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KYB

Thrn, it's a Fact

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Overview

6.2 The adjoint of a linear operator

Ex6.2.9

Let $M : \mathcal{P}_2 \to \mathcal{P}_3$ be defined by $M(p) = q$, where $q(x) = xp(x)$. Find M^*, assuming that the $L^2(0,1)$ inner product is imposed on both $\mathcal{P}_2$ and $\mathcal{P}_3$.

6.4 The projection theorem

Ex6.4.1

Let $A \in \mathbb{R}^{m\times n}$.

(a) Prove that $\mathcal{N}(A^T A) = \mathcal{N}(A)$.

(b) If A has full rank($\mathrm{rank}(A) = n$), then $A^T A$ is invertible.

(c) If A has full rank, then $Ax = y$ has a unique least-squares solution for each $y \in \mathbb{R}^m$, namely, $x = (A^T A)^{-1} A^T y$.

Ex6.4.2

Let $A \in \mathbb{R}^{m\times n}$ and $y \in \mathbb{R}^m$ be given, and assume that A has full rank. Then $\{A_1, \cdots, A_n\}$ is a basis for $\mathrm{col}(A)$. Show that $A^T A$ is the Gram matrix for $\{A_1, \cdots, A_n\}$.

Ex6.4.11

Let $A \in \mathbb{R}^{m \times n}$ and $y \in \mathbb{R}^m$ be given. Prove that $A^T A x = A^T y$ always has a solution.

Ex6.2.11

Let X and U be finite-dimensional inner product spaces over $\mathbb{R}$, and suppose $T : X \to U$ is linear. Defines $S : \mathcal{R}(T^*) \to \mathcal{R}(T)$ by $S(x) = T(x)$.

Ex6.4.13

Assume $A \in \mathbb{R}^{m\times n}$. Ex6.2.11 implies that $\mathrm{col}(A^T) \subset \mathbb{R}^n$ and $\mathrm{col}(A) \subset \mathbb{R}^m$ are isomorhpic. That exercise shows that $S : \mathrm{col}(A^T) \to \mathrm{col}(A)$ defined by $S(x) = Ax$ is an isomorphism.

(a) Show that $R : \mathrm{col}(A) \to \mathrm{col}(A^T)$ defined by $R(y) = A^T y$ is another isomorphism.

(b) Show that for each $y \in \mathbb{R}^m$, $A^T Ax = A^T y$ has a solution $x \in \mathbb{R}^n$.

Ex6.4.14

Let $A \in \mathbb{R}^{m \times n}$ and $y \in \mathbb{R}^m$. Show that there is a unique solution $\overline{x}$ to $A^T Ax = A^T y$ that also belongs to $\mathrm{col}(A^T)$.

Ex6.4.15

Let $A \in \mathbb{R}^{m\times n}$, where $m < n$ and $\mathrm{rank} A = m$. Let $y \in \mathbb{R}^m$.

(a) Prove that $Ax = y$ has infinitely many solutions.

(b) Prove that AA^T is invertible.

(c) Let $S = \{x \in \mathbb{R}^n : Ax = y\}$, and define $\overline{x} = A^T(AA^T)^{-1}y$. Prove that $\overline{x} \in S$.

Ex6.4.16

Suppose V is a finie-dimensional inner product space over $\mathbb{R}$, S is a finite-dimensional subspace of V, and $P : V \to V$ is defines by $P(v) = \mathrm{proj}_S v$ for all $v \in V$. We call P the orthogonal projection operator onto S.

(a) Prove that P is linear.

(b) Prove that $P^2 = P$.

(c) Prove that $P^* = P$.

Ex6.4.17

suppose V is a finite-dimensional inner product space over $\mathbb{R}$, and assume that $P : V \to V$ satisfies $P^2 = P$ and $P^* = P$. Prove that there exists a subspace S of V such thatr $P(v) = \mathrm{proj}_S v$ for all $v \in V$.

Ex6.4.19

Let V be a vector space over a field F, and let $P : V \to V$. If $P^2 = P$, then we say P is a <u>projection operator</u>.

(a) Prove that if P is a projection operator, then so is $I - P$.

