Top10 Countability and Separation Axioms

KYB

Thrn, it's a Fact

mathrnfact@gmail.com

May 18, 2020

Overview

Definition

Definition (countable basis at x)

if there is cble coll $\mathcal{B}$ of nbd of x s.t. $\forall$ nbd of x contains at least one of ele of $\mathcal{B}$

Definition (first cble axiom)

if X has cble basis at each x

Definition (second cble axiom)

if X has a cble basis

Definition (dense)

$A \subset X$ is dense if $\bar{A} = X$

Theorem

Theorem (30.1)

- *(a) $A \subset X$. If $\exists \{x_n\}$ in A s.t. $x_n \to x$, then $x \in \bar{A}$. The converse holds if X is first-cble.*
- *(b) $f : X \to Y$ fct. If f is conti, then for every convergent seq $x_n \to x$ in X, $f(x_n) \to f(x)$. The converse holds if X is first-cble.*

Theorem (30.2)

- *subspace, cble product of first(second)-cble is again first(secon)-cble.*

Theorem (30.3)

Sps X has a cble basis. Then:

- *(a) Every open covering of X contains a cble subcoll covering X.*
- *(b) There exists a cble subset of X that is dense in X.*

Ex30.2

Show that if X has a cble basis $\{B_n\}$, then every basis $\mathcal{C}$ for X contains a cble basis for X.

Ex30.3

Let X has a cble basis; let A be an ucble subset of X. Show that uncountably many points of A are limit points of A.

Ex30.4

Show that every compact metrizable space X has a cble basis.

Ex30.5

(a) Show that every metrizable space with a cble dense subset has a cble basis.
(b) Show that every metrizable Lindelöf space has a cble basis.
(Lindelöf space : every open covering contains a cble subcovering)

Ex30.9

Let A be a closed subspace of X. Show that if X is Lindelöf, then A is Lindelöf.

Show by a example that if X has a cble dense subset, A need not have a cble dense subset.

Ex30.14

Show that if X is Lindelöf and Y is compact, then $X \times Y$ is Lindelöf.

Definition

Definition

Suppose X is $T1$-space.

- Hausdorff
- Regular
- Normal

Theorem

Lemma (31.1)

Suppose X is $T1$-space.

- *(a) X is regular iff given pt x and nbd U of x, there is a nbd V of x s.t. $\bar{V} \subset U$.*
- *(b) X is normal iff given closed set A and open set U containing A, there is an open set V containing A s.t. $\bar{V} \subset U$*

Theorem (31.2)

- *(a) subspace of Haus if Haus. product of Haus is Haus.*
- *(b) subspace of regular is regular. product of regular is regular.*

Excercise

Ex.31.1

Show that if X is regular, every pair of points of X have neighborhoods whose closures are disjoint.

Ex.31.2

Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.

Ex.31.5

Let $f, g : X \to Y$ be conti; assume that Y is Haus. Show that $\{x \mid f(x) = g(x)\}$ is closed in X.

Ex.31.6

Let $p : X \to Y$ be a closed conti surj map. Show that if X is normal, then so is Y.

Theorem

Theorem (32.1)

Every regular space with a countable basis is normal.

Theorem

Theorem (32.2)

Every metrizable space is normal.

Theorem

Theorem (32.3)

Every compact Hausdorff if normal.

Theorem

Theorem (32.4)

Every well-ordered set X is normal in the order topology.

Exercise

Ex32.1

Show that a closed subspace of a normal space is normal.

Ex32.2

Show that if $\prod X_\alpha$ is Haus(or regular, or normal), then so is X_α.

Ex32.3

Show that every locally compact Hausdorff space is regular.

Ex32.4

Show that every regular Lindelöf space is normal.

Ex32.6

A space X is said to be completely normal if every subspace of X is normal. Show that X is completely normal iff for every pair A, B of separated sets in X(that is, $\bar{A} \cap B = \varnothing$ and $A \cap \bar{B} = \varnothing$), there exist disjoint open sets containing them.

Ex32.9(Challenging)

Theorem

If J is uncountable, then $\mathbb{R}^J$ is not normal.

Proof) Let $X = (\mathbb{Z}_+)^J$. Since X is closed subsapce of $\mathbb{R}^J$, it will suffice to show that X is not normal.

Ex32.9

(a) If $\boldsymbol{x} \in X$ and if B is a finite subset of J, let $U(\boldsymbol{x}, B)$ denote the set consisting of all those elements $\boldsymbol{y}$ of X such that $\boldsymbol{y}(\alpha) = \boldsymbol{x}(\alpha)$ for $\alpha \in B$. Show that the set $U(\boldsymbol{x}, B)$ are a basis for X.

Ex32.9

(b) Define P_n to be the subset of X consisting of those $\boldsymbol{x}$ such that on the set $J - \boldsymbol{x}^{-1}(n)$, the map $\boldsymbol{x}$ is injective. Show that P_1 and P_2 are closed and disjoint.

Ex32.9

(c) Suppose U and V are open sets contatining P_1 and P_2, respectively. Given a sequence $\alpha_1, \cdots$ of distinct elements of J, and a sequence $0 = n_0 < n_1 < \cdots$ of integers,for $i \geq 1$ let us set $B_i \{a_1, \cdots, \alpha_{n_i}\}$ and define $\boldsymbol{x}_i \in X$ by the equations $\boldsymbol{x}_i(\alpha_j) = j$ for $1 \leq j \leq n_{i-1}$, $\boldsymbol{x}_i(\alpha) = 1$ for all other values of α. Show that one can choose the sequences α_j and n_j so that for each i, one has the inclusion $U(\boldsymbol{x}_i, B_i) \subset U$.

Ex32.9

(d) Let A be the set $\{\alpha_1, \cdots\}$ construceted in (c). Define $\boldsymbol{y} : J \to \mathbb{Z}_+$ by the equations $\boldsymbol{y}(\alpha_j) = j$ for $\alpha_j \in A$, $\boldsymbol{y}(\alpha) = 2$ for all other values of α. Choose B so that $U(\boldsymbol{y}, B) \subset V$. Then choose i so that $B \cap A$ is contained in the set B_i. Show that $U(\boldsymbol{x}_{i+1}, B_{i+1}) \cap U(\boldsymbol{y}, B)$ is not empty.

The End