COT 4521: Introduction to Computational Geometry

The Art Gallery Problem

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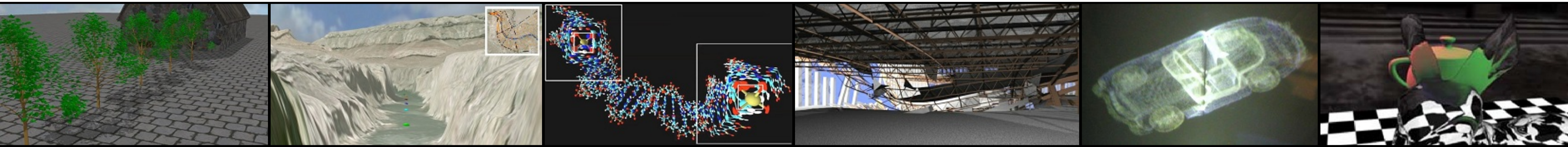
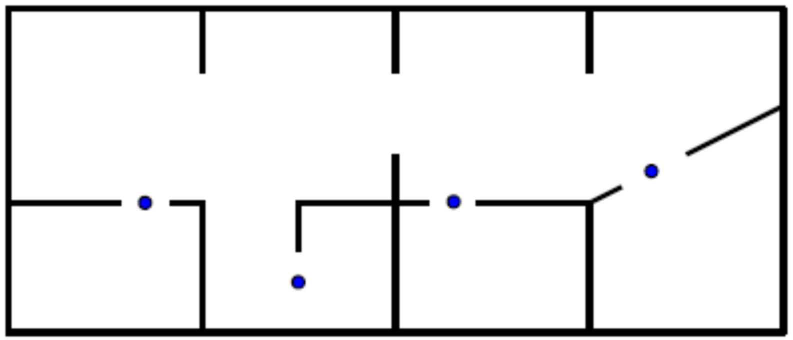
Some slides from Valentina Korzhova



The Art Gallery Problem

- The Art Gallery Problem: how many cameras we need to guard a given gallery so that every point is seen, and how we decide to place them?



- In geometry terminology: How many points are needed in a simple polygon with n vertices so that every point in the polygon is seen?

The Art Gallery Problem

- This problem was posed by Victor Klee in 1973
- A guard of the gallery corresponds to a point on the polygonomial floor plan.
- Guards can see in every direction, with a full range of visibility
- The optimization problem is computationally difficult

The Art Gallery Problems

- In a simple polygon P, a point X is said to be *visible* from a point Y (or, vice versa) whenever the line segment XY does not intersect with the exterior of P

$$P: XY \subseteq P$$

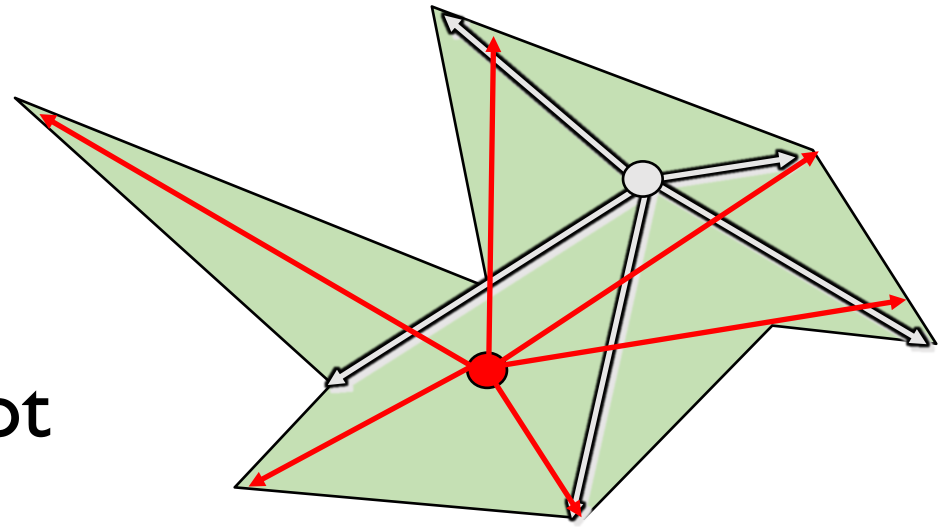
- Vertices of P are considered non-blockers of visibility

- Visibility: 2π range

THE ART GALLERY PROBLEMS

- CONSIDER A ROOM WHOSE FLOOR IS POLYGON OF N VERTICES, HOW MANY POINT LIGHTS (CAMERAS) ARE NEEDED TO LIGHT THE WHOLE ROOM?

- A SET OF LIGHTS IS SAID TO COVER A POLYGON IF EVERY POINT IN THE POLYGON IS LIGHTED.
  - Assume the lights themselves are not sources of shadows

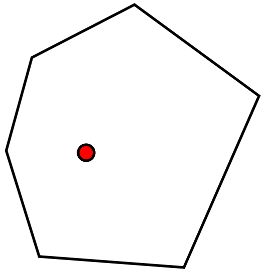
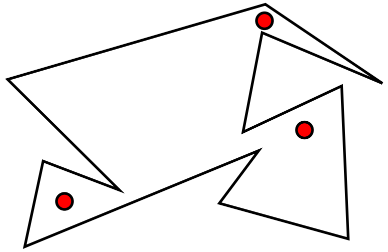
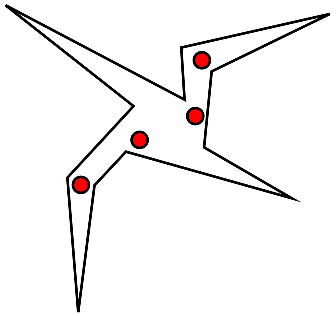


Guarding a simple polygon

- Given a simple polygon P with n vertices, find the *minimum number of guards* required for every point of P to be visible from some guard

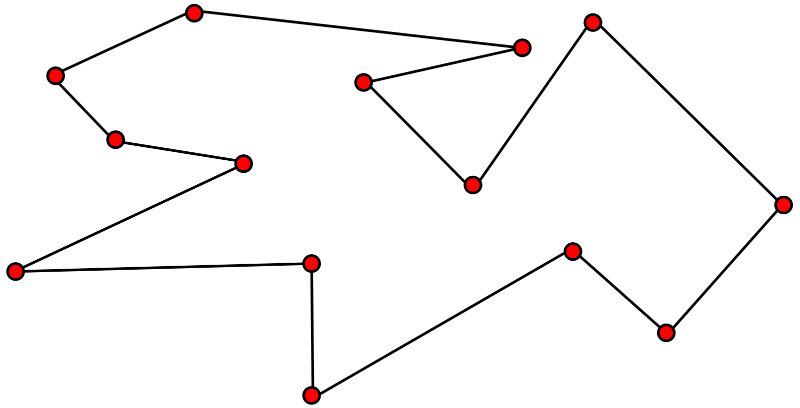
- Assume that every guard can view 360 degrees around it

- How many lights we need to place to guard a simple polygon?
  - One guard is both necessary and sufficient for any convex polygon

