$$c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k$$

is called a *linear combination of* $\mathbf{x}_1, \cdots, \mathbf{x}_k$. If $S \subset \mathbb{R}^n$ and if E is the set of all linear combinations of elements of S, we say S *spans* E, or that E *is the span of* S.

(c) A set consisting of vectors $\mathbf{x}_1, \cdots, \mathbf{x}_k$ is said to be *independent* if the relation $c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$ implies that $c_1 = \cdots = c_k = 0$. Otherwise $\{\mathbf{x}_1, \cdots, \mathbf{x}_k\}$ is said to be *dependent*.

(d) If a vector space X contains an indepedent set of r vectors but contains no independent set of $r + 1$ vectors, we say that X *has dimension* r, and write $\dim X = r$.

(e) An independent subset of a vector space X which spans X is called a *basis* of X.

Linear Transformations

Observe

- Every span is a vector space.
- The set $\{\mathbf{0}\}$ is a vector space and its dimension is 0.
- If $B = \{\mathbf{x}_1, \cdots, \mathbf{x}_k\}$ is a basis of X, then every $\mathbf{x} \in X$ has a unique representation of the form $\mathbf{x} = \sum c_j \mathbf{x}_j$. Such a representation exists since B spans X, and it is unique since B is indepdent. The numbers $c_1, \cdots, c_k$ are called the *coordinates of* $\mathbf{x}$ with respect to the basis B.
- The most familar example of a basis is the set $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$. If $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} = (x_1, \cdots, x_n)$, then $\mathbf{x} = \sum x_j \mathbf{e}_j$.

Linear Transformations

Theorem

Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then $\dim X \leq r$.

Corollary

$\dim \mathbb{R}^n = n$.

Linear Transformations

Theorem

Suppose X is a vector space, and $\dim X = n$.

(a) *A set E of n vectors in X spans X if and only if E is independent.*

(b) *X has a basis, and every basis consists of n vectors.*

(c) *If $1 \leq r \leq n$ and $\{\mathbf{y}_1, \cdots, \mathbf{y}_r\}$ is an independet set in X, then X has a basis containing $\{\mathbf{y}_1, \cdots, \mathbf{y}_r\}$.*

Linear Transformations

Definition

A mapping A of vector space X into a vector space Y is said to be a *linear transformation* if

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2, \quad A(c\mathbf{x}) = cA\mathbf{x}.$$

for all $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in X$ and for all scalars c.

Linear Transformations

Observe

- $A\mathbf{0} = \mathbf{0}$ if A is linear.
- Any linear transformation A of X into Y is completley determined by its action on any basis.

Linear Transformations

- If A is a linear operator on X which (i) ono-to-one and (ii) maps X onto X, we say that A is *invertible*.
- In this case, we can define an operaor A^{-1} on X by requiring that $A^{-1}(A\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$.

Theorem

A linear operator A on a finite-dimensional vector space X is one-to-one if and only if the range of A is all of X.

Linear Transformations

Definition

(a) Let $L(X, Y)$ be the et of all linear transformations of the vector space X into the vector space Y. If $X = Y$, simply write $L(X)$. If $A_1, A_2 \in L(X, Y)$ and c_1, c_2 are scalars, $c_1A_1 + c_2A_2$ by

$$(c_1A_1 + c_2A_2)\mathbf{x} = c_1A_1\mathbf{x} + c_2A_2\mathbf{x}.$$

Then $c_1A_1 + c_2A_2 \in L(X, Y)$ (so $L(X, Y)$ is also vector space.)

(b) If X, Y, Z are vector spaces, and if $A \in L(X, Y)$ and $B \in L(Y, Z)$, we define their *product* BA to be the somposition of A and B:

$$(BA)\mathbf{x} = B(A\mathbf{x}).$$

Then $BA \in L(X, Z)$.

(c) For $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, define the *norm* $\|A\|$ of A to be the sup of all numbers $|A\mathbf{x}|$, where $\mathbf{x}$ ranges over all vectors in $\mathbb{R}^n$ with $|\mathbf{x}| \leq 1$.

Linear Transformations

Remark

- $|A\mathbf{x}| \leq \|A\||\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$.
- If λ such that $|A\mathbf{x}| \leq \lambda|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$, then $\|A\| \leq \lambda$.

