

COT 4521: INTRODUCTION TO COMPUTATIONAL GEOMETRY



Polygons

Paul Rosen
Assistant Professor
University of South Florida

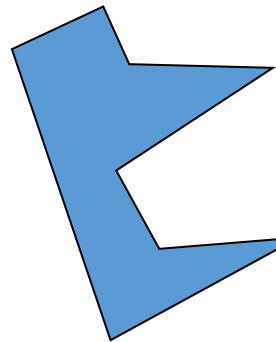
Some slides from Valentina Korzhova



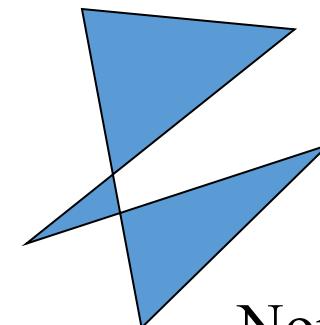
POLYGON

- **POLYGON IS A REGION OF A PLANE BOUNDED BY A FINITE COLLECTION OF LINE SEGMENTS FORMING A SIMPLE CLOSED CURVE.**

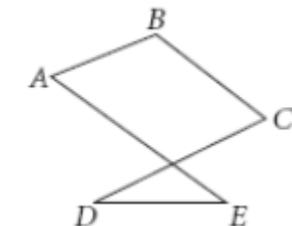
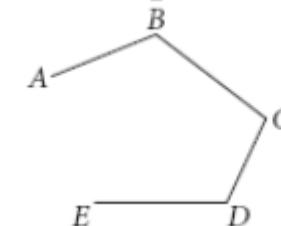
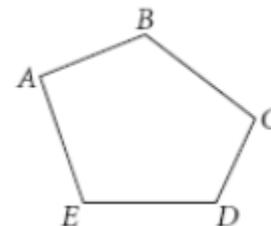
Boundary



Simple

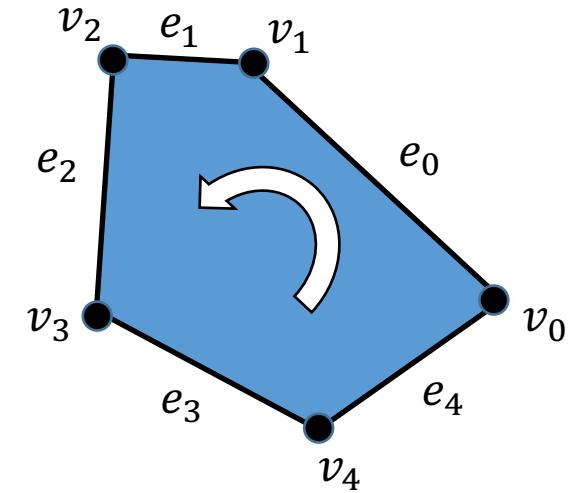


Non-simple



POLYGON

- EDGES – THE LINE SEGMENTS $(e_0, e_1, \dots, e_{n-1})$
- VERTICES – THE POINTS WHERE ADJACENT EDGES MEET
 - Start at any vertex and list the vertices consecutively in a counterclockwise direction $(v_0, v_1, \dots, v_{n-1})$
- ANGLES
 - Name by angle naming convention
 - $\angle v_0, \angle v_1, \dots, \angle v_{n-1}$



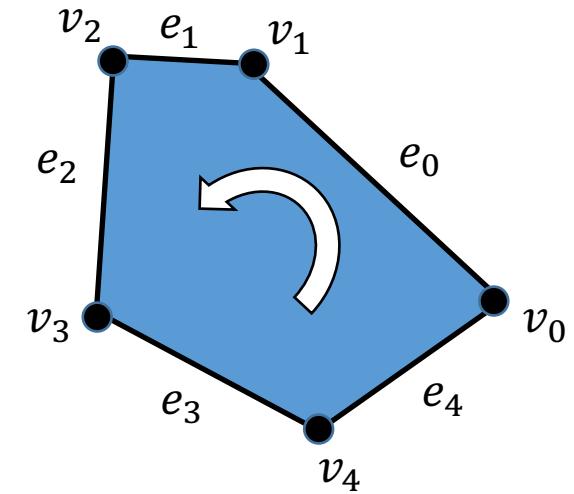
PROPERTIES OF POLYGON

- THE LINE SEGMENTS THAT MAKE-UP A POLYGON (CALLED SIDES OR EDGES) MEET ONLY AT THEIR ENDPOINTS, CALLED VERTICES (SINGULAR: VERTEX) OR LESS FORMALLY "CORNERS"
- EXACTLY TWO EDGES MEET AT EVERY VERTEX
- THE NUMBER OF EDGES ALWAYS EQUALS THE NUMBER OF VERTICES.
- TWO EDGES MEETING AT A CORNER ARE REQUIRED TO FORM AN ANGLE THAT IS NOT STRAIGHT (180°); OTHERWISE, THE LINE SEGMENTS WILL BE CONSIDERED PARTS OF A SINGLE EDGE



FORMAL DEFINITION OF A SIMPLE POLYGON

- FORMALLY, WE ARE GIVEN N VERTICES (I.E., POINTS) v_0, v_1, \dots, v_{n-1} , THE CHAIN FORMED BY $v_0v_1 \dots v_{n-1}$ IS A SIMPLE POLYGON IFF
 - The segments $e_0 = v_0v_1, \dots, e_{n-2} = v_{n-2}v_{n-1}$, and $e_{n-1} = v_{n-1}v_0$ are disjoint in their interior
 - Consecutive segments intersect only in their endpoints. Namely $e_i \cap e_{i+1} = v_{i+1}$ for $i = 0, \dots, n - 2$ and $e_{n-1} \cap e_0 = v_0$
 - Non adjacent segments do not intersect $e_i \cap e_j = \emptyset$, for $\forall j \neq i + 1$
- WE WORK mod n .
 - Namely $v_i = v_i \text{ mod } n$



