# COT 4521: Introduction to Computational Geometry

## The Art Gallery Problem

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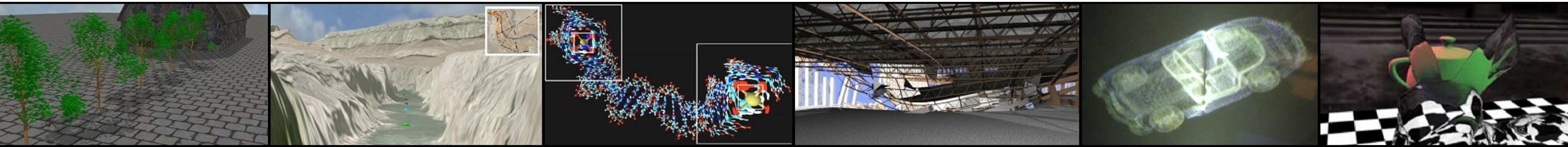
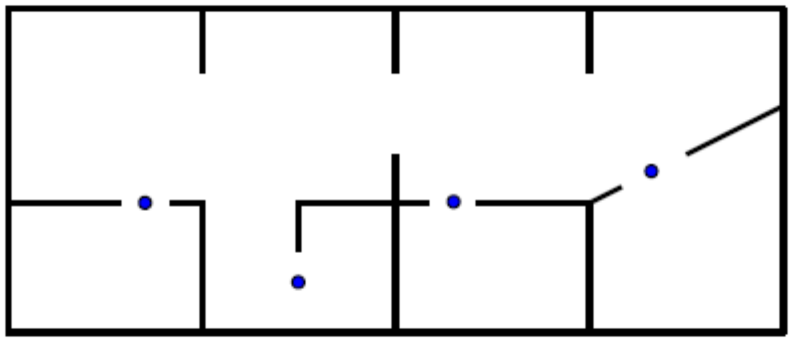
Some slides from Valentina Korzhova



# The Art Gallery Problem

- The Art Gallery Problem: how many cameras we need to guard a given gallery so that every point is seen, and how we decide to place them?



- In geometry terminology: How many points are needed in a simple polygon with n vertices so that every point in the polygon is seen?

# The Art Gallery Problem

- This problem was posed by Victor Klee in 1973
- A guard of the gallery corresponds to a point on the polygonomial floor plan.
- Guards can see in every direction, with a full range of visibility
- The optimization problem is computationally difficult

# The Art Gallery Problems

- In a simple polygon $P$, a point $X$ is said to be *visible* from a point Y (or, vice versa) whenever the line segment $XY$ does not intersect with the exterior of $P$

$$P: XY \subseteq P$$

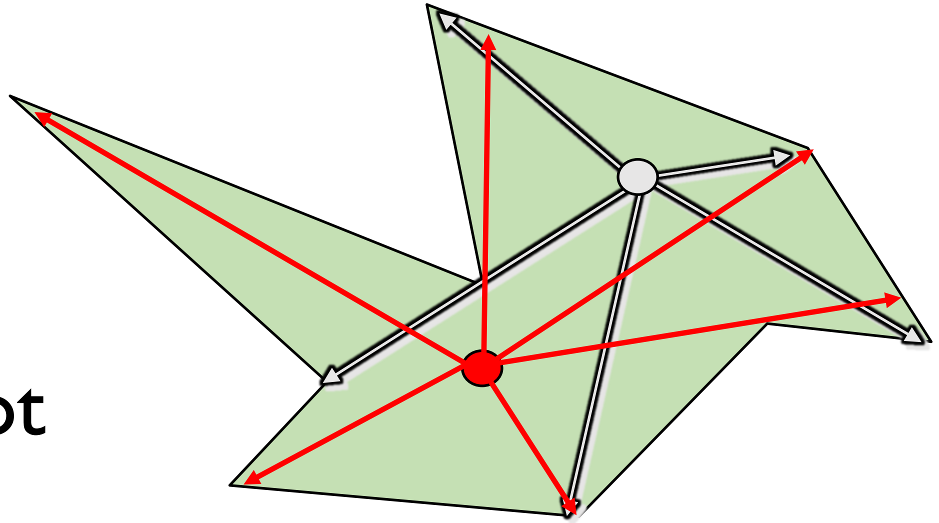
- Vertices of $P$ are considered non-blockers of visibility

- Visibility: $2\pi$ range

# The Art Gallery Problems

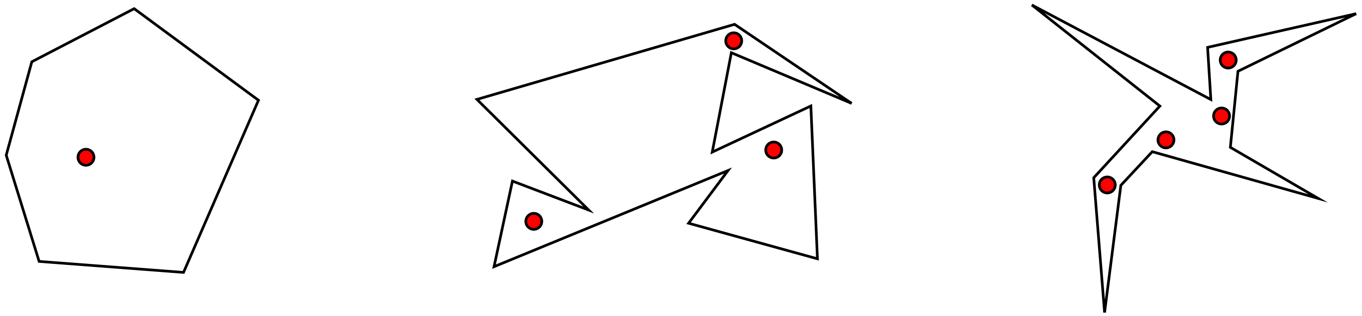
- Consider a room whose floor is polygon of n vertices, how many point lights (cameras) are needed to light the whole room?

- A set of lights is said to cover a polygon if every point in the polygon is lighted.
  - Assume the lights themselves are not sources of shadows



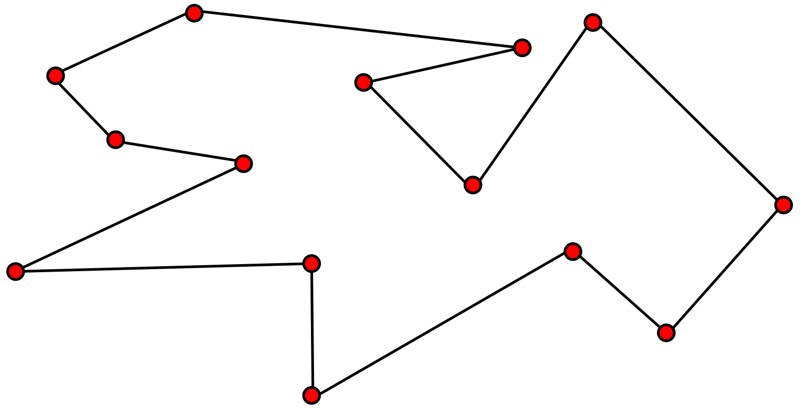
# Guarding a simple polygon

- Given a simple polygon P with n vertices, find the *minimum number of guards* required for every point of P to be visible from some guard
- Assume that every guard can view 360 degrees around it
- How many lights we need to place to guard a simple polygon?
  - One guard is both necessary and sufficient for any convex polygon