Theorem

(a) *If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $\|A\| < \infty$ and A is uniformly continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^m$.*

(b) *If $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ and c is scalar, then*

$$\|A + B\| \leq \|A\| + \|B\|, \quad \|cA\| = |c|\|A\|.$$

With the distance between A and B defined as $\|A - B\|$, $L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space.

(c) *If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$, then*

$$\|BA\| \leq \|B\|\|A\|.$$

Linear Transformations

Theorem

Let Ω be the set of all invertible linear operators on $\mathbb{R}^n$.

(a) *If $A \in \Omega$, $B \in L(\mathbb{R}^n)$, and*

$$\|B - A\| \cdot \left\|A^{-1}\right\| < 1,$$

then $B \in \Omega$.

(b) *Ω is an open subset of $L(\mathbb{R}^n)$, and the mapping $A \to A^{-1}$ is conitnuous on Ω.*

Linear Transformations

Matrices

Suppose $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ and $\{\mathbf{y}_1, \cdots, \mathbf{y}_m\}$ are bases of vector spaces X and Y, respectively. Then every $A \in L(X, Y)$ determines a set of numbers a_{ij} such that

$$A\mathbf{x}_j = \sum_{i=1}^{n} a_{ij}\mathbf{y}_i.$$

It is conveinent to visualize these numbers in a rectangle array of m rows and n columns, called an m by n matrix:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Linear Transformations

Matrices

If $\mathbf{x} = \sum c_j \mathbf{x}_j$, then

$$A\mathbf{x} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} c_j \right) \mathbf{y}_i.$$

Thus the coordinated of $A\mathbf{x}$ are $\sum_j a_{ij} c_j$.

Now, suppose that an m by n matrix is given, with real entries a_{ij}. If A is defined by $A\mathbf{x}_j = \sum_i a_{ij} \mathbf{y}_i$, then $A \in L(X, Y)$. Thus there is a natural 1-1 correspondence between $L(X, Y)$ and the set of all m by n matrices.

Linear Transformations

Matrices

If Z is a third vector space, with basis $\{\mathbf{z}_1, \cdots, \mathbf{z}_p\}$, and if

$$B\mathbf{y}_i = \sum_k b_{ki}\mathbf{z}_k, \quad (BA)\mathbf{x}_j = \sum_k c_{kj}\mathbf{z}_k,$$

then $A \in L(X,Y)$, $B \in L(Y,Z)$, $BA \in L(X,Z)$, and since

$$B(A\mathbf{x}_j) = \sum_k \left(\sum_i b_{ki}a_{ij} \right) \mathbf{z}_k,$$

the independence of $\{\mathbf{z}_1, \cdots, \mathbf{z}_p\}$ implies that

$$c_{kj} = \sum_i b_{ki}a_{ij}.$$

This gives the rule of product of two matrices A and B.

Linear Transformations

Matrices

Finally, suppose $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ and $\{\mathbf{y}_1, \cdots, \mathbf{y}_m\}$ are standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, and A is given $A\mathbf{x} = \sum_i (\sum_j a_{ij} c_j)\mathbf{y}_i$. The Schwarz inequality shows that

$$|A\mathbf{x}|^2 = \sum_i \sum_j a_{ij}^2 |\mathbf{x}|^2.$$

Thus

$$\|A\| \leq \left(\sum_i \sum_j a_{ij}^2 \right)^{1/2}.$$

Remark

If S is a metric space, if $a_{11}, \cdots, a_{mn}$ are real continuous functions on S, and if, for each $p \in S$, A_p is the linear transformation of $\mathbb{R}^n$ to $\mathbb{R}^m$ whose matrix has entries $a_{ij}(p)$, then the mapping $p \to A_p$ is a continuous mapping of S into $L(\mathbb{R}^n, \mathbb{R}^m)$.

Exercises

Ex 9.1

If S is a nonempty subset of a vector space X, prove that the span of S is a vector space.

Exercises

Ex 9.2

Prove that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible.

Exercises

Ex 9.3

Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.

Exercises

Ex 9.4

Prove that null spaces and ranges of linear transformations are vector spaces.

Exercises

Ex 9.5

Prove that to every $A \in L(\mathbb{R}^n, \mathbb{R}^1)$ corresponds a unique $\mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $\|A\| = |\mathbf{y}|$.