Sufficient number of guards for any polygon of n vertices

- How many guards are sufficient to cover any n-vertex simple polygon?
  - By placing a guard at every vertex, any n-vertex simple polygon can be trivially guarded with n guards — loose upper bound



Maximum over minimum formulation
Formal definition

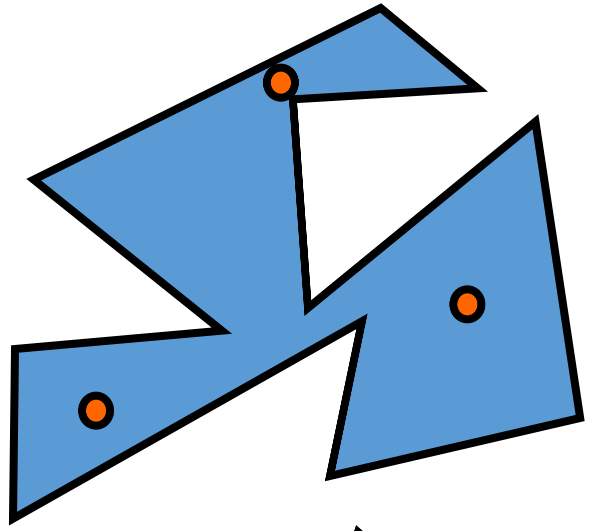
- Let $g(P_N)$ be the smallest number of lights need to cover a particular polygon of N sides.

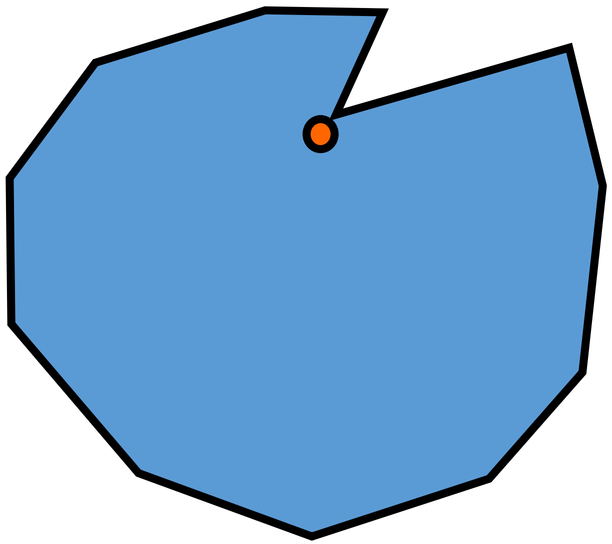
$$\boxed{g(P_N) = \min_S |\{S : S \text{ covers } P\}|}$$

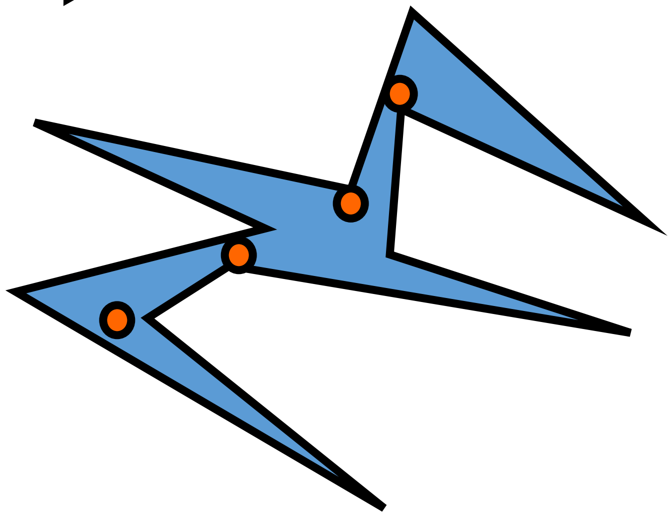
  - S is the set of points where the lights are located
- What is the max of $g(P_N)$ over all P_N?

$$\boxed{G(N) = \max_{P_N} g(P_N)}$$

HOW MANY LIGHTS ARE NEEDED?

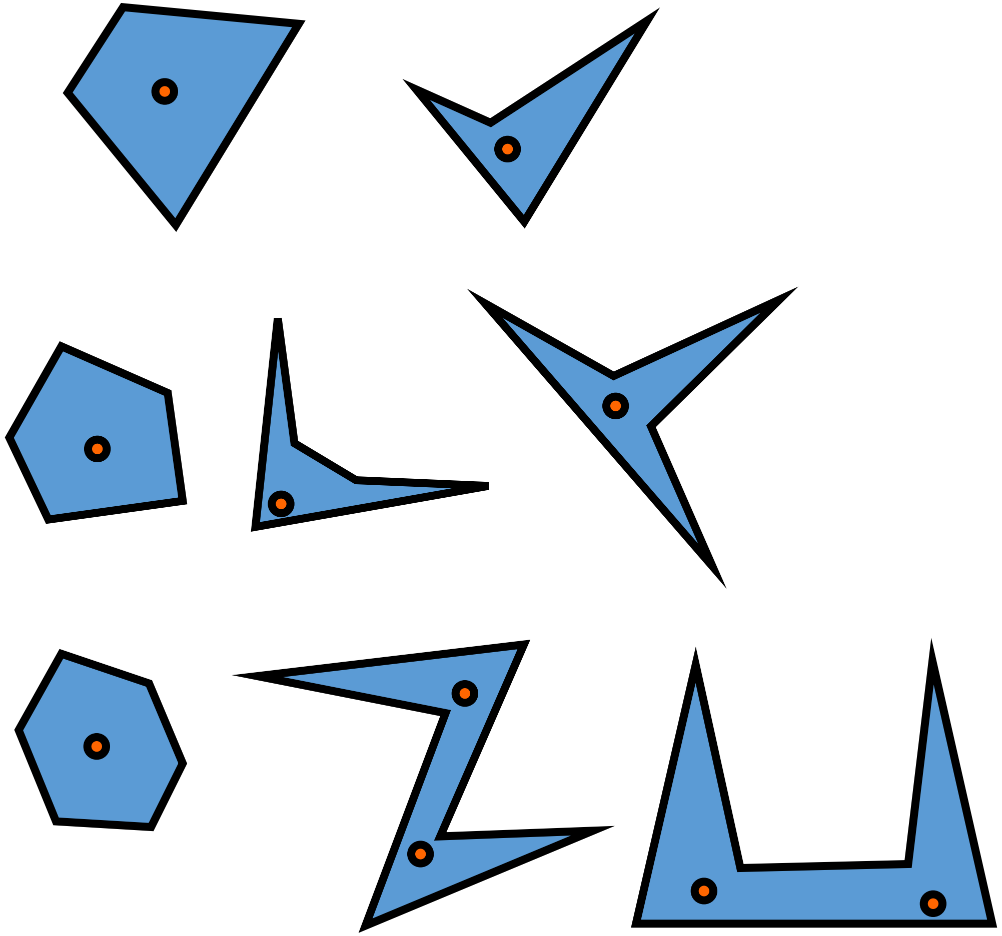






What is the maximum of the minimum number of lights needed to cover a 12 sided polygon?

