Top14
Compactness in Metric
Pointwise and Compact Convergence

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Thrn, it's a Fact

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Overview

Theorem

There exists a continuous map $f : I \to I^2$ whose image fills up the entire square I^2.

Comment

We say a continuous map $f : I \to X$ is a *curve* in X. In this sense, a curve may not be a line.

Definition

A metric space (X, d) is said to be **totally bounded** if for every $\epsilon > 0$, there is a finite covering of X by ϵ-balls.

Theorem (45.1)

A metric space (X, d) is compact if and only if it is complete and totally bounded.

Theorem 27.3

A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the euclidean metric d or the square metric ρ.

Definition

Let (Y, d) be a metric space. Let $\mathcal{F} \subset \mathcal{C}(X, Y)$. If $x_0 \in X$, the set $\mathcal{F}$ is said to be **equicontinuous at** x_0 if given $\epsilon > 0$, there is a neighborhood U of x_0 such that for all $x \in U$ and for all $f \in \mathcal{F}$,

$$d(f(x), f(x_0)) < \epsilon.$$

If $\mathcal{F}$ is equicontinuous at x_0 for each $x_0 \in X$, it is said simply to be **equicontinuous**.

Lemma (45.2)

Let X be a space; let (Y,d) be a metric space. If $\mathcal{F} \subset \mathcal{C}(X,Y)$ is totally bounded undet the uniform metric corresponding to d, then $\mathcal{F}$ is equicontinuous under d.

Lemma (45.3)

Let X be a space; let (Y,d) be a metric space; assume X and Y are compact. If $\mathcal{F} \subset \mathcal{C}(X,Y)$ is equicontinuous under d, then $\mathcal{F}$ is totally bounded under the uniform and $\sup$ metrics corresponding to d.

Definition

If (Y, d) is a metric space, a subset $\mathcal{F} \subset \mathcal{C}(X, Y)$ is said to be **pointwise bounded** under d if for each $a \in X$, the subset

$$\mathcal{F}_a = \{f(a) : f \in \mathcal{F}\}$$

of Y is bounded under d.

Theorem (45.4 Ascoli's theorem, classical version)

Let X be a compact space; let $(\mathbb{R}^n, d)$ denote euclidean space in either the square metric or the euclidean metric; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding uniform topology. $\mathcal{F} \subset \mathcal{C}(X, \mathbb{R}^n)$ has compact closure if and only if $\mathcal{F}$ is equicontinuous and pointwise bounded under d.

Corollary (45.5)

Let X be compact; let d denote either the square metric or the euclidean metric on $\mathbb{R}^n$; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding uniform topology. $\mathcal{F} \subset \mathcal{C}(X, \mathbb{R}^n)$ is compact if and only if it is closed, bounded under the $\sup$ metric ρ, and equicontinuous under d.

Ex45.1

If X_n is metrizable with metric d_n, then

$$D(\boldsymbol{x}, \boldsymbol{y}) = \sup\{\bar{d}_i(x_i, y_i)/i\}$$

is a metric for the product space $X = \prod X_n$. Show that X is totally bounded under D if each X_n is totally bounded under d_n. Conclude without using the Tychonoff theorem that a countable product of compact metrizable spaces is compact.

Proof

Recall that

- (X, D) is compact if and only if it is complete and totally bounded.
- For given sequence $\{\boldsymbol{x}_n\}$ of X, $\boldsymbol{x}_n \to \boldsymbol{x}$ if and only if $\pi_i(\boldsymbol{x}_n) \to \pi_i(\boldsymbol{x})$ for each i.
- If $\{\boldsymbol{x}_n\}$ is Cauchy sequence, $\{\pi_i(\boldsymbol{x}_n)\}$ is also Cauchy sequence.(P.265 Theorem 43.4)

Definition

Given a point x of the set X and an open set U of Y, let

$$S(x,U) = \{f : f \in Y^X \text{ and } f(x) \in U\}.$$

The sets $S(x,U)$ are a subbasis for topology on Y^X, which is called the **topology of pointwise convergence**, or the **point-open topology**.

For $f \in Y^X$, there are $(x_1,U_1),\cdots,(x_k,U_k)$ such that

$$f \in \bigcap_{i=1}^{k} S(x_i,U_i).$$

Note

The topology of pointwise convergence on Y^X is nothing but the product topology.

