COT 4521: Introduction to Computational Geometry

<u>Polygons</u>

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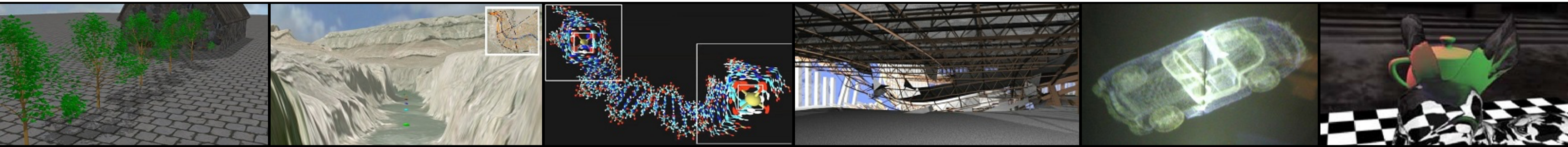
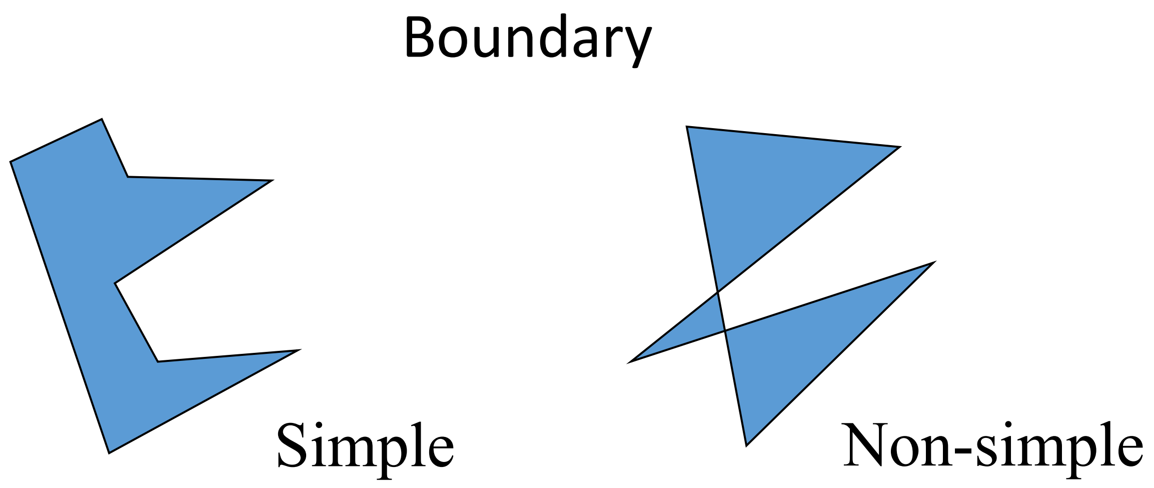
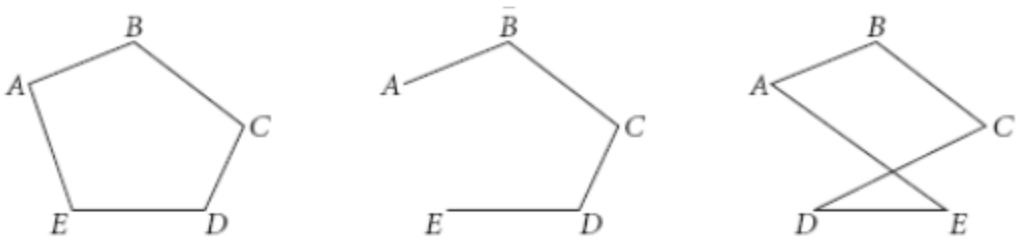
Some slides from Valentina Korzhova



POLYGON

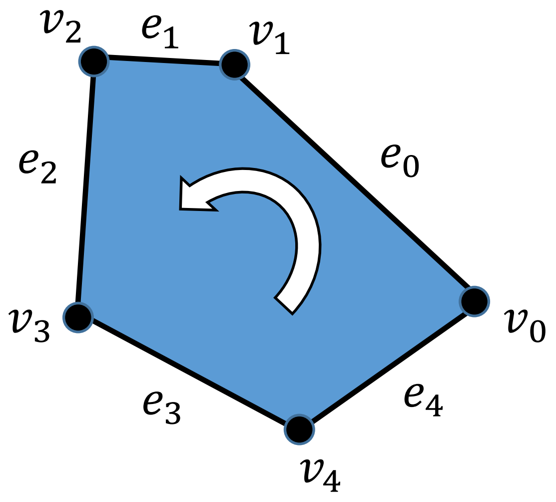
- **POLYGON** IS A REGION OF A PLANE BOUNDED BY A FINITE COLLECTION OF LINE SEGMENTS FORMING A SIMPLE CLOSED CURVE.





Polygon

- Edges – the line segments $(e_0, e_1, \cdots, e_{n-1})$
- Vertices – the points where adjacent edges meet
  - Start at any vertex and list the vertices consecutively in a counterclockwise direction $(v_0, v_1, \ldots, v_{n-1})$
- Angles
  - Name by angle naming convention
  - $\measuredangle v_0, \measuredangle v_1, \ldots, \measuredangle v_{n-1}$

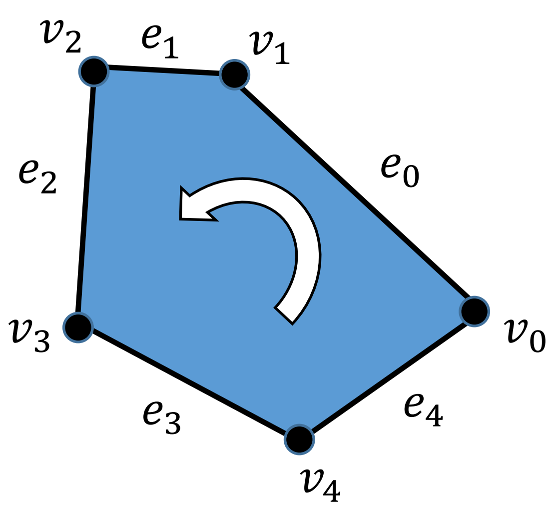


Properties of polygon

- The line segments that make-up a polygon (called sides or edges) meet only at their endpoints, called vertices (singular: vertex) or less formally "corners"
- Exactly two edges meet at every vertex
- The number of edges always equals the number of vertices.
- Two edges meeting at a corner are required to form an angle that is not straight (180°); otherwise, the line segments will be considered parts of a single edge

Formal Definition of a Simple Polygon

- Formally, we are given n vertices (i.e., points) $v0, v1, \ldots, v_{n-1}$, the chain formed by $v0v1 \cdots v_{n-1}$ is a simple polygon iff
  - The segments $e0 = v0v1, \ldots, e_{n-2} = v_{n-2}v_{n-1}$, and $e_{n-1} = v_{n-1}v0$ are disjoint in their interior
  - Consecutive segments intersect only in their endpoints. Namely $ei \cap e_{i+1} = v_{i+1}$ for $i = 0, \ldots, n-2$ and $e_{n-1} \cap e0 = v0$
  - Non adjacent segments do not intersect $ei \cap ej = \varnothing$, for $\forall j \neq i + 1$
- We work $mod\ n$.
  - Namely $v_i = v_i\ mod\ n$



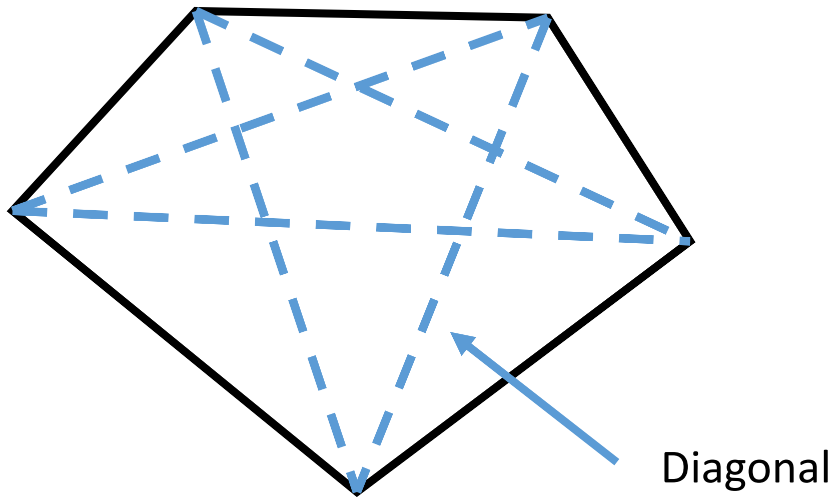
Diagonals

- Definition: The diagonal of a polygon is a line segment linking two non-adjacent vertices, interior to the polygon, and not blocked by portion of polygon's boundary

- Lemma: The segment $s = v_i v_j$ is a diagonal of P iff
  1. For all edges e of P that are not incident to either $v_i or\ v_j$, s and e do not intersect: $s \cap e = \emptyset$;
  2. s is internal to P in neighborhood of v_i and v_j

DIAGONALS

- DEFINITION: THE DIAGONAL OF A POLYGON IS A LINE SEGMENT LINKING TWO NON-ADJACENT VERTICES, INTERIOR TO THE POLYGON, AND NOT BLOCKED BY PORTION OF POLYGON'S BOUNDARY



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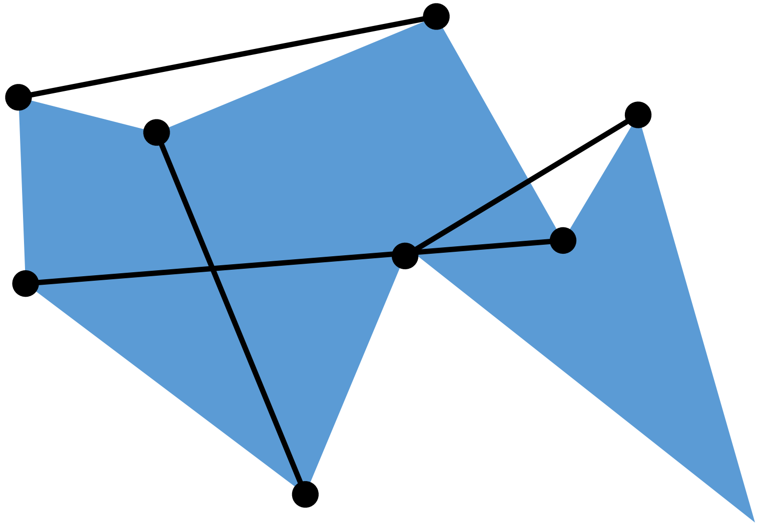
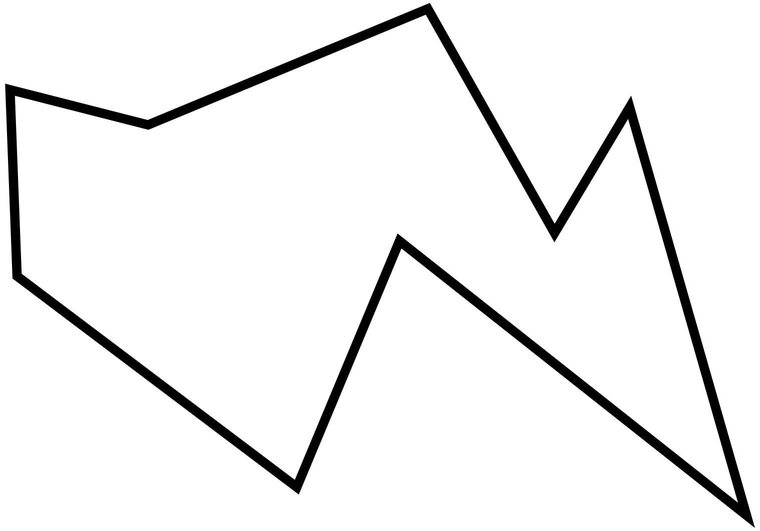


which are legal diagonals?