# Sufficient number of guards for any polygon of n vertices

- How many guards are sufficient to cover any n-vertex simple polygon?
  - By placing a guard at every vertex, any n-vertex simple polygon can be trivially guarded with n guards — loose upper bound



# Maximum over minimum formulation
# Formal definition

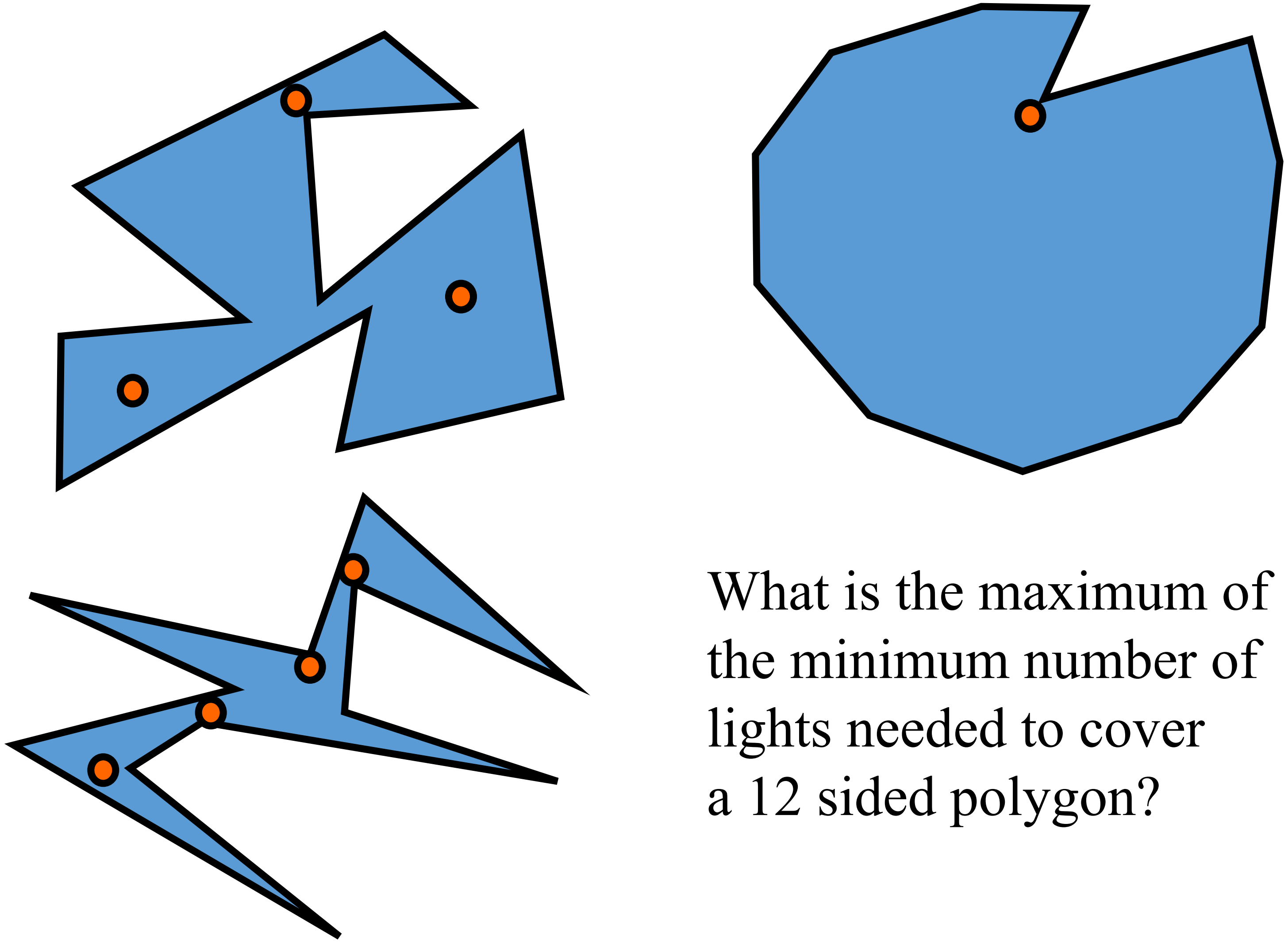
- Let $g(P_N)$ be the smallest number of lights need to cover a particular polygon of N sides.

$$\boxed{g(P_N) = \min_S |\{S : S \text{ covers } P\}|}$$

  - S is the set of points where the lights are located
- What is the max of $g(P_N)$ over all $P_N$?

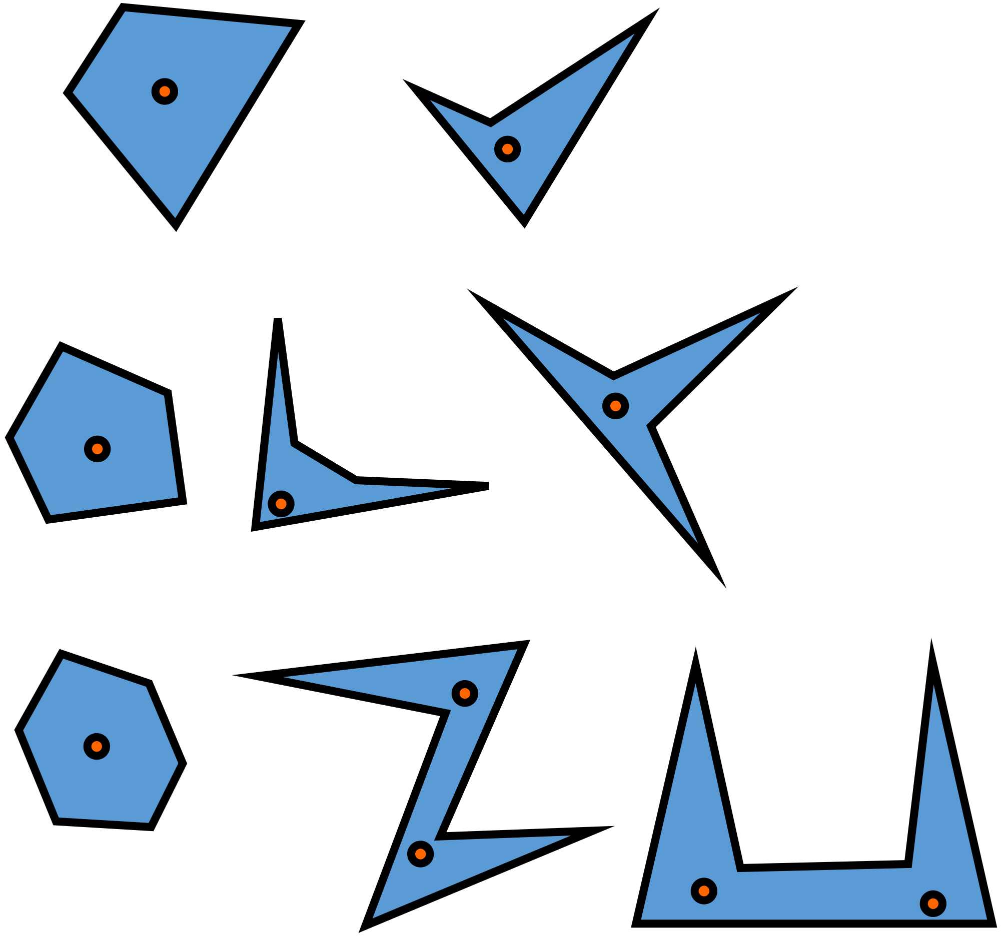
$$\boxed{G(N) = \max_{P_N} g(P_N)}$$

# How many lights are needed?



What is the maximum of the minimum number of lights needed to cover a 12 sided polygon?

