COT 4521: INTRODUCTION TO COMPUTATIONAL GEOMETRY

Introduction and Preliminaries

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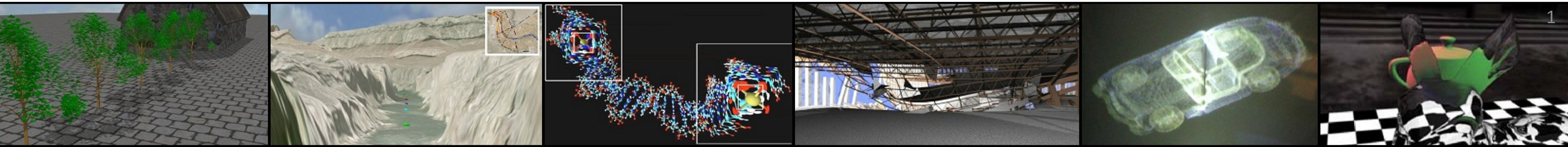
Some slides from Valentina Korzhova



2004

Problem Domain

Computer Graphics

Problem Solving Approach

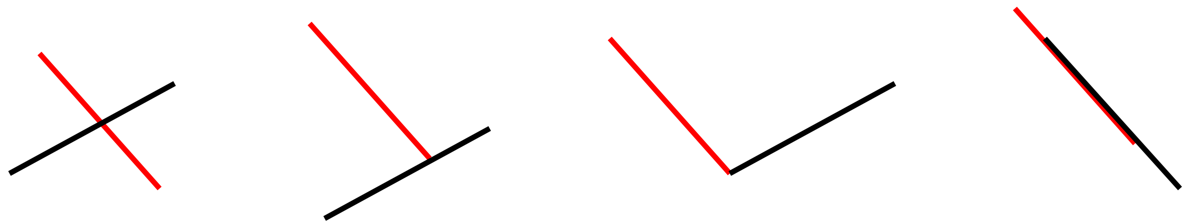
Geometric Approaches

Introduction

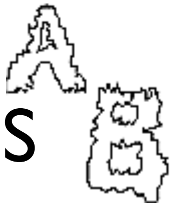
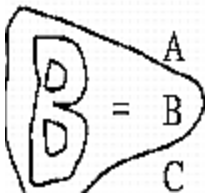
- What is computational geometry?
  - A subfield of the Analysis of Algorithms
  - Design of efficient algorithms and data structures for geometrical problems
- Started in mid 70's
  - started out by developing solid theoretical foundations, but became more and more applied over the last years
- Deals with geometrical structures
  - Points, lines, line segments, vectors, planes, etc.
- We'll focus mostly on 2-dimensional geometry, but 3D will be covered when possible

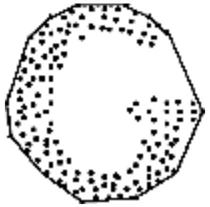
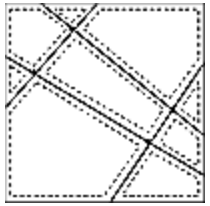
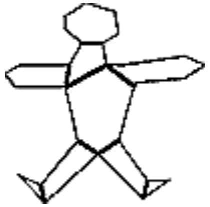
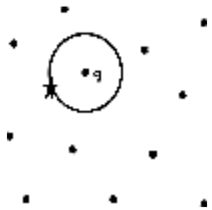
What makes Computational Geometry problems so Interesting/annoying?

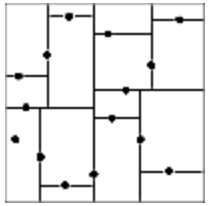
- Complexity
  - Curse of dimensionality
- Precision error
  - Avoid floating-point computations whenever possible
- Degeneracy
  - Boundary cases
  - For example, imagine how two line segments can intersect



Typical Computational Geometry Problems

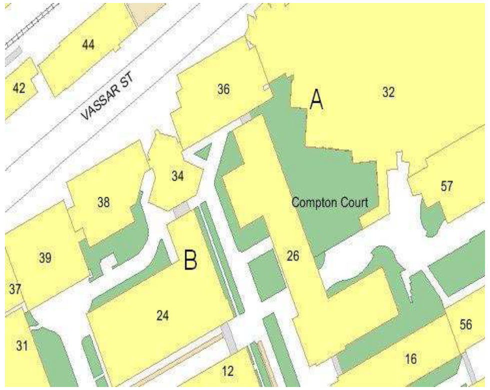
- Voronoi diagram 
- simplifying polygons 
- shape similarity 

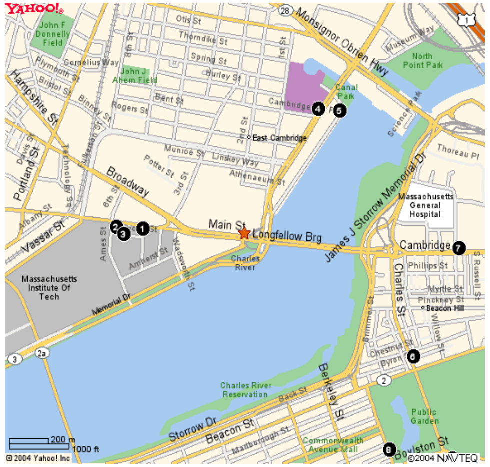
- convex hull 
- maintaining line arrangements 
- polygon partitioning 
- nearest neighbor search 

- kd-trees 

Applications

- Computer Graphics
- Robotics motion planning
- Geographic Information
- CAD/CAM
- Collision detection
- Computer vision
- Molecular modeling
- VLSI design
- Data Mining, Machine Learning, and Visualization



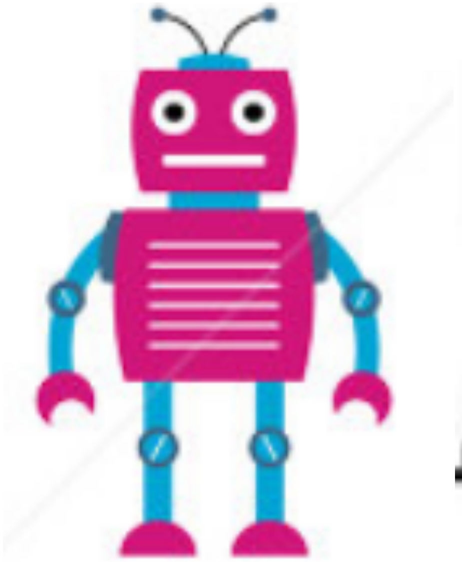


Applications of Computational Geometry

- Computer Graphics
  - For 2-dimensional graphics, typical questions involve the intersection of certain primitives, determining the subset of primitives that lie within a particular region, etc.
  - For 3-dimensional graphics, hidden surface removal

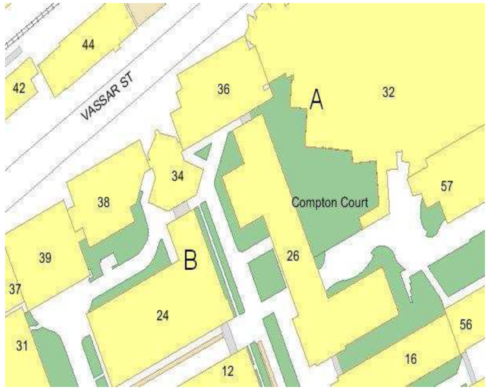
Applications of Computational Geometry

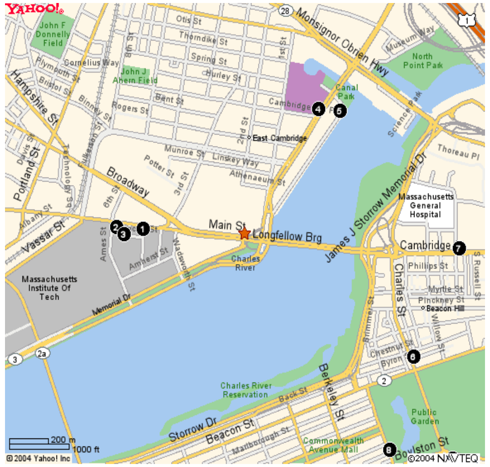
- Robotic motion planning
  - The geometric problem arise at many places because robots are geometric objects that operate in a 3-dimensional space



Applications of Computational Geometry

- Geographic Information system (GIS)
  - Stores geographical data
  - Can be used to extract information about certain regions and to obtain information between different types of data



