

# **Graphs**

**It's all about relationships**

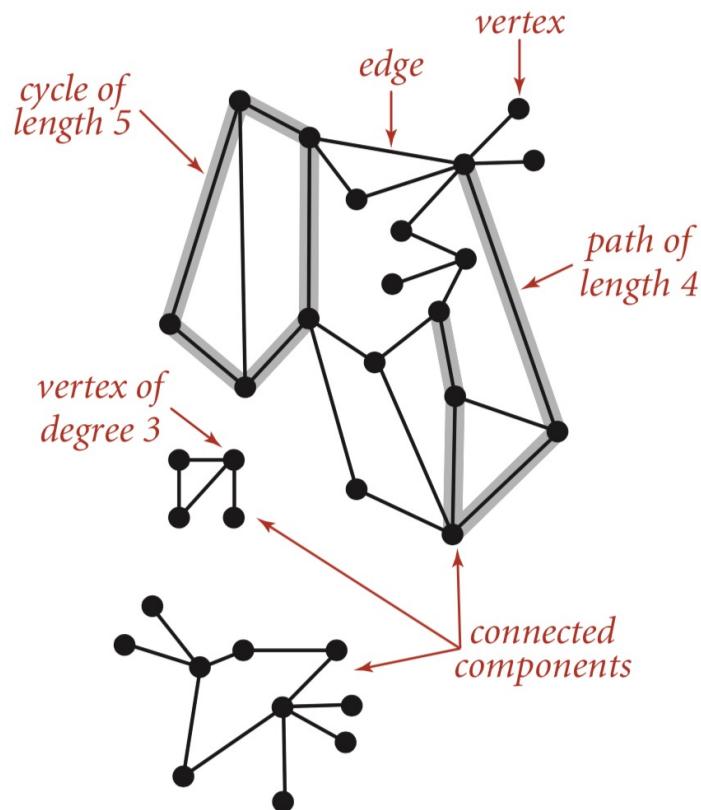
Terence Parr  
MSDS program  
**University of San Francisco**



# What's a graph?

- A graph is a collection of connected element pairs,  $u \rightarrow v$  or  $u - v$
- As with a tree, a graph is the aggregate of nodes/edges
- Nodes can be email addresses, map locations, documents, tasks to perform, URLs on the web, customers, computers on network, friends, observations, sensors, states in markov chain, ...
- Terms: *nodes* or *vertices* connected with *edges*, which can have labels; e.g., recall the Trie graph with labeled edges
- *Directed* graphs have arrows as edges but *undirected* use lines
- For  $n$  nodes, num directed edges is  $\geq 0$  and  $\leq n^2$  since  $n$  nodes could connect to all other nodes and itself; if no self cycles then the max is  $n(n - 1)$

