Topology

Thrn
it's a Fact

튜터링

9. Loc.Compact

Claim For any nonempty open $\mathcal{U}$, $\exists V \subset \mathcal{U}$ s.t. $A_n \cap \overline{V} = \phi$, $V \neq \phi$ open.

① Since $\text{Int} A_n \neq \phi$, $\exists y \in \mathcal{U} - A_n \to$ fix y.

② Using Haus, find $\underset{x}{\underset{\Downarrow}{U_x}} \cap \underset{y}{\underset{\Downarrow}{V_x}} = \phi$. if $V_x \not\subset \mathcal{U}$, replace it by $V_x \cap \mathcal{U}$

③ Using cpt choose $x_1, \cdots, x_k$ s.t. $A_n \subset \bigsqcup_{i=1}^{k} U_{x_i}$ and let $V_i = V_{x_i}$

④ Define $V = \prod_{i=1}^{k} V_i$. $\to y \in V$ & V is nonempty open & $V \subset \mathcal{U}$

⑤ Check $A_n \cap \overline{V} = \phi$.

Sps $A \cap B = \emptyset$. Then $\bar{A} \cap B = A \cap \bar{B} = \emptyset$

open

□ Let $x \in \bar{A} \cap B$. Since $A \cap B = \emptyset$, $x \in B$ & it is a lim pt of A.

Since B is open containing x, $B \cap A \neq \emptyset$. (contradiction).

- limit point compact : $\forall$ infinite subset has a lim pt.

- seq compact : $\forall (x_n) \rightsquigarrow \exists (x_{n_k})$ s.t. it is convergent.

$$\text{cpt} \underset{\Longleftarrow}{\Longrightarrow} \text{L cpt} \Longleftrightarrow \text{seq cpt}$$

if metrizable

Ex 28.1 Find an infinite subset of $[0,1]^{\omega}$ uniform s.t. no lim pt.

□ $a_n \in [0,1]^{\omega}$ s.t. $(a_n)_i = \begin{cases} 0 & \text{if } i \neq n \\ 1 & \text{if } i = n \end{cases}$

Ex 28.2 $[0,1] \subseteq \mathbb{R}_\ell$ is not lc cpt.

□ $\{1 - \frac{1}{n} \mid n \in \mathbb{Z}_+\} = A$

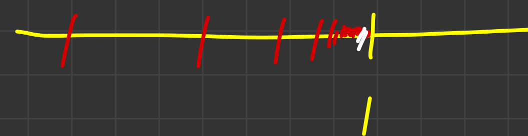
$\underbrace{\forall_a \ \mathbb{R}_\ell}$

$\exists U \cap A = \emptyset$

$\underline{\mathbb{R}}$

$\underbrace{\lim 1 - \frac{1}{n} = 1}$

$[1, 2)$



Ex 28.3 X : lc cpt

(a) If $f: X \to Y$ conti, $f(X)$ lc cpt ?

□ Example 1 $\underbrace{\{a, b\}}_{\text{top'l } \{\emptyset, \{a,b\}\}} \times \mathbb{Z}_+ \xrightarrow[\pi_2]{} \mathbb{Z}_+$

(b) If $A \underset{\text{cld}}{\subseteq} X$, A is lc cpt!

□ Let $B \subset A$ infinite $\to$ $\exists x \in A$ s.t. x is a lim of B.

(c) X is subs of Haus $Z \to X$ is closed in Z ?

□ Example 2

$Z = S_\Omega$, $X = [0, \Omega)$

minimal well-ordered uncountable set

$[0, \Omega)$

countable

$A = \{ x \in X \mid \forall n \in \mathbb{Z}_+, x_n < x \}$

$x < y$

$x \in [0, y)$, $y \in (y, \Omega)$

open

$y \in S_\Omega \iff y < \Omega$

Ordinal number 서수

order 순서 WOP every subset has minimal elt

$$\begin{cases} \phi = 0 \\ 1 = \{\phi\} \\ 2 = \{0, 1\} \\ \vdots \\ n+1 = n \cup \{n\} \end{cases}$$

a set A is Inductive set if

$\phi \in A$ & $x \in A \rightarrow x+1 \in A$.

$$\mathcal{I} = \{A \text{ inductive}\}$$

$$\bigcap_{A \in \mathcal{I}} A \leftarrow \text{minimal inductive set}$$

natural number

order $\quad x < y \iff x \in y$ $\quad$ WOP

$n < n+1 \iff n+1 = n \cup \{n\}$

$$\begin{matrix} n+1 \\ \cup\!\!| \\ n \end{matrix}$$

fin. number $\quad$ ordinal number.