# $\underline{G(N) = ?}$



$$1 \le G(N) \le N$$

$$G(3) = 1$$

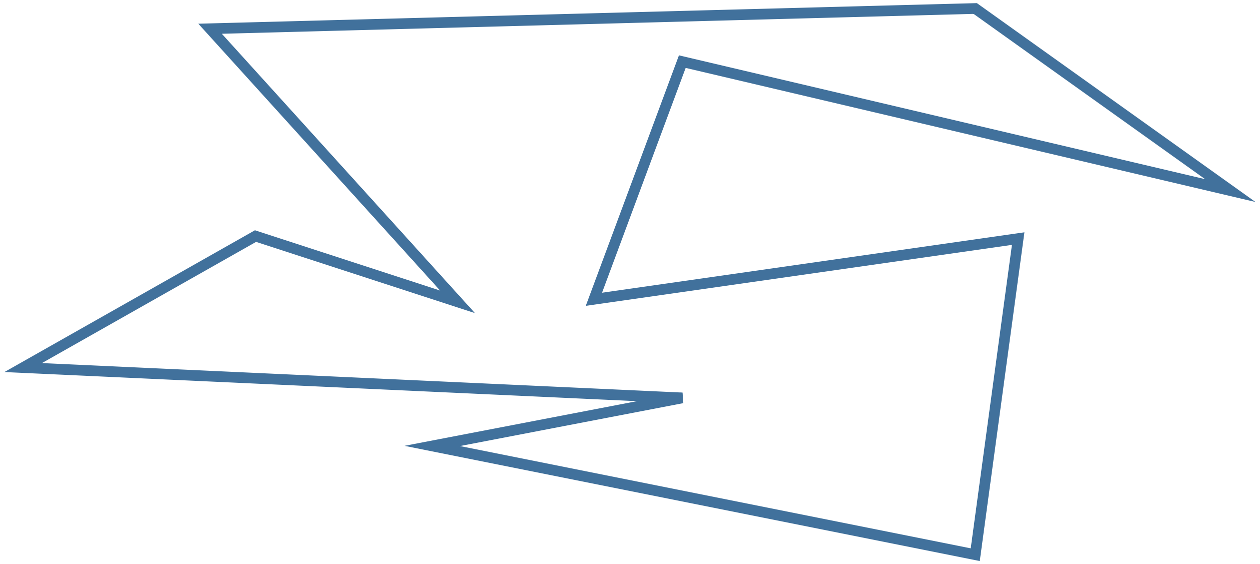
$$G(4) = 1$$

$$G(5) = 1$$

$$G(6) = 2$$

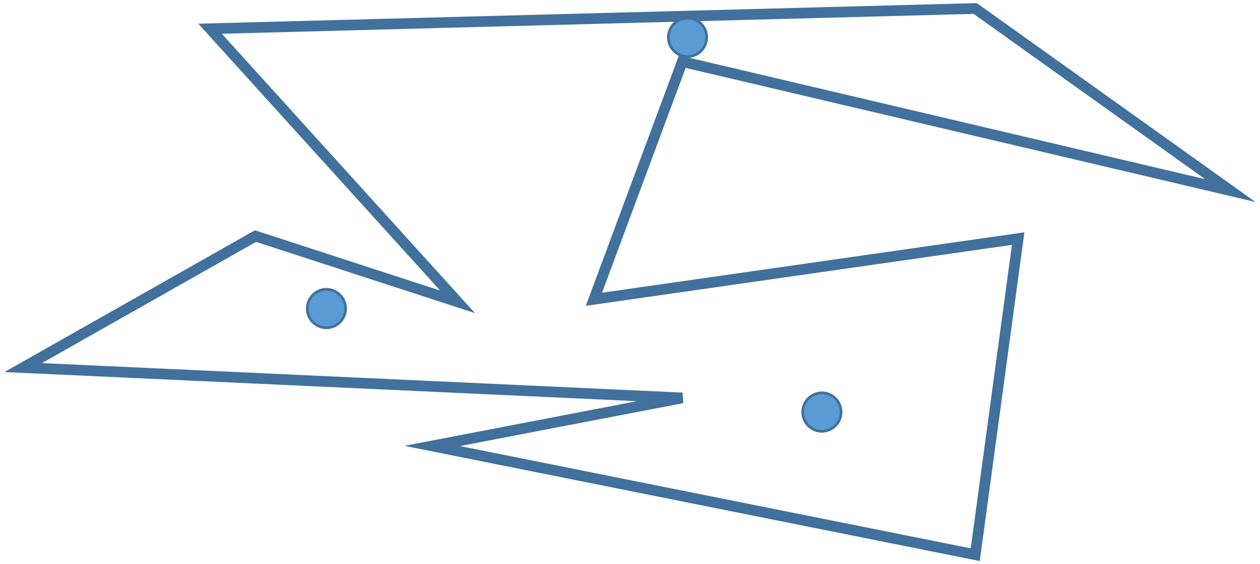
# Maximum over minimum formulation

- How many lights (cameras) needed (n=12)



# Maximum over minimum formulation: Quiz

- How many lights (cameras) needed (n=12)



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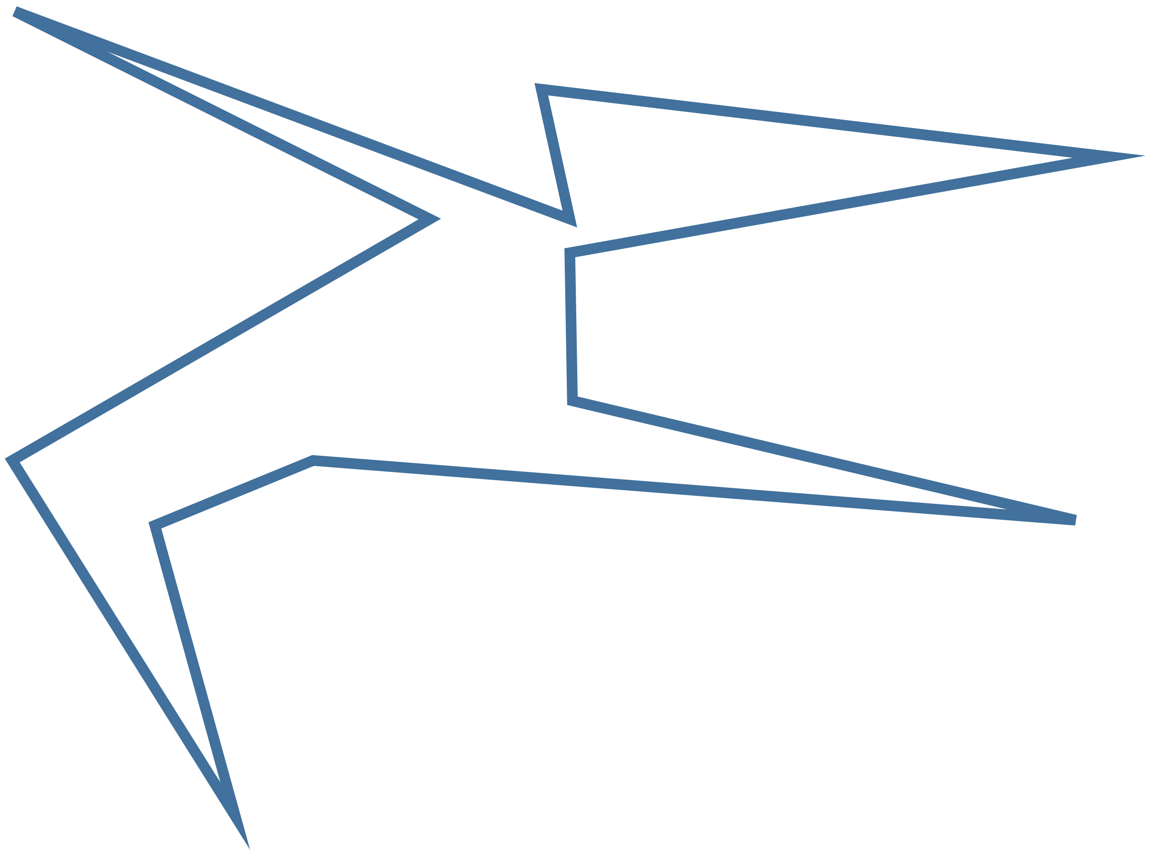