$G(N) = ?$



$$1 \leq G(N) \leq N$$

$$G(3) = 1$$

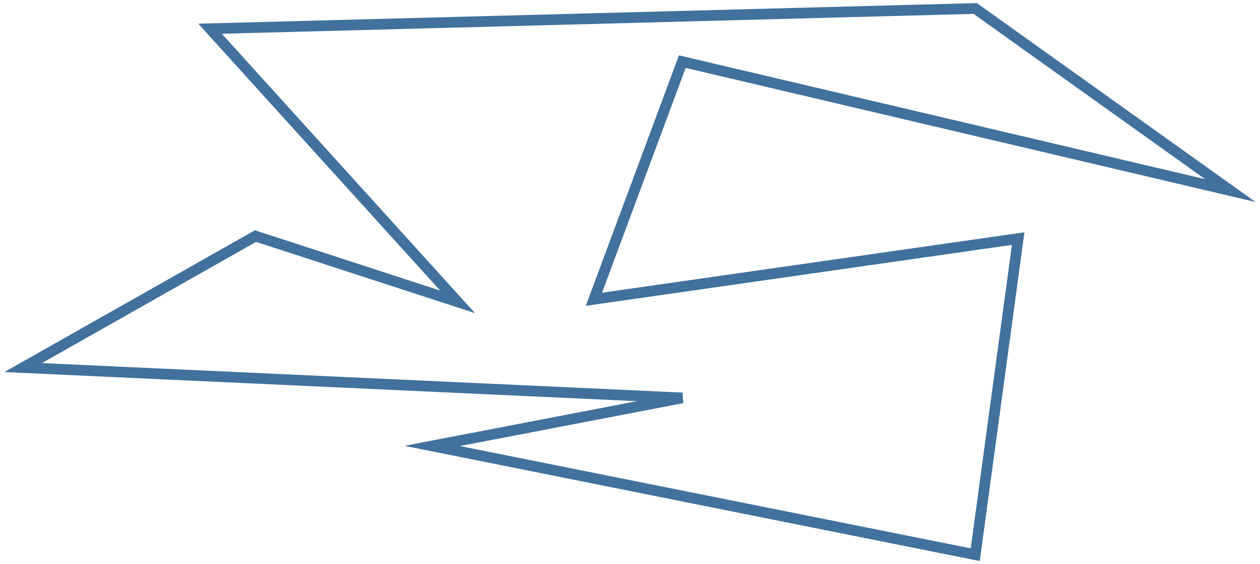
$$G(4) = 1$$

$$G(5) = 1$$

$$G(6) = 2$$

Maximum over minimum formulation: Quiz

- How many lights (cameras) needed (n=12)



Maximum over minimum formulation: Quiz

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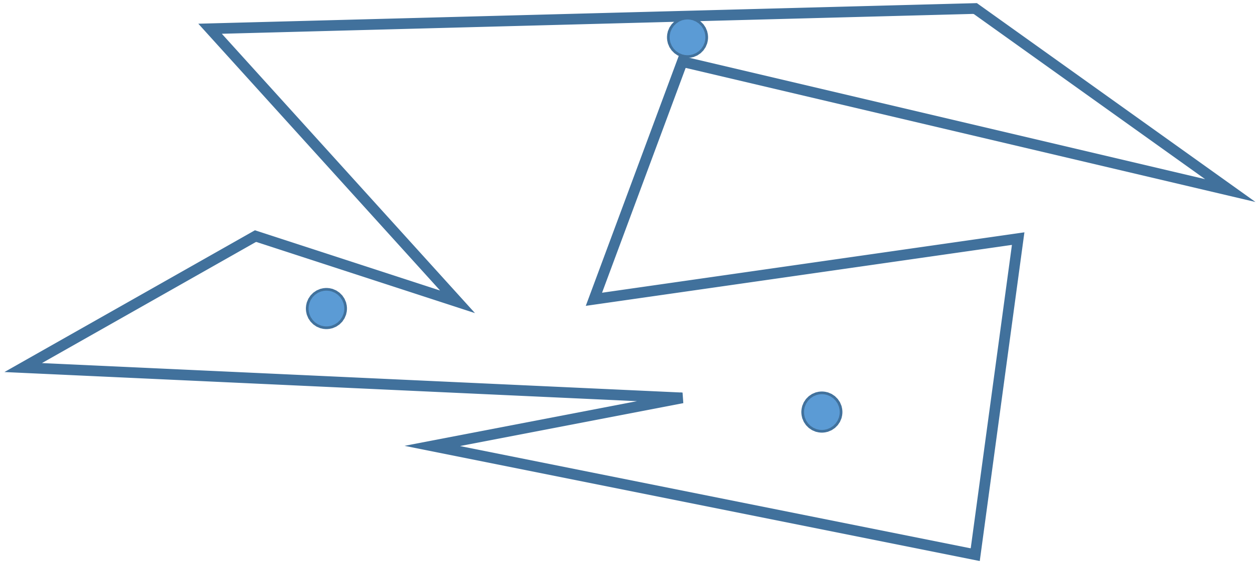

Maximum over minimum formulation: Quiz

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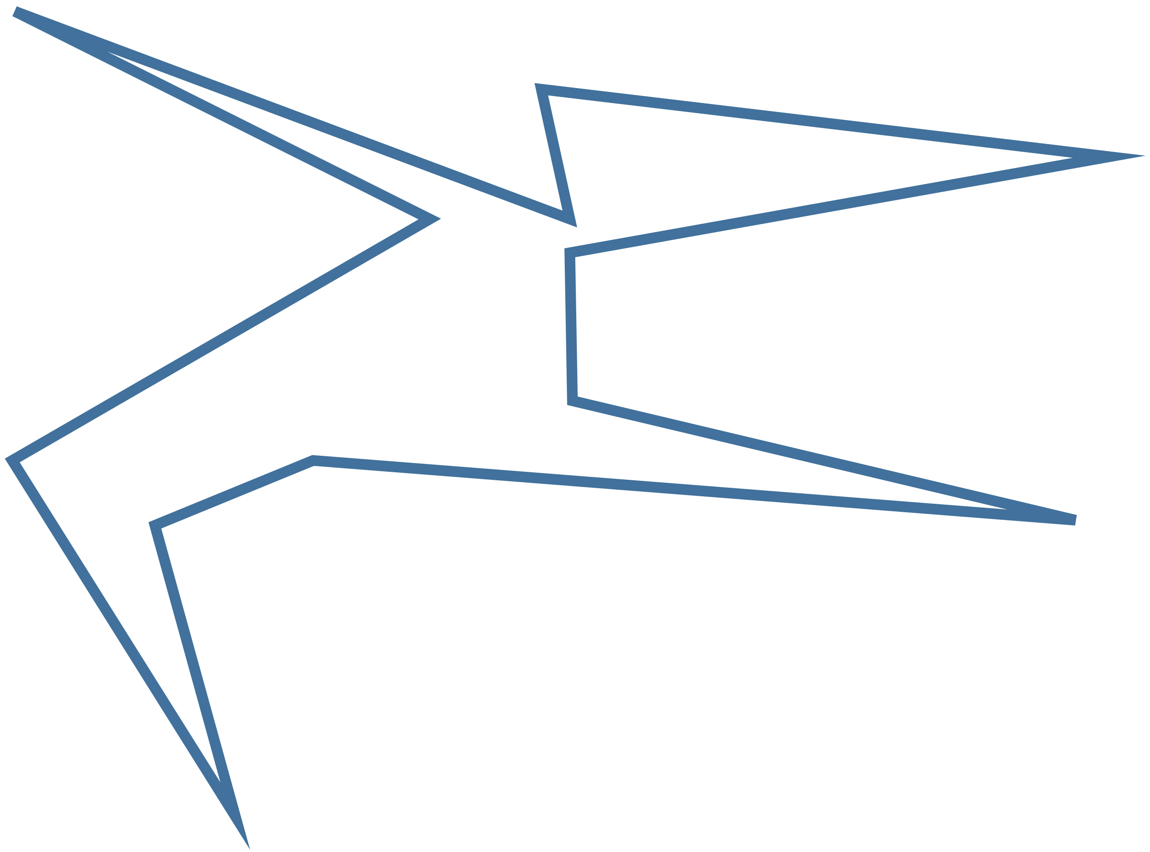

