LA5 Linear Operator Equations

KYB

Thrn, it's a Fact

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Overview

Ch3. Linear Operators
3.4 Linear Operator Equations
3.5 Existence and Uniqueness of Solutions
The First Isomorphism Theorem

Observe

- Every linear operator $L : F^n \to F^m$, there is $A \in F^{m\times n}$ such that $L(x) = Ax$.
- What is A? Let $\{e_i\}$ be the standard basis for F^n. Define

$$A_i = L(e_i),$$

and then

$$L(\sum x_i e_i) = \sum x_i L(e_i) = \sum x_i A_i = Ax.$$

Ex 3.2.4

Fix $r \in \mathbb{R}$, and let $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L(x, y) = (x + ry, y)$. Then L is linear. Find the matrix representing of L.

Ex 3.2.6

Let $w = \alpha + i\beta$ be a fixed complex number and define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = wz$.

(a) Regarding as a vector space over $\mathbb{C}$, prove that f is linear.

(b) Regard the set $\mathbb{C}$ as identical with $\mathbb{R}^2$, writing (x, y) for $x + iy$. Represent the function f by multiplication by a 2×2 matrix.

Definition

Let X, Y be any sets, and let $f : X \to Y$ be a function.

1. f is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
2. f is surjective if for each $y \in Y$, there is $x \in X$ such that $f(x) = y$.
3. f is bijective if f is both injective and surjective.

Theorem

Let $f : X \to Y$ be a function. Then f is bijective if and only if there is $g : Y \to X$ such that $f \circ g(y) = y$ and $g \circ f(x) = x$. Furthermore, such g is unique.

Ex 3.3.4

Let X, Y, Z be sets, and $f : X \to Y$, $g : Y \to Z$ be bijective functions. Show that $g \circ f$ is bijective and find $(g \circ f)^{-1}$.

Ex 3.3.5

Let X, Y, and Z be sets, and suppose $f : X \to Y$, $g : Y \to Z$ are given functions.

(a) f and $g \circ f$ invertible $\implies$ g invertible?

Injective:

Surjective:

(b) g and $g \circ f$ invertible $\implies$ f invertible?

Injective:

Surjective:

(c) $g \circ f$ invertible $\implies$ f, g invertible?

Definition

A linear map $L : X \to Y$ is called an isomorphism if L is bijective.

Check

L^{-1} is linear.

Definition

Let X, Y be vector spaces. X and Y are isomorphic if there is an isomorphism $L : X \to Y$.

Theorem

Let X, Y, Z be vector spaces over F. Suppose $X \cong Y$ and $Y \cong Z$. Then $X \cong Z$.

Theorem

Suppose X and Y are both n-dimensional. Then $X \cong Y$.

Remark

Let X be an n-dimensional vector space with basis $\mathcal{X} = \{x_1, \cdots, x_n\}$. Define $E_{\mathcal{X}} : X \to F^n$ by $E_{\mathcal{X}}(x_i) = e_i$ and $E_{\mathcal{X}}(\sum \alpha_i x_i) = \sum \alpha_i e_i$. Then $E_{\mathcal{X}}$ is an isomorphism. Note that $E_{\mathcal{X}}$ depends on a basis.

Notation

$E_{\mathcal{X}}(x) = [x]_{\mathcal{X}}$, i.e., $[\cdot]_{\mathcal{X}} : X \to F^n$.

Observe

Suppose $L : (X, \mathcal{X}) \to (U, \mathcal{U})$ be linear map where $\dim X = n$, $\dim U = m$.

(1) What is A?

(2) What if bases change?

$$\begin{array}{ccc} F^n & \xrightarrow{A} & F^m \\ E_{\mathcal{X}} \uparrow & & \uparrow E_{\mathcal{U}} \\ X & \xrightarrow[L]{} & Y \end{array}$$

Notation

$$A = [L]_{\mathcal{X},\mathcal{U}}.$$

Observe

(1) What is $[L]_{\mathcal{X},\mathcal{U}}$?
Let $\{e_1, \cdots, e_n\}$ be the standard basis for F^n.

