Analysis
- PMA 13 -

KYB

Thrn, it's a Fact

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Overview

Equicontinuous Families of Functions

Definition

Let $\{f_n\}$ be a sequence of functions defined on a set E.

- We say that $\{f_n\}$ is pointwise bounded on E if for every $x \in E$, there exists a finite-valued function ϕ defined on E such that
$$|f_n(x)| < \phi(x)$$
for all $x \in E$.
- We say that $\{f_n\}$ is uniformly bounded on E if there exists a number M such that
$$|f_n(x)| < M$$
for all $x \in E$.

Equicontinuous Families of Functions

Example

Let $f_n(x) = \sin nx$ for $0 \le x \le 2\pi$ and for all n. Then there is no convergent subsequence.

Example

Let $f_n(x) = \frac{x^2}{x^2+(1-nx)^2}$ for $0 \le x \le 1$. Then $\{f_n\}$ is uniformly bounded on $[0,1]$ and $\lim_{n\to\infty} f_n(x) = 0$ for all x but $f_n\left(\frac{1}{n}\right) = 1$ for all n. So there is no subsequence converging uniformly on $[0,1]$.

Equicontinuous Families of Functions

Definition

A family $\mathscr{F}$ of complex functions f defined on a set E in a metric space X is said to be equicontinuous on E if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

whenever $d(x, y) < \delta$, $x, y \in E$, and $f \in \mathscr{F}$.

Equicontinuous Families of Functions

Theorem

If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

Equicontinuous Families of Functions

Theorem

If K is a compact metric space, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \cdots$ and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.

Equicontinuous Families of Functions

Theorem

If K is compact, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \cdots$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then

(a) *$\{f_n\}$ is uniformly bounded on K,*

(b) *$\{f_n\}$ contains a uniformly convergent subsequence.*

Exercises

Ex 7.1

Prove that every uniformly convergent sequence of bounded function is uniformly bounded.

Exercises

Ex 7.11

Suppose $\{f_n\}$, $\{g_n\}$ are defined on E, and

(a) $\sum f_n$ has uniformly bounded partial sums;

(b) $g_n \to 0$ uniformly on E;

(c) $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$.

Prove that $\sum f_n g_n$ converges uniformly on E.

Exercises

Ex 7.13

Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on $\mathbb{R}$ with $0 \leq f_n(x) \leq 1$ for all x and all n.

(a) Prove that there is a function f and a sequence $\{n_k\}$ such that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x)$$

for every $x \in \mathbb{R}$.

(b) If, moreover, f is continuous, prove that $f_{n_k} \to f$ uniformly on compact sets.

Exercises

Ex 7.15

Suppose f is a real continuous function on $\mathbb{R}$, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \cdots$, and $\{f_n\}$ is equicontinuous on $[0, 1]$. What conclusion can you draw about f?

Exercises

Ex 7.16

Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

Exercises

Ex 7.17

- Define the notions of uniform convergence and equicontinuity for mappings into any metric space.
- Theorems 7.9 and 7.12 are valid for mappings into any metric space.
- Theorems 7.8 and 7.11 are valid for mappings into any compete metric space.
- Theorems 7.10, 7.16, 7.17, 7.24 and 7.25 hold for vector-valued functions, that is, for mappings into any $\mathbb{R}^k$.

Exercises

Definition

We say that a sequence of functions $\{f_n\}$ converges uniformly on E to a function f if for every $\epsilon > 0$, there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \epsilon$$

for all $x \in E$.

Theorem (7.9)

Suppose

$$\lim_{n\to\infty} f_n(x) = f(x) \quad (x \in E).$$

Put

Theorem (7.12)

If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.

Exercises

Theorem (7.8)

The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E if and only if for every $\epsilon > 0$ there exists an integer N such that $m, n \geq N$, $x \in E$ implies

$$|f_n(x) - f_m(x)| \leq \epsilon.$$

Theorem (7.11)

Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that

$$\lim_{t \to x} f_n(t) = A_n.$$

Then $\{A_n\}$ converges, and

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n.$$

Exercises

Theorem (7.10)

Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose

$$|f_n(x)| \le M_n.$$

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Theorem (7.16)

Let α be monotonically increasing on $[a,b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$, and suppose $f_n \to f$ uniformly on $[a,b]$. Then $f \in \mathscr{R}(\alpha)$ and

$$\int_a^b f \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha.$$

Theorem (7.17)

Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and

$$f'(x) = \lim_{n\to\infty} f_m'(x).$$

Exercises

Theorem (7.24)

If K is a compact metric space, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \cdots$ and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.

