Analysis of Algorithms

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Lecture 6

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Dynamic Programming

Reading: chapter 6

Review

For each of the following recurrences, give an expression for the runtime T(n) if the recurrence can be solved by the Master Theorem. Otherwise, indicate that the Master Theorem does not apply.

1. $T(n) = 16\ T(n/4) + 5\ n^3 + \log n$
2. $T(n) = 4\ T(n/2) + n^2 \log n$
3. $T(n) = 4\ T(n/8) - n^2$
4. $T(n) = 2^n\ T(n/2) + n$
5. $T(n) = 0.2\ T(n/2) + n \log n$

Review

Design a new Mergesort algorithm in which instead of splitting the input array in half we split it in the ratio 1:3.

Write down the recurrence relation for the number of comparisons.

What is the runtime complexity of this algorithm?

REVIEW QUESTIONS

1. (**T/F**) For a divide-and-conquer algorithm, it is possible that the dividing step takes asymptotically longer time than the combining step.
2. (**T/F**) A divide-and-conquer algorithm acting on an input size of n can have a lower bound less than $\Theta(n \log n)$.
3. (**T/F**) There exist some problems that can be efficiently solved by a divide-and-conquer algorithm but cannot be solved by a greedy algorithm.
4. (**T/F**) It is possible for a divide-and-conquer algorithm to have an exponential runtime.
5. (**T/F**) A divide-and-conquer algorithm is always recursive.
6. (**T/F**) The master theorem can be applied to the following recurrence: $T(n) = 1.2\, T(n/2) + n$.
7. (**T/F**) The master theorem can be applied to the following recurrence: $T(n) = 9\, T(n/3) - n^2 \log n + n$.
8. (**T/F**) Karatsuba's algorithm reduces the number of multiplications from four to three.
9. (**T/F**) The runtime complexity of mergesort can be asymptotically improved by recursively splitting an array into three parts (rather than into two parts).

Fibonacci Numbers

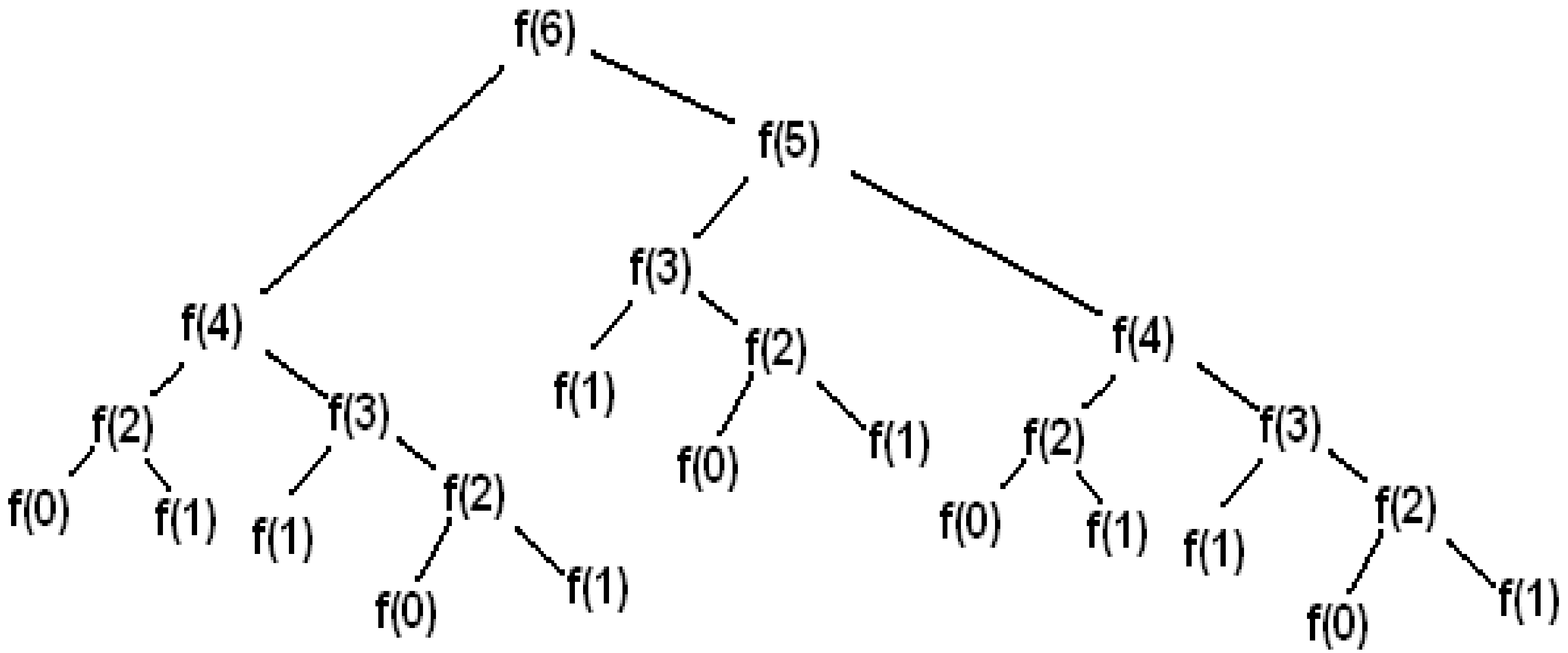
Fibonacci number F_n is defined as the sum of two previous Fibonacci numbers:

$$F_n = F_{n-1} + F_{n-2}$$

$$F_0 = F_1 = 1$$

Design a divide & conquer algorithm to compute Fibonacci numbers. What is its runtime complexity?

Overlapping Subproblems



Fibonacci Numbers: $F_n = F_{n-1} + F_{n-2}$

Memoization

```
int  table [50];    //initialize to zero
table[0] = table[1] = 1;

int  fib(int  n)
{
          if  (table[n] != 0) return  table[n];
          else
          table[n] = fib(n-1) + fib(n-2);

          return  table[n];
}
```

Runtime complexity?

Tabulation

```
int  table [n];

void fib(int n)
{
          table[0] = table[1] = 1;
          for(int k = 2; k < n; k++)
             table[k] = table[k-1] + table[k-2];

          return;
}
```

Two Approaches

```
int  table [n];
table[0] = table[1] = 1;

int  fib(int  n)
{
   if  (table[n] != 0)
      return  table[n];
   else
      table[n] = fib(n-1) + fib(n-2);

   return  table[n];
}
```

Memoization:
a top-down approach.

```
int  table [n];

int[]  fib(int n)
{
   table[0] = table[1] = 1;
   for(int k = 2; k < n; k++)
      table[k]=table[k-1]+table[k-2];

   return  table;
}
```

Tabulation:
a bottom-up approach.

Dynamic Programming

General approach: in order to solve a larger problem, we solve smaller subproblems and store their values in a table.

DP is applicable when the subproblems are greatly overlap. Compare with Mergesort.

DP is not greedy either. DP tries <u>every</u> choice before solving the problem. It is much more expensive than greedy.

DP can be implemented by means of memoization or tabulation.

Dynamic Programming

Optimal substructure means that the solution can be obtained by the combination of optimal solutions to its subproblems. Such optimal substructures are usually described recursively.

Overlapping subproblems means that the space of subproblems must be small, so an algorithm solving the problem should solve the same subproblems over and over again.