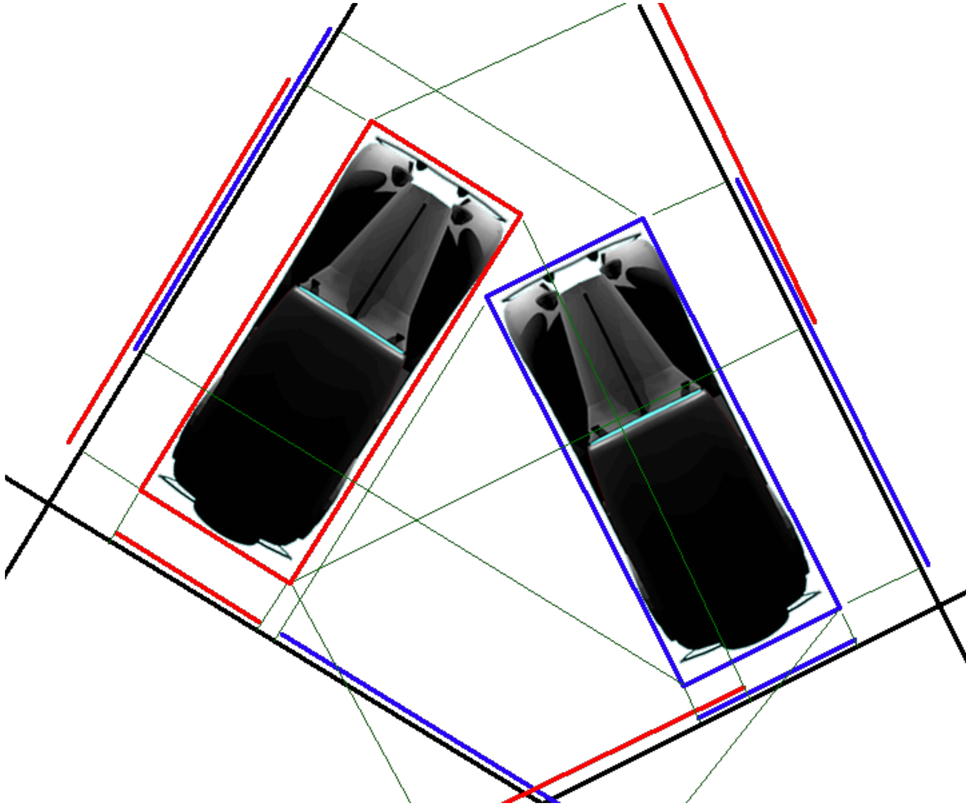


Applications of Computational Geometry

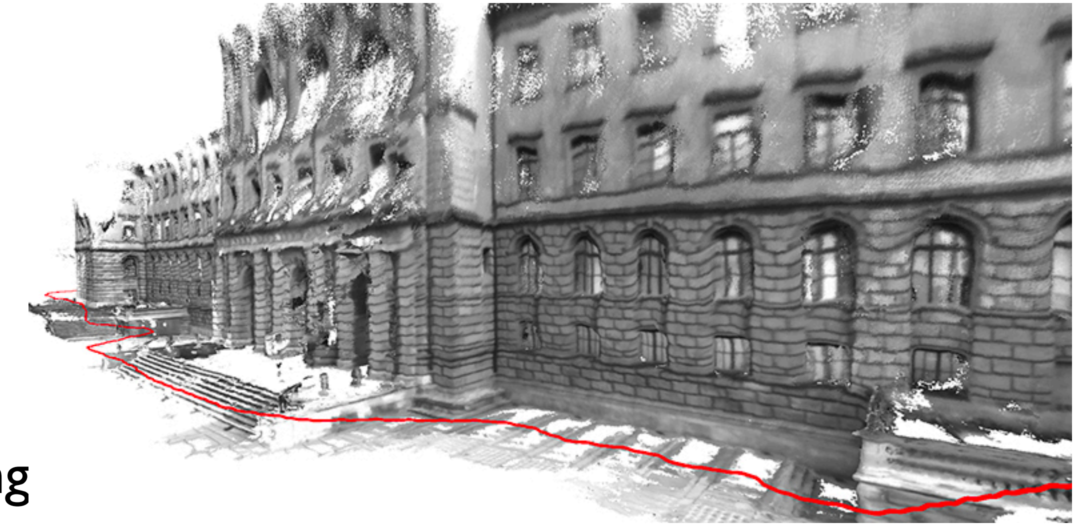
- CAD/CAM
  - Computer-Aided Design
    - Deal with intersections and unions of objects, with decompositions objects and object boundaries into simpler shapes, and with visualizing the designed products
  - Computer-Aided Manufacturing
    - Computer controls the actual manufacturing process

Applications of Computational Geometry

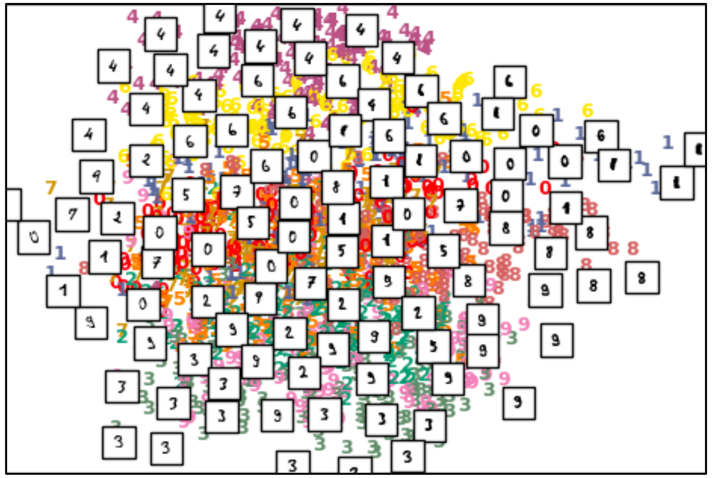
- Collision detection
  - Check whether two (possibly complicated) 3d objects intersect!
    - Approximate the objects by simple ones that enclose them (bounding volumes)
    - Popular bounding volumes: boxes, spheres, ellipsoids,...
    - If bounding volumes don't intersect, the objects don't intersect, either
    - *Only* if bounding volumes intersect, apply more expensive intersection test(s)



APPLICATIONS OF COMPUTATIONAL GEOMETRY

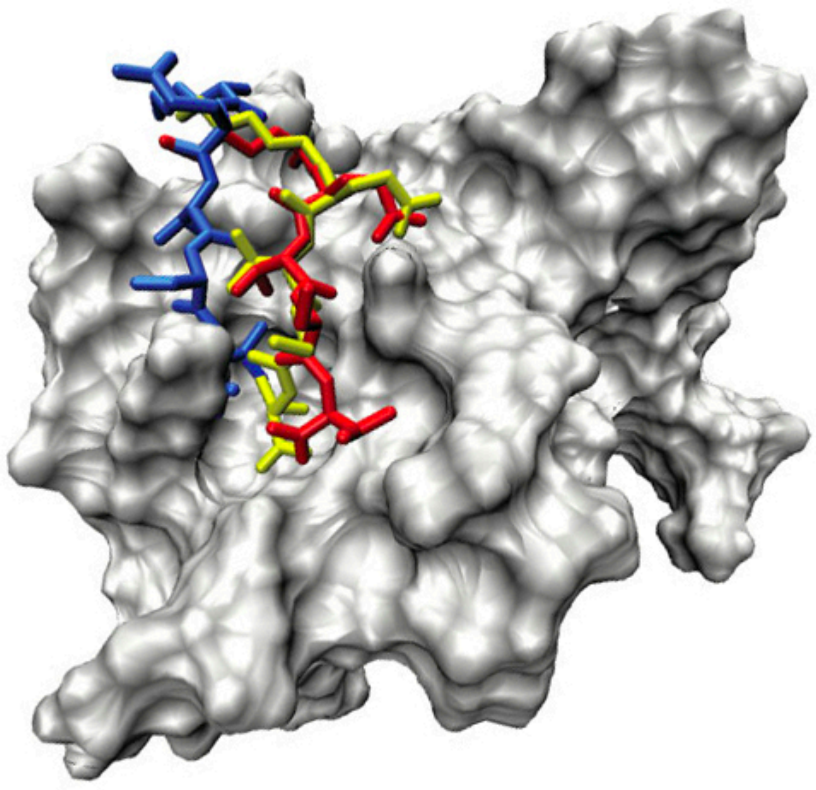


- COMPUTER VISION
  - Surface Reconstruction
    - Step 1: Scan the object (3d laser scanner)
    - Step 2: Create a triangulation
    - Step 3: process the triangulation (rendering *smooth* surface in $\boldsymbol{R}^3$
    - Major Computational Geometry task: Create a "good" triangulation
  - Pattern recognition
    - Example: an optical character recognition system



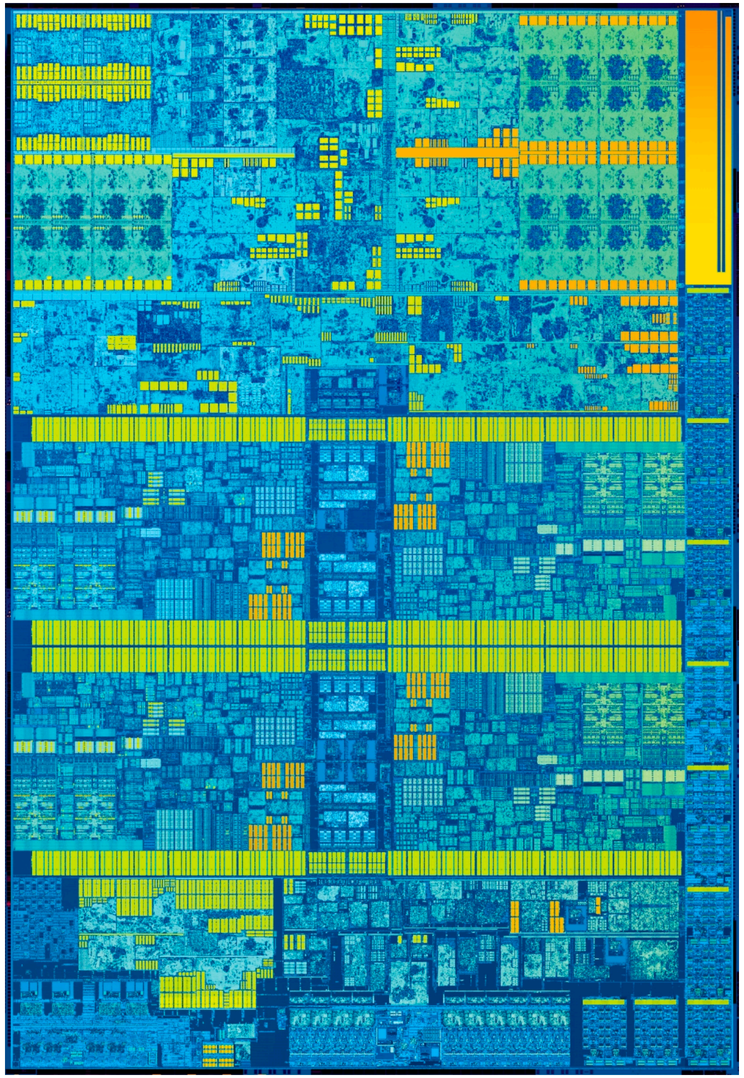
Applications of Computational Geometry

- Molecular modeling
  - Typical questions are to compute the union of the atom balls to obtain the molecule surface or to compute where two molecules can touch each other



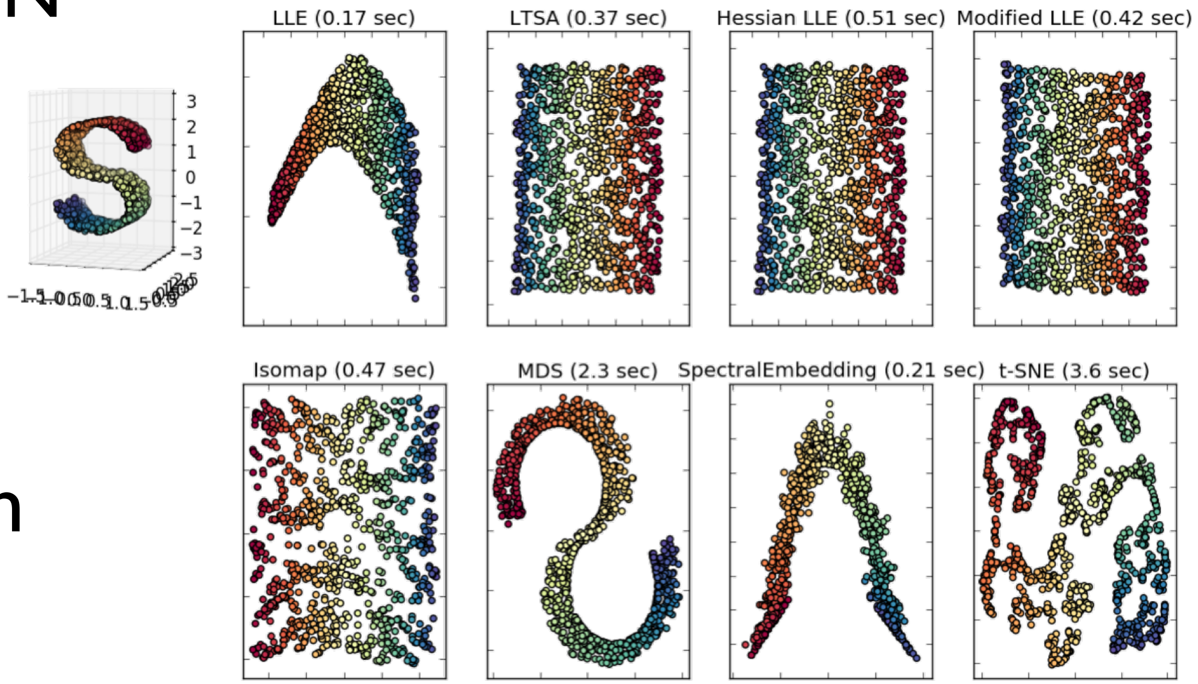
Applications of Computational Geometry

- VLSI design
  - The process of creating an integrated circuit (IC) by combining thousands of transistors into a single chip