Efficient Diagonal Finding

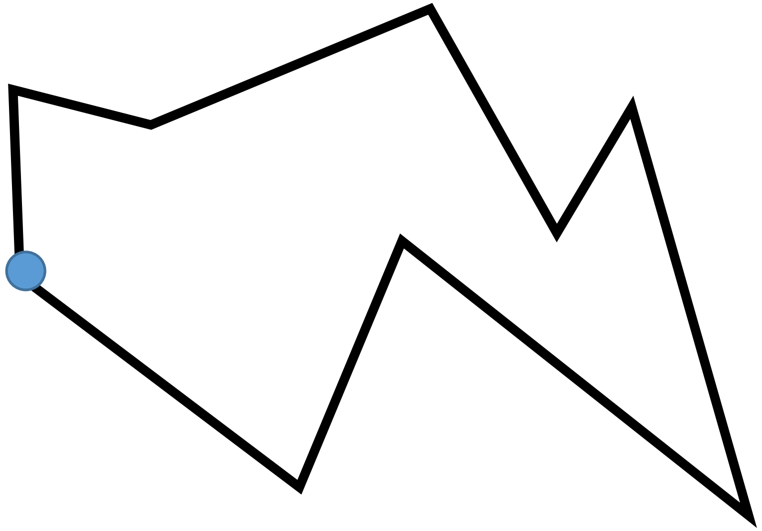
- Brute force
  - Generate each potential diagonal—$O(n^2)$ possible diagonals
  - Check each diagonal against the boundary to determine
    - If it is inside or outside
    - If it intersects the boundary
  - Total performance $O(n^3)$



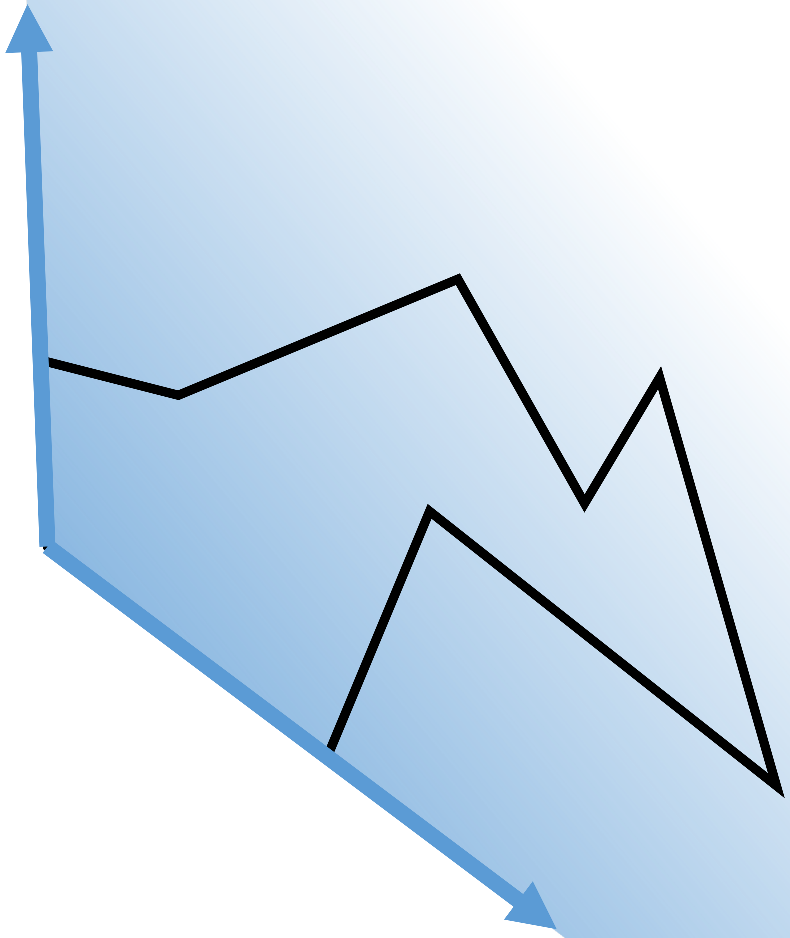
- Can we do Better?

Efficient Diagonal Finding

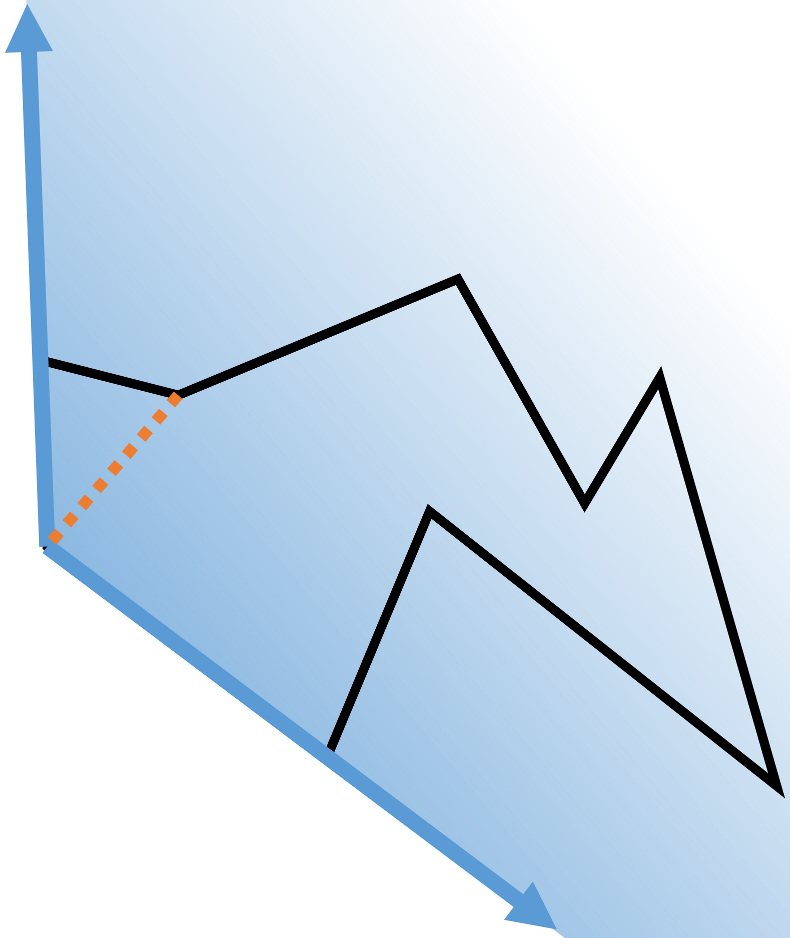
- Variation on a sweep algorithm
  - From a given vertex, sweep both clockwise and counterclockwise, ordered by the distance of points
  - Retain a notion of "valid space" for diagonals



Efficient Diagonal Finding



Efficient Diagonal Finding



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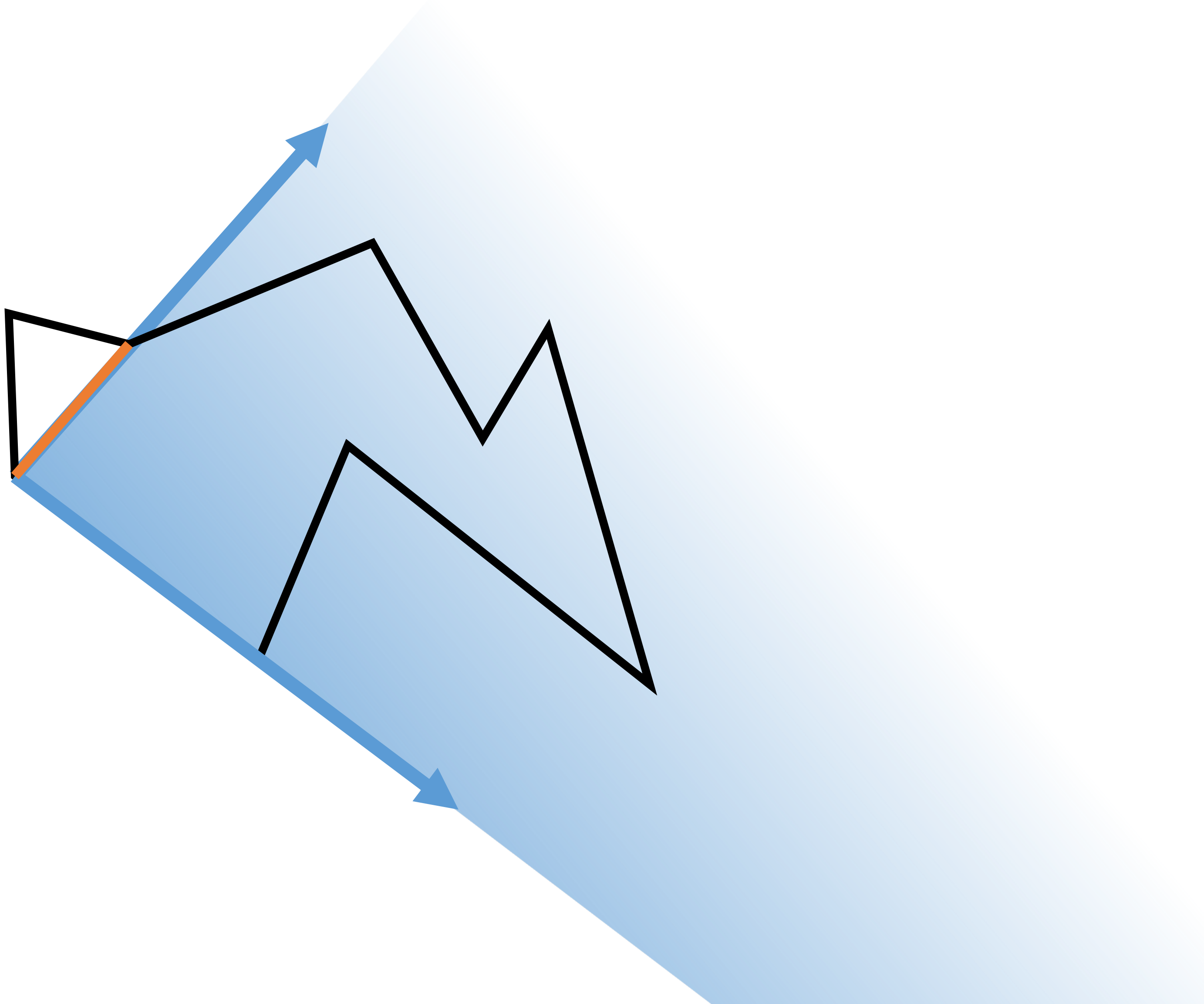
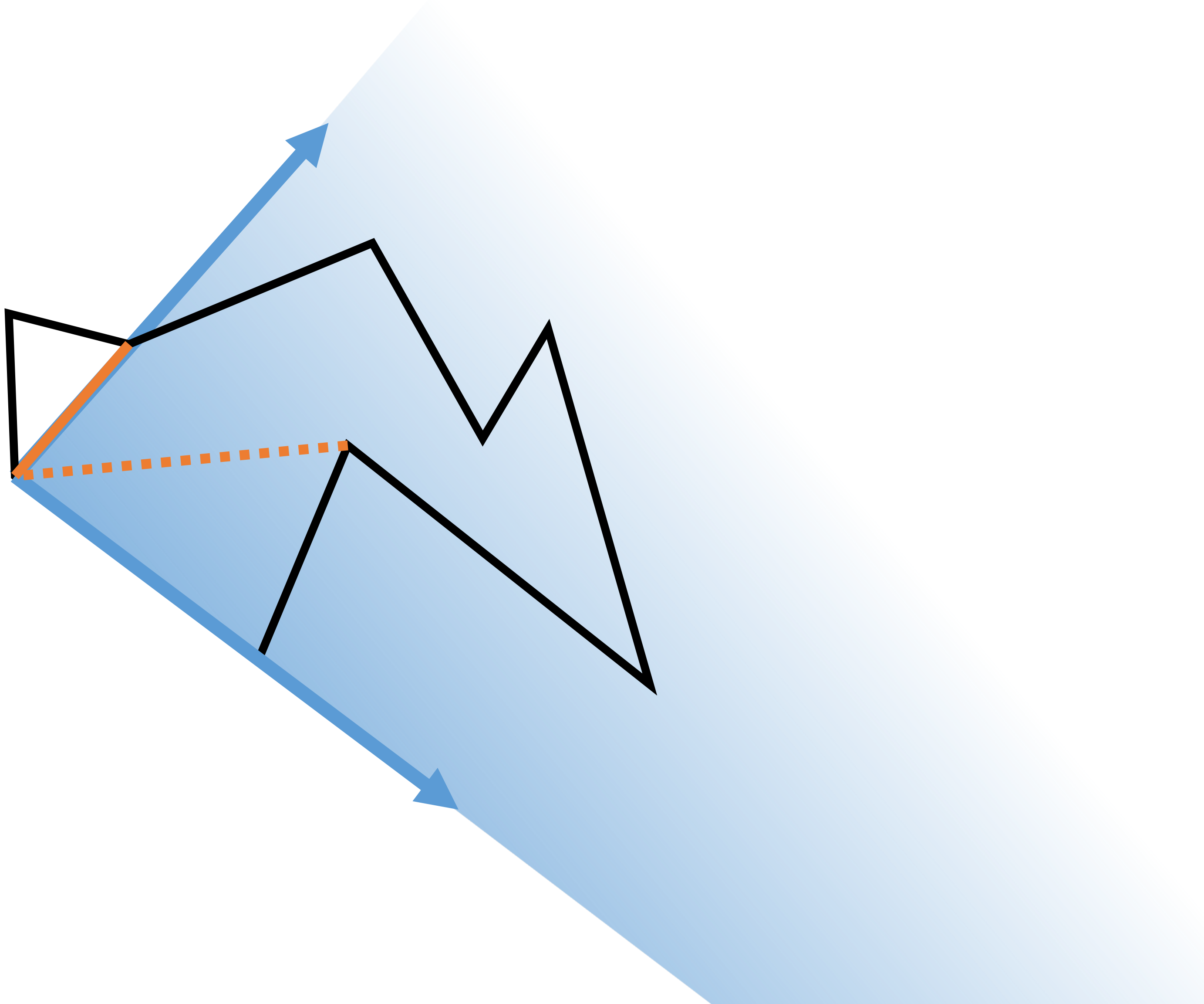