Differentiation

Preliminaries

If f is a real function with domain $(a,b) \subset \mathbb{R}$ and if $x \in (a,b)$, then $f'(x)$ is usually defined to be the real number

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided that this limits exists. Thus

$$f(x+h) - f(x) = f'(x)h + r(h)$$

where the remainder $r(h)$ is small, in the sense that

$$\lim_{h \to 0} \frac{r(h)}{h} = 0.$$

Differentiation

Preliminaries

Consider a function $\mathbf{f}$ that maps $(a,b) \subset \mathbb{R}$ into $\mathbb{R}^m$. In that case, $\mathbf{f}'(x)$ was defined to be that vector $\mathbf{y} \in \mathbb{R}^m$ for which

$$\lim_{h\to 0}\left(\frac{\mathbf{f}'(x+h)-\mathbf{f}(x)}{h}-\mathbf{y}\right)=\mathbf{0}.$$

We can again rewrite this in the form

$$\mathbf{f}(x+h)-\mathbf{f}(x)=h\mathbf{y}+\mathbf{r}(h),$$

where $\mathbf{r}(h)/h \to \mathbf{0}$ as $h \to 0$.

If $\mathbf{f}$ is a differentialbe mapping of $(a,b) \subset \mathbb{R}$ into $\mathbb{R}^m$, and if $x \in (a,b)$, then $\mathbf{f}'(x)$ is the linear transformation of $\mathbb{R}$ into $\mathbb{R}^m$ that satisfies

$$\lim_{h\to 0}\frac{|\mathbf{f}(x+h)-\mathbf{f}(x)-\mathbf{f}'(x)h|}{|h|}=0.$$

Differentiation

Definition

Suppose E is an open set in $\mathbb{R}^n$, $\mathbf{f}$ maps E into $\mathbb{R}^m$, and $\mathbf{x} \in E$. If there exists a linear transformation A of $\mathbb{R}^n$ into $\mathbb{R}^m$ such that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{|\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) - A\mathbf{h}|}{|\mathbf{h}|} = 0,$$

then we say thet $\mathbf{f}$ is *differentiable* at $\mathbf{x}$, and write

$$\mathbf{f}'(\mathbf{x}) = A.$$

If $\mathbf{f}$ is differentiable at every $\mathbf{x} \in E$, we say that $\mathbf{f}$ is *differentiable* in E.

Differentiation

Theorem

Suppose E is an open set in $\mathbb{R}^n$, $\mathbf{f}$ maps E into $\mathbb{R}^m$, $\mathbf{x} \in E$, and $\mathbf{f}$ is differentiable at $\mathbf{x}$ with $A = A_1$ and with $A = A_2$. Then $A_1 = A_2$.

Differentiation

Remark

(a) If $\mathbf{f}$ is differentiable at $\mathbf{x}$,

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) = \mathbf{f}'(\mathbf{x})\mathbf{h} + \mathbf{r}(\mathbf{h})$$

where

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|\mathbf{r}(\mathbf{h})|}{|\mathbf{h}|}$$

(b) Suppose $\mathbf{f}$ is differentiable in E. Then for every $\mathbf{x} \in E$, $\mathbf{f}'(\mathbf{x})$ is a function: $\mathbf{f}'$ maps E into $L(\mathbb{R}^n, \mathbb{R}^m)$.

(c) $\mathbf{f}$ is continuous at any point at which $\mathbf{f}$ is differentiable.

(d) The derivative $\mathbf{f}'$ is often called the *differential* of $\mathbf{f}$ at $\mathbf{x}$, or the *total derivative* of $\mathbf{f}$ at $\mathbf{x}$.

Differentiation

Example

If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and if $\mathbf{x} \in \mathbb{R}^n$, then

$$A'(\mathbf{x}) = A.$$

Differentiation

Theorem (The Chain Rule)

Suppose E is an open set in $\mathbb{R}^n$, $\mathbf{f}$ maps E into $\mathbb{R}^m$, $\mathbf{f}$ is differentiable at $\mathbf{x}_0 \in E$, $\mathbf{g}$ maps an open set containing $\mathbf{f}(E)$ into $\mathbb{R}^k$, and $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{x}_0)$. Then the mapping $\mathbf{F}$ of E into $\mathbb{R}^k$ defined by

$$\mathbf{F}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$$

is differentiable at $\mathbf{x}_0$, and

$$\mathbf{F}'(\mathbf{x}_0) = \mathbf{g}'(\mathbf{f}(\mathbf{x}_0))\mathbf{f}'(\mathbf{x}_0).$$

