8.3 *A requirement exists to simulate in a digital computer an analogue system with the following normalized characteristic:*

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Obtain a suitable transfer function using:

(1) the impulse invariant method, and
(2) the bilinear transform method.

Assume a sampling frequency of 5 kHz and a 3 dB cutoff frequency of 1 kHz.

Solution

(1) The transfer function of the denormalised analog filter is obtained by replacing s by s/α (where α is the cutoff frequency relative to the sampling frequency, i.e $\alpha = 2\pi \times 1/5 = 1.25395$):

$$H'(s) = H(s)\Big|_{s=\frac{s}{\alpha}} = \frac{\alpha^2}{s^2 + \sqrt{2}\alpha s + \alpha^2}$$

For convenience, H(s) will be written in the general form:

$$H'(s) = \frac{B_0}{s^2 + A_1 s + A_0}$$

To apply the impulse invariant method, the s-plane transfer function is first expanded into partial fractions as described in section 8.6.

$$H'(s) = \frac{B_0}{s^2 + A_1 s + A_0} = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2}$$

where c_1 and c_2 are partial fraction coefficients and p_1 and p_2 are the poles of H'(s), given by:

$$p_{1,2} = -\frac{A_1}{2} \pm \left[\left(\frac{A_1}{2}\right)^2 - A_0\right]^{\frac{1}{2}}$$

$$p_r = -\frac{A_1}{2}; \quad p_i = \left\{-\left[\left(\frac{A_1}{2}\right)^2 - A_0\right]\right\}^{\frac{1}{2}}$$

Multiplying both sides of H'(s) by $(s - p_1)(s - p_2)$ and equating coefficients of s and the constant terms we have:

$$B_0 = -(c_1 p_2 + c_2 p_1)$$

$$0 = c_1 + c_2$$

Solving for c_1 and c_2 we have: $c_1 = \dfrac{B_0}{p_1 - p_2}$ and $c_2 = -c_1$

Applying the impulse invariant transformation to Equation we obtain the transfer function:

$$H(z) = \frac{c_1}{1 - e^{p_1 T} z^{-1}} + \frac{c_2}{1 - e^{p_2 T} z^{-1}}$$

$$= \frac{2c_r - [c_r \cos(p_i T) + c_i \sin(p_i T)] 2e^{p_r T} z^{-1}}{1 - 2e^{p_r T} \cos(p_i T) z^{-1} + e^{2p_r T} z^{-2}} \quad \text{(see Equation 8A.8).}$$

where p_r and p_i pr and pi are the real and imaginary parts of p_1, c_r and c_i the real and imaginary parts of c_1. From the expressions for p_1 above, the real and imaginary parts of p_1 are:

$$p_r = -\frac{A_1}{2}; \quad p_i = \left\{ -\left[\left(\frac{A_1}{2} \right)^2 - A_0 \right] \right\}^{\frac{1}{2}}$$

The real and imaginary parts of c_1 are: $c_r = 0$; $c_i = -\frac{B_0}{2p_i} j$. Now for the problem, $A_0 = 1.25395$, $A_1 = \sqrt{2}\alpha =$; $B_0 = \alpha^2$. Using these values in the equation above, we have:

$p_r = -0.888576$, $p_i = 0.888576$, $2c_r = 0$, $c_i = -0.888576j$
$[c_r \cos(p_i T) + c_i \sin(p_i T)] 2e^{p_r T} = 0.567258$, $2e^{p_r T} \cos(p_i T) = 0.518588$ and
$e^{2p_r T} = 0.169119$ (T=1 is assumed). Using these values leads to:

$$H(z) = \frac{0.567258 z^{-1}}{1 - 0.5185889 z^{-1} + 0.169119 z^{-2}}$$

(2) The pre-warped cutoff frequency, $\omega_p' = \tan(\omega_p T/2) = \tan(\pi \times 1000/5000) = 0.72654$. The scaled s-plane transfer function, $H'(s)$, is:

$$H'(s) = H(s)|_{s=s/\omega'_p} = \frac{1}{(s/\omega'_p)^2 + \sqrt{2}\alpha s/\omega'_p + 1}$$

$$= \frac{\omega_p'^2}{s^2 + \sqrt{2}\omega'_p s + \omega_p'^2} = \frac{0.52786}{s^2 + 1.02748s + 0.52786}$$

Applying the BZT and simplifying gives

$$H(z) = H'(s)\Big|_{s=\frac{z-1}{z+1}} = \frac{0.52786}{\left(\frac{z-1}{z+1}\right)^2 + 1.02748\left(\frac{z-1}{z+1}\right) + 0.52786}$$

$$= \frac{0.20657(1 - 2z^{-1} + z^{-2})}{1 - 0.36953z^{-1} + 0.19582z^{-2}}$$

8.4 Determine, using the BZT method, the transfer function and difference equation for the digital equivalent of the resistance-capacitance (RC) filter shown in Figure 8.43 Assume a sampling frequency of 150 Hz and a cut off frequency of 30 Hz.

Solution

The normalized transfer function for the RC filter is $H(s) = \dfrac{1}{s+1}$. The cutoff frequency for the digital filter is $\omega_p = 2\pi \times 30 rad$. The cutoff frequency for the equivalent analog filter, after pre-warping, is $\omega'_p = \tan(\omega_p T/2)$. With $T = 1/150Hz$, $\omega'_p = \tan(\pi/5) = 0.7265$. The denormalized analog filter transfer function is obtained from H(s) as:

$$H'(s) = H(s)\,|_{s=s/0.7265} = \frac{1}{(s/0.7265)+1} = \frac{0.7265}{s+0.7265}$$

$$H(z) = H'(s)\,|_{s=(z-1)/(z+1)} = \frac{0.7265}{\left(\dfrac{z-1}{z+1}\right)+0.7265} = \frac{0.7265(z+1)}{(z-1)+0.7265(z+1)}$$

$$= \frac{0.4208(1+z^{-1})}{1-0.1584z^{-1}}$$

The difference equation may be written by inspection of

H(z) $y(n) = 0.1584y(n-1) + 0.4208[x(n) + x(n-1)]$.