Analysis
- PMA 12 -

KYB

Thrn, it's a Fact

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Overview

Discussion of Main Problem

Definition

- Suppose $\{f_n\}$ is a sequence of functions defined on a set E, and suppose that the sequence that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can then define a function f by

$$f(x) = \lim_{n \to \infty} f_n(x).$$

  We say that $\{f_n\}$ converges on E and that f is the limit of $\{f_n\}$. Sometimes we shall say that "$\{f_n\}$ converges to f pointwise on E".

- If $\sum f_n(x)$ converges for every $x \in E$ and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x),$$

  the function f is called the sum of the series $\sum f_n$.

Discussion of Main Problem

Example (Double Sequences)

For m, n, let $s_{m,n} = \frac{m}{m+n}$. Then for every fixed n,

$$\lim_{m\to\infty} s_{m,n} = 1,$$

so that

$$\lim_{n\to\infty} \lim_{m\to\infty} s_{m,n} = 1.$$

On the other hand, for every fixed m,

$$\lim_{n\to\infty} s_{m,n} = 0,$$

so that

$$\lim_{m\to\infty} \lim_{n\to\infty} s_{m,n} = 0.$$

For $m = n$,

$$\lim_{m\to\infty} s_{m,m} = \frac{1}{2} \cdots$$

Discussion of Main Problem

Example

Let $f_n(x) = \frac{x^2}{(1+x^2)^n}$ and consider $f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$. Then

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 + x^2 & x \neq 0 \end{cases}.$$

So a convergent series of continuous functions may have a discontinuous sum.

Discussion of Main Problem

Let $f_m(x) = \lim_{n\to\infty}(\cos m!\pi x)^{2n}$.

- When $m!x$ is an integer, $f_m(x) = 1$. For all other values of x, $f_m(x) = 0$.

$$f_m(x) = \begin{cases} 1 & m!x \text{ integer} \\ 0 & \text{otherwise} \end{cases}$$

- So for irrational x, $f_m(x) = 0$ for all m. For rational x, $m!x$ is an integer for some m. Hence

$$\lim_{m\to\infty}\lim_{n\to\infty}(\cos m!\pi x)^{2n} = \begin{cases} 0 & x \text{ irrational} \\ 1 & x \text{ rational} \end{cases}$$

We have obtained an everywhere discontinuous limit function, which is not Riemann-integrable.

Discussion of Main Problem

Example

Let $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ and $f(x) = \lim_{n\to\infty} f_n(x) = 0$. Then $f'(x) = 0$, and $f_n'(x) = \sqrt{n}\cos nx$, so that $\{f_n'\}$ does not converges to f'.

Discussion of Main Problem

Example

Let $f_n(x) = n^2 x(1-x^2)^n$ for $0 \le x \le 1$.

- For $0 < x \le 1$, $\lim_{n\to\infty} f_n(x) = 0$
- $\lim_{n\to\infty} f_n(0) = 0$.

So $\lim_{n\to\infty} f_n(x) = 0$ for $0 \le x \le 1$. And

$$\int_0^1 x(1-x^2)^n \, dx = \frac{1}{2n+2}.$$

Thus

$$\int_0^1 f_n(x)\, dx = \frac{n^2}{2n+2} \to \infty \text{ as } n \to \infty.$$

Example

Let $f_n(x) = nx(1-x^2)^n$ for $0 \le x \le 1$. Then

$$\int_0^1 f_n(x)\, dx = \frac{n}{2n+2} \to \frac{1}{2} \text{ as } n \to \infty.$$

Uniform Convergence

Definition

- We say that a sequence of functions $\{f_n\}$ converges uniformly on E to a function f if for every $\epsilon > 0$, there is an integer N such that $n \geq N$ implies
$$|f_n(x) - f(x)| \leq \epsilon$$
for all $x \in E$.
- We say that the series $\sum f_n$ converges uniformly on E if the sequence $\{s_n\}$ of partial sums defined by
$$\sum_{i=1}^{n} f_i(x) = s_n(x)$$
converges uniformly on E.

Remark

- 'Uniformly convergent sequence' is 'pointwise convergent'.

Uniform Convergence

Theorem

The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E if and only if for every $\epsilon > 0$ there exists an integer N such that $m, n \geq N$, $x \in E$ implies

$$|f_n(x) - f_m(x)| \leq \epsilon.$$

Uniform Convergence

Theorem

Suppose

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all $x \in E$. Put

$$M_n = \sup_{x\in E} |f_n(x) - f(x)|.$$

Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

Theorem

Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose

$$|f_n(x)| \leq M_n.$$

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Uniform Convergence and Continuity

Theorem

Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that

$$\lim_{t \to x} f_n(t) = A_n.$$

Then $\{A_n\}$ converges, and

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n.$$

Uniform Convergence and Continuity

Theorem

If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.

