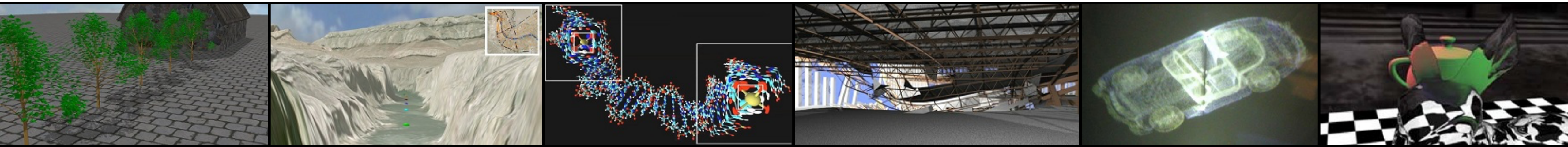
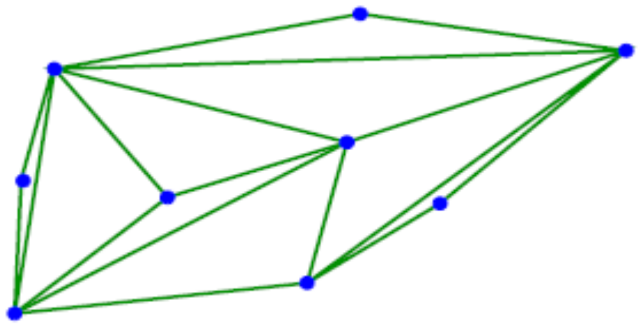
# COT 4521: Introduction to Computational Geometry

## Delaunay Triangulation


Paul Rosen
Assistant Professor
University of South Florida

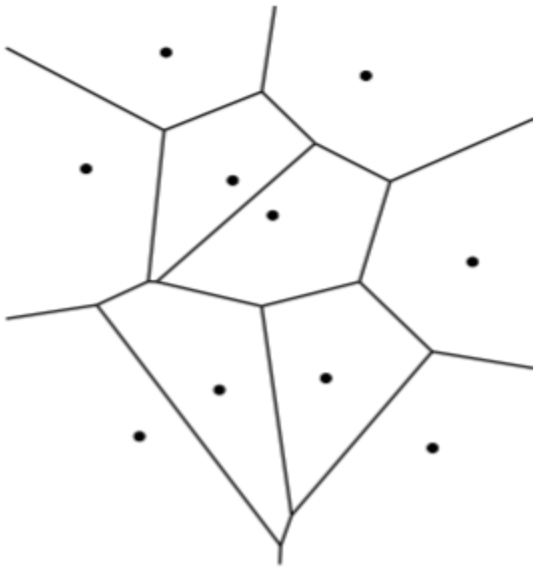
# TRIANGULATION OF A POINT-SET

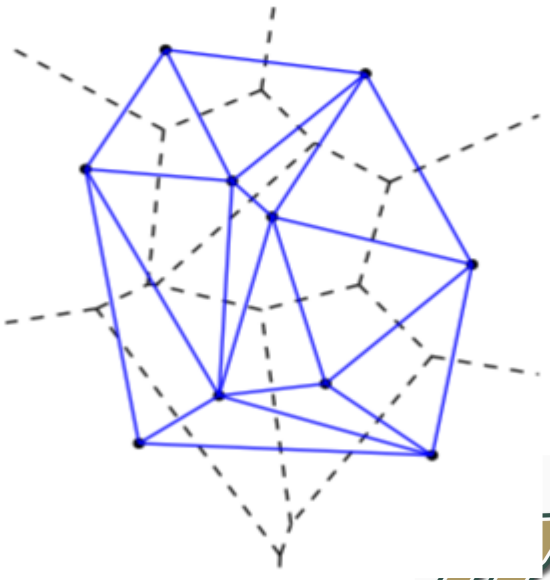


- DEFINITION (POINT-SET TRIANGULATION)
  - Given a set *S* of *n* points in $R^2$, a triangulation of S is a planar graph with vertex set *S*, such that all the bounded faces are triangles, and these faces form a partition of the convex hull *CH*(*S*) of *S*.