(b) Let P be a projection operator, and define

$$S = \mathcal{R}(P), T = \mathcal{R}(I - P).$$

i. Prove that $S \cap T = \{0\}$.
ii. Find $\ker(P)$ and $\ker(I - P)$.
iii. Prove that for any $v \in V$, there exist $s \in S$, $t \in T$ such that $v = s + t$.

6.5 The Gram-Schmidt process

Theorem (293, the Gram-Schmidt process)

Let V be an inner product space over a fielf $\mathbb{R}$, and suppose $\{u_1, \cdots, u_n\}$ is a basis for V. Let $\{\hat{u}_1, \cdots, \hat{u}_n\}$ be defined by

$$\hat{u}_1 = u_1$$

$$\hat{u}_{k+1} = u_{k+1} - \operatorname{proj}_{S_k} u_{k+1}$$

where $S_k = \operatorname{span}\{u_1, \cdots, u_k\}$. Then $\{\hat{u}_1, \cdots, \hat{u}_n\}$ is an orthogonal set and $\operatorname{span}\{u_1, \cdots, u_n\} = \operatorname{span}\{\hat{u}_1, \cdots, \hat{u}_n\}$

Thus every finite-dimensional inner product space over $\mathbb{R}$ has an orthonormal basis.

Ex6.5.5

(a) Find the best cubic approximation, in the $L^2(-1,1)$ norm, to the function $f(x) = e^x$.

(b) Find an orthogonal basis for $\mathcal{P}_3$ under the $L^2(-1,1)$ inner product.

(c) Repeat the calculations of (a) using the orthogonal basis in place of the standard basis.

(a)

Using the standard basis $\{1, x, x^2, x^3\}$ for $\mathcal{P}_3$,

$$G_{ij} = \int_{-1}^{1} x^i x^j \, dx = \frac{1 - (-1)^{i+j+1}}{i+j+1}.$$

$$G = \begin{bmatrix} 2 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{5} \\ \frac{2}{3} & 0 & \frac{2}{5} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{2}{7} \end{bmatrix}$$

$$b_0 = \int_{-1}^{1} e^x \, dx = e - e^{-1} \quad b_1 = \int_{-1}^{1} x e^x \, dx = 2e^{-1}$$

$$b_2 = \int_{-1}^{1} x^2 e^x \, dx = e - 5e^{-1} \quad b_3 = \int_{-1}^{1} x^3 e^x \, dx = -2e + 16e^{-1}$$

(a)

$$G^{-1} = \begin{bmatrix} \frac{9}{8} & 0 & -\frac{15}{8} & 0 \\ 0 & \frac{75}{8} & 0 & -\frac{105}{8} \\ -\frac{15}{8} & 0 & \frac{45}{8} & 0 \\ 0 & -\frac{105}{8} & 0 & \frac{175}{8} \end{bmatrix}$$

$$G^{-1}b = \begin{bmatrix} -\frac{3}{4}e + \frac{33}{4}e^{-1} \\ \frac{105}{4}e - \frac{765}{4}e^{-1} \\ \frac{15}{4}e - \frac{105}{4}e^{-1} \\ -\frac{175}{4}e + \frac{1295}{4}e^{-1} \end{bmatrix}$$

(b)

$$\{1, x, x^2 - \tfrac{1}{3}, x^3 - \tfrac{3}{5}x\}.$$

(c)

$$G = \operatorname{diag}(2, \tfrac{2}{3}, \tfrac{8}{45}, \tfrac{-46}{525})$$

$$\langle 1, e^x \rangle = e - e^{-1} \quad \langle x, e^x \rangle = 2e^{-1}$$
$$\left\langle x^2 - \tfrac{1}{3}, e^x \right\rangle = \tfrac{2}{3}e - \tfrac{14}{3}e^{-1} \quad \left\langle x^3 - \tfrac{3}{5}x, e^x \right\rangle = -2e + \tfrac{74}{5}e^{-1}.$$

6.6 Orthogonal complements

Definition

Let V be aninner product spave over $\mathbb{R}$, and S be a nonempty subset of V. The orthogonal complement of S is the set

$$S^{\perp} = \{u \in V : \langle u, s \rangle = 0 \text{ for all } s \in S\}.$$

Theorem (299)

$S^{\perp}$ is a subspace of V.