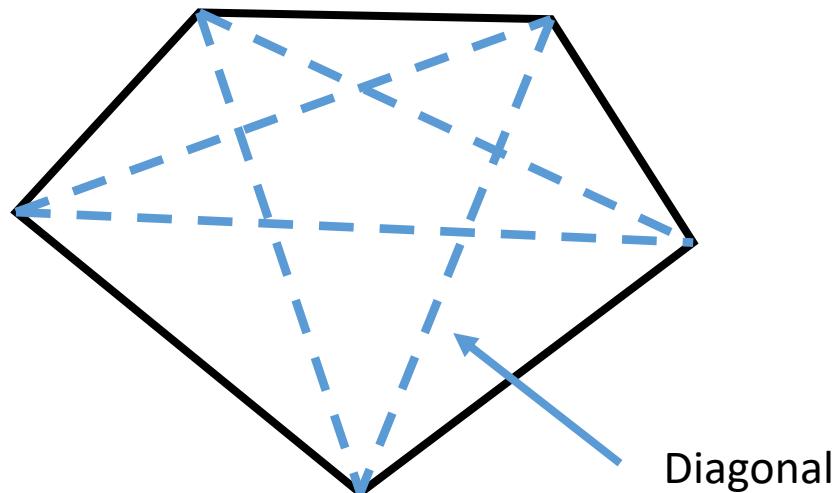
DIAGONALS

- DEFINITION: THE DIAGONAL OF A POLYGON IS A LINE SEGMENT LINKING TWO NON-ADJACENT VERTICES, INTERIOR TO THE POLYGON, AND NOT BLOCKED BY PORTION OF POLYGON'S BOUNDARY
- LEMMA: THE SEGMENT $s = v_i v_j$ IS A DIAGONAL OF P IFF
 1. For all edges e of P that are not incident to either v_i or v_j , s and e do not intersect: $s \cap e = \emptyset$;
 2. s is internal to P in neighborhood of v_i and v_j



DIAGONALS

- DEFINITION: THE DIAGONAL OF A POLYGON IS A LINE SEGMENT LINKING TWO NON-ADJACENT VERTICES, INTERIOR TO THE POLYGON, AND NOT BLOCKED BY PORTION OF POLYGON'S BOUNDARY

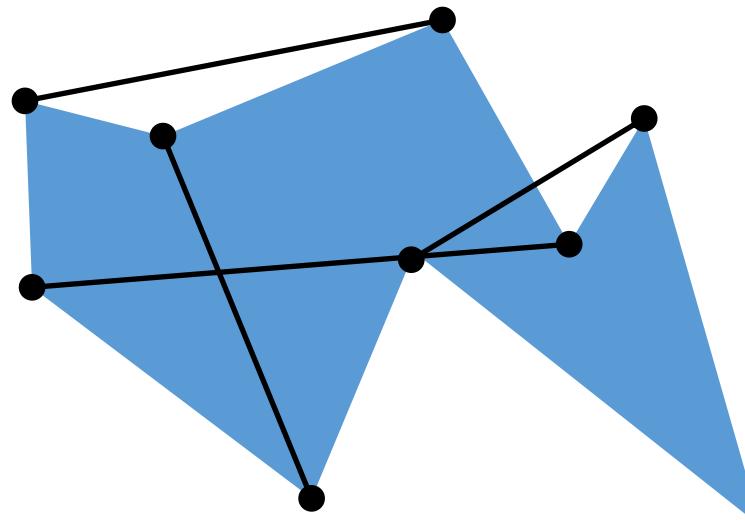


Diagonal



DIAGONALS

- DEFINITION: THE DIAGONAL OF A POLYGON IS A LINE SEGMENT LINKING TWO NON-ADJACENT VERTICES, INTERIOR TO THE POLYGON, AND NOT BLOCKED BY PORTION OF POLYGON'S BOUNDARY

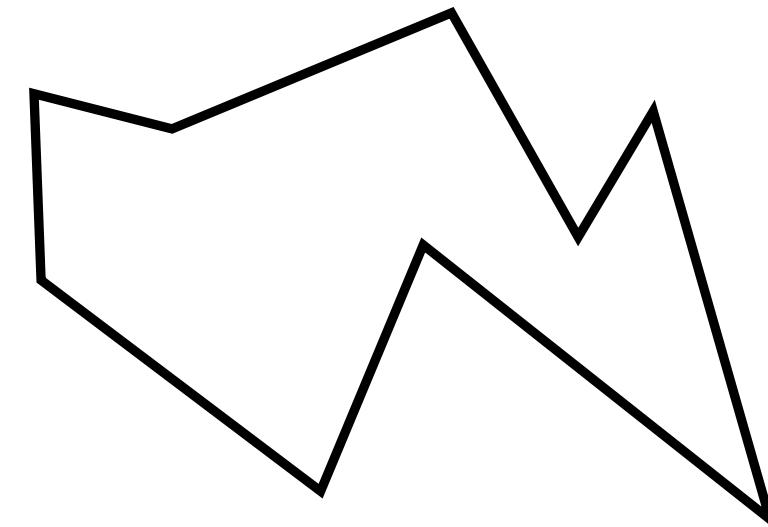


which are legal diagonals?



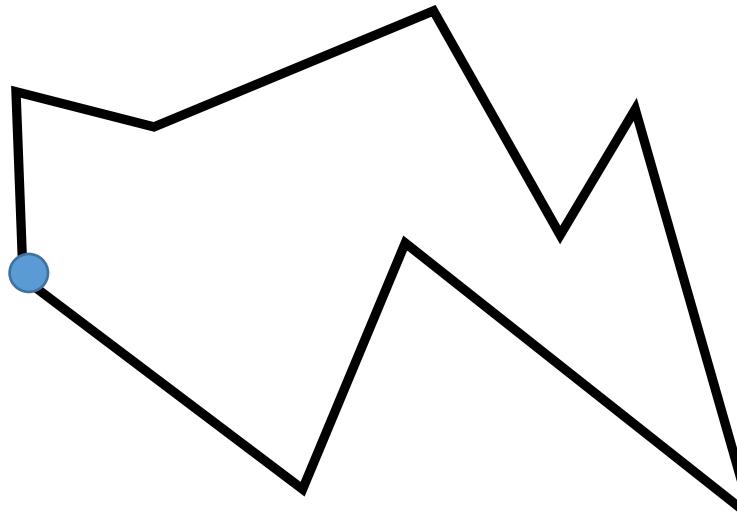
EFFICIENT DIAGONAL FINDING

- BRUTE FORCE
 - Generate each potential diagonal— $O(n^2)$ possible diagonals
 - Check each diagonal against the boundary to determine
 - If it is inside or outside
 - If it intersects the boundary
 - Total performance $O(n^3)$
- CAN WE DO BETTER?