Differentiation

Partial Derivatives

Consider a function $\mathbf{f} : E \subset \mathbb{R}^n \to \mathbb{R}$. Let $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ and $\{\mathbf{u}_1, \cdots, \mathbf{u}_m\}$ be the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$. The *components* of $\mathbf{f}$ are the real functions $f_1, \cdots, f_m$ defined by

$$\mathbf{f}(\mathbf{x}) = \sum_{i=1}^{m} f_i(\mathbf{x})\mathbf{u}_j,$$

or, equivalently, by $f_i(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}_i$.
For $\mathbf{x} \in E$, we define

$$(D_j f_i)(\mathbf{x}) = \lim_{t \to 0} \frac{f_i(\mathbf{x} + t\mathbf{e}_j) - f_i(\mathbf{x})}{t},$$

provided the limit exists. Writing $f_i(x_1, \cdots, x_n)$ in place of $f_i(\mathbf{x})$, we see that $D_j f_i$ is the derivative of f_i with respect to x_j, keeping the other variables fixed. The notation $\frac{\partial f_i}{\partial x_j}$ is often used in place of $D_j f_i$, and it is called a *partial derivative*.

Differentiation

Theorem

Suppose $\mathbf{f}$ maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and $\mathbf{f}$ is differentiable at a point $\mathbf{x} \in E$. Then the partial derivatives $(D_i f_j)(\mathbf{x})$ exists, and

$$\mathbf{f}'(\mathbf{x})\mathbf{e}_j = \sum_{i=1}^{m} (D_j f_i)(\mathbf{x})\mathbf{u}_i.$$

Remark

Let $[\mathbf{f}'(\mathbf{x}]$ be the matrix that represents $\mathbf{f}'(\mathbf{x})$ with respect to our standard bases. Then $\mathbf{f}'(\mathbf{x})\mathbf{e}_j$ is the jth column vector of $[\mathbf{f}'(\mathbf{x})]$, and

$$[\mathbf{f}'(\mathbf{x})] = \begin{bmatrix} (D_1 f_1)(\mathbf{x}) & \cdots & (D_n f_1)(\mathbf{x}) \\ \cdots & \cdots & \cdots \\ (D_1 f_m)(\mathbf{x}) & \cdots & (D_n f_m)(\mathbf{x}) \end{bmatrix}.$$

Differentiation

Example

Let γ be a differentiable mapping of the segment $(a, b) \subset \mathbb{R}$ into an open set $E \subset \mathbb{R}^n$. Let f be a real-valued differentiable function with domain E. Thus f is differentiable mapping of E into $\mathbb{R}$. Define

$$g(t) = f(\gamma(t)).$$

The chain rule asserts then that

$$g'(t) = f'(\gamma(t))\gamma'(t).$$

Then $g'(t)$ can be regarded as a real number.

$$g'(t) = \sum_{i=1}^{n} (D_i f)(\gamma(t))\gamma_i'(t).$$

Differentiation

Example

Associate with each $\mathbf{x} \in E$ a vector, the so-called "gradient" of f at $\mathbf{x}$, defined by

$$(\nabla f)(\mathbf{x}) = \sum_{i=1}^{n} (D_i f)(\mathbf{x})\mathbf{e}_i.$$

Since $\gamma'(t) = \sum_{i=1}^{n} \gamma_i'(t)\mathbf{e}_i$,

$$g'(t) = (\nabla f)(\gamma(t)) \cdot \gamma'(t).$$

Differentiation

Example

Let us now fix an $\mathbf{x} \in E$, let $\mathbf{u} \in \mathbb{R}^n$ be a unit vector, and specialize γ so that $\gamma(t) = \mathbf{x} + t\mathbf{u}$ for $-\infty < t < \infty$. Then $\gamma'(t) = \mathbf{u}$ for every t. Hence

$$g'(0) = (\nabla f)(\mathbf{x}) \cdot \mathbf{u}.$$

On the other hand,

$$g(t) - g(0) = f(\mathbf{x} + y\mathbf{u}) - f(\mathbf{x}).$$

Hence

$$\lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x})}{t} = (\nabla f)(\mathbf{x}) \cdot \mathbf{u}$$

The limit is called the *directional derivative* of f at $\mathbf{x}$, denoted by $(D_{\mathbf{u}} f)(\mathbf{x})$.

