Algebraic Topology
- Dunkin's Torus 1 -

KYB

Thrn, it's a Fact

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The Fundamental Group

- Homotopy of Paths

Text : Topology/James Raymond Munkres –2nd ed

선수지식

- 필요할때 앞의 내용(50장까지)을 찾아가서 공부할 수 있는 수준
- Group과 group homomorphism의 정의를 알고 있는 상황 (혹은 선형대수에대한 경험이 있는 경우)
- Quotient Topology를 이해 하고 있어야함

Summary of General Topology

- 공간의 일반화(Topology)
- 공간의 성질(Compactness, Connectedness, Countability, Separation Axioms, Metrizable, etc.)

Goal

- 공간의 분류

Poincaré Conjecture

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

Homotopy of Paths

Definition

- If f and f' are continuous maps of the space X into the space Y, we say that f is *homotopic* to f' if there is a continuous map $F : X \times I \to Y$ such that
$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = f'(x)$$
for each x. (Here $I = [0, 1]$.)
- The map F is called a *homotopy* between f and f'.
- If f is homotopic to f', we write $f \simeq f'$.
- If $f \simeq f'$ and f' is a constant map, we say that f is *nullhomotopic*.

Definition

- Two paths f and f', mapping the interval I into X, are said to be *path homotopic* if they have the same initial point x_0 and the same final point x_1 and if there is a continuous map $F : I \times I \to X$ such that
$$\begin{aligned} F(s, 0) = f(s) \quad &\text{and} \quad F(s, 1) = f'(s), \\ F(0, t) = x_0 \quad &\text{and} \quad F(1, t) = x_1. \end{aligned}$$
for each $s \in I$ and each $t \in I$.
- We call F a path homotopy between f and f'.
- If f is path homotopic to f', we write $f \simeq_p f'$.

Lemma (51.1)

The relations $\simeq$ and $\simeq_{\mathrm{p}}$ are equivalence relations.

Example

Let f and g be any two maps of a space X into $\mathbb{R}^2$.

$$F(x, t) = (1 - t)f(x) + tg(x)$$

is a homotopy between them. It is called a *straight-line homotopy*.
More generally, let A be any convex subspace of $\mathbb{R}^n$. Then any two paths f, g in A from x_0 to x_1 are path homotopic in A.

Example

Let X denote the *punctured plane*, $\mathbb{R}^2 - \{\mathbf{0}\}$(or simply $\mathbb{R}^2 - \mathbf{0}$).

$$\begin{aligned} f(s) &= (\cos \pi s, \sin \pi s), \\ g(s) &= (\cos \pi s, 2 \sin \pi s), \\ h(s) &= (\cos \pi s, -\sin \pi s) \end{aligned}$$

Homotopy of Paths

Definition

If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the *product* $f * g$ of f and g to be the path h given by the equations

$$h(s) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ g(2s-1) & s \in [\frac{1}{2}, 1]. \end{cases}$$

The function h is well-defined and continuous, by the pasting lemma; it is a path in X from x_0 to x_2.

Definition

The product on paths induces a well-defined operation on path-homotopy classes, defined by the equation

$$[f] * [g] = [f * g].$$

Theorem (51.2)

The operation $*$ has the following properties:

(1) *(Associativity) If $[f] * ([g] * [h])$ is defined, so is $([f] * [g]) * [h]$, and they are equal.*

(2) *(Right and left identities) Given $x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all of I to the point x. If f is a path in X from x_0 to x_1, then*

$$[f] * [e_{x_1}] = [f] \quad \textit{and} \quad [e_{x_0}] * [f] = [f].$$

(3) *(Inverse) Given the path f in X from x_0 to x_1, let $\bar{f}$ be the path defined by $\bar{f}(s) = f(1 - s)$. It is called the reverse of f. Then*

$$[f] * [\bar{f}] = [e_{x_0}] \quad \textit{and} \quad [\bar{f}] * [f] = [e_{x_1}].$$

Theorem (51.3)

Let f *be a path in* X*, and let* $a_0, \cdots, a_n$ *be numbers such that* $0 = a_0 < a_1 < \cdots < a_n = 1$*. Let* $f_i : I \to X$ *be the path that equals the positive linear map of* I *onto* $[a_{i-1}, a_i]$ *followed by* f*. Then*

$$[f] = [f_1] * [f_2] * \cdots * [f_n].$$

Ex 51.1

Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Ex 51.2

Given spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps of X into Y.

(a) Show that for any X, the set $[X, I]$ has a single element.

(b) Show that if Y is path connected, the set $[I, Y]$ has a single element.

Ex 51.3

A space X is said to be *contractible* if the identity map $i_X : X \to X$ is nulhomotopic.

(a) Show that I and $\mathbb{R}$ are contractible.

(b) Show that a contractible space is path connected.

(c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.

(d) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.