Consider a product topology $\prod X_\alpha$ for $\{X_\alpha = Y\}_{\alpha \in J}$ and $J = X$. Then $\boldsymbol{x} \in \prod X_\alpha$ is a function $\boldsymbol{x} : X \to Y$ such that $\boldsymbol{x}(\alpha) = x_\alpha$. For (α, U), $S(\alpha, U)$ is the set of all $\boldsymbol{x} : X \to Y$ such that $\boldsymbol{x}(\alpha) \in U$. Then

$$S(\alpha, U) = \pi_\alpha^{-1}(U).$$

Theorem (46.1)

A sequence f_n of functions converges to the function f in the topology of pointwise convergence if and only if for each $x \in X$, the sequence $f_n(x)$ of points of Y converges to the point $f(x)$.

Concept of Convergence

Let $f_n : X \to \mathbb{R}$ be a sequence of function and $f : X \to \mathbb{R}$ be a function.

- If for each x, $f_n(x) \to f(x)$, we say $f_n \to f$ **pointwise**.
- If for any $\epsilon > 0$ there is N such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$ and $n \geq N$, we say $f_n \to f$ **uniformly**.
- If $\int |f_n(x) - f(x)|^p dx \to 0$, $f_n \to f$ in L^p **sense**.
- ect.

Definition

(Y, d). Given $f \in Y^X$, a compact subspace C of X, and a number $\epsilon > 0$, let $B_C(f, \epsilon)$ denote the set of all those elements $g \in Y^X$ for which

$$\sup\{d(f(x), g(x)) | x \in C\} < \epsilon.$$

Then $B_C(f, \epsilon)$ form a basis for a topology on Y^X. Is is called the **topology of compact convergence**.

Theorem (46.2)

$f_n : X \to Y$ converges to f in the topology of compact convergence if and only if for each compact subspace C of X, $f_n\big|_C$ converges uniformly to $f\big|_C$.

Definition

X is said to be **compactly generated** if it satisfies the following condition:

- A set A is open in X if $A \cap C$ is open in C for each compact subspace C of X.

Equivalently, B is closed in X if $B \cap C$ is closed in C for each compact C.

Lemma (46.3)

If X is locally compact, or if X satisfies the first countability axtiom, then X is compactly generated.

Proof

Suppose X is locally compact. Let $A \cap C$ be open in C for evey compact subspace C of X. Given $x \in A$, choose a nbd U of x that lies in a compact subspace C of X. Since $A \cap C$ is open in C, $A \cap U$ is open in U, and hence open in X. Then $A \cap U$ is a nbd of x contained in A, so that A is open in X.

Suppose that X satisfies the first countability axtiom. Let $B \cap C$ be closed in C for each compact subspace C of X. Let $x \in \overline{B}$. Since X has a countable basis at x, there is a sequence (x_n) of points of B converging to x. The subspace

$$C = \{x\} \cup \{x_n : n \in \mathbb{Z}_+\}$$

is compact, so that $B \cap C$ is closed in C. Since $B \cap C$ contains x_n for every n, it contains x as well. Therefore $x \in B$.

Lemma (46.4)

If X is compactly generated, then $f : X \to Y$ is continuous if for each compact subspace C of X, the restricted function $f\big|_C$ is continuous.

Theorem (46.5)

LEt X be a compactly generated space: let (Y,d) be a metric soace. Then $\mathcal{C}(X,Y)$ is closed in Y^X in the topology of compact convergence.

Theorem (43.6)

X : top'l. (Y,d) metric.

- *$\mathcal{C}(X,Y)$ is closed in Y^X under $\bar{\rho}$.*
- *$\mathcal{B}(X,Y)$ is closed in Y^X under $\bar{\rho}$.*

Therefore, if Y is complete, these spaces are complete in $\bar{\rho}$.

Corollary (46.6)

Let X be a compactly generated space; let (Y, d) be a metric space. If a sequence of continuous functions $f_n : X \to Y$ converges to f in the topology of compact convergence, then f is continuous.

