LA6 Linear Operator Equations, FTLA

KYB

Thrn, it's a Fact

mathrnfact@gmail.com

July 17, 2021

Overview

Ex 3.3.17

(c) $\mathcal{L}(X, U) \cong F^{m \times n}$.

- $\mathcal{L}(F,F) \cong F$
- $\mathcal{L}(X,F) := X^*$ (algebraic) dual space of X

$$\mathcal{L}(X,F) \cong \mathcal{L}(F^n,F) \cong (\mathcal{L}(F,F))^n$$

- $\mathcal{L}(F,U) \cong \mathcal{L}(F,F^m) \cong (\mathcal{L}(F,F))^m$
- $\mathcal{L}(X,U) \cong \mathcal{L}(F^n,F^m) \cong \mathcal{L}(F^n,F)^m \cong (\mathcal{L}(F,F)^n)^m \cong F^{mn}$

Ex 3.4.9

Let V be a vector space over a eld F, let $\hat{x} \in V$, and let S be a subspace of V. Prove that if $\tilde{x} \in \hat{x} + S$, then $\tilde{x} + S = \hat{x} + S$. Interpret this result in terms of the solution set $\hat{x} + \ker(T)$ for a linear operator equation $T(x) = b$.

Goal: The First Isomorphism Theorem

If $L : X \to U$ is linear, then there is an isomorphism $\tilde{L} : X/\ker L \to \mathcal{R}(L)$.

Ex 3.4.13

Let V be a vector space over a eld F, and let S be a proper subspace of V. Prove that the relation $\sim$ dened by $u \sim v$ if and only if $uv \in S$ is an equivalence relation on V.

Ex 3.4.14

For any vector $u \in V$, let $[u]$ denote the equivalence class of u under $\sim$. We denote the set of all equivalence classes by V/S and define addition and scalar multiplication on V/S by

$$[u] + [v] = [u + v], \quad \alpha[u] = [\alpha u].$$

(a) Show that $+, \cdot$ are well-defined.

(b) Show that V/S is a vector space over F.

Ex 3.4.14 (a)

$+ : V/S \times V/S \to V/S$ is well-defined.

Ex 3.4.14 (a)

$\cdot : F \times V/S \to V/S$ is well-defined.

Ex 3.4.14 (b)

V/S is a vector space over F.

Ex 3.4.15

Now let X and U be vector spaces over a field F, and let $L : X \to U$ be a linear operator. Define $T : X/\ker(L) \to \mathcal{R}(L)$ by

$$T([x]) = L(x) \text{ for all } [a] \in X/\ker(L).$$

(a) Prove that T is a well-defined linear operator.

(b) Prove that T is an isomorphism.

(c) Let $u \in \mathcal{R}(L)$ be given, and let $\hat{x} \in X$ be a solution to $L(x) = u$. In terms of $X/\ker(L)$, what is the solution set to $L(x) = u$? How can you describe $X/\ker(L)$ in terms of linear operator equations of the form $L(x) = v$, $v \in \mathcal{R}(L)$?

Ex 3.4.15(a)

1. T is well-defined.
2. T is linear.

Ex 3.4.15(b)

T is an isomorphism.

Ex 3.4.15(c)

$[\hat{x}]$ is the solution set of $L(x) = u$, where $u = L(\hat{x})$.

Observation

For $L(x) = u$,

1. Existence
2. Uniqueness

1. There is a solution to $L(x) = b$ if and only if $b \in \mathcal{R}(L)$
2. Let $b \in \mathcal{R}(L)$. A solution to $L(x) = b$ is unique if and only if $\ker(L) = \{0\}$.

1. $\text{nullity}(T) = \dim(\ker(L))$
2. T is singular $\iff \ker T \neq \{0\} \iff \text{nullity}(T) \geq 1$.
3. T is nonsingular $\iff \ker T = \{0\} \iff \text{nullity}(T) = 0 \iff T$ is injective.

Theorem

Let $T : X \to U$ be an injective linear operator. Then $\dim X \leq \dim U$.