# Undirected graph, terms

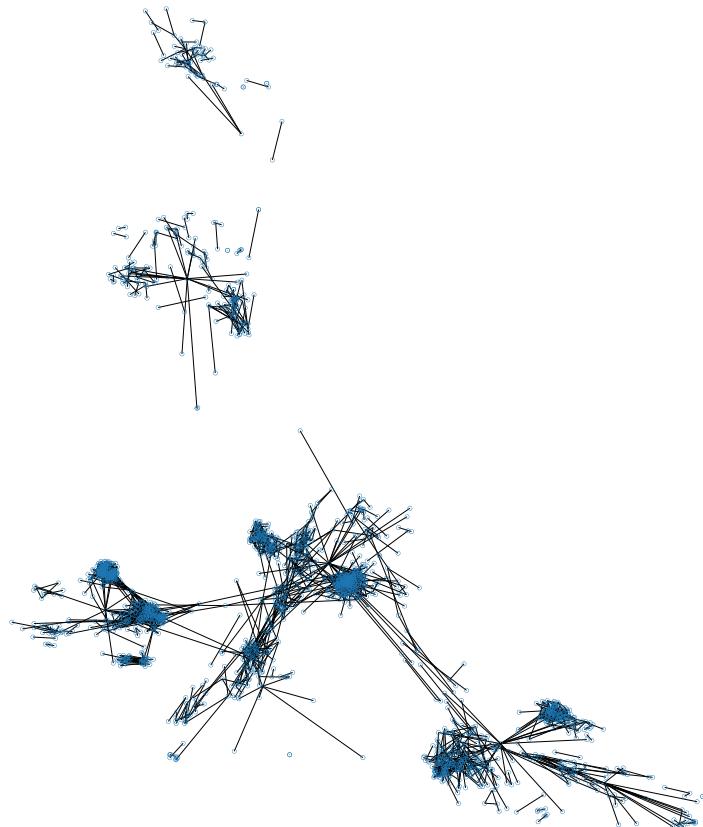


From: Algorithms book by Robert Sedgewick and Kevin Wayne

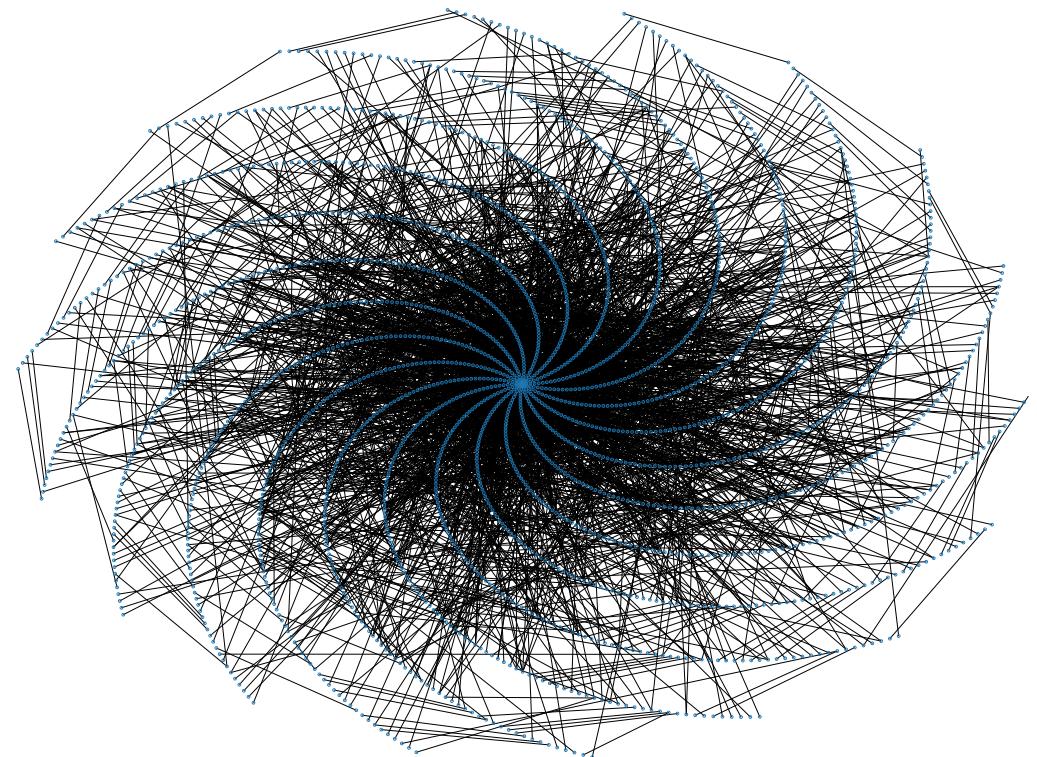
Uses networkx

# Facebook friend network, different layouts

Fruchterman Reingold



spiral



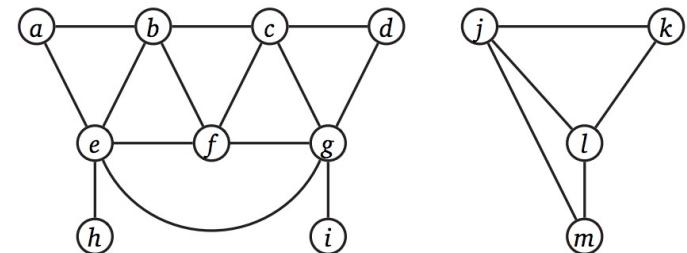
# Common questions

- Is q reachable from p?
- How many edges are on paths between p and q?
- Is graph connected? (reach any p from any q)
- Is graph cyclic? (p reaches p traversing at least one edge)
- Which nodes are within k edges of node p? (neighborhood)
- What is shortest path (num edges) from p to q?
- What is shortest path using edge weights? [beyond scope of 689]
- Traveling salesman problem [beyond scope of 689]  
[https://en.wikipedia.org/wiki/Travelling\\_salesman\\_problem](https://en.wikipedia.org/wiki/Travelling_salesman_problem)

# Adjacency matrix implementations

- Adjacency matrix,  $n \times n$  matrix of  $\{0,1\}$  if unlabeled or  $\{\text{labels}\}$  if edges are labeled; undirected matrices are symmetric
- Wastes space for sparse edges; use sparse matrix
- Fast to access arbitrary node's edges if we have unique node IDs

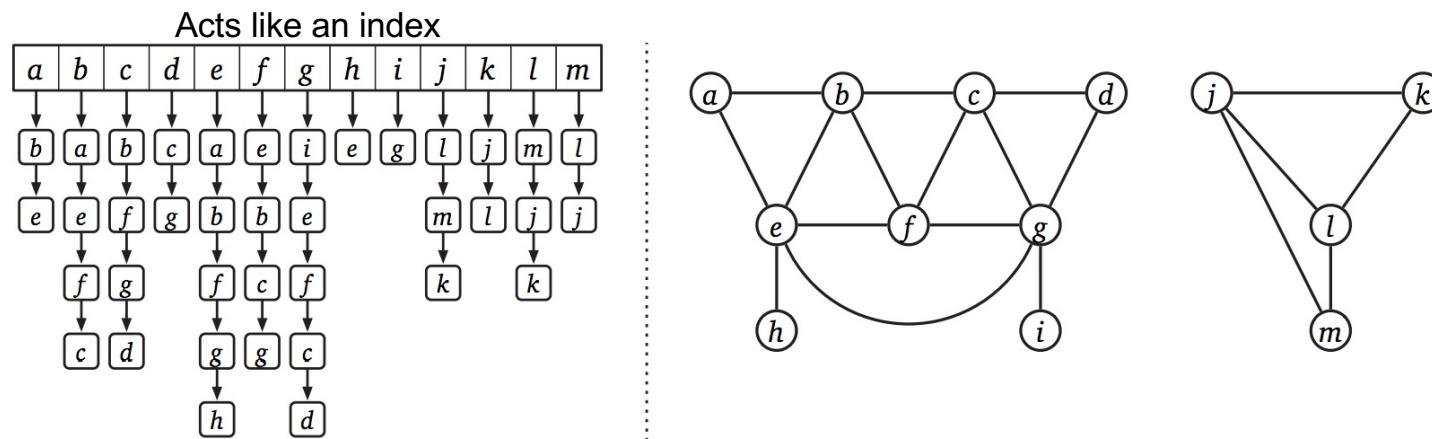
	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>i</i>	<i>j</i>	<i>k</i>	<i>l</i>	<i>m</i>
<i>a</i>	0	1	0	0	1	0	0	0	0	0	0	0	0
<i>b</i>	1	0	1	0	1	1	0	0	0	0	0	0	0
<i>c</i>	0	1	0	1	0	1	1	0	0	0	0	0	0
<i>d</i>	0	0	1	0	0	0	1	0	0	0	0	0	0
<i>e</i>	1	1	0	0	0	1	1	1	0	0	0	0	0
<i>f</i>	0	1	1	0	1	0	1	0	0	0	0	0	0
<i>g</i>	0	0	1	1	1	1	0	0	1	0	0	0	0
<i>h</i>	0	0	0	0	1	0	0	0	0	0	0	0	0
<i>i</i>	0	0	0	0	0	0	1	0	0	0	0	0	0
<i>j</i>	0	0	0	0	0	0	0	0	0	1	1	1	1
<i>k</i>	0	0	0	0	0	0	0	0	0	1	0	1	0
<i>l</i>	0	0	0	0	0	0	0	0	1	1	0	1	0
<i>m</i>	0	0	0	0	0	0	0	0	1	0	1	0	0



**Figure 5.11.** An adjacency matrix for our example graph.

# Adjacency list implementations

- List of edge lists for nodes
- Fast arbitrary node access for numbered nodes, space efficient

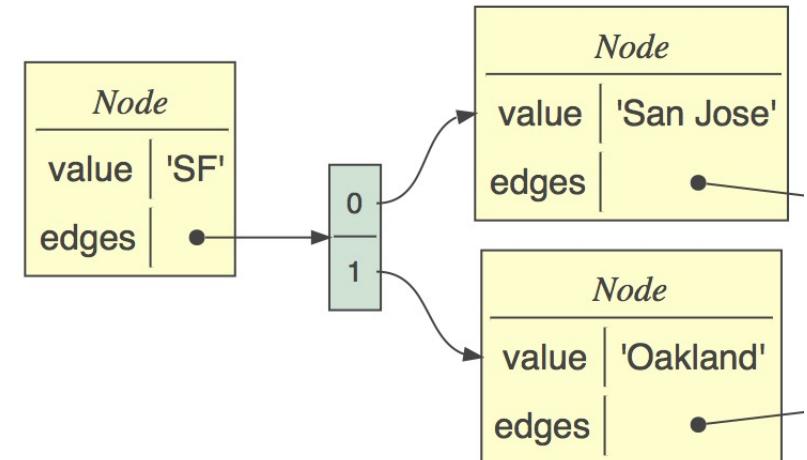


**Figure 5.9.** An adjacency list for our example graph.

# Connected nodes implementation

- Common implementation due to nice mapping to objects
- Each node has info about that node and its edge list
- Use list or dictionary index if you need to access nodes directly

```
class Node:  
    def __init__(self, value):  
        self.value = value  
        self.edges = []  
    def edge(self, target:Node):  
        self.edges.append(target)
```