# Maximum over minimum formulation: Quiz

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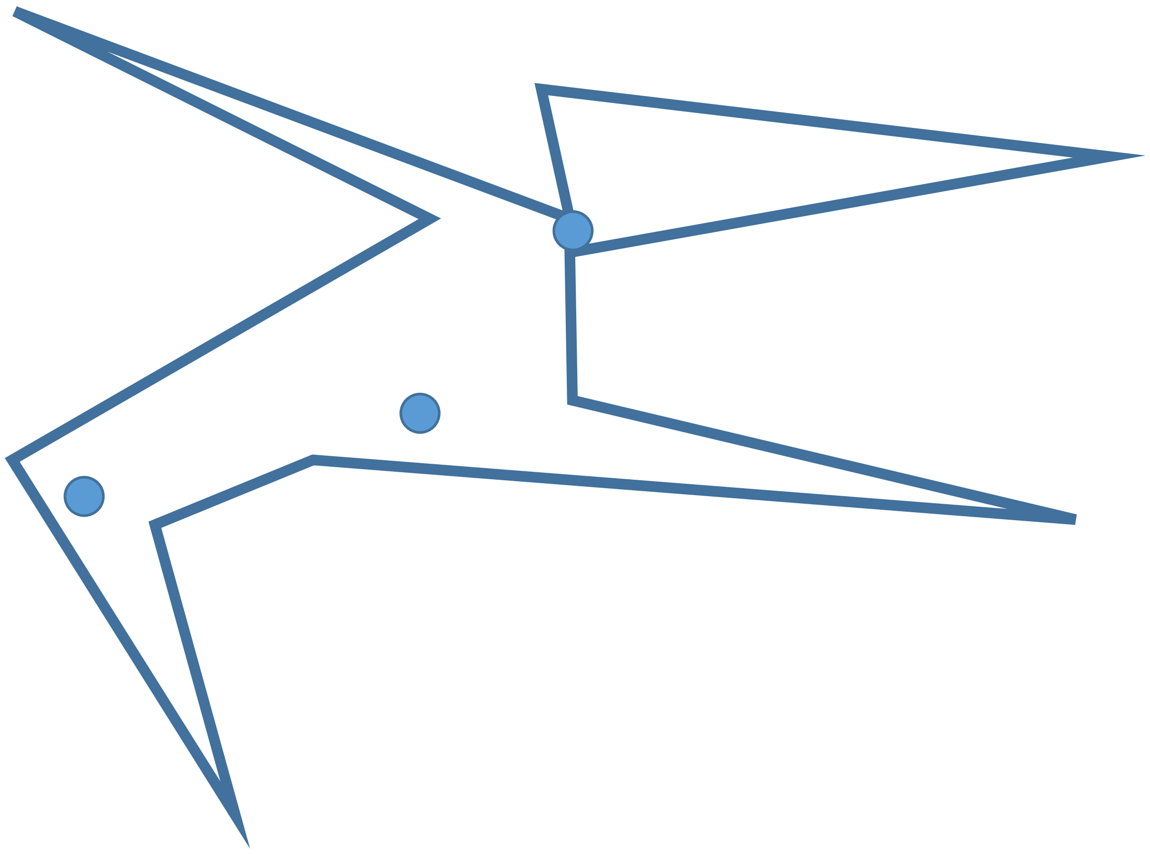
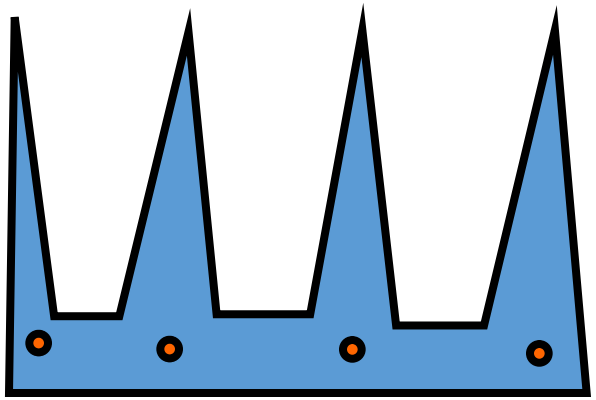


# $G(N) = \ldots.$

- Chvatal's comb
  - $G(12) = 4$

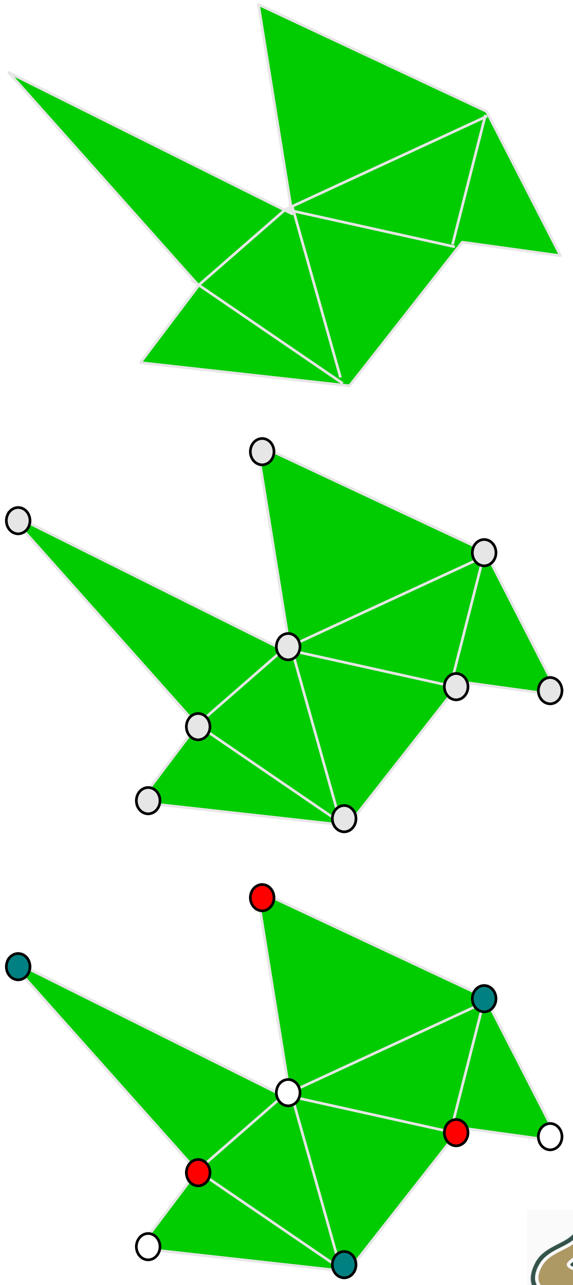


- Can it be that $G(N) = \left\lfloor \frac{N}{3} \right\rfloor$?

