Algebraic Topology
- Dunkin's Torus 7 -

KYB

Thrn, it's a Fact

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Overview

Theorem (59.1)

Suppose $X = U \cup V$, *where* U *and* V *are open sets of* X. *Suppose that* $U \cap V$ *is path connected, and that* $x_0 \in U \cap V$. *Let* i *and* j *be the inclusion mappings of* U *and* V, *respectively, into* X. *Then the images of the induces homomorphisms*

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \quad \textit{and} \quad j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

generated $\pi_1(X, x_0)$.

This theorem is a special case of a famous theorem of topology called the *Seifert-van Kampen theorem.*

Proof, Step 1

Step 1 : There is a subdivision $a_0 < a_1 < \cdots < a_n$ of the unit interval such that $f(a_i) \in U \cap V$ and $f([a_{i-1}, a_i])$ is contained either in U or in V, for each i.

The Fundamental Group of S^n

Proof, Step 2

Step 2 : given any loop f in X bases at x_0, it is path homotopic to a product of the form $g_1 * \cdots * g_n$, where g_i is a loop in X based at x_0 that lies either in U or in V.

The Fundamental Group of S^n

Corollary (59.2)

Suppose $X = U \cup V$, where U and V are open sets of X; suppose $U \cap V$ is nonempty and path connected. If U and V are simply connected, then X is simply connected.

Theorem (59.3)

If $n \geqslant 2$, the n-sphere S^n is simply connected.

Recall

A *surface* is a Hausdorff space with a countable basis, each point of which has a neighborhood that is homeomorphic with an open subset of $\mathbb{R}^2$.

Recall

If A and B are groups with operation $\cdot$, then the cartesian product $A \times B$ is given a group structure by using the operation

$$(a \times b) \cdot (a' \times b') = (a \cdot a') \times (b \cdot b').$$

If $h : C \to A$ and $k : C \to B$ are group homomorphisms, then the map $\Phi : C \to A \times B$ defined by $\Phi(c) = h(c) \times k(c)$ is a group homomorphism.

Fundamental Groups of Some Surfaces

Theorem (60.1)

$\pi_1(X \times Y, x_0 \times y_0)$ is isomorphic with $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Corollary (60.2)

The fundamental group of the torus $T = S^1 \times S^1$ is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.

Fundamental Groups of Some Surfaces

Definition

The *projective plane* P^2 is the quotient space obtained from S^2 by identifying each point x of S^2 with its antipodal point $-x$.

Theorem (60.3)

The projective plane P^2 *is a compact surface, and the quotient map* $p : S^2 \to P^2$ *is a covering map.*

Corollary (60.4)

$\pi_1(P^2, y)$ is a group of order 2.

Fundamental Groups of Some Surfaces

Lemma (60.5)

The fundamental group of the figure eight is not abelian.

Theorem (60.6)

The fundamental group of the double torus is not abelian.

Fundamental Groups of Some Surfaces

Theorem (60.7)

The 2-sphere, torus, projective plane, and double torus are topologically distinct.

Lemma (61.1)

Let C be a compact surface of S^2; let b be a point of $S^2 - C$; and let h be a homeomorphism of $S^2 - b$ with $\mathbb{R}^2$. Suppose U is a component of $S^2 - C$.

- *If U does not contain b, then $h(U)$ is a bounded component of $\mathbb{R}^2 - h(C)$.*
- *If U contains b, then $h(U - b)$ is the unbounded component of $\mathbb{R}^2 - h(C)$.*

In particular, if $S^2 - C$ has n components, then $\mathbb{R}^2 - h(C)$ has n components.

Lemma (61.2, Nulhomotopy lemma)

Let a and b be points of S^2. Let A be a compact space, and let

$$f : A \to S^2 - a - b$$

be a continuous map. If a and b lie in the same component of $S^1 - f(A)$, then f is nulhomotopic.

