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Solutions to Selected Problems in Chapter 2

Sampling and aliasing control

2-1 *What do you understand by each of the following:*

(a) *Nyquist frequency?*
(b) *Nyquist rate?*
(c) *Sampling rate?*
(b) *Sampling frequency?*

(a) This is half the sampling frequency (it is the 'foldover' frequency in the spectrum of the sampled data).
(b) The term used to describe the sampling frequency when we sample at close to the minimum allowable sampling frequency or rate (i.e. the sampling frequency is close to $2f_{\max}$).
(c) This is the same as the sampling frequency. It is a more appropriate term when data is already in a digital form (e.g. in multirate systems or encoders/decoders).
(d) The frequency/rate at which values of signals are measured (this should satisfy the sampling theorem).

2.2 A signal has a spectrum depicted in Figure 2.45. Determine the minimum sampling frequency to avoid aliasing. Assume the signal is sampled at a rate of 16 kHz, and sketch the spectrum of the sampled signal in the interval $\pm 16kHz$ *. Indicate the relevant frequencies including the foldover frequency in your sketch.*

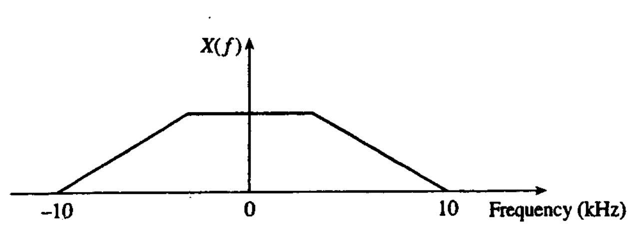
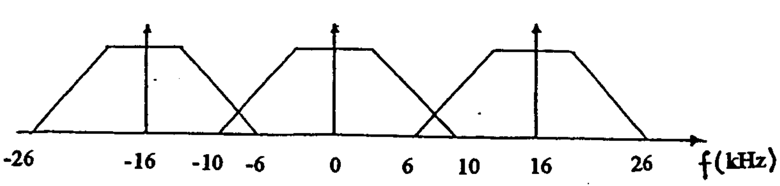


Figure 2.45

The minimum sampling frequency is $2f_{\max} = 2\times 10kHz = 20kHz$. The spectrum of the sampled signal is depicted in the figure below.



2.3 Explain why sampling theorem considerations alone are not sufficient for establishing the actual sampling frequencies used in practical DSP systems.

According to the sampling theorem, the sampling frequency must satisfy the relation:

$$F_s \geq 2f_{\max}$$

In practice, we need to take into account the fact that the ADC resolution (with its inherent S/N) establishes a noise floor. Thus, the choice of the anti-aliasing filter and sampling frequency should be such that frequencies above the Nyquist frequency are attenuated to a level not detectable by the ADC; the non-ideal response of practical anti-aliasing filters means that aliasing is nearly always present due to significant energy outside of the highest frequency of interest and/or noise which is wideband. Thus, we should specify the level of tolerable aliasing; the cost and errors associated with the analogue parts (e.g. amplitude and phase errors in the anti-aliasing filters and sample and hold circuits). In all but the simplest cases, the minimum rate specified by the sampling theorem is not sufficient. It is nearly always necessary to oversample.

2.4 Explain clearly the role of the anti-aliasing filter and the anti-aliasing filter in real-time DSP systems. Why are the requirements of the two filters often the same in DSP systems?

The main role of the anti-aliasing filter is to bandlimit the analogue input signal to reduce aliasing from signal components and/or noise outside the band of interest. The anti-image filter smooths out the steps in the DAC outputs after DSP thereby removing the unwanted high frequency components centred on multiples of the sampling frequency when the data is converted into analogue. To reconstruct the analogue signal after DSP the update rate must obey the sampling theorem just as at the input.

2.5 The requirements for the analog input section of a certain real-time DSP system are:

frequency band of interest	*0 – 4 kHz*
maximum permissible passband ripple	$\leq 0.5dB$
stopband attenuation	$\geq 50dB$

Determine the minimum order of an anti-aliasing filter with Butterworth characteristics and a suitable sampling frequency to satisfy the requirements.

The amplitude response of an n^th^ order Butterworth filter is given by:

$$A(f) = \frac{1}{\sqrt{1+\left(\frac{f}{f_c}\right)^{2n}}}$$

and is depicted in S2.5a below. To keep the amplitude error within the specifications, then at the bandedge (4 kHz) the deviation should satisfy the relation:

$$20\log\left[1+\left(\frac{4}{f_c}\right)^{2n}\right]^{\frac{1}{2}} \leq 0.5dB \qquad (1.1)$$

After bandlimiting and sampling, the signal spectrum (assume a wideband input) is as shown in Figure S2.5b. The sampling frequency should be such that the aliasing level is at least 50 dB down on all signal components in the band of interest. Taking the bandedge frequency as the limiting case, we have:

$$20\log\left[1+\left(\frac{F_s-4}{f_c}\right)^{2n}\right]^{\frac{1}{2}} \geq 50dB \qquad (1.2)$$

We may solve Equations (1.1) and (1.2) iteratively, i.e. find the values of f_c (for n=2, 3, 4 and 5) using Equation 1.1. Then find the corresponding values of F_s by using Equation 1.2. A choice of practical values for n and F_s can then be made.

For convenience, Equations 1.1 and 1.2 are re-written as

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$$f_c = \frac{4}{\left[\left(10^{\frac{0.5}{20}}\right)^2 - 1\right]^{\frac{1}{2n}}} ; \qquad F_s = f_c\left[\left(10^{\frac{50}{20}}\right)^2 - 1\right]^{\frac{1}{2n}} + 4$$

where f_c and F_s are in kHz. The values for n, f_c and F_s are given in Table S2.5.

Filter order, n	Cutoff frequency, f_c (kHz)	Sampling frequency, F_s (kHz)
2	6.77	124.35
3	5.68	42.7
4	5.2	27.95
5	4.94	19.61
6	4.77	16.44

Table S2.5

From the table n=3 or 4 should be adequate as either leads to a relatively low sampling frequency. Higher values of n are less attractive as they imply higher order filters which are expensive (in terms of design effort, component count and errors).

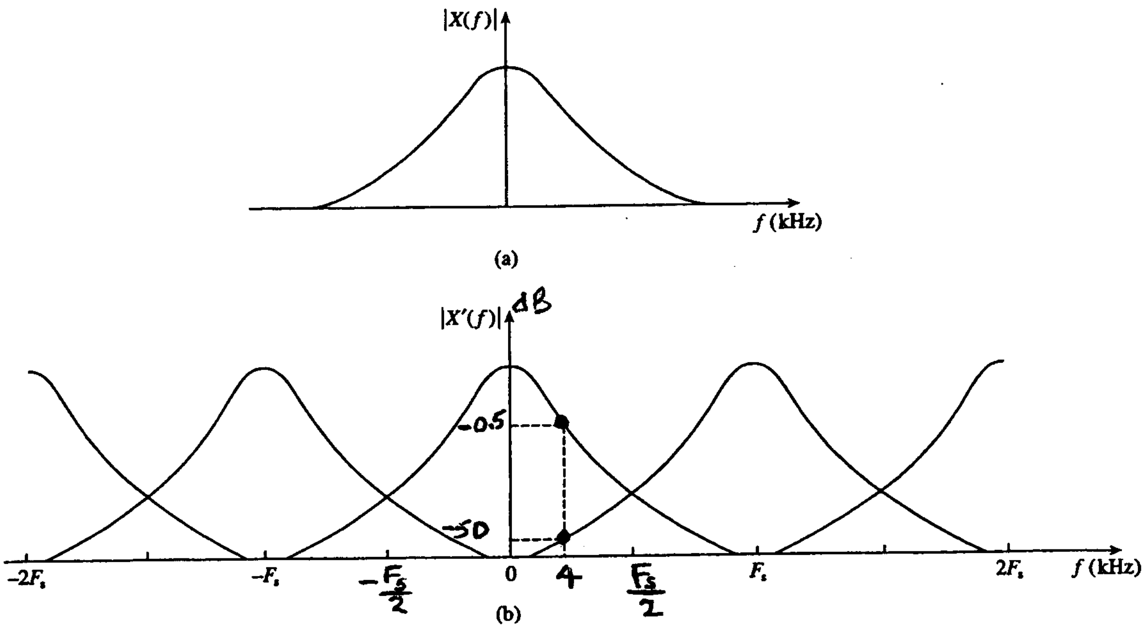

Figure S2.5: Spectrum of signal (a) at the output of the analog filter and (b) after sampling

2.6 The analog input to a real-time DSP system is digitized with a 16-bit ADC in the bipolar mode. The peak-to-peak amplitude of the input signal is in the range $\pm 10V$ *and lies in the band 0 – 10 kHz. Estimate the minimum*

(1) stopband attenuation, $A_{\min}$, *for the anti-aliasing filter, and*

(2) sampling frequency, F_s, *to keep the aliasing error in the passband to just below the quantization noise level (assume a sixth order Butterworth filter is used for the anti-aliasing filtering).*

The figure for the sampled signal has the form depicted in Figure S2.6 below.

