$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/M}. \tag{3.54}$$

Define a new sequence $x_e[n]$ obtained from $x[n]$ by augmenting with $M - N$ zero-valued samples:

$$x_e[n] = \begin{cases} x[n], & 0 \le n \le N-1, \\ 0, & N \le n \le M-1. \end{cases} \tag{3.55}$$

Making use of $x_e[n]$ in Eq. (3.54) we arrive at

$$X(e^{j\omega_k}) = \sum_{n=0}^{M-1} x_e[n]e^{-j2\pi kn/M}, \tag{3.56}$$

which is seen to be an M-point DFT $X_e[k]$ of the length-M sequence $x_e[n]$. The DFT $X_e[k]$ can be computed very efficiently using the FFT algorithm if M is an integer power of 2.

The MATLAB function `freqz`, described in Section 3.1.6, employs the above approach to evaluate the frequency response of a rational DTFT expressed as a rational function in $e^{-j\omega}$ at a prescribed set of discrete frequencies. It computes the DFTs of the numerator and the denominator separately by considering each as finite-length sequences, and then expresses the ratio of the DFT samples at each frequency point to evaluate the DTFT.

3.4 Discrete Fourier Transform Properties

Like the DTFT, the DFT also satisfies a number of properties that are useful in signal processing applications. Some of these properties are essentially identical to those of the DTFT, while some others are somewhat different. A summary of the DFT properties are included in Tables 3.5, 3.6, and 3.7. Their proofs are again quite straightforward and have been left as exercises. Most of these properties can also be verified using MATLAB. We discuss next those properties that are different from their counterparts for the DTFT.

3.4.1 Circular Shift of a Sequence

This property is analogous to the time-shifting property of the DTFT as given in Table 3.2, but with a subtle difference. Let us consider length-N sequences defined for $0 \le n \le N-1$. Such sequences have sample values equal to zero for $n < 0$ and $n \ge N$. If $x[n]$ is such a sequence, then, for any arbitrary integer n_o, the shifted sequence $x_1[n] = x[n - n_o]$ is no longer defined for the range $0 \le n \le N-1$. We therefore need to define another type of a shift that will always keep the shifted sequence in the range $0 \le n \le N-1$. This is achieved by defining a new type of shift, called the *circular shift*, using a modulo operation according to[4]

$$x_c[n] = x\left[\langle n - n_o\rangle_N\right]. \tag{3.57}$$

For $n_o > 0$ (right circular shift), the above equation implies

$$x_c[n] = \begin{cases} x[n-n_o], & \text{for } n_o \le n \le N-1, \\ x[N-n_o+n], & \text{for } 0 \le n < n_o. \end{cases} \tag{3.58}$$

The concept of a circular shift of a finite-length sequence is illustrated in Figure 3.10. Figure 3.10(a) shows a length-6 sequence $x[n]$. Figure 3.10(b) shows its circularly shifted version shifted to the right by

[4] $\langle m\rangle_N = m$ modulo N.

Table 3.5: General properties of the DFT.

Type of Property	Length-N Sequence	N-point DFT
	$g[n]$ $h[n]$	$G[k]$ $H[k]$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular time-shifting	$g[\langle n - n_o \rangle_N]$	$W_N^{kn_o} G[k]$
Circular frequency-shifting	$W_N^{-k_o n} g[n]$	$G[\langle k - k_o \rangle_N]$
Duality	$G[n]$	$Ng[\langle -k \rangle_N]$
N-point circular convolution	$\sum_{m=0}^{N-1} g[m]h[\langle n - m \rangle_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N} \sum_{m=0}^{N-1} G[m]H[\langle k - m \rangle_N]$
Parseval's relation	$\sum_{n=0}^{N-1} \lvert x[n] \rvert^2 = \frac{1}{N} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2$	

Table 3.6: Symmetry properties of the DFT of a complex sequence.

Length-N Sequence	N-point DFT
$x[n]$	$X[k]$
$x^*[n]$	$X^*[\langle -k \rangle_N]$
$x^*[\langle -n \rangle_N]$	$X^*[k]$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{pcs}}[k] = \frac{1}{2}\{X[\langle k \rangle_N] + X^*[\langle -k \rangle_N]\}$
$j\,\mathrm{Im}\{x[n]\}$	$X_{\mathrm{pca}}[k] = \frac{1}{2}\{X[\langle k \rangle_N] - X^*[\langle -k \rangle_N]\}$
$x_{\mathrm{pcs}}[n]$	$\mathrm{Re}\{X[k]\}$
$x_{\mathrm{pca}}[n]$	$j\,\mathrm{Im}\{X[k]\}$

Note: $x_{\mathrm{pcs}}[n]$ and $x_{\mathrm{pca}}[n]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $x[n]$, respectively. Likewise, $X_{\mathrm{pcs}}[k]$ and $X_{\mathrm{pca}}[k]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $X[k]$, respectively.

