Analysis
- PMA 11 -

KYB

Thrn, it's a Fact

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February 15, 2021

Overview

Definition and Existence of the Integral

Definition

Let $[a, b]$ be a given interval. A partition P of $[a, b]$ is a finite set of points $x_0, \cdots, x_n$, where

$$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b.$$

We write $\Delta x_i = x_i - x_{i-1}$ for $i = 1, \cdots, n$.

Definition and Existence of the Integral

Upper and Lower Sums

Suppose f is a bounded real function defined on $[a, b]$. Given partition, put

$$\begin{aligned} M_i &= \sup f(x) & , x_{i-1} \le x \le x_i \\ m_i &= \inf f(x) & , x_{i-1} \le x \le x_i \\ U(P, f) &= \sum_{i=1}^{n} M_i \Delta x_i, \\ L(P, f) &= \sum_{i=1}^{n} m_i \Delta x_i \end{aligned}$$

and

$$\overline{\int_a^b} f\, dx = \inf_P U(P, f), \quad \underline{\int_a^b} f\, dx = \sup_P L(P, f),$$

where inf and sup are taken over all partitions P of $[a, b]$. If they are equal, we say f is Riemann-integrable on $[a, b]$, and we write $f \in \mathscr{R}$,and denote the common value by

$$\int_a^b f(x)\, dx.$$

Definition and Existence of the Integral

Remark

Since f is bounded, there is m and M such that $m \leq f \leq M$. Hence, for every P,

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a),$$

so $L(P,f)$ and $U(P,f)$ form a bounded set (over P). This shows that the upper and lower integrals are defined for every bounded function f.

Definition and Existence of the Integral

Definition (The Riemann-Stieltjes Integral)

Let α be a monotonically increasing function on $[a,b]$. Given partition P, we write $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. Then $\Delta\alpha_i \geq 0$. For any real function f which is bounded on $[a,b]$, we put

$$
\begin{aligned}
M_i &= \sup f(x) && , x_{i-1} \leq x \leq x_i \\
m_i &= \inf f(x) && , x_{i-1} \leq x \leq x_i \\
U(P,f,\alpha) &= \sum_{i=1}^{n} M_i \Delta x_i, \\
L(P,f,\alpha) &= \sum_{i=1}^{n} m_i \Delta x_i
\end{aligned}
$$

and

$$\overline{\int_a^b} f\, d\alpha = \inf_P U(P,f,\alpha), \quad \underline{\int_a^b} f\, d\alpha = \sup_P L(P,f,\alpha),$$

If they are equal, we write $f \in \mathscr{R}(\alpha)$, and denote

$$\int_a^b f(x)\, d\alpha(x).$$

Note that α need not be continuous.

Definition and Existence of the Integral

Definition

We say the partition P^* is a refinement of P if $P^* \supset P$. Given two partitions, P_1 and P_2, we say that P^* is their common refinement if $P^* = P_1 \cup P_2$.

Theorem

If P^* is a refinement of P, then

$$L(P, f, \alpha) \leq L(P^*, f, \alpha), \quad U(P^*, f, \alpha) \leq U(P, f, \alpha).$$

Definition and Existence of the Integral

Theorem

$$\underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha.$$

Theorem

$f \in \mathscr{R}(\alpha)$ on $[a,b]$ if and only if for every $\epsilon > 0$, there exists a partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon.$$

Definition and Existence of the Integral

Theorem

(a) *If* $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ *for some* P *and some* ϵ*, then* $U(P^*, f, \alpha) - L(P^*, f, \alpha) < \epsilon$ *for every refinement* P^* *of* P*.*

(b) *If* $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ *for* $P = \{x_0, \cdots, x_n\}$ *and if* s_i, t_i *are arbitrary points in* $[x_{i-1}, x_i]$*, then*

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon$$

(c) *If* $f \in \mathscr{R}(\alpha)$ *and the hypotheses of (b) hold, then*

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\, d\alpha \right| < \epsilon.$$

Definition and Existence of the Integral

Theorem

If f is continuous on $[a, b]$, then $f \in \mathscr{R}(\alpha)$ on $[a, b]$.