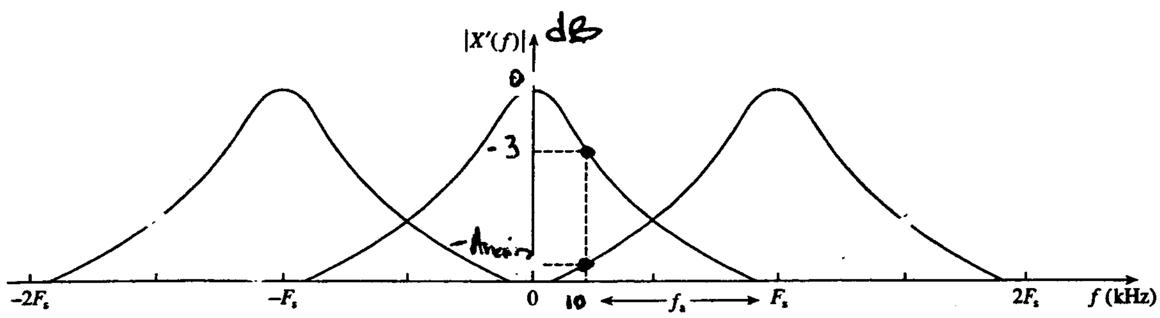

Figure S2.6

From section 2.3.1.3, $A_{\min} = 20\log\left(\sqrt{1.5}\times 2^{B}\right) = 98.09dB$

Assume $f_c = 10kHz$, *then the worst case aliasing due to the image components centred on* F_s is given by

$$20\log\left[1+\left(\frac{f_a}{f_c}\right)^{12}\right]^{\frac{1}{2}} = 98.09,\ i.e.\ \ f_a = f_c\left[\left(10^{\frac{98.09}{20}}\right)^{2}-1\right]^{\frac{1}{12}} = 65.68kHz$$

Thus, $F_s = (65.67 + 10)kHz = 75.68kHz$.

2.7 An analog signal with a uniform power spectral density is bandlimited by a filter with the following amplitude response:

$$|H(f)| = \frac{1}{\left[1+(f/f_c)^{6}\right]^{1/2}}$$

where $f_c = 3.4kHz$. *The signal is digitized using a linear 8-bit ADC. Determine the minimum sampling frequency so that the maximum aliasing error is less than the quantization error level in the pass band.*

From equation 2.2 in the main text and with B=8, the minimum stopband attenuation for the filter is 49.93 dB. The aliasing frequency at the bandedge is:

$$f_a = f_c\left[\left(10^{\frac{49.93}{20}}\right)^{2}-1\right]^{\frac{1}{6}} = 23.1kHz$$

Thus, the sampling frequency, $F_s = 23.1 + 3.4kHz = 26.5kHz$.

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2.8 The analogue input signal to a real-time DSP system is bandlimited to 30 Hz with an analog filter having a third-order Butterworth characteristic before it is digitized. If the aliasing error due to sampling is to be less than 1% of the signal level in the passband, determine the minimum sampling frequency, F_s, required for the system.
If the signal, after digitization and processing, was converted back to analog what will be the average error introduced by the aperture effect at 30 Hz? Assume that the input signal was digitized using an ideal sampler and ADC, but was recovered using a zero-order hold DAC. A common sampling frequency of 256 Hz at the input and output may be assumed.

At 30 Hz, the normalised signal level is 0.7071, and so the target aliasing level at 30 Hz is:

$\leq 0.7071 \times \frac{1}{100} = 0.007071$. Thus,

$$0.007071 \leq \frac{1}{\sqrt{1+\left(\frac{f_a}{30}\right)^6}}$$

From which $f_a = 156.29Hz$.

Thus, $F_s(\min) \geq f_a + f_c = 156.29 + 30 = 186.29Hz$. If we assume $F_s = 256Hz$, then at 30 Hz, $\frac{\sin x}{x} = 0.97756$, where $x = \omega T = 0.368rad$. The average error $= (1 - 0.97756) \times 100\% = 2.244\% \equiv 0.197dB$.

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2.9 The front-end of a real-time DSP system is depicted in the Figure 2.46. Assume a wide band input signal.

(a) Sketch the spectrum of the signal before sampling (point A) and after sampling (point B) between the range $\pm \frac{F_s}{2}$.

(b) Determine the signal and aliasing error levels at 15 kHz and the Nyquist frequency (i.e. 30 kHz).

(c) Determine the minimum sampling frequency, F_s *(min), to give a signal-to-aliasing error level of 10: 1 at 15 kHz. State any assumptions made.*

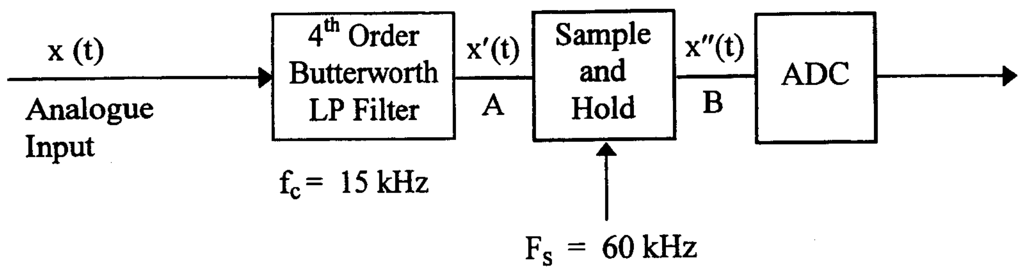

Figure 2.46.

Solution

(a) Sketches of the spectrum of the signal before and after sampling are shown below. We note that the shape of each spectrum is governed by the equation of the response of the Butterworth filter, i.e.

$$|H(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{f_c}\right)^8}}$$

(b) Signal spectrum at the output of the filter is equal to the product of the signal spectrum and the response of the filter, i.e. $X(f)|H(f)|$. For a wide band input, the spectrum X(f) is essentially flat. If we assume that both X(f) and H(f) have a maximum value of 1 (i.e normalised), then the signal levels before sampling (at the output of the filter) and after sampling (at the output of the sample and hold) are governed by the shape of the analogue filter.

Thus, at 15 kHz, $f_c = 15kHz$, the normalised signal level (from the equation above) is simply 0.707 (i.e. $1/\sqrt{2}$). This is point (1) in Figure S2.9b. The aliasing error level (point 2 in Figure S2.9b) is given by:

$$\text{Aliasing level, } X_a = \frac{1}{\sqrt{1+\left(\frac{45}{10}\right)^8}} = 0.012$$

The Nyquist frequency is 30 kHz (i.e. half the sampling frequency). This is the cross over point in Figure S2.9b (point 3) and so the signal and aliasing error levels are the same. Signal and aliasing levels at 30 kHz, are each (using the Butterworth equation, with $f = 30kHz$ and $f_c = 15kHz$) equal to 0.062.

(c) At 15 kHz, the signal level is 0.707. A ratio of 10:1 between the signal level and aliasing level implies an aliasing level of 0.0707. The image component that causes aliasing is governed by the Butterworth equation.

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Thus, from $\dfrac{1}{\sqrt{1+\left(\dfrac{f}{15}\right)^{8}}} = 0.0707$, we find that $f = 29.07kHz$.

This corresponds to the aliasing frequency at 15 kHz, i.e. f_a, in Figure S2.9b above. Thus, the sampling frequency, $F_s = f_a + 15 = 44.07kHz$

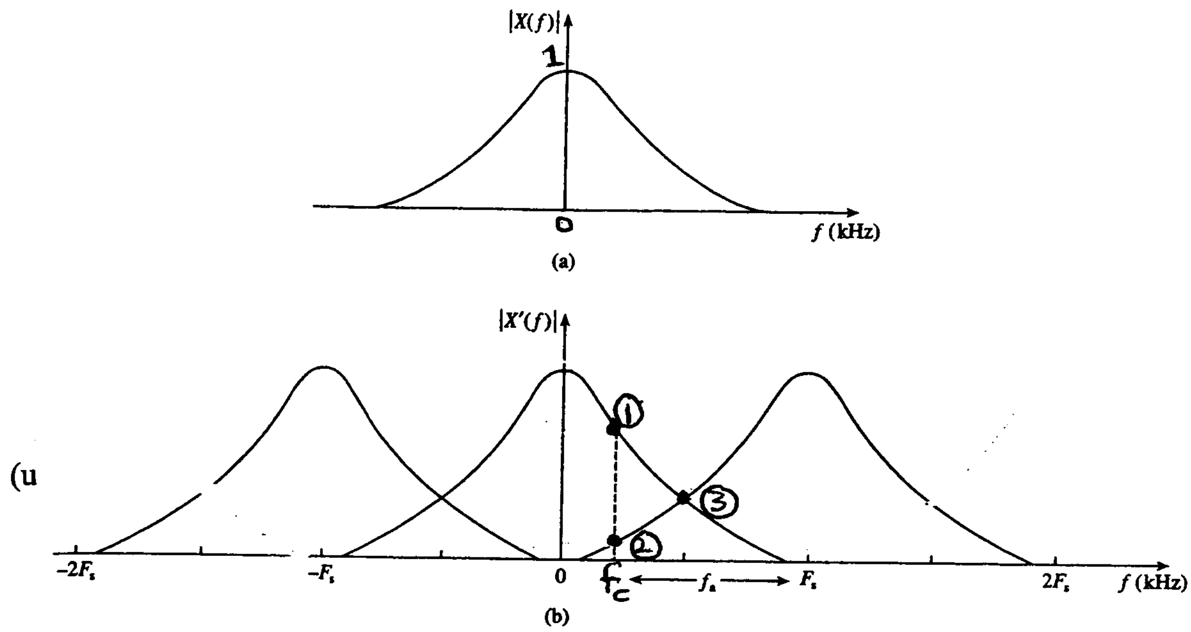

Figure S2.9 Spectrum of signal (a) at the output of the analog filter and (b) after sampling

2.10 Figure 2.47 depicts a real-time DSP system. Assuming that the frequency band of interest extends from 0 to 100 Hz and that a 16-bit, bipolar, ADC is used, estimate:

(1) the minimum stop band attenuation, $A_{\min}$, for the anti-aliasing filter,

(2) minimum sampling frequency, F_s, and

(3) the level of the aliasing error relative to signal level in the pass band for the estimated $A_{\min}$ and F_s.

Sketch and label the spectrum of the signal at the output of the analogue filter, assuming a wideband signal at the input, and that of the signal after sampling.

Solution

(1) From analysis of the magnitude frequency response of a practical anti-aliasing filter as in Example 2.3 in the main text, the minimum stop band attenuation, A_{min}, is given by (for a sine wave input):

$$A_{min} = 20 \log (\sqrt{1.5} \times 2^{B})\,\text{dB}$$

$= 98.09$ dB.