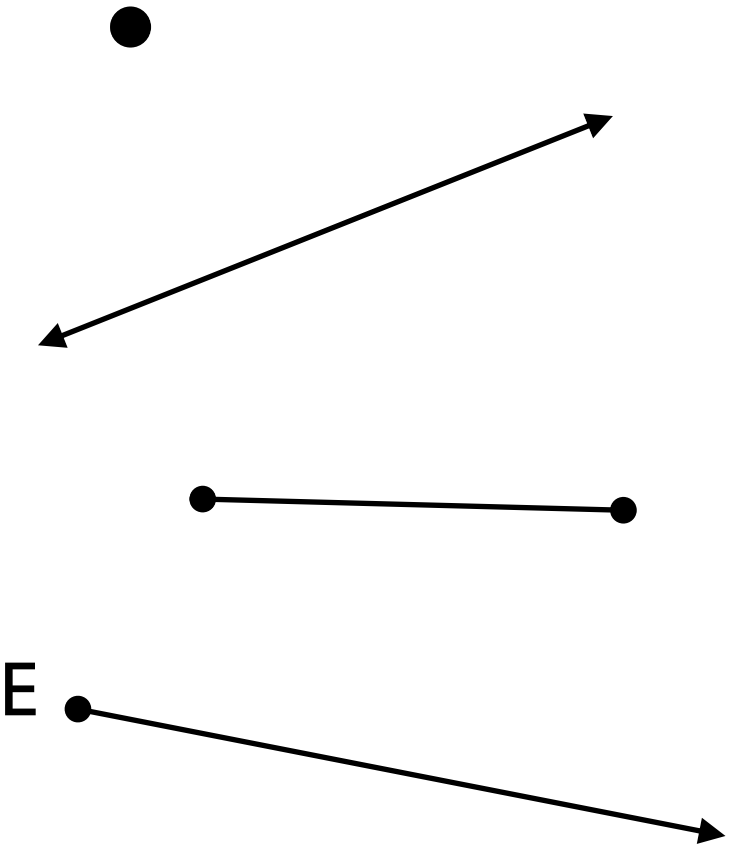
APPLICATIONS OF COMPUTATIONAL GEOMETRY

- DATA MINING, MACHINE LEARNING, AND VISUALIZATION
  - Many algorithms for operations such as clustering, dimension reduction, classification, and segmentation are based upon geometric approaches.

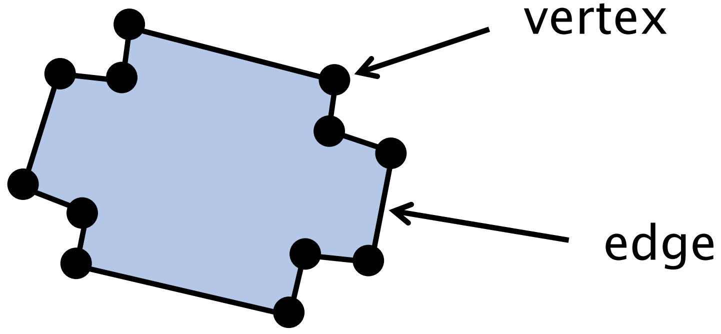


Basic definitions

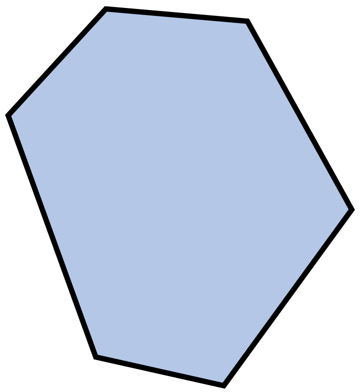
- Point—Specified by two coordinates (x,y)
- Line—Extends to infinity in both directions
- Line segment—Specified by two endpoints
- Ray—Extends to infinity in one direction



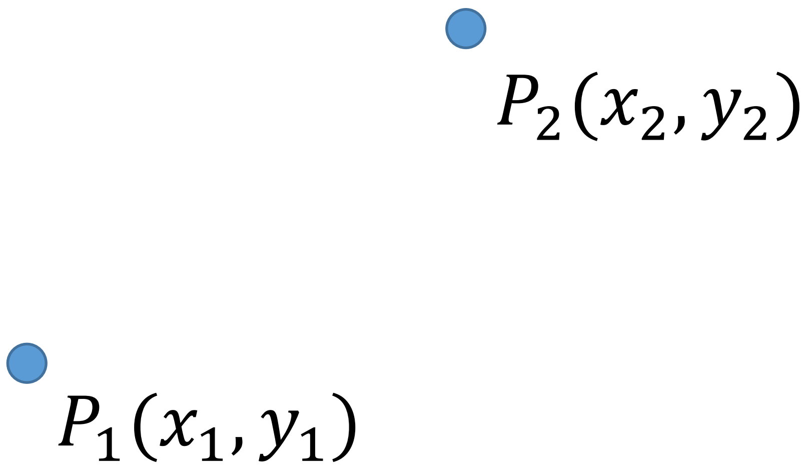
BASIC DEFINITIONS



- POLYGON
  - We assume edges do not cross
- CONVEX POLYGON
  - Every interior angle is at most 180 degrees
  - Precise definition of *convex*: For any two points inside the polygon, the line segment joining them lies entirely inside the polygon (we'll cover this later)

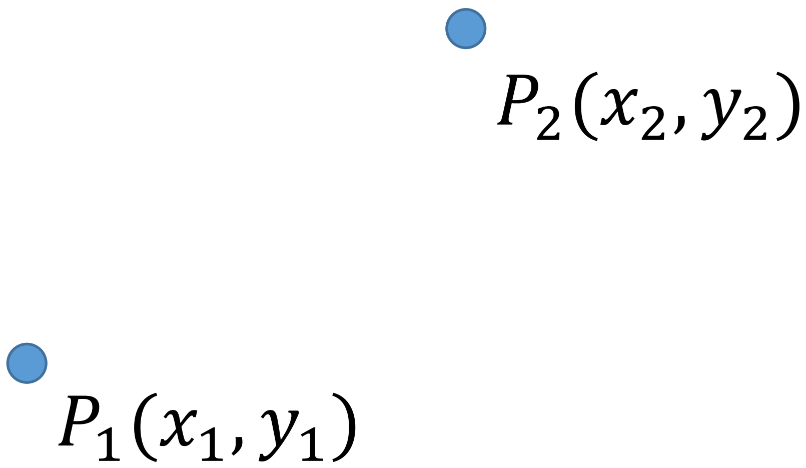


Points



Euclidean Distance

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



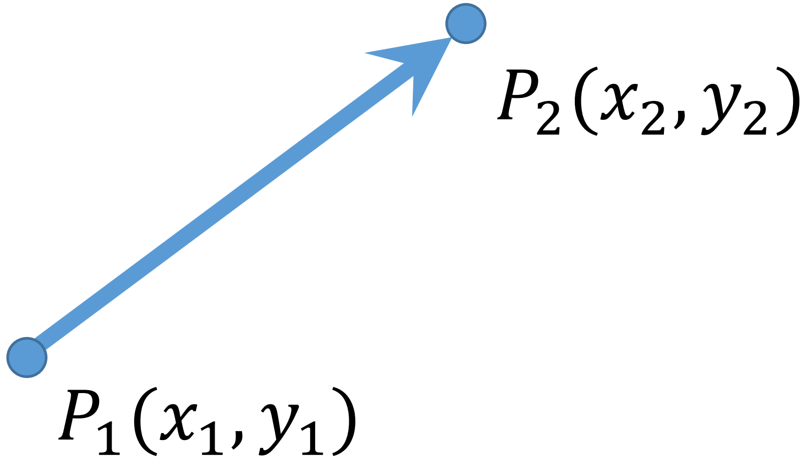
Euclidean Distance

- Square root is kind of slow and imprecise
- If we only need to check whether the distance is less than some certain length, say R

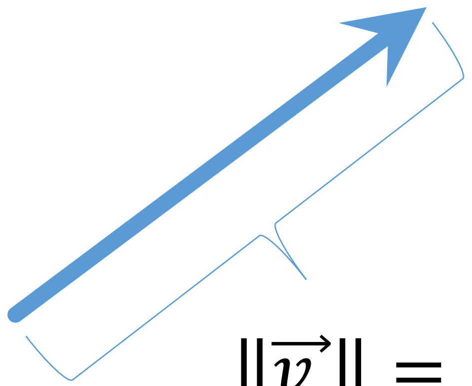
$$\text{if } \left((x_2 - x_1)^2 + (y_2 - y_1)^2 < R^2\right) \ldots$$