# Delaunay Triangulations

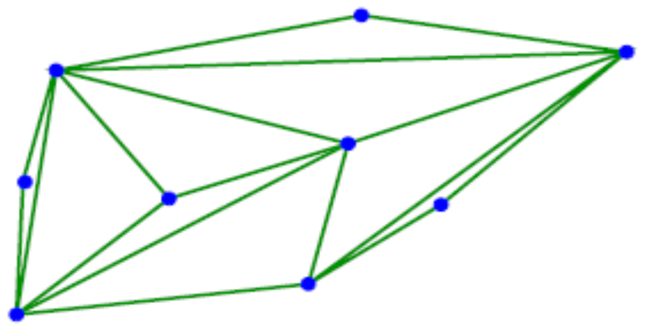
- The Voronoi diagram $Vor(P)$ is the subdivision of the plane into Voronoi cells $V(p)$ for all $p \in P$
- In 1934 Delaunay proved that when the dual graph is drawn with straight lines, it produces a planar triangulation of the Voronoi sites $P$, now called the Delaunay Triangulation

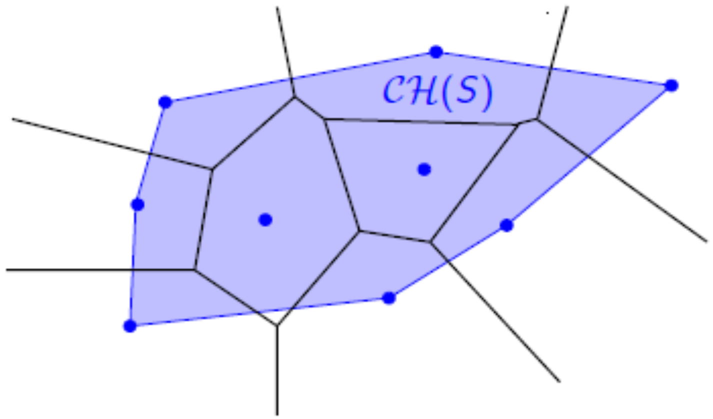


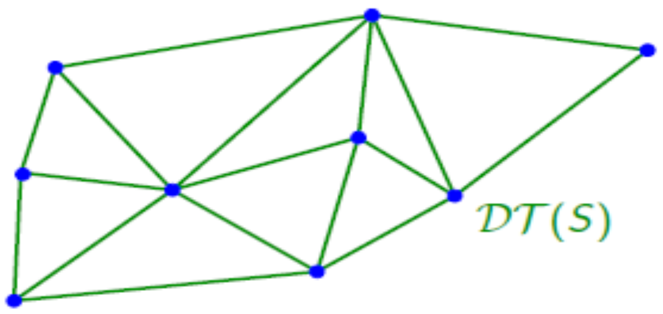


# The Delaunay Triangulation

- The Delaunay triangulation of the same set.
- It has many interesting properties.





# The Delaunay Triangulation

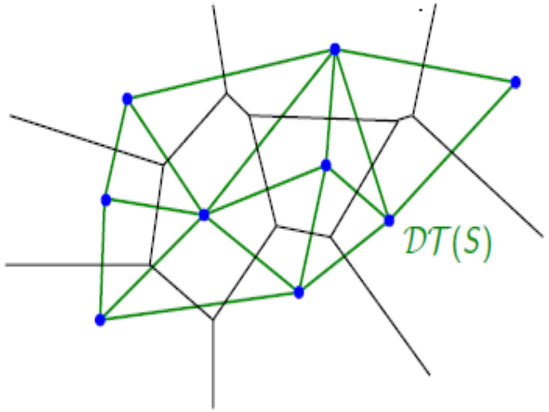
- Let S be a set of n points in $R^2$. We assume general position in the sense that no 4 points in S are co-circular. The Delaunay triangulation DT of S is the dual graph of the Voronoi diagram of S such that:
  - Each vertex DT($s_i$) is located at the corresponding site $s_i$
  - The edges of DT(S) are straight line segments.