(2) The signal spectrum, before and after sampling (ignoring higher order image frequencies), shown in Figure S2.14.

From $$A_{min} = 98.09 = 20 \log\left[1 + \left(\frac{f_a}{fc}\right)^6\right]^{\frac{1}{2}}$$

We find that $f_a = 1.68\ kHz$. If we assume that $f_a = f'_{max} = 1.68kHz$ (see Figure S2.14b) then we have $F_s = 2f'_{max} = 3.37kHz$

If we wish the band edge frequency to be attenuated to just below the quantisation noise level, then the sampling frequency can be reduced to 1.68kHz + 100 Hz = 1.78 kHz (see Figure S2.14c).

(3) For the higher sampling frequency, the aliasing level at 4 kHz is 8.7×10^{-7} *and* the aliasing level relative to signal level at 4 kHz = 6.21×10^{-7}. For the lower sampling rate the aliasing level is 1.24×10^{-5} and the aliasing level relative to signal level is 8.8×10^{-6}

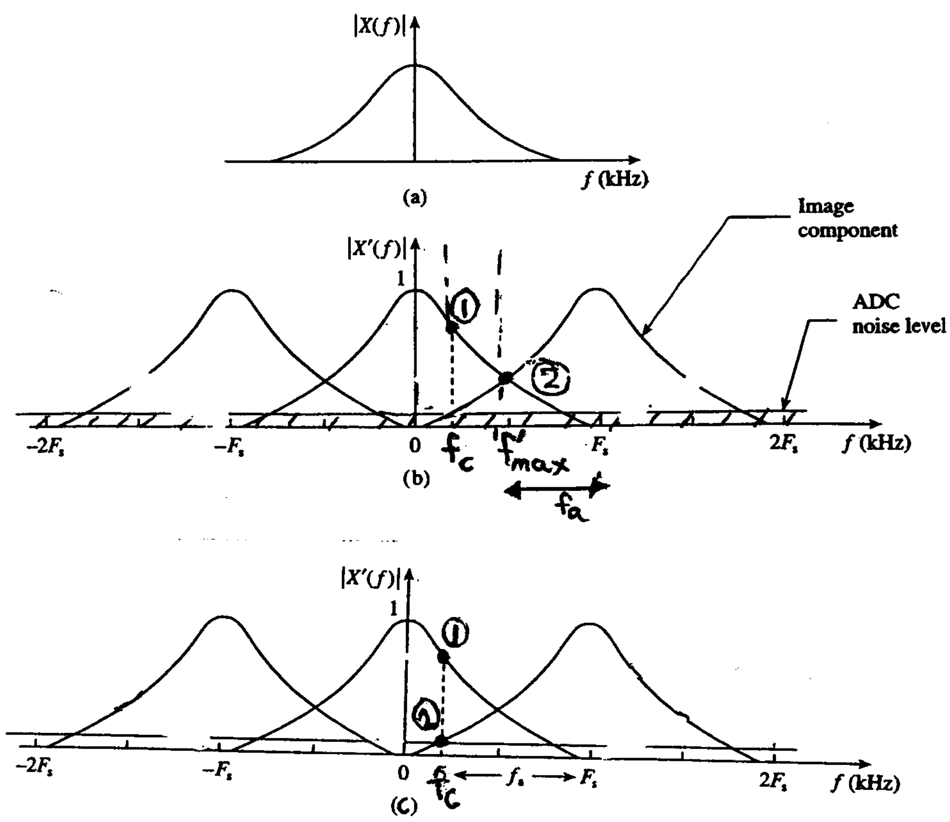

Figure S2.14

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2.11 (a) Discuss, briefly, three main factors that determine the level of the aliasing error in practical DSP systems. Point out the relevance of each to aliasing control.

(b) An analogue signal with uniform power spectrum density is bandlimited by an anti-aliasing filter with a magnitude-frequency response characterised by the following equation:

$$\frac{1}{\sqrt{1+\left(\frac{f}{f_c}\right)^8}}$$

where $f_c = 40$ Hz. The signal is digitised with a linear 12-bit, bipolar, ADC. Determine the:

(1) minimum sampling frequency to keep the maximum aliasing error in the passband to no greater than the quantisation error level
(2) maximum passband signal level, in dB, relative to the ADC quantisation noised floor.

State any reasonable assumptions made.

(c) Write down the equation for the band pass sampling theorem. Explain why the theorem is of interest in digital communication systems.

(d) Starting with the equation in (c), derive an expression for the theoretical minimum sampling rate for a band pass signal. Assume that the ratio of the upper band edge frequency to the bandwidth of the signal is an integer. Comment on why the theoretical minimum sampling rate should not be used in practice.

Solution

(a) Main factors that determine the level of aliasing error and their relevance are covered in the main text. They include:

sampling frequency – determines the frequency separation between the signal band and the image components and hence the extent of aliasing;

anti-aliasing filter – bandlimits the input signal to reduce the signal/noise levels outside of the band of interest and hence limits the level of aliasing;

ADC resolution – introduces quantisation noise which establishes the input noise floor. Places a limit to how the anti-aliasing filter should reduce the out of band signals (attenuation to just below the quantisation noise level is sufficient).

b(1) The sampling frequency should be chosen such that the anti-aliasing filter attenuates the aliasing error folded back into the passband to less than the maximum rms quantisation level of the ADC so that they are not detectable by the ADC. Following the method described in Example 2.4 in the main text and assuming a sinewave input, with peak amplitude of A (which just fills the ADC input range), then At 5 kHz, the maximum aliasing error folded back is:

$$\frac{A}{\sqrt{2}}X\frac{1}{\left[1+\left(\frac{fa}{5}\right)^6\right]^{\frac{1}{2}}}=\frac{A}{\sqrt{3}\times 2^B}$$

With B = 12 bits, we can solve for f_a, the aliasing frequency and hence the sampling frequency, F_S: f_a = 336.6 Hz; $F_S = f_a + 40 = 376.6$ Hz

b(2) Maximum signal relative to ADC noise floor (from Example 2.4) is: $20 \log\left(\sqrt{1.5} \times 2^B\right)$ =74dB

(c) and (d) – see Problem 2.17.

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Bandpass undersampling

2.12 *The front-end of the receiver for a multichannel communication system is depicted in Figure 2.48a. The received signal has the spectrum shown in Figure 2.48b, with the channel numbers indicated. A bandpass filter is used to isolate the signal in the desired channel before the signal is digitised at the lowest possible rate.*

Assume an ideal bandpass filter with the following characteristics:

$$H(f)=1, \; 10kHz \le f \le 20kHz$$

$$0 \quad \text{otherwise}$$

(a) (i) Determine the minimum theoretical sampling frequency.
(ii) Sketch the spectrum of the signal before sampling (point A) and after sampling (point B).

(b) Repeat parts (i) and (ii) for a bandpass filter that passes channel No.2.

Solution

(a) (i) The minimum theoretical sampling frequency is 2 x 10 kHz, i.e. 20 kHz.

(ii) The spectrum at point A (output of the bandpass filter) is simply the signal spectrum for channel 1 (Figure S2.12a).

The spectrum at point B (i.e. after sampling) may be obtained by convolving the spectrum of the signal at the output of the bandpass filter (Figure S2.12a) and the spectrum of the sampling function (Figure S2.12b). This gives Figure S2.12c.

(b) (i) The sampling frequency remains at 20 kHz.

(ii) Proceeding as in part a, the spectrum at points A and B in this case are shown in Figures 2.12d and e, respectively.

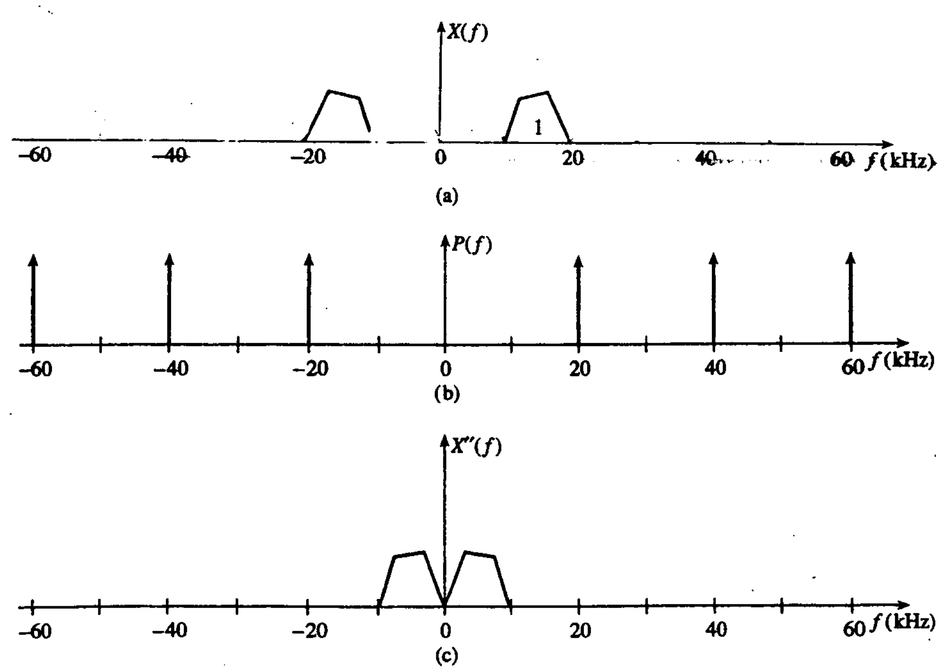

Figure S2.12 (a) Output of the bandpass filter. (b) The sampling function. (c) Output of sampler.