1. $e_i^n = [x_i]_{\mathcal{X}} = E_{\mathcal{X}}(x_i)$
2. $([L]_{\mathcal{X},\mathcal{U}})_i = [L]_{\mathcal{X},\mathcal{U}} e_i^n = [L]_{\mathcal{X},\mathcal{U}}[x_i]_{\mathcal{X}} = [L(x_i)]_{\mathcal{U}}$. Hence,

$$[L]_{\mathcal{X},\mathcal{U}} = [[L(x_1)]_{\mathcal{U}}|\cdots|[L(x_n)]_{\mathcal{U}}]$$

Observe

(2) What if bases change? Then $[L]_{\mathcal{X},\mathcal{U}}$ will be changed.
Idea)

$$\begin{array}{ccc} (X, \mathcal{X}_1) & \xrightarrow{E_{\mathcal{X}_1}} & F^n \\ \downarrow & & \downarrow M^X_{1,2} \\ (X, \mathcal{X}_2) & \xrightarrow{E_{\mathcal{X}_2}} & F^n \end{array}$$

Similarly, we can find $M^U_{1,2}$. Then

$$\begin{array}{ccc} F^n & \xrightarrow{[L]_{\mathcal{X}_2,\mathcal{U}_2}} & F^m \\ E_{\mathcal{X}_2}\uparrow & & \uparrow E_{\mathcal{U}_2} \\ X & \xrightarrow{L} & U \\ E_{\mathcal{X}_1}\downarrow & & \downarrow E_{\mathcal{U}_1} \\ F^n & \xrightarrow[{[L]_{\mathcal{X}_1,\mathcal{U}_1}}]{} & F^m \end{array}$$

Hence, $[L]_{\mathcal{X}_2,\mathcal{U}_2} = (M^Y_{1,2})[L]_{\mathcal{X}_1,\mathcal{U}_1}(M^X_{1,2})^{-1}$.

Ex 3.3.7

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L(x) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x$. Let $\mathcal{X} = \{(1,1),(1,2)\}$.

(1) $[L]_{\mathcal{S},\mathcal{S}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(2) Find $[L]_{\mathcal{X},\mathcal{X}}$.

Ex 3.3.7

- $[L]_{\mathcal{S},\mathcal{S}}$.

$$[L]_{\mathcal{S},\mathcal{S}} = [[L(e_1)]_{\mathcal{S}}|[L(e_2)]_{\mathcal{S}}] = A.$$

$$(\mathbb{R}^2, \mathcal{S}) \to (\mathbb{R}^2, \mathcal{S})$$
$$e_i \mapsto L(e_i) = \square e_1 + \triangle e_2 = \begin{bmatrix} \square \\ \triangle \end{bmatrix}$$

- $[L]_{\mathcal{X},\mathcal{X}}$

$$[L]_{\mathcal{X},\mathcal{X}} = \begin{bmatrix} {\color{red}2} & {\color{blue}3} \\ {\color{red}0} & {\color{blue}0} \end{bmatrix}$$

because

$$x_1 = (1,1) \mapsto (2,2) = {\color{red}2}(1,1) + {\color{red}0}(1,2)$$
$$x_2 = (1,2) \mapsto (3,3) = {\color{blue}3}(1,1) + {\color{blue}0}(1,2)$$

Ex 3.3.7

- $[L]_{\mathcal{X},\mathcal{S}}$

$$[L]_{\mathcal{X},\mathcal{S}} = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

because

$$\begin{aligned} x_1 = (1,1) &\mapsto (2,2) = 2(1,0) + 2(0,1) \\ x_2 = (1,2) &\mapsto (3,3) = 3(1,0) + 3(0,1) \end{aligned}$$

- $[L]_{\mathcal{S},\mathcal{X}}$

$$[L]_{\mathcal{S},\mathcal{X}} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

because

$$\begin{aligned} e_1 = (1,0) &\mapsto (1,1) = 1(1,1) + 0(1,2) \\ e_2 = (0,1) &\mapsto (1,1) = 1(1,1) + 0(1,2) \end{aligned}$$

Ex 3.3.8

Let $I : \mathbb{R}^2 \to \mathbb{R}^3$ by $I(x) = x$. Let

$$\mathcal{S} = \{(1,0,0),(0,1,0),(0,0,1)\}, \mathcal{X} = \{(1,1,1),(0,1,1),(0,0,1)\}.$$

Then

$$[I]_{\mathcal{X},\mathcal{S}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$\begin{aligned}
(1,1,1) &\mapsto 1\cdot(1,0,0) + 1\cdot(0,1,0) + 1\cdot(0,0,1) \\
(0,1,1) &\mapsto 0\cdot(1,0,0) + 1\cdot(0,1,0) + 1\cdot(0,0,1) \\
(0,0,1) &\mapsto 0\cdot(1,0,0) + 0\cdot(0,1,0) + 1\cdot(0,0,1)
\end{aligned}$$

$$[I]_{\mathcal{S},\mathcal{X}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

$$\begin{aligned}
(1,0,0) &\mapsto 1\cdot(1,0,0) + (-1)\cdot(0,1,0) + 0\cdot(0,0,1) \\
(0,1,0) &\mapsto 0\cdot(1,0,0) + 1\cdot(0,1,0) + (-1)\cdot(0,0,1) \\
(0,0,1) &\mapsto 0\cdot(1,0,0) + 0\cdot(0,1,0) + 1\cdot(0,0,1)
\end{aligned}$$