Efficient Diagonal Finding



Efficient Diagonal Finding

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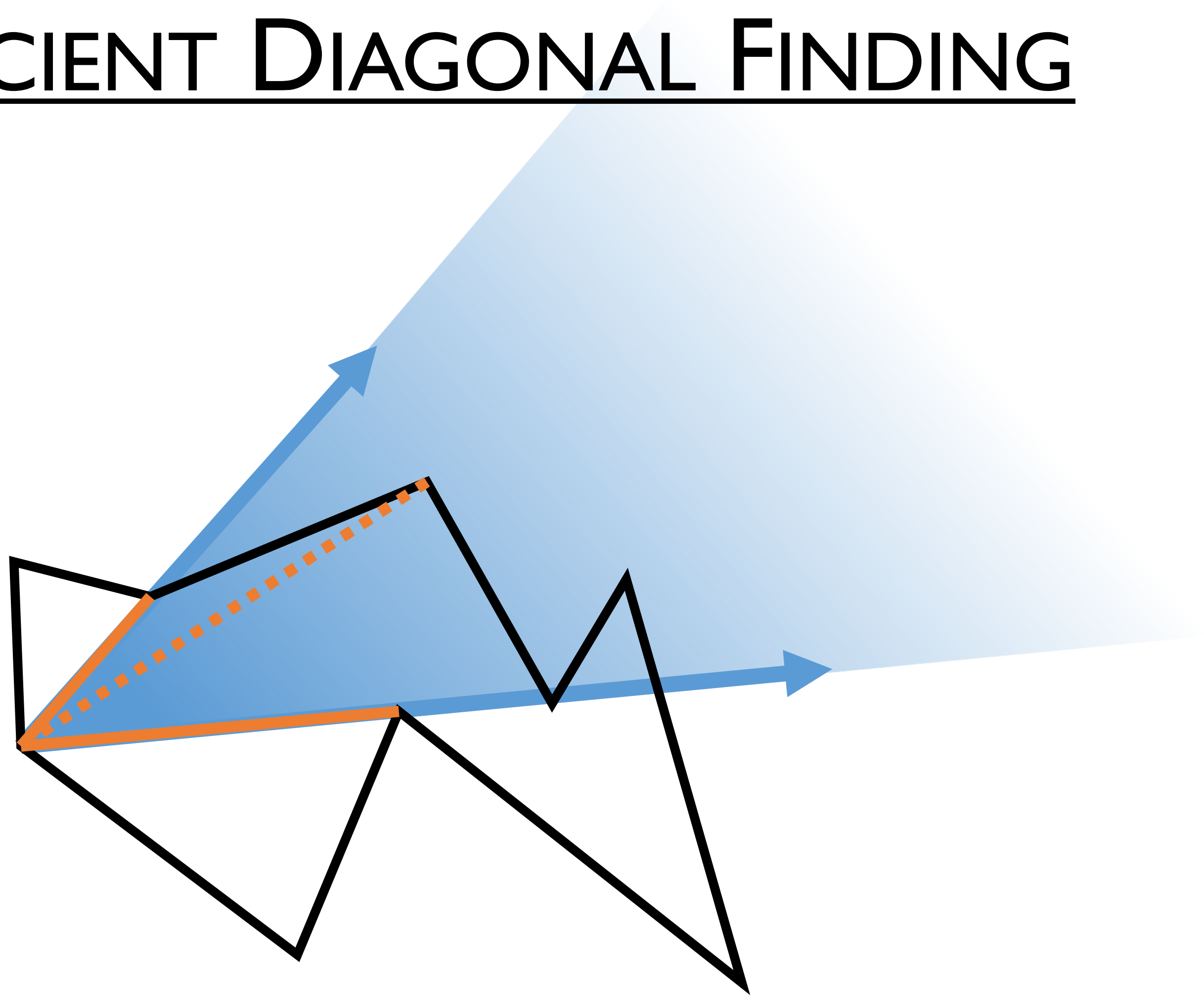
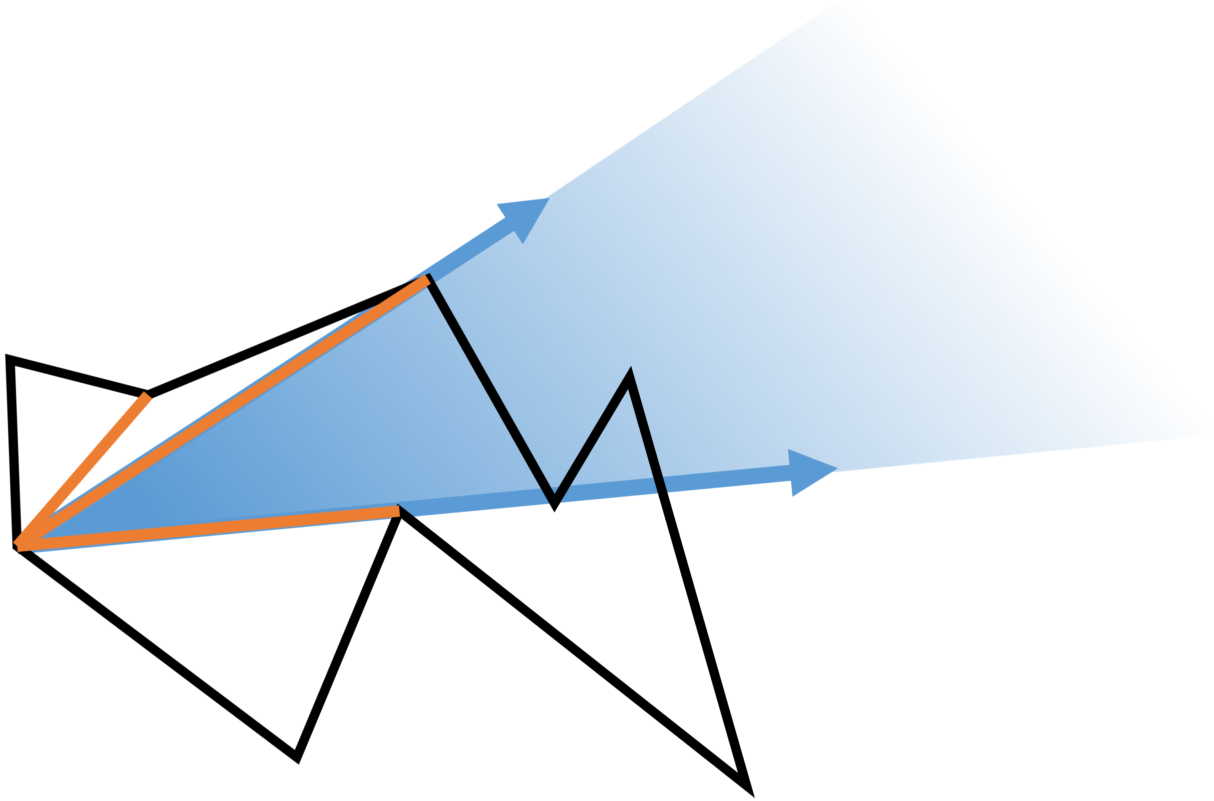
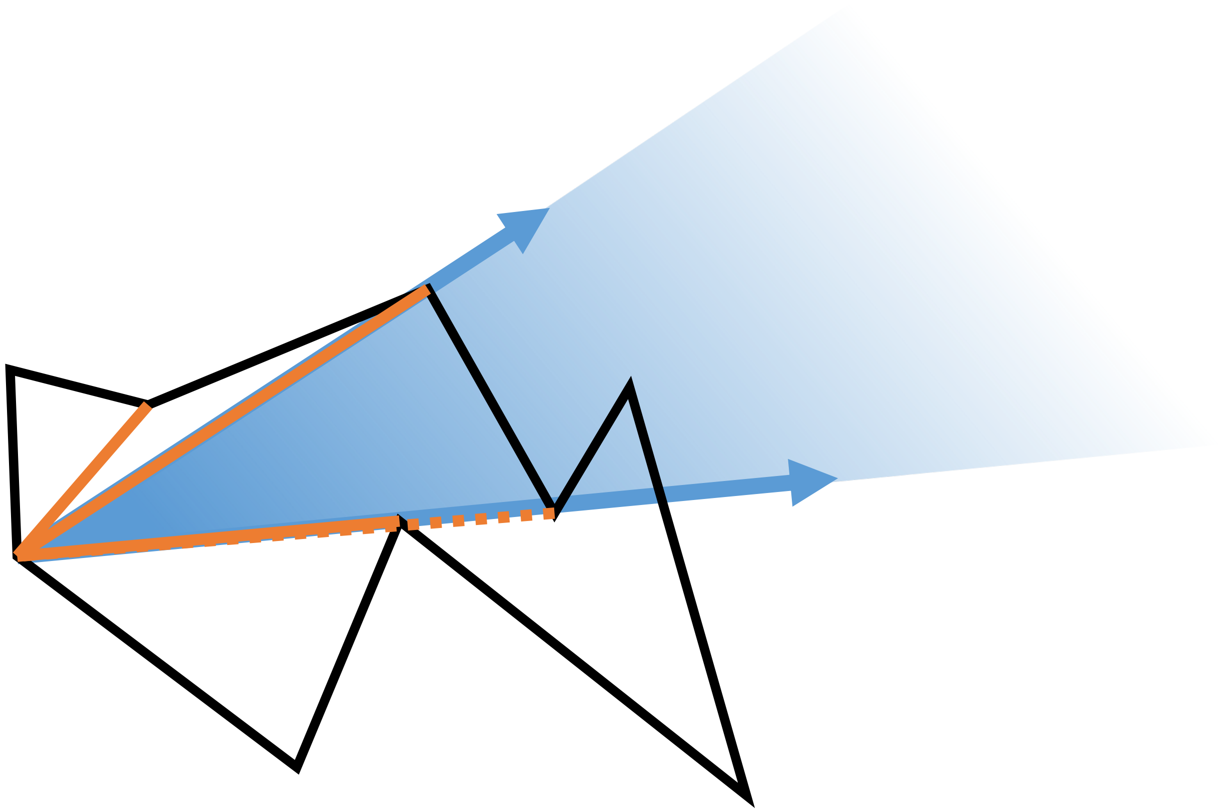