VECTORS

$$\overrightarrow{v_{12}} = P_2 - P_1$$
$$= < x_2 - x_1, y_2 - y_1 >$$

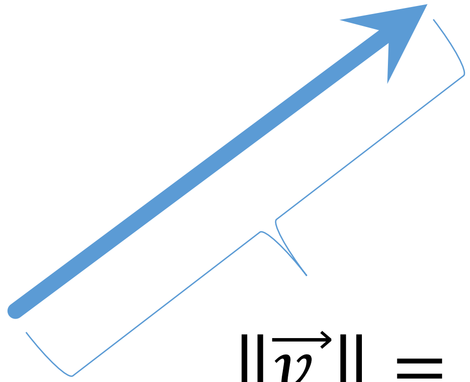


Vector Length/Magnitude



$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2}$$

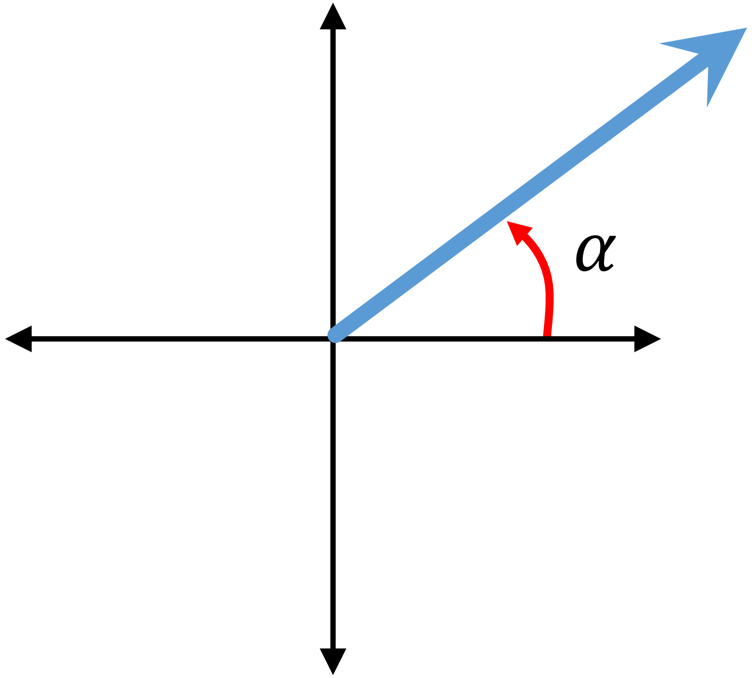
VECTOR LENGTH/MAGNITUDE



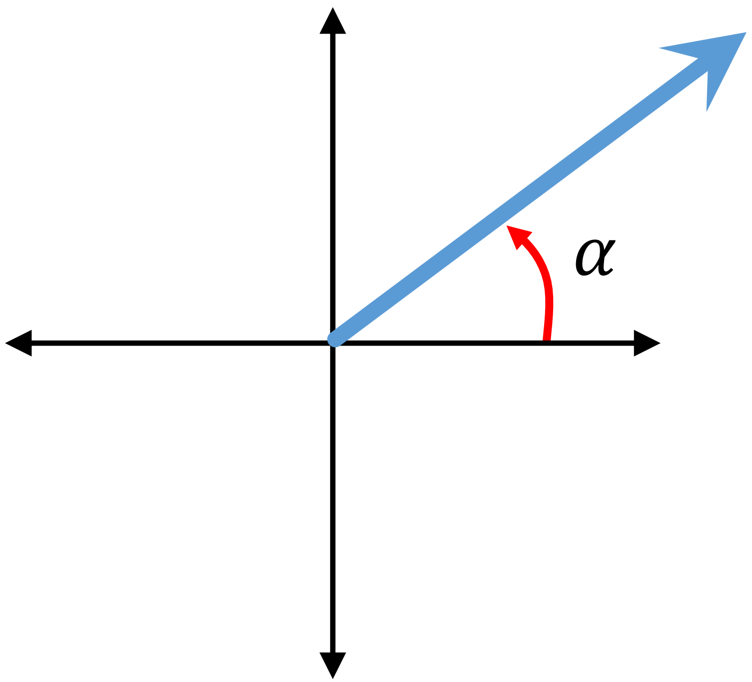
$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2}$$

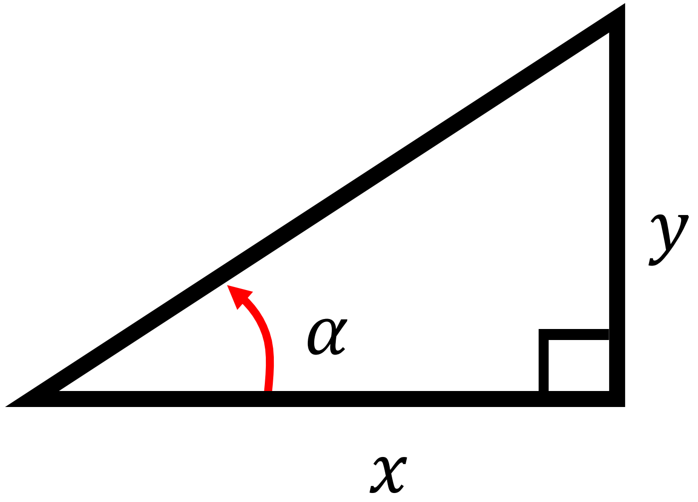
Equivalent to Euclidean distance…

Vector Angle



Vector Angle





$$\tan(\alpha) = \frac{y}{x}$$

Vector Angle

$$\alpha = \tan^{-1} \frac{y}{x}$$

or

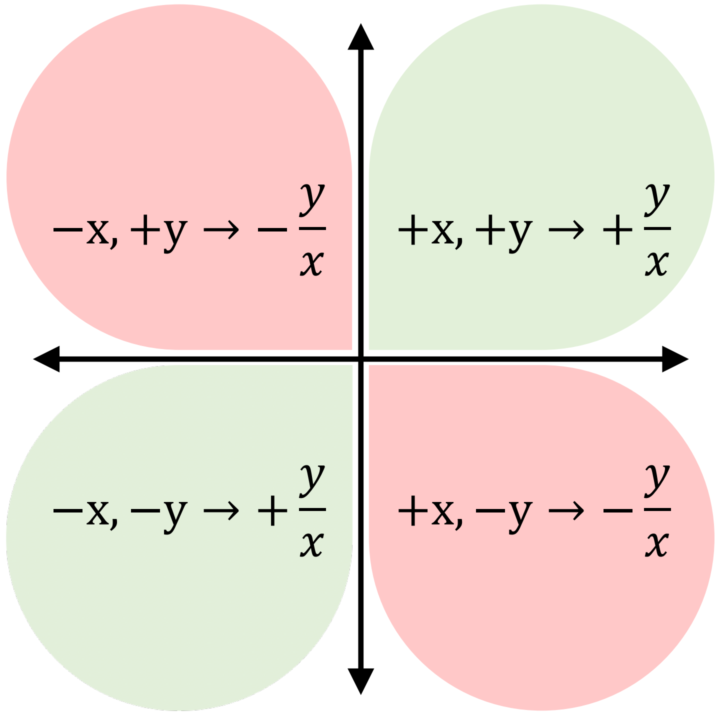
$$\alpha = \operatorname{atan}\left(\frac{y}{x}\right)$$

(results in radians, not degrees)

Problems?

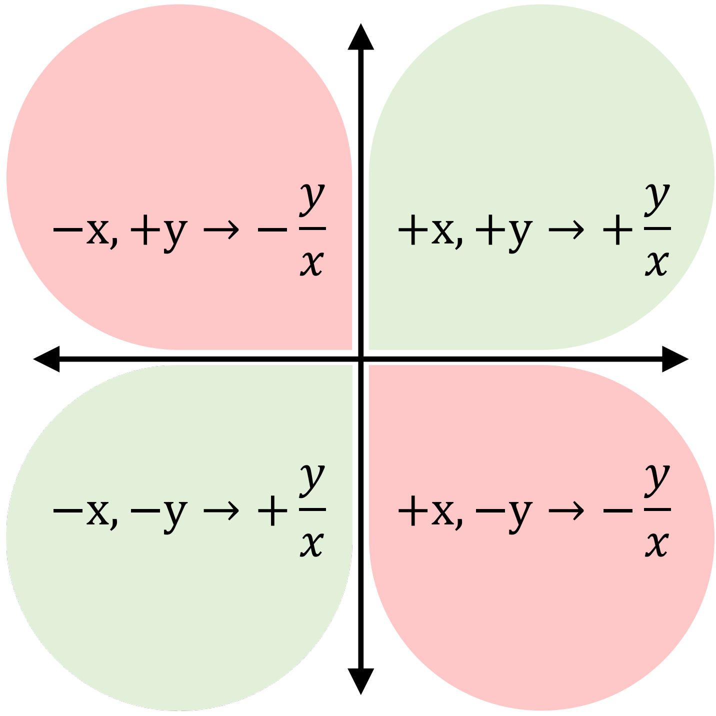
Vector Angle

- Problem 1: Division by zero
  - When α is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$
- Problem 2: $\frac{y}{x}$ doesn't give a 1-to-1 mapping

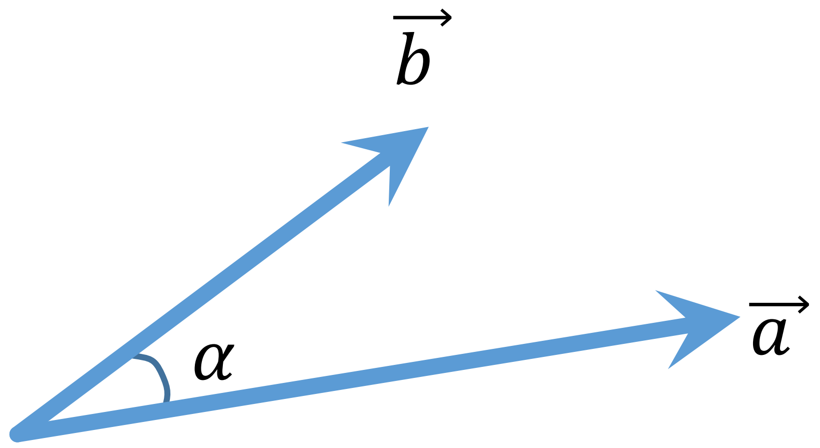


Vector Angle

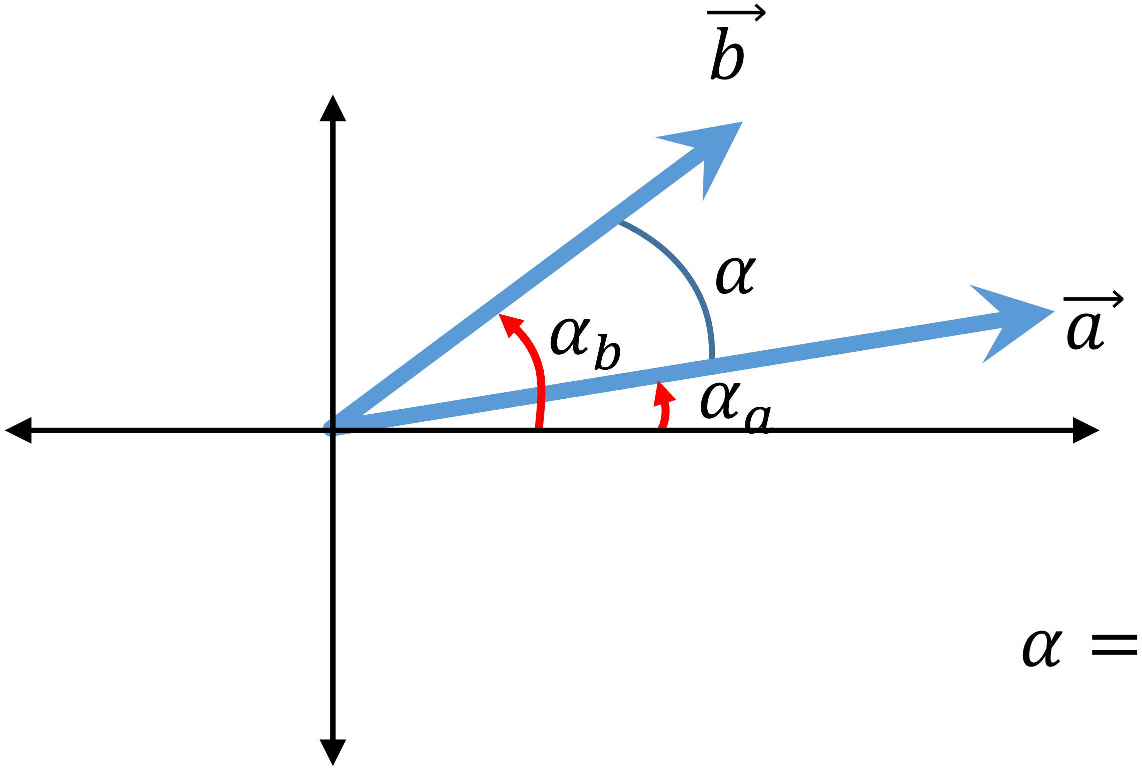
- Problem 1: Division by zero
  - When α is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$
- Problem 2: $\frac{y}{x}$ doesn't give a 1-to-1 mapping
- Solution: $\boldsymbol{\alpha = \mathrm{atan}\,2(y, x)}$
  - Note: the arguments are $(\mathbf{y}, \boldsymbol{x})$, not (x, y)!!!



