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Thrn, it's a Fact

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Overview

Proposition

Let V be a normed vector space over $\mathbb{R}$. Suppose C is a nonempty subset of V. V is closed iff for each sequence $\{x_k\}$ in C which converges to $x \in V$, $x \in C$.

Proof.

($\Longrightarrow$). Let $\{x_k\}$ be a sequence in C converging to $x \in V$. Then for any $\epsilon > 0$, $B_\epsilon(x) \cap C$ is not empty. Since C is closed, $x \notin V - C$, or $x \in C$.

($\Longleftarrow$) Suppose C is not closed. Then there is $x \in V - C$ such that for all $\epsilon > 0$, $B_\epsilon(x) \cap C \neq \varnothing$. For each k, we can choose x_k so that

$$x_k \in B_{1/k}(x) \cap C, \text{ or } \|x_k - x\| < \frac{1}{k}.$$

Then $x_k \to x$. By the assumption, $x \in C$, a contradiction. $\square$

Recall

- A topology $\mathcal{T}$ on X is a collection of subsets of X satisfying
  1. $\varnothing, X \in \mathcal{T}$;
  2. if $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$;
  3. if $\{U_\alpha : \alpha \in J\} \subset \mathcal{T}$, then $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}$.
- A basis $\mathcal{B}$ on X is a collection of subsets of X satisfying
  1. for each $x \in X$, there is $B \in \mathcal{B}$ such that $x \in B$;
  2. if $B_1, B_2 \in \mathcal{B}$ intersects, then for each $x \in B_1 \cap B_2$ there is $B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subset B_1 \cap B_2.$$

Definition

A subbasis $\mathcal{S}$ is a collection of subsets of X satisfying

$$X = \bigcup_{S \in \mathcal{S}} S.$$

Using a subbasis, we can construct a basis as follows:

$$B \in \mathcal{B} \iff \exists S_1, \cdots, S_k \in \mathcal{S} \text{ such that } B = S_1 \cap \cdots \cap S_k.$$

Remark

Every topology is itself a basis and every basis is itself a subbasis. In general, the converses does not hold.

Definition

Let X be a topological space and suppose $\{x_k\}$ is a sequence in X. Let $x \in X$. x_k converges to x if for each open set U containing x, there is N such that $x_n \in U$ for all $n \geq N$.
If $\mathcal{B}$ is a (topological) basis for X, it suffices to check for basis elements instead for all open sets.

Example

1. For a normed vector space V, this definition is equivalent to the usual definition.
2. Let X be infinite set and topolize it by given co-finite topology, i.e. $U \subset X$ is open iff $U = \varnothing$ or $X - U$ is finite. Then for any sequence $\{x_k\}$ converges to any point of X.
3. We know that every nontrivial vector space over $\mathbb{R}$ is infinite. If U is open in co-finite topology, so is in normed space. But the converse does not hold. In this sense, we can weaken the topology (deleting open sets).

Theorem

Let H be an inner product space over $\mathbb{R}$, and let $\overline{B}$ be the closed unit ball in H. Then $\overline{B}$ is sequentially compact if and only if H is finite-dimensional.

Proof

It suffices to show that if H is infinite-dimensional then $\overline{B}$ is not sequentially compact.
Let x_1 be any vector of norm 1 in H. Suppose $\{x_1, \cdots, x_n\}$ are an orthonormal set in H. Let $S_n = \mathrm{span}\{x_1, \cdots, x_n\}$. Choose $y \notin H - S_n$ (we can always find such y because H is infinite dimensional). Define $w = y - \mathrm{proj}_{S_n} y$ and $x_{n+1} = w/\|w\|$. Then $\{x_1, \cdots, x_n, x_{n+1}\}$ is also orthonormal. Thus we can construct a sequence $\{x_k\}$ in $\overline{B}$ which can not converge to any vector in H.

Corollary

If H is an infinite-dimensional Hilbert space and S is a subset of H with a nonempty interior, then S is not sequentially compact.

Weak topology

Definition

Let H be a Hilbert space over $\mathbb{R}$. The weak topology on H is the weakest topology such that each $f \in H^*$ is still continuous. If a sequence in H converges with respect to the weak topology, then it is said to converge weakly or to be weakly convergent.

