LA7 Derterminant

KYB

Thrn, it's a Fact

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Overview

Ex 3.6.13

Let F be a field and suppose $A \in F^{m\times n}$, $B \in F^{n\times p}$. Prove that $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$.

Ex 3.6.15

Prove that a strictly diagonally dominant matrix $A \in \mathbb{C}^{n\times n}$ is nonsingular.

Proof

Claim) $\mathcal{N}A = \{0\}$.

Suppose not. Then there exists $x \in \mathbb{C}^n$ such that $x \neq 0$ and $Ax = 0$.
(continued)

Proof, continued

Since $x \neq 0$, there exists i such that $|x_i| = \max\{|x_1|, \cdots, |x_n|\} > 0$, then $\frac{|x_j|}{absx_i} \leq 1$.

$$(Ax)_i = \sum_j x_j A_{ij} = 0 \implies -x_i A_{ii} = \sum_{j \neq i} x_j A_{ij}$$

$$\implies |A_{ii}| = \left| \sum_{j \neq i} -\frac{x_j}{x_i} A_{ij} \right| \leq \sum_{j \neq i} \frac{|x_j|}{|x_i|} A_{ij} \leq \sum_{j \neq i} |A_{ij}| < |A_{ii}| \text{(contradiction)}$$

$\mathcal{N}(A) = \{0\}$.

A is invertible.

Ex 3.6.16

Let X and U be vector spaces over a field F, and let $T : X \to U$ be linear.

(a) There exists a left inverse of S of $T \iff T$ is injective.

(b) There exists a right inverse of S of $T \iff T$ is surjective.

Ex 3.6.17

Let $A \in F^{m\times n}$ and $B \in F^{n\times m}$.

- left inverse of A : $BA = I_n$
- right inverse of A : $AB = I_m$

(a) There exists a left inverse of B of $A \iff \mathcal{N}(A) = \{0\}$.

(b) There exists a right inverse of B of $A \iff \operatorname{col}(A) = F^m$.

Ex 3.6.23 $\sim$ 26) Change of Coordinate

1. Let $\mathcal{X}, \mathcal{Y}$ be two bases of X, and let $x \in X$.

$$[x]_{\mathcal{X}} \mapsto [x]_{\mathcal{Y}} = C[x]_{\mathcal{X}}$$

2. $L : X \to X \implies [L]_{\mathcal{X},\mathcal{X}} = \underbrace{\qquad}[L]_{\mathcal{Y},\mathcal{Y}}\underbrace{\qquad}.$
3. $T : X \to U \implies [L]_{\mathcal{X},\mathcal{U}} = \underbrace{\qquad}[L]_{\mathcal{Y},\mathcal{V}}\underbrace{\qquad}.$

Goal: $Ax = b$

Good Cases:

(1) If $A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, $x = [b_i/\lambda_i]$

(2) If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & \\ & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix}$, $\begin{array}{cccc} a_{11}x_1 + & \cdots & +a_{1n}x_n & = & b_1 \\ & & \vdots & & \vdots \\ & & a_{nn} & = & b_n \end{array}$

Goal: $Ax = b$

(3) If there is $X \in F^{n\times n}$ such that X is invertible and A is diagonal or triangular,

$$Ax = b \iff AXx' = b \implies x = Xx'$$

(4)

$$\begin{aligned} AX &= [AX_1|\cdots|AX_n] = [\lambda_1 X_1|\cdots|\lambda_n X_n] \\ &= [X_1|\cdots|X_n]\,\mathrm{Diag}(\lambda_1,\cdots,\lambda_n) = XD. \\ &\implies A = XDX^{-1} \implies AX = b = XDX^{-1}x \implies x = XD^{-1}X^{-1}b \end{aligned}$$

Determinant

Determinant is a generalization of (signed) volume.

1. $\{e_1, \cdots, e_n\} \implies \det(e_1, \cdots, e_n) = 1$
2. $$\det(A_1, \cdots, \lambda A_i, \cdots, A_n) = \lambda \det(A_1, \cdots, A_n)$$
   $$\det(A_1, \cdots, A_i + \lambda A_j, \cdots, A_n) = \det(A_1, \cdots, A_n)$$

$$\begin{aligned}\det(A_1, \cdots, A_n) &= \det(\sum A_{i1}e_i, \sum A_{i2}e_i, \cdots, \sum A_{in}e_i) \\ &= \sum_{i_1}\sum_{i_2}\cdots\sum_{i_n} A_{i_1 1}A_{i_2 2}\cdots A_{i_n n}\det(e_{i_1}, \cdots, e_{i_n})\end{aligned}$$

$\det(e_{i_1}, \cdots, e_{i_n}) \neq 0$ 이려면, $\{i_1, \cdots, i_n\} = \{1, \cdots, n\}$.

Properties of $\det(A)$

1. $\det(\cdots, A_i + \sum_{j \neq i} \lambda_j A_k, \cdots) = \det(A)$.
2. $\det(\cdots, 0, \cdots) = 0$
3. If $\{A_1, \cdots, A_n\}$ is linearly dependent, then $\det(A_1, \cdots, A_n) = 0$.
4. $\det(\cdots, A_j, \cdots, A_i, \cdots) = -\det(A_1, \cdots, A_n)$.
5. $\det(\cdots, A_i + B, \cdots) = \det(A_1, \cdots, A_n) + \det(A_1, \cdots, B, \cdots, A_n)$.