Differentiation

Example

If f and $\mathbf{x}$ are fixed, but $\mathbf{u}$ varies, then $(D_{\mathbf{u}}f)(\mathbf{x})$ attains its maximum when $\mathbf{u}$ is a positive scalar multiple of $(\nabla f)(\mathbf{x})$.
If $\mathbf{u} = \sum u_i \mathbf{e}_i$, then

$$(D_{\mathbf{u}}f)(\mathbf{x}) = \sum_{i=1}^{n} (D_i f)(\mathbf{x}) u_i.$$

Differentiation

Theorem

Suppose $\mathbf{f}$ maps a convex open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, $\mathbf{f}$ is differentiable in E, and there is a real number M such that

$$\left\|\mathbf{f}'(\mathbf{x})\right\| \le M$$

for every $\mathbf{x} \in E$. Then

$$|\mathbf{f}(\mathbf{b}) - \mathbf{f}(\mathbf{a})| \le M|\mathbf{b} - \mathbf{a}|$$

for every $\mathbf{a}, \mathbf{b} \in E$.

Corollary

If, in addition, $\mathbf{f}'(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in E$, then $\mathbf{f}$ is constant.

Differentiation

Definition

- A differentiable mapping $\mathbf{f}$ of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is said to be *continuously differentiable* in E if $\mathbf{f}'$ is a continuous mapping of E into $L(\mathbb{R}^n, \mathbb{R}^m)$.
- If this is so, we also say that $\mathbf{f}$ is a $\mathscr{C}'$-mapping, or that $\mathbf{f} \in \mathscr{C}(E)$.

Differentiation

Theorem

Suppose $\mathbf{f}$ *maps an open set* $E \subset \mathbb{R}^n$ *into* $\mathbb{R}^m$. *Then* $\mathbf{f} \in \mathscr{C}'(E)$ *if and only if the partial derivatives* $D_j f_i$ *exist and are continuous on* E *for* $1 \leq i \leq m$, $1 \leq j \leq n$.

Exercises

Ex 9.6

Let

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \end{cases}$$

Prove that $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ exist at every point of $\mathbb{R}^2$, although f is not continuous at $(0,0)$.

Exercises

Ex 9.7

Suppose that f a real-valued function defined in an open set $E \subset \mathbb{R}^n$, and that the partial derivatives $D_1 f, \cdots, D_n f$ are bounded in E. Prove that f is continuous in E.

Exercises

Ex 9.8

Suppose that f is a differentiable real function in an open set $E \subset \mathbb{R}^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = 0$.

Exercises

Ex 9.9

If $\mathbf{f}$ is a differentiable mapping of a *connected* open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and if $\mathbf{f}'(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $\mathbf{f}$ is constant in E.

Exercises

Ex 9.10

If f is a real function defined in a convex open set $E \subset \mathbb{R}^n$, such that $(D_1 f)(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $f(x)$ depends only on $x_2, ..., x_n$.

Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like a horseshoe, the statement may be false.

Exercises

Ex 9.11

If f and g are differentiable real functions in $\mathbb{R}^n$, prove that

$$\nabla(fg) = f\nabla g + g\nabla f$$

and that $\nabla(1/f) = -f^{-2}\nabla f$ whenever $f \neq 0$.

Exercises

Ex 9.12

Fix two real numbers a and b, $0 < a < b$. Define a mapping $\mathbf{f} = (f_1, f_2, f_3)$ of $\mathbb{R}^2$ into $\mathbb{R}^3$ by

$$\begin{aligned} f_1(s,t) &= (b + a\cos s)\cos t \\ f_2(s,t) &= (b + a\cos s)\sin t \\ f_3(s,t) &= a\sin s. \end{aligned}$$

Describe the range K of $\mathbf{f}$.

(a) Show that there are exactly 4 points $\mathbf{p} \in K$ such that $(\nabla f_1)(\mathbf{f}^{-1}(\mathbf{p})) = \mathbf{0}$.

Exercises

Ex 9.12

Fix two real numbers a and b, $0 < a < b$. Define a mapping $\mathbf{f}(f_1, f_2, f_3)$ of $\mathbb{R}^2$ into $\mathbb{R}^3$ by

$$f_1(s,t) = (b + a\cos s)\cos t$$
$$f_2(s,t) = (b + a\cos s)\sin t$$
$$f_3(s,t) = a\sin s.$$

Describe the range K of $\mathbf{f}$.

