Analysis
- PMA 18 -

KYB

Thrn, it's a Fact

mathrnfact@gmail.com

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Overview

Determinants

Definition

- If $(j_1, \cdots, j_n)$ is an ordered n-tuples of integers, define

$$s(j_1, \cdots, j_n) = \prod_{p<q} \operatorname{sgn}(j_q - j_p).$$

- Let $[A]$ be the matrix of a linear operator A on $\mathbb{R}^n$, relative to the standard basis $\{\mathbf{e}_1, \cdots \mathbf{e}_n\}$, with entries $a(i, j)$ in the ith row and jth column. The *determinant* of $[A]$ is defined to be the number

$$\det[A] = \sum s(j_1, \cdots, j_n) a(1, j_1) a(2, j_2) \cdots a(n, j_n)$$

  where the sum extends over all ordered n-tuples of integers $(j_1, \cdots, j_n)$ with $1 \le j_r \le n$.
- The column vectors $\mathbf{x}_j$ of $[A]$ are

$$\mathbf{x}_j = \sum_{i=1}^{n} a(i, j)\mathbf{e}_i \quad 1 \le j \le n.$$

  If we write

$$\det(\mathbf{x}_1, \cdots, \mathbf{x}_n) = \det[A],$$

  $\det$ is now a real function on the set of all ordered n-tuples of vectors in $\mathbb{R}^n$.

Determinants

Theorem (9.34)

(a) *If I is the identity operator on $\mathbb{R}^n$, then*

$$\det[I] = \det(\mathbf{e}_1, \cdots, \mathbf{e}_n) = 1.$$

(b) $\det$ *is a linear function of each of the column vectors $\mathbf{x}_j$, if the others are held fixed.*

(c) *If $[A]_1$ is obtained from $[A]$ by interchanging two columns, then $\det[A]_1 = -\det[A]$.*

(d) *If $[A]$ has two equal columns, then $\det[A] = 0$.*

Determinants

Theorem (9.35)

If $[A]$ *and* $[B]$ *are* n *by* n *matrices, then*

$$\det([B][A]) = \det[B]\det[A].$$

Theorem (9.36)

A linear operator A *on* $\mathbb{R}^n$ *is invertible if and only if* $\det[A] \neq 0$*.*

Determinants

Remark

Suppose $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ and $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ are bases in $\mathbb{R}^n$. Every linear operator A on $\mathbb{R}^n$ determines matrices $[A]$ and $[A]_U$, with entries a_{ij} and α_{ij}, given by

$$A\mathbf{e}_j = \sum_i a_{ij}\mathbf{e}_i, \quad A\mathbf{u}_j = \sum_i \alpha_{ij}\mathbf{u}_i.$$

If $\mathbf{u}_j = B\mathbf{e}_j = \sum b_{ij}\mathbf{e}_i$, then $A\mathbf{u}_j$ is equal to

$$\sum_k \alpha_{kj} B\mathbf{e}_k = \sum_k \alpha_{kj} \sum_i b_{ik}\mathbf{e}_i = \sum_i \left(\sum_k b_{ik}\alpha_{kj}\right)\mathbf{e}_i,$$

and also to

$$AB\mathbf{e}_j = A\sum_k b_{kj}\mathbf{e}_k = \sum_i \left(\sum_k a_{ik}b_{kj}\right)\mathbf{e}_i.$$

Thus $\sum b_{ik}\alpha_{kj} = \sum a_{ik}b_{kj}$, or

$$[B][A]_U = [A][B].$$

Since B is invertible, $\det[B] \neq 0$. Hence,

$$\det[A]_U = \det[A].$$

The determinant of the matrix of a linear operator does therefore not depend on the basis which is used to construct the matrix.

Determinants

Jacobians(9.38)

If $\mathbf{f}$ maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, and if $\mathbf{f}$ is differentiable at a point $\mathbf{x} \in E$, the determinant of the linear operator $\mathbf{f}'(\mathbf{x})$ is called the Jacobian of $\mathbf{f}$ at $\mathbf{x}$.

$$J_{\mathbf{f}}(\mathbf{x}) = \det \mathbf{f}'(\mathbf{x}).$$

We shall also use the notation

$$\frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)}$$

for $J_{\mathbf{f}}(\mathbf{x})$ if $(y_1, \cdots, y_n) = \mathbf{f}(x_1, \cdots, x_n)$.

Derivatives of Higher Order

Definition

- Suppose f is a real function defined in an open set $E \subset \mathbb{R}^n$, with partial derivatives $D_1 f, \cdots, D_n f$. If the functions $D_j f$ are themselves differentiable, then the second-order partial derivatives of f are defined by

$$D_{ij} f = D_i D_j f.$$

- If all these functions $D_{ij} f$ are continuous in E, we say that f is of class $\mathscr{C}''$ in E, or that $f \in \mathscr{C}''(E)$.
- A mapping $\mathbf{f}$ of E into $\mathbb{R}^n$ is said to be of class $\mathscr{C}''$ if each component of $\mathbf{f}$ is of class $\mathscr{C}''$.