# Maximum over minimum formulation

- **Theorem** (Art Gallery Theorem). For a simple polygon with n vertices, $\lfloor n/3 \rfloor$ cameras are occasionally necessary and always sufficient to have every point in the polygon visible from at least one of the cameras
  - Sufficiency of n
    - Certainly at least one camera is needed—lower bound on $G(n)$: $1 \leq G(n)$
    - An upper bound on $G(n)$: $G(n) \leq n$
  - The first proof that $G(n) = \lfloor n/3 \rfloor$ was due to Ghvatal (1975)
  - We will present Fiske's proof of sufficiency of $\lfloor n/3 \rfloor$ guards for any n-sided polygon

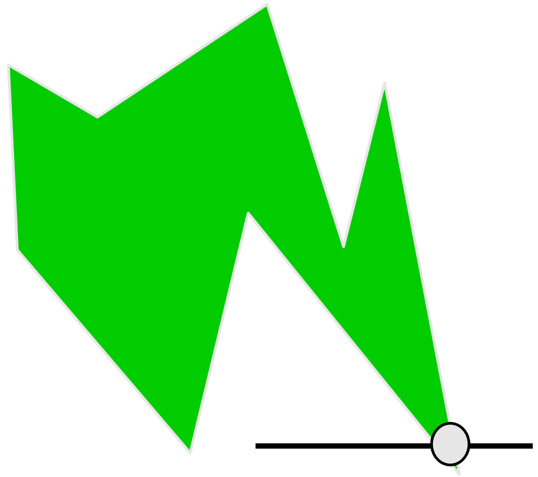
# FISKE' PROOF

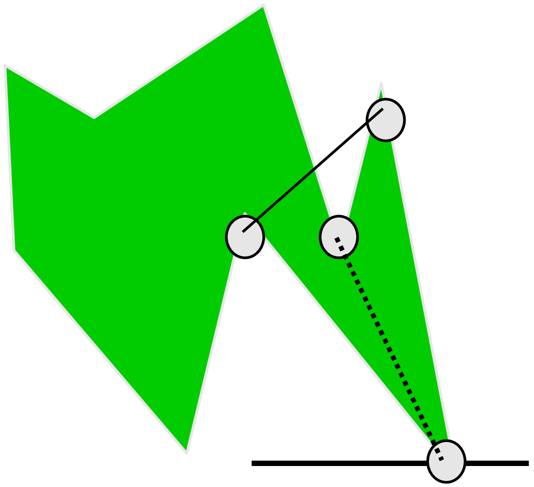


- GIVEN ARBITRARY N-VERTEX P:
  - Triangulate P using diagonals
  - Color the vertices of triangulation graph G
  - G can be 3-colored
  - Place lights at same colored nodes
  - Guaranteed to light the whole polygon P

# Triangulation Theory: Existence of a Diagonal

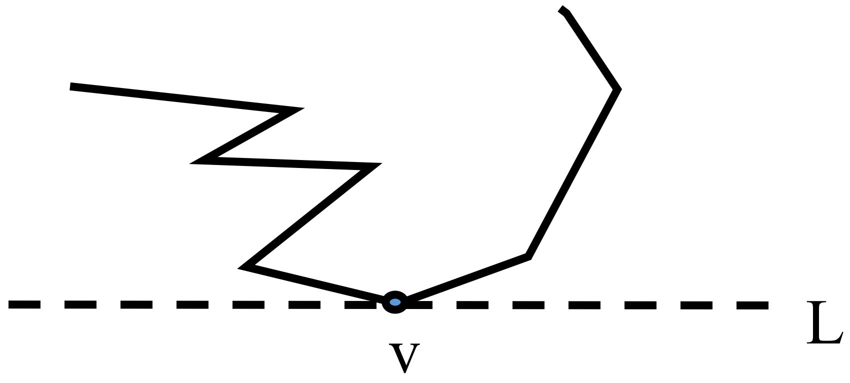
- Every polygon must have $\geq \ 1$ *strictly* convex vertex (no collinearity)
- Every polygon of $n \ \geq \ 4$ vertices has a diagonal
- Every n-vertex polygon P may be partitioned into triangles by adding $(\geq 0)$ diagonals [proof by induction using diagonals]





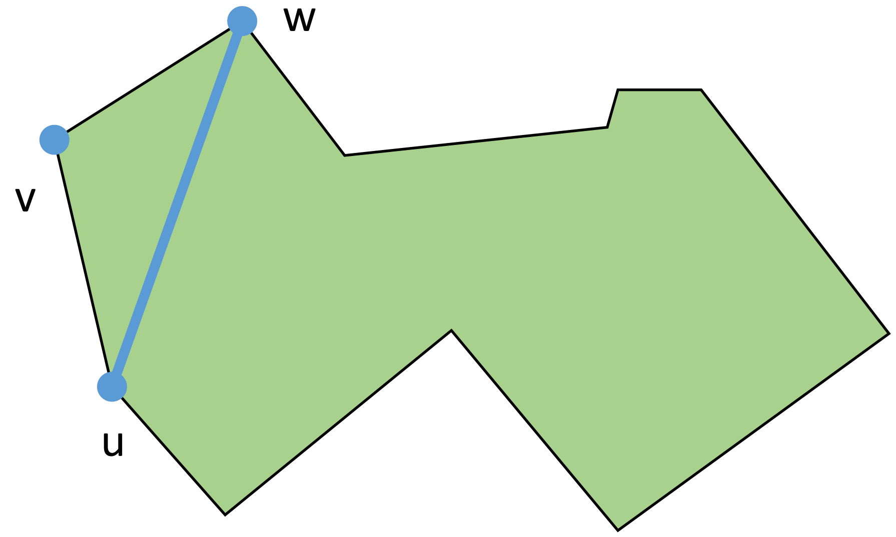
# TRIANGULATION THEORY OF POLYGON

- LEMMA: EVERY POLYGON MUST HAVE AT LEAST ONE STRICTLY CONVEX VERTEX.
- PROOF:
  - If the edges of polygon oriented in a counter-clockwise traversal, then a convex vertex is a left turn, and reflex vertex is right turn and interior of the polygon is always to the left
  - Let L is the line through the lowest vertex v (y-coordinate)
    - The interior of the polygon must be above
    - The edges following v must be above L
    - The walker make the left turn at v, thus v is convex



