CSCI 570 Homework 4
Spring 2023

Due Date: Mar. 27, 2023 at 11:59 P.M.

1. You are given the following graph G. Each edge is labeled with the capacity of that edge.

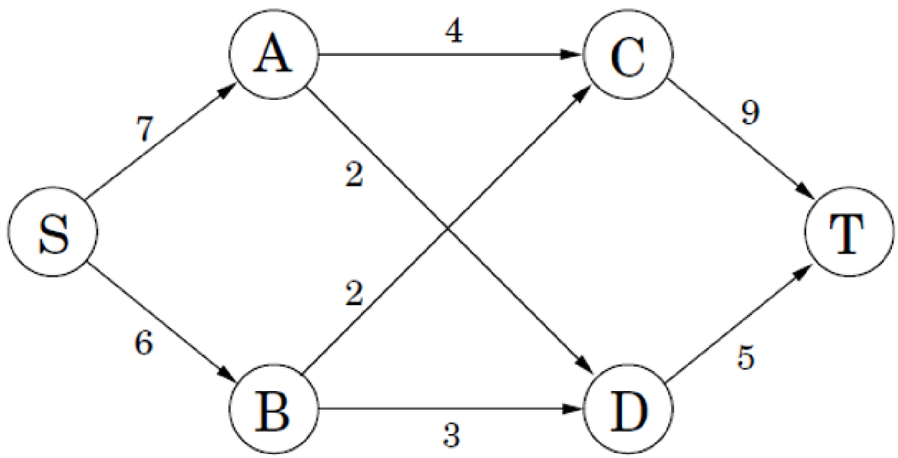


Figure 1: Graph G

   (a) Find a max-flow in G using the Ford-Fulkerson algorithm. Draw the residual graph G_f corresponding to the max flow. You do not need to show all intermediate steps.

   (b) Find the max-flow value and a min-cut.

   (c) Prove or disprove that increasing the capacity of an edge that belongs to a min cut will always result in increasing the maximum flow.

2. Consider a set of mobile computing clients in a certain town who each need to be connected to one of several possible base stations. We'll suppose there are n clients, with the position of each client specified by its (x, y) coordinates in the plane. There are also k base stations; the position of each of these is specified by (x, y) coordinates as well. For each client, we wish to connect it to exactly one of the base stations. Our choice of connections is constrained in the following ways. There is a range parameter R which means that a client can only be connected to a base station that is within distance R. There is also a load parameter L which means that no more than L clients can be connected to any single base station. Given the positions of a set of clients and a set of base stations, as well as the range and load parameters, decide whether every client can be connected simultaneously to a base station.

   (a) Describe how to construct a flow network

   (b) Make a claim of how the original problem is related to the max-flow problem.

   (c) Prove the above claim in both directions

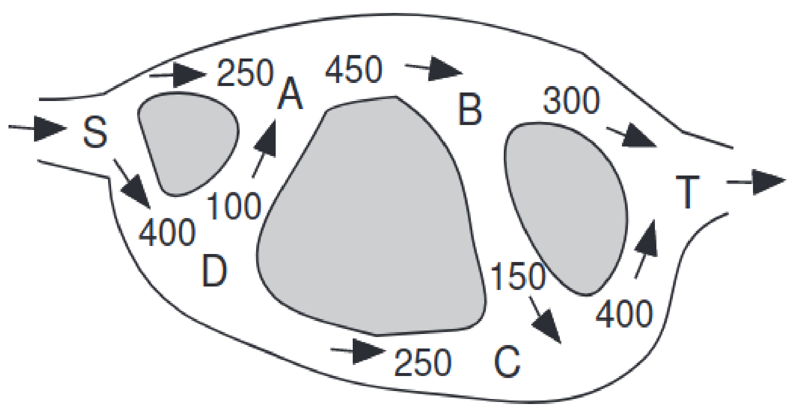
3. There are N students at the USC Viterbi school who want to celebrate the Indian festival of colour called Holi. There are M unique colors (call the set of colors C) available at the USC Bookstore and i^{th} color has c_i packets left at the store (e.g there are c_1 packets left of color 1, c_2 packets left of color 2, and so on till c_m packets of color M). The i^{th} student has a set $F_i \subseteq C$ representing their favourite colors and wishes to buy a total of b_i packets from their favourite color set (it doesn't matter which colors out of F_i as long as the total number of packets is b_i). However, to ensure fair availability of all colors, the USC Bookstore restricts each student to buy a maximum of 2 packets of the same color. Design an algorithm that determines if all the students can have their wish granted.

   (a) Describe how to construct a flow network

   (b) Make a claim of how the original problem is related to the max-flow problem.