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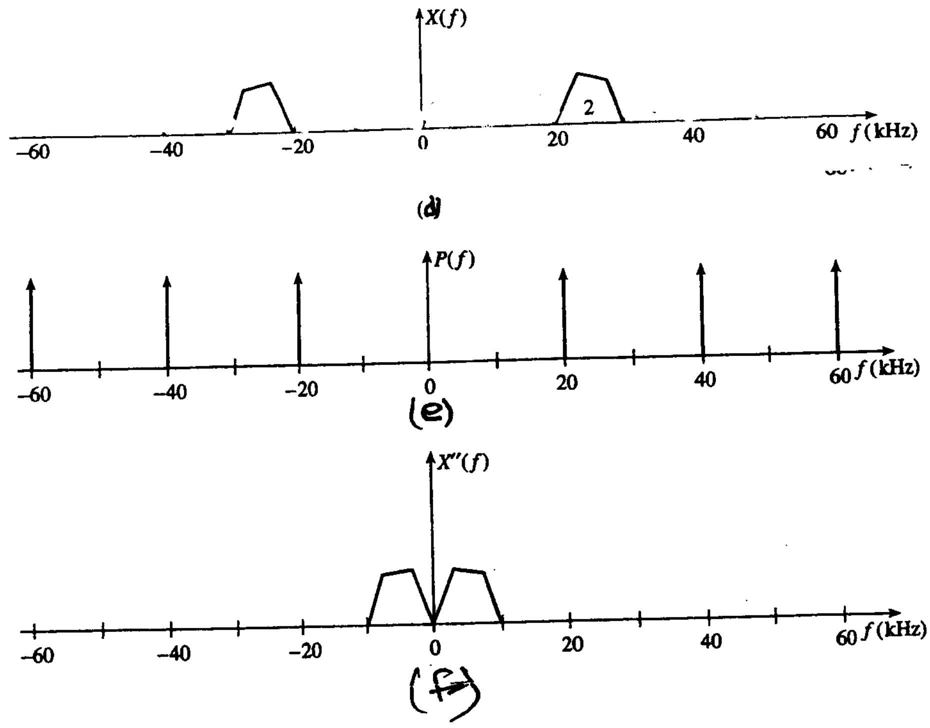

Figure S2.12 - continued

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2.13 *The spectrum of a narrowband signal is depicted in Figure 2.49. Obtain and sketch the spectrum of the sample signal, in the range* $\pm\frac{F_s}{2}$, *for each of the following three cases:*

(i) $\frac{f_H}{B}=3$

(ii) $\frac{f_H}{B}=4$

(iii) $\frac{f_H}{B}=4.5$

Assume that the bandwidth of the signal, B = 5 kHz, and that the signal is sampled at the rate of 2 B in each case.

Solution

(i) For a signal with a bandwidth of 5 kHz, $f_H = 15kHz$. The spectrum of the signal is shown in Figure S2.13a. The sampling frequency is 2B, i.e. 10 kHz. Following the method described in Example 2.7, the spectrum of the sampled signal is obtained by graphically convolving the spectrum of the signal, Figure S2.13a, and that of the sampling function, Figure S2.13b.

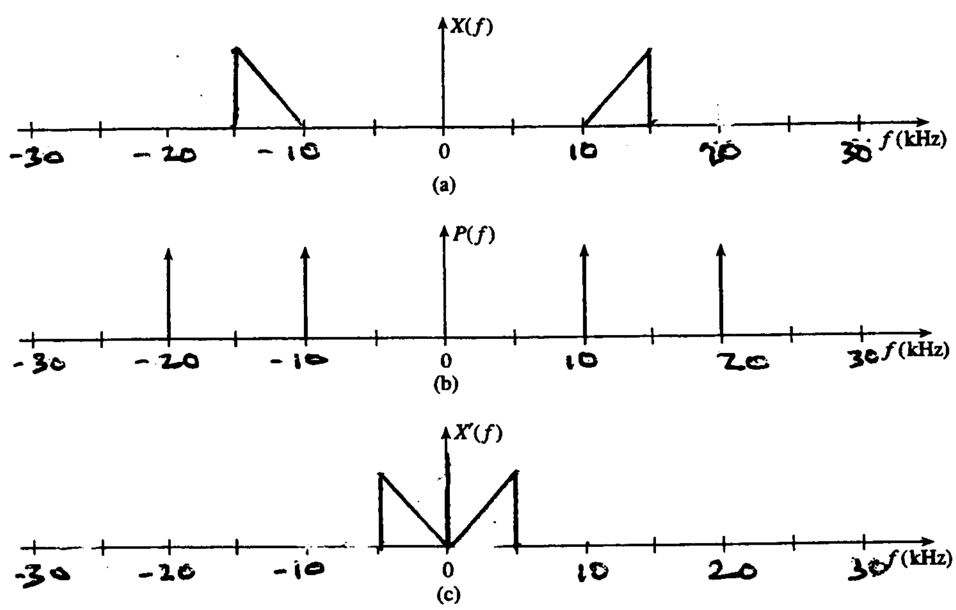

Figure S2.13

(ii) In this case, the upper edge frequency, $f_H = 20kHz$. The sampling frequency remains at 10 kHz. The spectra of the signal, the sampling function and the sampled signal are shown Figures S2.13d, e and f.

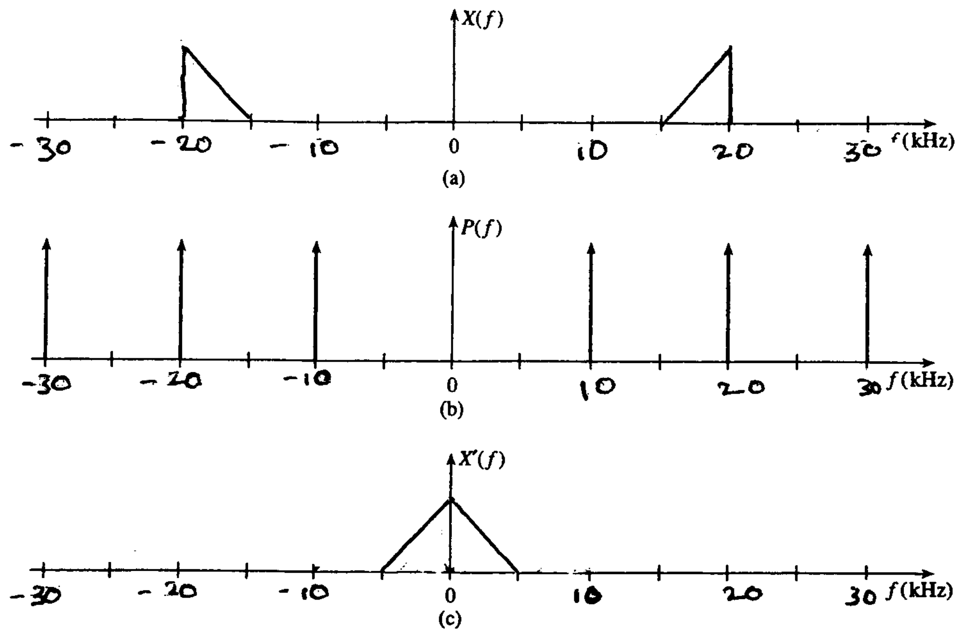

Figure S2.13 continued

(iii) In this case, the upper edge frequency, $f_H = 22.5kHz$. The sampling frequency remains at 10 kHz. The spectra of the signal, the sampling function and the sampled signal are shown Figures S2.13g, h and i.

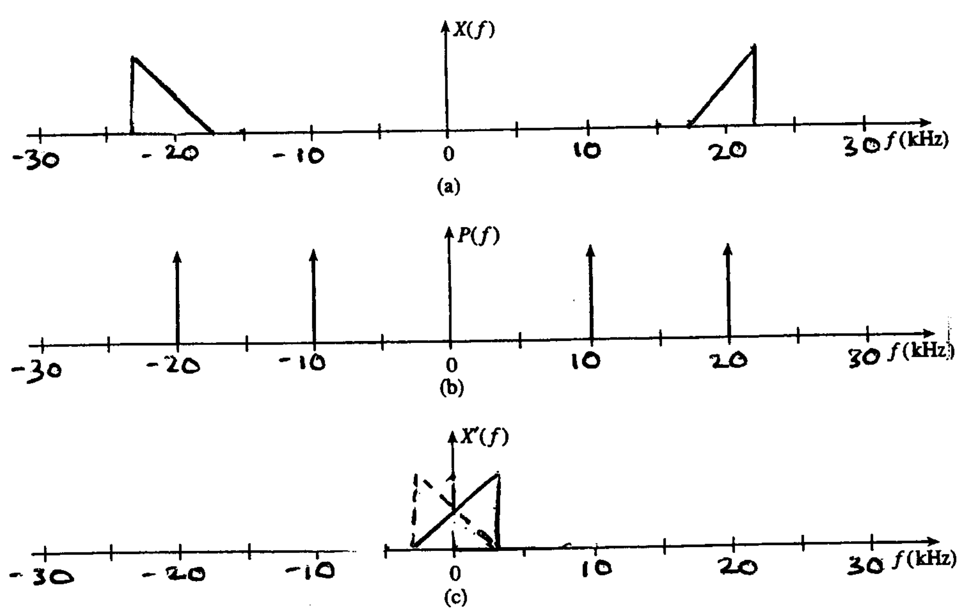

Figure S2.13 continued.

2.14 *(a) Determine the minimum theoretical sampling rate, Fs, to avoid aliasing for a bandpass signal with frequency components in the interval $20MHz < f < 30MHz$. Justify your answer and explain why the minimum theoretical sampling rate should not be used in practice.*

(b) Assume that the bandpass signal in (a) has a spectrum depicted in Figure 2.50. Determine the edge frequencies of all the frequency bands (including the image components) of the signal after sampling in the interval $\pm 2F_s$. Sketch and clearly label the spectrum of the sampled signal in that interval on the graph paper provided.

(c) Calculate the allowable range of sampling rates to avoid aliasing if the analog bandpass signal is augmented by a 5 kHz guardband at either band edge.

