Modules

KYB

Thrn, it's a Fact

mathrnfact@gmail.com

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Overview

Direct Products

Let $\{M_i : i \in I\}$ be a collection of R-modules. Then the Cartesian product of $\{M_i\}$ is the set $\prod M_i$ of all choice functions $f : I \to \bigcup M_i$. Write $(m_i)_{i \in I}$ instead for f where $m_i = f(i)$. Define $(m_i) + (m_i')$ and $r(m_i)$ by

$$(m_i) + (m_i') = (m_i + m_i'), r(m_i) = (rm_i).$$

Then $\prod M_i$ has an R-module structure.

Direct Sums

Consider a subset M of $\prod M_i$ such that the set of all $f \in \prod M_i$ such that $\{i : f(i) \neq 0\}$ is finite. Then for $f, g \in M$ and for $r \in R$, $f + g, rf \in M$. Hence M is a submodule of $\prod M_i$. We call M the direct sum of $\{M_i\}$, denoted by $\bigoplus M_i$.

Recall

If I is finite, then $M_1 \times \cdots \times M_n \cong M_1 \oplus \cdots \oplus M_n$.

Proposition

Let I be a nonempty index set and for each $i \in I$, let N_i be a submodule of M. The following are equivalent:

(1) $\sum N_i \cong \oplus N_i$;

(2) *For any finite subset* $\{i_1, \cdots, i_k\}$ *of* I, $N_{i_1} \cap (N_{i_2} + \cdots + N_{i_k}) = 0$;

(3) *For any finite subset* $\{i_1, \cdots, i_k\}$ *of* I, $N_{i_1} + \cdots + N_{i_k} = N_{i_1} \oplus \cdots \oplus N_{i_k}$;

(4) *For every* $x \in \sum M_i$, *there are unique elements* $a_i \in N_i$ *for all* $i \in I$ *such that all but only finite number of* a_i *are zero and* $x = \sum a_i$,

where $\sum M_i$ *is the submodule generated by all* M_i.

Proposition

Suppose F is a free R-module with basis A. Then $F \cong \bigoplus_{a\in A} R$ (copy of R's).

Corollary

Let $\{F_i : i \in I\}$ be a collection of free R-modules. Then $\bigoplus F_i$ is again free.

Example (Direct products and direct sums may differ)

Let $I = \mathbb{Z}^+$. Then $\prod_{i\in I} \mathbb{Z} \neq \bigoplus_{i\in I} \mathbb{Z}$ because the latter is free but the formal is not.

Proposition

Let M_i, M, N_i, N be R-modules.

(1) $\mathrm{Hom}_R(M, \prod N_i) \cong \prod \mathrm{Hom}_R(M, N_i)$.

(2) $\mathrm{Hom}_R(\bigoplus M_i, N) \cong \prod \mathrm{Hom}_R(M_i, N)$.

Proposition

Let M be a right R-module and N_i be left R-modules. Then

$$M \otimes \left(\bigoplus N_i \right) \cong \bigoplus (M \otimes N_i)$$

as groups. If M is (S, R)-bimodule, then the above map is an S-module isomorphism.

The End