Derivatives of Higher Order

Theorem (9.40)

Suppose f is defined in an open set $E \subset \mathbb{R}^2$, and $D_1 f$ and $D_{21} f$ exists at every point of E. Suppose $Q \subset E$ is a closed rectangle with sides parallel to the coordinate axes, having (a,b) and $(a+h, b+k)$ as opposite vertices $(h \neq 0, k \neq 0)$. Put

$$\Delta(f,Q) = f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b).$$

Then there is a point (x,y) in the interior of Q such that

$$\Delta(f,Q) = hk(D_{21}f)(x,y).$$

Derivatives of Higher Order

Theorem (9.41)

Suppose f is defined in an open set $E \subset \mathbb{R}^2$, suppose that $D_1 f$, $D_{21} f$, and $D_2 f$ exists at every point of E, and $D_{21} f$ is continuous at some point $(a, b) \in E$.
Then $D_{12} f$ exists at (a, b) and $(D_{12} f)(a, b) = (D_{21} f)(a, b)$.

Corollary

$D_{21} f = D_{12} f$ *if* $f \in \mathscr{C}''(E)$.

Differentiation of Integrals

Theorem (9.42)

Suppose

(a) $\varphi(x,t)$ *is defined for* $a \leq x \leq b$, $c \leq t \leq d$;

(b) α *is an increasing function on* $[a,b]$;

(c) $\varphi^t \in \mathscr{R}(\alpha)$ *for every* $t \in [c,d]$;

(d) $c < s < d$, *and to every* $\epsilon > 0$ *corresponds a* $\delta > 0$ *such that*

$$|(D_2\varphi)(x,t) - (D_2\varphi)(x,s)| < \epsilon$$

for all $x \in [a,b]$ *and for all* $t \in (s-\delta, s+\delta)$.

Define

$$f(t) = \int_a^b \varphi(x,t) d\alpha(x) \quad c \leq t \leq d.$$

Then $(F_2\varphi)^s \in \mathscr{R}(\alpha)$, $f'(s)$ *exists, and*

$$f'(s) = \int_a^b (D_2\varphi)(x,s)\, d\alpha(x).$$

Differentiation of Integrals

Example

Define

$$f(t) = \int_{-\infty}^{\infty} e^{-x^2} \cos(xt)\, dx$$

and

$$g(t) = -\int_{-\infty}^{\infty} xe^{-x^2} \sin(xt)\, dx,$$

for $-\infty < t < \infty$.
Then f is differentiable and $f'(t) = g(t)$.

Exercises

Ex 9.26

Show that the existence (and even the continuity) of $D_{12}f$ does not imply the existence of $D_1 f$.

Exercises

Ex 9.27

Put $f(0,0) = 0$, and

$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if $(x,y) \neq (0,0)$. Prove that

(a) f, $D_1 f$, $D_2 f$ are continuous in $\mathbb{R}^2$.

(b) $D_{12} f$ and $D_{21} f$ exist at every point of $\mathbb{R}^2$, and are continuous except at $(0,0)$.

(c) $(D_{12} f)(0,0) = 1$, and $(D_{21} f)(0,0) = -1$.

Exercises

Ex 9.28

For $t \geq 0$, put

$$\varphi(x,t) = \begin{cases} x & 0 \leq x \leq \sqrt{t} \\ -x + 2\sqrt{t} & \sqrt{t} \leq x \leq 2\sqrt{t} \\ 0 & \text{otherwise,} \end{cases}$$

and put $\varphi(x,t) = -\varphi(x,|t|)$ if $t < 0$. Show that φ is continuous on $\mathbb{R}^2$, and

$$(D_2\varphi)(x,0) = 0$$

for all x. Define

$$f(t) = \int_{-1}^{1} \varphi(x,t)\, dx.$$

Show that $f(t) = t$ if $|t| < \frac{1}{4}$. Hence

$$f'(0) \neq \int_{-1}^{1} (D_2\varphi)(x,0)\, dx.$$

Exercises

Ex 9.30, Taylor Series

Let $f \in \mathscr{C}^{(m)}(E)$, where E is an open subset of $\mathbb{R}^n$. Fix $\mathbf{a} \in E$, and suppose $\mathbf{x} \in \mathbb{R}^n$ is so closed to $\mathbf{0}$ that the points

$$\mathbf{p}(t) = \mathbf{a} + t\mathbf{x}$$

lie in E whenever $0 \le t \le 1$. Define

$$h(t) = f(\mathbf{p}(t))$$

for all $t \in \mathbb{R}$ for which $\mathbf{p}(t) \in E$.

Exercises

Ex 9.30, Taylor Series

(a) For $1 \leq k \leq m$, show that

$$h^{(k)}(t) = \sum (D_{i_1 \cdots i_k} f)(\mathbf{p}(t)) x_{i_1} \cdots x_{i_k}.$$

The sum extend over all ordered k-tuples $(i_1, \cdots, i_k)$ in which each i_j is one of the integers $1, \cdots, n$.

Exercises

Ex 9.30, Taylor Series

(b) By the Taylor's theorem,

$$h(1) = \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!} + \frac{h^{(m)}(t)}{m!}$$

for some $t \in (0,1)$. Use this to prove Taylor's theorem in n variables by showing that the formula

$$f(\mathbf{a}+\mathbf{x}) = \sum_{k=0}^{m-1} \frac{1}{k!} \sum (D_{i_1 \cdots i_k} f)(\mathbf{a}) x_{i_1} \cdots x_{i_k} + r(\mathbf{x})$$

where

$$\lim_{\mathbf{x} \to \mathbf{0}} \frac{r(\mathbf{x})}{|\mathbf{x}|^{m-1}} = 0.$$

The End