EFFICIENT DIAGONAL FINDING

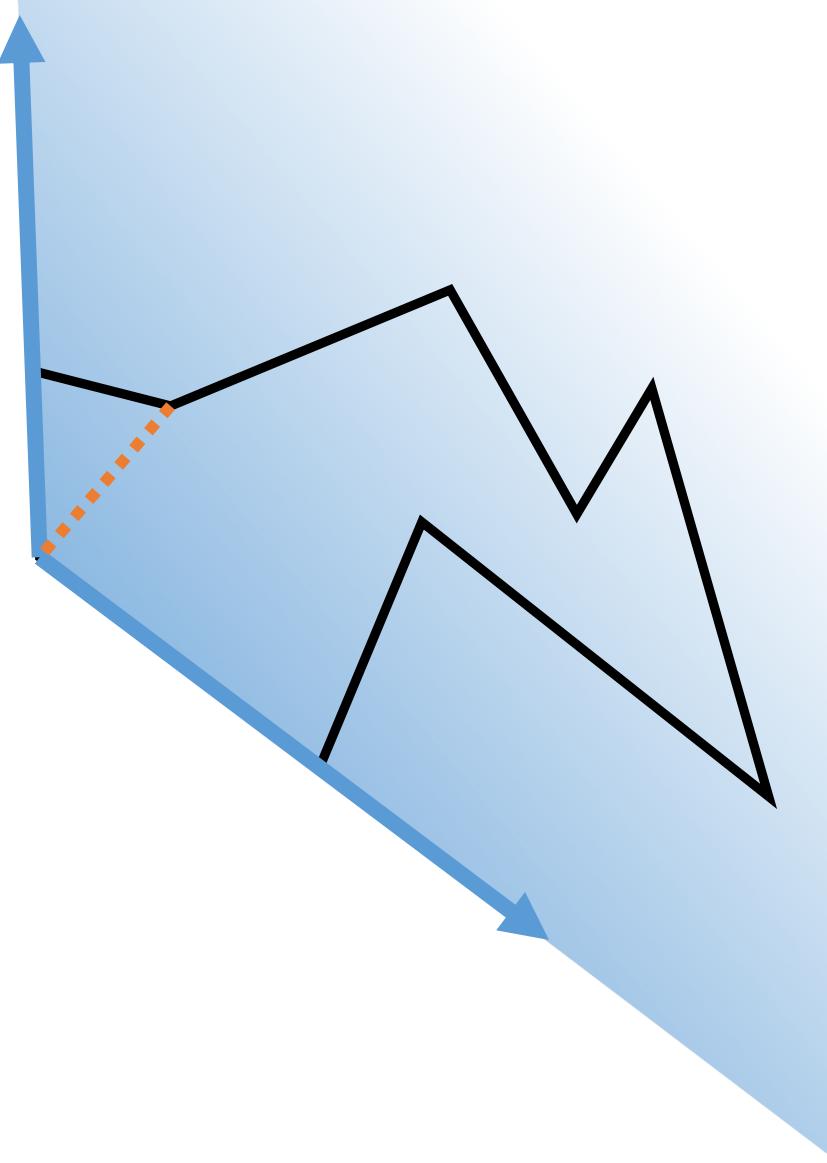
- VARIATION ON A SWEEP ALGORITHM
 - From a given vertex, sweep both clockwise and counterclockwise, ordered by the distance of points
 - Retain a notion of “valid space” for diagonals