Theorem (460)

Let H be a Hilbert space over $\mathbb{R}$, and let $\{x_k\}$ be a sequence in H. Then $\{x_k\}$ converges weakly to $x \in H$ if and only if

$$f(x_k) \to f(x) \text{ for all } f \in H^*.$$

By the Riesz representation theorem, we can eqivalently say that $\{x_k\}$ converges weakly to $x \in H$ if and only if

$$\langle x_k, u\rangle_H \to \langle x, u\rangle_H \text{ for all } u \in H.$$

Proof of Theorem 460

($\Longrightarrow$) Since every $f \in H^*$ is still continuous in the weak topology, $f(x_k) \to f(x)$.

($\Longleftarrow$) The weakest topology on H such that every $f \in H^*$ is still continuous means this topology induced by a subbasis $\mathcal{S}$ such that

$$\mathcal{S} = \{f^{-1}(V) : V \text{ is open in } \mathbb{R} \text{ and } f \in H^*\}.$$

(continued)

Proof

Let U be an open in H containing x. Then there $f_1, \cdots, f_k \in H^*$ and open sets $V_1, \cdots, V_k$ in $\mathbb{R}$ such that

$$x \in f_1^{-1}(V_1) \cap \cdots \cap f_k^{-1}(V_k) \subset U.$$

Since $f_i(x_k) \to f_i(x)$ for all i, there is N_i such that

$$|f_i(x_n) - f_i(x)| < \epsilon \text{ for all } n \geq N_i.$$

Choose $\epsilon_i > 0$ so that $(f_i(x) - \epsilon_i, f_i(x) + \epsilon_i) \subset V_i$. Then $x_n \in f_i^{-1}(V_i)$ for all $n \geq N_i$. Take $N = \max\{N_1, \cdots, N_k\}$, and the for all $n \geq N$, $x_n \in U$. Hence $x_k \to x$ in the weak topology on H.

Example

Consider l^2. In l^2, $\{e_k\}$ is not a convergent sequence in the norm topology. But, given any $u \in l^2$, we have $\langle e_k, u\rangle_{l^2} = u_k$ and $u_k \to 0$. It follows that $\{e_k\}$ converges weakly to the zero vector in l^2.
To distinguish convergence in the norm topology from weak convergence, we say $x_k \to x$ strongly or $x_k \to x$ in norm.

For net, see MSG-net

Theorem (462)

Let H be a Hilbert space over $\mathbb{R}$. Then the closed unit ball $\overline{B}$ is sequentially compact in the weak topology.

Corollary (463)

Let H be a Hilbert space over $\mathbb{R}$, and let S be a closed and bounded subset of H. If $\{x_k\}$ is a sequence in S, then there exists a subsequence $\{x_{k_j}\}$ and a vector $x \in H$ such that $x_{k_j} \to x$ weakly.

Ex 10.4.2

Prove Corollary 463.

Proof

Let L be a bound of S, i.e. for all $x \in S$, $\|x\| < M$. Then

$$S' = \{x/M : x \in S\} \subset \overline{B}.$$

Thus $\{x_k/M\}$ has a subsequence $\{x_{k_j}/M\}$ and a vector x such that $x_{k_j}/M \to x$ weakly. Hence $x_{k_j} \to Mx$ weakly.

Theorem (464)

Let H be a Hilbert space over $\mathbb{R}$, and suppose $\{x_k\}$ is a sequence in H converging strongly to $x \in H$. Then $x_k \to x$ weakly.

Proof

Let $y \in H$.

$$|\langle x_k, y\rangle - \langle x, y\rangle| = |\langle x_k - x, y\rangle| \leq \|x_k - x\|\|y\|.$$

Since $x_k \to x$ strongly, $\|x_k - x\|\|y\| \to 0$ as $k \to \infty$. Thus $\langle x_k, y\rangle \to \langle x, y\rangle$ as $k \to \infty$.

Theorem (465)

Let H be a Hilbert space over $\mathbb{R}$, and suppose $\{x_k\}$ is a sequence in H. If $x_k \to x \in H$ weakly and $\|x_k\| \to \|x\|$, then $x_k \to x$ strongly.

