LA4 Linear Operator, Matrix

KYB

Thrn, it's a Fact

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Overview

Ex 2.6.11, 2.7.13

Let F be a finite field with $|F| = q$. Consider $\dim(\mathcal{P}_n(F))$.

(1) If $n \leq q-1$, then $\{1, x, \cdots, x^n\}$ is linearly independent. So $\dim(\mathcal{P}_n(F)) = n+1$.

Proof.

Suppose $a_0 + a_1 x + \cdots + a_n x^n = 0$. Since every nonzero polynomial has at most "n" roots, $a_0 = a_1 = \cdots = a_n = 0$. $\square$

Ex 2.6.11, 2.7.13

Let F be a finite field with $|F| = q$. Consider $\dim(\mathcal{P}_n(F))$.

(2) If $n \geq q$, then $\{1, x, \cdots, x^{q-1}\}$ is linearly independent. So $\dim(\mathcal{P}_n(F)) \geq q$. In fact, $\dim(\mathcal{P}_n(F)) = q$.

Proof.

Suppose $F = \{\alpha_1, \cdots, \alpha_q\}$. Then any function $f : F \to F$ is determined by

$$(f(\alpha_1), \cdots, f(\alpha_q)) \in F^q.$$

Thus there are at most $|F^q|$ many polynomials. On the other hand, $\dim(\mathcal{P}_n(F)) \geq q$ implies there are at least $|F^q|$ many polynomials. Hence $\dim(\mathcal{P}_n(F)) = q$.
(Note that every $v \in F^q$ can be identified with a function $v : F \to F$ by $v(\alpha_i) = v_i$). $\square$

Linear Operator

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y).$$

Example

1. $C^1(\mathbb{R}) \to C^0(\mathbb{R})$ by $f \mapsto \frac{df}{dx}$.
2. $C^0[0,1] \to \mathbb{R}$ by $f \mapsto \int_0^1 f(x)dx$.

$m \times n$ matrix, $F^{m\times n}$

$$\begin{array}{cc} & \begin{array}{ccc} & n & \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{cc} & \rightarrow \\ & \rightarrow \\ m & \vdots \\ & \rightarrow \end{array} & \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ A_{21} & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} = A, \end{array}$$

$A_{ij} \in \mathbb{R}.$

$$A_j = \begin{bmatrix} A_{1i} \\ \vdots \\ A_{mi} \end{bmatrix} = (A_{1i}, \cdots, A_{mi}) \in F^m, \quad r_i = [A_{i1}|\cdots|A_{in}]$$

$m \times n$ matrix, $F^{m\times n}$

$$Ax = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 A_1 + \cdots + x_n A_n$$

$$\Longrightarrow \quad Ax = \sum_{i=1}^{n} x_i A_i, \quad (Ax)_i = \sum_{j=1}^{n} A_{ij} x_j$$

For $A \in F^{m\times}$, $B \in F^{n\times l}$,

$$AB = [AB_1 | \cdots | AB_l] \in F^{m\times l}.$$

$$A^T = F^{n\times m}, \text{ and } (A^T)_{ij} = A_{ji}.$$

Ex 3.1.8

Let $L : \mathcal{P}_n \to \mathcal{P}_{2n-1}$ be defined by $L(p) = pp'$. Is L linear or nonlinear?.

Ex 3.1.10

(a) $A \in \mathbb{R}^{3\times 3}$, $b \in \mathbb{R}^4$,

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 5 & -11 & -15 \\ -2 & 6 & -1 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 23 \\ -14 \end{bmatrix}.$$

Ex 3.1.10

(b) $A \in \mathbb{Z}^{3\times 3}$, $b \in \mathbb{Z}^4$,

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Ex 3.1.12

Give a formula for $(AB)_{ij}$, assuming $A \in F^{m\times n}$, $B \in F^{n\times p}$.

Theorem

Let X and U be vector spaces over a field F.