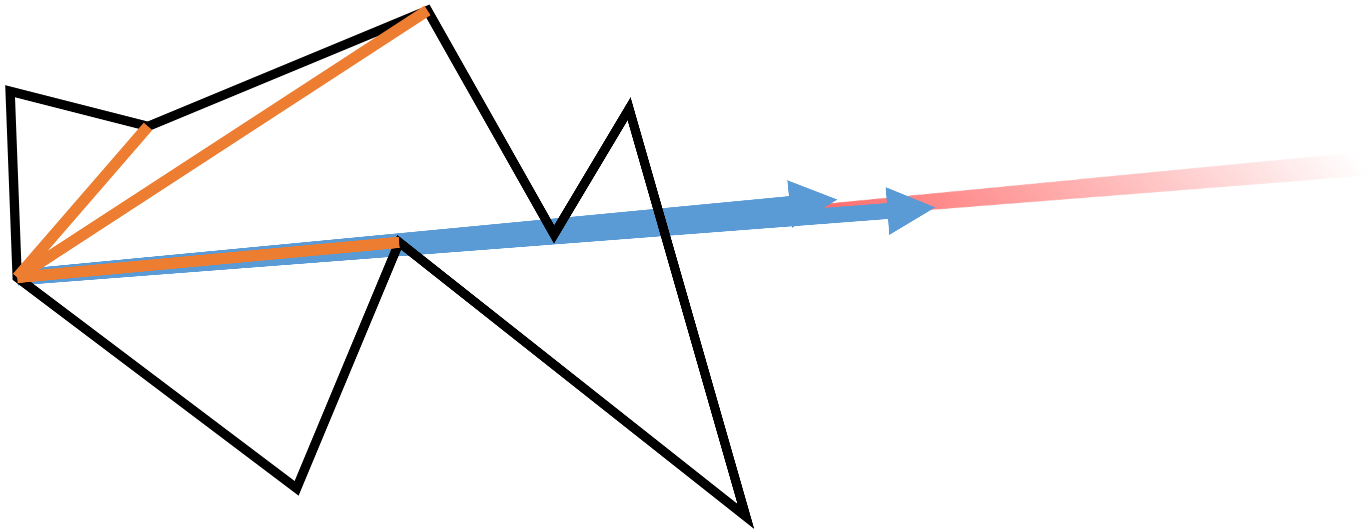
Efficient Diagonal Finding



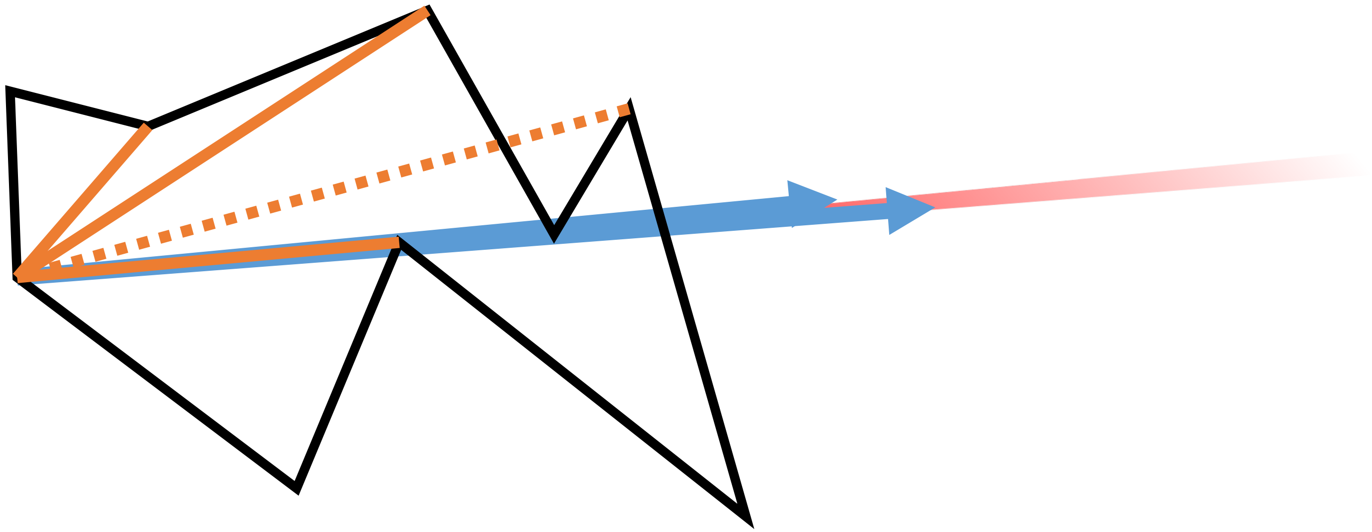
Efficient Diagonal Finding



Efficient Diagonal Finding



Efficient Diagonal Finding



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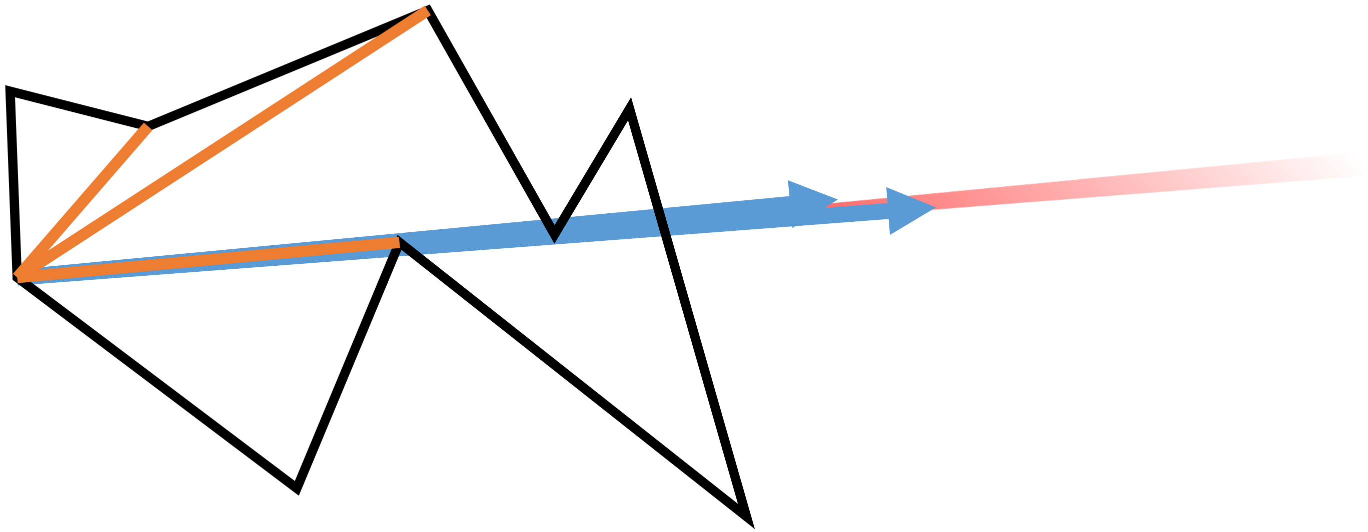
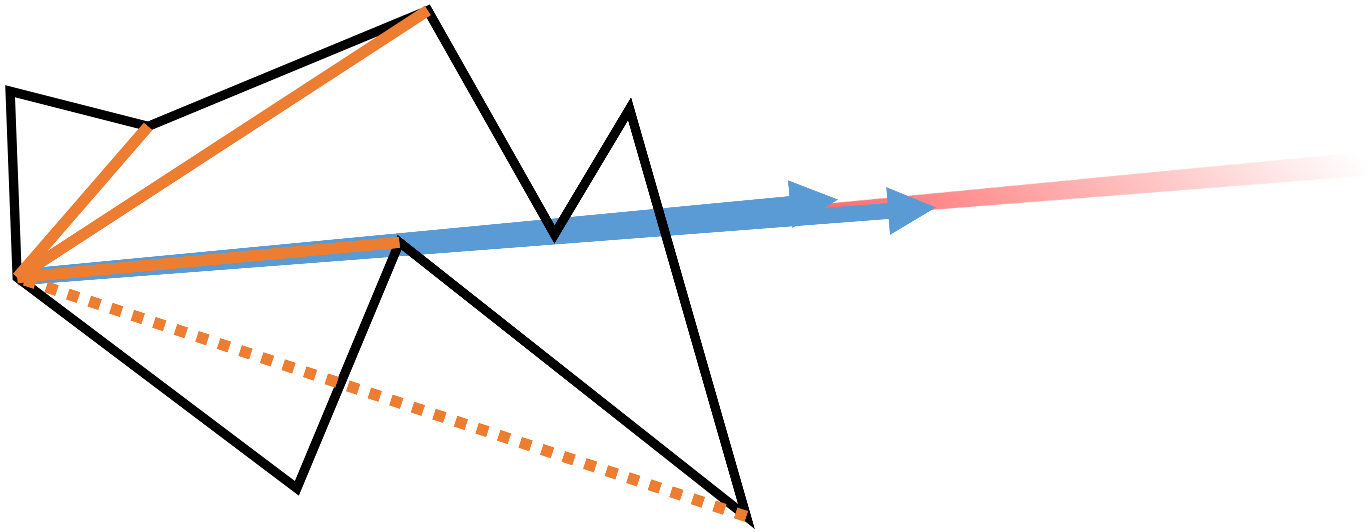