Maximum over minimum formulation: Quiz

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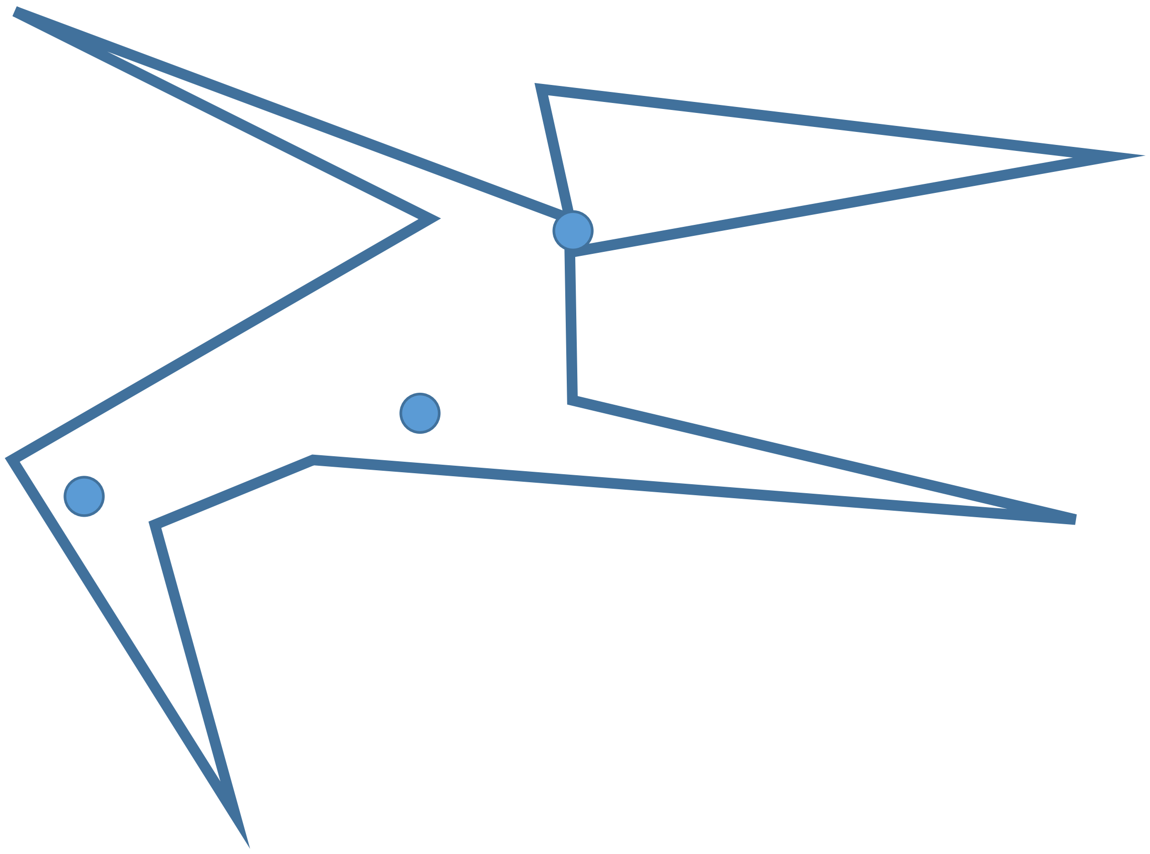
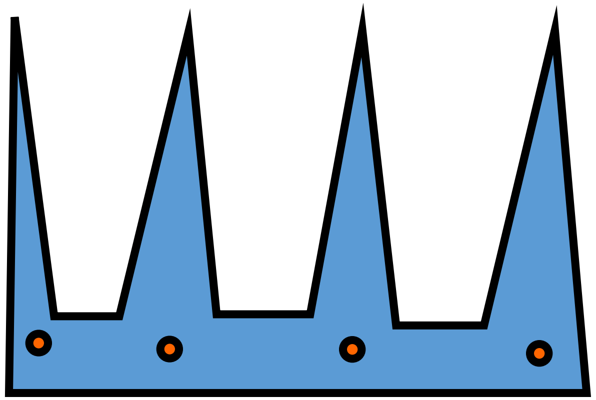


$G(N) = \ldots.$

- CHVATAL'S COMB
  - $G(12) = 4$



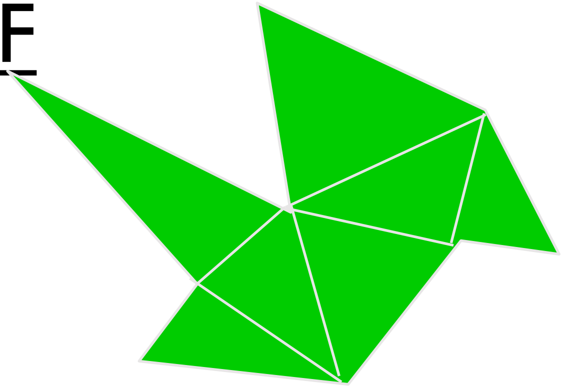
- CAN IT BE THAT $G(N) = \left\lfloor \frac{N}{3} \right\rfloor$?