Angle Between 2 Vectors

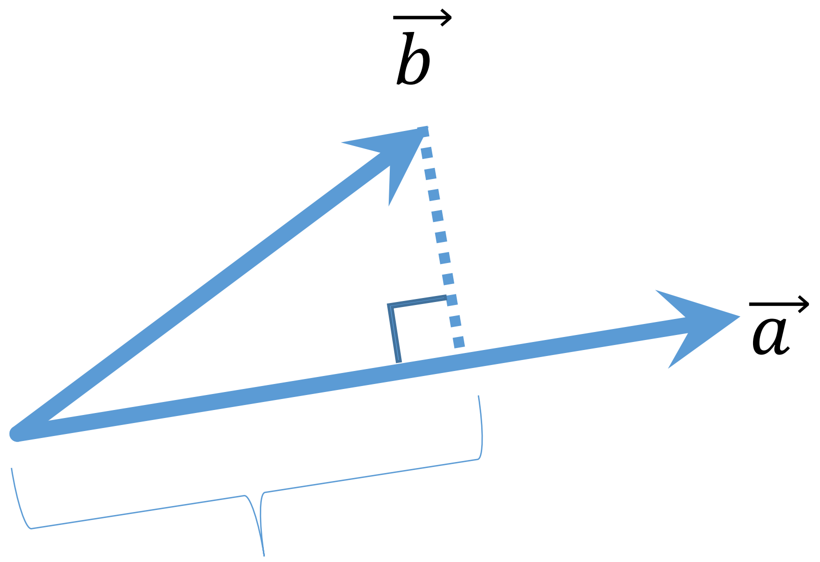


Angle Between 2 Vectors



$$\alpha = \alpha_b - \alpha_a$$

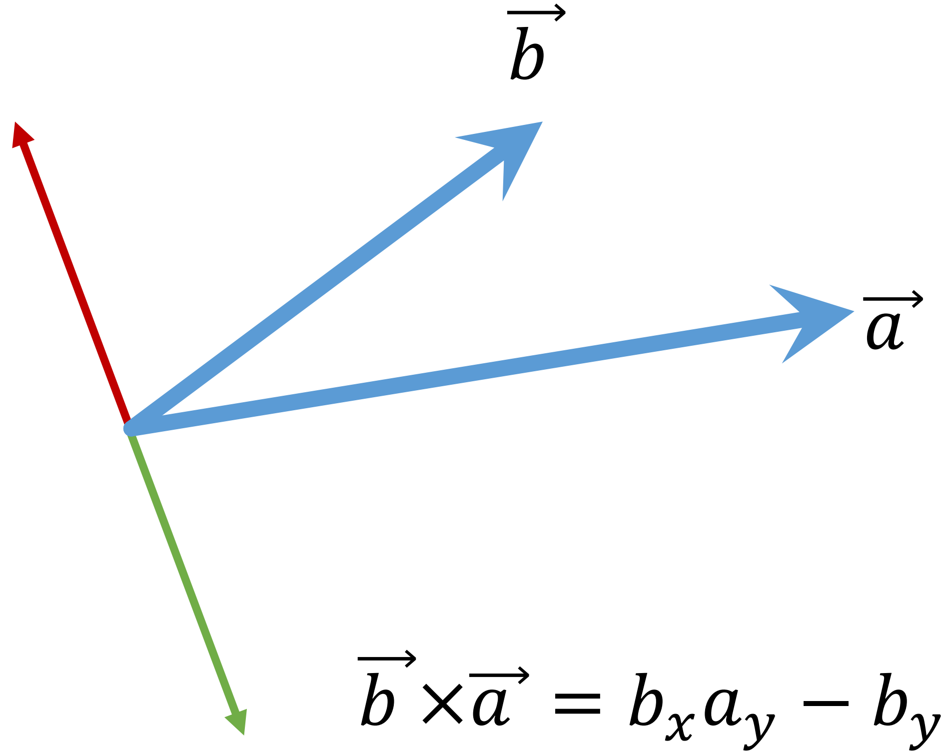
Dot Product



$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y$$

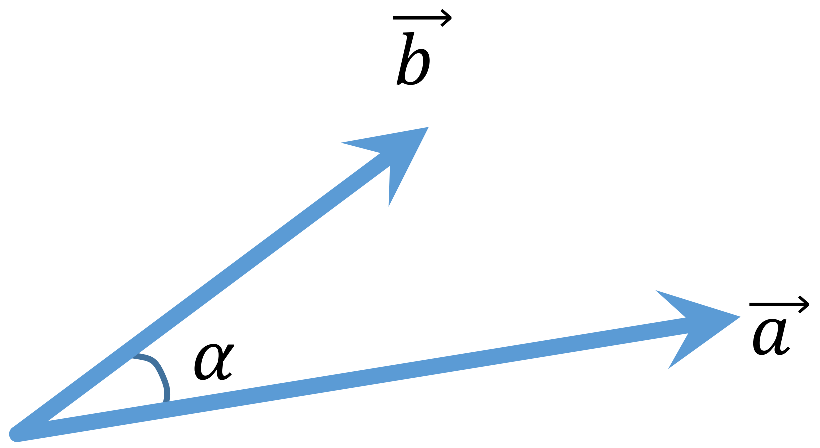
Cross Product

$$\vec{a} \times \vec{b} = a_x b_y - a_y b_x$$



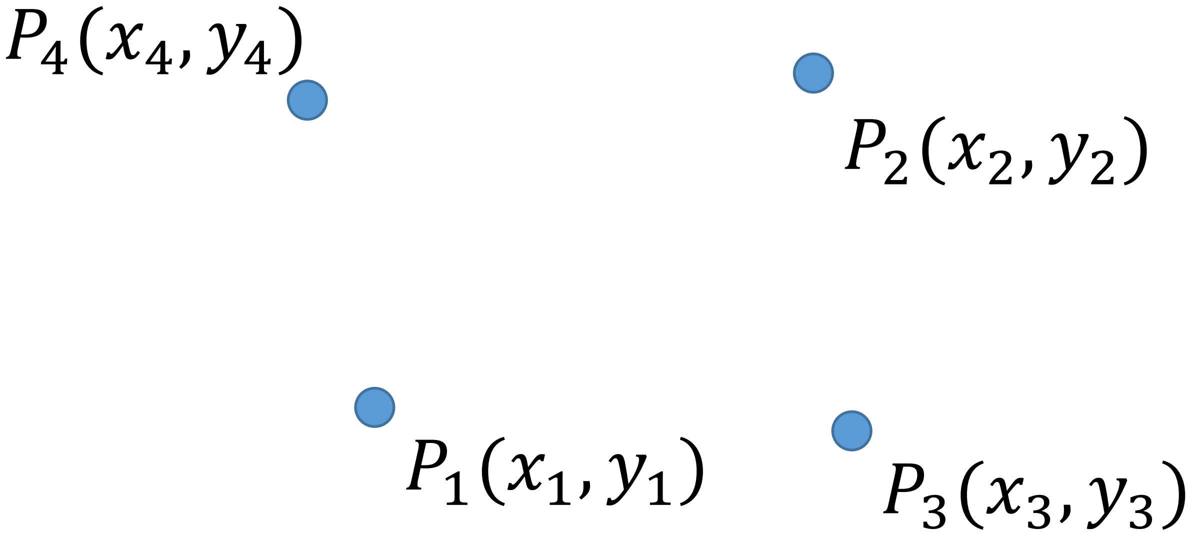
$$\vec{b} \times \vec{a} = b_x a_y - b_y a_x$$

Angle Between 2 Vectors



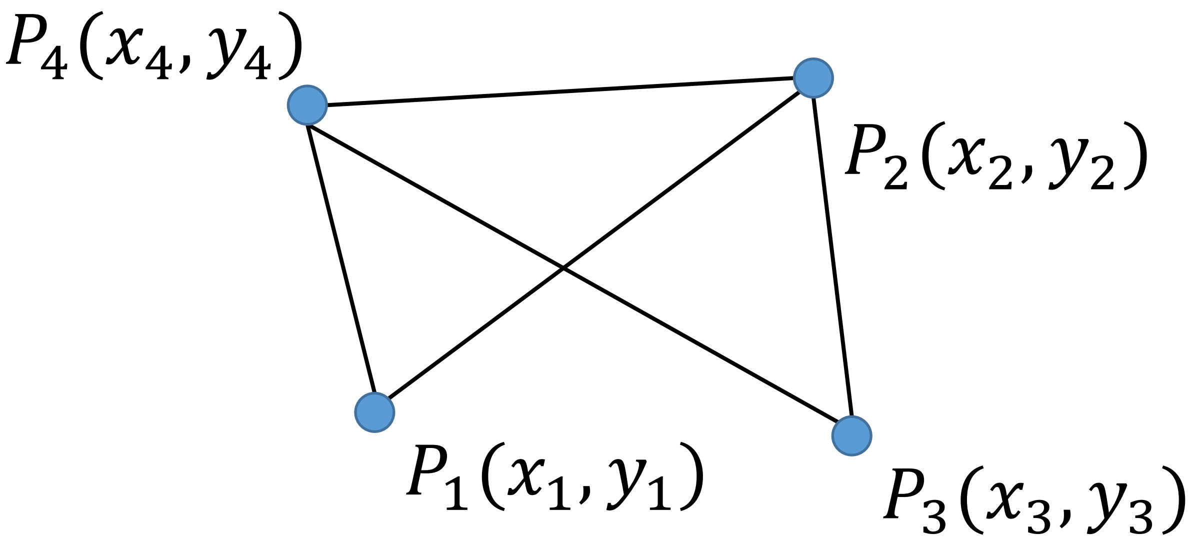
$$\cos\alpha = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|} \qquad \sin\alpha = \frac{\|\vec{a}\times\vec{b}\|}{\|\vec{a}\|\|\vec{b}\|}$$