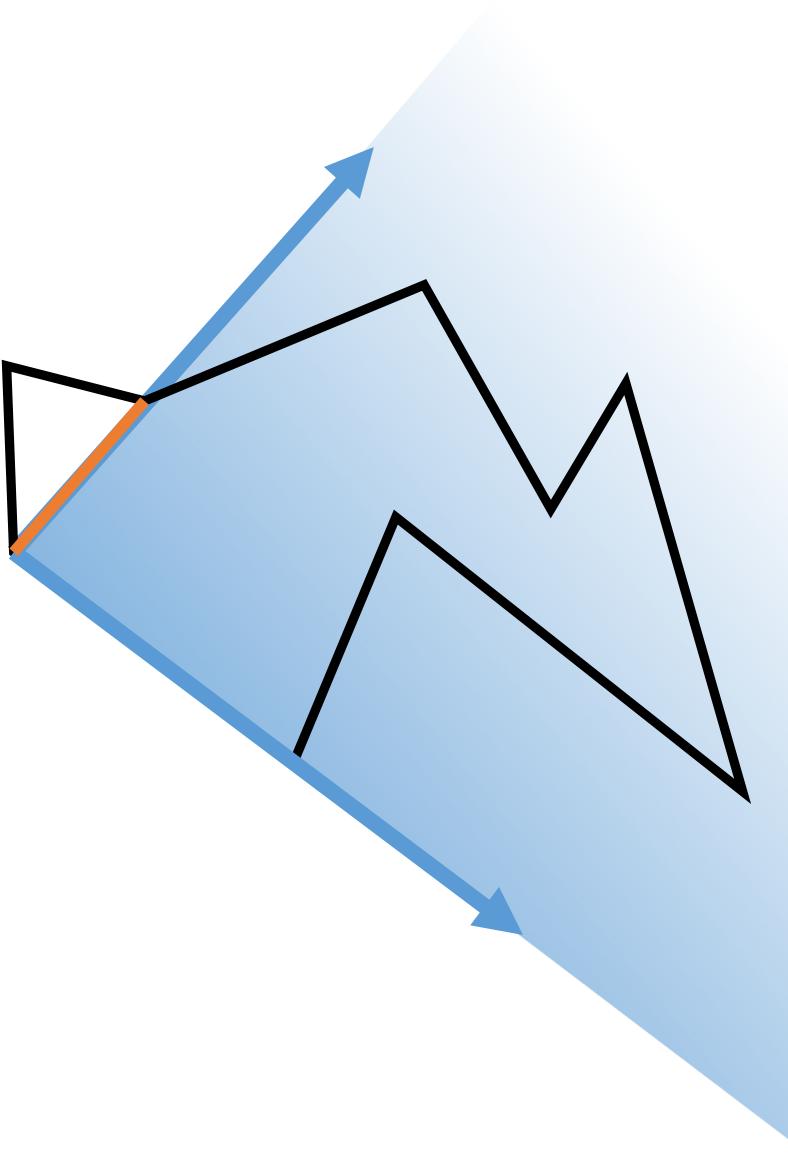
EFFICIENT DIAGONAL FINDING



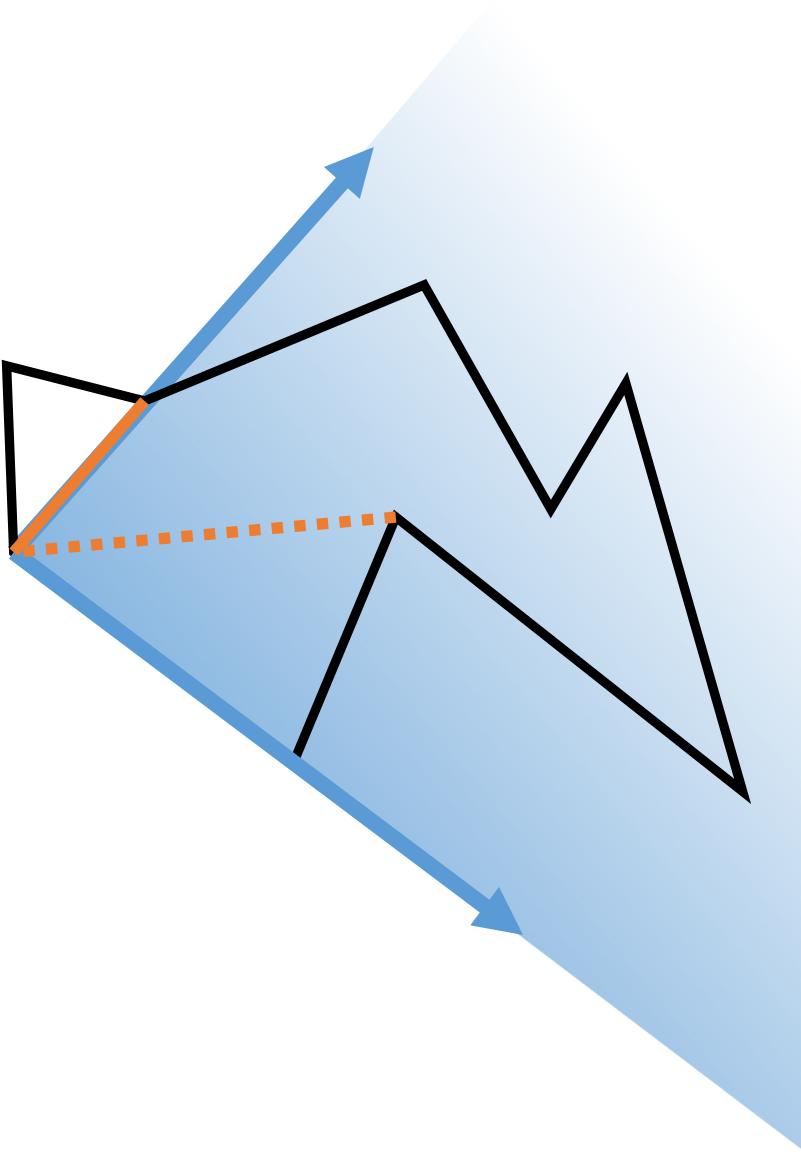
EFFICIENT DIAGONAL FINDING



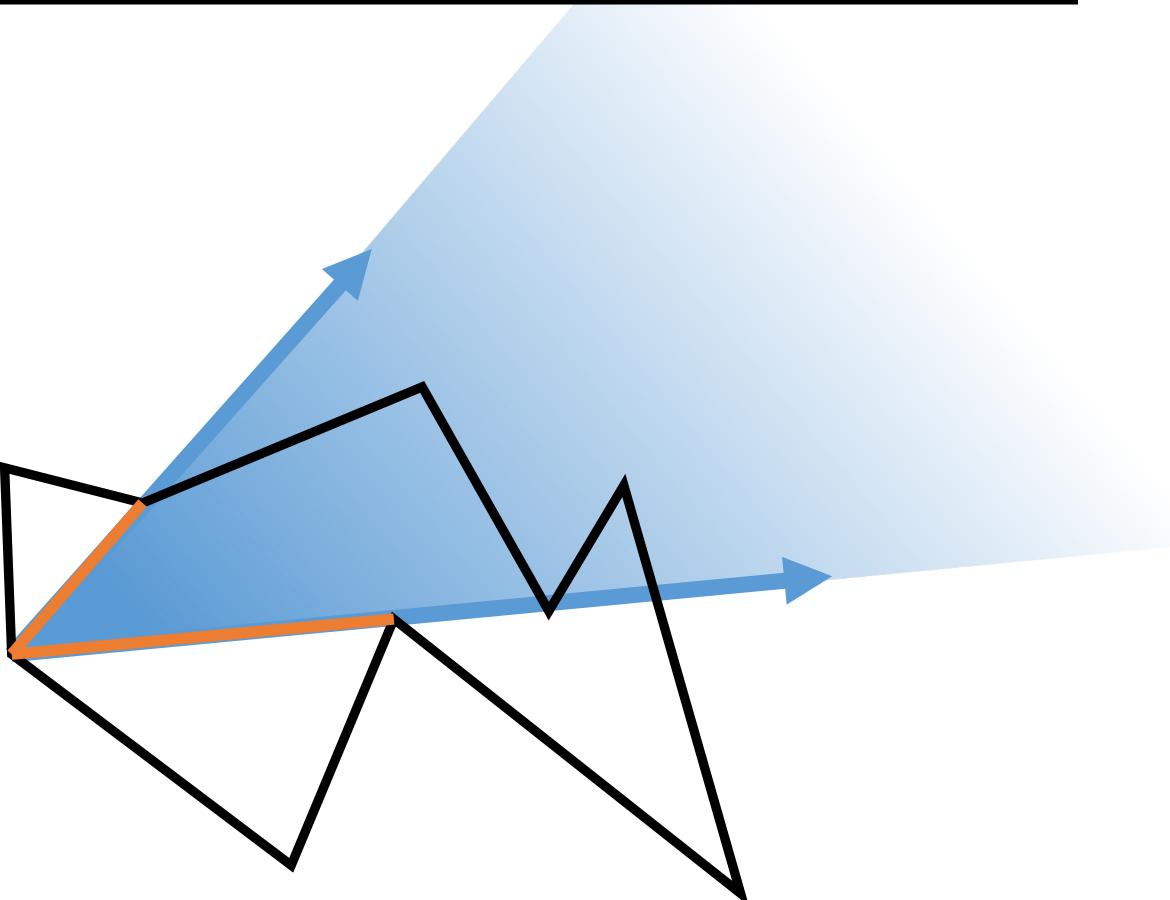
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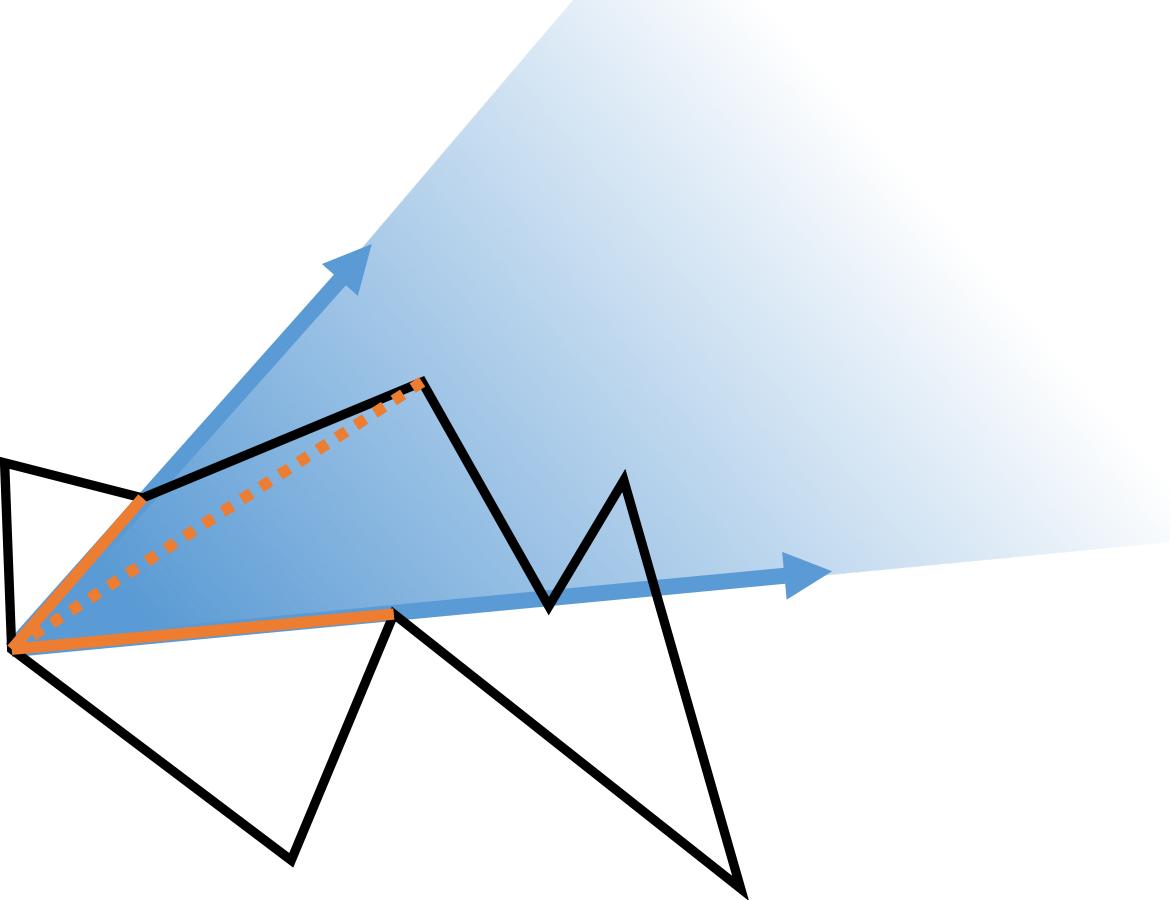