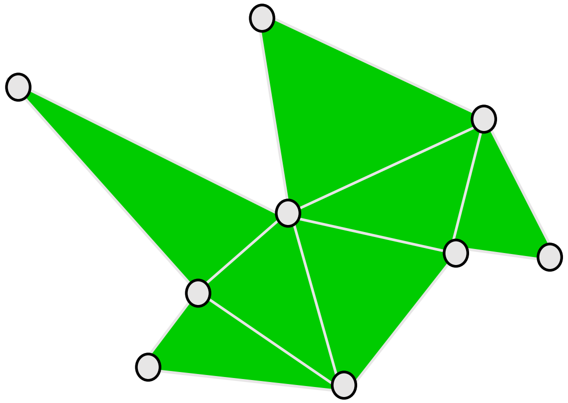
MAXIMUM OVER MINIMUM FORMULATION

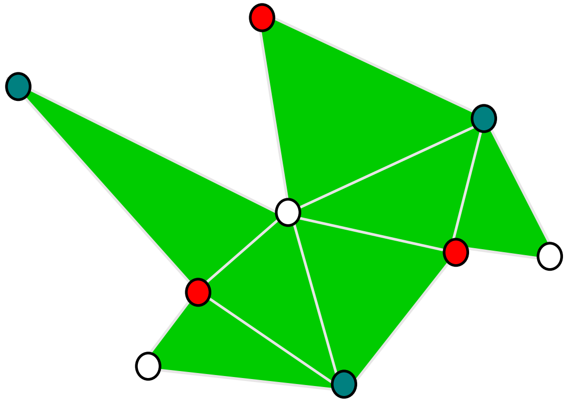
- **THEOREM** (ART GALLERY THEOREM). FOR A SIMPLE POLYGON WITH N VERTICES, $\lfloor n/3 \rfloor$ CAMERAS ARE OCCASIONALLY NECESSARY AND ALWAYS SUFFICIENT TO HAVE EVERY POINT IN THE POLYGON VISIBLE FROM AT LEAST ONE OF THE CAMERAS
  - Sufficiency of n
    - Certainly at least one camera is needed—lower bound on $G(n)$: $1 \leq G(n)$
    - An upper bound on $G(n)$: $G(n) \leq n$
  - The first proof that $G(n) = \lfloor n/3 \rfloor$ was due to Ghvatal (1975)
  - We will present Fiske's proof of sufficiency of $\lfloor n/3 \rfloor$ guards for any n-sided polygon

FISKE’ PROOF

- GIVEN ARBITRARY N-VERTEX P:
  - Triangulate P using diagonals
  - Color the vertices of triangulation graph G
  - G can be 3-colored (proof later)
  - Place lights at similarly colored nodes
  - Guaranteed to light the whole polygon P





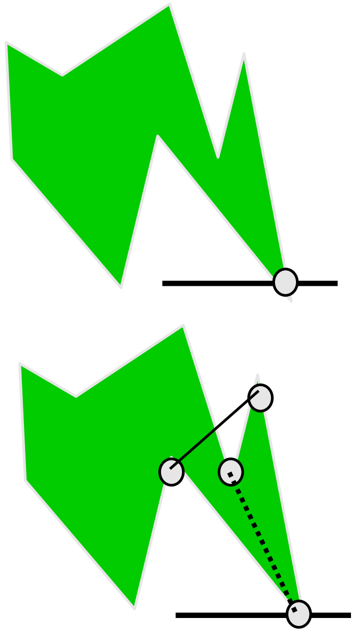


Triangulation Theory

- Existence of a Diagonal
- Properties of Triangulations
- Triangulation Dual
- 3-Coloring Proof

Triangulation Theory: Existence of a Diagonal

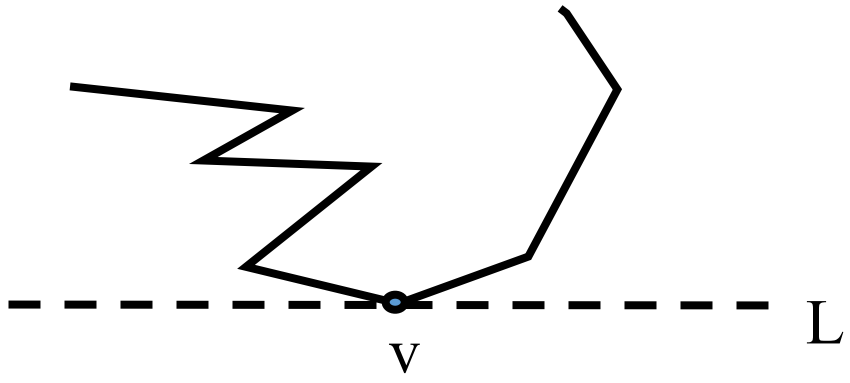
- Every polygon must have ≥ 1 *strictly* convex vertex (no collinearity)
- Every polygon of $n \geq 4$ vertices has a diagonal
- Every n-vertex polygon P may be partitioned into triangles by adding (≥ 0) diagonals [proof by induction using diagonals]



TRIANGULATION THEORY OF POLYGON

- LEMMA: EVERY POLYGON MUST HAVE AT LEAST ONE STRICTLY CONVEX VERTEX.

- PROOF:
  - If the edges of polygon oriented in a counter-clockwise traversal, then a convex vertex is a left turn, and reflex vertex is right turn and interior of the polygon is always to the left
  - Let L is the line through the lowest vertex v (y-coordinate)
    - The interior of the polygon must be above
    - The edges following v must be above L
    - The walker make the left turn at v, thus v is convex



Triangulation Theory: Properties

- Lemma: An internal diagonal exists between any two nonadjacent vertices of a polygon P if and only if P is convex polygon.
- Proof: The proof consists of two parts, both established by contradiction.

TRIANGULATION THEORY: PROPERTIES

- **THEOREM:** THE NUMBER OF DISTINCT TRIANGULATIONS OF A CONVEX POLYGON WITH n VERTICES IS THE CATALAN NUMBER

$$C_n = \frac{1}{n-1}\binom{2(n-2)}{n-2}$$

Proof: Let P_n be a convex polygon with vertices labeled from 1 to n counterclockwise. Let τ_n be the set of triangulation of P_n with t_n elements.

Let ϕ be the map from τ_n to τ_{n-1}

Triangulation Theory: Properties

- Theorem: Let P be a simple polygon with n vertices. The number of triangulations of P is between 1 and C_n.

Brute Force Triangulation

- **Theorem:** Every polygon P of n vertices can be partitioned into triangle by the addition of(zero or more) diagonals.
  - Complexity of diagonal-based algorithm:
    - $O(n^2)$ - # of diagonal candidates
    - $O(n)$ testing each of neighborhoods
    - Repeating this $O(n^3)$ computation for each of the n-3 diagonals yields $O(n^4)$

Triangulation Theory

- Every polygon P of N vertices can be partitioned into triangles by the addition of zero or more diagonals. (Induction proof)
  - Base case: N = 3 (triangle)
  - Assumption: Let it be true for < N sided polygon
  - Any N sided polygon can be partitioned into two polygons of less then N sides each by adding a diagonal, each of which can be partitioned by using premise 2 above
  - Thus, it is true for all N.