Table 3.7: Symmetry properties of the DFT of a real sequence.

Length-N Sequence	**N-point DFT**
$x[n]$	$X[k] = \text{Re}\{X[k]\} + j\,\text{Im}\{X[k]\}$
$x_{\text{pe}}[n]$	$\text{Re}\{X[k]\}$
$x_{\text{po}}[n]$	$j\,\text{Im}\{X[k]\}$
Symmetry relations	$X[k] = X^*[\langle -k\rangle_N]$ $\text{Re}\,X[k] = \text{Re}\,X[\langle -k\rangle_N]$ $\text{Im}\,X[k] = -\,\text{Im}\,X[\langle -k\rangle_N]$ $\lvert X[k]\rvert = \lvert X[\langle -k\rangle_N]\rvert$ $\arg X[k] = -\arg X[\langle -k\rangle_N]$

Note: $x_{\text{pe}}[n]$ and $x_{\text{po}}[n]$ are the periodic even and periodic odd parts of $x[n]$, respectively.

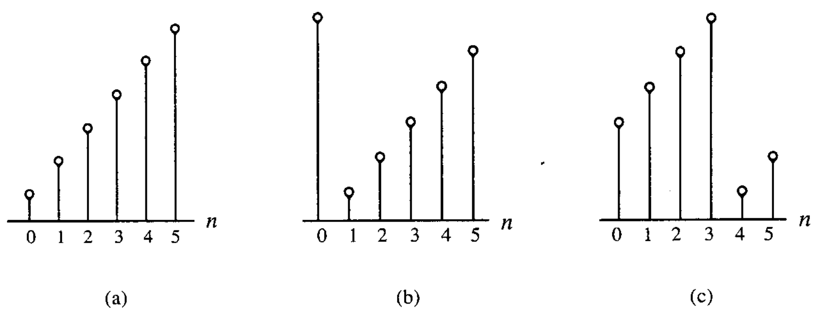

Figure 3.10: Illustration of a circular shift of a finite-length sequence. (a) $x[n]$, (b) $x[\langle n-1\rangle_6] = x[\langle n+5\rangle_6]$, and (c) $x[\langle n-4\rangle_6] = x[\langle n+2\rangle_6]$.

1 sample period or, equivalently, shifted to the left by 5 sample periods. Likewise, Figure 3.10(c) depicts its circularly shifted version shifted to the right by 4 sample periods or, equivalently, shifted to the left by 2 sample periods.

As can be seen from Figure 3.10(b) and (c), a right circular shift by n_o is equivalent to a left circular shift by $N - n_o$ sample periods. It should be noted that a circular shift by an integer number n_o greater than N is equivalent to a circular shift by $\langle n_o\rangle_N$.

If we view the length-N sequence displayed on the circumference of a cylinder at N equally spaced points, then the circular shift operation can be considered as a clockwise or anticlockwise rotation of the sequence by n_o sample spacings on the cylinder.

Figure 3.11: Two length-4 sequences.

3.4.2 Circular Convolution

This property is analogous to the linear convolution of Eq. (2.64), but with a subtle difference. Consider two length-N sequences, $g[n]$ and $h[n]$, respectively. Their linear convolution results in a length-$(2N-1)$ sequence $y_L[n]$ given by

$$y_L[n] = \sum_{m=0}^{N-1} g[m]h[n-m], \qquad 0 \le n \le 2N-2, \tag{3.59}$$

where we have assumed that both N-length sequences have been zero-padded to extend their lengths to $2N-1$.[5] The longer length of $y_L[n]$ results from the time-reversal of the sequence $h[n]$ and its linear shifting to the right. The first nonzero value of $y_L[n]$ is $y_L[0] = g[0]h[0]$, and the last nonzero value of $y_L[n]$ is $y_L[2N-2] = g[N-1]h[N-1]$.

To develop a convolution-like operation resulting in a length-N sequence $y_C[n]$, we need to define a circular time-reversal and then apply a circular time-shift. The resulting operation, called a *circular convolution*, is defined below:[6]

$$y_C[n] = \sum_{m=0}^{N-1} g[m]h\left[\langle n-m\rangle_N\right]. \tag{3.60}$$

Since the above operation involves two length-N sequences, it is often referred to as an *N-point circular convolution*, denoted as

$$y_C[n] = g[n] \circledN h[n]. \tag{3.61}$$

Like the linear convolution, the circular convolution is commutative (Problem 3.65), i.e.,

$$g[n] \circledN h[n] = h[n] \circledN g[n]. \tag{3.62}$$

We illustrate the concept of circular convolution through several examples.