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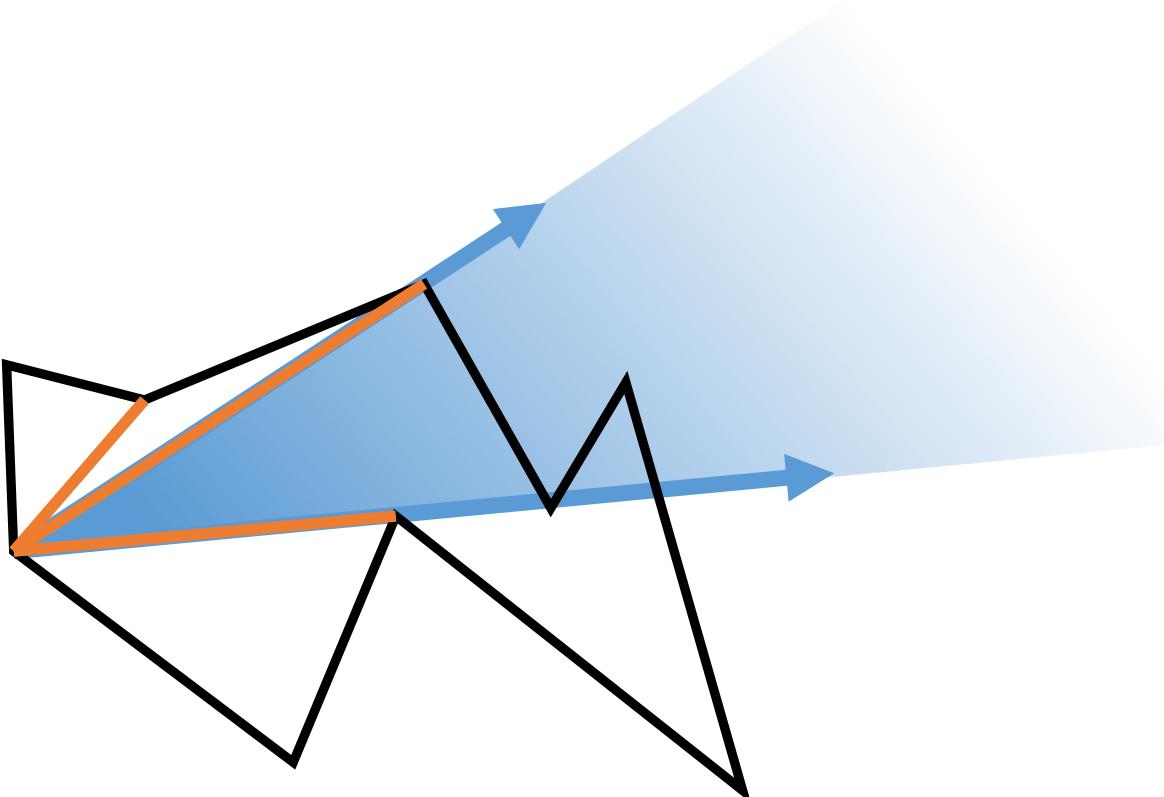
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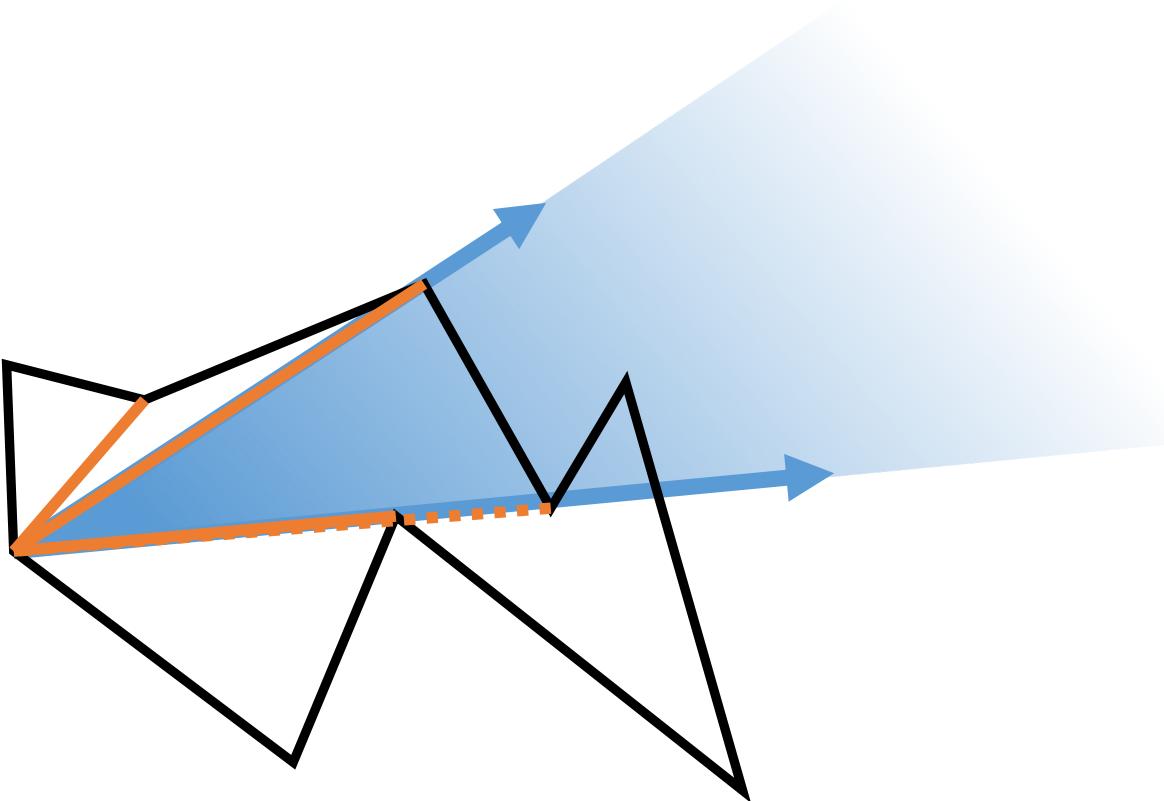
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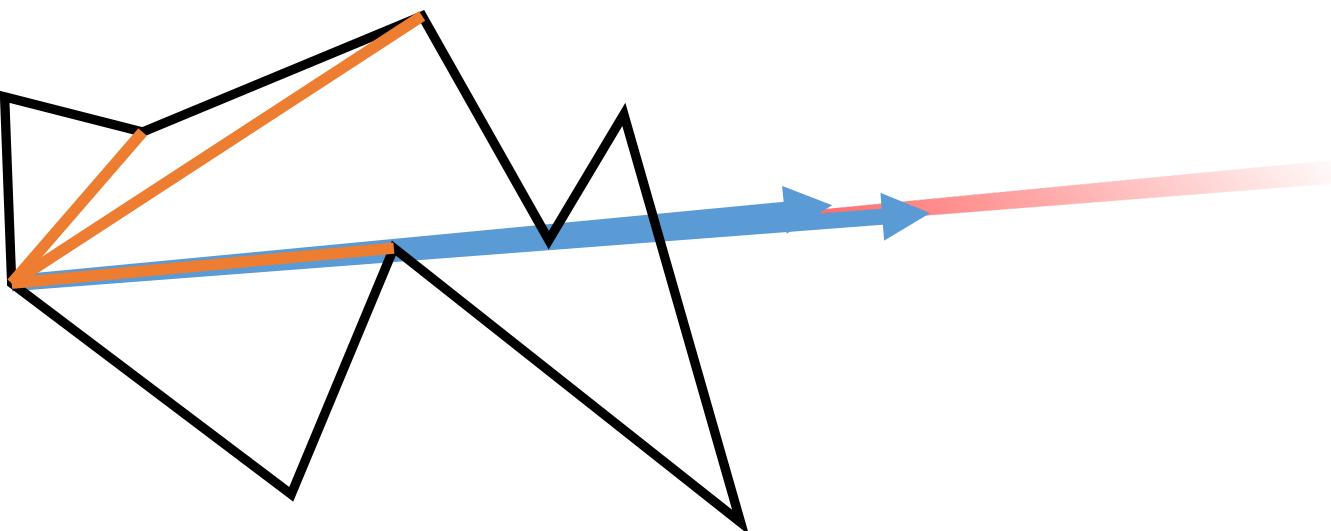