Efficient Diagonal Finding



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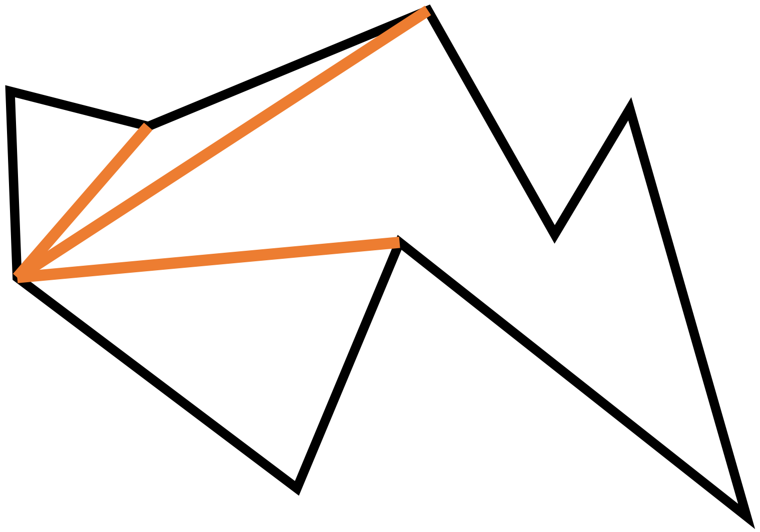
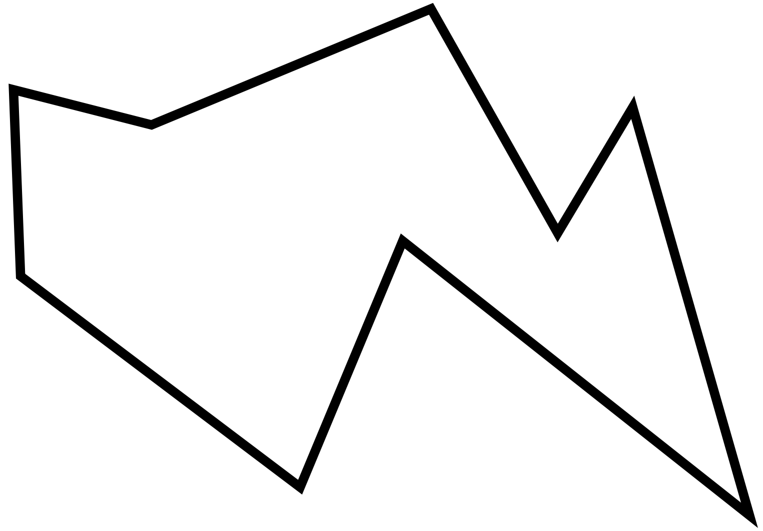


Efficient Diagonal Finding

- Analysis
  - Preprocessing: None
  - Query: Worst case $O(n^2 \log n)$; $O(n \log n)$ per vertex
  - Storage: $O(n)$



Triangulation Theory: Properties

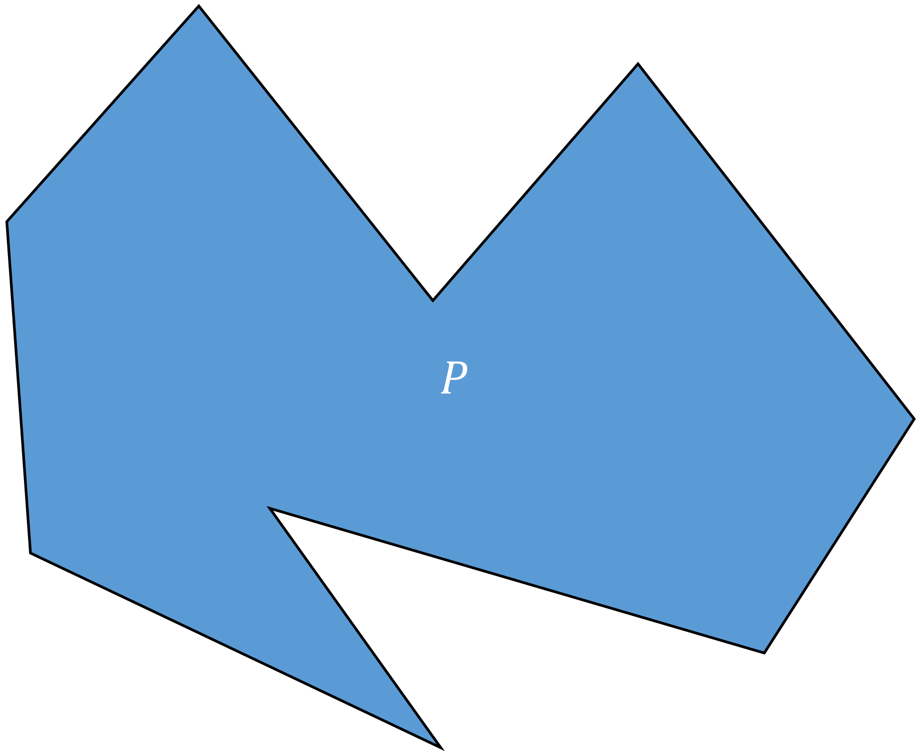
- Lemma: An internal diagonal exists between any two nonadjacent vertices of a polygon P if and only if P is convex polygon.
- Proof: The proof consists of two parts, both established by contradiction.

Classification of Polygons by the Number of Edges
Most Common Polygons

Number of Sides	Name
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
8	Octagon
9	Nonagon
10	Decagon
12	Dodecagon
n	n-gon

Sum of Angles

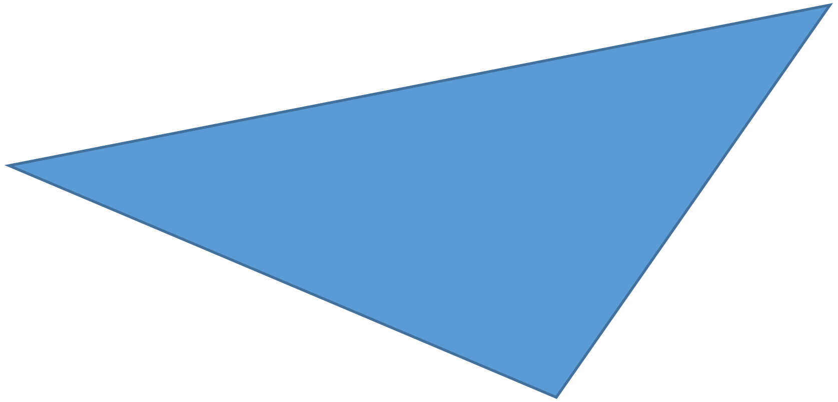
- The sum of the internal angles of a polygon of n vertices is?



Sum of Angles

- Use the Triangle Angle-Sum Theorem to find the sum of the measures of the angles of a polygon.

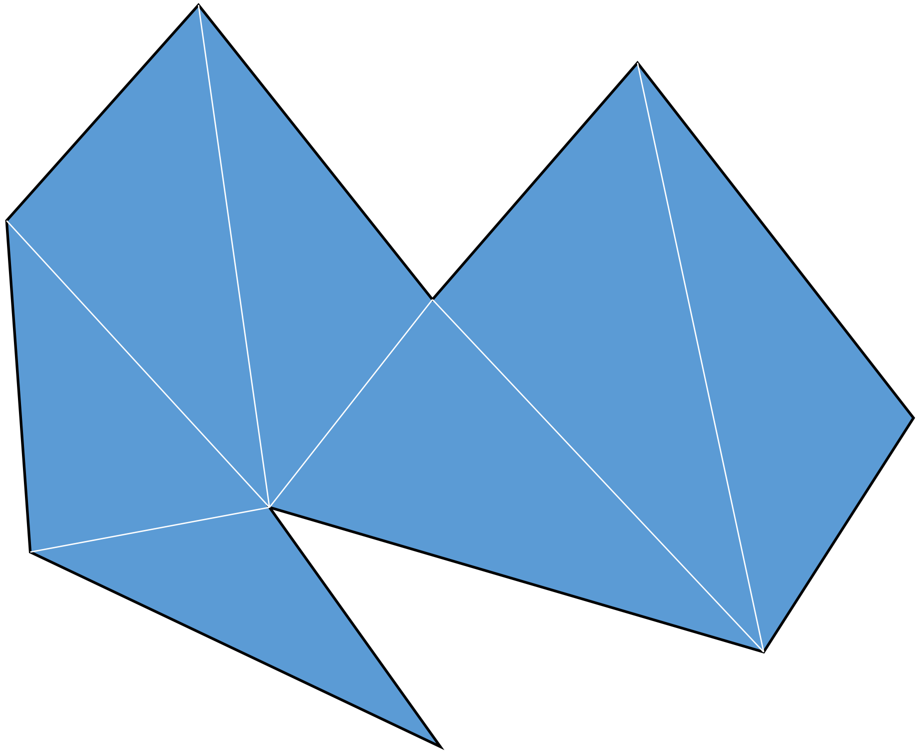
- Triangle Angle-Sum Theorem
  - The sum of the measures of the angles of a triangle measure 180°



Sum of Angles

- Theorem: The sum of the measures of the internal angles of an n-gon is (n - 2) *180.

- Proof by induction



Triangulation Theory

- **Theorem**: Every triangulation of an n-vertex polygon P uses $n - 3$ diagonals and consists of $n - 2$ triangles.
  - Proof by induction:
    - Base case $N = 3$
    - Assume true for any polygon $< N$ sides
    - Given a N sided polygon partition it into two ($N1$ and $N2$) by adding a diagonal

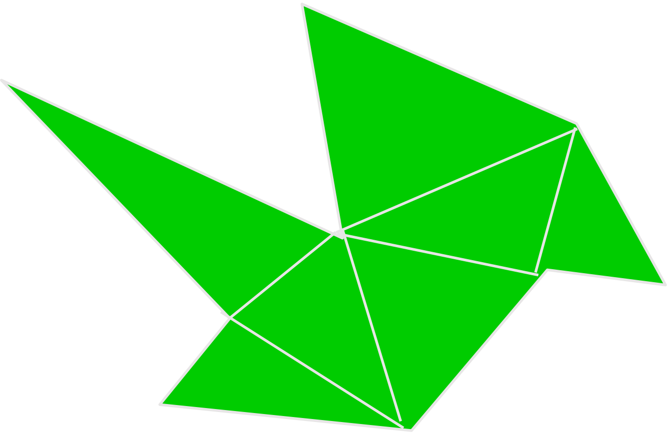
      Total number of diagonals:

      $(N1 - 3) + (N2 - 3) + 1 = (N1 + N2 - 2) - 3 = N - 3$

      Total number of triangles:

      $(N1 - 2) + (N2 - 2) = (N1 + N2) - 4 = N + 2 - 4 = N - 2$

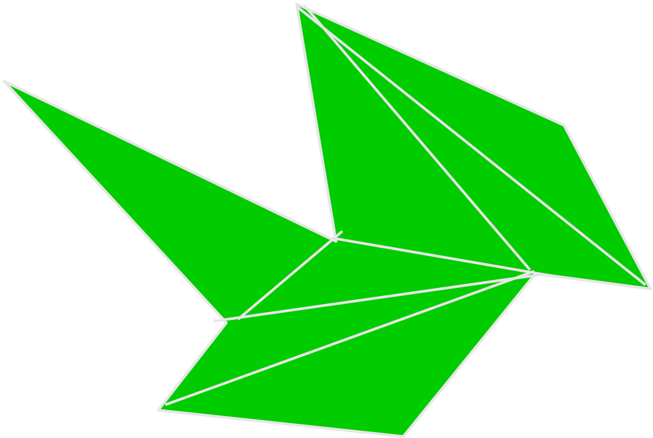
Triangulation Example



9 vertices

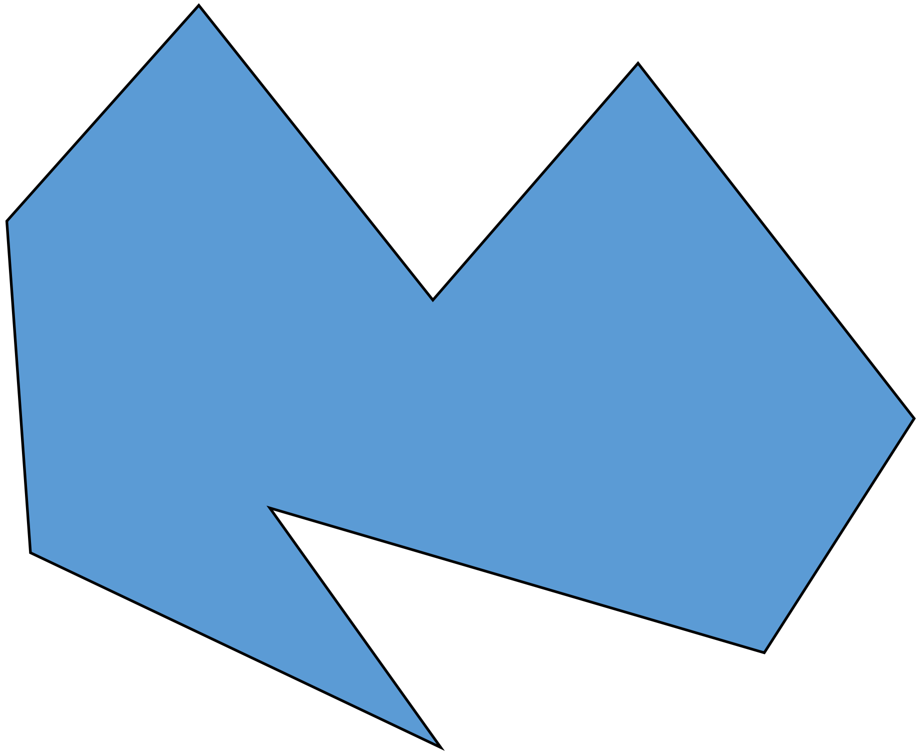
7 triangles

6 diagonals

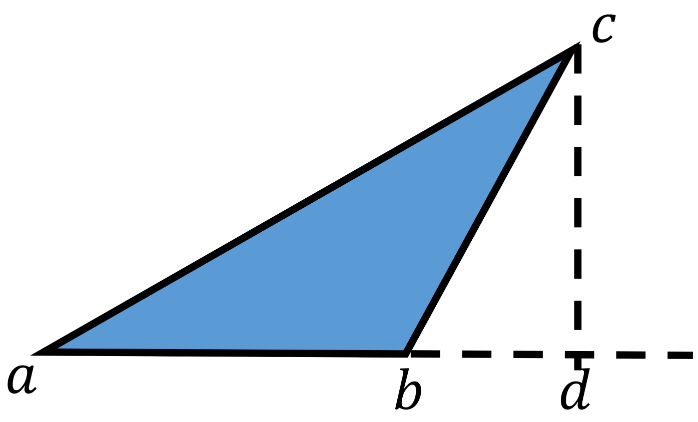


Area of a Polygon

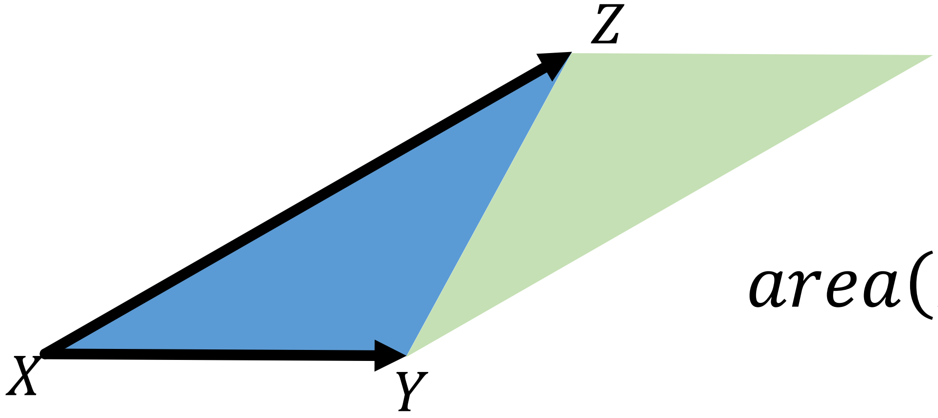
- What is the area of the polygon?



AREA OF A TRIANGLE



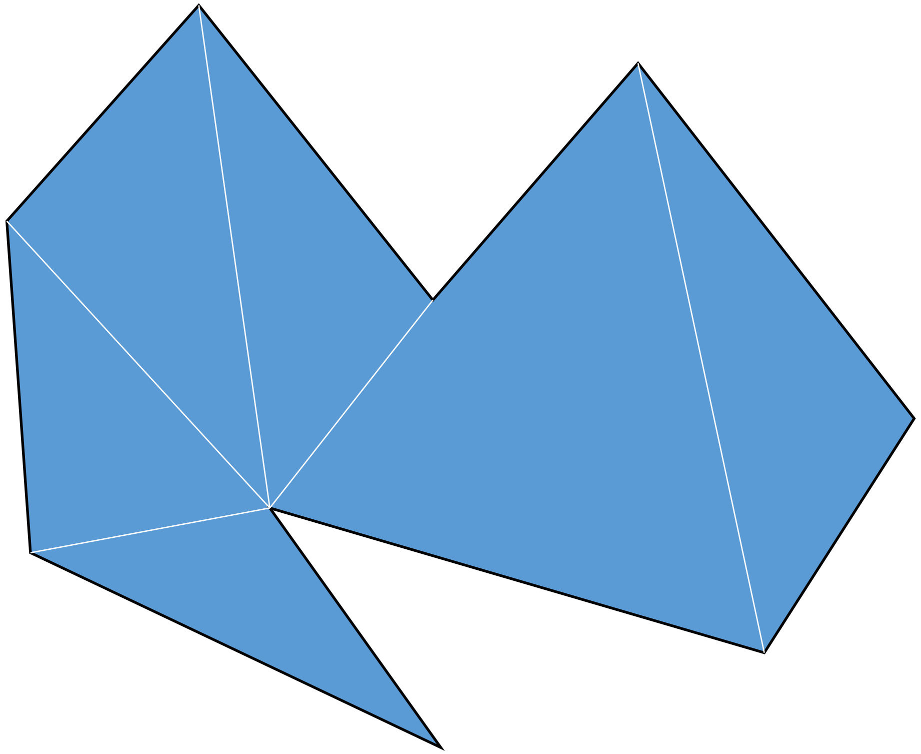
$$area = 0.5|a - b||c - d|$$



$$area(blue + green) = \left|\left(\overrightarrow{b - a}\right) \times \left(\overrightarrow{c - a}\right)\right|$$

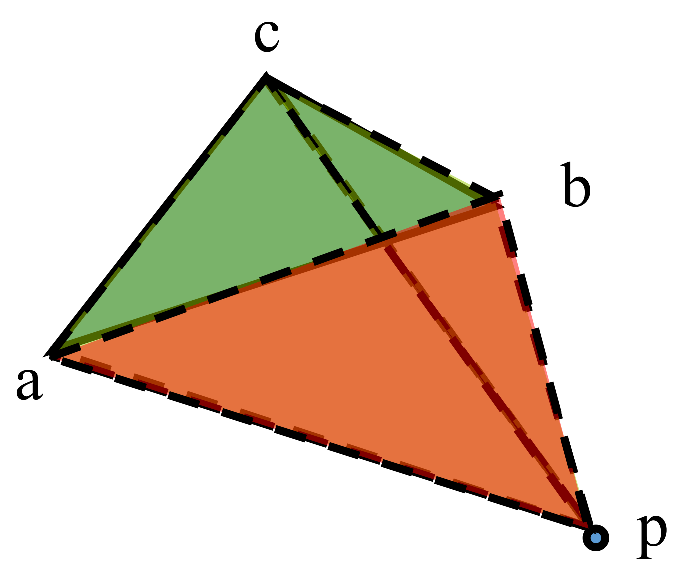
Sum of Areas

- Area of the polygon can be found by adding the areas of a triangulation



ANOTHER WAY OF COMPUTING AREA

$$Area\ (T) = Area(p, b, c) + Area(p, c, a) + Area(p, a, b)$$



Area of Polygon

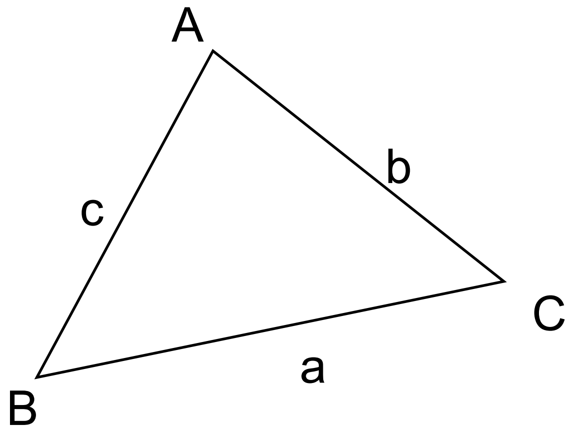
- Theorem: Let a polygon (convex or non-convex) P have vertices $v_0, v_1, \ldots, v_{n-1}$

$$Area(P) = A(p, v_0, v_1) + A(p, v_1, v_2) + A(p, v_2, v_3) + \cdots + A(p, v_{n-2}, v_{n-1}) + A(p, v_{n-1}, v_0)$$

- Proof is by induction

AREA OF TRIANGLE

- HERON'S FORMULA:
  - $Area = sqrt\left(s(s-a)(s-b)(s-c)\right)$
  - where $s = (a+b+c)/2$ is the semiperimeter



AREA OF TRIANGLE

- WHAT IF ONLY THE VERTICES OF THE TRIANGLE ARE GIVEN?
- GIVEN 3 VERTICES $(x_1, y_1), (x_2, y_2), (x_3, y_3)$

$$Area = \frac{|x_1 * y_2 + x_2 * y_3 + x_3 * y_1 - x_2 * y_1 - x_3 * y_2 - x_1 * y_3|}{2}$$

- Note: abs can be omitted if the vertices are in **counterclockwise** order. If the vertices are in clockwise order, the difference evaluates to a negative quantity

Area of triangle

- That hard-to-memorize expression can be written this way:

- Area = ½ *

$$\begin{vmatrix} x1 & y1 \\ x2 & y2 \\ x3 & y3 \\ x1 & y1 \end{vmatrix}$$

-

+

AREA OF CONVEX POLYGON

- IT TURNS OUT THE PREVIOUS FORMULA STILL WORKS!