Theorem

Let X be a space; let (Y, d) be a metric space. For the function space Y^X, one has the following includsions of topologies:

$$\text{(uniform)} \supset \text{(compact convergence)} \supset \text{(pointwise convergence)}.$$

If X is compact, the first two coincide.
If X is discrete, the second two coincide.

Now observe $\mathcal{C}(X,Y)$ not whole Y^X.

Definition

If C is a compact subspace of X and U is an open subset of Y, define

$$S(C,U) = \{f : f \in \mathcal{C}(X,Y) \text{ and } f(C) \subset U\}.$$

$S(C,U)$ form a subbasis for a topology on $\mathcal{C}(X,Y)$ that is called the **compact-open topology**.

By the definition,

$$\text{(compact-open)} \supset \text{(pointwise convergence)}.$$

Theorem (46.8)

(Y, d). On the set $\mathcal{C}(X, Y)$, the compact-open topology and the topology of compact convergence coincide.

Corollary (46.8)

Let Y be a metric space. The compact convergence topology on $\mathcal{C}(X, Y)$ does not depend on the metric of Y. Therefore if X is compact, the uniform topology on $\mathcal{C}(X, Y)$ does not depend on the metric of Y.

Observe

In vector calculus, we define two concepts of the product of two vectors.

- Dot Product : $\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$
- Cross Product : $\boldsymbol{x} \times \boldsymbol{y} = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$.

Roughly speaking, the result of dot product is a number and the result of cross product is a vector.
In this sence, consider $\mathcal{C}(\mathbb{R}, \mathbb{R})$. For $r \in \mathbb{R}$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, we already showed that rf is again continuous function.(view as a cross product).
On the other hand, $f(r)$ is a number.(view as a dot product).

Definition

$$e : X \times \mathcal{C}(X,Y) \to Y$$

by $e(x,f) = f(x)$ is called the **evaluation map**.

Theorem (46.10)

Let X be locally compact Hausdorff; let $\mathcal{C}(X,Y)$ have the compact open topology. Then the evaluation map is continuous.

Proof

Let $(x,f) \in X \times \mathcal{C}(X,Y)$ and V be open in Y such that $e(x,f) \in V$. Since f is continuous and X is locally compact Hausdorff, there is nbd U of x such that $f(\overline{U}) \subset V$. Then $U \times S(\overline{U},V)$ is an open set containing (x,f). If (x',f') belongs to this set, $e(x',f') = f'(x') \in V$, as desired.

Definition

Given a function $f : X \times Z \to Y$, there is a corresponding function $F : Z \to \mathcal{C}(X,Y)$, defined by the equation

$$(F(z))(x) = f(x,z).$$

Conversely, given $F : Z \to \mathcal{C}(X,Y)$, this equation defines a corresponding function $f : X \times Z \to Y$. We say that F is the map of Z into $\mathcal{C}(X,Y)$ that is **induced** by f.

$$\mathrm{Map}(X \times Z, Y) \rightleftarrows \mathrm{Map}(Z, \mathcal{C}(X,Y))$$

Theorem (46.11)

Give $\mathcal{C}(X,Y)$ the compact-open topology. If $f : X \times Z \to Y$ is continuous, then so is the induced function $F : Z \to \mathcal{C}(X,Y)$. The converse holds if X is locally compact Hausdorff.

Definition

Let $f, g : X \to Y$ be continuous. $h : X \times [0,1] \to Y$ is called a **homotopy** if

- h is continuous
- $h(x,0) = f(x)$, $h(x,1) = g(x)$ for all $x \in X$.

If there is a homotopy between f and g, we say f and g are homotopic.
A homotopy h induces a map

$$H : [0,1] \to \mathcal{C}(X,Y)$$

Definition

Let $f, g : [0,1] \to X$ be paths with $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$.
$h : [0,1] \times [0,1] \to X$ is called a **path-homotopy** if

- h is a homotopy of f and g
- $h(0,t) = x_0$ and $h(1,t) = x_1$ for all $t \in [0,1]$.

Ex46.3

Show that $\mathcal{B}(\mathbb{R},\mathbb{R})$ is closed in $\mathbb{R}^{\mathbb{R}}$ in the uniform topology, but not in the topology of compact convergence.

Proof

1)(P.267 Theorem 43.6)
2) $f_n(x) = \min\{x^2, n\}$ converges to $f(x) = x^2$.