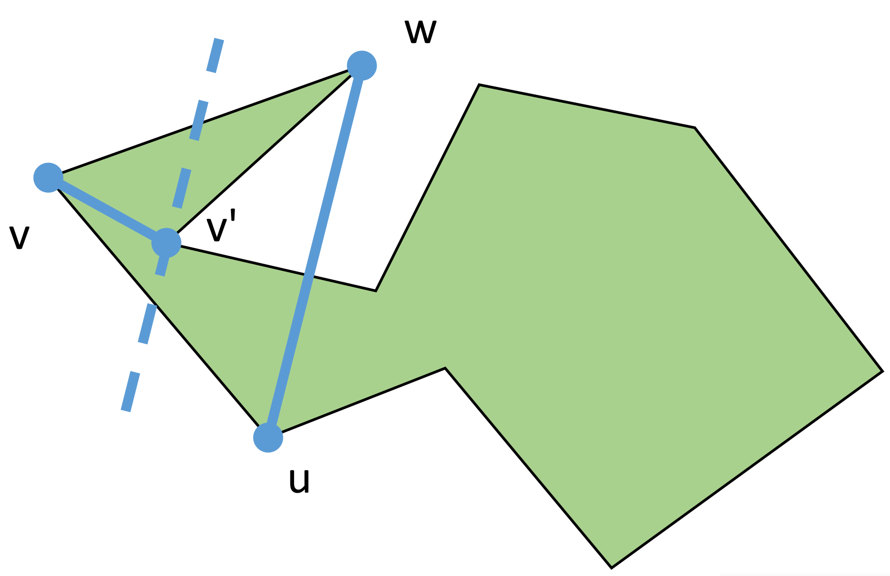
# Existence of a Diagonal

- Lemma: every polygon P with more than three vertices has a diagonal
- Proof:
  - Let $v$ be the leftmost vertex of P.
  - Let $u$ and $w$ be its neighbors.
  - If $uw$ is a diagonal we are done



# Existence of a Diagonal

- If $uw$ is not a diagonal, let $v'$ be the vertex in triangle $(u, v, w)$ that is farthest from $uw$
- Then $vv'$ is a diagonal: if an edge was crossing it, one of its endpoints would be farther from $uw$ and inside $(u, v, w)$



# Triangulation Theory: Properties

- Lemma: An internal diagonal exists between any two nonadjacent vertices of a polygon P if and only if P is convex polygon.
- Proof: The proof consists of two parts, both established by contradiction.

# TRIANGULATION THEORY: PROPERTIES

- **THEOREM:** THE NUMBER OF DISTINCT TRIANGULATIONS OF A CONVEX POLYGON WITH $n$ VERTICES IS THE CATALAN NUMBER

$$C_n = \frac{1}{n-1}\binom{2(n-2)}{n-2}$$

Proof: Let $P_n$ be a convex polygon with vertices labeled from 1 to $n$ counterclockwise. Let $\tau_n$ be the set of triangulation of $P_n$ with $t_n$ elements.

Let $\phi$ be the map from $\tau_n$ to $\tau_{n-1}$

# Triangulation Theory: Properties

- Theorem: Let P be a simple polygon with n vertices. The number of triangulations of P is between 1 and $C_n$.

# Brute Force Triangulation

- **Theorem:** Every polygon P of n vertices can be partitioned into triangle by the addition of (zero or more) diagonals.
  - Complexity of diagonal-based algorithm:
    - $O(n^2)$ - # of diagonal candidates
    - $O(n)$ testing each of neighborhoods
    - Repeating this $O(n^3)$ computation for each of the n-3 diagonals yields $O(n^4)$

# Triangulation Theory

- Every polygon P of N vertices can be partitioned into triangles by the addition of zero or more diagonals. (Induction proof)
  - Base case: N = 3 (triangle)
  - Assumption: Let it be true for < N sided polygon
  - Any N sided polygon can be partitioned into two polygons of less then N sides each by adding a diagonal, each of which can be partitioned by using premise 2 above
  - Thus, it is true for all N.

# TRIANGULATION THEORY

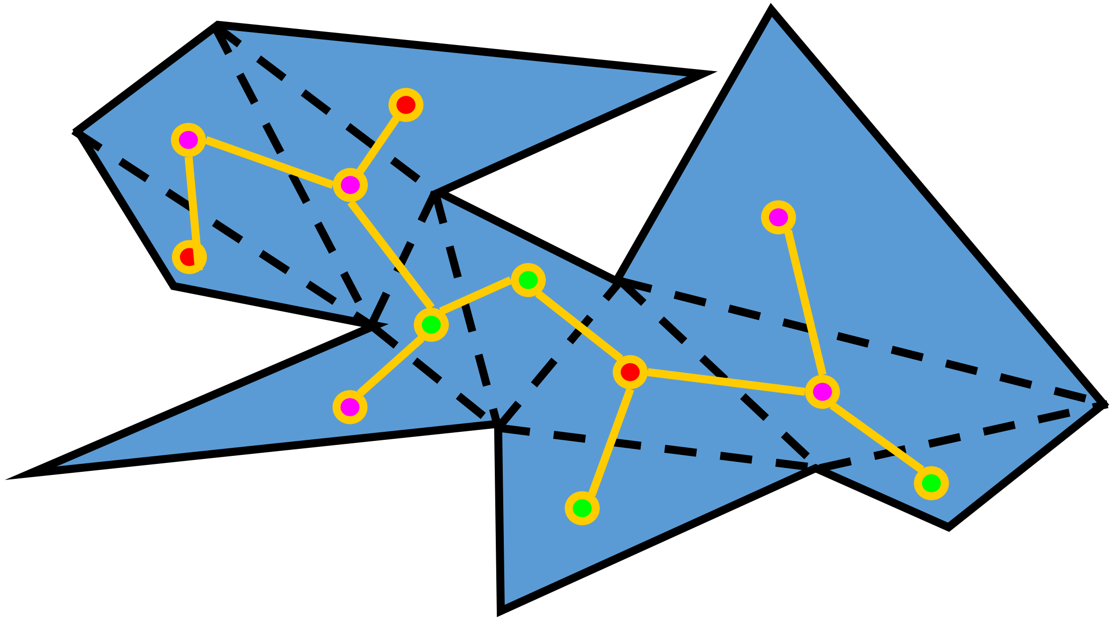
- ANY DIAGONAL CUTS $P$ INTO TWO SIMPLE SUBPOLYGONS $P_1$ AND $P_2$
- LET $m_1$ BE THE NUMBER OF VERTICES OF $P_1$ AND $m_2$ THE NUMBER OF VERTICES OF $P_2$
- BOTH $m_1$ AND $m_2$ MUST BE SMALLER THAN $n$
  - So by induction $P_1$ and $P_2$ can be triangulated
  - Hence, $P$ can be triangulated as well

# Triangulation Theory

- ANY TRIANGULATION OF $P$ CONSISTS OF $n-2$ TRIANGLES.
  - Consider an arbitrary diagonal in some triangulation $T_P$
  - The diagonal cuts P into two subpolygons with $m_1$ and $m_2$ vertices
  - Every vertex of $P$ occurs in exact1y one of the two subpolygons, except for the vertices defining the diagonal, which occur in both subpolygons. Hence, $m_1 + m_2 = n + 2$.
  - By induction, any triangulation of $P_i$ consists of $m_i - 2$ triangles, which implies that $T_P$ consists of $(m_1 - 2) + (m_2 - 2) = n - 2$ triangles.