Uniform Convergence and Continuity

Theorem

Suppose K is compact, and

(a) $\{f_n\}$ *is a sequence of continuous functions on* K,

(b) $\{f_n\}$ *converges pointwise to a continuous function* f *on* K

(c) $f_n(x) \geq f_{n+1}(x)$ *for all* $x \in K$.

Then $f_n \to f$ *uniformly on* K.

Uniform Convergence and Continuity

Definition

If X is a metric pace, $\mathscr{C}(X)$ will denote the set of all complex-valued, continuous, bounded functions with domain X.

Remark

We associate with each $f \in \mathscr{C}(X)$ its supremum norm

$$\|f\| = \sup_{x\in X} |f(x)|.$$

Then

(1) $\|f\| \geq 0$; $\|f\| = 0 \iff f = 0$.

(2) $\|\alpha f\| = |\alpha|\|f\|$.

(3) $\|f + g\| \leq \|f\| + \|g\|$.

Thus $\|\cdot\|$ is a norm on $\mathscr{C}(X)$, and it induces a metric ρ defined by $\rho(f, g) = \|f - g\|$. Hence $\mathscr{C}(X)$ is a metric space.

Remark

A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathscr{C}(X)$ if and only if $f_n \to f$ uniformly on X.

Uniform Convergence and Continuity

Theorem
$(\mathscr{C}(X), \rho)$ *is a complete metric space.*

Uniform Convergence and Integration

Theorem

Let α be monotonically increasing on $[a,b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$, and suppose $f_n \to f$ uniformly on $[a,b]$. Then $f \in \mathscr{R}(\alpha)$ and

$$\int_a^b f \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha.$$

Corollary

If $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$ and if $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is the series converging uniformly on $[a,b]$, then

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha.$$

Uniform Convergence and Differentiation

Theorem

Suppose $\{f_n\}$ *is a sequence of functions, differentiable on* $[a, b]$ *and such that* $\{f_n(x_0)\}$ *converges for some* x_0 *on* $[a, b]$*. If* $\{f_n'\}$ *converges uniformly on* $[a, b]$*, then* $\{f_n\}$ *converges uniformly on* $[a, b]$*, to a function* f*, and*

$$f'(x) = \lim_{n\to\infty} f_m'(x).$$

Uniform Convergence and Differentiation

Theorem

There exists a real continuous function on the real line which is nowhere differentiable.

Exercises

Ex 7.2

If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

Exercises

Ex 7.3

Construct sequences $\{f_n\}, \{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

Exercises

Ex 7.4

Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}.$$

- For what values of x does the series converge absolutely?
- On what intervals does it converge uniformly?
- On what intervals does it fail to converge uniformly?
- Is f continuous wherever the series converges?
- If f bounded?

Exercises

Ex 7.5

Let

$$f_n(x) = \begin{cases} 0 & x < \frac{1}{n+1} \\ \sin^2 \frac{\pi}{x} & \frac{1}{n+1} \le x \le \frac{1}{n} \\ 0 & \frac{1}{n} < x. \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolutely convergence, even for all x, does not imply uniform convergence.

Exercises

Ex 7.6

Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Exercises

Ex 7.7

For $n = 1, 2, 3, \cdots$, x real, put

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n \to \infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

Exercises

Ex 7.8

If

$$I(x) = \begin{cases} 0 & x \leq 0, \\ 1 & x > 0, \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n), \quad a \leq x \leq b$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

Exercises

Ex 7.9

Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

Exercises

Ex 7.10

Letting (x) denote the fractional part of the real number, that is, $(x) = x - [x]$, consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}.$$

Find all discontinuities of f, and show that form a countable dense set. Show that f is nevertheless Riemann-integrable on every bounded interval.

Exercises

Ex 7.12

Suppose g and f_n are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \to f$ uniformly on every compact subset of $(0, \infty)$, and

$$\int_0^\infty g(x)\, dx < \infty.$$

Prove that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx = \int_0^\infty f(x)\, dx.$$

Exercises

Ex 7.14, Space Filling Curve

Let f be a continuous real function on $\mathbb{R}$ with the following properties:

- $0 \leq f(t) \leq 1,\ f(t+2) = f(t)$ for every t
- $f(t) = \begin{cases} 0 & 0 \leq t \leq \frac{1}{3} \\ 1 & \frac{2}{3} \leq t \leq 1 \end{cases}$

Put $\Phi(t) = (x(t), y(t))$, where

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), \quad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t).$$

Prove that Φ is continuous and that Φ maps $I = [0, 1]$ onto the unit square $I^2 \subset \mathbb{R}^2$. In fact, Φ maps the Cantor set onto I^2.

The End