# Properties of Delaunay Triangulations

- The planar Voronoi diagram and the Delaunay triangulation are duals in a graph theoretical sense
  - Voronoi vertices correspond to Delaunay triangles
  - Node of $DT(P)$ corresponds to Voronoi regions
  - Edges of both types correspond by definition.

# The Delaunay Triangulation

- DT over VD
  - Face of DT(S) ⇔ vertex of VD(S)
  - Node of DT(S) ⇔ sites of VD(S)
  - Edges of DT(S) ⇔ edges of VD(S)
  - Boundary of DT(S): convex hull
  - Interior of each DT(S) face do not contain any cite

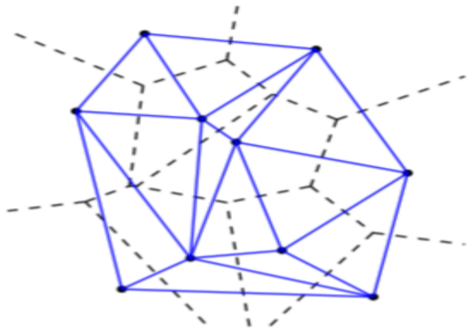


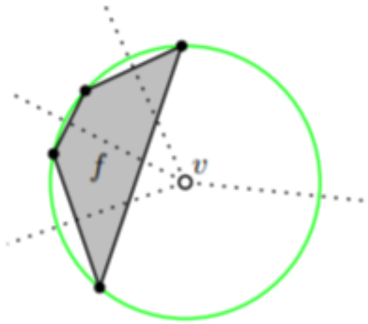
# Properties of Delaunay Triangulations

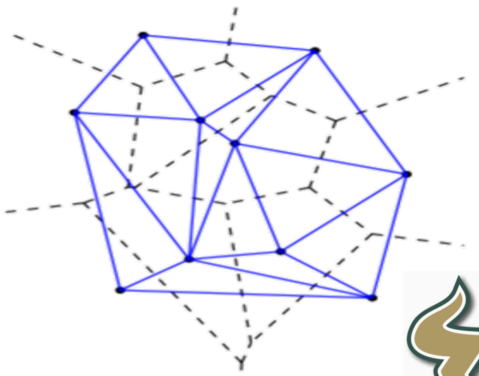
- The duality immediately implies upper bounds of $3n - 6$ and of $2n - 5$ on the number of Delaunay edges and triangles, respectively.
- The Delaunay triangulation and its duality to Voronoi diagrams generalize to higher dimensions in an obvious way.

# Properties of Delaunay Triangulations

- $DT(P)$ is the straight-line dual of $VD(P)$. This is by definition.
- $DT(P)$ is a triangulation if no four points of $P$ are co-circular: Every face is a triangle. This is a Delaunay's theorem. The faces of $DT(P)$ are called Delaunay triangles
- Each face (triangle) of $DT(P)$ corresponds to a vertex of $VD(P)$
- Each edge of $DT(P)$ corresponds to an edge of $VD(P)$