Ex46.5

Consider the sequence of functions $f_n : (-1, 1) \to \mathbb{R}$, defined by

$$f_n(X) = \sum_{k=1}^{n} kx^k.$$

(a) Show that f_n converges in the topology of compact convergence; conclude that the limit function is continuous.
(b) Show that f_n does not converges in the uniform topology.

Proof

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\left(\sum_{k=0}^{\infty} x^k\right)' = \sum_{k=1}^{\infty} kx^{k-1}$$

$$f(x) = \sum_{k=1}^{\infty} kx^k = x\sum_{k=1}^{\infty} kx^{k-1} = \frac{x}{(1-x)^2}$$

Thus f_n converges pointwise.

Proof

Every compact subspace C of $(0,1)$ is contained in $[-a,a]$ for some $0<a<1$. For $x \in [-a,a]$,

$$\begin{aligned}
|f(x)-f_n(x)| &= \left|\sum_{k=1}^{\infty} kx^k - \sum_{k=1}^{n} kx^k\right| = \left|\sum_{k=n+1}^{\infty} kx^k\right| \\
&\leq \sum_{k=n+1}^{\infty} ka^k = |f(a)-f_n(a)| \\
&< \epsilon
\end{aligned}$$

Thus f_n converges in the topology of compact convergence; moreover for each compact subspace C, $f_n\big|_C \to f\big|_C$ uniformly, thus $f\big|_C$ is continuous. Hence f is continuous.
On the other hand $f(x)$ is not uniform continuous, thus f_n does not converge to f in the uniform topology.

Definition

We say A has **empty interior** if A contains no open set of X other than the empty set. Equivalently, $X - A$ is dense in X.

Definition

X is said to be a **Baire space** if the following condition holds:

- Given any countable collection $\{A_n\}$ of closed sets of X each of which has empty interior in X, their union $\bigcup A_n$ also has empty interior in X.

Example

$\mathbb{Q}$ is not a Baire space. Every one-point set in $\mathbb{Q}$ is closed and has empty interior in $\mathbb{Q}$, but $\bigcup_{q \in \mathbb{Q}} \{q\} = \mathbb{Q}$ has non empty interior.
On the other hand, $\mathbb{Z}_+$ is a Baire space because it has no subsets having empty interior.

Definition

We say a subset A of X is of the **first category** in X if it was contained in the union of a countable collection of closed sets of X having empty interiors in X; otherwise, it was said to be of the **second category** in X.

Then X is a Baire space if and only if every nonempty open set in X is of the second category.

Lemma (48.1)

X is a Baire space if and only if given any countable collection $\{U_n\}$ of open sets in X, each of which is dense in X, their intersection $\bigcap U_n$ is also dense in X.

Theorem (48.2 Baire category theorem)

If X is a compact Hausdorff space or complete metric space, then X is a Baire space.

Theorem (48.3)

Let $C_1 \supset C_2 \supset \cdots$ be a nested sequence of nonempty closed sets in the complete metric space X. If diam$C_n \to 0$, then $\bigcap C_n \neq \varnothing$.

Lemma (48.4)

Any open subspace Y of a Baire space X is itself a Baire space.

Theorem (48.5)

Let X be a space; let (Y, d) be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for all $x \in X$, where $f : X \to Y$. If X is a Baire sapce, the set of points at which f is continuous is dense in X.

Ex48.1

Let X equal the countable union $\bigcup B_n$. Show that if X is a nonempty Baire space, at leash one of the sets $\overline{B}_n$ has a nonempty interior.

Ex48.6

Show that the irrationals are a Baire space.

Ex48.8

If f_n is a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$, show that f is continuous at uncountably many points of $\mathbb{R}$.

Ex48.9

Let $g : \mathbb{Z}_+ \to \mathbb{Q}$ be a bijective function; let $x_n = g(n)$. Define $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$\begin{cases} f(x_n) = 1/n & \text{for } x_n \in \mathbb{Q} \\ f(x) = 0 & \text{for } x \notin \mathbb{Q} \end{cases}$$

Show that f is continuous at each irrational and discontinuous at each rational. Can you find a sequence of continuous functions f_n converging to f?