Proof

Main idea

1. $\mathcal{R}(T)$ is a subspace of U.
2. If $\{x_1, \cdots, x_n\}$ is a basis of X, then $\operatorname{span}\{T(x_1), \cdots, T(x_n)\} = \mathcal{R}(T)$.

1. $\operatorname{rank}(T) = \dim(\mathcal{R}(L))$
2. If $\dim X = \operatorname{rank} T$, T is full rank.
3. T is surjective $\iff \operatorname{rank} T = \dim U \iff \mathcal{R}T = U$.

Theorem

If T is surjective, then $\dim X \geq \dim U$.

Ex 3.5.5

$M : \mathbb{R}^n \to \mathbb{R}^3$ by

$$M(x) = \begin{bmatrix} x_1 + 3x_2 - x_3 - x_4 \\ 2x_1 + 7x_2 - 2x_3 - 3x_4 \\ 3x_1 + 8x_2 - 3x_3 - 16x_4 \end{bmatrix}$$

Find rank and nullity.

Proof

$$\begin{bmatrix} 1 & 3 & -1 & -1 \\ 2 & 7 & -2 & -3 \\ 3 & 8 & -3 & -16 \end{bmatrix} x \Rightarrow \mathcal{R}(M) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -16 \end{bmatrix} \right\}$$

Elementary "column" operation $\Longrightarrow$ range does not change!

$$\begin{bmatrix} 1 & 3 & -1 & -1 \\ 2 & 7 & -2 & -3 \\ 3 & 8 & -3 & -16 \end{bmatrix}$$

Elementary "row" operation $\implies$ kernel does not change!

$$\begin{bmatrix} 1 & 3 & -1 & -1 \\ 2 & 7 & -2 & -3 \\ 3 & 8 & -3 & -16 \end{bmatrix}$$

Ex 3.5.10

Is the following statement a theorem?

Let X and U be vector spaces over a eld F, and let $T : X \to U$ be linear. Then $\{x_1, x_2, \cdots, x_n\} \subset X$ is linearly independent if and only if $\{T(x_1), T(x_2), \cdots, T(x_n)\} \subset U$ is linearly independent.

If it is, prove it. If it is not, give a counterexample.

Ex 3.5.11

Suppose X and U are vector spaces over a eld F, with U nite-dimensional, and $L : X \to U$ is linear. Let $\{u_1, u_2, \cdots, u_m\}$ be a basis for U and assume that, for each j, $L(x) = u_j$ has a solution $x \in X$. Prove that L is surjective.

Proof

1. Let $y \in Y$.
2. Let x_j be a solution to $L(x) = u_j$. Then $\mathrm{span}\{L(x_1), \cdots, L(x_m)\} \subseteq \mathcal{R}(L) \subseteq U$.

Ex 3.5.13

Suppose $T : X \to U$ is an injective linear operator. Prove T defines (induces) an isomorphism between X and a subspace of U.

Proof

$\ker T = \{0\}$. By the First Isomorphism Theorem,

$$\begin{aligned} X/\{0\} &\to \mathcal{R}(T) \\ [x] &\mapsto T(x) \end{aligned}$$

is an isomorphism.

Claim) $X/\{0\} \cong X$.

1. $[x] = \{x\}$
2. $X/\ker T \to X$ by $[x] \mapsto x$ is an isomorphism.

Ex 3.5.14

Let V be a vector space over a eld F, and let S, T be subspaces of V. Since S and T are vector spaces in their own right, we can dene the product $S \times T$ (see Ex 2.2.15). We can also dene the subspace $S + T$ of V (see Ex 2.3.21). Dene a linear operator $L : S \times T \to S + T$ by $L((s, t)) = s + t$.

(a) Prove that $\ker(L)$ is isomorphic to $S \cap T$ and find an isomorphism.

(b) Suppose $S \cap T = \{0\}$. Prove that $S \times T$ is isomorphism to $S + T$ and that L is an isomorphism.

Fundamental Theorem of Linear Algebra

If $T : X \to U$ is linear, then $\dim X = \operatorname{nullity} T + \operatorname{rank} T$.

