Algebraic Topology
- Dunkin's Torus 4 -

KYB

Thrn, it's a Fact

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September 14, 2021

Overview

The Fundamental Group

- The Fundamental Group of the Circle

Definition

Let $p : E \to B$ be a map. If f is a continuous mapping of some space X into B, a *lifting* of f is a map $\widetilde{f} : X \to E$ such that $p \circ \widetilde{f} = f$.

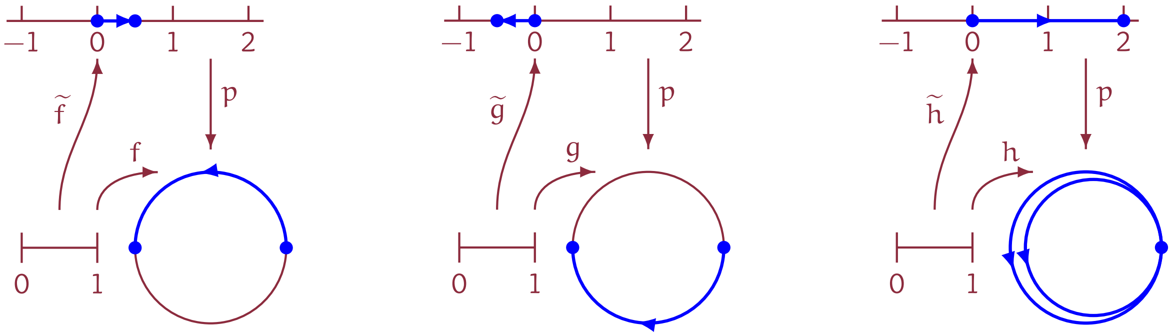
$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

The Fundamental Group of the Circle

Example

Consider the covering $p : \mathbb{R} \to S^1$.

- $f : [0, 1] \to S^1$ given by $f(s) = (\cos \pi s, \sin \pi s)$ lifts to the path $\widetilde{f}(s) = s/2$.
- $g : [0, 1] \to S^1$ given by $f(s) = (\cos \pi s, -\sin \pi s)$ lifts to the path $\widetilde{g}(s) = -s/2$.
- $h : [0, 1] \to S^1$ given by $f(s) = (\cos 4\pi s, \sin 4\pi s)$ lifts to the path $\widetilde{h}(s) = 2s$.



The Fundamental Group of the Circle

Lemma (54.1)

Let $p : E \to B$ *be a covering map, let* $p(e_0) = b_0$*. Any path* $f : [0, 1] \to B$ *beginning at* b_0 *has a unique lifting to a path* $\widetilde{f}$ *in* E *beginning at* e_0*.*

Lemma (The Lebesgue number lemma)

Let $\mathcal{A}$ *be an open covering of the metric space* (X, d)*. If* X *is compact, there is a* $\delta > 0$ *such that for each subset of* X *having diameter less than* δ*, there exists an element of* $\mathcal{A}$ *containing it.*

Lemma (54.2)

Let $p : E \to B$ *be a covering map; let* $p(e_0) = b_0$*. Let the map* $F : I \times I \to B$ *be continuous, with* $F(0,0) = b_0$*. There is a unique lifting of* F *to a continuous map* $\widetilde{F} : I \times I \to E$ *such that* $\widetilde{F}(0,0) = e_0$*. If* F *is a path homotopy, then* $\widetilde{F}$ *is a path homotopy.*

Lemma (54.3)

Let $p : E \to B$ *be a covering map; let* $p(e_0) = b_0$*. Let* f *and* g *be two paths in* B *from* b_0 *to* b_1*, let* $\widetilde{f}$ *and* $\widetilde{g}$ *be their respective liftings to paths in* E *beginning at* e_0*. If* f *and* g *are path homotopic, then* $\widetilde{f}$ *and* $\widetilde{g}$ *end at the same point of* E *and are path homotopic.*

Definition

Let $p : E \to B$ be a covering map; let $b_0 \in B$. Choose e_0 so that $p(e_0) = b_0$. Given an element $[f]$ of $\pi_1(B, b_0)$, let $\widetilde{f}$ be the lifting of f to a path in E that begins at e_0. Let $\phi([f])$ denote the end point $\widetilde{f}(1)$ of $\widetilde{f}$. Then ϕ is a well-defined set map

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0).$$

We call ϕ the *lifting correspondence* derived from the covering map p. It depends of course on the choice of the point e_0.

Theorem (54.4)

Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0).$$

is surjective. If E is simply connected, it is bijective.

Theorem (54.5)

The fundamental group of S^1 is isomorphic to the additive group of integers.

The Fundamental Group of the Circle

Definition

Let G be a group; let x be an element of G. We denote the inverse of x by x^{-1}.

- x^n denote the n-fold product of x with itself
- x^{-n} denotes the n-fold product of x^{-1} with itself
- x^0 denotes the identity element of G.

If the set of all elements of the form x^m for $m \in \mathbb{Z}$, equals G, then G is said to be a *cyclic* group, and x is said to be a *generator* of G.

- The cardinality of a group is called the *order* of the group.
- A group is cyclic of infinite order if and only if it is isomorphic to the additive group of integers
- A group is cyclic of order k if and only if it is isomorphic to the group $\mathbb{Z}/k$ of integers modulo k.

Remark

Note that if x is a generator of the infinite cyclic group G, and y is an element of the arbitrary group H, then there is a unique homomorphism h of G into H such that $h(x) = y$; it is defined by setting $h(x^n) = y^n$ for all n.

Theorem (54.6)

Let $p : E \to B$ *be a covering map; let* $p(e_0) = b_0$.

(a) *The homomorphism* $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ *is a monomorphism.*

(b) *Let* $H = p_*(\pi_1(E, e_0))$. *The lifting correspondence* ϕ *induces an injective map*

$$\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$$

of the collection of right cosets of H *into* $p^{-1}(b_0)$*, which is bijective if* E *is path connected.*

(c) *If* f *is a loop in* B *based at* b_0*, then* $[f] \in H$ *if and only if* f *lifts to a loop in* E *based at* e_0.

Ex 54.3

Let $p : E \to B$ be a covering map. Let α and β be paths in B with $\alpha(1) = \beta(0)$; let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be liftings of them such that $\widetilde{\alpha}(1) = \widetilde{\beta}(0)$. Show that $\widetilde{\alpha} * \widetilde{\beta}$ is a lifting of $\alpha * \beta$.

Ex 54.6

Consider the maps $g, h : S^1 \to S^1$ given by $g(z) = z^n$ and $h(z) = 1/z^n$. Compute the induced homomorphisms g_*, h_* of the infinite cyclic group $\pi_1(S^1, b_0)$ into itself

Ex 54.8

Let $p : E \to B$ be a covering map, with E path connected. Show that if B is simply connected, them p is a homeomorphism.