Proof

$$\begin{aligned}\|x_k - x\|^2 = \langle x_k - x, x_k - x\rangle &= \langle x_k, x_k\rangle - 2\langle x_k, x\rangle + \langle x, x\rangle \\ &= \|x_k\|^2 - 2\langle x_k, x\rangle + \|x\|^2 \\ &\to \|x\|^2 - 2\langle x, x\rangle + \|x\|^2 = 0.\end{aligned}$$

Definition

Let $\{\alpha_k\}$ be a sequence of real numbsers. The limit inferior of $\{\alpha_k\}$ is defined by

$$\liminf_{k\to\infty} = \lim_{k\to\infty} \inf\{\alpha_l : l \geq k\} = \sup_{k\geq 1} \inf_{l\geq k}\{\alpha_l : l \geq k\}$$

Similarly, we define the limit superior of $\{\alpha_k\}$ as

$$\limsup_{k\to\infty} = \lim_{k\to\infty} \sup\{\alpha_l : l \geq k\} \inf_{k\geq 1} \sup_{l\geq k}\{\alpha_l : l \geq k\}$$

Remark

It is possible that $\inf\{\alpha_k : k \geq 1\} = -\infty$. In this case, $\inf\{\alpha_l : l \geq k\} = -\infty$ for all k. Suppose $-\infty < \inf\{\alpha_k : k \geq 1\}$, $\{\inf\{\alpha_l : l \geq k\}\}$ is a monotonically increasing sequence of real numbers. So $\liminf_{k\to\infty} \alpha_k$ is always exists, possibly $\pm\infty$.

Theorem (467)

Let $\{\alpha_k\}$ be a sequence of real numbers.

1. *There exists a subsequence $\{\alpha_{k_j}\}$ such that*
$$\lim_{j\to\infty} \alpha_{k_j} = \liminf_{k\to\infty} \alpha_k$$
2. *There exists a subsequence $\{\alpha_{k_j}\}$ such that*
$$\lim_{j\to\infty} \alpha_{k_j} = \limsup_{k\to\infty} \alpha_k$$
3. *If $\{\alpha_{k_k}\}$ is any convergent subsequence of $\{\alpha_k\}$, then*
$$\liminf_{k\to\infty} \alpha_k \le \lim_{j\to\infty} \alpha_{k_j} \le \limsup_{k\to\infty} \alpha_k.$$
4. *If $\lim_{k\to\infty} \alpha_k$ exists, then*
$$\liminf_{k\to\infty} \alpha_k = \lim_{j\to\infty} \alpha_k = \limsup_{k\to\infty} \alpha_k.$$

Proof

1 and 2. Let $\alpha = \liminf_{k\to\infty} \alpha_k$. We may assume $\alpha > -\infty$. Since $\inf\{\alpha_l : l \geq k\}$ is a monotonically increasing sequence, for each j we can find j such that

$$\alpha \leq \inf\{\alpha_l : l \geq j\} < \alpha + \frac{1}{j}.$$

For given $\{k_1 < \cdots < k_{j-1}\}$, we can find k_j such that $k_j > k_{j-1}$ and

$$\alpha \leq \alpha_{k_j} < \alpha + \frac{1}{j}.$$

Then $\{\alpha_{k_j}\}$ is a subsequence converging to α. In the same way, you can find a subsequence converging to $\limsup_{k\to\infty} \alpha_k$
(continued)

Proof

3. Suppose $\{\alpha_{k_j}\}$ is a convergent subsequence of $\{\alpha_k\}$. Let $\alpha = \lim_{j\to\infty} \alpha_{k_j}$. Then for any $m > 0$, there is N such that

$$\left|\alpha_{k_j} - \alpha\right| < \frac{1}{m} \text{ for all } j \geq N.$$

For all $j \geq N$, $\alpha_{k_j} < \alpha + \frac{1}{m}$. Then

$$\inf\{\alpha_l : l \geq k_N\} \leq \alpha_{k_j} < \alpha + \frac{1}{m}.$$

Now letting $N \to \infty$, we get

$$\liminf_{N\to\infty}\{\alpha_l : l \geq N\} \leq \alpha + \frac{1}{m}.$$

But m is arbitrary, and thus $\liminf_{k\to\infty}\{\alpha_l : l \geq k\} \leq \alpha$. Similarly, you can show that $\alpha \leq \limsup_{k\to\infty}\{\alpha_l : l \geq k\}$.