Definition and Existence of the Integral

Theorem

If f is monotonic on $[a, b]$ and if α is monotonic continuous on $[a, b]$, then $f \in \mathscr{R}(\alpha)$.

Definition and Existence of the Integral

Theorem

Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathscr{R}(\alpha)$.

Definition and Existence of the Integral

Theorem

Suppose $f \in \mathscr{R}(\alpha)$ *on* $[a, b]$, $m \leq f \leq M$, ϕ *is continuous on* $[m, M]$, *and* $h(x) = \phi(f(x))$ *on* $[a, b]$. *Then* $h \in \mathscr{R}(\alpha)$ *on* $[a, b]$.

Properties of the Integral

Theorem

(a) *If* $f_1, f_2 \in \mathscr{R}(\alpha)$ *on* $[a,b]$, *then* $f_1 + f_2 \in \mathscr{R}(\alpha)$, $cf \in \mathscr{R}(\alpha)$ *for every constant* c, *and*

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha, \quad \int_a^b cf\, d\alpha = c \int_a^b f\, d\alpha$$

(b) *If* $f_1 \leq f_2$ *on* $[a,b]$, *then* $\int_a^b f_1\, d\alpha \leq \int_a^b f_2\, d\alpha$.

(c) *If* $f \in \mathscr{R}(\alpha)$ *on* $[a,b]$ *and if* $a < c < b$, *then* $f \in \mathscr{R}(\alpha)$ *on* $[a,c]$ *and on* $[c,b]$ *and,*

$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha.$$

(d) *If* $f \in \mathscr{R}$ *on* $[a,b]$ *and if* $|f(x)| \leq M$ *on* $[a,b]$, *then* $\left| \int_a^b f\, d\alpha \right| \leq M[\alpha(b) - \alpha(a)]$.

(e) *If* $f \in \alpha$ *and* $f \in \alpha$, *then* $f \in \mathscr{R}(\alpha_1 + \alpha_2)$, *and*

$$\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2;$$

if $f \in \mathscr{R}(\alpha$ *and* c *is a positive constant, then* $f \in \mathscr{R}(c\alpha)$ *and*

$$\int_a^b f\, d(c\alpha) = c \int_a^b f\, d\alpha.$$

Properties of the Integral

Theorem

If $f, g \in \mathscr{R}(\alpha)$ on $[a, b]$, then

(a) $fg \in \mathscr{R}(\alpha)$;

(b) $|f| \in \mathscr{R}(\alpha)$ *and* $\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha$.

Properties of the Integral

Definition

The unit step function I is defined by

$$I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Theorem

If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s, and $\alpha(x) = I(x - s)$, then

$$\int_a^b f \, d\alpha = f(s).$$

Properties of the Integral

Theorem

Suppose $c_n \geq 0$ *for* $1, 2, 3, c \ldots$, $\sum c_n$ *converges,* $\{s_n\}$ *is a sequence of distinct points in* (a, b)*, and*

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n).$$

Let f *be continuous on* $[a, b]$*, then*

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n).$$

Properties of the Integral

Theorem

Assume α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$. In that case,

$$\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)\, dx.$$

Properties of the Integral

Theorem (Change of Variable)

Suppose φ *is a strictly increasing continuous function that maps an interval* $[A, B]$ *onto* $[a, b]$*. Suppose* α *is monotonically increasing on* $[a, b]$ *and* $f \in \mathscr{R}(\alpha)$ *on* $[a, b]$*. Define* β *and* g *on* $[A, B]$ *by*

$$\beta(y) = \alpha(\varphi(y)), \quad g(y) = f(\varphi(y)).$$

Then $g \in \mathscr{R}(\beta)$ *and*

$$\int_A^B g\, d\beta = \int_a^b f\, d\alpha.$$

Integration and Differentiation

Theorem

Let $f \in \mathscr{R}$ on $[a,b]$. For $a \leq x \leq b$, put

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is continuous on $[a,b]$; furthermore, if f is continuous at a point x_0 of $[a,b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

Integration and Differentiation

Theorem (The Fundamental Theorem of Calculus)