   (c) Prove the above claim in both directions

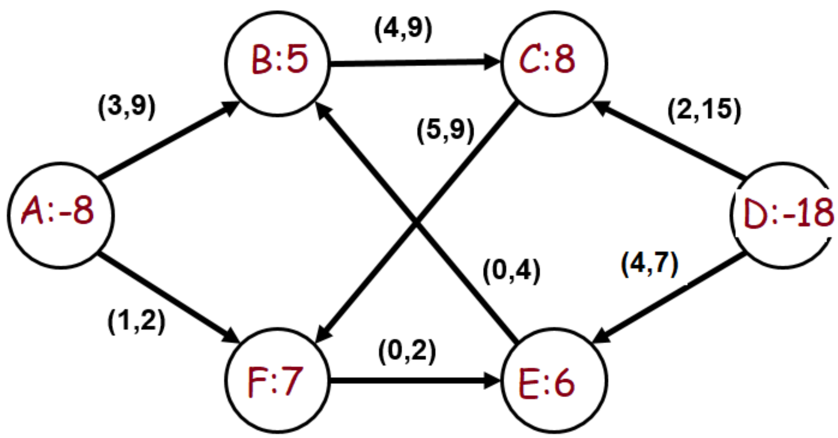
4. In a museum S and T denote the entrance and exits of a hall. There are exhibits placed at the positions A, B, C, D. The hall has three gardens as shown in gray in the figure. There are hallways around the gardens that can support the movement of a specified number of people per hour which are marked in the figure. For example, people wanting to visit exhibit D can enter at S and go to D via the hallway which supports 400 visitors to pass through every hour. Similarly the hallway from exhibit B to C support 150 visitors per hour. The museum wants to attract 6000 visitors every day and closes when the target is reached for that day. If the museum opens at 8 am on a specific day, when is the earliest it can close on that day to support 6000 visitors?



5. In addition to the edge capacity constraints, we introduce vertex capacities b_i constraints for each vertex v_i in a network-flow problem. The total amount of flow passing via vertex v_i cannot exceed b_i. Describe an algorithm to solve the vertex capacity constrained max-flow problem. Hint: Use reduction to edge capacity network flow problem.

6. The edge connectivity of an undirected graph is the minimum number of edges whose removal disconnects the graph. Describe an algorithm to compute the edge connectivity of an undirected graph with n vertices and m edges in $O(m^2 n)$ time.

7. In the network below, the demand values are shown on vertices. Lower bounds on flow and edge capacities are shown as (lower bound, capacity) for each edge.



Answer the following questions:

(a) Remove the lower bounds on each edge. Write down the new demands on each vertex A, B, C, D, E, F in this order.

(b) Solve the circulation problem without lower bounds. Write down the max-flow value.

(c) Is there is a feasible circulation in the original graph? Explain your answer.

8. Consider a student dormitory with n students and n rooms. A student can only be assigned to one room. Each room has the capacity to hold either one or two students. Each student has a subset of rooms as their possible choice. We also need to make sure that there is at least one student assigned to each room. Give a polynomial time algorithm that determines whether a feasible assignment of students to rooms is possible that meets all of the above constraints. If there is a feasible assignment, describe how your solution can identify which student is assigned to which room.

9. Suppose that you have just bought a new computer and you want to install software on that. Specifically, two companies, which you can think of like Microsoft and Apple, are trying to sell their own copy of n different products, like Operation System. Spread Sheet, Web Browser. For each product i, $i \in \{1, 2, \ldots, n\}$, we have

   - the price $p_i \geq 0$ that Microsoft charges and the price $p'_i \geq 0$ that Apple charges.
   - the quality $q_i \geq 0$ of Microsoft version and the quality $q'_i \geq 0$ of Apple version.

   For example, Apple may provide a better Web Browser Safari, but Microsoft a better Word Processor. You want to assemble your favorite computer by installing exactly one copy of each of the n products, e.g. you want to buy one operating system, one Web Browser, one Word Processor, etc. However, you don't want to spend too much money on that. Therefore, your goal is to maximize the quality minus total price.

   However, as you may know, the products of different companies may not be compatible. More concretely, for each product pair (i, j), we will suffer a penalty $\tau_{ij} \geq 0$ if we install product i of Microsoft and product j of Apple. Note that τ_{ij} may not be equal to τ_{ji} just because Apple's Safari does not work well on Microsoft Windows doesn't mean that Microsoft's Edge does not work well in Mac-OS. We assume that products are always compatible internally, which means that there is no penalty for installing two products from the same company. All pairwise penalties will be subtracted from the total quality of the system.

   Your task is then to give a polynomial-time algorithm for computing which product i to purchase from which of the two companies (Apple and Microsoft) for all $i \in \{1, 2, \ldots, n\}$, to maximize the total system quality (including the penalties) minus the total price. Prove the correctness of your algorithm.

   (a) Describe how to model this problem as a min-cut problem.

   (b) Make a claim of how the original problem is related to the min-cut problem.

(c) Prove the above claim in both directions

10. **Online Questions.** Please go to DEN (https://courses.uscden.net/) and take the online portion of your assignment.