Storing Points



Point List
(x_1, y_1)
(x_2, y_2)
(x_3, y_3)
(x_4, y_4)
…
(x_n, y_n)

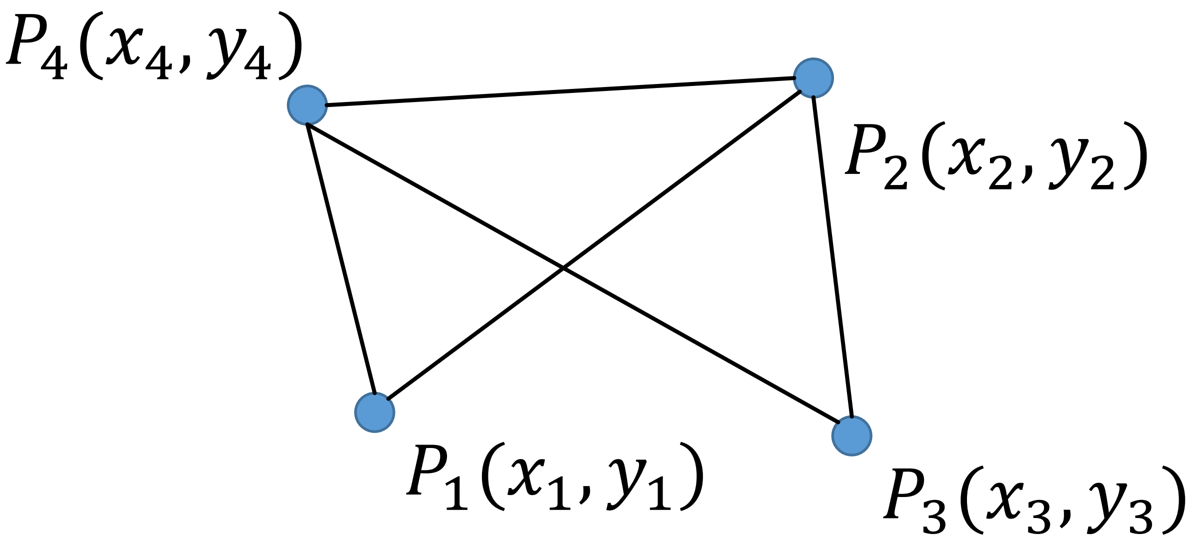
Storing Edges as Indices

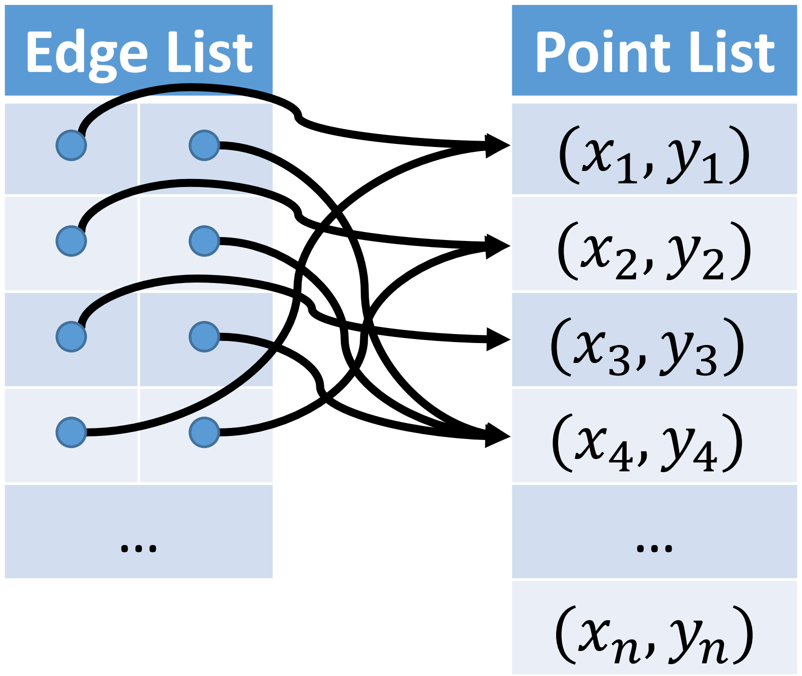


Edge List	
0	3
1	3
2	3
0	1
…	

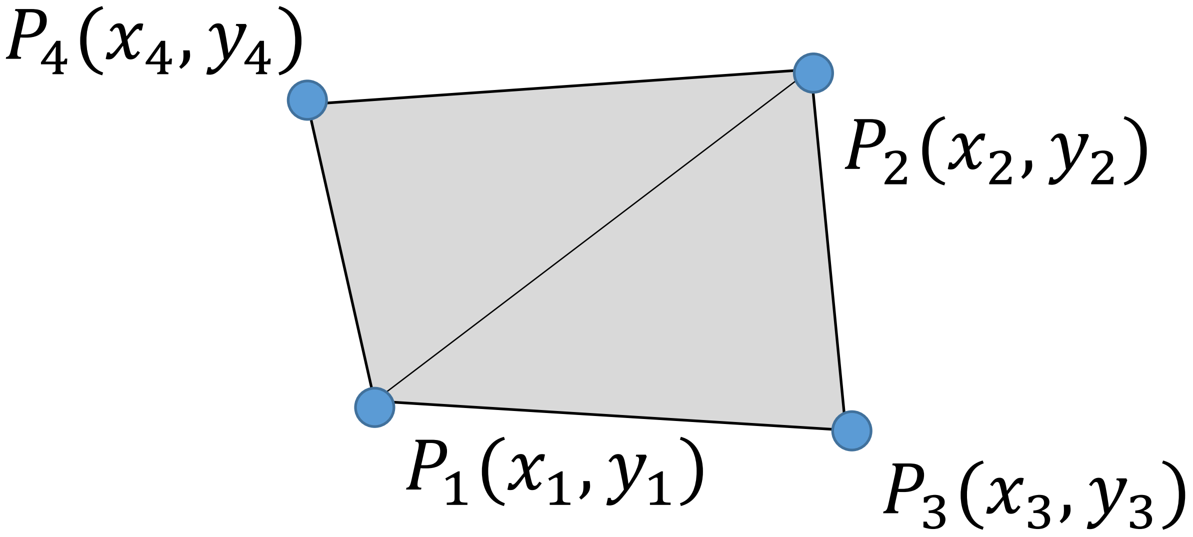
Point List	
0	(x_1, y_1)
1	(x_2, y_2)
2	(x_3, y_3)
3	(x_4, y_4)
…	…
n-1	(x_n, y_n)

Storing Points as Pointers





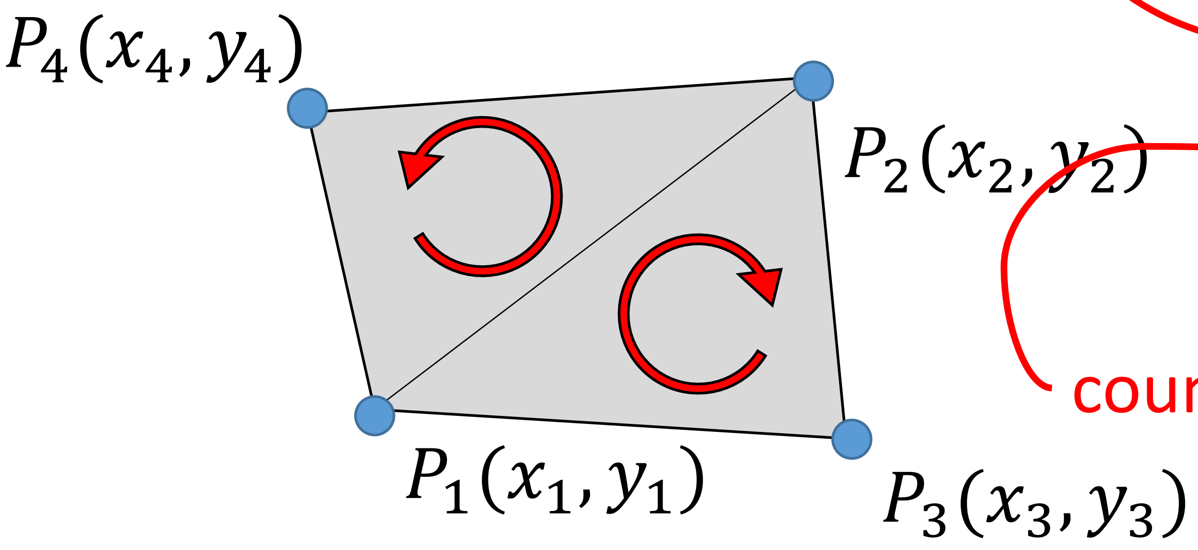
Storing Triangles with Indices



Triangle List		
0	1	2
0	1	3
…		

Point List	
0	(x_1, y_1)
1	(x_2, y_2)
2	(x_3, y_3)
3	(x_4, y_4)
…	…
n-1	(x_n, y_n)

STORING TRIANGLES WITH INDICES



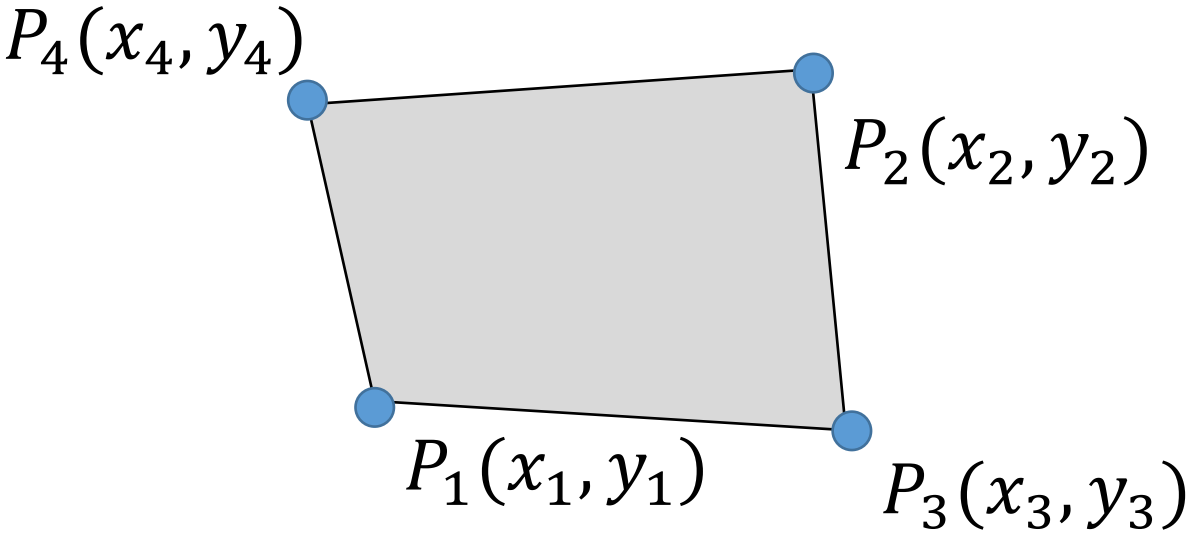
clockwise

counterclockwise

Triangle List		
0	1	2
0	1	3
…		

Point List	
0	(x_1, y_1)
1	(x_2, y_2)
2	(x_3, y_3)
3	(x_4, y_4)
…	…
n-1	(x_n, y_n)