Ex 3.3.17

$F^{m\times n} \cong F^{mn}$ as follows: Map $A \in F^{m\times n}$ to $a \in F^{mn}$ by

$$\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \mapsto (A_{11}, \cdots, A_{1n}, A_{21}, \cdots, A_{mn})$$

$a_k = A_{ij}$ where $k = n(i-1) + j$.

Ex 3.3.20

Let X, U be finite dimensional vector spaces over F with bases

$$\mathcal{X} = \{x_1, \cdots, x_n\}, \mathcal{U} = \{u_1, \cdots, u_m\}.$$

Let $A \in F^{m \times n}$. Prove that there is a unique linear map $L : X \to U$ such that $[L]_{\mathcal{X},\mathcal{U}} = A$.

Proof

(1) Existence: Let $\{e_1^n, \cdots, e_n^n\}$, $\{e_1^m, \cdots, e_m^m\}$ be the standard bases for F^n and F^m, respectively. Define $L : X \to U$ as follows:

- $L(x_i) = Ae_i^m$
- For $x = \sum_1^n \alpha_i x_i$, $L(x) = \sum_1^n \alpha_i L(x_i)$.

Since $\mathcal{X}$ is a basis, L is well defined. Check that L is a linear map. Then $[L]_{\mathcal{X},\mathcal{U}} = A$.

Ex 3.3.20

Let X, U be finite dimensional vector spaces over F with bases

$$\mathcal{X} = \{x_1, \cdots, x_n\}, \mathcal{U} = \{u_1, \cdots, u_m\}.$$

Let $A \in F^{m\times n}$. Prove that there is a unique linear map $L : X \to U$ such that $[L]_{\mathcal{X},\mathcal{U}} = A$.

Proof

(2) Uniqueness: Let $T : X \to U$ be linear such that $[T]_{\mathcal{X},\mathcal{U}} = A$. Then

$$[T(x_i)]_{\mathcal{U}} = [T]_{\mathcal{X},\mathcal{U}}[x_i]_{\mathcal{X}} = Ae_i = [L]_{\mathcal{X},\mathcal{U}}[x_i]_{\mathcal{X}} = [L(x_i)]_{\mathcal{U}}$$

Thus $T(x_i) = L(x_i)$ for all i. Now let $x = \sum \alpha_i x_i$. Then

$$T(x) = T(\sum \alpha_i x_i) = \sum \alpha_i T(x_i) = \alpha_i L(x_i) = L(\sum \alpha_i x_i) = L(x).$$

Hence, $T = L$.

Definition

Let X, U be vector spaces over a field F, and let $L : X \to U$ be linear.

- $\ker(L) = \{x \in X : L(x) = 0\}$.
- $\mathcal{R}(L) = \{L(x) : x \in X\} = \{u \in U : L(x) = u \text{ for some } x \in X\}$.

Theorem

$\ker(L)$ *is a subspace of* X *and* $\mathcal{R}(L)$ *is a subspace of* U*.*

Goal

Find a solution of $L(x) = b$.

Example

ODE: Consider $u(t)x'' + v(t)x' + w(t)x = f(t)$ where

- $u, v, w, f : \mathbb{R} \to \mathbb{R}$ are continuous
- $x : \mathbb{R} \to \mathbb{R}$ by $t \mapsto x(t)$ belongs to $C^2(\mathbb{R})$.

Let $L = u\frac{d^2}{dt^2} + v\frac{d}{dt} + w$. Then $L : C^2(\mathbb{R}) \to C(\mathbb{R}$ is a linear operator. We want to solve $L(x) = f$.

Lemma

Let $L : X \to U$ be linear. Suppose $u \in U$. If $\hat{x}$ solves to $L(x) = u$, then for all $y \in \ker L$, $\hat{x} + y$ solve to $L(x) = u$. In this case, $\hat{x} + y$ is called a general solution to $L(x) = u$.

Lemma

If x_1, x_2 are solutions to $L(x) = u$, then $x_1 - x_2 \in \ker L$.