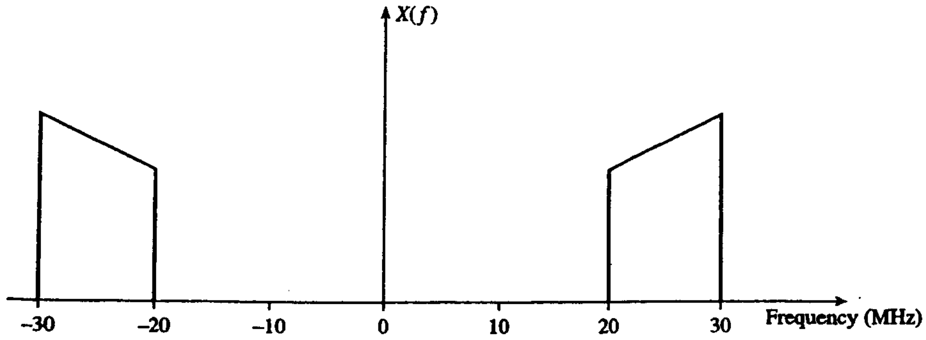

Figure 2.50

Solution

(a) The signal band is integer-positioned, since f_H / B is an integer. Thus, $F_s(\min) = 2B = 20MHz$. Sampling generates images of the baseband centred at multiples of the sampling frequency. In bandpass sampling, advantage is taken of this by undersampling and taking an appropriate image band translated to the low frequency end, in stead of the original bandpass signal. Fs(min) is valid only for integer bands. It is not practical since imperfections (e.g. in variations in sampling clock, anti-aliasing filter characteristics) would lead to, for example, a sampling rate that is not permissible and aliasing errors.

(b) The spectrum of the sampled signal and hence the bandedge frequencies can be determined by graphically convolving the spectrum of the input signal (Figure S2.14a) and that of the sampling function (Figure S2.14b). The outcome is shown in Figure S2.14c.

(c) With symmetrical guardbands, the overall band is now $(10 + 2\Delta B)MHz$. From the bandpass sampling theorem, the range of allowable rates is:

$$\frac{2f_H}{n} \le F_S \le \frac{2f_L}{n-1}$$

With $n = \max\_\text{int}(30{,}005/10010) = 3$. Thus, the range of allowable rate is 20003.3 – 19995 kHz = 8.3 kHz.

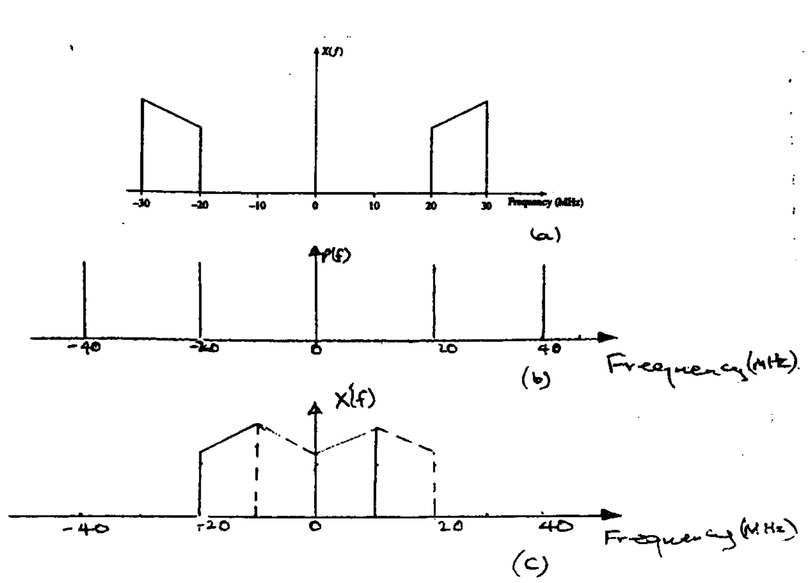

Figure S2.14

2.15 *(a) Explain briefly the principles of band pass under sampling technique. Comment on the benefits of the techniques in practice.*

(b) A digital radio receiver uses an IF (intermediate Frequency) of 50 kHz in the second stage.
(i) Determine the minimum sampling frequency, Fs, for the system to avoid aliasing if the IF signal bandwidth is 6 kHz.
(ii) Sketch and label the spectrum of the sampled signal in the interval, $\pm F_s$. Explain clearly how you obtained the spectrum of the sampled signal and comment on its shape.

Assume that integer band sampling technique is used and that the signal spectrum at the second IF stage is as shown in Figure 2.51.

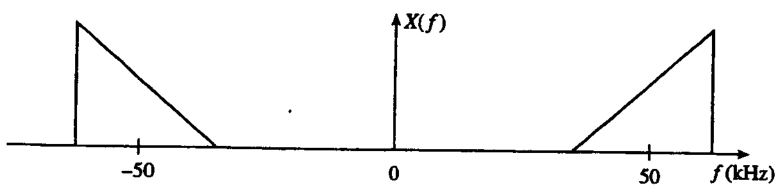

Figure 2.51

Solution

(a) The main points are covered in the main text. In summary, in bandpass sampling, a bandpass signal may be sampled at twice the signal bandwidth to translate it to low frequency so that the rate still satisfies the sampling theorem and avoids aliasing. The benefits include that it allows sampling of a narrow band high frequency signal at a much reduced rate (compared to twice the highest frequency component), and still avoid aliasing. This leads to reduced hardware complexity and costs.

b(i) The ratio of the highest frequency, f_H , to the bandwidth, B, is 53/6 = 8.83 kHz. This is not an integer and so does not satisfy the condition for alias-free sampling at 2B. To avoid aliasing, it will be necessary to extend the bandwidth, e.g. by reducing the lower band edge frequency to f_H:

$$f'_H = \left(\frac{n-1}{n}\right) f_h \qquad \text{where } n \le \frac{f_H}{B} \text{ (an integer).}$$

with n=8, the new lower band edge frequency, $f'_H = 46.375kHz$, and the new bandwidth, B'=6.625 kHz. Thus, the minimum sampling frequency is 2B'=13.25 kHz.

(ii) The spectrum of the sampled signal can be obtained by convolving the spectrum of the sampling function and that of the IF signal. This leads to Figure S2.15.

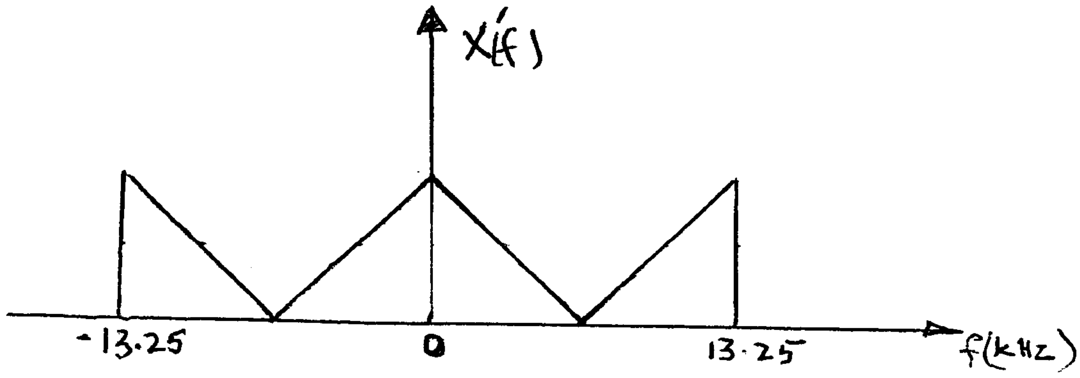

Figure S2.15

2.16. (a) The spectrum at the second IF stage of a digital receiver is depicted in Figure 2.52a, where the IF frequency is 2.976 MHz. Show, with the aid of appropriate sketches, that the IF signal can be sampled at a rate of 128 kHz without aliasing.

(b) Show, with the aid of appropriate sketches, that if the IF frequency is 3 MHz there will be an aliased output if the IF signal is the sampled signal at 128 kHz.

Solution

(a) The signal bandwidth is 3.008-2.944 MHz = 64 kHz and the rratio of the upper band edge frequency to the signal bandwidth is 3008/64 = 47. Since this is an integer, sampling at a rate of 128 kHz should, in theory at least, produce no aliasing. The spectrum of the sampling function will have components at multiples of the sampling frequency, i.e. at kF_s, $\pm1,\pm2,\pm3\ldots$ including at $\pm2.944MHz$ and $\pm3.072Mz$ (see Figure S2.16b). Graphical convolution of the sampling function with the spectrum of the signal (Figure S2.16a) leads to Figure S2.16c.

(b) If the IF is centred at 3 MHz, the upper edge frequency becomes 3MHz + 32 kHz = 3.032 MHz. The ratio of the upper frequency to the bandwidth is no longer an integer (3032/64 = 47.375). The sampled spectrum can be obtained by graphical convolution . In this case, both the negative and positive frequency components of the signal contribute simultaneously to the final results.

