# spmodel: Spatial Modeling in R – Supplementary Material

A Preprint

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## Abstract

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## 1 Estimation

### 1.1 Likelihood-based Estimation

Minus twice a profiled Gaussian log-likelihood, denoted $-2l(\boldsymbol{\theta}|\mathbf{y})$ is given by

$$-2l(\boldsymbol{\theta}|\mathbf{y}) = \ln|\boldsymbol{\Sigma}| + (\mathbf{y} - \mathbf{X}\tilde{\beta})^\intercal \boldsymbol{\Sigma}^{-1}(\mathbf{y} - \mathbf{X}\tilde{\beta}) + n\ln 2\pi, \tag{1}$$

where $\tilde{\boldsymbol{\beta}} = (\mathbf{X}^\intercal \boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}^\intercal \boldsymbol{\Sigma}^{-1}\mathbf{y}$. Minimizing Equation 1 yields $\hat{\boldsymbol{\theta}}_{ml}$, the maximum likelihood estimates for $\boldsymbol{\theta}$. Then a closed for solution exists for $\hat{\boldsymbol{\beta}}_{ml}$, the maximum likelihood estimates for $\boldsymbol{\beta}$: $\hat{\boldsymbol{\beta}}_{ml} = \tilde{\boldsymbol{\beta}}_{ml}$, where $\tilde{\boldsymbol{\beta}}_{ml}$ is $\tilde{\boldsymbol{\beta}}$ evaluated at $\hat{\boldsymbol{\theta}}_{ml}$. Unfortunately $\hat{\boldsymbol{\theta}}_{ml}$ can be badly biased for $\boldsymbol{\theta}$ (especially for small sample sizes), which impacts the estimation of $\boldsymbol{\beta}$ (Patterson and Thompson 1971). This bias occurs due to the simultaneous estimation of $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$ To reduce this bias, restricted maximum likelihood estimation (REML) emerged (Patterson and Thompson 1971; Harville 1977; Wolfinger, Tobias, and Sall 1994). It can be shown that integrating $\boldsymbol{\beta}$ out of a Gaussian likelihood yields the restricted Gaussian likelihood used in REML estimation. Minus twice a restricted Gaussian log-likelihood, denoted $-2l_R(\boldsymbol{\theta}|\mathbf{y})$ is given by

$$-2l_R(\boldsymbol{\theta}|\mathbf{y}) = -2l(\boldsymbol{\theta}|\mathbf{y}) + \ln|\mathbf{X}^\intercal \boldsymbol{\Sigma}^{-1}\mathbf{X}| - p\ln 2\pi. \tag{2}$$

Minimizing Equation 2 yields $\hat{\boldsymbol{\theta}}_{reml}$, the restricted maximum likelihood estimates for $\boldsymbol{\theta}$. Then a closed for solution exists for $\hat{\boldsymbol{\beta}}_{reml}$, the restricted maximum likelihood estimates for $\boldsymbol{\beta}$: $\hat{\boldsymbol{\beta}}_{reml} = \tilde{\boldsymbol{\beta}}_{reml}$, where $\tilde{\boldsymbol{\beta}}_{reml}$ is $\tilde{\boldsymbol{\beta}}$ evaluated at $\hat{\boldsymbol{\theta}}_{reml}$.

When all variance parameters are unknown, the overall variance, $\sigma^2$, can be profiled out of Equation 1 and Equation 2. This reduces the number of parameters requiring optimization by one, which can dramatically

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reduce estimation time. For example, profiling $\sigma^2$ out of Equation 1 yields

$$-2l^*(\boldsymbol{\theta}^*|\mathbf{y}) = \ln|\boldsymbol{\Sigma}^*| + n\ln[(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}})^\intercal\boldsymbol{\Sigma}^{-1}(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}})] + n + n\ln 2\pi/n. \tag{3}$$

After finding $\hat{\boldsymbol{\theta}}^*_{ml}$ a closed form solution for $\hat{\sigma}^2_{ml}$ exists: $\hat{\sigma}^2_{ml} = [(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}})^\intercal\boldsymbol{\Sigma}^{-1}(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}})]/n$. Then $\hat{\boldsymbol{\theta}}^*_{ml}$ is combined with $\hat{\sigma}^2_{ml}$ to yield $\hat{\boldsymbol{\theta}}_{ml}$ and subsequently $\hat{\boldsymbol{\beta}}_{ml}$

Next describe REML adjustments

### 1.2 Semivariogram-based Estimation

#### 1.2.1 Weighted Least Squares

(Cressie 1985, 1993)

#### 1.2.2 Composite Likelihood

(Curriero and Lele 1999)

## 2 Hypothesis Testing

### 2.1 The General Linear Hypothesis Test

### 2.2 Contrasts

## 3 Random Effects

### 3.1 BLUPs

## References

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