Definition

For two subsets S, T of a vector space U and $x \in U$,

- $S + T = \{s + t : s \in S, t \in T\}$
- $x + T = \{x + t : t \in T\}$.

Ex 3.4.2

Let $L : C^2(\mathbb{R}) \to C(\mathbb{R})$ be a linear differential operator, and let f in $C(\mathbb{R})$ be defined by $f(t) = 2(1-t)e^t$. Suppose $x_1(t) = t^2 e^t$ and $x_2(t) = (t^2+1)e^t$ are solutions of $L(x) = f$. Find two more solutions of $L(x) = f$.

Ex 3.4.6

(a) Let X and U be vector spaces over a field F, and let $L : X \to U$ be linear. Suppose that $b, c \in U$, $y \in X$ is a solution to $L(x) = b$, and $z \in X$ is a solution to $L(x) = c$. Find a solution to $L(x) = \beta b + \gamma c$, where $\beta, \gamma \in F$.

Ex 3.4.7

Let $A \in \mathbb{Z}_2^{3\times 3}$, $b \in \mathbb{Z}_2^3$ be defined by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

List all solutions to $Ax = 0$ and $Ax = b$. Check $\hat{x} + \ker L$.

Ex 3.4.7

(1) Find $\hat{x}$ such that $L(\hat{x}) = b$.

Proof.

$\hat{x} = (1, 0, 1)$. $\square$

(2) Find $\ker L$.

Proof.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{cases} x_1 + x_2 \phantom{{}+x_3} = 0 \\ x_1 \phantom{{}+x_2} + x_3 = 0 \\ \phantom{x_1+{}} x_2 + x_3 = 0 \end{cases}$$

Then $x_1 = x_2 = x_3$. Thus $\ker L = \{(0,0,0), (1,1,1)\}$. Hence

$$\hat{x} + \ker L = \{(1,0,1), (0,1,0)\}.$$

$\square$

Definition

Let A and B be any sets.

- A relation R of A and B is a subset of $A \times B$.
- If $(a, b) \in R$, we write aRb.

Example

$=$, $<$ (for $A = B$), a function $f : A \to B$, etc.

Observe

Consider '$=$' on a set X.

(1) For all $x \in X$, $x = x$ (Reflexive)

(2) If $x = y$, then $y = x$ (Symmetric)

(3) If $x = y$ and $y = z$, then $x = z$ (Transitive)

We want to generalize "$=$", say "equivalence relation".

Definition

Let $\sim$ be a relation on X such that

(1) For all $x \in X$, $x \sim x$ (Reflexive)

(2) If $x \sim y$, then $y \sim x$ (Symmetric)

(3) If $x \sim y$ and $y \sim z$, then $x \sim z$ (Transitive)

Then $\sim$ is called an equivalence relation.

Example

For $\mathbb{Z}$ and $n \in \mathbb{Z}$, define a relation $\sim$ as follows:

$$a \sim b \iff n|b-a \iff \exists k \in \mathbb{Z} \text{ such that } b-a=kn.$$

(1) For all $a \in \mathbb{Z}$, $a-a=0 \cdot n$. So $a \sim a$.

(2) If $a \sim b$ with $b-a=kn$, then $a-b=(-k)n$. So $b \sim a$.

(3) If $a \sim b$ with $b-a=kn$ and $b \sim c$ with $c-b=ln$, then $c-b=(l+k)n$. So $a \sim c$.

Definition

For an equivalence relation $\sim$ on X and $x \in X$, define a set $[x]$ by

$$[x] = \{y \in X : y \sim x\}.$$

$[x]$ is called a equivalence class of x.

Properties of Equivalence Class

(1) $X = \bigcup_{x \in X} [x]$.

(2) If $[x] \cap [y] \neq \varnothing$, then $[x] = [y]$. (need not $x = y$.)

Proof.

(1) Clearly, $\bigcup_{x \in X} [x] \subset X$. Let $y \in X$. Since $y \sim y$, $y \in [y]$. So $\bigcup_{x \in X} [x] \supset X$.

(2) Suppose $[x] \cap [y] \neq \varnothing$. Then there is $z \in [x] \cap [y]$. So $z \sim x$ and $z \sim y$. Then $x \sim z$ and $z \sim y$ implies $x \sim y$. Thus $x \in [y]$. If $w \in [x]$, $w \sim x$ and $x \sim y$ implies, $w \sim y$. So $[x] \subset [y]$. In the same reason, $[y] \subset [x]$. Hence $[x] = [y]$.