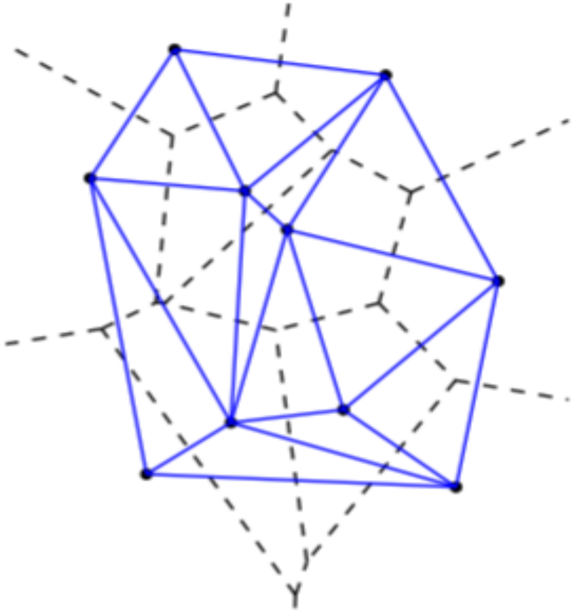




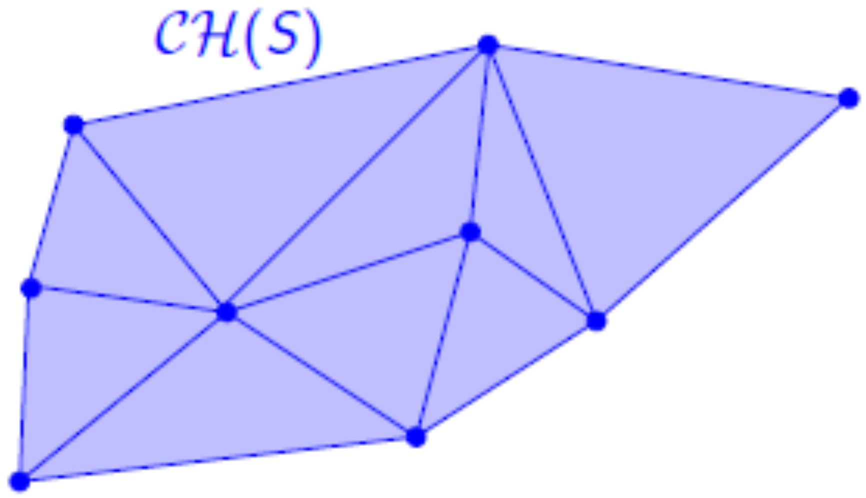
# Properties of Delaunay Triangulations

- Each node of $DT(P)$ corresponds to a region of $\mathrm{VD}(P)$
- The boundary of $DT(P)$ is the convex hull of sites
- The interior of each (triangle) face of $DT(P)$ contains no sites



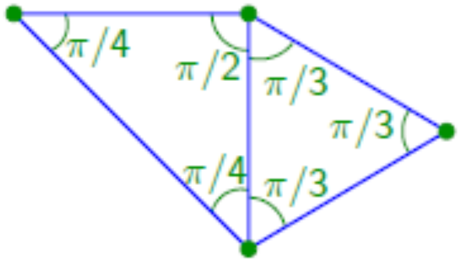
# Convex Hull

- The boundary of $DT(S)$ is convex hull of the sites.



# Triangulation Maximizing the Minimum Angle

- Let T be a triangulation of S
- Angle sequence $\theta(T)$: Sequence of all the angles of the triangle of T in non-decreasing order
- Example: Angle Sequence