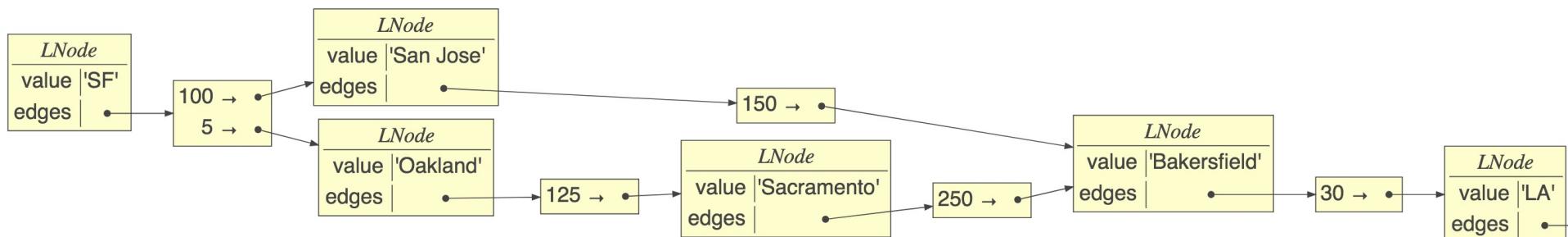


# Implementation with labels

```
class LNode:  
    def __init__(self, value):  
        self.value = value  
        self.edges = {}  
    def edge(self, label, target):  
        self.edges[label] = target
```

Edge->node is a dictionary, not list

sf.edge(100,sj)  
sj.edge(150,baker)  
...



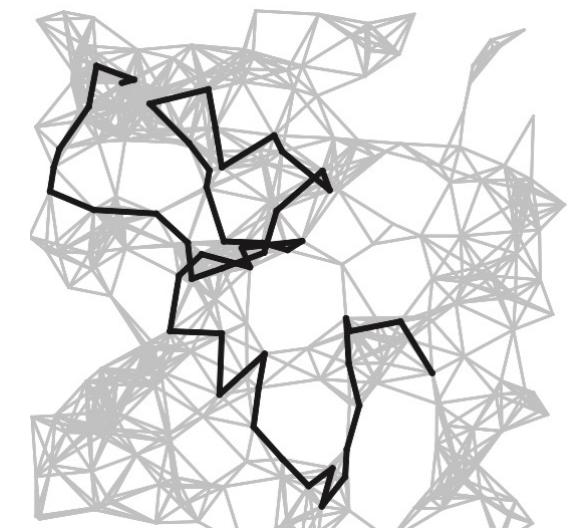
# Depth-first search (review)

```
RECURSIVEDFS( $v$ ):  
    if  $v$  is unmarked  
        mark  $v$   
        for each edge  $vw$   
            RECURSIVEDFS( $w$ )
```

- The fundamental algorithm for answering graph questions
- Visits all reachable nodes from  $p$ , avoiding cycles
- Go deep first

```
def walk_graph( $p$ :Node, visited=set()):  
    if  $p$  in visited: return  
    visited.add( $p$ )  
    for  $q$  in  $p$ .edges:  
        walk_graph( $q$ , visited)
```

$O(n,m) = n + m$ , for  $n$  **nodes**,  $m$  **edges**;  $m$  can be  $n^2$   
(counts node visits, edge traversals)



Algorithms book by Sedgewick, Wayne



UNIVERSITY OF SAN FRANCISCO

# Is there a cycle (p can reach p)?

- E.g., can I get back to starting spot via one-way streets?
- If we run into starting node in visited set, return True;  
**blue** means different than plain walk:

```
def iscyclic(p:Node) -> bool:  
    return iscyclic_(p,p,set())  
  
def iscyclic_(start:Node, p:Node, visited) -> bool:  
    if p in visited:  
        if p is start: return True # we find start?  
        return False # can't loop forever so don't chase  
    visited.add(p)  
    for q in p.edges:  
        c = iscyclic_(start, q, visited)  
        if c: return True # found it, we can stop  
    return False
```



<https://github.com/parrt/msds689/blob/master/notes/graphs.ipynb>

# Find set of nodes p can reach

- E.g., can we reach city y from city x (hint: Juneau not accessible)?
- Need two sets, one for avoiding cycles, another for reached nodes
- If we used visited for both, then p would also appear to reach itself, which might not be true; we add start node to visited but doesn't mean we can return to start via any path in graph

```
def reachable(p:Node) -> set:  
    reaches = set();  
    reachable_(p, reaches, set())  
    return reaches  
  
def reachable_(p:Node, reaches:set, visited:set):  
    if p in visited: return  
    visited.add(p)  
    for q in p.edges:  
        reaches.add(q) # add only if we traverse  
        reachable_(q, reaches, visited)
```



UNIVERSITY OF SAN FRANCISCO

# Find set of nodes p can reach, track depth

- E.g., how many hops from person A to B in social network?
- Track node->depth map, not just set of nodes