If $f \in \mathscr{R}$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Integration and Differentiation

Theorem (Integration by Parts)

Suppose F and G are differentiable functions on $[a,b]$, $F' = f \in \mathscr{R}$, and $G' = g \in \mathscr{R}$. Then

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx.$$

Integration of Vector-Valued Functions

Definition

Let $f_1, \cdots, f_k$ be real functions on $[a, b]$, and let $\mathbf{f} = (f_1, \cdots, f_k)$ be the corresponding mapping of $[a, b]$ into $\mathbb{R}^k$. If α increases monotonically on $[a, b]$, to say that $\mathbf{f} \in \mathscr{R}(\alpha)$ means that $f_j \in \mathscr{R}(\alpha)$ for $j = 1, \cdots, k$. If this is the case, we define

$$\int_a^b \mathbf{f}\, d\alpha = \left(\int_a^b f_1\, d\alpha, \cdots, \int_a^b f_k\, d\alpha \right).$$

Theorem

If $\mathbf{f}$ and $\mathbf{F}$ map $[a, b]$ into $\mathbb{R}^k$, if $\mathbf{f} \in \mathscr{R}$, and if $\mathbf{F}' = \mathbf{f}$, then

$$\int_a^b \mathbf{f}(t)\, dt = \mathbf{F}(b) - \mathbf{F}(a).$$

Integration of Vector-Valued Functions

Theorem

If $\mathbf{f}$ *maps* $[a,b]$ *into* $\mathbb{R}^k$ *and if* $\mathbf{f} \in \mathscr{R}(\alpha)$ *for some monotonically increasing function* α *on* $[a,b]$*, then* $|\mathbf{f}| \in \mathscr{R}(\alpha)$*, and*

$$\left| \int_a^b \mathbf{f}\, d\alpha \right| \leq \int_a^b |\mathbf{f}|\, d\alpha.$$

Rectifiable Curves

Definition

A continuous mapping γ of an interval $[a, b]$ into $\mathbb{R}^k$ is called a curve in $\mathbb{R}^k$. To emphasize the parameter interval $[a, b]$, we may also say that γ is a curve on $[a, b]$.

- If γ is one-to-one, γ is called an arc.
- If $\gamma(a) = \gamma(b)$, γ is said to be a closed curve.

Observe

We associate to each partition $P = \{x_0, \cdots, x_n\}$ of $[a, b]$ and to each curve γ on $[a, b]$ the number

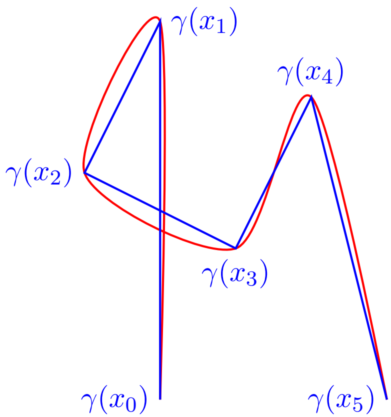
$$\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|.$$

$\Lambda(P, \gamma)$ is the length of a polygonal path with vertices at $\gamma(x_0), \cdots, \gamma(x_n)$, in this order. As our partition becomes finer and finer, this polygon approaches the range of γ more and more closely. This makes it seem reasonable to define the length of γ as

$$\Lambda(\gamma) = \sup_P \Lambda(P, \gamma).$$

- If $\Lambda(\gamma) < \infty$, we say that γ is rectifiable.

Rectifiable Curves



Rectifiable Curves

Theorem

If γ' is continuous on $[a, b]$, then γ is rectifiable, and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)| \, dt.$$

Exercises

Ex 6.1

Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is cotinuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f \, d\alpha = 0$.

Exercises

Ex 6.2

Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Exercises

Ex 6.3

Define three functions $\beta_1, \beta_2, \beta_3$ as follows:

$$\beta_j(x) = 0 \text{ if } x < 0, \quad \beta_j(x) = 1 \text{ if } x > 0$$

and

$$\beta_1(0) = 0, \quad \beta_2(0) = 1, \quad \beta_3(0) = \frac{1}{2}.$$

Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then $\int f\, d\beta_1 = f(0)$.