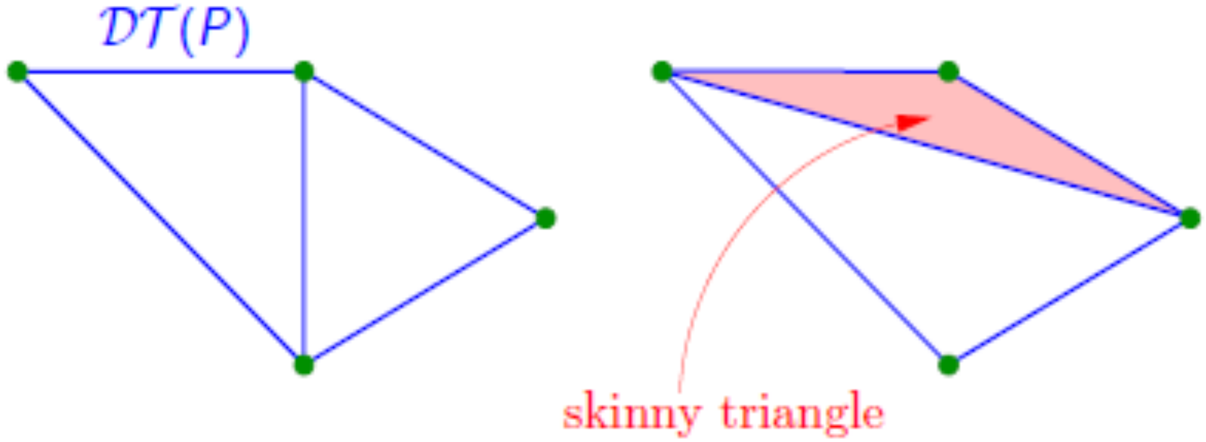



$\Theta(\mathcal{T}) = (\pi/4, \pi/4, \pi/3, \pi/3, \pi/3, \pi/2)$

- Comparison: Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulation of S. We compare $\Theta(\mathcal{T})$ $and$ $\theta(\mathcal{T}')$ in lexicographical order
- Example: {1,1,3,4,5} <{1,2,5,6,7}

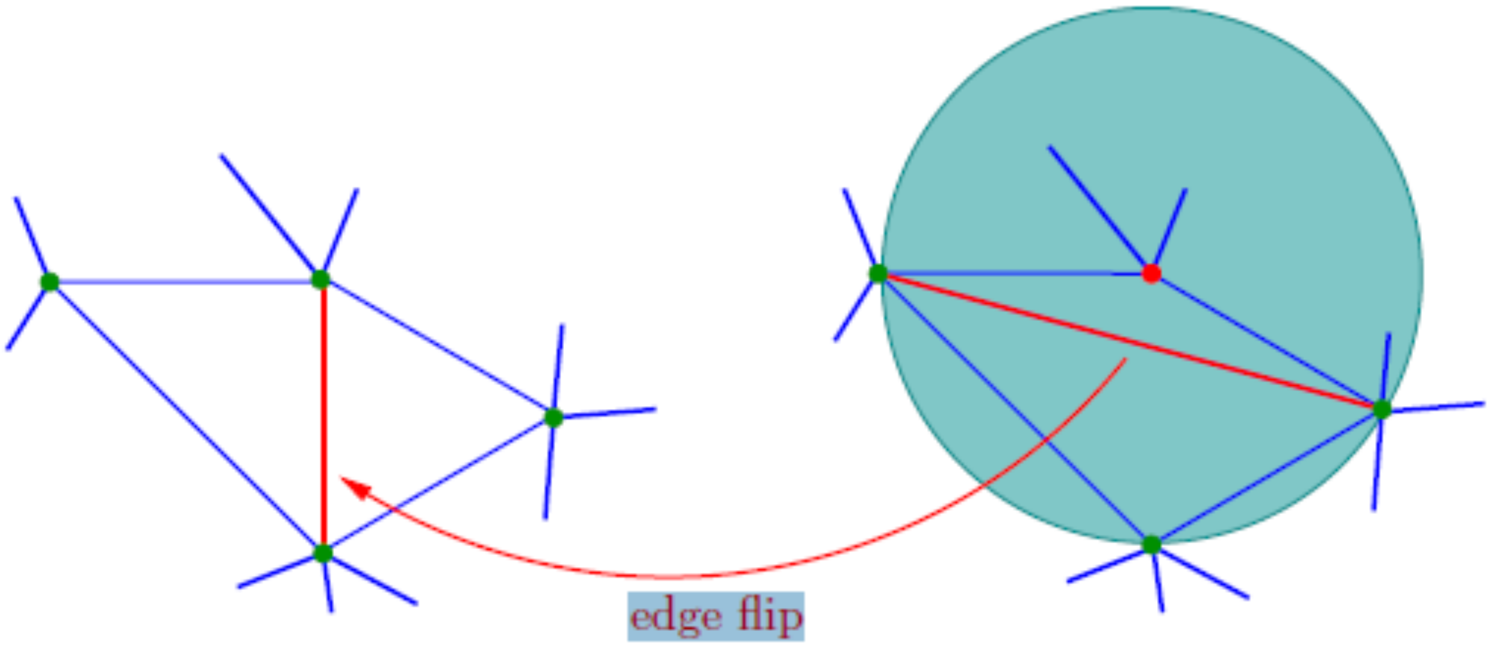
# Optimality of the Delaunay Triangulation

