CSCI 570 Homework 1

Due Date: Jan. 26, 2023 at 11:59 P.M.

1. Arrange the following functions in increasing order of growth rate with $g(n)$ following $f(n)$ in your list if and only if $f(n) = O(g(n))$

$$\sqrt{n}^{\sqrt{n}}, n^{\log n}, n\log(\log n), n^{\frac{1}{\log n}}, 2^{\log n}, \log^2 n, (\log n)^{\sqrt{\log n}}.$$

2. The Fibonacci sequence $F_1, F_2, F_3 \ldots$, is defined as follows. $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. For example, $F_3 = F_1 + F_2 = 2$ and $F_4 = F_3 + F_2 = 3$. Prove by induction that $F_n = \frac{a^n - b^n}{\sqrt{5}}$, where $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$.

3. Let T be a breadth-first search tree of a connected graph G. Let (x, y) be an edge of G, where x and y are nodes in G. Prove by contradiction that in T the level of x and the level of y differ by at most 1.

4. We have a connected graph $G = (V, E)$, and a specific vertex $u \in V$. Suppose we compute a depth-first search tree rooted at u, and obtain a tree T that includes all nodes of G. Suppose we then compute a breadth-first search tree rooted at u, and obtain the same tree T. Prove by contradiction that G has the same structure as T, that is, G cannot contain any edges that do not belong to T.

5. You have been given an unweighted, undirected, and connected graph G = (V, E). Device an algorithm to determine maximum of the shortest paths' lengths between all pairs of nodes in graph G (Also called diameter of the graph). Also, determine the time complexity of your algorithm and justify your answer.

6. You have been given an unweighted and undirected graph G = (V, E), and an edge $e \in E$.

   a. Your task is to devise an algorithm to determine whether the graph G has a cycle containing that specific given edge e. Also, determine the time complexity of your algorithm and justify your answer. Note: To become eligible for full credits on this problem, the running time of your algorithm should be bounded by $O(V + E)$.

b. Will your algorithm still work if the graph is unweighted but directed? If not, how would you modify your algorithm to make it work on an unweighted-directed graph.

7. Use prove by induction to show that in any binary tree the number of nodes with two children is exactly one less than the number of leaves.

8. This problem is divided into following subsections:
   a. Consider a sequence of numbers: 4, 13, 7, 15, 21, 24,10 Construct a binomial min-heap H1 by reading the above numbers from left to right. Draw all the intermediate binomial heaps as well as the final binomial heap H1. Illustrate your work clearly and concisely.
   b. Repeat a. for another sequence of numbers: 11,12,21,24,18,13,16 to construct another binomial min-heap H2.
   c. Merge H1 and H2 to construct the final binomial heap H. Draw all the intermediate binomial heaps (while merging H1 and H2) as well as the final binomial heap H.

9. Consider an array on which the following operations can be performed:
   addLast(e): An element 'e' is added at the end of the array.
   deleteEveryThird(): Removes every third element in the list i.e. removes the first, fourth, eighth, etc., elements of the array.

   You may assume that addLast(e) has a cost 1, and deleteEveryThird() has a cost equals to the number of elements in the list. What is the amortized cost of addLast and deleteEveryThird operations? Consider the worst sequence of operations. Justify your answer with proper explanation using accounting method.

10. Given a sequence of n operations, suppose the i-th operation cost $i + j$ if $i = 2^j$ for some integer j; otherwise, the cost is 1. Prove that the amortized cost per operation is $O(1)$.

11. **Online Questions.** Please go to DEN (https://courses.uscden.net/) and take the online portion of your assignment.