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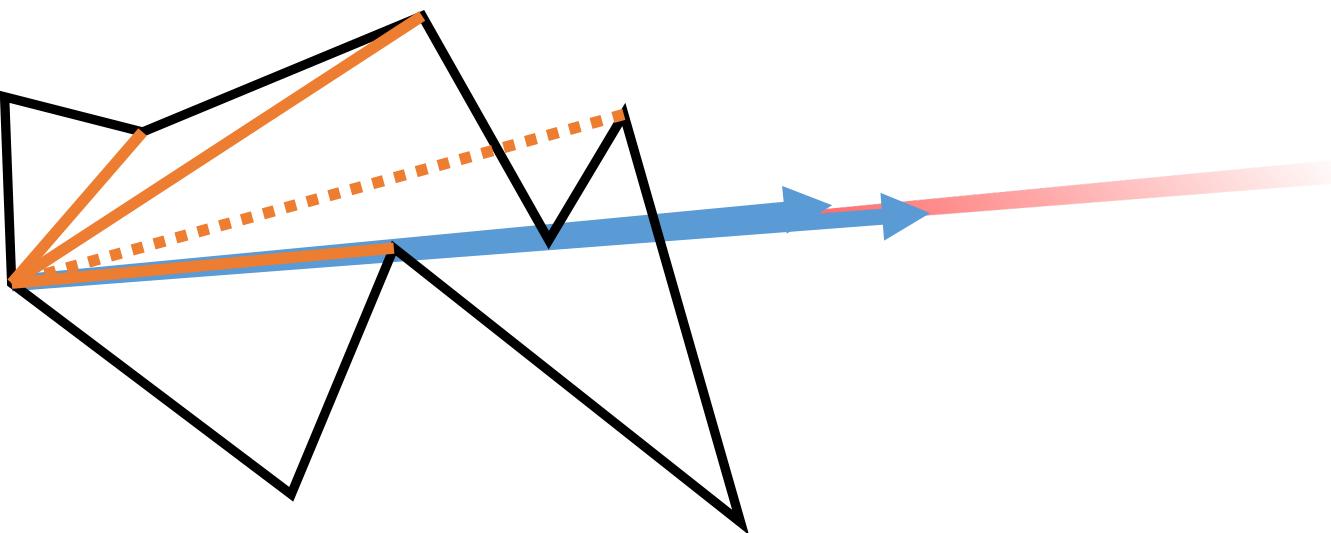
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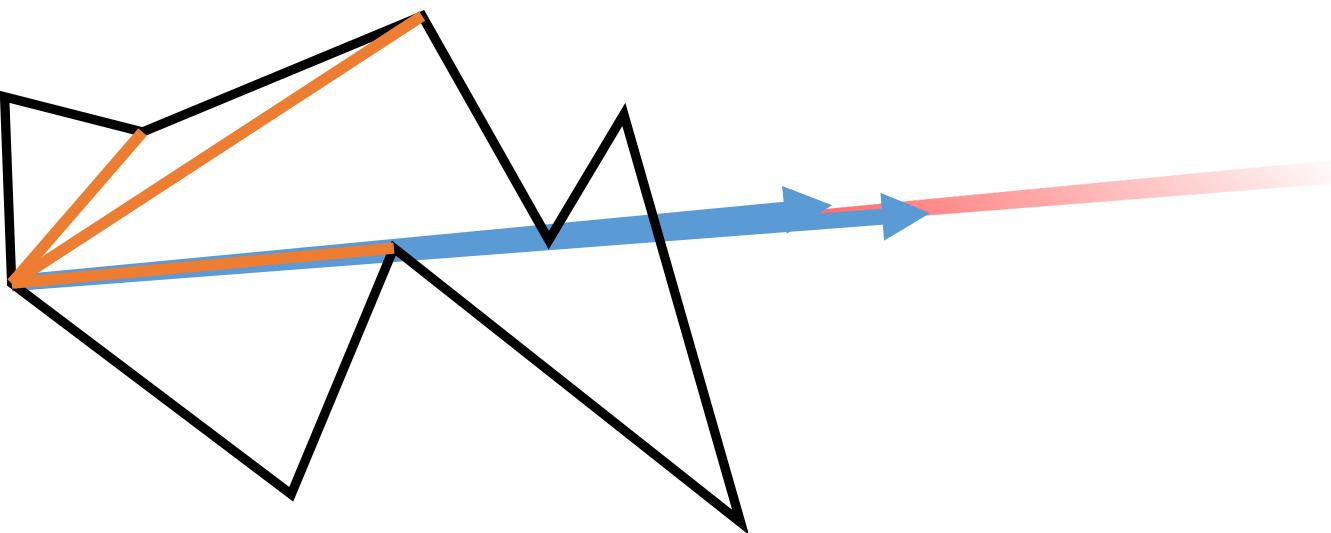
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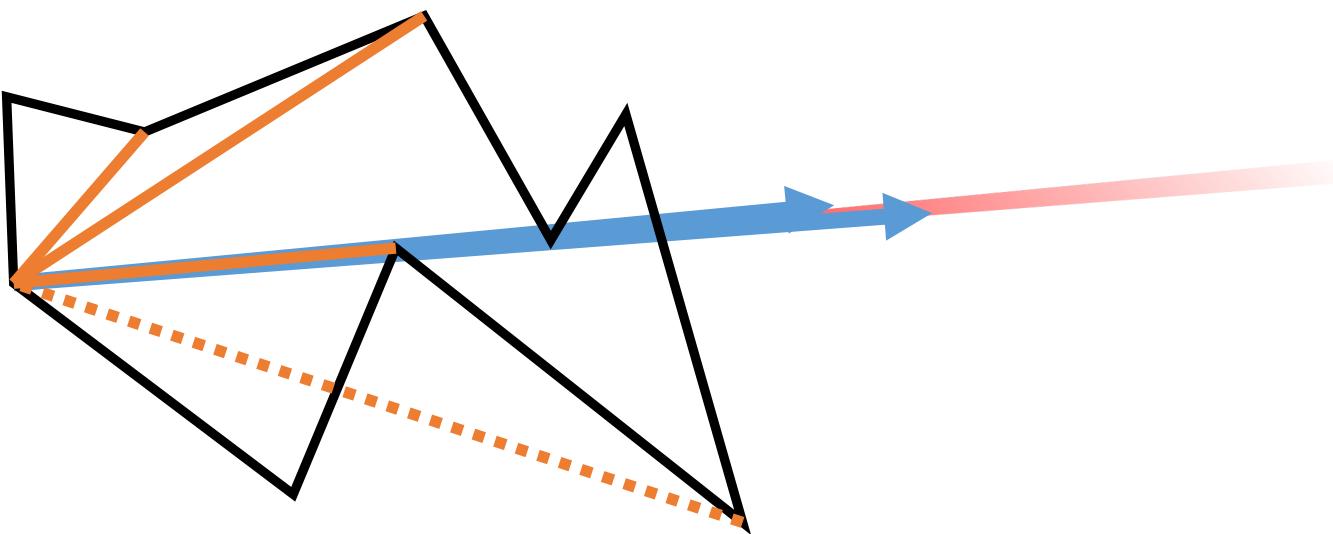
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