Proof

4. Since $\lim_{k\to\infty} \alpha_k$ exists, for any subsequence $\{\alpha_{k_j}\}$,

$$\lim_{j\to\infty} \alpha_{k_j} = \lim_{k\to\infty} \alpha_k.$$

Now by 1 and 2,

$$\liminf_{k\to\infty} \alpha_k = \lim_{j\to\infty} \alpha_k = \limsup_{k\to\infty} \alpha_k.$$

Theorem (468)

Let H be a Hilbert space over $\mathbb{R}$, and let $\{x_k\}$ be a sequence in H converging weakly to $x \in H$. Then

$$\|x\| \leq \liminf_{k\to\infty} \|x_k\|.$$

Proof

We may assume $x \neq 0$. Let $\{x_{k_j}\}$ be a subsequence of $\{x_k\}$ such that

$$\lim_{j\to\infty} \|x_{k_j}\| = \liminf_{k\to\infty} \|x_k\|.$$

Since $\langle x_{k_j}, x\rangle \leq \|x_{k_j}\|\|x\|$,

$$\|x\|^2 = \lim_{j\to\infty} \langle x_{k_j}, x\rangle \leq \lim_{j\to\infty} \|x_{k_j}\|\|x\| = \|x\| \liminf_{k\to\infty} \|x_k\|.$$

Definition

Let X be a normed vector space and let $f : X \to \mathbb{R}$.

1. We say that f is lower semicontinuous at x if
$$f(x) \leq \liminf_{k\to\infty} f(x_k)$$
for all sequences $x_k \to x$.
2. We say that f is upper semicontinuous at x if
$$f(x) \geq \limsup_{k\to\infty} f(x_k)$$
for all sequences $x_k \to x$.

Theorem (470)

Let H be a Hilbert space over $\mathbb{R}$, and let S be a closed and bounded subset of H. If S is also closed with respect to the weak topology, then there exists $\overline{x} \in S$ such that

$$\|\overline{x}\| = \inf\{\|x\| : x \in S\}.$$

Proof

Let $\{x_k\} \subset S$ be a minimizing sequence:

$$\lim_{k\to\infty} \|x_k\| = \inf\{\|x\| : x \in S\}.$$

Since S is closed and bounded, there exists a subsequence $\{x_{k_j}\}$ and $\overline{x} \in H$ such that $x_{k_j} \to \overline{x}$ weakly. Since S is weakly closed, $\overline{x} \in S$. Then

$$\|\overline{x}\| \leq \liminf_{j\to\infty} \left\|x_{k_j}\right\| = \lim_{j\to\infty} \left\|x_{k_j}\right\| = \inf\{\|x\| : x \in S\} \leq \|\overline{x}\|.$$

Convexity

Definition

Let V be a vector space over $\mathbb{R}$, and let C be a subset of V. We say that C is convex if and only if

$$x, y \in C, \alpha \in [0,1] \implies (1-\alpha)x + \alpha y \in C.$$

Definition

Let C be a convex subset of a vector space V over $\mathbb{R}$. We say that $f : C \to \mathbb{R}$ is a convex function if

$$f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y) \text{ for all } x, y \in C, \alpha \in [0,1].$$

Theorem

Let H be a Hilbert space over $\mathbb{R}$, and let C be a nonempty, closed, convex subset of H. For any $x \in H$, there exists a unique $\overline{x} \in C$ such that

$$\|x - \overline{x}\| = \inf\{\|x - z\| : z \in C\}.$$

Moreover, $\overline{x}$ is the unique vector in C satisfying

$$\langle x - \overline{x}, z - \overline{x}\rangle \leq 0 \text{ for all } z \in C.$$

Theorem (474)

Let H be a Hilbert space over $\mathbb{R}$, and let C be a closed convex subset of H. Then C is weakly sequentially closed; that is, if $\{x_k\} \subset C$ converges weakly to $x \in H$, then $x \in C$.