1. *If $L : X \to U$ is linear and $\alpha \in F$, then αL is also linear.*
2. *If $L, M : X \to U$ are linear, then so is $L + M$.*

Remark

- *Let $\mathcal{L}(X, U)$ be the set of all linear operator from X to U. Then it is a vector space over F.*
- *For a vector space X, $\mathcal{L}(X, F)$ is called the (algebraic) "dual space" of X, and write $X^* = \mathcal{L}(X, F)$.*

The Matrix of a Linear Operator on Euclidean Spaces

Let $L : F^n \to F^m$ be linear. We can find a matrix $A \in F^{m\times n}$ such that $L(x) = Ax$ as follows:

$$L(e_i) = a_{i1}e_2 + \cdots + a_{im}e_m$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \ddots & \vdots \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Then $L(e_i) = A_i$ and

$$L\left(\sum_{i=1}^{n} \alpha_i e_i\right) = \sum_{1}^{n} \alpha_i L(e_i) = \sum_{1}^{n} \alpha_i A_i = A \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = Ax$$

Hence, $L(x) = Ax$.

Associated Matrix with Bases

Let $L : X \to U$ be linear and let $\mathcal{X} = \{x_1, \cdots, x_n\}$, $\mathcal{U} = \{u_1, \cdots, u_m\}$ be bases for X and U, respectively. Then

$$L(x) = L\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \sum_{i=1}^{n} \alpha_i L(x_i)$$

and

$$L(x_i) = \sum_{j=1}^{m} \beta_{ij} u_j.$$

Associated Matrix with Bases

Let $E_X : X \to F^n$ and $E_U : U \to F^m$ by $E_X(x_i) = e_i$, $E_U(u_j) = e_j$. Then

$$
\begin{tikzcd}
x_i \arrow[d, mapsto] & X \arrow[r, "L"] \arrow[d, "E_X"'] & U \arrow[d, "E_U"] & u_j \arrow[d, mapsto] \\
e_i & F^n \arrow[r, "A"'] & F^m & e_j
\end{tikzcd}
$$

So

$$L = E_Y^{-1} \circ A \circ E_X$$

Associated Matrix with Bases

Main Idea: $(X, \mathcal{X})$, $(U, \mathcal{U})$, $\dim X = n$, $\dim U = m$.

$$L\left(\sum_1^n \alpha_i x_i\right) = \sum_1^m \beta_j u_j \implies F^n \longrightarrow F^m$$

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \mapsto \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$$

Then there is a unique $A \in F^{m \times n}$ such that

$$A \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$$

Write $A = [L]_{\mathcal{X},\mathcal{U}}$. A depends on the choices of bases

Remark

Every linear map is fully determined by basis elements. If for all $x_i \in \mathcal{B}$ $L(x_i) = y_i$, then for all $x \in X$, $L(x)$ is determined uniquely.

$$[L]_{\mathcal{X},\mathcal{U}} = [L(x_1)| \cdots |L(x_n)] \,.$$

Ex 3.2.1

Let A be an $m \times n$ matrix with real entries, $n > m$. Prove that $Ax = 0$ has a nontrivial solution $x \in \mathbb{R}^n$.

Ex 3.2.2

Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation of angle θ about the origin.

(a) Give a geometric argument that R is linear.

(b) Find the matrix A such that $R(x) = Ax$ for all $x \in \mathbb{R}^2$.

Ex 3.2.9, Convolution

Let $x \in \mathbb{R}^N$ be denoted as $x = (x_0, x_1, \cdots, x_{N-1})$. Given $x, y \in \mathbb{R}^N$, the convolution of x and y is the vector $x * y \in \mathbb{R}^N$ defined by

$$(x * y)_n = \sum_{m=0}^{N-1} x_m y_{n-m}$$

In this formula, if $n - m < 0$, we take $y_{n-m} = y_{N+n-m}$. Prove that if $y \in \mathbb{R}^N$ is fixed, then the mapping $x \mapsto x * y$ is linear. Find the matrix representing this operator.

Definition

Let X and Y be any sets and let $f : X \to Y$ be a function.

1. f is injective if and only if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
2. f is surjective if and only if for each $y \in Y$, there exits $x \in X$ such that $f(x) = y$.
3. If f is both injective and surjective, then f is called bijective.