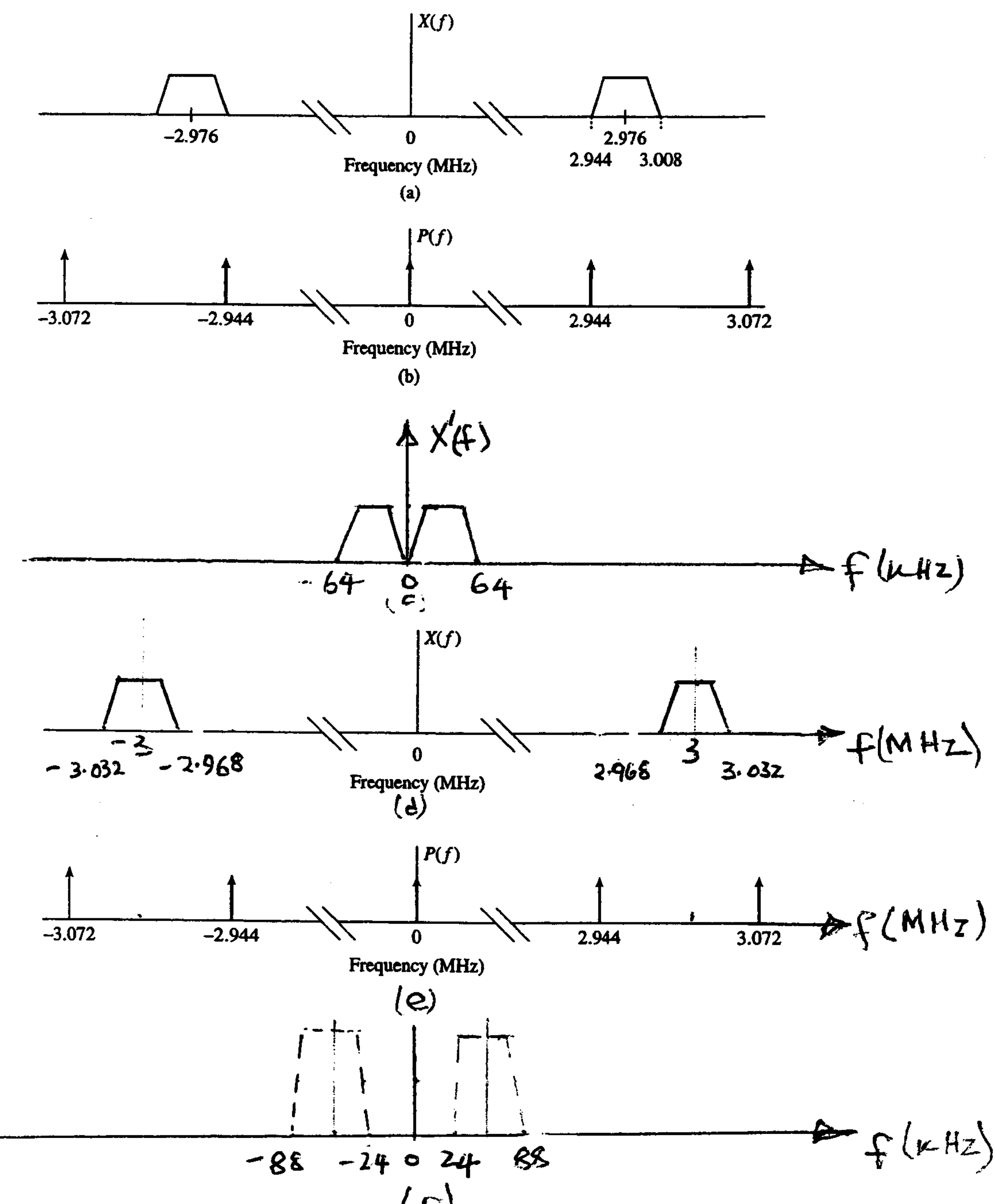

Figure S2.16

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2.17(a) Write down the equation of the bandpass sampling theorem. Explain why the bandpass sampling theorem is of interest in digital communication.

(b) Starting with the equation, derive an expression for the theoretical minimum sampling rate for a bandpass signal. Asssume that the ration of the upper bandedge frequency to the bandwidth of the signal is an integer. Comment on why the theoretical minimum sampling rate may be inappropriate in practice.

Solution

(a) $\frac{2f_H}{n} \le F_S \le \frac{2f_L}{n-1}$ $n = \frac{f_H}{B}$ *(n is an integer, rounded up to the nearest integer).*

In communications systems, signals of interest may only occupy a small proportion of the available band. The theorem provides a way of sampling such narrow band, high frequency, signal at a much reduced rate and still avoid aliasing which simplifies system design.

(b) For the problem, $\frac{f_H}{B} = n$ is an integer and so $f_H = nB$. Substituting for f_H, the left hand side in the bandpass sampling equation becomes 2B.

Noting that $f_H = f_L + B$, then $f_L = f_H - B = B(n-1)$. Substituting for f_L, the RHS of the bandpass sampling equation is also 2B. Thus, $F_s(\min) = 2B$.

In practice, imperfections in the sampling clock and the non-ideal characteristics of the bandpass filter used to isolate the desired band will mean that the theoretical sampling rate is inappropriate. A guard band may be included before determining the minimum sampling rate.

Quantisation noise in A/D Conversion

2.18 A real-time DSP system uses a linear 16-bit ADC in the bipolar mode, with an input range of $\pm 5V$. What is the maximum quantization error? Calculate the theoretical maximum SQNR, in decibels, for the system.

Solution

For an ADC with B-binary digits, the quantization stepsize, q, is $q = \frac{V_{fs}}{2^B - 1} \approx \frac{V_{fs}}{2^B}$

where V_{fs} is the full scale range of the ADC. With B=16, and V_{fs} =10 volts, $q = \frac{10}{2^{16}} = 0.152mV$. The maximum quantization error (assuming rounding) = $\frac{q}{2} = 76\mu V$. The theoretical maximum SQNR = 6.02 + 1.76 = 98.08 dB.

2.19 A sinusoidal signal with peak-to-peak amplitude of 5V is digitized with a 16-bit ADC. Assuming linear quantization, determine:

(1) the quantization step size, and
(2) the rms signal-to-noise ratio.

State any assumptions.

(1). Quantization stepsize = $q = \frac{V_{fs}}{2^B - 1} = 0.076mV$

(2). SQNR (rms) = $\frac{A}{\sqrt{2}} \Big/ \sqrt{\frac{q^2}{12}} = 1.22 \times 2^B = 98.08dB$, where we have set A =2.5V and B=16.

2.20 The analog input to a DSP system is digitized at a rate of 100 kHz with uniform quantization. Assuming a sine wave input with a peak-to-peak amplitude of $\pm 5V$, find the minimum number of bits for the ADC to achieve a SQNR of at least 90 dB. State any reasonable assumptions.

Solution

From SQNR = $1.76 + 6.02dB$, $B = (90 - 1.76)/6.02 = 14.65 \approx 15bits$

2.21 Show that the signal-to-quantization noise ratio of a linear ADC is given by:

$$SQNR = 6.02B + 4.77 - 20\log(A/\sigma_x)dB.$$

where B is the number of ADC bits, $\pm A$ is the input range of the ADC, and σ_x is the rms value of the input signal. Determine the SQNR if the resolution of the ADC is 16 bits and the input is:

(1). a sine wave signal, and
(2). a signal with an rms value of $A/4$.

State any assumptions.

Solution

For a B-bit ADC with an input range of $\pm A$, the quantization step size is:

$$q = \frac{2A}{2^B - 1} \approx \frac{2A}{2^B}$$

Assuming the errors are random with zero mean and uniformly distributed in the range $\pm\frac{q}{2}$, the quantization noise power (i.e. variance) is given by:

$$E\left[e^2(n)\right] = \sigma^2 = \frac{q^2}{12}$$

where E[.] is the expectation. The signal-to-quantization noise ratio (SQNR) is given by:

$$SQNR = \frac{\sigma_x^2}{q^2/12} \approx \frac{12\sigma_x^2}{\left(\frac{2A}{2^B}\right)^2} = \frac{3\sigma_x^2 \times 2^{2B}}{A^2}$$

$$= 10\log\left(3 \times 2^{2B} \times \frac{\sigma_x^2}{A^2}\right) = 6.02B + 4.771 - 20\log\left(\frac{A}{\sigma_x}\right)$$

where σ_x^2 = mean square value of the signal.

(1) For a sine wave input with peak amplitude A, that just fills the ADC range

$\sigma_x = \frac{A}{\sqrt{2}}$ and so $20\log\left(\frac{A}{\sigma_x}\right) = 3.01dB$. Thus, SQNR = 6.02B + 1.7 dB = 98.02 dB (B=16).

(2) $SQNR = 6.02B + 4.77 - 20\log\left(\frac{A}{A/4}\right) = 89.05dB.$

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2.22 *The analog input signal to a B-bit ADC has an rms value of σ_x (V). The input range of the ADC is adjusted to the range $\pm 3\sigma_x$ (V). Find an expression for the SQNR, in decibels, for the converter. State any reasonable assumptions.*

Solution

$SQNR = \frac{\sigma_x^2}{\sigma_e^2}$; For a B-bit converter, $q = \frac{V_{fs}}{2^B - 1} = \frac{6\sigma_x}{2^B}$

The noise power, $\sigma_e^2 = \frac{q^2}{12}$. ***Thus, SQNR =*** $\sigma_x^2 \Big/ \left(36 \times \sigma_x^2 \Big/ 2^{2B} \times 12\right) = (6.02B - 0.477)dB.$

Over-sampling in A/D Conversion – aliasing and quantization noise control

2.23 *(a) Explain, with the aid of a suitable sketch, the principles of oversampling techniques and how it can be used to increase the effective resolution of Nyquist rate analogue-to-digital converters.*

(b) A digital audio system uses oversampling techniques and an 8-bit, bipolar, Nyquist rate converter to digitise an analogue input signal which has frequency components in the range 0-4 kHz. Estimate the effective resolution, in bits, of the converter if the sampling rate is 40 MHz. Show clearly how you obtained your answer. Comment on the practical problems associated with this approach.

Solution

(a) See section 2.3.2 for details.

(b) The in-band quantisation noise is reduced by the oversampling ratio, i.e. by:

$$\frac{40{,}000}{2 \times 4} = 5000.$$

From $\frac{\sigma_1^2}{\sigma_2^2} = \frac{2^{-2(B_1-1)}}{2^{-2(B_2-1)}} = 2^{2(B_2-B_1)} = 5000$ and with B_1 =8 bits, we find that the resolution of the ADC, B_2 is about 14 bits.

Over-sampling techniques on their own may not be economically feasible to achieve the desired resolution using low resolution ADC because often it requires very high sampling rates which may not be supportable by current technology.

As is evident in the previous examples, over sampling techniques on their own may not be economically feasible for achieving the desired resolution using low resolution ADCs because often it requires very high sampling frequencies which may not be supported by current technology.

In practice, over sampling is combined with noise shaping to shift the quantisation noise to higher frequencies well outside the signal band, where it can be filtered out.

