Modules

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Thrn, it's a Fact

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Overview

Generation of Modules

In this chapter, a ring has 1 and a “module” means “left module.”

Definition

Let M be and R-module and let $N_1, \cdots, N_n$ be submodules of M.

1. $N_1 + \cdots + N_n = \{a_1 + \cdots + a_n : a_i \in N_i\}$.
2. For any subset A of M, $RA = \{r_1 a_1 + \cdots + r_m a_m : r_i \in R, a_i \in A\}$.
   - if $A = \varnothing$, $RA = \{0\}$.
   - if $A = \{a_1, \cdots, a_n\}$, $RA = Ra_1 + \cdots + Ra_n$.
   - if N is a submodule such that $N = RA$, A is called a generating set for N.
3. A submodule N of M is finitely generated if there is some finite subset A of M such that $N = RA$.
4. A submodule N of M is cyclic if there exists an element $a \in M$ such that $N = Ra$.

Remark

- A R-linear combination of A is a element x of M such that

$$x = r_1 a_1 + \cdots + r_k a_k$$

  for some $r_1, \cdots, r_k \in R$ and $a_1, \cdots, a_k \in A$. In this sense, RA is the set of all R-linear combinations of A.
- RA is the smallest submodule of M containing A.
- $N_1 + \cdots + N_n = R(N_1 \cup \cdots \cup N_n)$.
- If $N_i = RA_i$, $N_1 + \cdots + N_n = R(A_1 \cup \cdots \cup A_n)$.

Example

1. Let $R = \mathbb{Z}$ and let M be a $\mathbb{Z}$-module. For $a \in M$, $\mathbb{Z}a$ is the cyclic subgroup of M generated by a.
2. Every ring R is a cyclic module, $R = R1$.
3. Submodules of a finitely generated module need not be finitely generated.
4. Let R be a ring with 1 and $M = R^n$. For each i, let $e_i = (\cdots, 1, \cdots)$. Then

$$(s_1, \cdots, s_n) = \sum_{i=1}^{n} s_i e_i.$$

So M is generated by $\{e_1, \cdots, e_n\}$.

Ditect sums and direct products

Definition

Let $M_1, \cdots, M_k$ be a collection of R-modules. The collection of k-tuples $(m_1, \cdots, m_k)$ where $m_i \in M_i$ with addition and multiplication with R defined componentwise is called the direct product of $M_1, \cdots, M_k$, denoted by $M_1 \times \cdots \times M_k$.

Remark

If $M_i = R$, $M_1 \times \cdots \times M_k = R^k$.

Proposition-definition

Let $N_1, \cdots, N_k$ be submodules of the R-module M. Then the following are equivalent:

1. The map $\pi : N_1 \times \cdots \times N_k \to N_1 + \cdots + N_k$ defined by
$$\pi(a_1, \cdots, a_k) = a_1 + \cdots + a_k$$
is an isomorphism.
2. $N_j \cap (\cdots + N_{j-1} + N_{j+1} \cdots) = 0$ for all j.
3. Every $x \in N_1 + \cdots + N_k$ can be written uniquely in the form $a_1 + \cdots + a_k$ with $a_i \in N_i$.

If $N_1, \cdots, N_k$ satisfies one of them, we call $N_1 + \cdots + N_k$ the direct sum of $N_1, \cdots, N_k$ denoted by

$$N_1 + \cdots + N_k = N_1 \oplus \cdots \oplus N_k.$$

Proof

$1 \implies 2.$

Proof

$2 \implies 3.$

Proof

$3 \implies 1.$

Free modules

Definition

An R-module F is said to be free on the subset A of F if for every nonzero element x of F, there exist unique nonzero elements $r_1, \cdots, r_n \in R$ and $a_1, \cdots, a_n \in A$ such that

$$x = r_1 a_1 + \cdots r_n a_n.$$

A is called a basis for F.

Remark

- In $N_1 \oplus N_2$, each element can be written uniquely as $n_1 + n_2$; here the uniqueness refers to the module elements n_1, n_2.
- If free modules, the uniqueness is on the ring elements as well as the module elements.

Let $R = \mathbb{Z}$ and $N_1 = N_2 = \mathbb{Z}/2\mathbb{Z}$. Then $N_1 \oplus N_2$ has a unique representation in the form $n_1 + n_2$ but $n_1 = 3n_1 = \cdots$. Thus $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is not a free-module on the set $\{(1,0),(0,1)\}$ (in fact, on any set).

Theorem (The Universal Property of Free R-modules)

For any set A there is a free R-module $F(A)$ on the set A and $F(A)$ satisfies the following universal property:

- *if M is any R-module and $\varphi : A \to M$ is any map of sets, then there is a unique R-module homomorphism $\Phi : F(A) \to M$ such that $\Phi(a) = \varphi(a)$ for all $a \in A$, that is, the following diagram commutes.*

$$\begin{tikzcd} A \arrow[r, "\text{inclusion}"] \arrow[dr, "\varphi"'] & F(A) \arrow[d, dashed, "\Phi"] \arrow[dl, phantom, "\circlearrowright"] \\ & M \end{tikzcd}$$

When A is the finite set $\{a_1, \cdots, a_n\}$, $F(A) = Ra_1 \oplus Ra_2 \oplus \cdots \oplus Ra_n \cong R^n$.

Proof

(Uniqueness)

Proof

(Existence)

Remark

For given set A, $F(A)$ may not be unique. But $F(A)$ is unique up to isomorphic.

Corollary

1. *If* F_1 *and* F_2 *are free modules on the same set* A, *there is a unique isomorphism between* F_1 *and* F_2 *which is the identity map on* A.
2. *If* F *is any free* R*-module with basis* A, *then* $F \cong F(A)$. *In particular,* F *enjoys the same universal property with respect to* A *as* $F(A)$ *does in above theorem.*

Ex 1.

Prove if M is a finitely generated R-module that is generated by n elements, then every quotient of M may generated n (or fewer) elements. Deduce that quotients of cyclic modules are cyclic.

Ex 2.

Let N be a submodule of M. Prove that if both M/N and N are finitely generated then so is M.

Ex 3.

Let R be a commutative ring and let A, B, and M be R-modules. Then

(a) $\mathrm{Hom}_R(A \times B, M) \cong \mathrm{Hom}_R(A, M) \times \mathrm{Hom}_R(B, M)$.

(b) $\mathrm{Hom}_R(M, A \times B) \cong \mathrm{Hom}_R(M, A) \times \mathrm{Hom}_R(M, B)$.

Ex 4.

Let R be a commutative ring and let F be a free R-module of finite rank. Prove that $\mathrm{Hom}_R(F, R) \cong F$.

Ex 5.

Let R be a commutative ring and let F be a free R-module of rank n. Prove that

$$\operatorname{Hom}_R(F, M) \cong M \times \cdots \times M \text{ (} n \text{ times).}$$

Ex 6.

Show that any direct sum of free R-modules is free.

The End