Remark

- $f : X \to Y$ is bijective if and only if there is $g : Y \to X$ such that $f \circ g(y) = y$ and $g \circ f(x) = x$ for all $x \in X$, $y \in Y$. Define $g = f^{-1}$ and we say "f is invertible".
- If $L : X \to U$ is an invertible linear operator, then L^{-1} is again linear. In this case, we say 1) "L is an isomorphism" and 2) X is isomorphic to U.

Theorem

Suppose X and Y are n-dimensional vector spaces over a field F. Then $X \cong Y$.

Remark

Notation)

$$x \in X \implies [x]_{\mathcal{X}} := E_X(x), u \in U \implies [u]_{\mathcal{U}} := E_U(u)$$
$$A = [L]_{\mathcal{X},\mathcal{U}}, [L]_{\mathcal{X},\mathcal{U}}[x]_{\mathcal{X}} = [L(x)]_{\mathcal{U}}.$$

$$\begin{tikzcd}
F^n \arrow[r, "A"] & F^m \\
X \arrow[u, "E_X"] \arrow[r, "L"'] & U \arrow[u, "E_U"']
\end{tikzcd}$$

Remark

$\mathcal{X} = \{x_1, \cdots, x_n\}, \mathcal{U} = \{u_1, \cdots, u_m\}$.

(1) $[x_i]_{\mathcal{X}} = E_X(x_i) = e_i^n$, $[u_j]_{\mathcal{U}} = E_U(u_j) = e_j^m$.

(2) $[L]_{\mathcal{X},\mathcal{U}} = A = [A_1|\cdots|A_n]$.

$$A_i = Ae_i = A[x_i]_{\mathcal{X}} = [L]_{\mathcal{X},\mathcal{U}}[x_i]_{\mathcal{X}} = [L(x_i)]_{\mathcal{U}}.$$

Hence,

$$[L]_{\mathcal{X},\mathcal{U}} = [[L(x_1)]_{\mathcal{U}}|\cdots|[L(x_n)]_{\mathcal{U}}]$$

Ex 3.3.7

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $L(x) = Ax$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let $\mathcal{S}$ be the standard basis, and let $\mathcal{X} = \{(1,1),(1,2)\}$. Find $[L]_{\mathcal{X},\mathcal{X}}$.

Ex 3.3.12

Let $\mathcal{X} = \{(1,0,0),(1,1,0),(1,1,1)\} \subset \mathbb{Z}_2^3$.

(b) Find $[x]_{\mathcal{X}}$ for arbitrary vector x in $\mathbb{Z}_2^3$.

Ex 3.3.13

Let $\mathcal{X} = \{(1,0,0),(1,1,0),(1,1,1)\} \subset \mathbb{Z}_2^3$, let $A \in \mathbb{Z}_2^{3\times 3}$ be defined by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

and define $L : \mathbb{Z}_2^3 \to \mathbb{Z}_2^3$ by $L(x) = Ax$. Find $[L]_{\mathcal{X},\mathcal{X}}$.

Ex 3.3.17

$$F^{m\times n} \cong F^{mn}.$$

Ex 3.3.5

Let X, Y, and Z be sets, and suppose $f : X \to Y$, $g : Y \to Z$ are given functions.

(a) f and $g \circ f$ invertible $\implies$ g invertible?

Injective:

Surjective:

(b) g and $g \circ f$ invertible $\implies$ f invertible?

Injective:

Surjective:

(c) $g \circ f$ invertible $\implies$ f, g invertible?

Definition

Let X, U be vector spaces over a field F, and let $L : X \to U$ be linear.

- $\ker(L) = \{x \in X : L(x) = 0\}$.
- $\mathcal{R}(L) = \{L(x) : x \in X\} = \{u \in U : L(x) = u \text{ for some } x \in X\}$.

Theorem

$\ker(L)$ *is a subspace of* X *and* $\mathcal{R}(L)$ *is a subspace of* U.

Remark

(1) L is injective if and only if $\ker(L) = \{0\}$.

(2) L is surjective if and only if $\mathcal{R}(L) = U$.

(3) If X and U are both n-dimensional, then

$$L \text{ is injective} \iff L \text{ is surjective} \iff L \text{ is bijective}$$
$$\iff \ker(L) = \{0\} \iff \mathcal{R}(L) = U$$

The End