Triangulation Theory

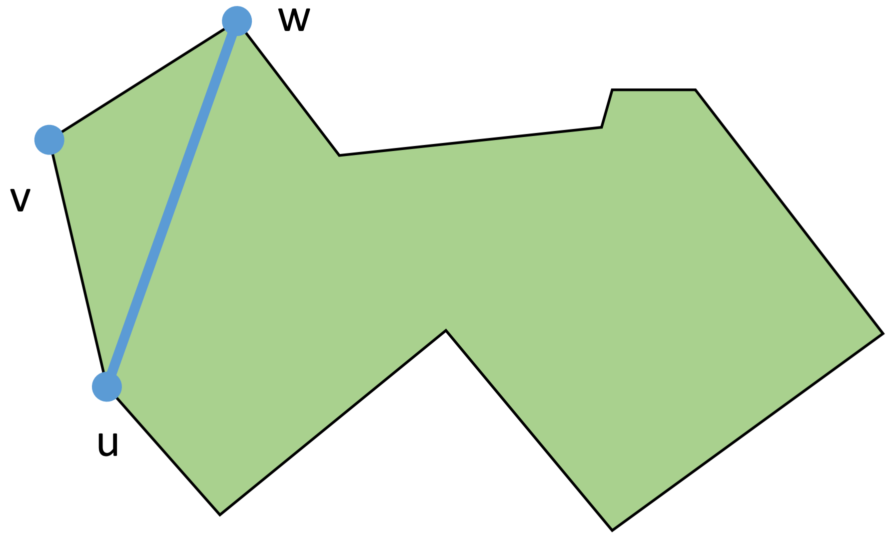
- Any diagonal cuts P into two simple subpolygons P_1 and P_2
- Let m_1 be the number of vertices of P_1 and m_2 the number of vertices of P_2
- Both m_1 and m_2 must be smaller than n
  - So by induction P_1 and P_2 can be triangulated
  - Hence, P can be triangulated as well

Triangulation Theory

- ANY TRIANGULATION OF P CONSISTS OF $n-2$ TRIANGLES.
  - Consider an arbitrary diagonal in some triangulation T_P
  - The diagonal cuts P into two subpolygons with m_1 and m_2 vertices
  - Every vertex of P occurs in exact1y one of the two subpolygons, except for the vertices defining the diagonal, which occur in both subpolygons. Hence, $m_1 + m_2 = n + 2$.
  - By induction, any triangulation of P_i consists of $m_i - 2$ triangles, which implies that T_P consists of $(m_1 - 2) + (m_2 - 2) = n - 2$ triangles.

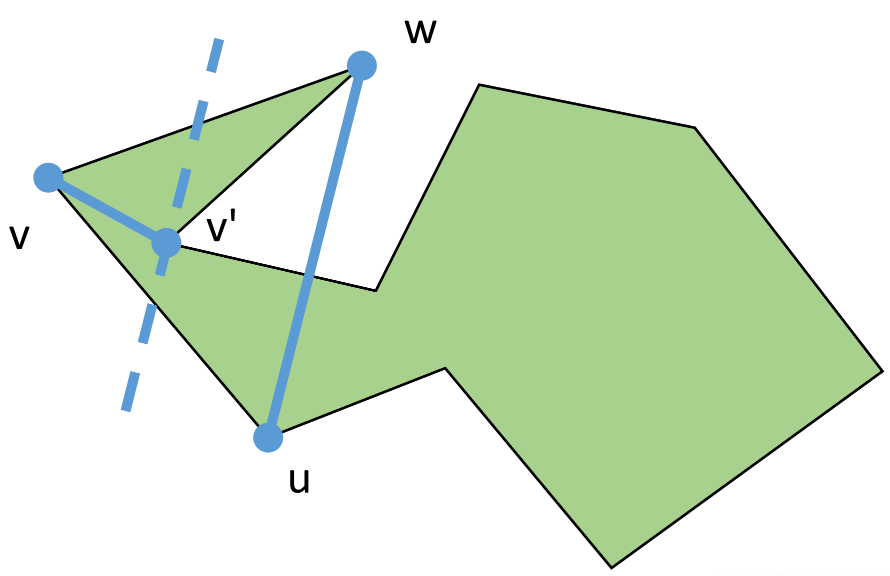
Existence of a Diagonal

- Lemma: every polygon P with more than three vertices has a diagonal
- Proof:
  - Let v be the leftmost vertex of P.
  - Let u and w be its neighbors.
  - If uw is a diagonal we are done



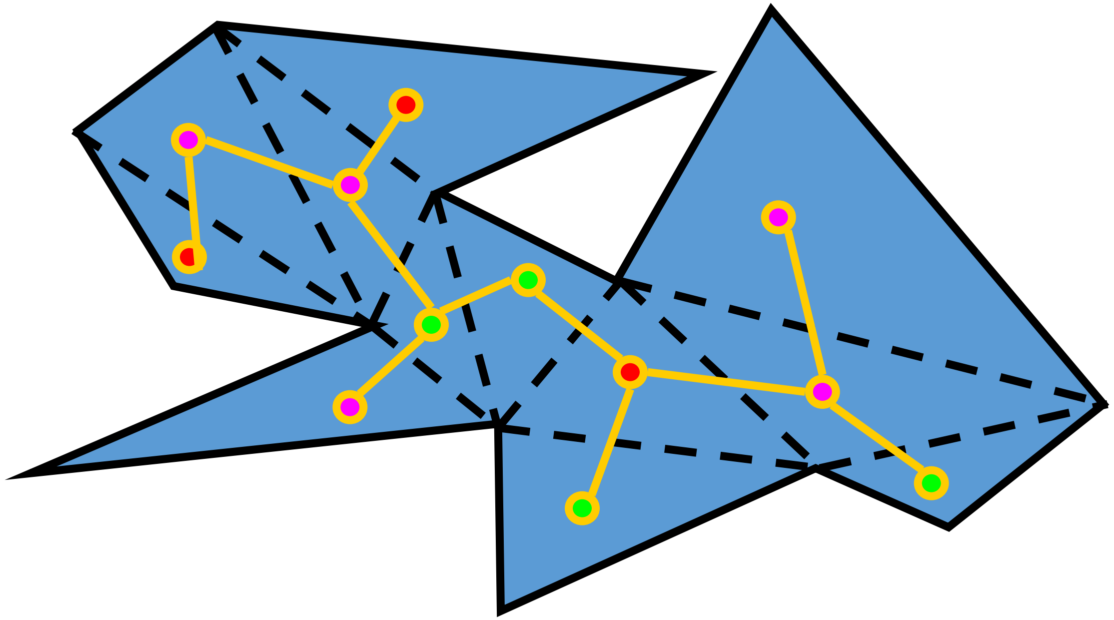
Existence of a Diagonal

- If uw is not a diagonal, let v' be the vertex in triangle (u, v, w) that is farthest from uw
- Then vv' is a diagonal: if an edge was crossing it, one of its endpoints would be farther from uw and inside (u, v, w)



