Algebraic Topology
- Dunkin's Torus 2 -

KYB

Thrn, it's a Fact

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The Fundamental Group

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Review of Groups

Let G and G' be groups.

- A *homomorphism* $f : G \to G'$ is a map such that $f(x \cdot y) = f(x) \cdot f(y)$ (preserving operations) for all $x, y \in G$.

Assume f is a homomorphism.

- $f(e) = e'$ where e and e' are identities of G and G', respectively
- $f(x^{-1}) = f(x)^{-1}$ where $^{-1}$ denotes the inverse.
- The *kernel* of f is the set $f^{-1}(e')$; it is a subgroup of G.
- The image of f is a subgroup of G'.
- An injective homomorphism is called a *monomorphism*.
- A surjective homomorphism is called a *epimorphism*.
- A bijective homomorphism is called an *isomorphism*.

Review of Groups

Let H be a subgroup of G.

- Fix $x \in G$. The set $xH = \{xh : h \in H\}$ is called a *left coset*;similarly Hx is called a *right coset*.
- The set of all left coset $\{xH : x \in G\}$ forms a partition of G; $\{Hx : x \in G\}$ forms also a partition of G.
- A subgroup H is called a *normal subgroup* of G is $xH = Hx$ for all $x \in G$.

Assume H is a normal subgroup of G, and denote $G/H = \{xH : x \in G\}$.

- We can define a well-defined operation on G/H by

$$(xH) \cdot (yH) = (x \cdot y)H,$$

  and this operation makes G/H a group. This group G/H is called the *quotient* of G by H.
- The map $f : G \to G/H$ by $f(x) = xH$ is an epimorphism with kernel H.
- Conversely, if $f : G \to G'$ is an epimorphism, then its kernel N is a normal subgroup of G, and f induces an isomorphism $G/N \to G'$ that carries xN to $f(x)$ for each $x \in G$.

In general, a subgroup is not a normal subgroup of G. In this case, we denote G/H the collection of right cosets of H in G.

Definition

Let X be a space; let x_0 be a point of X.

- A path in X that begins and ends at x_0 is called a *loop* bases at x_0.
- The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the *fundamental group* of X relative to the *base point* x_0. It is denoted by $\pi_1(X, x_0)$.

Example

- $\pi_1(\mathbb{R}^n, x_0)$ is the trivial group.
- More generally, if X is any convex subset of $\mathbb{R}^n$, then $\pi_1(X, x_0)$ is the trivial group.
- In particular, the *unit ball* $B^n = \{\mathbf{x} \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leqslant 1\}$ has trivial fundamental group.

Definition

Let α be a path in X from x_0 to x_1. We define a map

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

by the equation

$$\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha]$$

Theorem (52.1)

The map $\hat{\alpha}$ is a group isomorphism.

Corollary (52.2)

If X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_0)$ is isomorphism to $\pi_1(X, x_1)$.

Let C be the path component of X containing x_0. Then $\pi_1(C, x_0) = \pi_1(X, x_0)$, since all loops and homotopies in X that are based at x_0 must lie in C. Thus $\pi_1(X, x_0)$ depends on only the path component of X containing x_0. For this reason, it is usual to deal with only path-connected spaces when studying the fundamental group.

Remark

If X is path connected, all the groups $\pi_1(X, x)$ are isomorphic. So we want to say "the fundamental group of X" without mentioning of a base point x. But there is no natural way of identifying $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$; different paths α and β from x_0 to x_1 may give rise to different isomorphisms between these groups.

Definition

A space X is said to be *simply connected* if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial group for some $x_0 \in X$, and hence for every $x_0 \in X$. We often express the fact that $\pi_1(X, x_0)$ is the trivial group by writing $\pi_1(X, x_0) = 0$.

Lemma (52.3)

In a simply connected space X, any two paths having the same initial and final points are path homotopic.

Suppose $h : X \to Y$ is a continuous map such that $f(x_0) = y_0$, denoted by $h : (X, x_0) \to (Y, y_0)$. If f is a loop in X based at x_0, then $h \circ f : I \to Y$ is a loop in Y based at y_0. The correspondence $f \mapsto h \circ f$ gives rise to a map carrying $\pi_1(X, x_0)$ into $\pi_1(Y, y_0)$.

Definition

Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. Define

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

by the equation

$$h_*([f]) = [h \circ f].$$

Them map h_* is called the *homomorphism induced by* h, relative to the base point x_0.

Theorem (52.4)

If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

Corollary (52.5)

If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.

Ex 52.1

A subset A of $\mathbb{R}^n$ is said to be *star convex* if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A.

(a) Find a star convex set that is not convex.

(b) Show that if A is star convex, A is simply connected.

Ex 52.2

Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

Ex 52.3

Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

Ex 52.4

Let $A \subset X$; Suppose $r : X \to A$ is a continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a *retraction* of X onto A). If $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective.

Ex 52.5

Let A be a subspace of $\mathbb{R}^n$; let $h : (A, a_0) \to (Y, y_0)$. Show that if h is extendable to a continuous map of $\mathbb{R}^n$ into Y, then h_* is the trivial homomorphism. (the homomorphism that maps everything to the identity element).

Ex 52.6

Show that if X is path-connected, the homomorphism induced by a continuous map is independent of base point, up to isomorphisms of the groups involved. More precisely, let $h : X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$. Show that

$$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha}.$$

This equation expresses the fact that the following diagram of maps "commutes."

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\ \Big\downarrow{\scriptstyle \hat{\alpha}} & & \Big\downarrow{\scriptstyle \hat{\beta}} \\ \pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1) \end{array}$$

Ex 52.7

Let G be a topological group with operation $\cdot$ and identity element x_0. Let $\Omega(G, x_0)$ denote the set of all loops in G bases at x_0. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

(a) Show that this operation makes the set $\Omega(G, x_0)$ into a group.

(b) Show that this operation induces a group operation $\otimes$ on $\pi_1(G, x_0)$.

(c) Show that the two group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same.

(d) Show that $\pi_1(G, x_0)$ is abelian.