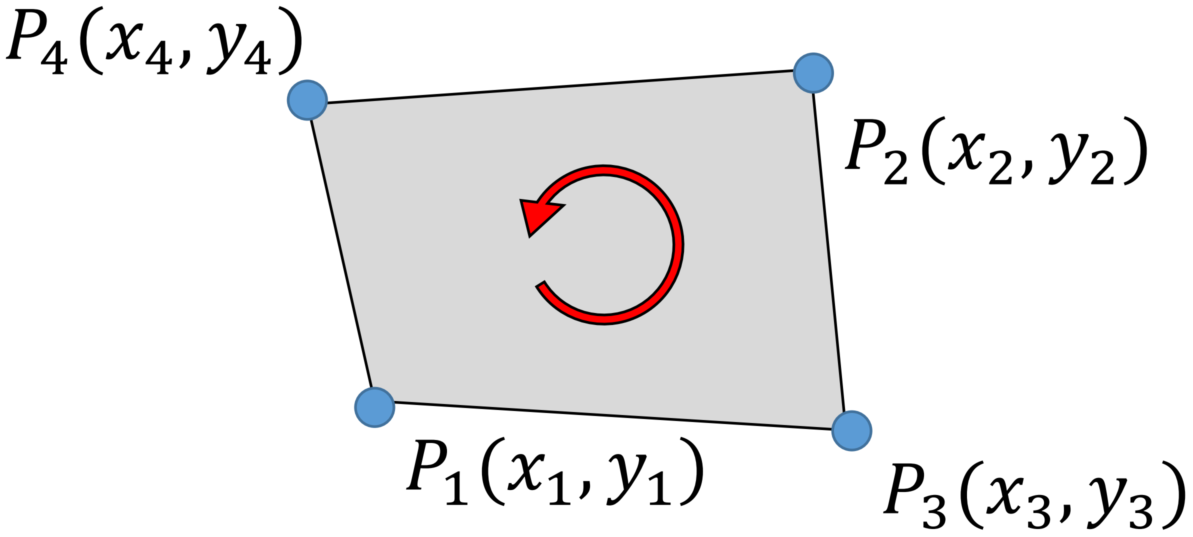
STORING POLYGONS WITH INDICES



Polygon List			
0	2	1	3
…			

Point List	
0	(x_1, y_1)
1	(x_2, y_2)
2	(x_3, y_3)
3	(x_4, y_4)
…	…
n-1	(x_n, y_n)

Storing Polygons with Indices



Polygon List			
0	2	1	3
…			

Point List	
0	(x_1, y_1)
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…	…
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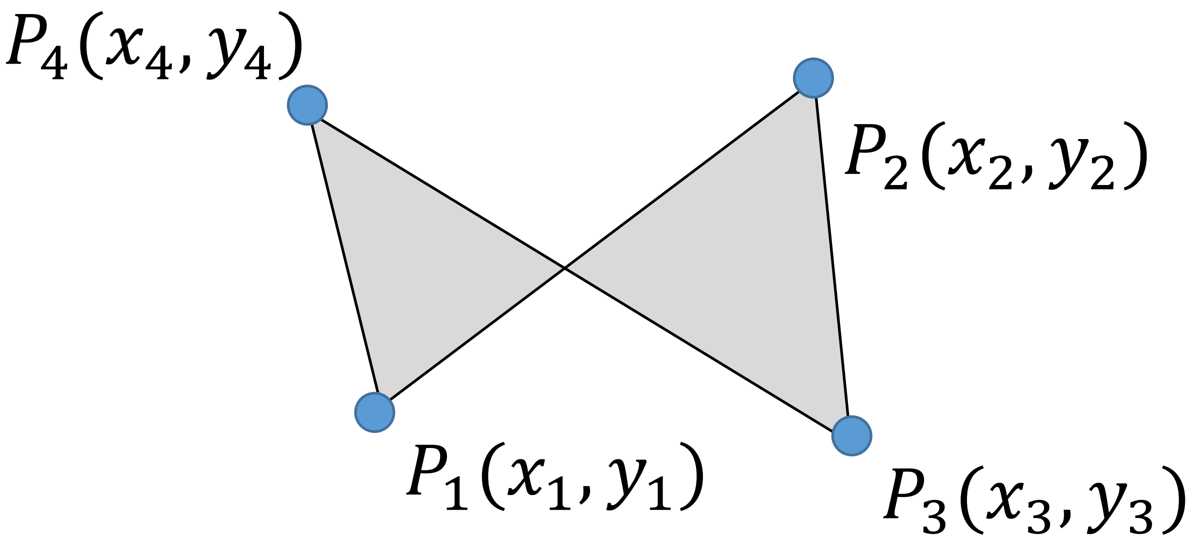
Storing Polygons with Indices

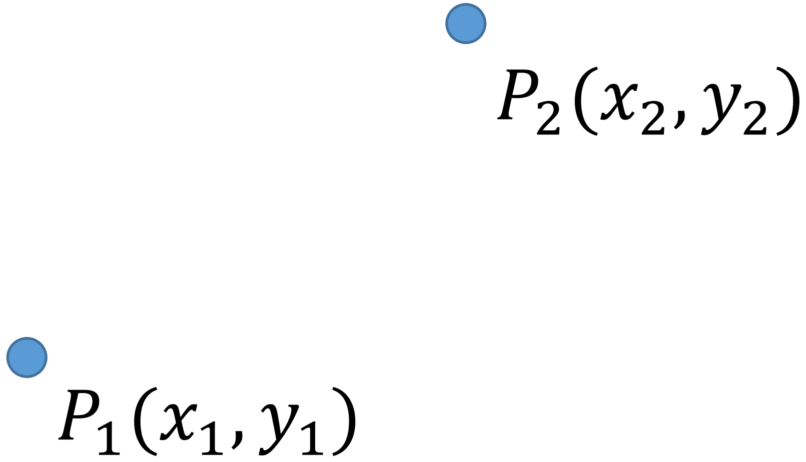


Polygon List			
0	1	2	3
…			

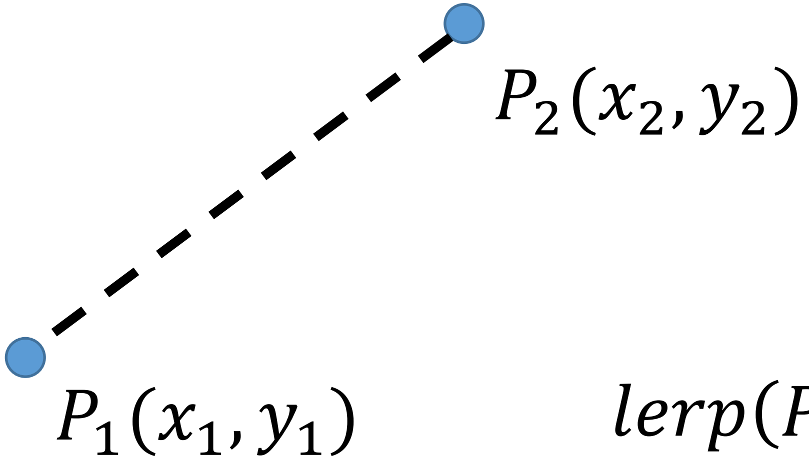
Take care with order

Point List	
0	(x_1, y_1)
1	(x_2, y_2)
2	(x_3, y_3)
3	(x_4, y_4)
…	…
n-1	(x_n, y_n)