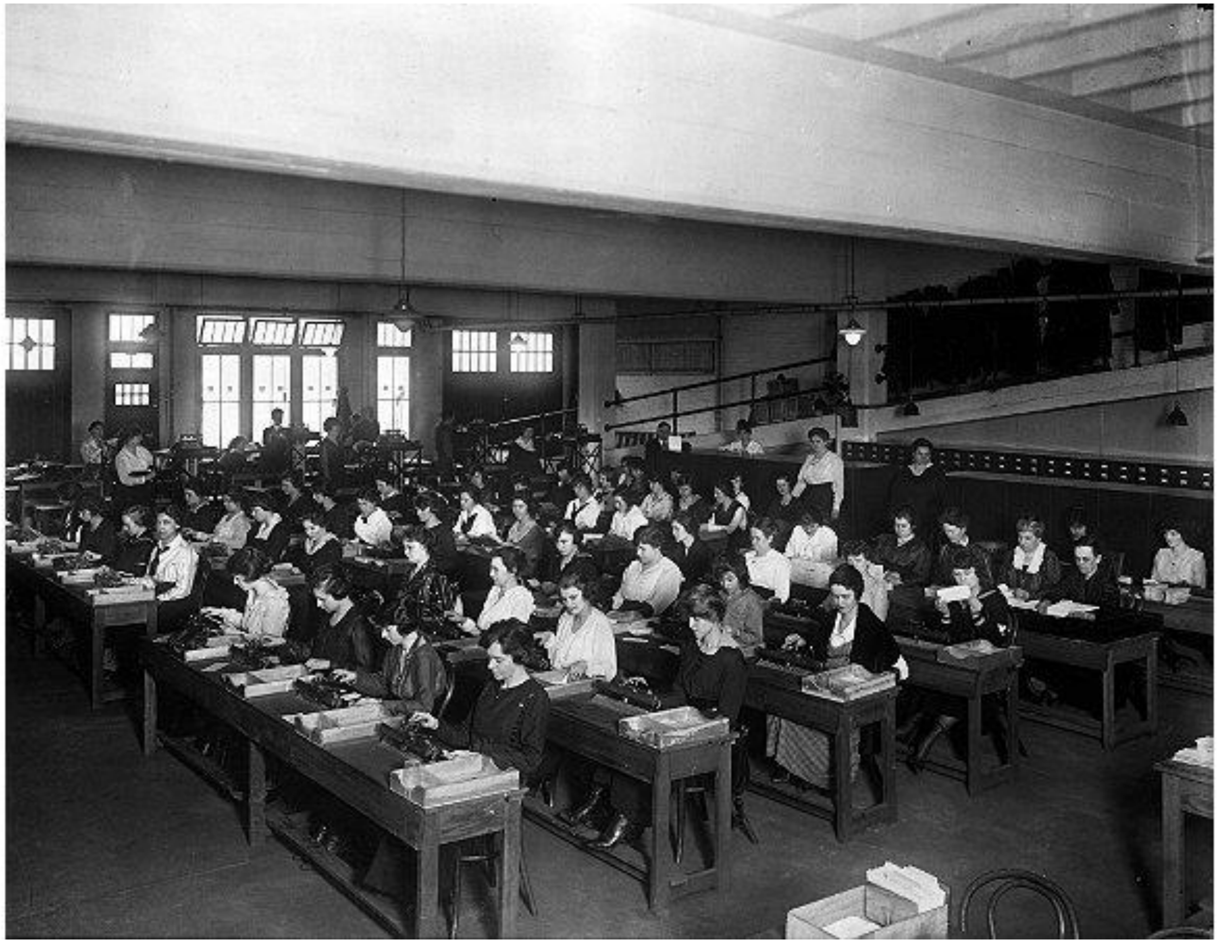
Dynamic Programming

The term dynamic programming was originally used in the 1950s by Richard Bellman.

The term <u>computer</u> (dated from 1613) meant a <u>person</u> performing mathematical calculations.

In the 30-50s those early computers were mostly *women* who used painstaking calculations on paper and later punch cards.

The earliest human computers



Who put a man to the moon?

0-1 Knapsack Problem

Given a set of n unbreakable *unique* items, each with a weight w_k and a value v_k, determine the subset of items such that the total weight is less or equal to a given knapsack capacity W and the total value is as large as possible.