Theorem (7.25)

If K is compact, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \cdots$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then

(a) *$\{f_n\}$ is uniformly bounded on K,*

(b) *$\{f_n\}$ contains a uniformly convergent subsequence.*

Exercises

Ex 7.18

Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t)\,dt \quad (a \le x \le b).$$

Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

Exercises

Ex 7.19

Let K be a compact metric space, let S be a subset of $\mathscr{C}(K)$. Prove that S is compact if and only if S is uniformly closed, pointwise bounded, and equicontinuous.

The Stone-Weierstrass Theorem

Theorem

If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomial P_n such that

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If f is real, the P_n may be taken real.

The Stone-Weierstrass Theorem

Proof, Step 1

We may assume that $[a, b] = [0, 1]$ and $f(0) = f(1) = 0$. Furthermore, we define $f(x) = 0$ for $x \notin [0, 1]$. Put $Q_n(x) = c_n(1 - x^2)^n$ where c_n satisfies

$$\int_{-1}^{1} Q_n(x)\, dx = 1.$$

Since

$$\begin{aligned}\int_{-1}^{1} (1 - x^2)^n\, dx = 2\int_{0}^{1} (1 - x^2)^n\, dx &\geq 2\int_{0}^{1/\sqrt{n}} (1 - x^2)^n\, dx \\ &\geq 2\int_{0}^{1/\sqrt{n}} (1 - nx^2)\, dx = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}},\end{aligned}$$

$c_n < \sqrt{n}$.

The Stone-Weierstrass Theorem

Proof, Step 2

For any $\delta > 0$, we have

$$Q_n(x) \le \sqrt{n}(1-\delta^2)^n \quad \delta \le |x| \le 1,$$

so that $Q_n \to 0$ uniformly on $\delta \le |x| \le 1$.
Set

$$P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)\,dt \quad 0 \le x \le 1.$$

By a simple change of variable,

$$P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)\,dt = \int_0^1 f(t)Q_n(t-x)\,dt,$$

and the last integral is clearly a polynomial in x. Thus $\{P_n\}$ is a sequence of polynomials, which are real if f is real.

The Stone-Weierstrass Theorem

Proof, Step 3

Given $\epsilon > 0$, we choose $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \frac{\epsilon}{2}$. Let $M = \sup |f(x)|$. Then

$$\begin{aligned}
|P_n(x) - f(x)| &= \left| \int_{-1}^{1} [f(x+t) - f(x)] Q_n(t)\, dt \right| \\
&\leq \int_{-1}^{1} |f(x+t) - f(x)| Q_n(t)\, dt \\
&\leq 2M \int_{-1}^{-\delta} Q_n(t)\, dt + \frac{\epsilon}{2} \int_{-\delta}^{\delta} Q_n(t)\, dt + 2M \int_{\delta}^{1} Q_n(t)\, dt \\
&\leq 4M\sqrt{n}(1-\delta^2)^n + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

□

The Stone-Weierstrass Theorem

Corollary

For every interval $[-a, a]$*, there is a sequence of real polynomials* P_n *such that* $P_n(0) = 0$ *and such that*

$$\lim_{n\to\infty} P_n(x) = |x|$$

uniformly on $[-a, a]$*.*

The Stone-Weierstrass Theorem

Definition

- A family $\mathscr{A}$ of complex functions defined on a set E is said to be an algebra if for all $f, g \in \mathscr{A}$ and for all $c \in \mathbb{C}$,
  (i) $f + g \in \mathscr{A}$;
  (ii) $fg \in \mathscr{A}$;
  (iii) $cf \in \mathscr{A}$.
- If $\mathscr{A}$ has the property that $f \in \mathscr{A}$ whenever $f_n \in \mathscr{A}$ and $f_n \to f$ uniformly on E, then $\mathscr{A}$ is said to be uniformly closed.
- Let $\mathscr{B}$ be the set of all functions which are limits of uniformly convergent sequences of members of $\mathscr{A}$. Then $\mathscr{B}$ is called the uniform closure of $\mathscr{A}$.

Example

The set of all polynomials is an algebra, and the Weierstrass theorem may be stated by saying that the set of continuous on $[a, b]$ is the uniform closure of the set of polynomials on $[a, b]$.

The Stone-Weierstrass Theorem

Theorem

Let $\mathscr{B}$ be the uniform closure of an algebra $\mathscr{A}$ of bounded functions. Then $\mathscr{B}$ is a uniformly closed algebra.

The Stone-Weierstrass Theorem

Definition

- Let $\mathscr{A}$ be a family of functions on a set E. Then $\mathscr{A}$ is said to separate points on E if every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.
- If to each $x \in E$ there corresponds a function $g \in \mathscr{A}$ such that $g(x) \neq 0$, we say that $\mathscr{A}$ vanishes at no point of E.