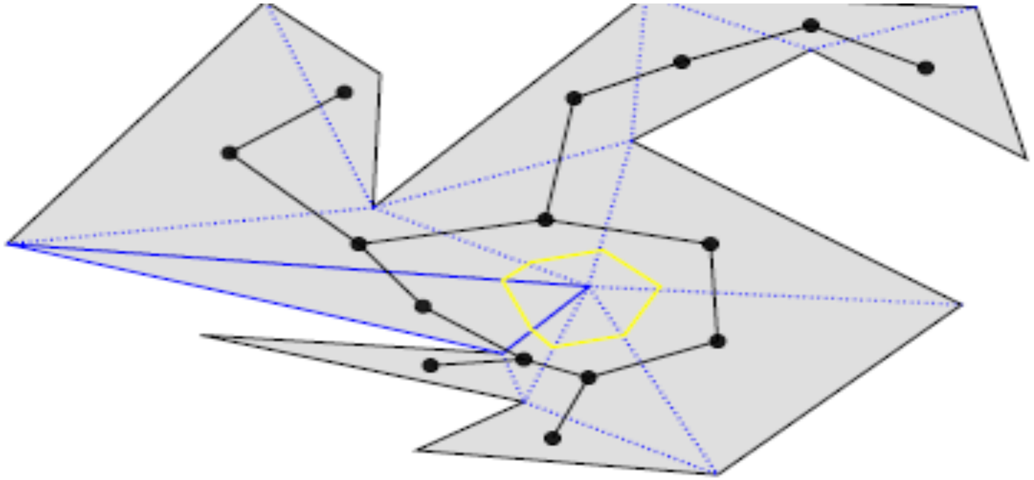
Triangulation Dual

- The dual T of a triangulation is a tree, with each node of degree at most three.
- Dual graph: each face gives a node; two nodes are connected if the faces are adjacent



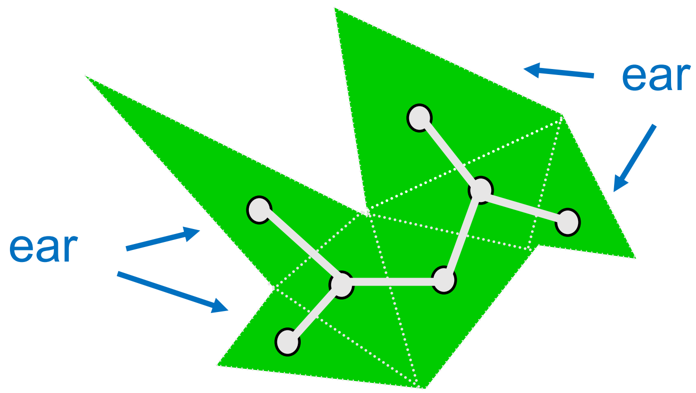
Properties of Triangulations

- Proof:
  - The degree three is immediate from the fact that every triangle have three sides.
  - If there is a cycle C in T it is easy to verify that...
  - There must be a vertex inside the polygon...



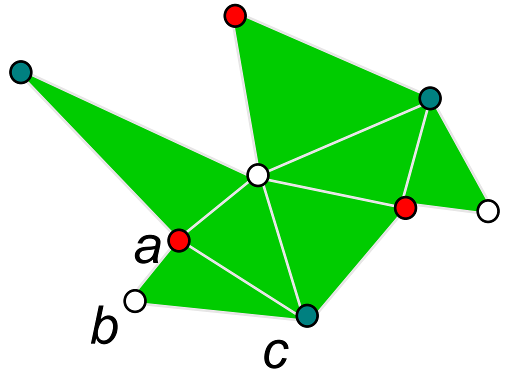
MEISTER'S TWO EARS THEOREM

- THREE CONSECUTIVE VERTICES, A, B, C FORM AN EAR IF AC IS A DIAGONAL
- "2-EARS" THEOREM: EVERY POLYGON OF $n \geq 4$ VERTICES HAS AT LEAST 2 NON-OVERLAPPING EARS.
  - The triangulation dual has at least 2 nodes
  - A tree of more than 2 nodes has at least 2 leaf nodes
  - Each leaf node corresponds to an ear.



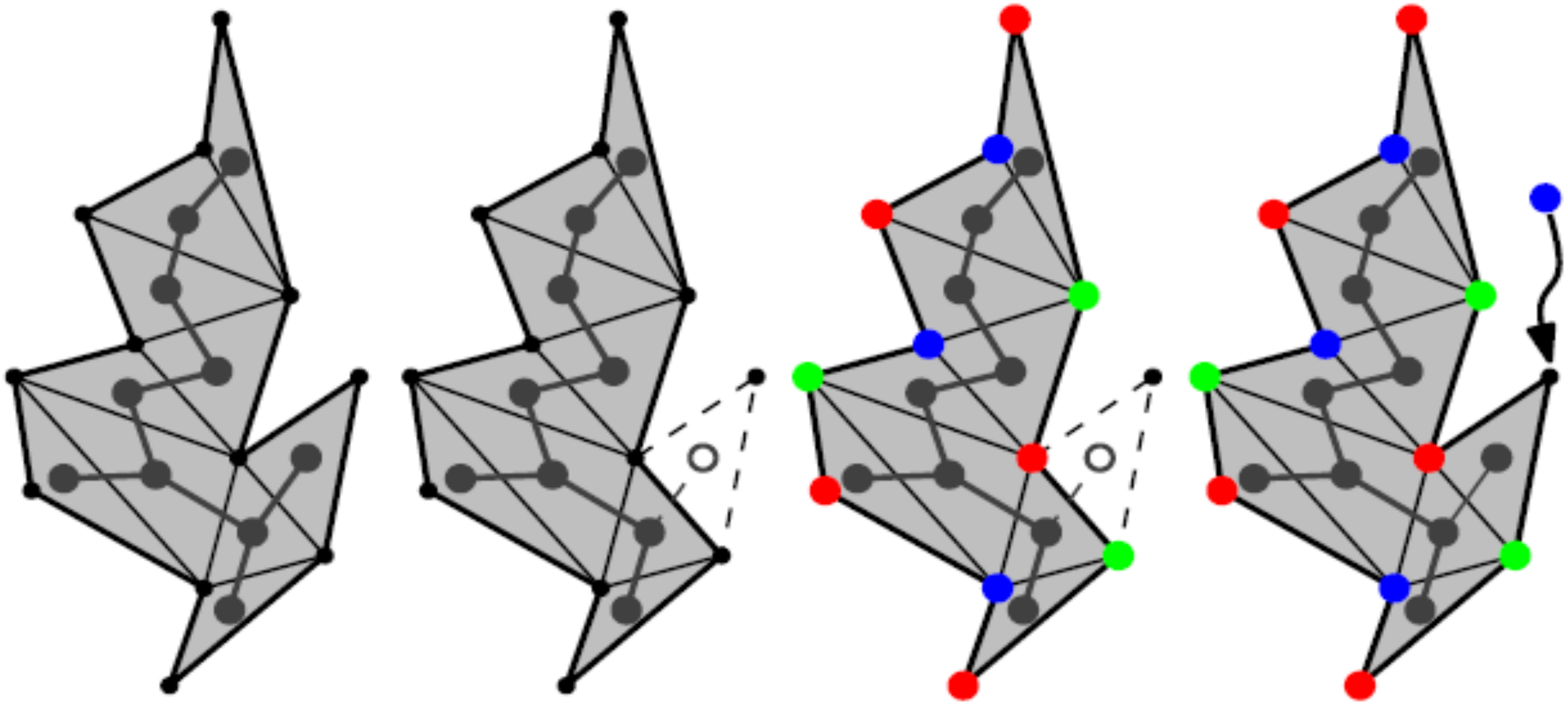
TRIANGULATION THEORY: 3-COLORING

- "2-EARS" THEOREM CAN BE USED TO EASILY PROVE 3-COLORABILITY OF TRIANGULATION GRAPHS
  - Induction on n
    - Base case: $n = 3$
    - For $n \geq 4$: 2-ears theorem guarantees that an ear abc exists apply inductive hypothesis to polygon P' without ear "reattaching" ear adds back in one vertex (w.l.o.g. b) color b whatever color a and c don't use result is a 3-coloring of P