Definition

If X is a connected space and $A \subset X$,

- we say that A *separates* X if $X - A$ is not connected;
- we say that A *separates* X *into* n *components* if $X - A$ has n components.

Definition

- An *arc* A is a space homeomorphic to the unit interval $[0, 1]$.
- The *end points* of A are two points p and q of A such that $A - p$ and $A - q$ are connected; the other points of A are called *interior points* of A.
- A *simple closed curve* is a space homeomorphic to the unit circle S^1.

The Jordan Separation Theorem

Theorem (61.3, The Jordan separation theorem)

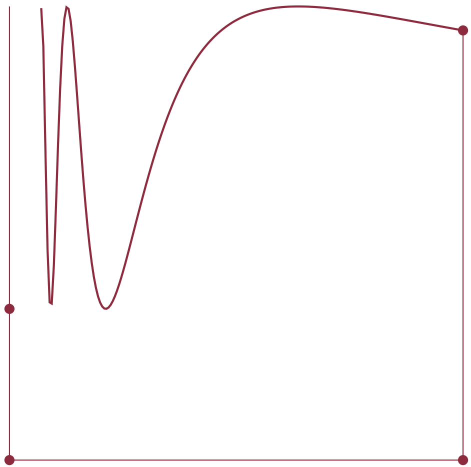
Let C be a simple closed curve in S^2. Then C separates S^2.

Theorem (61.4, A general separation theorem)

Let A_1 and A_2 be closed connected subsets of S^2 whose intersection consists of precisely two points a and b. Then the set $C = A_1 \cup A_2$ separates S^2.

Ex 61.2

Let A be the subset of $\mathbb{R}^2$ consisting of the union of the topologist's sine curve and the broken-line path from $(0,-1)$ to $(0,-2)$ to $(1,-2)$ to $(1, \sin 1)$. We call A the *closed topologist's sine curve*. Show that if C is a subspace of S^2 homeomorphic to the closed topologist's sine curve, then C separates S^2.



Theorem (63.1)

Let X be the union of two open sets U and V, such that $U \cap V$ can be written as the union of two disjoint open sets A and B. Assume that there is a path α in U from a point a of A to a point b of B, and there there is a path β in V from b to a. Let f be the loop $f = \alpha * \beta$.

(a) *The path-homotopy class $[f]$ generates an infinite cyclic subgroup of $\pi_1(X, a)$.*

(b) *If $\pi_1(X, a)$ is itself infinite cyclic, it generated by $[f]$.*

(c) *Assume there is a path γ in U from a to the point a' in A, and that there is a path δ in V from a' to a. Let g be the loop $g = \gamma * \delta$. Then the subgroups of $\pi_1(X, a)$ generated by $[f]$ and $[g]$ intersect in the identity element alone.*

Proof, Step 1

Let us take countably many copies of U and countably many copies of V, all disjoint, say

$$U \times (2n) \quad \text{and} \quad V \times (2n+1)$$

for all $z \in \mathbb{Z}$. Let Y denote the union of these spaces.
Identifying the points

$$x \times (2n) \quad \text{and} \quad x \times (2n-1) \quad \text{for } x \in A$$

and

$$x \times (2n) \quad \text{and} \quad x \times (2n+1) \quad \text{for } x \in B$$

Let $\pi : Y \to E$ be the quotient map. The map $\rho : Y \to X$ defined by $\rho(x \times m) = x$ induces a map $p : E \to X$.

- π is an open map.
- p is a covering map.

Proof, Step 2

For each n, let e_n be the point $\pi(a \times 2n)$ of E. Then e_n are distinct, and they constitute the set $p^{-1}(a)$. We define a lifting $\widetilde{f}_n$ of f that begins at e_n and ends at e_{n+1}.
Define

$$\widetilde{\alpha}_n(s) = \pi(\alpha(s) \times 2n)$$
$$\widetilde{\beta}_n(s) = \pi(\alpha(s) \times (2n+1))$$

and then $\widetilde{\alpha}_n$ and $\widetilde{\beta}_n$ are liftings of α and β, respectively, and $\widetilde{\alpha}_n * \widetilde{\beta}_n$ is defined. Set $\widetilde{f}_n = \widetilde{\alpha}_n * \widetilde{\beta}_n$ that begins e_n and ends at e_{n+1}.