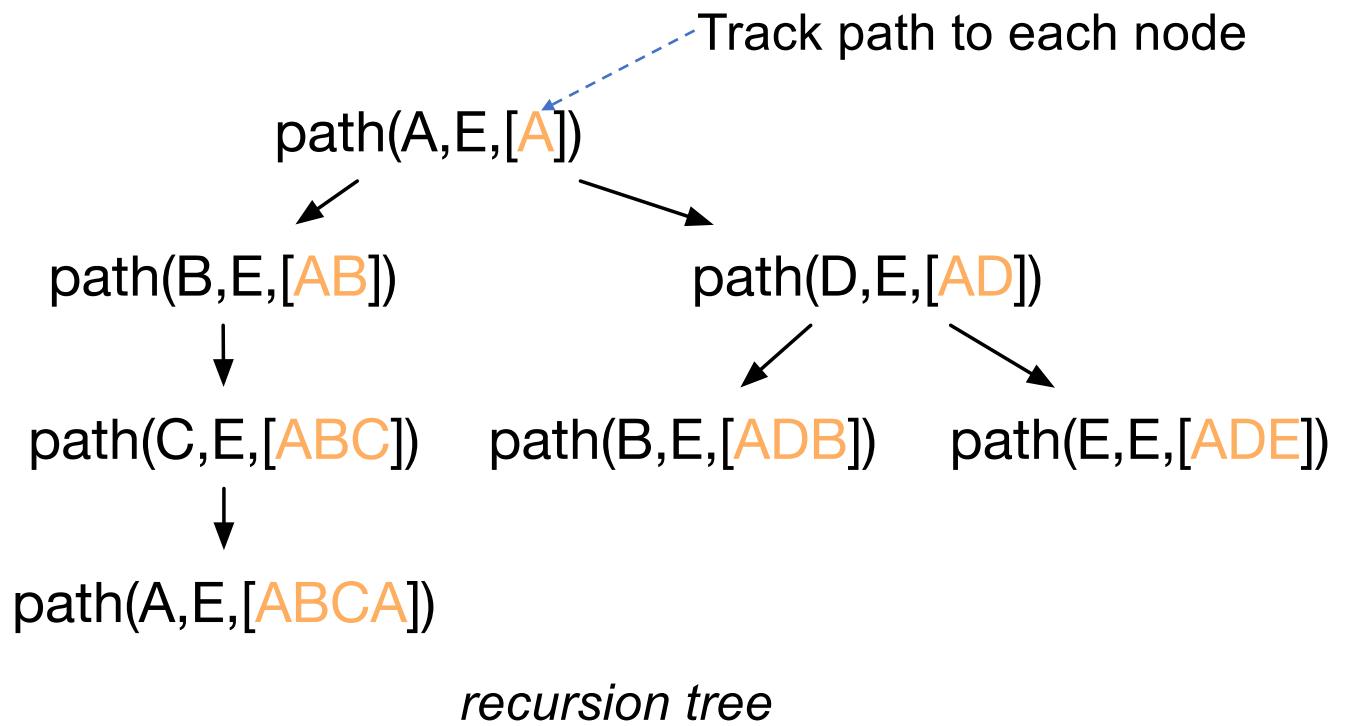
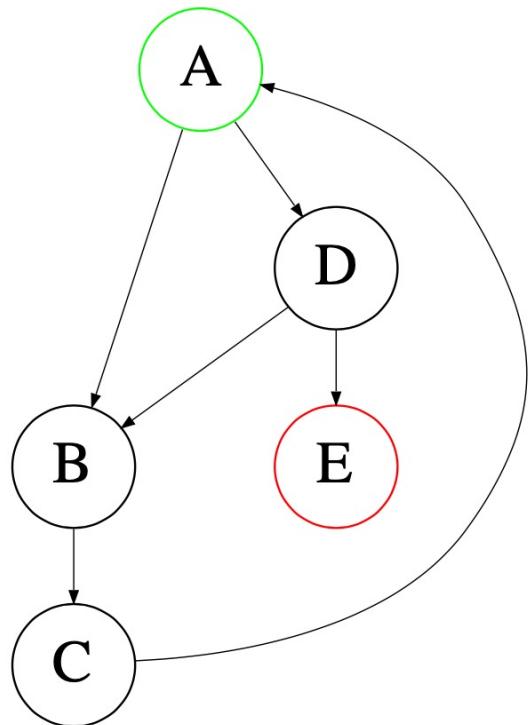
```
def depths(p:Node) -> dict:  
    reaches = {}  
    depths_(p, reaches, depth=0)  
    return reaches  
  
def depths_(p:Node, reaches:dict, depth:int):  
    if p in reaches: return  
    reaches[p] = depth # distance to p from start (could be 0)  
    for q in p.edges:  
        depths_(q, reaches, depth+1)
```

# Find neighborhood within k edges

- E.g., find all friends within k friend hops
- Track dict node->depth, stop when we reach depth

```
def neighbors(p:Node, k:int) -> dict:  
    reaches = dict()  
    neighbors_(p, k, reaches, depth=0)  
    return reaches  
  
def neighbors_(p:Node, k:int, reaches:dict, depth:int):  
    if p in reaches or depth>k: return # terminate at depth  
    reaches[p] = depth  
    for q in p.edges:  
        neighbors_(q, k, reaches, depth+1)
```

# Find first path from p to q

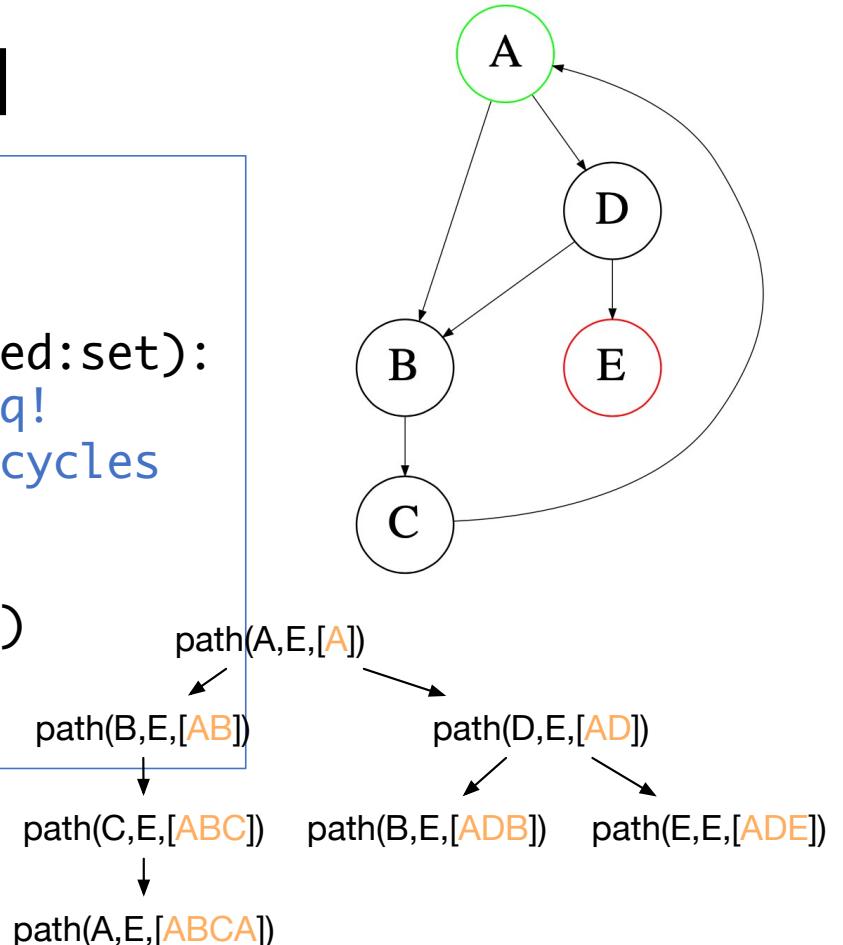


# Find first path from p to q

```
def path(p:Node, q:Node) -> list:  
    return path_(p, q, [p], set())
```

```
def path_(p:Node, q:Node, path:list, visited:set):  
    if p is q: return path          # found q!  
    if p in visited: return None   # avoid cycles  
    visited.add(p)  
    for e in p.edges:  
        pa = path_(e, q, path+[e], visited)  
        if pa is not None: return pa  
    return None
```