(b) State and prove a similar result for β_2.

(c) Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0.

(d) If f is continuous at 0, prove that $\int f\, d\beta_1 = \int f\, d\beta_2 = \int f\, d\beta_3 = f(0)$.

Exercises

Ex 6.4

If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a, b]$ for any $a < b$.

Exercises

Ex 6.5

Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathscr{R}$ on $[a, b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

Exercises

Ex 6.6

Let P be the Cantor set. Let f be a bounded real function on $[0, 1]$ which is continuous at every point outside P. Prove that $f \in \mathscr{R}$ on $[0, 1]$.

Exercises

Ex 6.7, Improper Integrals

Suppose f is a real function on $(0,1]$ and $f \in \mathscr{R}$ on $[c,1]$ for every $c > 0$. Define

$$\int_0^1 f(x)\,dx = \lim_{c \to 0} \int_c^1 f(x)\,dx$$

if this limit exists and is finite.

(a) If $f \in \mathscr{R}$ on $[0,1]$, show that this definition of the integral agrees with the old one.

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

Exercises

Ex 6.8, Integral Test

Suppose $f \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

if this limit exists and is finite. In this case, we say that the integral on the left converges. If it also converges after f has been replaced by $|f|$, it is said to converge absolutely.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that $\int_1^\infty f(x)\,dx$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges.

Exercises

Ex 6.9

Show that integration by parts can sometimes be applied to the improper integrals. Show that one of these integrals converges absolutely, but that the other does not.

Exercises

Ex 6.10, Hölder's Inequality

Let p and q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove that the following statements.

(a) If $u, v \geq 0$, then $uv \leq u^p/p + v^q/q$. Equality holds if and only if $u^p = v^q$.

(b) If $f, g \in \mathscr{R}(\alpha)$, $f, g \geq 0$, and $\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha$, then

$$\int_a^b f g \, d\alpha \leq 1.$$

(c) If f and g are complex functions in $\mathscr{R}(\alpha)$, then

$$\left| \int_a^b f g \, d\alpha \right| \leq \left(\int_a^b |f|^p \, d\alpha \right)^{1/p} \left(\int_a^b |g|^q \, d\alpha \right)^{1/q}$$

(d) Show that Hölder's inequality is also true for the improper integrals.

Exercises

Ex 6.11

Let α be a fixed increasing function $[a, b]$. For $u \in \mathscr{R}(\alpha)$, define

$$\|u\|_2 = \left(\int_a^b |u|^2 \right)^{1/2}.$$

Suppose $f, g, h \in \mathscr{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \le \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality.

Exercises

Ex 6.12

Suppose $f \in \mathscr{R}(\alpha)$ and $\epsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $\|f - g\|_2 < \epsilon$.

Exercises

Ex 6.13

Define

$$f(x) = \int_x^{x+1} \sin(t^2)\, dt.$$

(a) Prove that $|f(x)| < 1/x$ if $x > 0$.

(b) Prove that $2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$ where $|r(x)| < c/x$ and c is constant.

(c) Find the upper and lower limits of $xf(x)$, as $x \to \infty$.

(d) Does $\int_0^\infty \sin(t^2)\, dt$ converge?

Exercises

Ex 6.14

Define

$$f(x) = \int_x^{x+1} \sin(e^t)\, dt.$$

Show that $e^x|f(x)| < 2$ and that $e^x f(x) = \cos(e^x) - e^{-1}\cos(e^{x+1}) + r(x)$ where $|r(x)| < Ce^{-x}$ for some constant C.

Exercises

Ex 6.15

Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f^2(x)\,dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x)\,dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2\,dx \cdot \int_a^b x^2 f^2(x)\,dx > \frac{1}{4}.$$

Exercises, Riemann's zeta function

Ex 6.16

For $1 < s < \infty$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Prove that

(a) $\zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s+1}}\, dx$

(b) $\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{x - [x]}{x^{s+1}}\, dx.$

(c) Prove that the integral in (b) converges for all $s > 0$.

The End