(b) Determine the set of all $\mathbf{q} \in K$ such that $(\nabla f_3)(\mathbf{f}^{-1}(\mathbf{q})) = \mathbf{0}$.

Exercises

Ex 9.12

Fix two real numbers a and b, $0 < a < b$. Define a mapping $\mathbf{f}(f_1, f_2, f_3)$ of $\mathbb{R}^2$ into $\mathbb{R}^3$ by

$$f_1(s,t) = (b + a\cos s)\cos t$$
$$f_2(s,t) = (b + a\cos s)\sin t$$
$$f_3(s,t) = a\sin s.$$

Describe the range K of $\mathbf{f}$.

(c) Show that one of the points $\mathbf{p}$ founded in part (a) corresponds to a local maximum of f_1, one corresponds to a local minimum, and that the other two are neither (they are so-called "saddle points")

Exercises

Ex 9.12

Fix two real numbers a and b, $0 < a < b$. Define a mapping $\mathbf{f}(f_1, f_2, f_3)$ of $\mathbb{R}^2$ into $\mathbb{R}^3$ by

$$\begin{aligned} f_1(s,t) &= (b + a\cos s)\cos t \\ f_2(s,t) &= (b + a\cos s)\sin t \\ f_3(s,t) &= a\sin s. \end{aligned}$$

Describe the range K of $\mathbf{f}$.

(d) Let λ be an irrational real number, and define $\mathbf{g}(t) = \mathbf{f}(t, \lambda t)$. Prove that $\mathbf{g}$ is a 1-1 mapping of $\mathbb{R}$ onto a dense subset of K. Prove that

$$\left|\mathbf{g}'(t)\right|^2 = a^2 + \lambda^2 (b + a\cos t)^2.$$

Exercises

Ex 9.13

Suppose $\mathbf{f}$ is a differentiable mapping of $\mathbb{R}$ into $\mathbb{R}^3$ such that $|\mathbf{f}(t)| = 1$ for every t. Prove that $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$.

Exercises

Ex 9.14

Let

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \dfrac{x^3}{x^2+y^2} & (x,y) \neq (0,0) \end{cases}$$

(a) Prove that $D_1 f$ and $D_2 f$ are bounded functions in $\mathbb{R}^2$.

Exercises

Ex 9.14

Let

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \dfrac{x^3}{x^2+y^2} & (x,y) \neq (0,0) \end{cases}$$

(b) Let $\mathbf{u}$ be any vector in $\mathbb{R}^2$. Show that the directional derivative $(D_{\mathbf{u}} f)(0,0)$ exists, and that its absolute value is at most 1.

Exercises

Ex 9.14

Let

$$f(x,y)=\begin{cases}0 & (x,y)=(0,0)\\ \dfrac{x^3}{x^2+y^2} & (x,y)\neq(0,0)\end{cases}$$

(c) Let γ be a differential mapping of $\mathbb{R}$ into $\mathbb{R}^2$, with $\gamma(0)=(0,0)$ and $|\gamma'(t)|>0$. Put $g(t)=f(\gamma(t))$ and prove that g is differentiable for every $t\in\mathbb{R}$. If $\gamma\in\mathscr{C}'$, prove that $g\in\mathscr{C}'$.

Exercises

Ex 9.14
Let

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \dfrac{x^3}{x^2+y^2} & (x,y) \neq (0,0) \end{cases}$$

(d) In spite of this, prove that f is not differentiable at $(0,0)$.

Exercises

Ex 9.15

Let

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ x^2 + y^2 - 2x^2y - \dfrac{4x^6y^2}{(x^4+y^2)^2} & (x,y) \neq (0,0) \end{cases}$$

(a) Prove, for all $(x,y) \in \mathbb{R}^2$, that

$$4x^4y^2 \leq (x^4+y^2)^2.$$

Conclude that f is continuous,

Exercises

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(b) For $0 \leq \theta \leq 2\pi$, $-\infty < t < \infty$, define

$$g_\theta(t) = f(t\cos\theta, t\sin\theta).$$

Show that $g_\theta(0), g_\theta'(0) = 0, g_\theta''(0) = 2$. Each g_θ has therefore a strict local minimum at $t = 0$.

Exercises

Ex 9.15

Let

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(c) Show that $(0,0)$ is nevertheless not a local minimum for f, since $f(x,x^2) = -x^4$.

The End