- **Theorem**: Let S be a set of points in general position. Then the angle sequence of DT (S) is maximal among all triangulations of S.
  - So the Delaunay triangulation maximizes the minimum angle.
  - Intuition: Avoids skinny triangles.



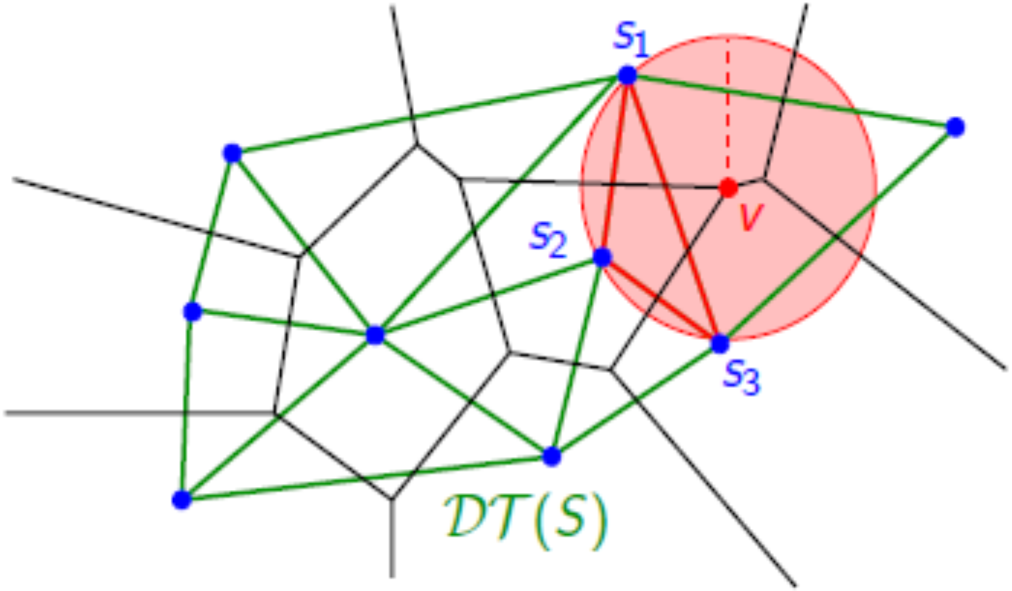
# OPTIMALITY OF THE DELAUNAY TRIANGULATION

- PROOF: IDEA
  - Flip edges to ensure the circumcircle property.
  - It increases the angle sequence.