Example (301)

Let $V = C[0,1]$ under the $L^2(0,1)$ inner product, and let

$$S = \left\{ v \in V : \int_0^1 v(x)dx = 0 \right\}.$$

We wish to determine $S^\perp$.

Lemma (302)

Let V be a finite-dimensional inner product space over $\mathbb{R}$, and let S and T be orthogonal subspaces of V. Then $S \cap V = \{0\}$.

Theorem (303)

Let S be a subspace of V. Then $(S^{\perp})^{\perp} = S$.

Lemma (304)

Let V be an inner product space over $\mathbb{R}$, and let S and T be orthogonal subspaces of V. Then $S + T = S \oplus T$.

Theorem (308)

Let X and U be finite-dimensional inner product spaves over $\mathbb{R}$, and let $T : X \to U$ be linear. Then

1. $\ker(T)^{\perp} = \mathcal{R}(T^*)$ *and* $\mathcal{R}(T^*)^{\perp} = \ker(T)$;
2. $\ker(T^*)^{\perp} = \mathcal{R}(T)$ *and* $\mathcal{R}(T)^{\perp} = \ker(T^*)$.

Ex6.6.11

Let V be a finite-dimensional inner product space over $\mathbb{R}$, and let S be a nonempty subset of V. Prove that $(S^{\perp})^{\perp} = \operatorname{span}(S)$.

The pseudoinverse of a matrix

Ex6.6.13 $\sim$ Ex6.6.22

6.7 Complex inner product spaces

Definition

Let V be a vector space over the field $\mathbb{C}$ and suppose $\langle u, v\rangle$ is a unique complex number for each $u, v \in V$. We say $\langle\cdot,\cdot\rangle$ is an inner product on V if it satisfies the following properties:

1. $\langle u, v\rangle = \overline{\langle v, u\rangle}$;
2. $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$
3. $\langle u, u\rangle \geq 0$, and $\langle u, u\rangle = 0$ iff $u = 0$.

Using 1 and 2, $\langle w, \alpha u + \beta v\rangle = \overline{\alpha}\langle w, u\rangle + \overline{\beta}\langle w, v\rangle$.

Theorem (313)

Let V be a vector space over $\mathbb{C}$, and let $\langle \cdot, \cdot \rangle$ be an inner product on V. Then

$$|\langle u, v\rangle| \leq \langle u, u\rangle^{1/2} \langle v, v\rangle^{1/2}.$$

Proof.

Let $\lambda = \langle u, v\rangle / \langle v, v\rangle$.

$$0 \leq \langle u - \lambda v, u - \lambda v\rangle.$$

□

Induced norm

$\langle \cdot, \cdot \rangle$ over $\mathbb{C}$ induces a norm $\|\cdot\|$.

Example (Complex Euclidean n-space)

For $\mathbb{C}^n$,

$$\langle u, v \rangle = \sum_{i=1}^{n} u_i \overline{v_i}$$

is an inner product.

Example (Complex $L^2(a, b)$)

For $f, g : [a, b] \to \mathbb{C}$,

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$$

is an inner product.

Definition

Let V be a complex inner product space.

- $u, v \in V$ are orthogonal if and only if $\langle u, v \rangle = 0$.
- $\{u_1, \cdots, u_n\} \subset V$ is an orthogonal set if and only if each u_j is nonzero and $\langle u_j, u_k \rangle = 0$ for $j \neq k$.
- If $\|u_j\| = 1$, the set is orthonormal set.

Theorem (315)

Let V be a complex inner product space, and suppose $u, v \in V$ satisfy $\langle u, v \rangle = 0$. Then

$$\|u \pm v\|^2 = \|u\|^2 + \|v\|^2.$$

The converse may not hold.

Proof.