By the First Isomorphism Theorem, $X/\ker(T) \to \mathcal{R}(T)$ is an isomorphism. Thus $\dim(X/\ker(T)) = \mathcal{R}(T)$.

$$\dim(X) = \text{nullity}(T) + \text{rank}(T) = \dim(\ker(T)) + \dim(X/\ker(T)).$$

Question

Suppose $\{[x_1], \cdots, [x_r]\}$ is a basis for $X/\ker(T)$. Then $\{x_1, \cdots, x_r\}$ is linearly independent.

Let $\{y_1, \cdots, y_k\}$ be a basis for $\ker(T)$. Then $\{x_1, \cdots, x_r, y_1, \cdots, y_k\}$ is a basis for X.

Quiz

For given a basis for X, can always we divide the basis by two parts,

$$\{\underbrace{\qquad\qquad}_{\text{kerenl}}, \underbrace{\qquad\qquad}_{\text{range}}\}?$$

- Suppose $\dim X = \dim U$ and $T : X \to U$.
  Then T is bijective $\iff$ T is injective $\iff$ T is surjective.
- Def) For $A \in F^{m \times n}$, $\text{col}(A), \mathcal{N}(A)$.
- Let $A \in F^{n \times n}$ and $\{x_1, \cdots, x_n\}$ be a basis.
  Then A is invertible $\iff$ $\{Ax_1, \cdots, Ax_n\}$ is a basis for F^n.

Ex 3.6.9

Let X and U be nite-dimensional vector spaces over a eld F, let $\mathcal{X} = \{x_1, x_2, \cdots, x_n\}$ and $\mathcal{U} = \{u_1, u_2, \cdots, u_m\}$ be bases for X and Y respectively, and let $T : X \to U$ be linear. Prove that T is invertible if and only if $[T]_{\mathcal{X},\mathcal{U}}$ is an invertible matrix.

Ex 3.6.12

Construct a different proof to Theorem 104, as follows: Choose vectors $x_1, \cdots, x_k$ in X such that $\{T(x_1), \cdots, T(x_k)\}$ is a basis for $\mathcal{R}(T)$, and choose a basis $\{y_1, \cdots, y_l\}$ for $\ker(T)$. Prove that $\{x_1, \cdots, x_k, y_1, \cdots, y_l\}$ is a basis for X.

Ex 3.6.13

Let F be a field and suppose $A \in F^{m\times n}$, $B \in F^{n\times p}$. Prove that $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$.

Ex 3.6.15

Prove that a strictly diagonally dominant matrix $A \in \mathbb{C}^{n\times n}$ is nonsingular.

Ex 3.6.16

Let X and U be vector spaces over a field F, and let $T : X \to U$ be linear.

(a) There exists a left inverse of S of $T \iff T$ is injective.

(b) There exists a right inverse of S of $T \iff T$ is surjective.

Ex 3.6.17

Let $A \in F^{m\times n}$ and $B \in F^{n\times m}$.

- left inverse of A : $BA = I_n$
- right inverse of A : $AB = I_m$

(a) There exists a left inverse of B of $A \iff \mathcal{N}(A) = \{0\}$.

(b) There exists a right inverse of B of $A \iff \operatorname{col}(A) = F^m$.

Ex 3.6.23 ∼ 26) Change of Coordinate

1. Let $\mathcal{X}, \mathcal{Y}$ be two bases of X, and let $x \in X$.

$$[x]_{\mathcal{X}} \mapsto [x]_{\mathcal{Y}} = C[x]_{\mathcal{X}}$$

2. $L : X \to X \implies [L]_{\mathcal{X},\mathcal{X}} = \underbrace{\qquad}[L]_{\mathcal{Y},\mathcal{Y}}\underbrace{\qquad}.$
3. $T : X \to U \implies [L]_{\mathcal{X},\mathcal{U}} = \underbrace{\qquad}[L]_{\mathcal{Y},\mathcal{V}}\underbrace{\qquad}.$

The End