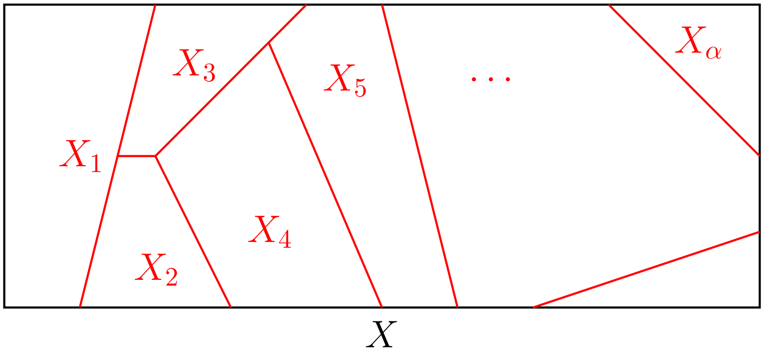
$\square$

Definition

Let X be a set. For some index set J and $\{X_\alpha : X_\alpha \subset X, \alpha \in J\}$. We call $\{X_\alpha\}_{\alpha \in J}$ a partition of X if

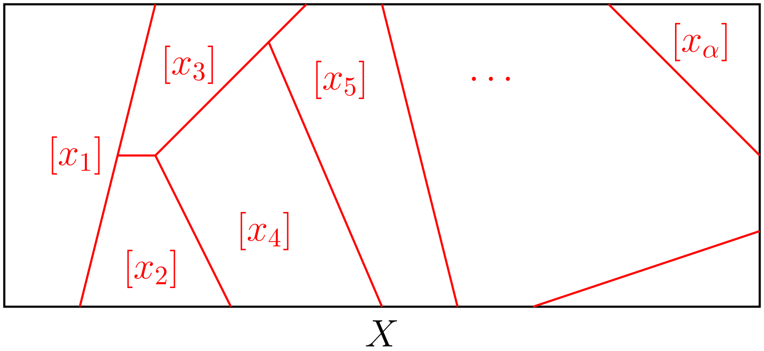
(1) $X = \bigcup_{\alpha \in J} X_\alpha$

(2) If $X_\alpha \cap X_\beta \neq \varnothing$, then $\alpha = \beta$.



Equivalence Relation and Partition

Suppose $\sim$ is an equivalence relation on X. We can choose $\{x_\alpha : x_\alpha \in X\}$ such that $\{[x_\alpha] : \alpha \in J\}$ is a partition of X. Note that for each equivalence relation, there is a corresponding partition, and vice versa.



Definition

Suppose $\sim$ is an equivalence relation on X. A quotient set of X by $\sim$ is a set $X/\sim$ of all equivalence class of X,

$$X/\sim = \{[x] : x \in X\}.$$

Remark

Suppose $x \sim y$. It may not $x = y$, but $[x] = [y]$. In this sense, an equivalence relation is a generalization of equality.

Example

Consider $\mathbb{Z}, \sim, n = 3$.

- $[0] = \{3k : k \in \mathbb{Z}\}$. So $[0] = [3k]$.
- $[1] = \{3k + 1 : k \in \mathbb{Z}\}$. So $[1] = [3k + 1]$.
- $[2] = \{3k + 2 : k \in \mathbb{Z}\}$. So $[0] = [3k + 2]$.

Write $[0] = 3\mathbb{Z}$, or $0 + 3\mathbb{Z}$, $[1] = 1 + 3\mathbb{Z}$, $[2] = 2 + 3\mathbb{Z}$.
In general, we can write $\mathbb{Z}/\sim = \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$.

$$\mathbb{Z}/n\mathbb{Z} = \{[0], [1], \cdots, [n-1]\} = \{\overline{0}, \overline{1}, \cdots, \overline{n-1}\}$$

The First Isomorphism Theorem

Let $L : X \to U$ be linear. Define a relation $\sim$ by

$$x_1 \sim x_2 \iff x_1 - x_2 \in \ker L.$$

Then $\sim$ is an equivalence relation and define $X/\ker L = X/\sim$. Write $[\hat{x}] = \hat{x} + \ker L$. Suppose $\hat{x}$ is a solution to $L(x) = b$. Then every vector in $\hat{x} + \ker L$ also solves $L(x) = b$. Thus we can define $\tilde{L} : X/\ker L \to \mathcal{R}(L)$ by $\tilde{L}([x]) = L(x)$. Then $\tilde{L}$ is an isomorphism.

The End