LA2 Topology

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Thrn, it's a Fact

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Overview

Definition (Topology)

Let X be a nonempty set. A topology $\mathcal{T}$ is a subset of $\mathcal{P}(X)$ satisfying:

- $\varnothing, X \in \mathcal{T}$;
- if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$;
- if $\{U_\alpha : \alpha \in J\} \subset \mathcal{T}$, then $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}$.

We call $(X, \mathcal{T})$ (simply X) a topological space. An element U in $\mathcal{T}$ is said to be open. Roughly speaking, a topology is a collection of open sets.

Example

- Any nonempty set X has the trivial topology $\mathcal{T}_1 = \{\varnothing, X\}$ and the discrete topology $\mathcal{T}_2 = \mathcal{P}(X)$.
- Let $X = \{a, b\}$ be two points set. Then $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}\}$ is a topology.
- Let X be an infinite set. Define $\mathcal{T} = \{U \subset X : U = \varnothing$ or $X - U$ is finite$\}$. Then $\mathcal{T}$ is a topology, say co-finite topology (or finite complement topology).
- Let X be an uncountable set. Define $\mathcal{T} = \{U \subset X : U = \varnothing$ or $X - U$ is countable (or finite)$\}$. Then $\mathcal{T}$ is a topology, say co-countable topology (or countable complement topology).

Definition (Basis)

Let X be a nonempty set. A basis $\mathcal{B}$ is a subset of $\mathcal{P}(X)$ satisfying:

- for each $x \in X$, there is $B \in \mathcal{B}$ such that $x \in B$;
- if $B_1, B_2 \in \mathcal{B}$ intersect, then for each $x \in B_1 \cap B_2$ there is $B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subset B_1 \cap B_2.$$

We call an element in $\mathcal{B}$ a basis element.
Using a basis $\mathcal{B}$, we can construct a topology $\mathcal{T}$ as follows:
$U \in \mathcal{T}$ iff for each $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$.

Example

- Every topology is itself a basis.
- Let $\mathcal{B}_p = \{B_{r,p}(x) : r > 0, x \in \mathbb{R}^n\}$ be the set of all open balls in $\mathbb{R}^n$. Then $\mathcal{B}_p$ is a basis. So there is a topology $\mathcal{T}$ induced from $\mathcal{B}_p$, say the standard topology on $\mathbb{R}$. Moreover, for all $p \geq 1$, every $\mathcal{B}_p$ induces the same topology, i.e. let $\mathcal{T}_p$ be the induced topology by $\mathcal{B}_p$. Then $\mathcal{T}_{p_1} = \mathcal{T}_{p_2}$.

Example

We can give several topologies on $\mathbb{R}$.

- $\mathcal{T}_1$: induced topology by $\mathcal{B}_1 = \{(a,b) : a < b\}$. (standard topology)
- $\mathcal{T}_2$: induced topology by $\mathcal{B}_2 = \{[a,b) : a < b\}$. (lower limit topology)
- $\mathcal{T}_3$: induced topology by $\mathcal{B}_3 = \{(a,b] : a < b\}$. (upper limit topology)
- $\mathcal{T}_4$: co-countable topology.
- $\mathcal{T}_5$: co-finite topology.

1. $\mathcal{T}_1 \subsetneq \mathcal{T}_2$ and $\mathcal{T}_1 \subsetneq \mathcal{T}_3$
2. $\mathcal{T}_4 \not\subset \mathcal{T}_i$ and $\mathcal{T}_4 \not\supset \mathcal{T}_i$ for $i = 1, 2, 3, 5$.
3. $\mathcal{T}_2 \not\subset \mathcal{T}_3$ and $\mathcal{T}_2 \not\supset \mathcal{T}_3$.
4. $\mathcal{T}_5 \subsetneq \mathcal{T}_i$ for $i = 1, 2, 3, 4$.

Definition

Let X be a set. A metric is a function $d : X \times X \to \mathbb{R}$ such that

- $d(x, y) \geq 0$ for all $x, y \in X$; and $d(x, y) = 0$ iff $x = y$;
- $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Given a metric d, define a ball of radius r centered at $x \in X$ by

$$B_d(x, r) = \{y \in X : d(x, y) < r\}.$$

The set of all balls in X forms a basis. The induced topology from a metric is called a metric space.

Example

- Let V be a normed vector space. Then $d(x,y) = \|x-y\|$ is a metric.
- Given a metric d, define $\overline{d}(x,y) = \min\{d(x,y),1\}$. Then $\overline{d}$ is a metric, say a bounded metric of d. $\overline{d}$ induces the same topology induced by d. Thus every norm induces a metric but the converse does not hold.
- For any set X, consider $\mathbb{R}^X$. We can define a metric on $\mathbb{R}^X$ by

$$\overline{\rho}(f,g) = \sup\{\overline{d}(f(x),g(x)) : x \in X\}.$$

- Suppose X is a nonempty compact subset of $\mathbb{R}$, for example a closed interval $[a,b]$. Then every continuous function $f : X \to \mathbb{R}$ is bounded. So we can define a metric on $C(X)$ by

$$\rho(f,g) = \sup\{d(f(x),g(x)) : x \in X\}.$$

Definition (Continuity)

Let X and Y be two topological spaces. Suppose $f : X \to Y$ is a function. f is continuous if for any open set V in Y, $f^{-1}(V)$ is open in X.
If f is bijective and f^{-1} is also continuous, then f is called a homeomorphism.

Example

Consider $\mathbb{R}$ with $\mathcal{T}_1$ and $\mathcal{T}_2$ Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x$.

- $f : (\mathbb{R}, \mathcal{T}_i) \to (\mathbb{R}, \mathcal{T}_i)$ is homeomorphism.
- $f : (\mathbb{R}, \mathcal{T}_1) \to (\mathbb{R}, \mathcal{T}_2)$ is not continuous.
- $f : (\mathbb{R}, \mathcal{T}_2) \to (\mathbb{R}, \mathcal{T}_1)$ is continuous but not homeomorphism.

Example

Let X be any set and let $(Y, \mathcal{T}_Y)$ be a topological space. Suppose $f : X \to Y$ is a surjective function. We can define a topology $\mathcal{T}_X$ on X by

$$\mathcal{T}_X = \{f^{-1}(V) : V \in \mathcal{T}_Y\}.$$

- $\varnothing = f^{-1}(\varnothing)$, $X = f^{-1}(Y)$.
- $f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2)$.
- $\bigcup f^{-1}(V_\alpha) = f^{-1}(\bigcup V_\alpha)$.

Then $\mathcal{T}_X$ is the smallest topology which makes f continuous.

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