Definition

Let V be a vector space over $\mathbb{R}$, ler S be a subset of V, and suppose $f : S \to \mathbb{R}$. The epigraph of f is the following subset of $S \times \mathbb{R}$:

$$\operatorname{epi}(f) = \{(x, r) \in V \times \mathbb{R} : f(x) \leq r\}.$$

Theorem (476)

Let V be a vector space over $\mathbb{R}$, and let C be a convex subset of V. A function $f : C \to \mathbb{R}$ is convex if and only if $\operatorname{epi}(f)$ is a convex set.

Theorem (477)

Let H be a Hilbert space over $\mathbb{R}$, let S be a closed subset of H, and let $f : S \to \mathbb{R}$. Then f is lower semicontinuous with respect to a given topology if and only if $\operatorname{epi}(f)$ is sequentially closed with respect to that topology.

Theorem (478)

Let H be a Hilbert space over $\mathbb{R}$, let C be a closed and bounded convex subset of H, and let $f : C \to \mathbb{R}$ be convex and lower semicontinuous. Then there exists $\overline{x} \in C$ such that

$$f(\overline{x}) = \inf\{f(x) : x \in C\}.$$

Ex 10.4.10

Let H be a Hilbert space, and let $\{u_k\}$ be an orthonormal sequence in H:

$$\langle u_j, u_k \rangle = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}.$$

(a) Prove Bessel's inequality: For all $x \in H$,

$$\sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2 < \infty.$$

(b) Using Bessel's inequality, prove that $\{u_k\}$ converges weakly to the zero vector.

Proof

(a) Let $S_n = \operatorname{span}\{u_1, \cdots, u_n\}$. Since $\{u_1, \cdots, u_n\}$ is an orthonormal basis for S_n,

$$\left\|\operatorname{proj}_{S_n} x\right\|^2 = \sum_{k=1}^{n} |\langle x, u_k\rangle|^2.$$

Let $x_n = \operatorname{proj}_{S_n} x$. Then $\langle x - x_n, x_n\rangle = 0$.

$$\|x\|^2 = \|x - x_n + x_n\|^2 = \|x - x_n\|^2 + \|x_n\|^2 \geq \|x_n\|^2.$$

Thus $\left\|\operatorname{proj}_{S_n} x\right\|^2 \leq \|x\|^2$ for all n, or

$$\sum_{k=1}^{\infty} |\langle x, u_k\rangle|^2 < \infty.$$

(b) Since $\sum_{k=1}^{\infty} |\langle x, u_k\rangle|^2$ converges, $\langle u_k, x\rangle \to 0 = \langle 0, x\rangle$ for all $x \in H$. Hence $u_k \to 0$ weakly.

Example

$L^2[0, 2\pi]$ is the Hilbert space with the inner product

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)dx.$$

Recall that $f(x) = g(x)$ in $L^2[0, 2\pi]$ means $\{x \in [0, 2\pi] : f(x) \neq g(x)\}$ is a measure zero set. Let $f_n(x) = \sin(nx)/\sqrt{\pi}$ $n \geq 1$ and $g_n(x) = \cos(nx)/\sqrt{\pi}$ $m \geq 1$ and $g_0(x) = \frac{1}{\sqrt{2\pi}}$. Then

$$\langle f_n, f_m\rangle = \delta_{mn} = \langle g_n, g_m\rangle, \langle f_n, g_m\rangle = 0.$$

Thus $\{f_n, g_m\}_{n\geq 1, m\geq 0}$ is an orthonormal set.
(continued)

Example

Let $\tilde{f}(x) = \sum_{n=1}^{\infty} a_n f_n + \sum_{m=0}^{\infty} b_m g_m$ where

$$a_n = \langle f, f_n \rangle, b_m = \langle f, g_m \rangle.$$

By the previous exercise, $a_n, b_m \to 0$ as $n, m \to \infty$ and

$$b_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = |\langle f, g_0 \rangle|^2 + \sum_{n=0}^{\infty} \left(|\langle f, f_n \rangle|^2 + |\langle f, g_n \rangle|^2 \right) < \infty.$$

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