Decision Tree

$$x_k = \begin{cases} 1, \text{item } k \text{ selected} \\ 0, \text{item } k \text{ not selected} \end{cases}$$

Subproblems

Recurrence Formula

Tracing the Algorithm

n = 4, W = 5

(weight, value) = (2,3), (3,4), (4,5), (5,6)

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0					
1,2	0					
1,2,3	0					
1,2,3,4	0					

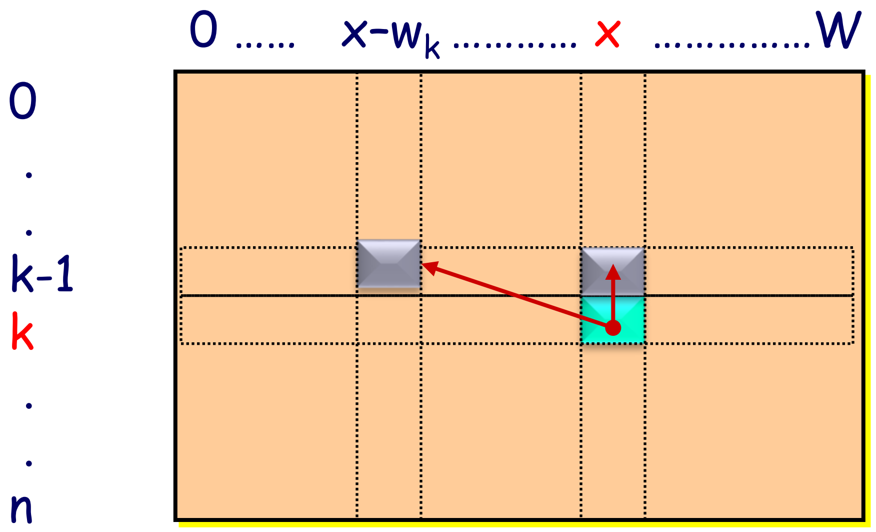
knapsack capacity

items

Pseudo-code

```
int knapsack(int W, int w[], int v[], int n) {
  int Opt[n+1][W+1];
  for (k = 0; k <= n; k++) {
     for (x = 0; x <= W; x++) {
        if (k==0 || x==0) Opt[k][x] = 0;
        if (w[x] > x) Opt[k][x] = Opt[k-1][x];
        else
          Opt[k][x] = max( v[k] + Opt[k-1][x - w[k-1]],
                                   Opt[k-1][x] );
     }
  }
  return Opt[n][W];
}
```

Complexity



Runtime Complexity?

Space Complexity?

Pseudo-Polynomial Runtime

Definition. A numeric algorithm runs in pseudo-polynomial time if its running time is polynomial in the numeric value of the input but is exponential in the length of the input.

0-1 Knapsack is pseudo-polynomial algorithm, $T(n) = \Theta(n \cdot W)$

How would you find the actual items?

The table built in the algorithm does not show the optimal items, but only the maximum value. How do we find which items give us that optimal value?

n = 4, W = 5
(weight, value) = (2, 3), (3, 4), (4, 5), (5, 6)

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

DP Approach

solve using the following four steps:

1. Define (in plain English) subproblems to be solved.
2. Write the recurrence relation for subproblems.
3. Write pseudo-code to compute the optimal value.
4. Compute the runtime of the above DP algorithm in terms of the input size.

Discussion Problem 1

You are to compute the minimum number of coins needed to make change for a given amount m. Assume that we have an unlimited supply of coins. All denominations d_k are sorted in ascending order:

$$1 = d_1 < d_2 < \dots < d_n$$

Longest Common Subsequence

We are given string S_1 of length n, and string S_2 of length m.

Our goal is to produce their longest common subsequence.

A *subsequence* is a subset of elements in the sequence taken in order (with strictly increasing indexes.) Or you may think as removing some characters from one string to get another.

Note, a subsequence is not a substring.