가장 작은 infin ordinal number $\omega = \mathbb{N}$

limit ordinal

$\omega + 1 = \omega \cup \{\omega\}$

$\omega + 2$

$\vdots$

$\omega + n$

$\omega + n$ $\quad$ cble

ucble ordinal number

Ω

$$\bigcup_{n<\omega} \omega + n \longrightarrow$$

minimal ucble ordinal

$<$ $\sim\theta$ top

S_Ω

$$\{[\phi, \alpha), (\alpha, \beta)\}$$

E28.4 Countably compact if $\forall \{U_n\}_{n=1}^{\infty}$ open cover $\to \exists \{U_{n_k}\}_{k=1}^{m}$ subcover

If X T_1 space, cble cpt $\iff$ lt cpt

$\square$ ($\Longleftarrow$) Sps $\{U_n\}_{n=1}^{\infty}$ open cover of X s.t. no fin. subcoll covers X.

Choose $x_n \notin U_1 \sqcup \cdots \sqcup U_n \longrightarrow \{x_1, \cdots\} \ \exists$ lim pt $x \in X$

$\to x \in U_n$ for some $n \to \forall k \geq n$, $x_k \notin U_n$ + $\underbrace{U_n - \{x_1, \cdots, x_{n-1}\}}_{\text{open}} \ni x$

($\Longrightarrow$) Same argument of Thm 28.1.

Let $A \subseteq X$ s.t. A has no lim. Choose $B \overset{\text{cble.}}{\subset} A$ $\quad$ A closed

$\to$ $\underline{A \text{ fin.}}$ $\quad$ B cld

U_n & B^c

Ex 28.6 (X,d): cpt, $f: X \longrightarrow X$ isometry $\Rightarrow$ f is bij.

□ ① f is inj. Easy $\qquad\qquad \{x \in X \mid d(x,a) > r\}$ $r \in \mathbb{R}_+$

② f is surj.

Sps not. $\exists a \notin f(X)$, choose $\varepsilon > 0$ s.t. $B(a,\varepsilon) \cap f(X) = \emptyset$

$x_1 = a$, $x_{n+1} = f(x_n)$ $\Rightarrow$ $m > n$, $d(x_m, x_n) = d(x_{m-n+1}, x_1) > \varepsilon$

$\Rightarrow \{x_1, \cdots\}$

infinite subset has no lim point. $\#$

Ex 28.7 (X, d), $f : X \to X$ satisfies $d(f(x), f(y)) < d(x, y)$ $\forall x \neq y$.

↑ shrinking map

If $\exists\, \alpha < 1$ s.t. $d(f(x), f(y)) \le \alpha\, d(x, y)$ $\forall x, y$ $\to$ f is contraction

fixed point x of f $\Rightarrow$ $f(x) = x$.

Goal: Find a fixed point.

- Check f is conti. (either f is shrinking map or f is contraction)

(a) f is a contraction & X is cpt $\to \exists!$ fixed pt.

□ i) Existence ii) Uniqueness

i) $A_n = f^n(X)$, $A = \bigcap_{n=1}^{\infty} A_n$. $A_1 \supset A_2 \supset \cdots\cdots \to$ F.I.P

& X cpt $\to f(X)$ cpt $\longrightarrow f(X)$ closed $\longrightarrow A_n$ closed

$\to A \neq \emptyset$. Let $a \in A$. $\to f(a) \in A$

Since X cpt $\to d(x,y) \le M$. $d(f(x), f(y)) \le \alpha d(x,y) \le \alpha M$

$$\Longrightarrow d(f^n(x), f^n(y)) \le \alpha^n M$$

$$\Rightarrow d(a, f(a)) \le \alpha^n M \ \forall n \to d(a, f(a)) = 0.$$

(b) If f shrinking map & X cpt $\to$ $\exists!$ fixed pt.

