Analysis
- PMA 4 -

KYB

Thrn, it's a Fact

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Overview

Compact Sets

Goal

- A subset of $\mathbb{R}^k$ is closed and bounded iff it is compact.

Compact Sets

Definition

Let E be a subset of a metric space X.

- A family $\{G_\alpha\}$ of open subsets in X is called an open cover of E if
$$E \subset \bigcup_\alpha G_\alpha.$$
- A subfamily $\{G_\beta\}$ of $\{G_\alpha\}$ is called an subcover of E if
$$E \subset \bigcup_\beta G_\beta.$$
- If $\{G_\beta\}$ is finite, say $\{G_{\alpha_1}, \cdots, G_{\alpha_n}\}$, it is called a finite subcover of E.

Compact Sets

Definition (Compact Set)

A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

Remark

Every finite subset is compact.

Compact Sets

Theorem

Suppose $K \subset Y \subset X$. *Then* K *is compact relative to* X *iff* K *is compact relative to* Y.

Compact Sets

Theorem

Compact subsets of metric spaces are closed.

Compact Sets

Theorem

Closed subsets of compact sets are compact.

Corollary

If F is closed and K is compact, then $F \cap K$ is compact.

Compact Sets

Theorem

If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.

Corollary

If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$, then $\bigcap_{n=1}^{\infty} K_n$ is not empty.

Compact Sets

Theorem

If E is an infinite subset of a compact set K, then E has a limit point in K.

Compact Sets

Theorem (The Nested Interval Theorem)

If $\{I_n\}$ is a sequence of intervals in $\mathbb{R}^1$, such that $I_n \supset I_{n+1}$, then $\bigcap_{n=1}^{\infty} I_n$ is not empty.

Compact Sets

Theorem

Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$, then $\bigcap_{n=1}^{\infty} I_n$ is not empty.

Theorem

Every k-cell is compact.

Compact Sets

Theorem

If a set E in $\mathbb{R}^k$ has one of the following three properties, then it has the other two:

(a) *E is closed and bounded.*

(b) *E is compact.*

(c) *Every infinite subset of E has a limit point in E.*

The equivalence of (a) and (b) is known as the Heine-Borel Theorem.

Compact Sets

Theorem (Weierstrass)

Every bounded infinite subset of $\mathbb{R}^k$ *has a limit point in* $\mathbb{R}^k$.

Ex 2.10

Let X be an infinite set. Define a metric d by

$$d(p,q) = \begin{cases} 1 & p \neq q \\ 0 & p = q \end{cases}.$$

Which subsets of X is compact?

Exercises

Ex 2.12

Prove that $K = \{0\} \cup \{\frac{1}{n} : n = 1, 2, \cdots\}$ is compact directly from the definition.

Exercises

Ex 2.13

Construct a compact set of real numbers whose limit points form a countable set.

Exercises

Ex 2.15

> Theorem) If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.

Show that the above theorem becomes false if the word "compact" is replaced by "closed" or by "bounded."

Exercises

Ex 2.16

Regard $\mathbb{Q}$ as a metric space with $d(p,q) = |p-q|$. Let $E = \{p \in \mathbb{Q} : 2 < p^2 < 3\}$. Show that E is closed and bounded in $\mathbb{Q}$, but is not compact. Is E open in $\mathbb{Q}$?

Perfect Sets

Goal

- Construct the Cantor Set.

Perfect Sets

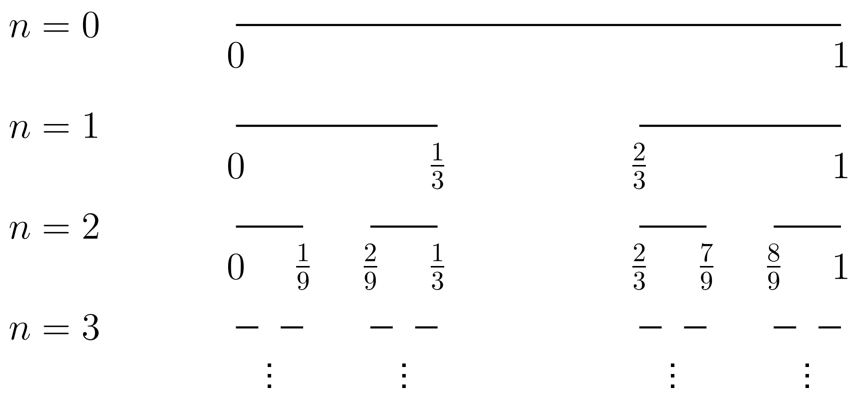
Theorem

Let P be a nonempty perfect set in $\mathbb{R}^k$. Then P is uncountable.

Corollary

Every interval $[a, b]$ $(a < b)$ is uncountable. In particular, $\mathbb{R}$ is uncountable.