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2.24 *(a) A requirement exists for a general purpose, multichannel (up to 64 channels) data acquisition system for collecting neurophysiological data. Each analogue channel is to be individually configured, by the user, to have a passband edge frequency between 0.5 Hz and 500 Hz, and a selectable sampling frequency in the range of 1 Hz to 5 kHz. In the passband, the maximum permissible ripple is 0.5 dB and the image components must be at least 40 dB below the signal components.*

Explain the strategy you would use to satisfy the above requirement. Your answer should address the following points:

(i) considerations for application-specific issues,

(ii) how oversampling techniques may be used in this application to satisfy the requirement in an efficient and economical way (in terms of cost/ component count).

(b) Assume that identical anti-aliasing filters are used for all the channels in the system in (a), each with the following Butterworth characteristics.

$$A(f) = \frac{1}{\sqrt{1+\left(\frac{f}{f_c}\right)^6}}$$

where f_c = 3 dB cutoff frequency of filter

Determine, with the aid of sketches of the spectrum of the data before and after sampling:

(i) the cutoff frequency, f_c,
(ii) a suitable common sampling frequency, F_s

Comment on your answer

Solution

a(i) To preserve information of clinical interest in the signals, both amplitude and phase distortions should be kept as low as possible. The time relationships between features across the channels should be preserved. The use of identical anti-aliasing filters, with reasonably good amplitude/phase responses is desirable.

(ii) To reduce component count/cost and size of the PCB for the system, all 64 channels should be fitted with identical, simple anti-aliasing filters. The channels should then be over sampled at a common fixed rate. The high, common sample rate can be reduced to the desired rates using multirate techniques. At least a 2nd order Butterworth filter should be used to avoid an excessive common sampling rate.

(b). From a consideration of the specifications and the spectrum of the data, before and after sampling we find that

(i) To keep within the specifications, the amplitude error between 0 and 500 Hz should satisfy the following criterion:

$$20\log\left[1+\left(\frac{500}{f_c}\right)^8\right]^{\frac{1}{2}} \le 0.5dB$$

where we have assumed a second order Butterworth filter with a cut off frequency of f_c.
Solving for f_c, we find that $f_c = 260.15Hz$.

(To allow for additional errors in subsequent stages and for convenience, a higher f_c may be used).

(ii) After band-limiting each channel, the spectrum of the sampled data has the form shown in Figure S2.24:

Now, F_s is chosen such that the aliasing error level is down by at least 40 dB at 500 Hz, i.e.

$$20\log\left[1+\left(\frac{F_s-500}{260.15}\right)^6\right]^{\frac{1}{2}} \geq 40dB$$

Solving for F_s gives, $F_s = 1.02kHz$.

Some care is required in the choice of the common sampling frequency, F_s, to allow for an efficient reduction in the sample rate. A possible choice is 2 kHz which allows the common rate to be reduced by simple integer factors.

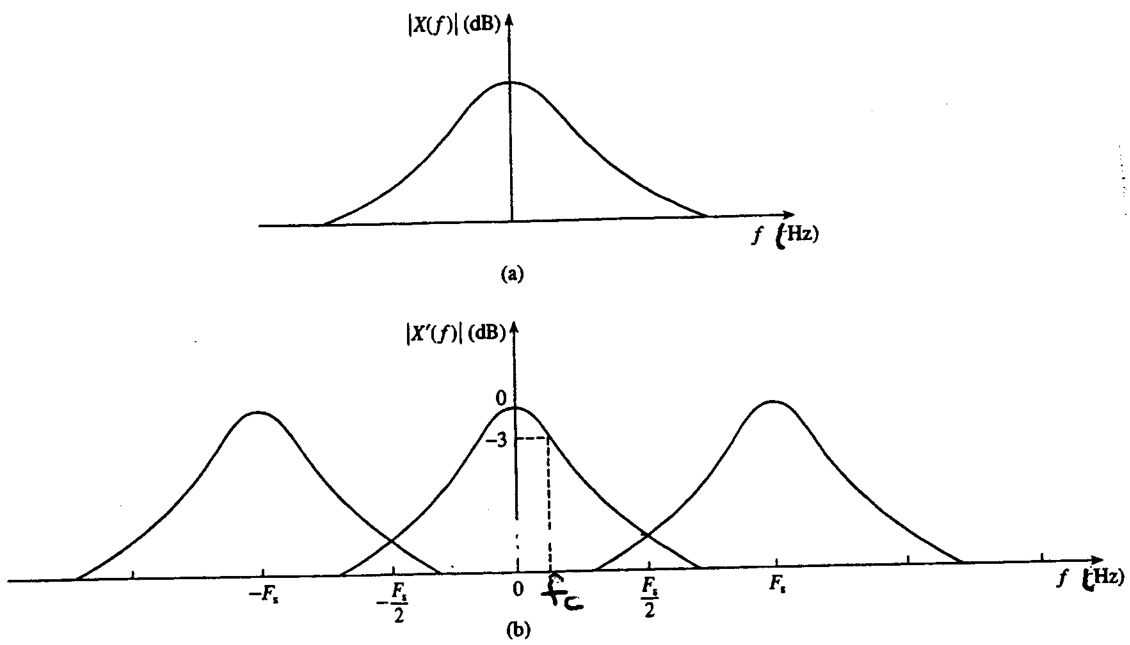

Figure S2.24

2.25 *An audio system handles signals with a baseband that extends from 0 to 20 kHz. Determine the over sampling ratio and the minimum sampling frequency that will be necessary to achieve a performance that would be obtained with a 16 bit ADC using an 8-bit bit converter.*

Solution

(a) At the Nyquist rate (i.e. $F_s = 2f_{\max}$), the normalised in-band quantisation noise power for the 8-bit and 16-bit converters are, respectively:

$$\sigma_1^2 = 2^{\frac{-2(B_1-1)}{12}} \quad \text{(where } B_1 = 8\text{)}$$

$$\sigma_2^2 = 2^{\frac{-2(B_2-1)}{12}} \quad \text{(where } B_2 = 16\text{).}$$

To achieve 16-bit performance with a 8-bit ADC, we will need to over-sample the input to the 8-bit converter to reduce the in-band quantisation noise power. The in-band quantisation noise power is reduced by the over sampling factor:

$$\sigma_1'^2 = \frac{2f_{\max}}{F_s}\sigma_1^2$$

Equating the new in-band quantisation noise to that for the 16-bit ADC, we have :

$$\frac{2f_{\max}}{F_s}\sigma_1^2 = \sigma_2^2$$

Thus, $\dfrac{2f_{\max}}{F_s} = \dfrac{\sigma_2^2}{\sigma_1^2} = \dfrac{2^{-2(B_2-1)}}{2^{-2(B_1-1)}} = 2^{-2(B_2-B_1)} = \dfrac{1}{65536}$

Thus, the oversampling ratio is given by $F_s/(2f_{\max}) = 65536$,i.e. $F_s = 2.621GHz$.

2.26 (a) In relation to "single-bit" ADCs, write brief explanatory notes on each of the following techniques:

(i) oversampling
(ii) noise spectrum shaping

(b) Why are "single bit" ADCs preferred to conventional successive approximation ADCs in high fidelity DSP systems?

(c). A digital signal processing system with analogue audio signal input in the range 0-20 kHz uses oversampling techniques and the first order delta-sigma modulator depicted in Figure 2.53 to digitise the analogue signal. Assume that the sampling frequency is 3.072 MHz, find the response of the noise shaping filter at 20 kHz.

Estimate the effective resolution, in bits, of the digitizer.

Solution

(a) (i) oversampling – sampling at a rate in excess of the Nyquist rate. Reduces in-band noise level bt spreading quantisation energy over a much wide frequency range; simplifies the requirements of the anti-aliasing filter.
(ii) Noise shaping – filtering/shaping noise spectrum by pushing most of the noise energy to higher frequencies and outside of the desired frequency band.
More details are in the main text.

(b) An estimate of the effective resolution can be obtained by a simplified analysis as follows:

The noise power is reduced by over-sampling and noise shaping. The reduction in noise power, due to oversampling, is given by the oversampling ratio.

The over-sampling ratio is:

$$\frac{F_s}{2f_{\max}} = \frac{3.072\times10^6}{2\times24\times10^3} = 76.8$$

That is a reduction of 18.85 dB in the quantisation noise power.

From the z-plane model of the delta sigma modulator, the transfer function seen by the quantisation noise is given by:

$$N(z) = \left(1 - z^{-1}\right)$$

This is essentially a highpass filter, with a double zero at d.c. It attenuates the noise component at the low frequency end. The magnitude response is given by:

$$\left|N(z)\right|^2_{z=e^{j\omega T}} = \left|\left(1 - e^{-j\omega T}\right)\right|^2$$

2/2-3

At $f = 20kHz$ (the band edge) and $F_s = 3.072MHz,\ \omega T = 2.34^o$ and

$$\left|N(e^{j\omega T})\right|^2 = 1.67\times 10^{-3}$$

This offers a reduction in SQNR of 27.76 dB. The effective wordlength of the ADC is determined mainly by the signal-to-noise ratio achieved through over-sampling and noise shaping. The overall reduction in SQNR is 46.6 dB. This corresponds to an effective ADC resolution of 7.4 bits (from SQNR = 6.02B+1.77 dB).

2.27 *A digital signal processing system, with analogue audio signal input in the range 0-20 kHz, uses oversampling techniques and the first order delta-sigma modulator to convert the analogue signal into a digital bit stream at a rate of 6.144 MHz. The z-plane model of the delta-sigma modulator is depicted in Figure 2.54.*

Determine the overall improvement in signal-to-quantisation noise ratio made possible by oversampling and noise shaping and hence estimate the effective resolution, in bits, of the digitizer.

Solution

An estimate of the effective resolution can be obtained by a simplified analysis as follows:

The noise power is reduced by over-sampling and noise shaping. The reduction in noise power, due to oversampling, is given by the oversampling ratio.

The over-sampling ratio is:

$$\frac{F_s}{2f_{max}} = \frac{6.144\times 10^6}{2\times 20\times 10^3} = 153.6$$

That is a reduction of 21.86 dB in the quantisation noise power.