Permuatation

Goal : $\det(A) = \sum_{\tau \in S_n} \operatorname{sgn}(\tau) A_{\tau(1)1}, \cdots, A_{\tau(n)n}.$

정의대로 $\det(A)$를 계산하는 것은 복잡

- 정의 : 이론적으로는 편함
- Permutation 이용 : 실제 계산 편리

S_n is the set of all bijective function from $\{1, \cdots, n\}$ to $\{1, \cdots, n\}$.
(In general, for any set X, the permutation set S_X is $\{$bijective functions $f : X \to X\}$)

- For $\tau \in S_n$, denote $\tau = (\tau(1), \cdots, \tau(n))$. If $\tau = (\tau_1, \cdots, \tau_n)$ where $\tau_i = \tau(i)$, $(\tau_1, \cdots, \tau_n)(i) = \tau_i$.
- Transpose if $(1, \cdots, j, \cdots, i, \cdots, n) = [i, j]$, or

$$[i, j](k) = \begin{cases} k & k \neq i, j \\ j & k = i \\ i & k = j \end{cases}$$

Fact

Every permutation can be written by a composition of transposes.
(주의! 유일하지는 않음, 길이가 다를 수 있음, 그러나 even, odd는 유지)

Example

$$(3\ 5\ 2\ 1\ 4) = [1\ 4][1\ 2][2\ 5][1\ 3] = [1\ 3][2\ 3][2\ 5][4\ 5]$$

Ex 4.1.5

$\tau(4, 3, 2, 1) \in S_4$.

Ex 4.1.7

$A \in F^{2\times 2}$, $\det(A) =?$

Ex 4.1.9

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Longrightarrow \left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array} \right) \Longrightarrow \left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & b_2 - \frac{a_{21}}{a_{11}} b_1 \end{array} \right)$$

Ex 4.1.10

If $\{A_1, \cdots, A_n\}$ is linearly depedent, $\det(\cdots, A_j + B, \cdots) = \det(\cdots, B, \cdots)$.

Ex 4.1.11

Let n be a positive integer, and let i and j be integers satisfying

$$1 \leq i, j \leq n, \quad i \neq j.$$

For any $\tau \in S_m$, define τ' by $\tau' = \tau[i, j]$. Finally, define $f : S_n \to S_n$ by $f(\tau) = \tau'$. Prove that f is a bijection.

Ex 4.1.12

Let $(j_1, \cdots, j_n) \in S_n$. What is

$$\det(A_{j_1}, \cdots, A_{j_n})$$

in terms of $\det(A_1, \cdots, A_n)$?

Ex 4.1.13, Another Definition of Determinants

- $\det(e_1, \cdots, e_n) = 1$
- $\det(\cdots, A_j, \cdots, A_i, \cdots) = -\det(\cdots, A_i, \cdots, A_j, \cdots)$
- $\det(\cdots, \lambda A_j, \cdots) = \lambda) \det(\cdots)$
- $\det(A_1, \cdots, A_j + B_j, \cdots, A_n) = \det(A_1, \cdots, A_j, \cdots, A_n) + \det(A_1, \cdots, B_j, \cdots, A_n)$

Properties of Determinants

- $\det(AB) = \det(A)\det(B)$
- If $\{A_1, \cdots, A_n\}$ is linearly independent, $\det(A) \neq 0$. So A is invertible if and only if $\det(A) \neq 0$.
- $\det(A^{-1}) = (\det(A))^{-1}$, $\det(A^T) = \det(A)$.

Ex 4.2.1

Prove or disprove.

If $A_{ii} =$ for all i, then $\det(A) = 0$.

Ex 4.2.2, Ex 4.2.3

Let $A \in F^{n \times n}$. A^T is singular iff A is singular iff $A^T A$ is singular.

Ex 4.2.4

$n > 0$.

(a) $\sigma(\tau^{-1}) = \sigma(\tau)$ for all $\tau \in S_n$.

(b) $f : S_n \to S_n$ by $f(\tau) = \tau^{-1}$ is bijective.

Ex 4.2.5

Suppose $A, X \in F^{n\times n}$ and X is invertible. Then $\det(X^{-1}AX) = \det(A)$.

Ex 4.2.6

$A, B \in F^{n\times n}$. AB is singular iff A is singular or B is singular.

Ex 4.2.7

$x_1, \cdots, x_n \in F^n$, $A \in F^{n\times n}$.

$$\det(Ax_1, \cdots, Ax_n) = \det(A)\det(x_1, \cdots, x_n).$$

Ex 4.2.8

$A \in F^{m\times n}$, $B \in F^{n\times m}$ and $m > n$. Then $\det(AB) = 0$.

Ex 4.2.9

If $m < n$, show by example that both $\det(AB) = 0$ and $\det(AB) \neq 0$ are possible.

The End