EXAMPLE 3.15 Determine the 4-point circular convolution of the two length-4 sequences $g[n]$ and $h[n]$ given by

$$g[n] = \{\underset{\uparrow}{1} \;\; 2 \;\; 0 \;\; 1\}, \qquad h[n] = \{\underset{\uparrow}{2} \;\; 2 \;\; 1 \;\; 1\}, \tag{3.63}$$

and sketched in Figure 3.11. The result is a length-4 sequence $y_C[n]$ given by

$$y_C[n] = g[n] \circled{4} h[n] = \sum_{m=0}^{3} g[m]h[\langle n-m\rangle_4], \qquad 0 \le n \le 3. \tag{3.64}$$

From Eq. (3.64), $y_C[0]$ is given by

[5]As indicated in Section 2.5.1, the sum of the indices of each sample product inside the summation is equal to the index of the sample being generated by the linear convolution operation.

[6]Note that here the sum of the indices of each sample product inside the summation modulo N is equal to the index of the sample being generated by the circular convolution operation.

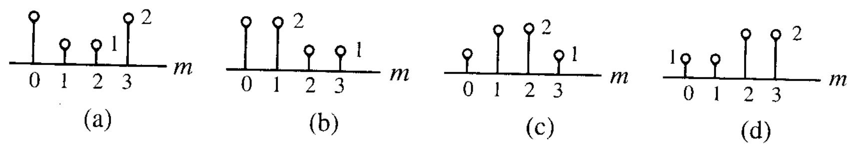

Figure 3.12: The circularly time reversed sequence and its circularly shifted versions: (a) $h[\langle -m\rangle_4]$, (b) $h[\langle 1-m\rangle_4]$, (c) $h[\langle 2-m\rangle_4]$, and (d) $h[\langle 3-m\rangle_4]$.

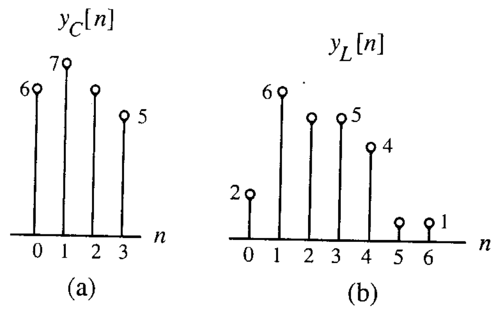

Figure 3.13: Results of convolution of the two sequences of Figure 3.11. (a) Circular convolution, and (b) linear convolution.

$$y_C[0] = \sum_{m=0}^{3} g[m]h[\langle -m\rangle_4]. \tag{3.65}$$

The circularly time reversed sequence $h[\langle -m\rangle_4]$ is shown in Figure 3.12(a). By forming the products of $g[m]$ with that of $h[\langle -m\rangle_4]$ for each value of m in the range $0 \le m < 3$ and summing the products, we arrive at

$$\begin{aligned} y_C[0] &= g[0]h[0] + g[1]h[3] + g[2]h[2] + g[3]h[1] \\ &= (1 \times 2) + (2 \times 1) + (0 \times 1) + (1 \times 2) = 6. \end{aligned} \tag{3.66}$$

Next, from Eq. (3.64) we compute $y_C[1]$ as

$$y_C[1] = \sum_{m=0}^{3} g[m]h[\langle 1-m\rangle_4]. \tag{3.67}$$

The sequence $h[\langle 1-m\rangle_4]$ obtained by circularly time shifting $h[\langle -m\rangle_4]$ to the right by one sample period is shown in Figure 3.12(b). Summing the products $g[m]h[\langle 1-m\rangle_4]$ for each value of m, we arrive at

$$\begin{aligned} y_C[1] &= g[0]h[1] + g[1]h[0] + g[2]h[3] + g[3]h[2] \\ &= (1 \times 2) + (2 \times 2) + (0 \times 1) + (1 \times 1) = 7. \end{aligned} \tag{3.68}$$

Continuing this process, we determine the remaining two samples of $y_C[n]$ as

$$\begin{aligned} y_C[2] &= g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[3] \\ &= (1 \times 1) + (2 \times 2) + (0 \times 2) + (1 \times 1) = 6. \end{aligned} \tag{3.69}$$

$$\begin{aligned} y_C[3] &= g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0] \\ &= (1 \times 1) + (2 \times 1) + (0 \times 2) + (1 \times 2) = 5. \end{aligned} \tag{3.70}$$

The length-4 sequence $y_C[n]$ obtained by a 4-point circular convolution of the two length-4 sequences $g[n]$ and $h[n]$ is sketched in Figure 3.13(a).