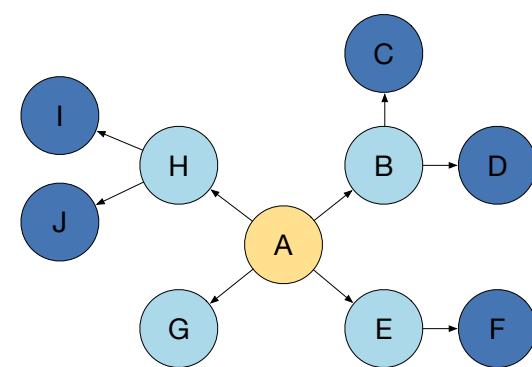
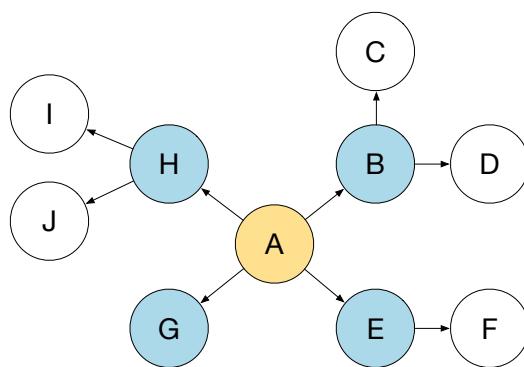
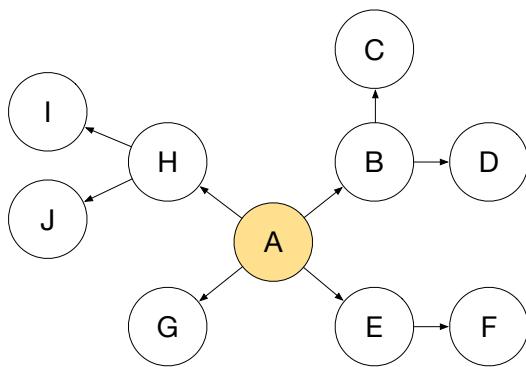
Must track path not just set of reachable nodes



<https://github.com/parrt/msds689/blob/master/notes/graphs.ipynb>

# Breadth-first search vs DFS

- Visit all children then grandchildren...



Todo: show worklist growing/shrinking here or next slide (animation)



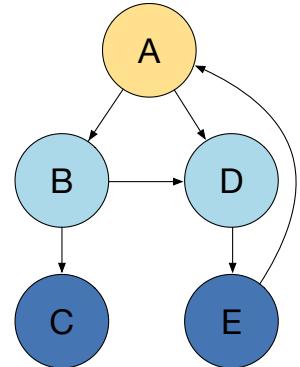
Algorithms book by Sedgewick, Wayne

# BFS implementation

- Maintains work list of nodes and visited set

BFS	DFS
Visit A	Visit A
Visit B	Visit B
Visit D	Visit C
Visit C	Visit D
Visit E	Visit E

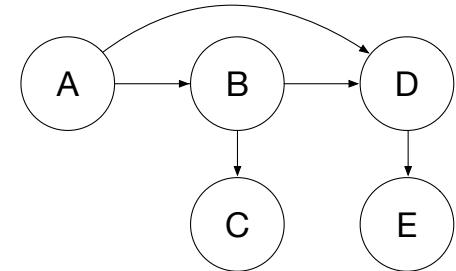
- Add to work list end, pull from front (queue)
- Nonrecursive*



```
def BFS(root:LNode):  
    visited = {root}  
    worklist = [root]  
    while len(worklist)>0:  
        p = worklist.pop(0) # dequeue  
        print(f“Visit {p}”)  
        for q in p.edges:  
            if q not in visited:  
                worklist.append(q)  
                visited.add(q)
```

# Find shortest path from p to q?

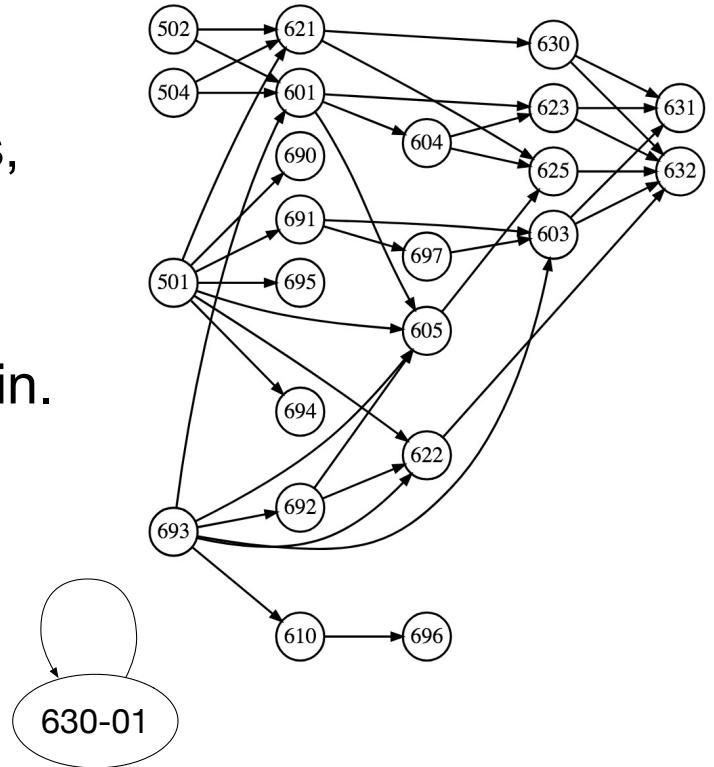
- BFS where work list is a list of paths not list of nodes
- Tail of path is where we left off working on it
- By searching all children before going deeper, we automatically find paths with shortest (unweighted) lengths



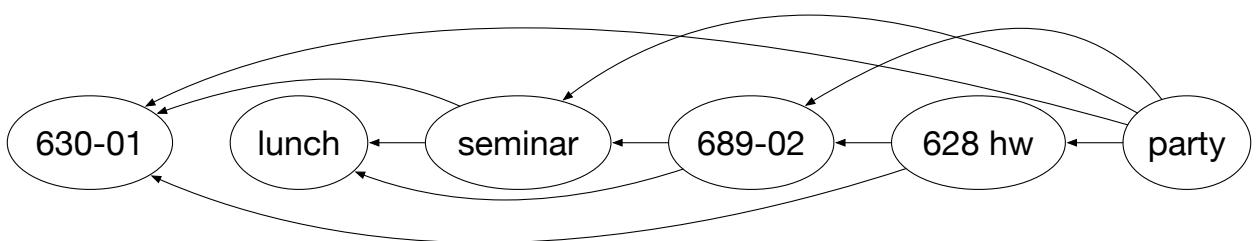
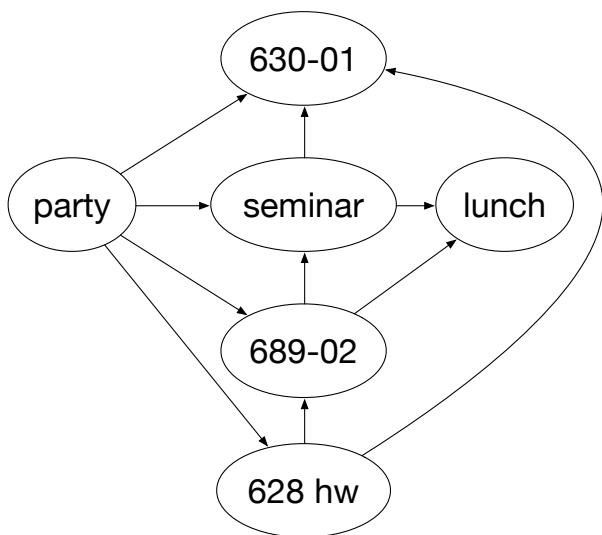
```
def shortest(root:Node, target:Node):  
    visited = {root}  
    worklist = [[root]] # list of paths  
    while len(worklist)>0:  
        path = worklist.pop(0)  
        p = path[-1] # tail of path  
        if p is target: return path  
        for q in p.edges:  
            if q not in visited:  
                worklist.append(path+[q])  
                visited.add(q)
```