- AREA = ½ *

$$\begin{vmatrix} x1 & y1 \\ x2 & y2 \\ x3 & y3 \\ x4 & y4 \\ x5 & y5 \\ x1 & y1 \end{vmatrix}$$

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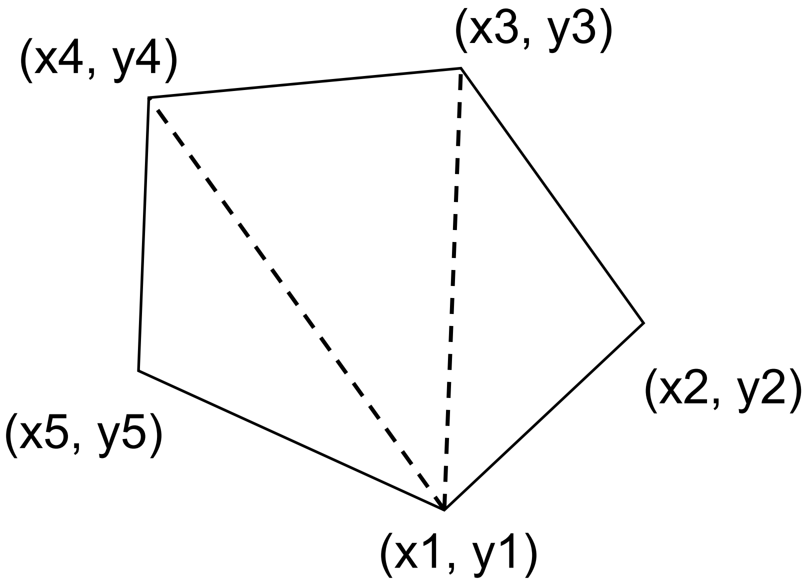
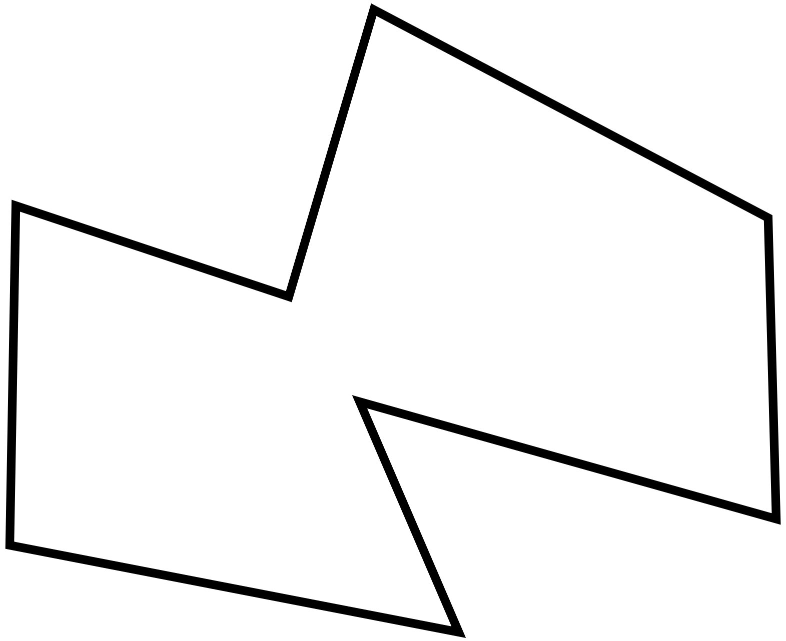


Area of (non-convex) polygon

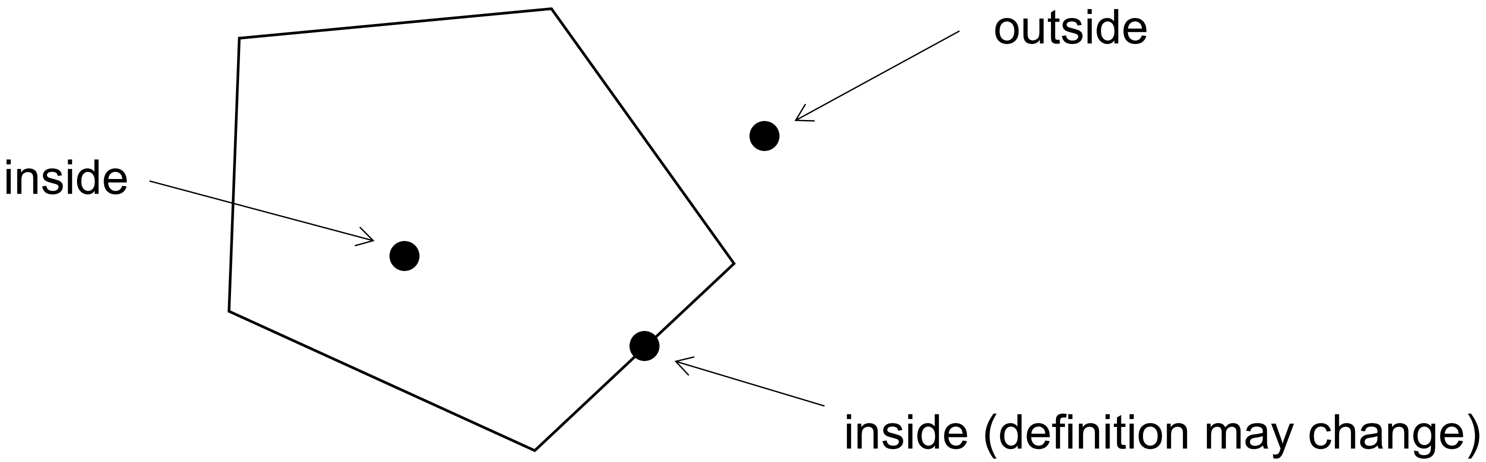
- Miraculously, the same formula still holds for non-convex polygons!

- Area = ½ * …



POINT INSIDE CONVEX POLYGON?

- GIVEN A CONVEX POLYGON AND A POINT, IS THE POINT CONTAINED INSIDE THE POLYGON?
  - Assume the vertices are given in **counterclockwise** order for convenience

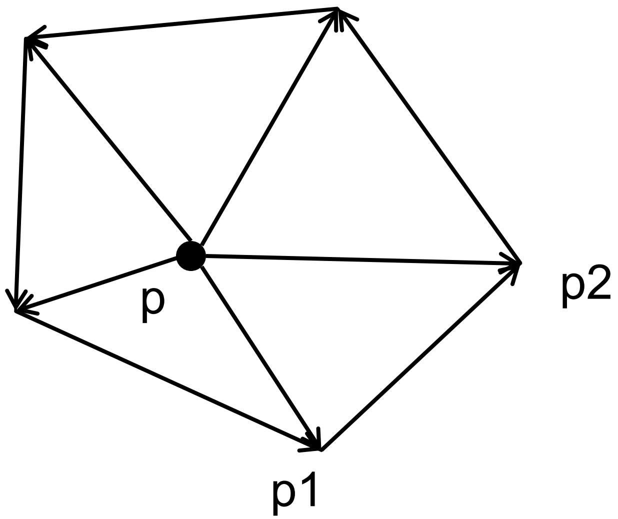


Detour – Is polygon convex?

- A quick question – how to tell if a polygon is convex?
- Answer: It is convex if and only if every turn (at every vertex) is a left turn
  - Whether a "straight" turn is allowed depends on the problem definition
- Our crossProd function is so useful

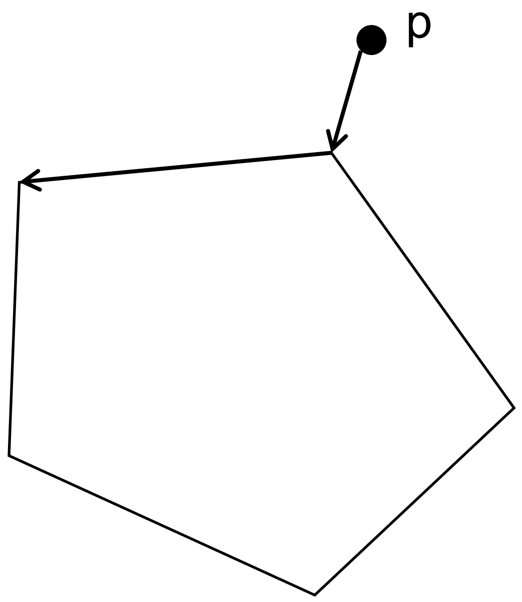
Point inside convex polygon?

- Consider the turn p → p1 → p2
- If p does lie inside the polygon, the turn must **not** be a right turn
- Also holds for other edges (mind the directions)



POINT INSIDE CONVEX POLYGON?

- CONVERSELY, IF P WAS OUTSIDE THE POLYGON, THERE WOULD BE A RIGHT TURN FOR SOME EDGE

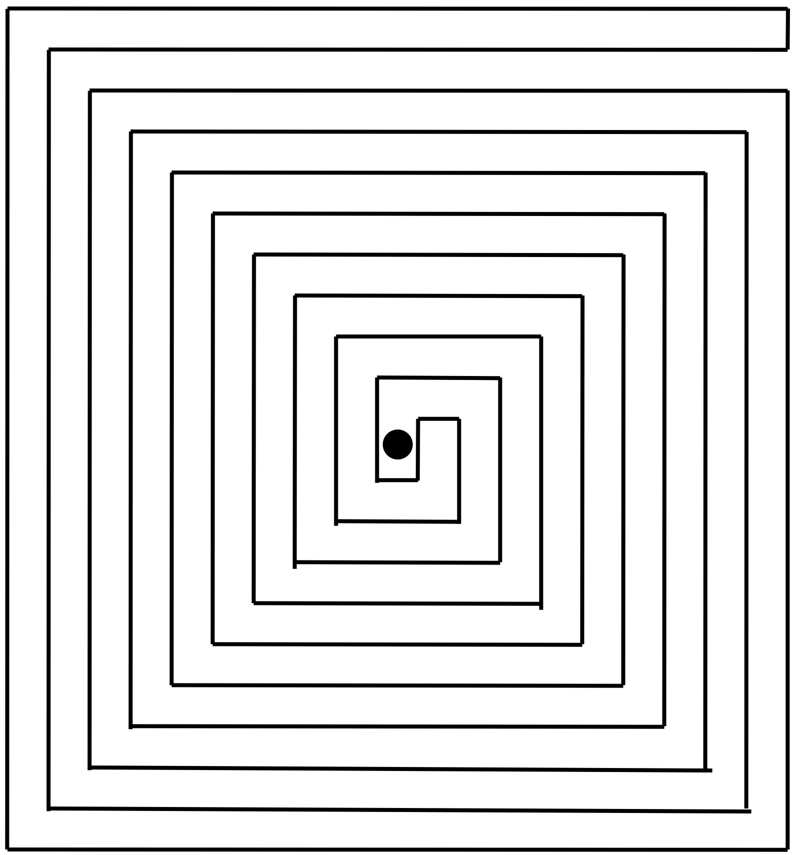


POINT INSIDE CONVEX POLYGON

- CONCLUSION: P IS INSIDE THE POLYGON IF AND ONLY IF IT MAKES A NON-LEFT TURN FOR EVERY EDGE (IN THE COUNTERCLOCKWISE DIRECTION)

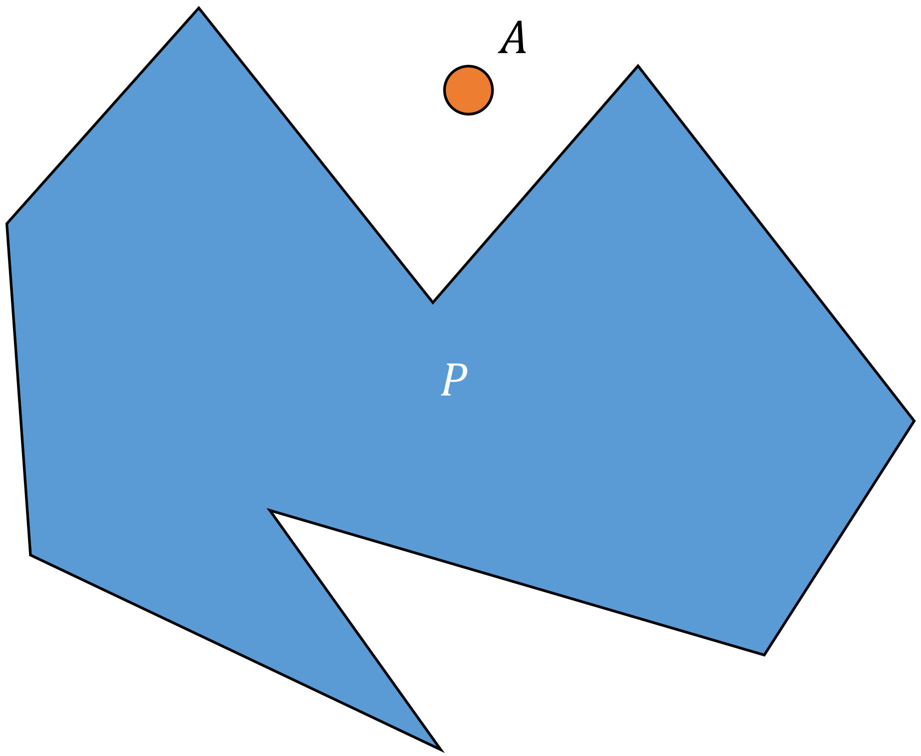
Point inside (non-convex) polygon

- Such a pain



Point inside a polygon test

- How can we tell if point A is inside of polygon P?