From the z-plane model of the delta sigma modulator, the transfer function seen by the quantisation noise is given by:

$$N(z) = \left(1 - z^{-1}\right)$$

This is essentially a highpass filter, with a double zero at d.c. It attenuates the noise component at the low frequency end. The magnitude response is given by:

$$\left|N(z)\right|^2_{z=e^{j\omega T}} = \left|\left(1 - e^{-j\omega T}\right)\right|^2$$

At $f = 20kHz$ (the band edge) and $F_s = 6.144MHz,\ \omega T = 1.17^o$ and

$$\left|N(e^{j\omega T})\right|^2 = 1.75\times 10^{-7}$$

This provides a reduction in SQNR of 67.57 dB. The effective wordlength of the ADC is determined mainly by the signal-to-noise ratio achieved through over-sampling and noise shaping. The overall reduction in SQNR is 89.43 dB. This corresponds to an effective ADC resolution of 14.6 bits (from SQNR = 6.02B+1.77 dB).

2.28 A digital signal processing system, with analogue audio signal input in the range 0-20 kHz, uses oversampling techniques and the first order delta-sigma modulator to convert the analogue signal into a digital bit stream at a rate of 6.144 MHz. The z-plane model of the delta-sigma modulator is depicted in Figure 2.56.

(i). Explain how the digital bit stream may be converted into a digital multibit stream at 92 kHz rate.
(ii). Determine the overall improvement in signal-to-quantisation noise ratio made possible by oversampling and noise shaping and hence estimate the effective resolution, in bits, of the digitizer.

Solution

(i). The single bit stream is converted to multi-bit words by decimation (down sampling process). The output of the DSM contains very small in-band quantisation noise, but very large out-of-band noise. The out-of-band noise is removed by lowpass digital filtering. Because of the high sampling rate, direct use of a digital filter is impractical. In stead, filtering is achieved by decimation which also serves to reduce the rate to the desired value. Typically, a 2-stage decimator would be used (factors of 16 and 4). After filtering, the resulting signal is a B-bit quantised data. The filtering serves to average out the high quantisation noise. Typically, the FIR coefficients of the decimating filter are represented by 16-24 bits.

(ii). An estimate of the effective resolution can be obtained by a simplified analysis as follows:

The noise power is reduced by over-sampling and noise shaping. The reduction in noise power, due to oversampling, is given by the oversampling ratio.

The over-sampling ratio is:

$$\frac{F_s}{2f_{\max}} = \frac{3.072\times10^6}{2\times24\times10^3} = 153.6$$

That is a reduction of 21.86 dB in the quantisation noise power.

From the z-plane model of the delta sigma modulator, the transfer function seen by the quantisation noise is given by:

$$N(z) = \left(1 - z^{-1}\right)$$

This is essentially a highpass filter, with a double zero at d.c. It attenuates the noise component at the low frequency end. The magnitude response is given by:

$$\left|N(z)\right|^2_{z=e^{j\omega T}} = \left|\left(1 - e^{-j\omega T}\right)\right|^2$$

At $f = 20kHz$ (the band edge) and $F_s = 6.144MHz,\ \omega T = 1.17^o$ and

$$\left|N(e^{j\omega T})\right|^2 = 4.18\times10^{-4}$$

This offers a reduction in SQNR of 33.78 dB. The effective wordlength of the ADC is determined mainly by the signal-to-noise ratio achieved through over-sampling and noise shaping. The overall reduction in SQNR is 55.65 dB. This corresponds to an effective ADC resolution of 8.9 bits (from SQNR = 6.02B+1.77 dB).

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D/A conversion and Sinx/x effects

2.29 The block diagram of a real-time DSP system with analogue output is depicted in Figure 2.56(a), and Figure 2.56(b) shows the baseband spectrum of the signal applied to the DAC. Sketch the spectrum of the signal at the output of the DAC in the interval 0-2F_s, where F_s is the sampling frequency. Determine the amplitudes of the signal components in your sketch. Assume a sampling frequency of 15 kHz.

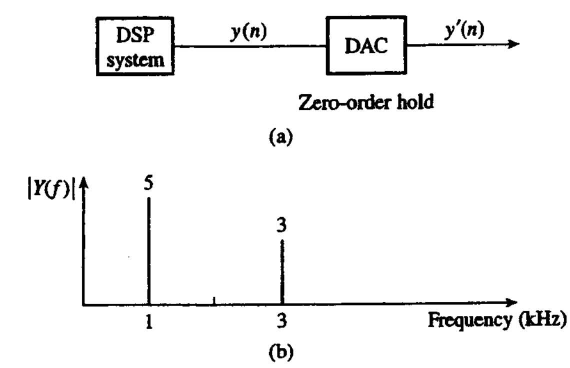
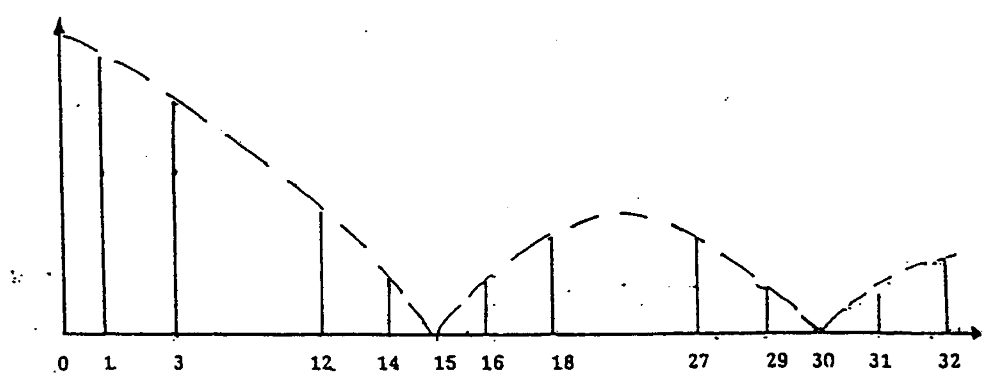


Figure 2.56 (a) Real-time DSP system with analog output, (b) Spectrum of signal applied to the DAC.

Solution

A sketch of the signal spectrum at the output of the DAC is shown in Figure S2.29 below.



The effect of the DAC is equivalent to convolution in the time domain or multiplication in the frequency domain. Thus, the amplitudes of the frequency components at the output of the DAC can be determined as follows:

@ 1 kHz $\quad \dfrac{5\times\sin(\pi f/15)}{(\pi f/15)}\Big|_{f=1kHz} = 5\times 0.99 = 4.96$ (N.B. $x = \dfrac{\omega T}{2} = \dfrac{\pi f}{F_s}$ radians)

@ 3 kHz $\quad \dfrac{3\times\sin(3\pi/15)}{(3\pi/15)} = 3\times 0.94 = 2.8$

The other values may be obtained in the same way.

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2.30 *A real-time DSP system uses a 16-bit processor, a 12-bit ADC with a conversion time of 15 μs, and a 12-bit DAC with a settling time of 500 ns. If the required DSP operation is the convolution summation given by:*

$$y(n) = \sum_{k=0}^{N-1} h(k)x(n-k)$$

where the variables have the usual meanings and the computation must be performed between samples, estimate the real-time capability of the system, stating any assumptions made.

The main task is to identify the computational requirements for the system and the associated time delays. For example, the convolution equation requires N multiplications, (N-1) additions and (N-1) delays or shifts. There will also be time delays for the A/D and D/A conversions, and for overheads (house keeping operations such as memory accesses to fetch instructions and or data. The sum of all delays defines the maximum sampling frequency that can be used.

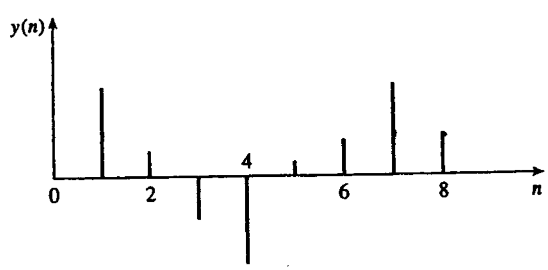
2.31 *The output of a digital-to-analog converter in response to a digital sequence is given by:*

$$y(n) = \sum_{n} y(n)h(t-nT)$$

where h(t) is the impulse response of the DAC and 1/T is the rate at which data is fed to the DAC. Assume that the DAC is a zero-order hold and h(t) a square pulse of duration T(s). Sketch the output of the DAC in response to the input sequence, y(n), shown in Figure 2.57. Show that the spectral shaping effect of the DAC on the signal spectrum can be compensated for by a digital filter with a spectrum of the form:

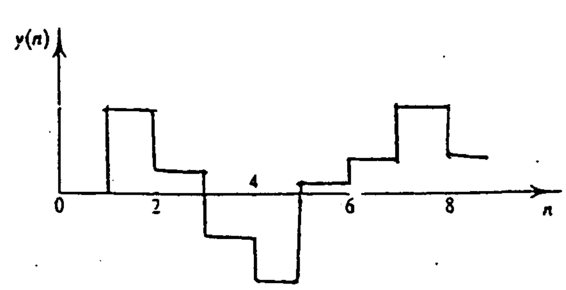
$$|H(\omega)| = \frac{\omega T}{2\sin(\omega T/2)}$$

Figure 2.57



Solution

A sketch of the output of the DAC in response to the input sequence is shown below.



2/27

By the convolution theorem, $Y'(\omega) = Y(\omega)H(\omega)$, i.e. the output signal spectrum, $Y(\omega)$, is modified by the frequency response of the DAC, $H(\omega)$. Assume h(t) is a pulse of width T, then:

$$H(\omega) = \frac{\sin(\omega T/2)}{\omega T/2}$$

Thus, if the output signal is first processed by a digital filter with an inverse response, this would compensate for the effect of the DAC response.

2.32 Critically examine the main constraints and errors introduced by analog/digital conversion processes in real-time digital signal processing, suggesting how each constraints or error may be reduced.

Constraints are summarised in section 2.10 of the main text.