The Cantor Set



Perfect Sets

The Cantor Set, Method 1

We will construct a sequence $\{a_{n,k}\}$ for $n = 0, 1, 2, \cdots$ and $k = 0, 1, \cdots, 2^{n+1} - 1$ as follows:

- Define $a_{0,0} = 0, a_{0,1} = 1$.
- Assume $a_{n-1,k}$'s are defined and define

$$a_{n,4k} = a_{n-1,2k}, a_{n,4k+1} = a_{n,4k} + \frac{1}{3^n}, a_{n,4k+2} = a_{n,4k} + \frac{2}{3^n}, a_{n,4k+3} = a_{n-1,2k+1}$$

$$\begin{tikzcd}
a_{n-1,2k} \arrow[rrr, dash, "1/3^{n-1}"] \arrow[d, dash] & & & a_{n-1,2k+1} \arrow[d, dash] \\
a_{n,4k} \arrow[r, dash, "1/3^n"] & a_{n,4k+1} \arrow[r, dash, "1/3^n"] & a_{n,4k+2} \arrow[r, dash, "1/3^n"] & a_{n,4k+3}
\end{tikzcd}$$

Let $I_{n,k} = [a_{n,2k}, a_{n,2k+1}]$ $(k = 0, 1, \cdots, 2^n - 1)$ and $E_n = \cup_k I_{n,k}$. For example,

$$E_0 = [0, 1], E_1 = [\tfrac{0}{3}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{3}{3}], E_2 = [\tfrac{0}{9}, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{3}{9}] \cup [\tfrac{6}{9}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, \tfrac{9}{9}]$$

The Cantor Set, Method 1

Then

(a) $E_1 \supset E_2 \supset E_3 \supset \cdots$;

(b) E_n is the union of 2^n intervals, each of length 3^{-n}

The set

$$P = \bigcap_{n=1}^{\infty} E_n$$

is called the Cantor set. Since each E_n is closed, so is P and since $P \subset [0, 1]$, P is bounded. Hence P is compact.

The Cantor Set, Method 2

$$C = \left\{ \sum_1^\infty a_j 3^{-j} : a_j = 0, 2 \right\}$$

Note that each $x \in [0,1]$ has a unique base-3 decimal expansion unless x is of the form $p3^{-k}$ for some integers p, k; for example

$$\frac{1}{3} = \frac{2}{3}\sum_1^\infty \left(\frac{1}{3}\right)^n = \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^3} + \cdots, \frac{2}{3} = \frac{1}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^3} + \cdots$$

In this case, we can choose an expansion such that $a_j \neq 1$ for all j. Otherwise,

$$a_1 = 1 \iff \tfrac{1}{3} < x < \tfrac{2}{3},$$
$$a_1 \neq 1 \text{ and } a_2 = 1 \iff \tfrac{1}{9} < x < \tfrac{2}{9} \text{ or } \tfrac{7}{9} < x < \tfrac{8}{9},$$

and so forth. Hence, $C = P$ is the Cantor set.

Perfect Sets

Properties of the Cantor Set (Folland)

- Let P be the Cantor set. Then there is no segment of the form

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$$

  has a point in common with P.
- There is no segment which is contained in P.
- Thus for $x \in [0,1]$, $x \in P$ iff x is an end point of for some $I_{n,k}$.
- P is perfect.
- Hence P is uncountable.

Ex 2.18

There is a nonempty perfect set without rationals.

Remark

- Cantor-Bendixson theorem says if F is an uncountable closed set, then $F = P \cup C$ where P is perfect and S is at most countable.
- Enumerate all rational, $\{q_n\}$, and let $G = \cup_1^\infty N_{2^{-n}}(q_n)$. Then $G \neq \mathbb{R}$ because $2\sum_1^\infty 2^{-n} = 2$. Thus $F = G^c$ is closed and uncountable. So there is a perfect set P such that $P \subset F$. Since $\mathbb{Q} \subset G$, $P(\subset F)$ contains no rational.

Connected Sets

Definition

- Two subsets A and B of a metric space X are said to be separated if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty.
- If a subset E of X is not a union of two nonempty separated sets, E is said to be connected.

Example

- $[0,1]$ and $(1,2)$ are not separated.
- $(0,1)$ and $(1,2)$ are separated.

Connected Sets

Theorem

A subset E of the real line $\mathbb{R}^1$ is connected iff it has the following property: if $x, y \in E$ and $x < z < y$, then $z \in E$.

Ex 2.19

(a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

(b) Prove the same for disjoint open sets.

(c) Fix $p \in X$, $\delta > 0$, define A and B to be the sets

$$A = \{q \in X : d(p,q) < \delta\};$$
$$B = \{q \in X : d(p,q) > \delta\}.$$

Prove that A and B are separated.

(d) Prove that every connected metric space with at least two points is uncountable.

The End