Topology

Thrn
it's a Fact

Properties of Number Systems
Order Relation

튜터링

<u>Def</u> Cartesian Product of A and B

$$A \times B = \{ (a, b) \mid a \in A, b \in B \}$$

↑ ordered pair

「 sometimes denote (a, b) by $a \times b$ (not multiplication !)

<u>Def</u> Relation of A and B

R is called a relation of A and B if $R \subseteq A \times B$.

if $(a, b) \in R$, denote aRb

Examples)

"$=$" $\subseteq A \times A$ s.t. "$=$" $= \{(a,a) \mid a \in A\}$

$<\ \subseteq \mathbb{R} \times \mathbb{R}$ s.t. $< \ = \{(a,b) \mid a<b,\ a,b \in \mathbb{R}\}$

"$\leq$", "$\subseteq$", "$\neq$", ...

Let f be a function from A to B.

Define $F = \{(a, f(a)) \mid a \in A\} \subseteq A \times B$.

$\rightarrow$ f is a relation

Def Order Relation on A

Let C be a relation on A s.t.

(1) $\forall x, y \in A$ w/ $x \neq y$, either xCy or yCx

(2) there is no $x \in A$ s.t. xAx

(3) if xCy and yCz, then xCz

We say C is an order relation (simple order, linear order)

Ex) $<, >$ order relations

$=, \leq, \geq, \subseteq, \supseteq$ not order relations

Some Properties of Number System $\mathbb{N}, \mathbb{Q}, \mathbb{R}$

natural number $\mathbb{N}$

(1) $0 \in \mathbb{N}$

successor of n

(2) $n \in \mathbb{N} \implies n+1 \in \mathbb{N}$

(3) $\mathbb{N}$ satisfies "the Well-ordering Property"

(2)* $n+m = n + \underbrace{1 + \cdots + 1}_{m \text{ times}}$ / $a < b$ iff $\exists\, m \in \mathbb{N}$ s.t. $a+m = b$

order relation

Def Let X be a set w/ order relation $<$. $(X,<)$

We say X has the well-ordering property if every nonempty subset U of X has a least elt.

⌜ $m \in U$ is a least element if $\forall x \in U$, $\underbrace{\text{either } m<x \text{ or } m=x}_{m \leq x}$ ⌟

$\mathbb{N}$ ✓

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ ×

Mathematical Induction

(i) $P(1)$ true

(ii) if $P(n)$ true, then $P(n+1)$ true.

Then $\forall n \in \mathbb{N}$, $P(n)$ true

$-n$: $n + (-n) = 0$ in $\mathbb{Z}$

$(X,<)$ Let $S \subseteq X$. We call S is bounded if

$\exists x \in X$ s.t. $\forall y \in S$, $y \leq x$.

In this case, we say x is an upper bound of S.

<u>Lemma</u> $\mathbb{N}$ is not bounded.

$\square$ Sps not, say N is an upper bound of $\mathbb{N}$.

Since $N \in \mathbb{N}$, $N+1 \in \mathbb{N}$. $\longrightarrow$ $N+1 \leq N$ $\lightning$

Rational Number $\mathbb{Q}$

(1) $^{\forall} q \in \mathbb{Q}$ has a form $q = \frac{n}{m}$ for some $n, m \in \mathbb{Z}$ w/ $m \neq 0$

(2) $^{\forall} p, q \in \mathbb{Q}$ w/ $p < q$, $\exists\, r \in \mathbb{Q}$ s.t. $p < r < q$ ($\mathbb{Q}$ is dense)

<u>Def</u> Let $(X, <)$. For $U \subseteq X$, we say $u \in X$ is an upper bound of U

if $\forall x \in U$, $x \leq u$ ⌜ $l \in X$ s.t. $\forall x \in U$ $l \leq x$

$\to$ l is a lower bound ⌟

<u>Def</u> Let s be an upper bound of U s.t.

if $u \in X$ is another upper bound of U, then $s \leq u$.

Then we say s is the least upper bound (or supremum)

denote $s = \sup U$.

⌜ $i = \inf U$ ⌟

Def Let $(X, <)$.

If every nonempty $U \subseteq X$ which has an upper bound has $\sup U \in X$, we say X has "the Least Upper Bound Property".

Ex) $\mathbb{Q}$ does not have the L.U.B.P

$$\sup \{x \in \mathbb{Q} \mid x^2 \leq 2\} = \sqrt{2}.$$

Note) $\mathbb{R}$ has the L.U.B.P (completeness)

$\mathbb{N}$: W.O.P

$\mathbb{Q}$: Dense

$\mathbb{R}$: Complete

- For any $x \in \mathbb{R}$, $\exists N \in \mathbb{N}$ s.t. $x < N$

- For any $\varepsilon \in \mathbb{R}$ w/ $\varepsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \varepsilon$

- For any $x, y \in \mathbb{R}$ w/ $x < y$, $\exists q \in \mathbb{Q}$ s.t. $x < q < y$

$$\bigcap_{n=1}^{\infty} \left(-1, \frac{1}{n}\right) = (-1, 0]$$

$$\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2\right) = (0, 2)$$

Def Let $(X, <)$ and $a<b$ in X.

$(a, b) = \{x \mid a<x<b\}$ is called an open interval in X

not ordered pair

if $(a,b) = \phi$, a : immediate predecessor of b
b : immediate successor of a

Ex) $X = \mathbb{N}$, $(n, n+1) = \phi$

Sps $S \subseteq X$ has a supremum.

$\to$ ① $\forall x \in S$, $x \leq \sup S$

② if u is an upper bound of S, $\sup S \leq u$.

$\mathbb{R}$ satisfies "the Least Upper Bound Property"

$\rotatebox{90}{$\subseteq$}$

$\forall X$ bounded above, X has $\sup X$.