Proof, Step 3

> Claim : $[f]$ generates an infinite cyclic subgroup of $\pi_1(X, a)$.

It suffices to show that if m is a positive integer, then $[f]^m$ is not the identity element.

Proof, Step 4

Claim : If $\pi_1(X, a)$ is infinite cyclic, it is generated by $[f]$.

Consider the lifting correspondence $\phi : \pi_1(X, a) \to p^{-1}(a)$. In Step 3, for each positive integer m, ϕ carries $[f]^m$ to the point e_m of $p^{-1}(a)$. Similarly, ϕ carries $[f]^{-m}$ to e_{-m}. Thus ϕ is surjective.

By Theorem 54.6, ϕ induces an injective map

$$\Phi : \pi_1(X, a)/H \to p^{-1}(a),$$

where $H = p_*(\pi_1(E, e_0))$; the map Φ is surjective because ϕ is surjective. Then H is the trivial group. Then ϕ is bijective.

Proof, Step 5

Given $g = \gamma * \delta$, define a lifting of g to E as follows:
Since γ is a path in U, we can define

$$\widetilde{\gamma}(s) = \pi(\gamma(s) \times 0);$$

since δ is a path in V, we can define

$$\widetilde{\delta}(s) = \pi(\delta(s) \times (-1)).$$

Then $\widetilde{\gamma}$ and are liftings of γ and δ. The product $\widetilde{g} = \widetilde{\gamma} * \widetilde{\delta}$ is defined and $\widetilde{g}$ is a loop in E.
Then m-fold product of f with itself lifts to a path that begins at e_0 and ends at e_m, while every product of g with itself lifts to a path beginning and ending at e_0. Hence $[f]^m \neq [g]^k$ for every nonzero m and k.

Theorem (63.2, A nonseparation theorem)

Let D *be an arc in* S^2*. Then* D *does not separate* S^2*.*

Theorem (63.3, A general nonseparation theorem)

Let D_1 *and* D_2 *be closed subsets of* S^2 *such that* $S^2 - D_1 \cap D_2$ *is simply connected. If neither* D_1 *nor* D_2 *separates* S^2*, then* $D_1 \cup D_2$ *does not separates* S^2*.*

Theorem (63.4, The Jordan curve theorem)

Let C be a simple closed curve in S^2. Then C separates S^2 into precisely two components W_1 and W_2. Each of the sets W_1 and W_2 has C as its boundary; that is $C = \overline{W}_i - W_i$ for $i = 1, 2$.

Theorem (63.5)

Let C_1 and C_2 be closed connected subsets of S^2 whose intersection consists of two points. If neither C_1 nor C_2 separates S^2, then $C_1 \cup C_2$ separates S^2 into precisely two components.

Ex 63.1

Let C_1 and C_2 be disjoint simple closed curves in S^2.

(a) Show that $S^2 - C_1 - C_2$ has precisely three components.

(b) Show that these three components have boundaries C_1 and C_2 and $C_1 \cup C_2$, respectively.

Ex 63.2

Let D be a closed connected subspaces of S^2 that separates S^2 into n-components.

(a) If A is an arc in S^2 whose intersection with D consists of one of its end points, show that $D \cup A$ separates S^2 into n components.

(b) If A is an arc in S^2 whose intersection with D consists of its end points, show that $D \cup A$ separates S^2 into $n + 1$ components.

(c) If C is a simple closed curve in S^2 that intersects D in a single point, show $D \cup C$ separates S^2 into $n + 1$ components.