# Topological sort (acyclic graphs)

- **Problem:** Find linear ordering of nodes in directed acyclic graph such that all constraints,  $u \rightarrow v$ , are satisfied where  $u$  depends on  $v$  so  $v$  must come before  $u$  OR  $u \rightarrow v$  mean  $u$  precedes  $v$
- Examples: task ordering or course prereq chain.
- E.g., 502 is prereq for 621 and 601... Find order we should take classes
- Sort is not usually unique
- Cycles are meaningless for dependencies; how can 630-01 be attended before itself?



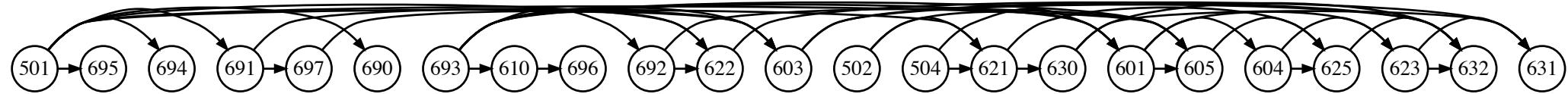
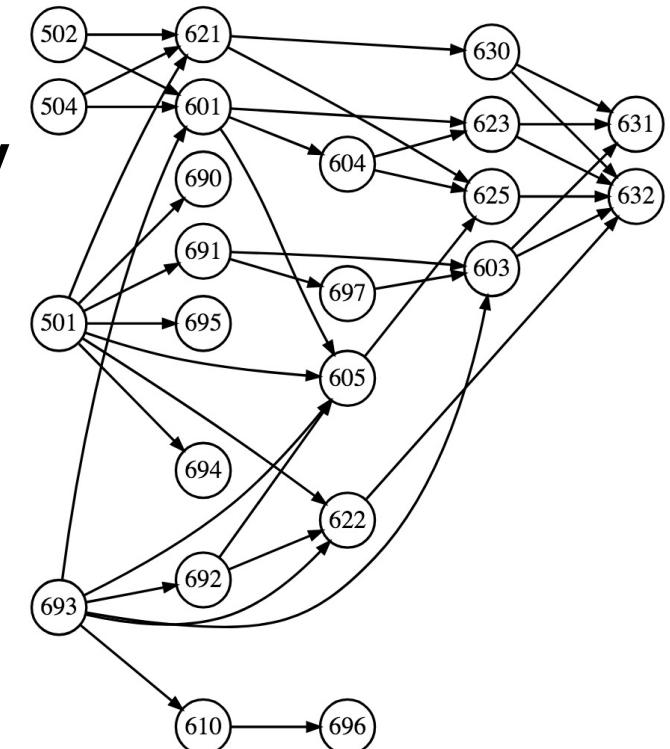
# Example topological sort (u depends v)



If u depends on v, any linear ordering where edges point to left is solution

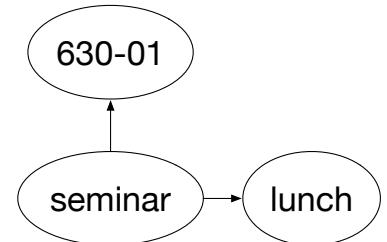
# Example where u precedes v

- In this case, edges must point to the right

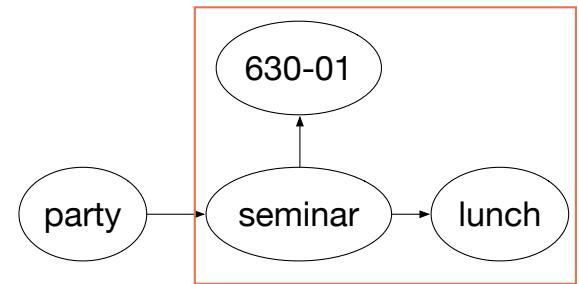


# How to approach the problem

- What order should we do these tasks (u depends v)?  
Think in terms of traversal order



- What if we add party goal?
- "Perform all children tasks then yourself"

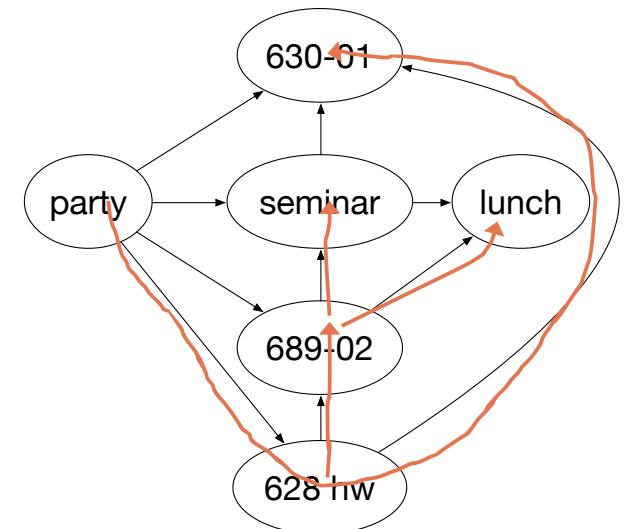


# DFS-based topo sort implementation

- Lots of very complex algorithms on the web (not sure why)
- Simplest solution: DFS-based topological sort
- A valid sort is just the post-order graph traversal if  $u$  depends  $v$ !
- Perform all children tasks then yourself
- If  $u$  precedes  $v$ , reverse the result of post-order traversal;  
See proof page 582 of Sedgewick/Wayne Algorithms book
- Well, we have to make sure to do DFS on all root nodes (nodes w/o incoming edges) but core is just DFS
- With one root, it's just postorder traversal via DFS

# Example walk through

- DFS starting with party:  
party -> 628 -> 630  
back out then hit 689 then lunch  
back out and hit seminar
- Postorder traversal processes/prints **after** visiting children:  
630-01, lunch, seminar, 689-02, 628, party
- Solution: 630-01, lunch, seminar, 689-02, 628 hw, party



# DFS postorder traversal

```
def postorder(p:Node, sorted:list, visited:set):  
    if p in visited: return  
    visited.add(p)  
    for q in p.edges:  
        postorder(q, sorted, visited)  
    sorted.append(p) # postorder done after kids
```

