ECE 447 Fall 2025

Lesson 30

Probability

UNITED STATES
AIR FORCE
ACADEMY

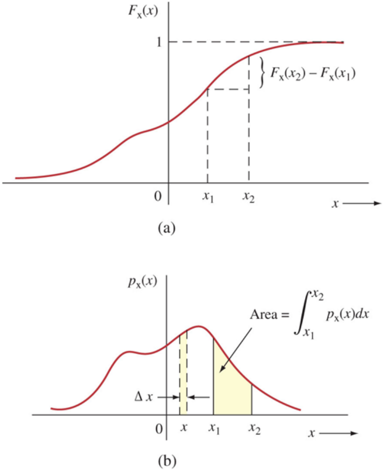
Schedule and Admin

- Schedule.
- Admin
  - **Lab 5**. PDF due 6 Nov (Lsn 32) to Gradescope.
  - **HW6**. Due 4 Nov (Lsn 31) to Gradescope.

CDF and PDF

- CDF: $F_{\mathrm{x}}(x) = P(\mathrm{x} \leqslant x)$
    - $F_{\mathrm{x}}(x) \geqslant 0$
    - $F_{\mathrm{x}}(\infty) = 1$
    - $F_{\mathrm{x}}(-\infty) = 0$
    - $F_{\mathrm{x}}(x_1) \leqslant F_{\mathrm{x}}(x_2)$ for $x_1 \leqslant x_2$

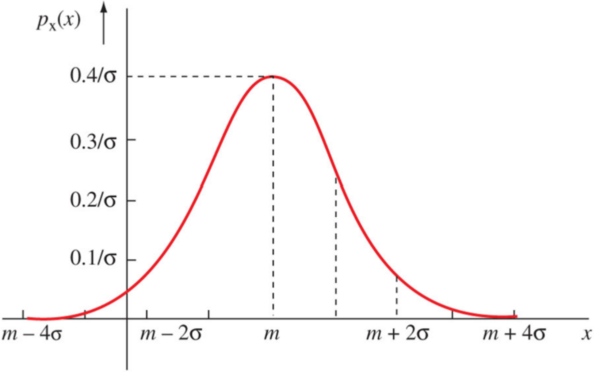
- PDF: $p_{\mathrm{x}}(x) = \frac{dF_{\mathrm{x}}(x)}{dx}$
    - $\int_{-\infty}^{\infty} p_{\mathrm{x}}(x)dx = 1$
    - $P(x_1 < x \leqslant x_2) = \int_{x_1}^{x_2} p_{\mathrm{x}}(x)dx$



Gaussian RV

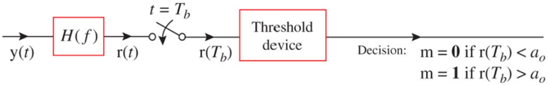
- Standard Gaussian has zero mean and unit variance, $x \sim \mathcal{N}(0, 1)$
- General equation for $x \sim \mathcal{N}(m, \sigma)$ is $p_{\mathrm{x}}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}$
- Typically evaluate using $Q(\cdot)$ functions - area under curve tail

$$P(\mathrm{x} > x) = Q\left(\frac{x-m}{\sigma}\right)$$

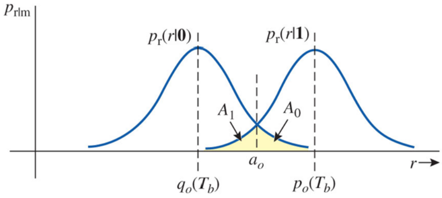


Gaussian RV in Communication Systems

- Recall (Lsn 25): only care about sample value (test statistic) at the receiver
- $r(T_b) = s_k(T_b) + n(T_b) \rightarrow s_k(T_b)$ deterministic, $n(T_b) \sim \mathcal{N}(0, \sigma)$
- How is $r(T_b)$ distributed? We need to know how to find the *expected value*, $\mathbb{E}[r(T_b)]$
- We'll come back to this problem...



(a)



(b)

Expected Value

- Aka average value aka mean value
- **For discrete RV:** $\bar{\mathrm{x}} = \mathbb{E}[\mathrm{x}] = \sum_i x_i P_{\mathrm{x}}(x_i)$
- **For continuous RV:** $\bar{\mathrm{x}} = \mathbb{E}[\mathrm{x}] = \int_{-\infty}^{\infty} x p_{\mathrm{x}}(x) dx$
- **Mean of a sum:** mean of the sum is equal to the sum of the means, $\overline{x+y} = \overline{x} + \overline{y}$
- **Mean of a product:** messy, unless independent - then $\overline{xy} = \overline{x} \cdot \overline{y}$

MOMENTS

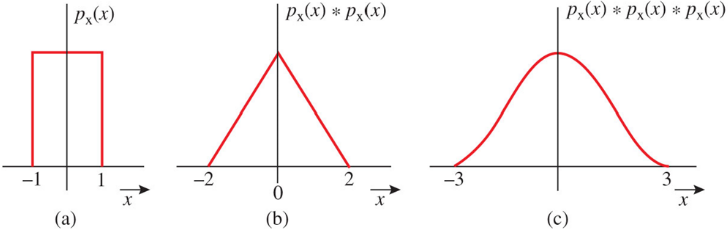
- The nth moment defined as the mean value of x^n
- **First moment:** expected value, $\mathbb{E}[\mathrm{x}] = \int_{-\infty}^{\infty} x p_{\mathrm{x}}(x) dx$
- **Central moments:** subtract out the mean first, $\mathbb{E}^n[(\mathrm{x}-\overline{\mathrm{x}})^n] = \int_{-\infty}^{\infty} (x-\overline{\mathrm{x}})^n p_{\mathrm{x}}(x) dx$
- **Second central moment:** variance, mean square value minus square of the mean:

$$\sigma_{\mathrm{x}}^2 = \mathbb{E}^2[(\mathrm{x}-\overline{\mathrm{x}})^2] = \int_{-\infty}^{\infty} (x-\overline{\mathrm{x}})^2 p_{\mathrm{x}}(x) dx = \overline{\mathrm{x}^2} - \overline{\mathrm{x}}^2$$

- Variance of a sum of independent RVs is equal to the sum of their variances
  - If x and y are independent RVs and $z = x + y$, then $\sigma_z^2 = \sigma_x^2 + \sigma_y^2$

Central Limit Theorem

- The sum of a large number n independent RVs with mean μ and variance σ^2 tends to be a Gaussian RV, regardless of the PDFs of the variables added
- Gaussian mean is $n\mu$
- Gaussian variance is σ^2/n
- Example 7.25 (similar to future HW problem, 7.7-1)
  - Note the problem defines a discrete RV, x_i
  - Use $\overline{\mathrm{x}} = \mathbb{E}[\mathrm{x}] = \sum_i x_i P_{\mathrm{x}}(x_i)$



Gaussian RV in Communication Systems

- Back to this problem!
- $r(T_b) = s_k(T_b) + n(T_b) \rightarrow s_k(T_b)$ deterministic, $n(T_b) \sim \mathcal{N}(0, \sigma)$
- $\mathbb{E}[r(T_b)] = \mathbb{E}[s_k(T_b) + n(T_b)] =$
- $\sigma_r^2 = \mathbb{E}[r^2(T_b)] - \mathbb{E}[r(T_b)]^2 =$