Intuition

A B A Z D C

B A C B A D

Subproblems

Recurrence

Example

S = ABAZDC
T = BACBAD

		B	A	C	B	A	D
	0	0	0	0	0	0	0
A	0						
B	0						
A	0						
Z	0						
D	0						
C	0						

Pseudo-code

```
int LCS(char[] S1, int n, char[] S2, int m)
{
int table[n+1, m+1];
table[0…n, 0] = table[0, 0…m] = 0; //init
for(i = 1; i <= n; i++)
  for(j = 1; j <= m; j++)
      if (S1[i] == S2[j])  table[i, j] = 1 + table[i-1, j-1]
      else
      table[i, j] = max(table[i, j-1], table[i-1, j]);

return table[n, m];
}
```

How much space do we need?

		B	A	C	B	A	D
	0	0	0	0	0	0	0
A	0	0	1	1	1	1	1
B	0	1	1	1	2	2	2
A	0	1	2	2	2	3	3
Z	0	1	2	2	2	3	3
D	0	1	2	2	2	3	4
C	0	1	2	3	3	3	4

How do we find the common sequence?

	0	0	0	0	0	0	0
	0	0	1	1	1	1	1
	0	1	1	1	2	2	2
	0	1	2	2	2	3	3
	0	1	2	2	2	3	3
	0	1	2	2	2	3	4
	0	1	2	3	3	3	4

Discussion Problem 2

A subsequence is palindromic if it reads the same left and right. Devise a DP algorithm that takes a string and returns the length of the longest palindromic subsequence (not necessarily contiguous) .

For example, the string

QRAECCETCAURP

has several palindromic subsequences, RACECAR is one of them.

Discussion Problem 3

You are to plan the fall 2023 schedule of classes. Suppose that you can sign up for as many classes as you want, and you'll have infinite amount of energy to handle all the classes, but you cannot have any time conflict between the lectures. Also assume that the problem reduces to planning your schedule of *one particular day*. Thus, consider one particular day of the week and all the classes happening on that day: $c_1, .., c_n$. Associated with each class c_i is a start time s_i and a finish time f_i and you also assign a score v_i to that class based on your interests and your program requirement. Assume $s_i < f_i$. You would like to choose a set of courses C for that day to *maximize* the total score. Devise an algorithm for planning your schedule.

Static Optimal Binary Search Tree

Build a binary search tree which gives a minimum search cost, assuming we know the frequencies p_i with which data k_i is accessed. The tree cannot be modified after it has been constructed.

Want to build a binary search tree with minimum expected search cost:

$$\text{Expected Cost} = \sum_{i=1}^{n} p_i \, \text{depth}(k_i)$$

Example

Consider 5 items

$$k_1 < k_2 < k_3 < k_4 < k_5$$

and their search probabilities

$$p_1 = 0.25,\ p_2 = 0.2,\ p_3 = 0.05,\ p_4 = 0.2,\ p_5 = 0.3.$$

Another possibility

$k_1 < k_2 < k_3 < k_4 < k_5$

$p_1 = 0.25$, $p_2 = 0.2$, $p_3 = 0.05$, $p_4 = 0.2$, $p_5 = 0.3$

Optimal Substructure

Recurrence Relation

Filling up the table

```
array p = [$p_1$, $p_2$, …, $p_n$]
set OPT[i, i-1] = 0, for $1 \le i \le n$
set OPT[i, i] = $p_i$, for $1 \le i \le n$

for(k = 1; k < n; k++)
   for(i = 1; i <= n-k, i++)
      j = i + k;
     OPT[i, j] =$p_i$+…+$p_j$+ $\min_r$(OPT[i,r-1]+OPT[r+1,j];
                                     ($i \le r \le j$)
return OPT[1,n];
```

Runtime Complexity-?

Example

n = 5

(prob,value)=(0.1,5), (0.3,6), (0.9,4), (0.3,3), (0.1,8)

		0	1	2	3	4	5
(0.1, 5)	1						
(0.3, 6)	2						
(0.9, 4)	3						
(0.3, 3)	4						
(0.1, 8)	5						