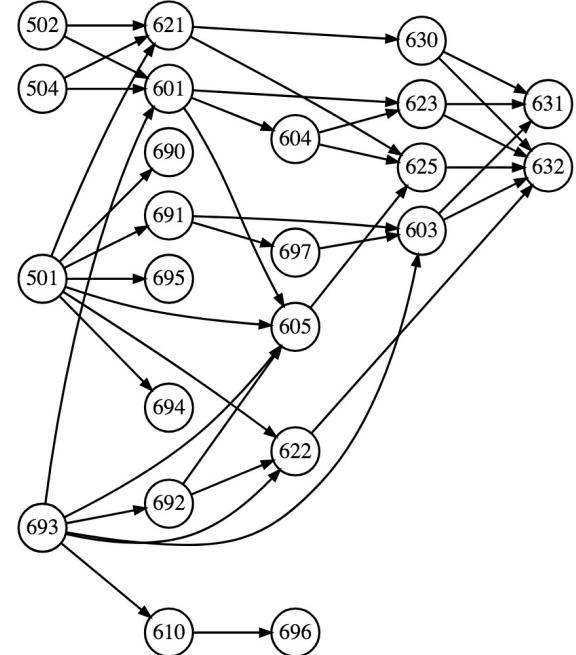
(recall: no cycles)

# With multiple roots, hit them all

```
# nodes is dict edge list mapping from->to
def toposort(nodes:dict):
    sorted = []
    visited = set()
    while len(visited) < len(nodes):
        todo = [node for node in nodes.values()
                if node not in visited]
        if len(todo)>0:
            anode = todo[0] # pick a node
            postorder(anode, sorted, visited)
    return reverse(sorted)
```

We reverse postorder here since u precedes v

```
MSDS = """501->690
502,504,693->601
601->604
601,604->623
...
..."""
```



# Summary

- Graphs are for showing relationships between elements
- DFS for finding a path or multiple paths or cycles  
(recursive backtracking to find all nodes)
- BFS for find shortest (in edges) path or neighborhood
- DFS postorder great for topo sort
- Recursive alg's all use **visited** set or similar to avoid cycles
- Non-recursive DFS: (use work list stack)
  - push targets in reverse order onto work list
  - pop last work list item for next node to process
- Non-recursive BFS: (use work list queue)
  - push targets in order onto work list
  - pull from first position

# Sample graph problems

# Exercise

- Given a directed graph, detect all direct or indirect cycles
- For  $p$  in nodes:  
    report `iscyclic(p)`

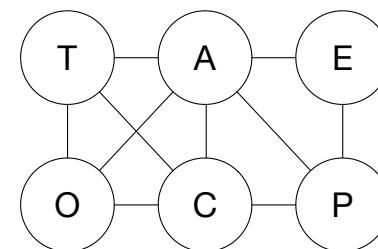
# Exercise

- Given 2 lists P,Q and function  $\text{conn}(p,q)=\text{true}$  if edge  $p \rightarrow q$  exists. P is origin (starting) nodes and Q destination nodes. Report True for  $P[i]$  reaches  $Q[i]$  directly or indirectly
- E.g., P, Q could be cities and  $\text{conn}(p,q)=\text{true}$  if direct flight  $p \rightarrow q$
- Create graph using  $\text{conn}(p,q)$  for all nodes in P and Q
- For each  $P[i]$ , see if  $Q[i]$  is in  $\text{reaches}(P[i])$  set.

# Exercise: Boggle

- Given  $m \times n$  matrix of letters. Find all English words possible by taking one adjacent step to another letter, starting with any letter; one occurrence of each letter per word; you're given a dictionary (`/usr/share/dict/words`)

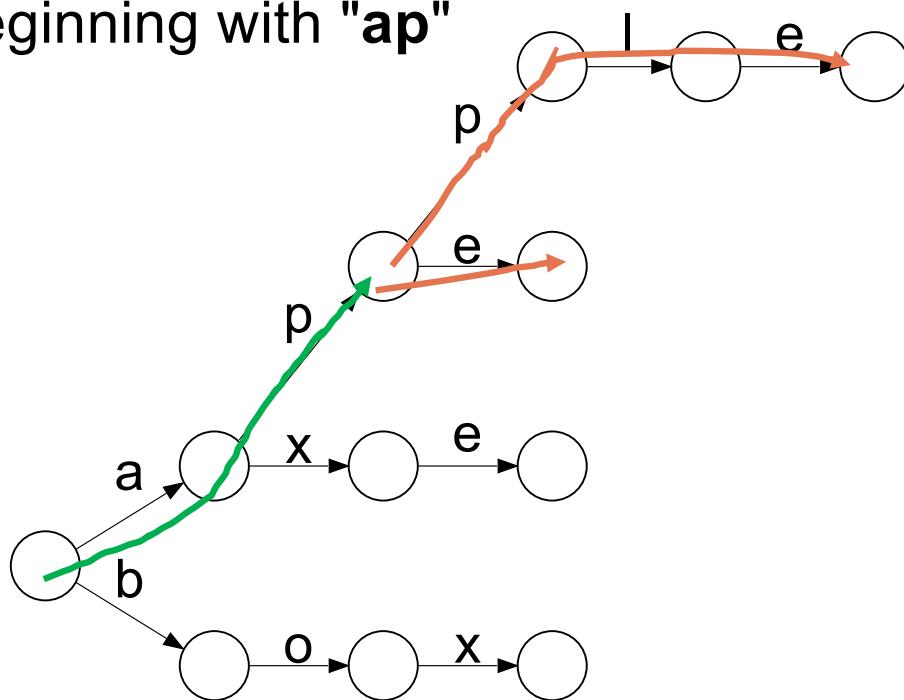
T	A	E
O	C	P



- For each node in graph, find all words
- For a specific starting node  $p$ , perform cycle-avoiding DFS; at each node  $q$ , look up word consisting of all letters on path from  $p$  to  $q$ , checking for duplicated letters

# Exercise: *Find all words with given prefix*

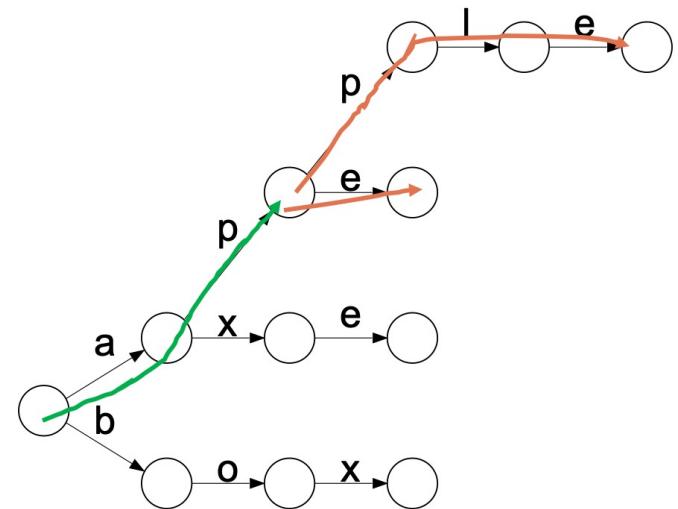
- Find words beginning with "ap"