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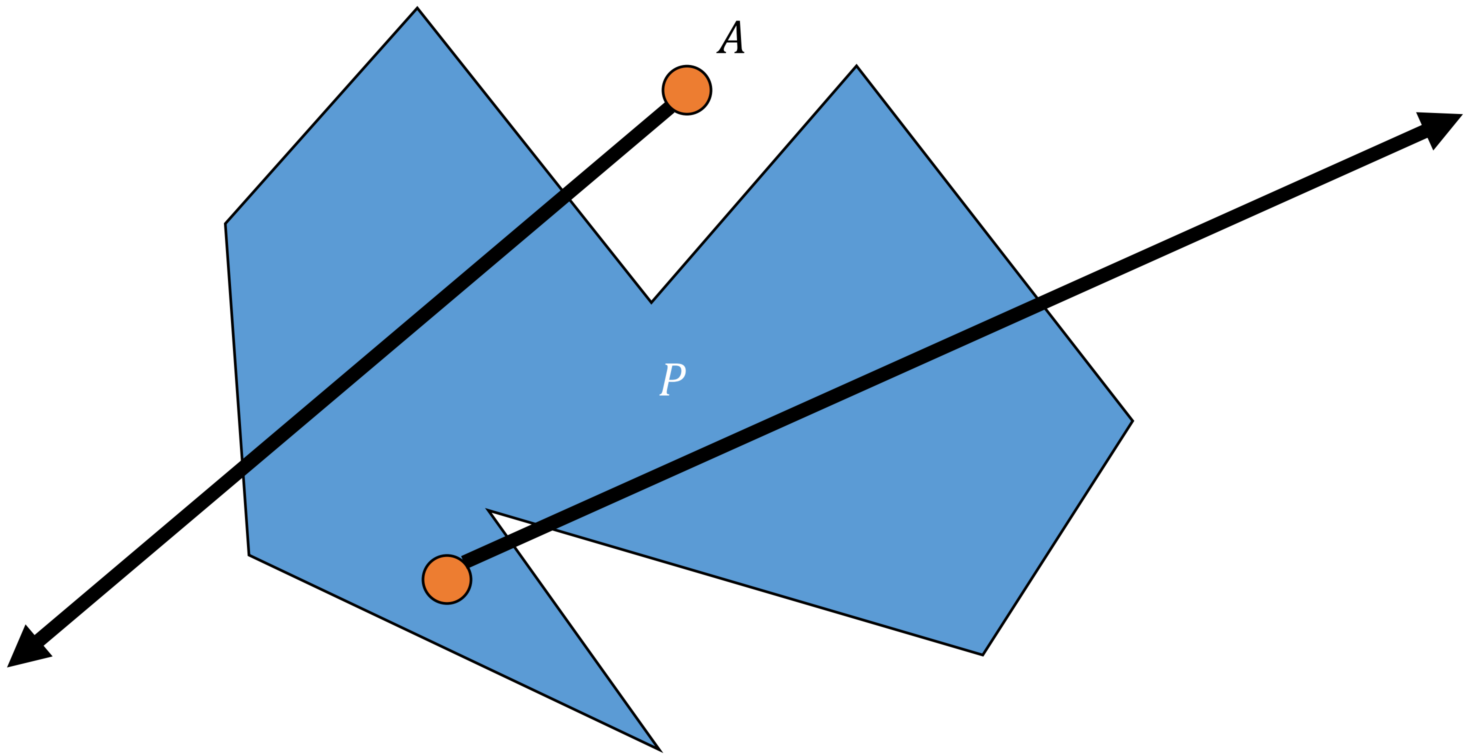
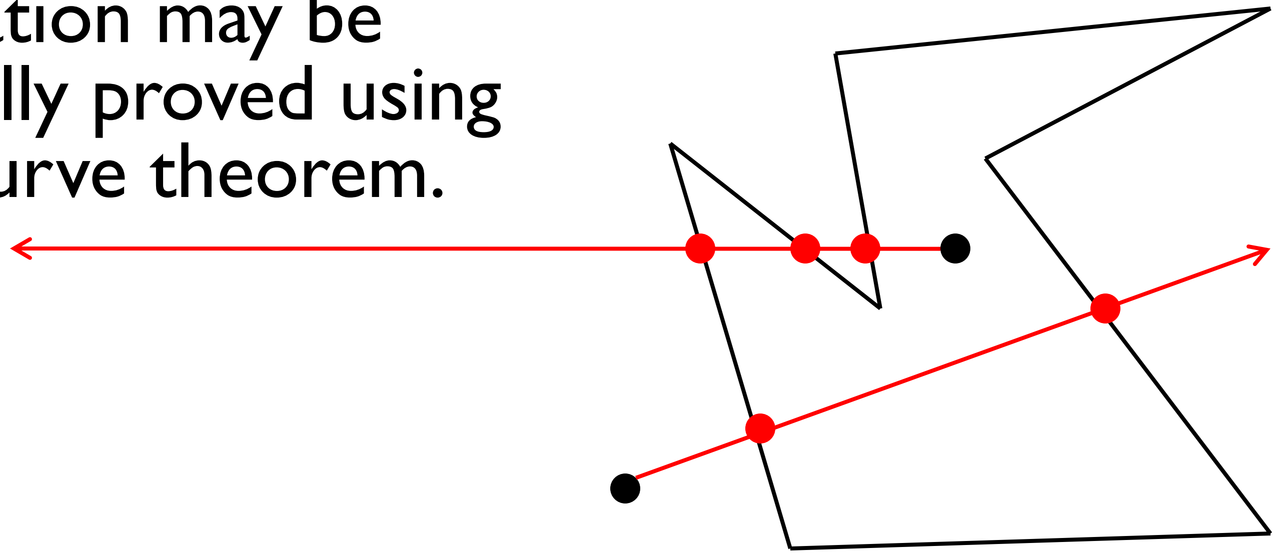


POINT INSIDE POLYGON

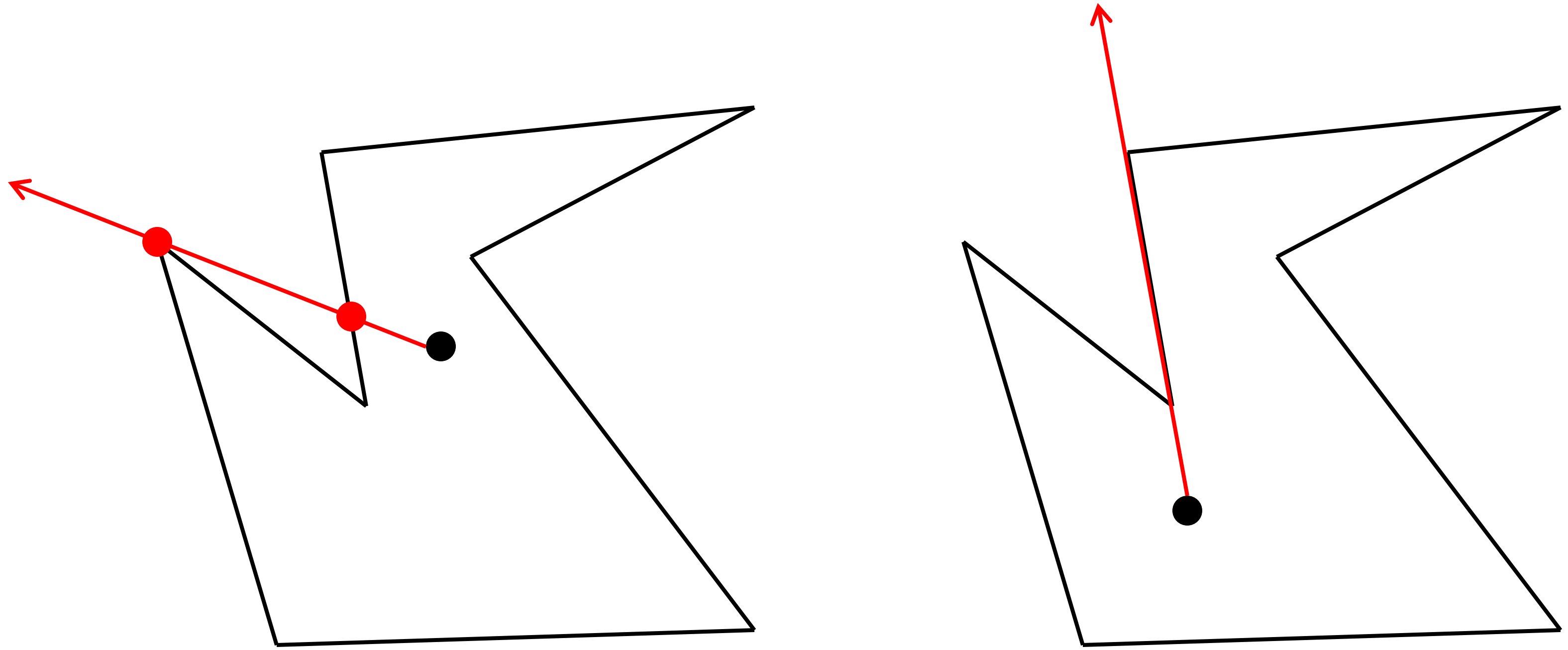
- EVEN–ODD RULE ALGORITHM
  - If a point A moves along a ray from infinity to A
  - If it crosses the boundary of a polygon, possibly several times, then it alternately goes from the outside to inside to outside
  - After every two "border crossings" the moving point goes outside.

  - This observation may be mathematically proved using the Jordan curve theorem.



Point inside polygon

- Problematic cases: Degenerate intersections



- Solution: Pick a random direction (i.e. random slope). If the ray hits a vertex of the polygon, pick a new direction. Repeat.

Point inside polygon

- Solution: Pick a random direction (i.e. random slope). If the ray hits a vertex of the polygon, pick a new direction. Repeat.