Linear Interpolation

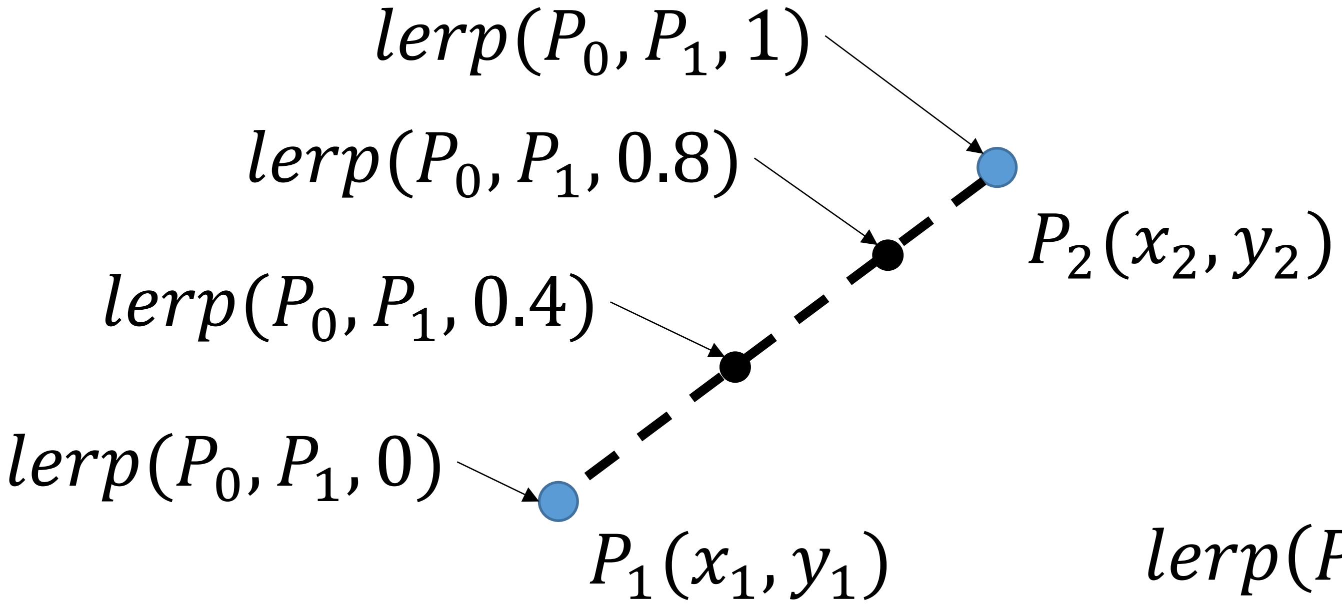


Linear Interpolation



$$lerp(P_1, P_2, a) = P_1(1 - a) + P_2 a$$

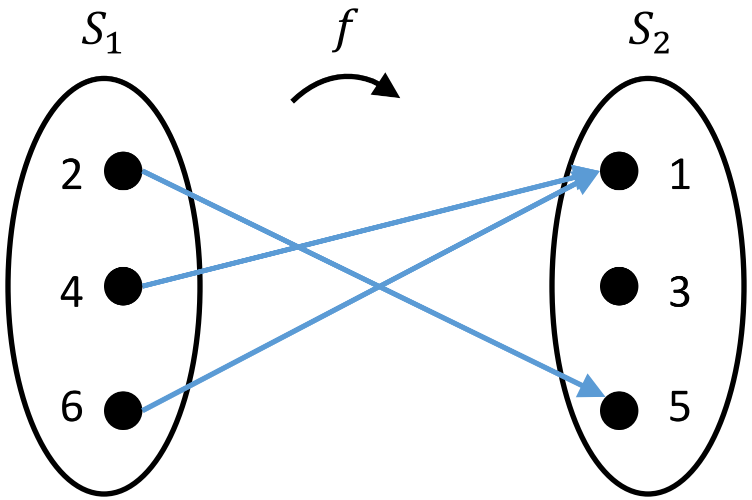
Linear Interpolation



$$lerp(P_1, P_2, a) = P_1(1 - a) + P_2 a$$

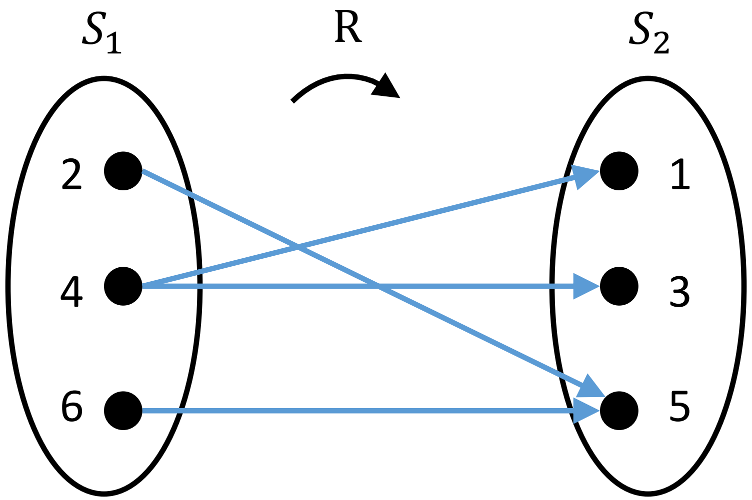
FUNCTIONS

- A FUNCTION IS A RULE THAT ASSIGNS TO ELEMENTS OF ONE SET A UNIQUE ELEMENT OF ANOTHER SET
  - $f: S_1 \rightarrow S_2,$
  - Where the domain of f is a subset of S_1 and the range of f is the subset of S_2
  - f is a total function on S_1 if the domain of f is all of S_1; otherwise f is a partial function



RELATIONS

- Some functions can be represented by a set of pairs $\{(x_1, y_1), (x_2, y_2), \ldots\}$, where x_i is an element in the domain of the function, and y_i is the corresponding value in its range
- For such a set to define a function, each x_i can occur at most once as the first element of a pair.
- If this is not satisfied, the set is called a relation.



FUNCTIONS

- THE BEHAVIOR OF FUNCTIONS:
  - Big O
    - f has order at most g
    - $f(n) \leq c|g(n)| \rightarrow f(n) = O\big(g(n)\big)$
  - Big Omega
    - f has order at least g
    - $|f(n)| \geq c|g(n)| \rightarrow f(n) = \Omega(g(n))$
  - Big Theta
    - f has the same order of magnitude as g
    - $c_1|g(n)| \leq |f(n)| \leq c_2|g(n)| \rightarrow f(n) = \Theta\big(g(n)\big)$

Function Examples

- $f(n) = 2n^2 + 4n$,
- $g(n) = n^3$,
- $h(n) = 9n^2 + 300$

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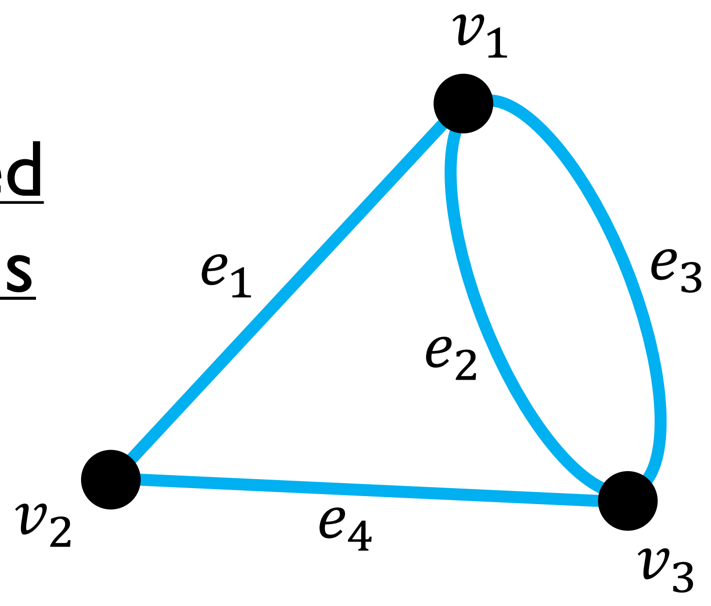
- $f(n) = O(g(n))$,
- $g(n) = \Omega(h(n))$,
- $f(n) = \Theta\big(h(n)\big)$

$$O(n) + O(n) = 2(O(n))?$$

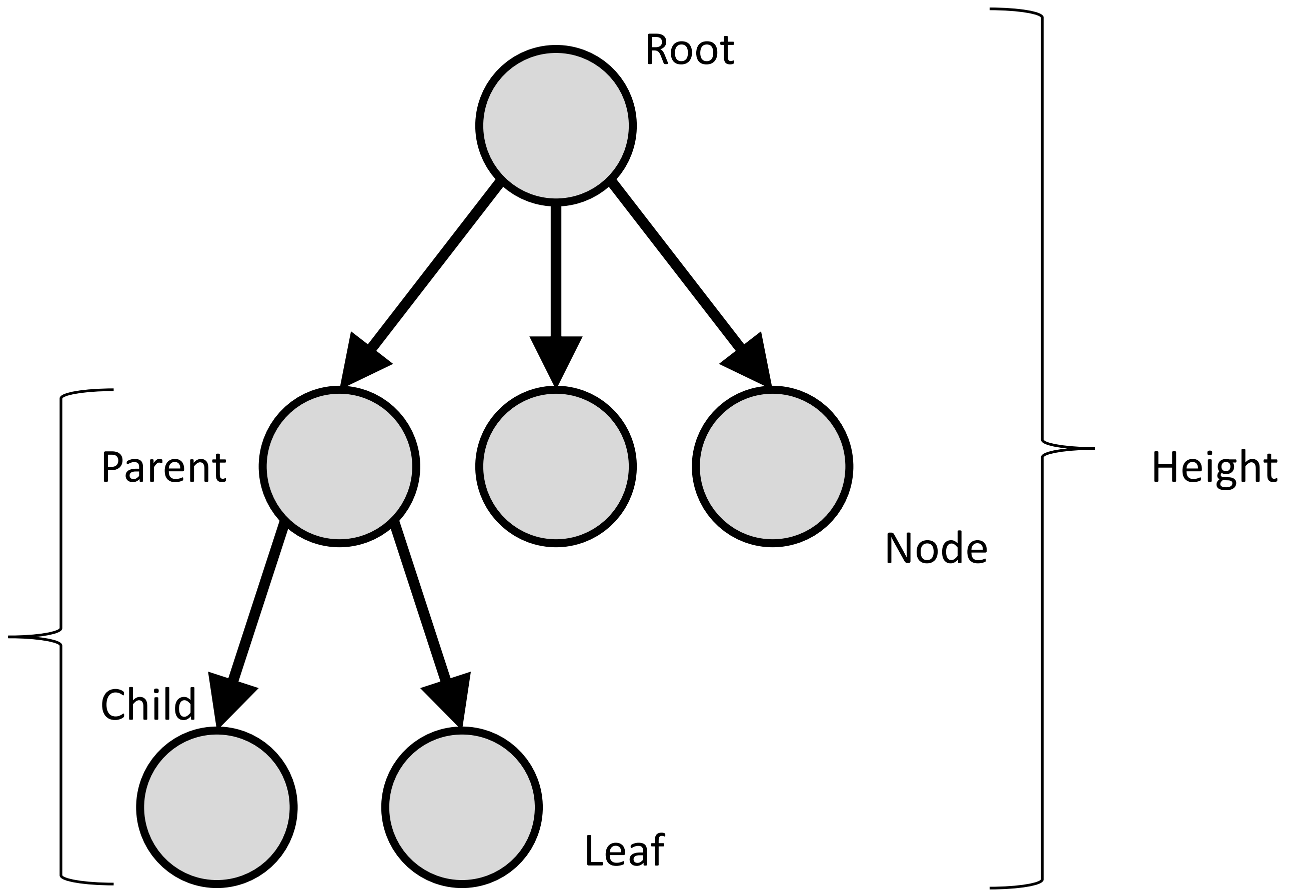
Graphs and Trees

- A graph is construct of two finite sets
  - Vertices, $V = \{v_1, v_2, \ldots, v_n\}$; Edges, E= $\{e_1, e_2, \ldots, e_m\}$
- A ***walk*** from v_i to v_n is a sequence of edges $(v_i, v_j), (v_j, v_k), \ldots, (v_m, v_n)$
- A ***path*** is a walk in which <u>no edge is repeated</u>
- A ***simple path*** is a path in which <u>no vertex is repeated</u>
- A ***cycle*** with base v_i is a walk from v_i to itself with <u>no repeated edges</u>
- A ***loop*** is an edge from a <u>vertex to itself</u>



TREES

- A tree is a directed graph that has no cycles, and that has one distinct vertex (the root)



Proof techniques

- A proof is a sequence of steps that lead from some known facts to the desired conclusion; each step must be obviously correct
- Proof by contradiction:
  - To prove some fact P, we show that "not P" is false
  - That is, we suppose "not P" and demonstrate that it leads to an obviously wrong result
  - E.g.: Prove that $\sqrt{2}$ is not rational. Suppose that is rational, that is $\sqrt{2} = \frac{m}{n}$, where n and m do not have common factors

Proof techniques

- Proof by induction
  - We show that some fact is true for every natural number n, using two arguments:
    - Base: It is true for $n = 1$ (or for some small number)
    - Step: If it is true for n, then it is true for $n + 1$
  - E.g.: prove that $1 + 2 + \cdots + n = n(n+1)/2$