□ Let $\exists x \in A$. (why?) $\checkmark$ $x \in A_{n+1}$

Let x_n s.t. $x = f^{n+1}(x_n)$. Let a be the limit of subseq of $\{\underbrace{f^n(x_n)}_{= y_n}\}$ (why?)

$$a = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f^{n_k}(x_{n_k}) \qquad f(\lim y_n) = \lim f(y_n)$$

$$f(a) = \lim_{k\to\infty} f^{n_k+1}(x_{n_k}) = \lim_{k\to\infty} x = x \in A \subset f(A)$$

$\to A = f(A)$

$$\underset{\text{cpt}}{d : A\times A} \longrightarrow \underset{\text{order}}{\mathbb{R}} \quad \to \exists (x,y)\in A \text{ s.t. } \max\{d(a,b)\} = d(x,y)$$

If $\neq$ $d(x,y) = d(f(x'), f(y')) < d(x',y')$ ↯

$\to d(x,y) = 0$. $\to A = \{a\}$.

- Locally Compact $x \in U \subset C \subset X$

  U: open in X, C: cpt

- One Point Compactification

$$X \text{ loc. cpt. Haus} \iff Y = X \cup \{\infty\} \text{ cpt Haus}$$

$\hookrightarrow$ unique up to homeo s.t. $f|_X = id_X$.

$$\mathcal{T}_Y = \mathcal{T}_X \cup \{ Y - C \mid C \underset{cpt}{\subseteq} X \}$$

Ex 29.3 $f : X \longrightarrow Y$ conti. ① $f(X)$ loc.cpt?

X: loc.cpt

② What if f is open map?

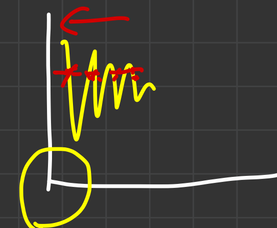
□ ② $x \in U \subset C \to f(x) \in f(U) \subset f(C)$.

① S : topologist's sine curve, $S \cup \{(0,0)\}$ ← not loc. cpt at $(0,0)$

$\{(x, \sin\frac{1}{x}) \mid x \in (0,1]\}$

∄ seq

$(0,1] \hookrightarrow S$



$\{-1\} \cup (0,1] \xrightarrow{\text{conti}} \{(0,0)\} \sqcup S$

$-1 \longmapsto (0,0)$

$-1 \neq x \longmapsto (x, \sin\frac{1}{x})$

loc.cpt

Ex 29.6 $S^1 \simeq \mathbb{R} \cup \{\infty\}$

$[0,1]$

In analysis, extended real $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\} \simeq [0,1]$

order preserving

$\mathbb{R} \xrightarrow{\tan^{-1} x} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow (-\pi, \pi) \longrightarrow S^1 - \{(1,0)\}$

$S^1 : x^2 + y^2 = 1$

$(\cos\theta, \sin\theta)$

$\overline{\mathbb{Z}_+} \simeq \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$

① $\mathbb{Z}_+$: loc. cpt Haus

② $\mathbb{Z}_+ \simeq \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$

③ one pt cpt of $\{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ is $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$

④ By Ex 29.5 , done.

Let $f \in X^{\omega}$ $\longleftrightarrow$ $f : \mathbb{Z}_+ \longrightarrow X$ conti

$\uparrow$ seq. $\qquad\qquad\qquad\qquad$ $\uparrow$ discrete

What is $\lim_{n\to\infty} f(n)$? $\rightsquigarrow$ $\overline{\mathbb{Z}_+} \longrightarrow X$

$\parallel L = f(\infty)$

$\updownarrow$

$\forall$ nbd V of L, $\exists N$ s.t. $f(x_n) \in V$ $\forall n \geq N$

$\{n \geq N\} \rightsquigarrow$ nbd of ∞

(Why? C is cpt in $\mathbb{Z}_+ \iff C$ is finite

$\mathcal{U}$ is nbd of $\infty \iff \mathcal{U} = \overline{\mathbb{Z}_+} - C$, C cpt in $\mathbb{Z}_+$)