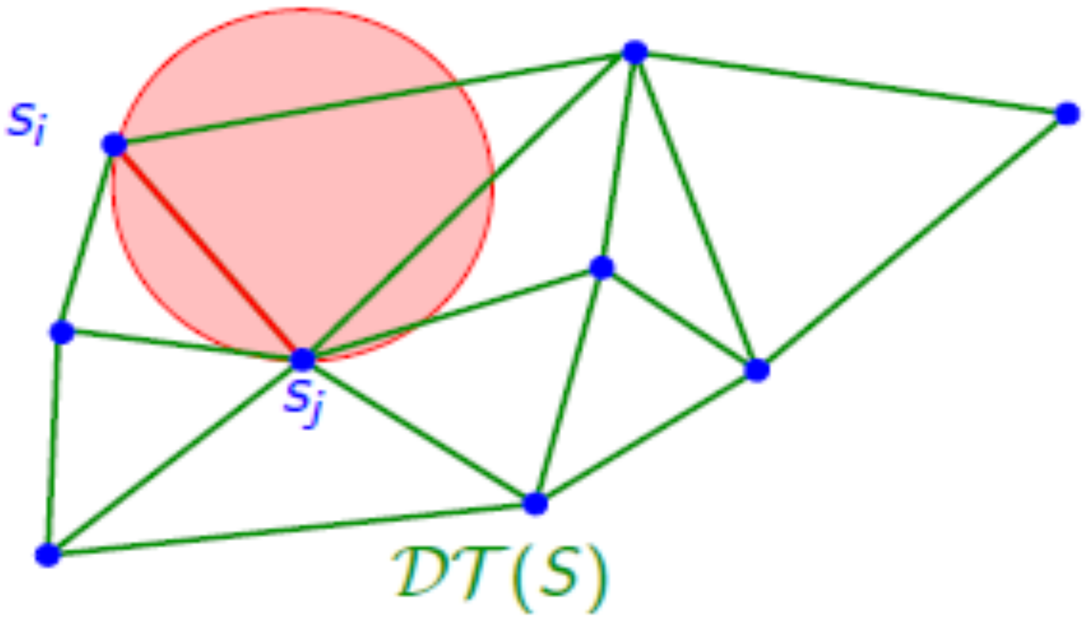
# Circumcircle Property

- Property (Circumcircle)
  - The circumcircle of any triangle in $DT(S)$ is empty. (It contains no site $s_i$ in its interior.)
- Proof: Let $s_1 s_2 s_3$ be a triangle in $DT(S)$, let v be the corresponding Voronoi vertex. Property of Voronoi vertices: the circle centered at $v$ through $s_1 s_2 s_3$ is empty.



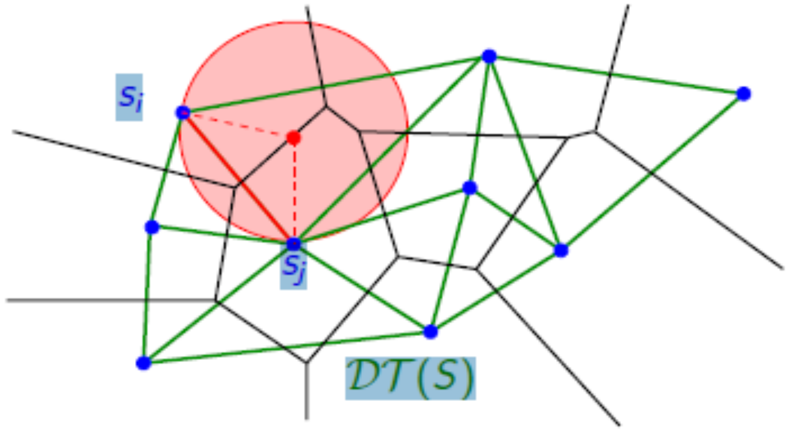
# EMPTY CIRCLE PROPERTY

- PROPERTY (EMPTY CIRCLE)
  - ($s_i$ $s_j$ ) is an edge of *DT* (*S*) iff there is an empty circle through $s_i$ and $s_j$ .



# Empty Circle Property

- Proof :
  - If $\overline{s_i, s_j}$ is a Delaunay edge then $V(s_i)\ and\ V(s_j)$ share the positive edge $e \in V(P)$.
  - Put the circle $C(x)$ with the center $x$ on the interior $e$ with the radius equal to the distance to $s_i$ or $s_j$. If circle is not empty then $x$ would be in $V(c)$ , but we know that $x$ is in $V(s_i)$ or $V(s_j)$



# DELAUNAY-IZATION OF A TRIANGULATION

- ANY TRIANGULATION OF THE CONVEX HULL CAN BE CONVERTED INTO A DELAUNAY TRIANGULATION BY REPEATEDLY TEST THE EMPTY CIRCLE PROPERTY.
- IF ANY EDGE "FAILS" THE TEST, IT IS SWAPPED WITH A NEW EDGE BETWEEN THE CONNECTING TRIANGLES