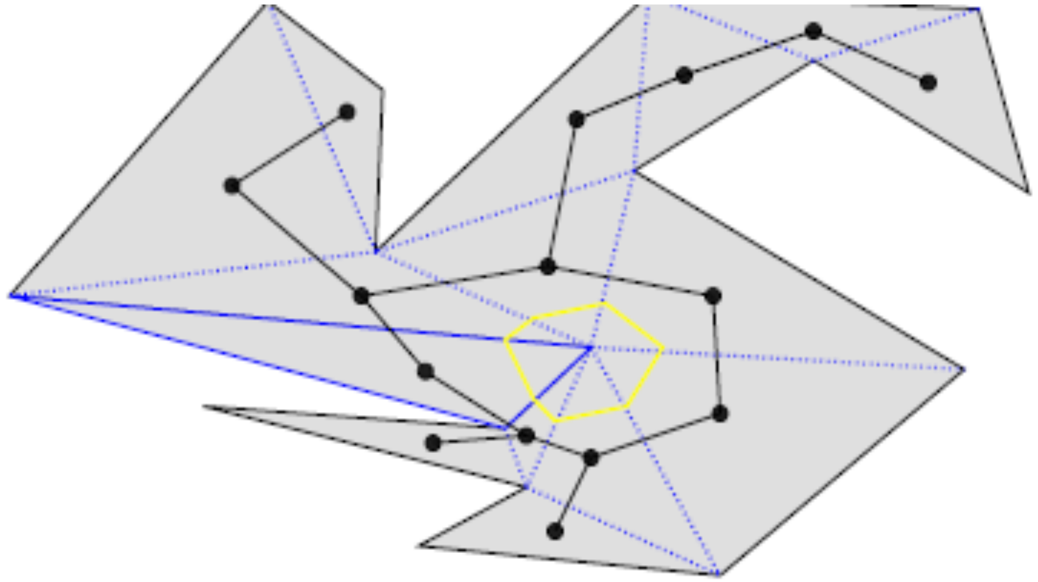
# Triangulation Dual

- The dual T of a triangulation is a tree, with each node of degree at most three.
- Dual graph: each face gives a node; two nodes are connected if the faces are adjacent



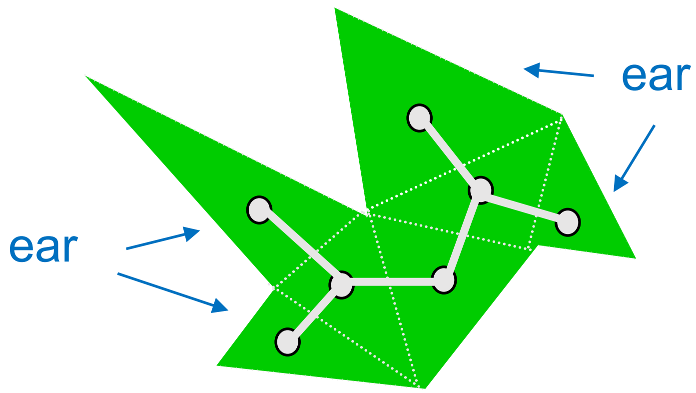
# Properties of triangulations

- Proof:
  - The degree three is immediate from the fact that every triangle have three sides.
  - If there is a cycle C in T it is easy to verify that...
  - There must be a vertex inside the polygon...



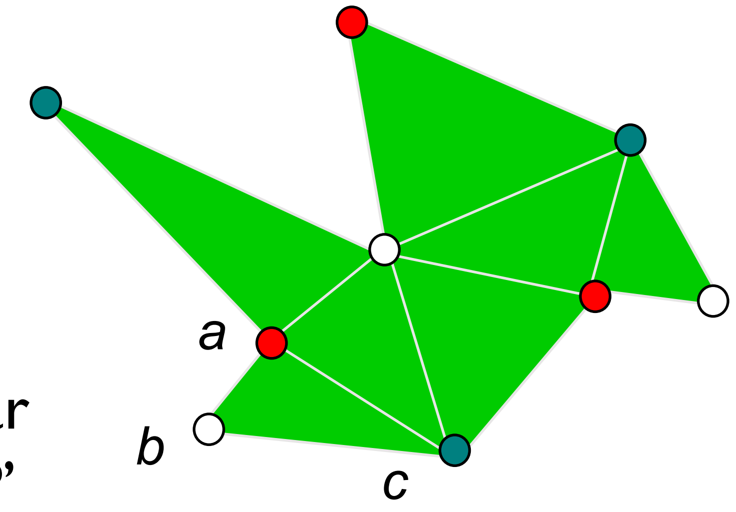
# Meister's Two Ears Theorem

- Three consecutive vertices, A, B, C form an ear if AC is a diagonal
- "2-Ears" Theorem: every polygon of $n \geq 4$ vertices has at least 2 non-overlapping ears.
  - The triangulation dual has at least 2 nodes
  - A tree of more than 2 nodes has at least 2 leaf nodes
  - Each leaf node corresponds to an ear.



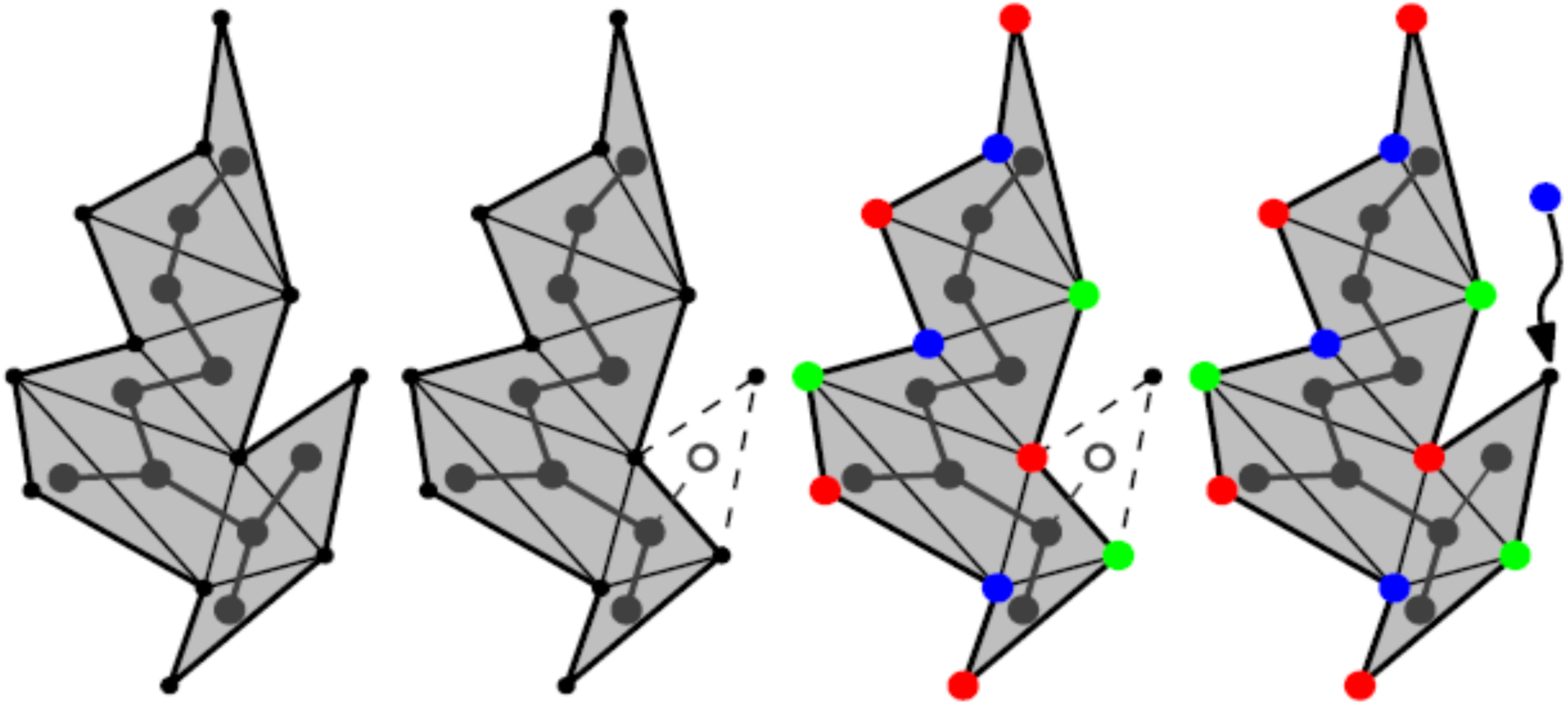
# TRIANGULATION THEORY: 3-COLORING

- "2-EARS" THEOREM CAN BE USED TO EASILY PROVE 3-COLORABILITY OF TRIANGULATION GRAPHS
  - Induction on $n$
    - Base case: $n = 3$
    - For $n \geq 4$: 2-ears theorem guarantees that an ear abc exists apply inductive hypothesis to polygon P' without ear "reattaching" ear adds back in one vertex (w.l.o.g. b) color b whatever color a and c don't use result is a 3-coloring of P