Example

- The algebra of all polynomials in one variable has these properties on $\mathbb{R}$.
- An algebra of all even polynomials does not separate points.

The Stone-Weierstrass Theorem

Theorem

Suppose $\mathscr{A}$ is an algebra of functions on a set E, $\mathscr{A}$ separates points on E, and $\mathscr{A}$ vanishes at no point of E. Suppose x_1, x_2 are distinct points of E, and c_1, c_2 are constants (real if $\mathscr{A}$ is a real algebra). Then $\mathscr{A}$ contains a function f such that

$$f(x_1) = c_1, \quad f(x_2) = c_2.$$

The Stone-Weierstrass Theorem

Theorem

Let $\mathscr{A}$ be an algebra of real continuous functions on a compact set K. If $\mathscr{A}$ separates on K and if $\mathscr{A}$ vanishes at no point of K, then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.

The Stone-Weierstrass Theorem

Step 1

If $f \in \mathscr{B}$, then $|f| \in \mathscr{B}$.

The Stone-Weierstrass Theorem

Step 2

If $f, g \in \mathscr{B}$, then $\max(f, g), \min(f, g) \in \mathscr{B}$.

The Stone-Weierstrass Theorem

Step 3

Given a real function f, continuous on K, a point $x \in K$, and $\epsilon > 0$, there exists a function $g_x \in \mathscr{B}$ such that $g_x(x) = f(x)$ and

$$g_x(t) > f(t) - \epsilon \quad t \in K.$$

The Stone-Weierstrass Theorem

Step 4

Given a real function f, continuous on K, and $\epsilon > 0$, there exists a function $h \in \mathscr{B}$ such that

$$|h(x) - f(x)| < \epsilon \quad x \in K.$$

Since $\mathscr{B}$ is uniformly closed, this statement is equivalent to the conclusion of the theorem.

The Stone-Weierstrass Theorem

Definition

A complex algebra $\mathscr{A}$ is said to be self-adjoint if for every $f \in \mathscr{A}$, $\overline{f} \in \mathscr{A}$, where $\overline{f}(x) = \overline{f(x)}$.

Theorem

Suppose $\mathscr{A}$ is a self-adjoint algebra of complex continuous functions on a compact set K, $\mathscr{A}$ separates points on K, and $\mathscr{A}$ vanishes at no point of K. Then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on K. In other words, $\mathscr{A}$ is dense in $\mathscr{C}(K)$.

Exercises

Ex 7.20

If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x)x^n \, dx = 0 \quad (n = 0, 1, 2, \cdots),$$

prove that $f(x) = 0$ on $[0,1]$.

Exercises

Ex 7.21

Let K be the unit circle in the complex plane, and let $\mathscr{A}$ be the algebra of all functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta} \quad (\theta \text{ real}).$$

Then $\mathscr{A}$ separates points on K and $\mathscr{A}$ vanishes at no point of K, but nevertheless there are continuous on K which are not in the uniform closure of $\mathscr{A}$.

Exercises

Ex 7.22

Assume $f \in \mathscr{R}(\alpha)$ on $[a, b]$, and prove that there are polynomials P_n such that

$$\lim_{n\to\infty} \int_a^b |f - P_n|^2 \, d\alpha = 0.$$

Exercises

Ex 7.23

Put $P_0 = 0$, and define, for $n = 0, 1, 2, \cdots$,

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Prove that

$$\lim_{n\to\infty} P_n(x) = |x|,$$

uniformly on $[-1, 1]$.

Exercises

Ex 7.24

Let X be a metric space, with metric d. Fix a point $a \in X$. Assign to each $p \in X$ the function f_p defined by

$$f_p(x) = d(x,p) - d(x,a) \quad (x \in X).$$

Prove that $|f_p(x)| \le d(p,a)$ for all $x \in X$, and therefore $f_p \in \mathscr{C}(X)$.
Prove that

$$\|f_p - f_q\| = d(p,q)$$

for all $p, q \in X$.
If $\Phi(p) = f_p$, it follows that Φ is an isometry of X onto $\Phi(X) \subset \mathscr{C}(X)$.
Let Y be the closure of $\Phi(X)$ in $\mathscr{C}(X)$. Show that Y is complete.(Hence, every metric space is a dense subset of a complete metric space.)

Exercises

Ex 7.25, Picard Interation

Suppose ϕ is a continuous bounded real function in the strip defined by $0 \leq x \leq 1, -\infty < y < \infty$. Prove that the initial-value problem

$$y' = \phi(x, y), \quad y(0) = c$$

has a solution.

The End