If $\|u + v\|^2 = \|u\|^2 + \|v\|^2$,

$$\begin{aligned} \|u + v\|^2 &= \|u\|^2 + \|v\|^2 + \langle u, v \rangle + \langle v, u \rangle \\ &\implies \langle u, v \rangle + \overline{\langle u, v \rangle} = 2\Re(\langle u, v \rangle) = 0 \end{aligned}$$

Consider $u = (i, 0)$ and $v = (1, 0)$. $u \cdot v = u_1\overline{v_1} + u_2\overline{v_2} = i$.

$$\|u + v\|^2 = \|(i + 1, 0)\|^2 = 2 = \|u\|^2 + \|v\|^2$$

$\square$

Definition

We can define the Gram matrix of a basis for a complex inner product space V by

$$G_{ij} = \langle u_j, u_i \rangle.$$

- $G_{ij} \neq \langle u_i, u_j \rangle$ in general.
- In the dot product, $u_j \cdot u_i = \overline{u_i^T} u_j = u_i^* u_j$.

Theorem (316, The projection theorem for complex inner space)

Let V be an complex inner product space over $\mathbb{C}$, and let S be a finite-dimensional subspace of V.

1. *For any $v \in V$, there is a unique $w \in S$ satisfying*

$$\|v - w\| = \min\{\|v - z\| : z \in S\}.$$

   In this case, we denote $w = \mathrm{proj}_S v$.

2. *$w \in S$ is the best approximation to v from S if and only if $\langle v - w, z\rangle = 0$ for all $z \in S$.*
3. *If $\{u_1, \cdots, u_n\}$ is a basis for S, then*

$$\mathrm{proj}_S v = \sum_{i=1}^{n} x_i u_i,$$

   where $x = (x_1, \cdots, x_n) \in \mathbb{C}^n$ is the unique solution to the equation $Gx = b$. G is the Gram matrix for the basis and $b_i = \langle v, u_i\rangle$.

Proof of Proj Thm

Fix $w \in S$. Then $y \in S$ if and only if $y = w + tz$ for some $t \in \mathbb{R}$ and $z \in S$. Consider

$$\begin{aligned}\|v-(w+tz)\|^2 &= \langle v-w-tz, v-w-tz\rangle \\ &= \langle v-w, v-w\rangle - t\langle z, v-w\rangle - t\langle v-w, z\rangle + t^2\langle z, z\rangle \\ &= \|v-w\|^2 - 2t\Re(\langle v-w, z\rangle) + t^2\|z\|^2\end{aligned}$$

Then $w = \text{proj}_S v$ if and only if $\Re\langle v-w, z\rangle = 0$ for all $z \in S$. Since $\Re(\langle v-w, iz\rangle) = \Re(-i\langle v-w, z\rangle) = -\Im\langle v-w, z\rangle$, $\Re\langle v-w, z\rangle = 0$ for all $z \in S$ if and only if $\langle v-w, z\rangle = 0$ for all $z \in S$ as desired.

Theorem (318, The adjoint of a linear operator)

Let V and W be finite-dimensional inner product spaves over $\mathbb{C}$, and let $L : V \to W$ be linear. Then there exists a unique linear operator $L^* : W \to V$ such that

$$\langle L(v), w \rangle_W = \langle v, L^*(w) \rangle_V .$$

Hermitian

Consider $L : \mathbb{C}^n \to \mathbb{C}^m$ defined by $L(x) = Ax$. Then

$$\begin{aligned} \langle L(x), y \rangle_m &= \langle Ax, y \rangle_m \\ &= \left\langle x, \overline{A}^T y \right\rangle_n . \end{aligned}$$

Thus $L^*(y) = \overline{A}^T y$. Define $A^* = \overline{A}^T$ and refer to A^* as the conjugate transpose of A.

Definition

- For $A \in \mathbb{R}^{n\times n}$(or $\mathbb{C}^{n\times n}$), A is symmetric if $A^T = A$.
- For $A \in \mathbb{C}^{n\times n}$, A is Hermitian if $A^* = A$.

Theorem (319)

Let $A \in \mathbb{C}^{n\times n}$ be Hermitian. Then $\langle Ax, x\rangle_{\mathbb{C}^n} \in \mathbb{R}$ for all $x \in \mathbb{C}^n$.

Ex6.7.3

Let

$$S = \left\{ e^{ik\pi x} : k \in \mathbb{Z} \right\}.$$

Prove that S is orthogonal under the complex $L^2(-1,1)$ inner product.

Ex6.7.13

All the eigenvalues of a Hermitian matrix are real.

The End