# Fiske' proof

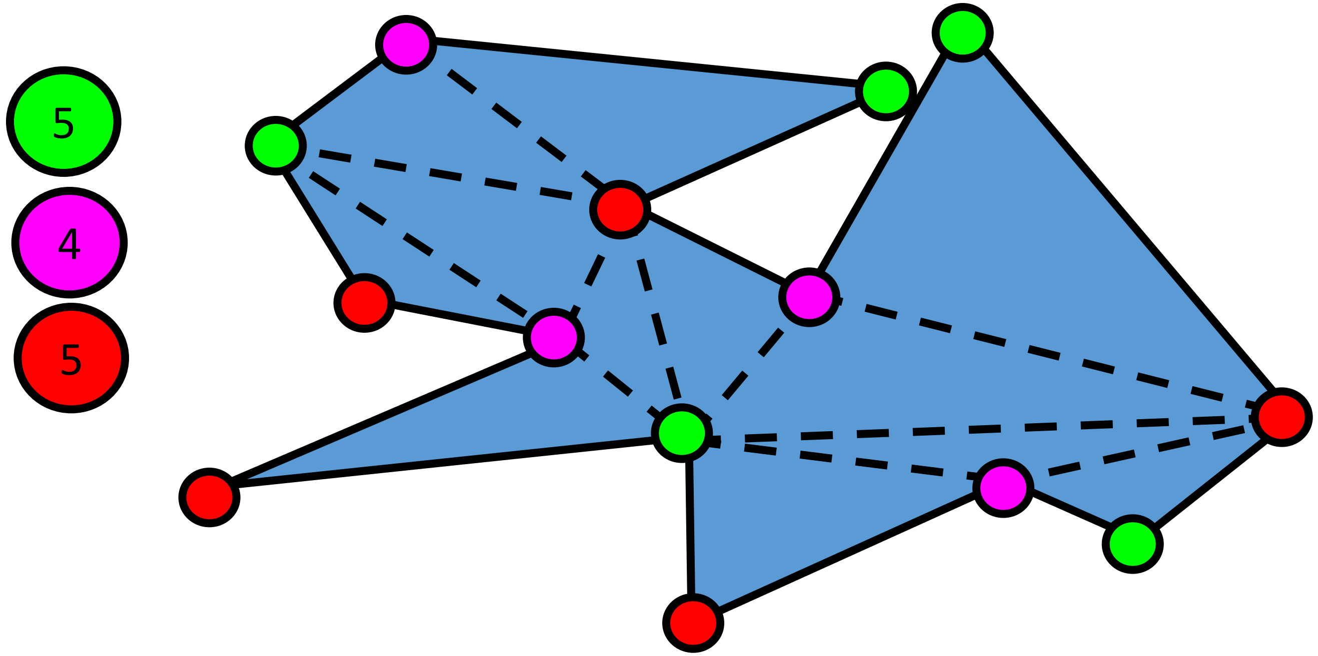
- 3 colors



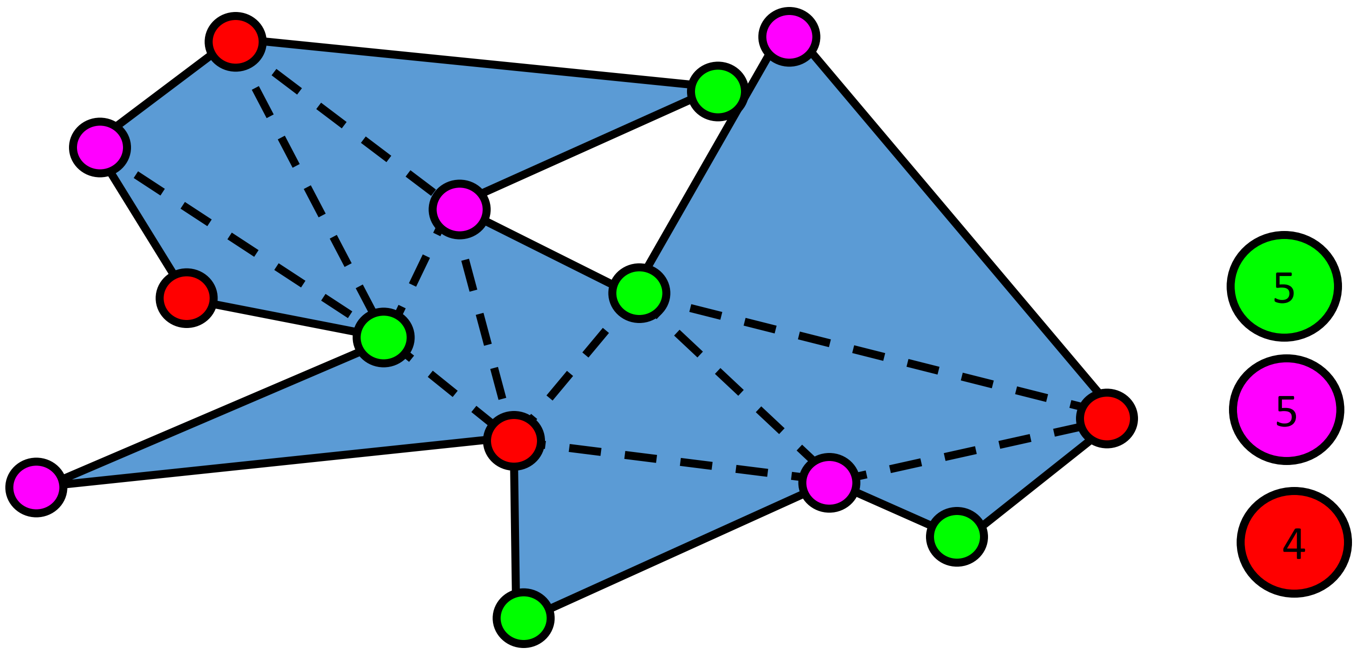
# Fiske' proof

- Apply the "pigeon-hole principle" – If $n$ objects are placed into k pigeon holes, then at least one hole must contain no more than $n/k$ objects

# 3 COLORS SUFFICE…



# 3 COLORS SUFFICE…



# Pigeon Hole Principle

- 3 holes (colors) and 14 pigeons (vertices) to go into them.
- There will always be one hole with less or equal to 14/3 pigeons
- Generalizing: for 3 colors and N vertices there will be a color that is used at most N/3 times. Place the light at those colors.

# Example