FISKE' PROOF

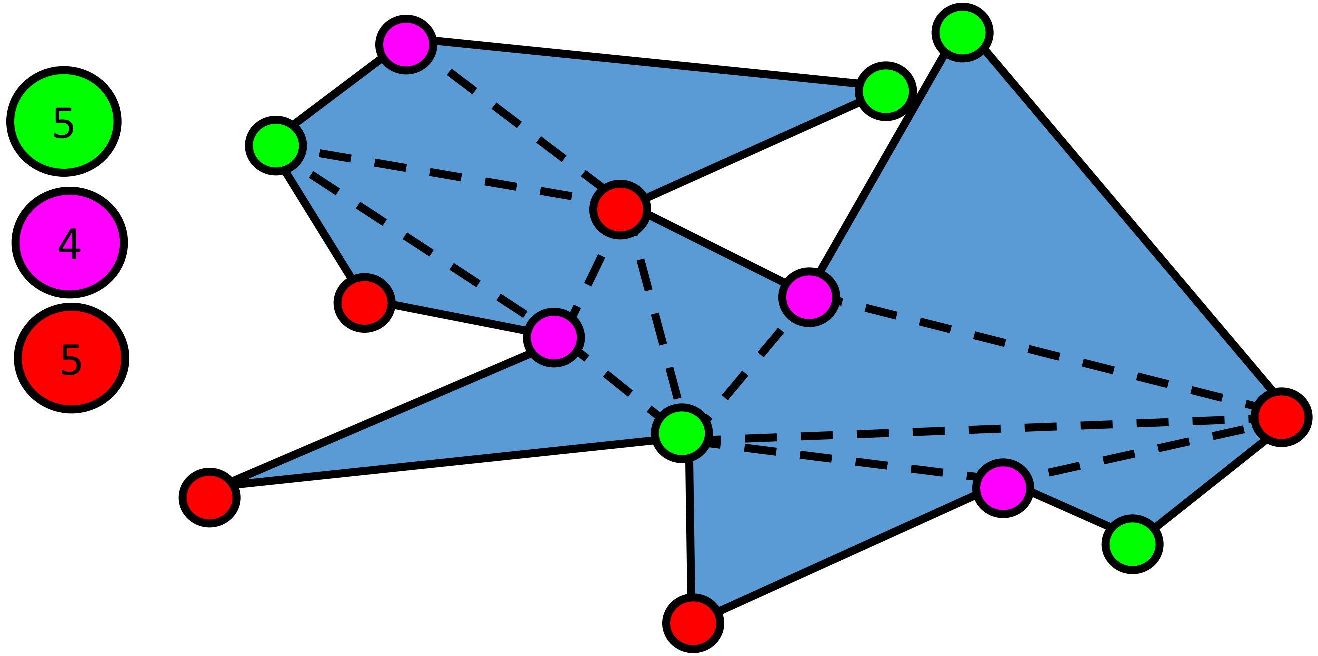
- 3 COLORS



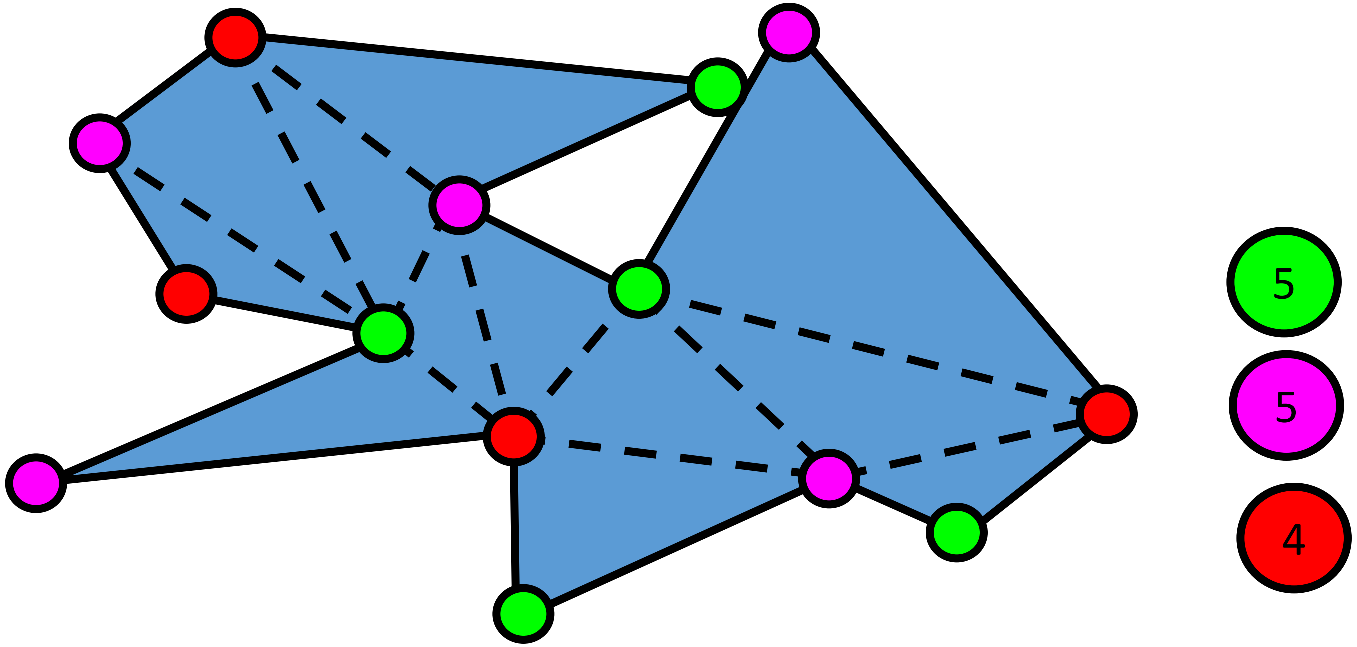
Fiske' proof

- Apply the "pigeon-hole principle" – If n objects are placed into k pigeon holes, then at least one hole must contain no more than n/k objects

3 COLORS SUFFICE…



3 COLORS SUFFICE…



Pigeon Hole Principle

- 3 holes (colors) and 14 pigeons (vertices) to go into them.
- There will always be one hole with less or equal to 14/3 pigeons
- Generalizing: for 3 colors and N vertices there will be a color that is used at most N/3 times. Place the light at those colors.