See [https://github.com/parrt/msds689/blob/master/notes/code/prefix\\_trie.py](https://github.com/parrt/msds689/blob/master/notes/code/prefix_trie.py)

## Exercise: *Find all words with given prefix*

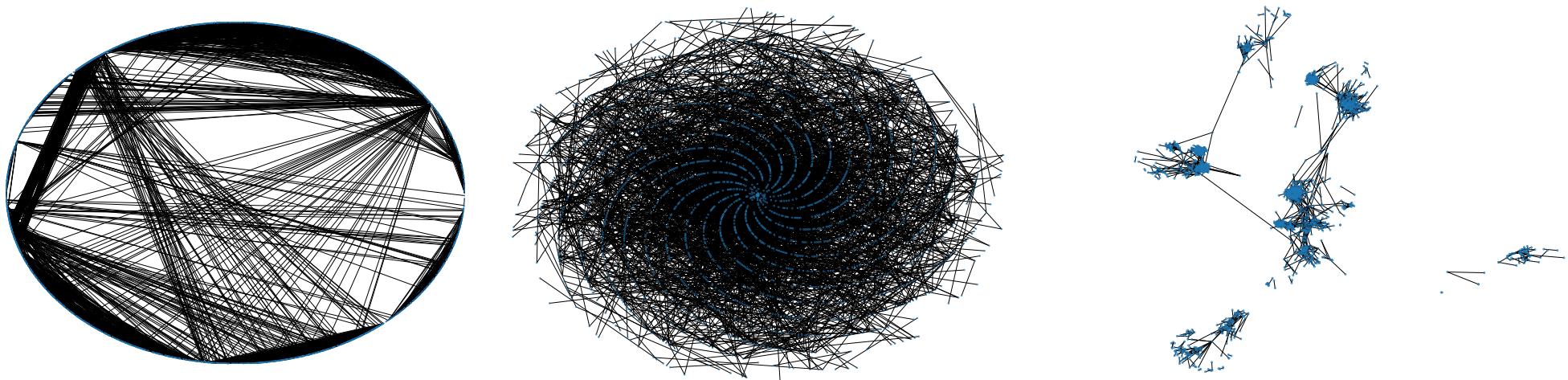
- Build a Trie with dictionary words
- Find node in trie reached by prefix; call it p (green line)
- Recursively, using depth first search, find all reachable leaf (stop/accept) nodes (orange lines)
- Pass a path string as arg down the recursion chain to incrementally build strings reachable from p  
`def suffixes_(p:TrieNode, path, paths): ...`
- There can be no cycle so we don't worry about "visited" arg, just track the current path and the overall list of paths we found



See [https://github.com/parrt/msds689/blob/master/notes/code/prefix\\_trie.py](https://github.com/parrt/msds689/blob/master/notes/code/prefix_trie.py)

# Coding exercise: Visualize: FB friend graph with networkx

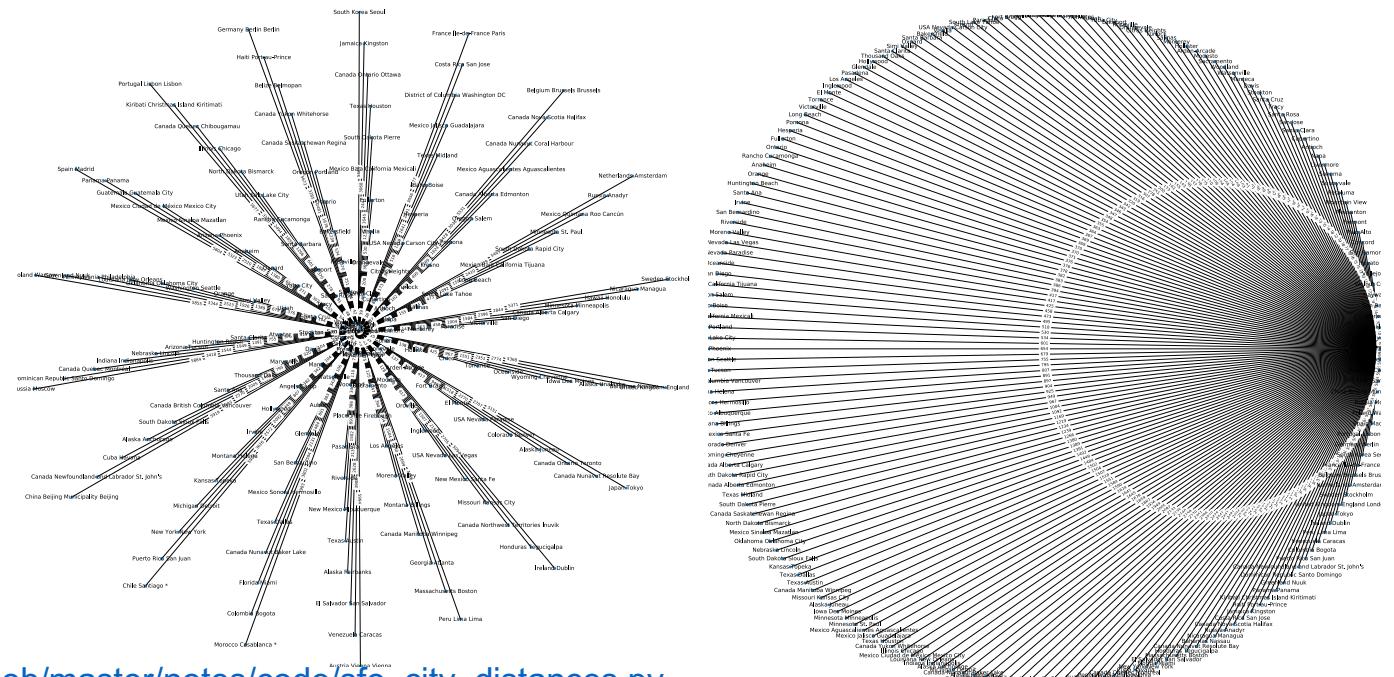
- pip install networkx
- download and uncompress [https://snap.stanford.edu/data/facebook\\_combined.txt.gz](https://snap.stanford.edu/data/facebook_combined.txt.gz)
- Get small sample of edges then do edge\_subgraph(), draw\_networkx\_edges(), draw\_networkx\_nodes()
- Need to pass positions of nodes for layout, such as circular\_layout()



See [https://github.com/parrt/msds689/blob/master/notes/code/viz\\_facebook.py](https://github.com/parrt/msds689/blob/master/notes/code/viz_facebook.py)

# Coding exercise networkx: distances from SF to other cities

- Download data: <https://github.com/parrt/msds689/blob/master/notes/code/distances.csv>
- Create graph showing SF to other cities using multiple layouts



See [https://github.com/parrt/msds689/blob/master/notes/code/sto\\_city\\_distances.py](https://github.com/parrt/msds689/blob/master/notes/code/sto_city_distances.py)