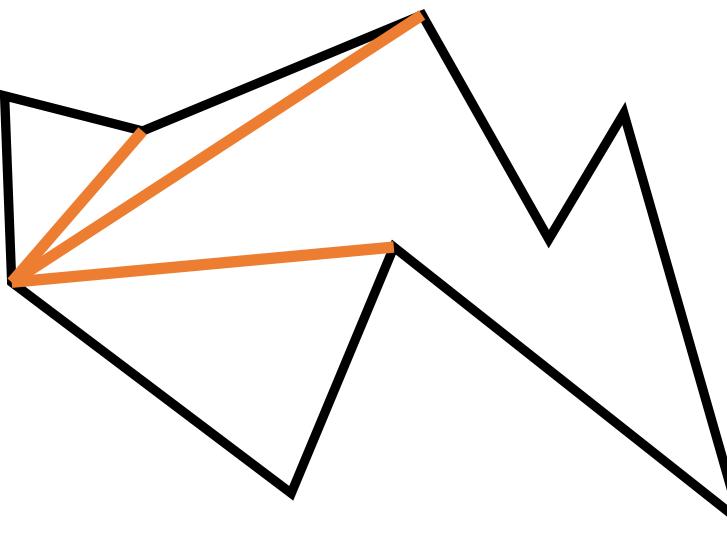
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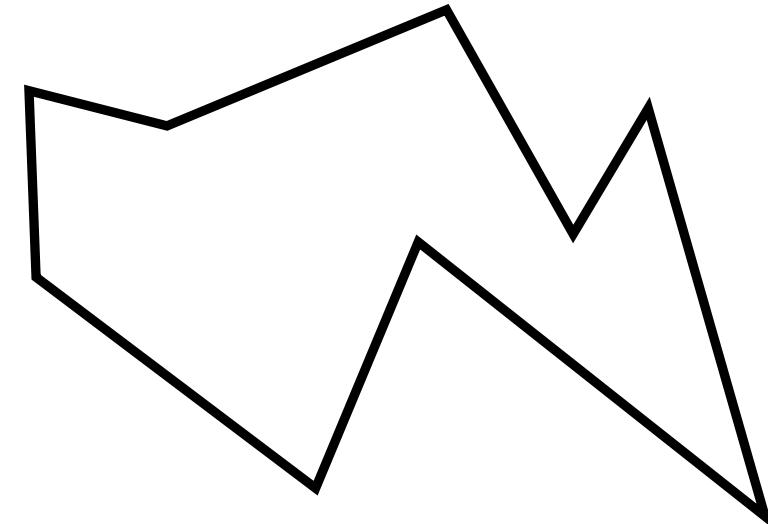


EFFICIENT DIAGONAL FINDING



EFFICIENT DIAGONAL FINDING

- ANALYSIS
 - Preprocessing: None
 - Query: Worst case $O(n^2)$; $O(n)$ per vertex
 - Storage: $O(n)$



TRIANGULATION THEORY: PROPERTIES

- LEMMA: AN INTERNAL DIAGONAL EXISTS BETWEEN ANY TWO NONADJACENT VERTICES OF A POLYGON P IF AND ONLY IF P IS CONVEX POLYGON.
- PROOF: THE PROOF CONSISTS OF TWO PARTS, BOTH ESTABLISHED BY CONTRADICTION.



CLASSIFICATION OF POLYGONS BY THE NUMBER OF EDGES

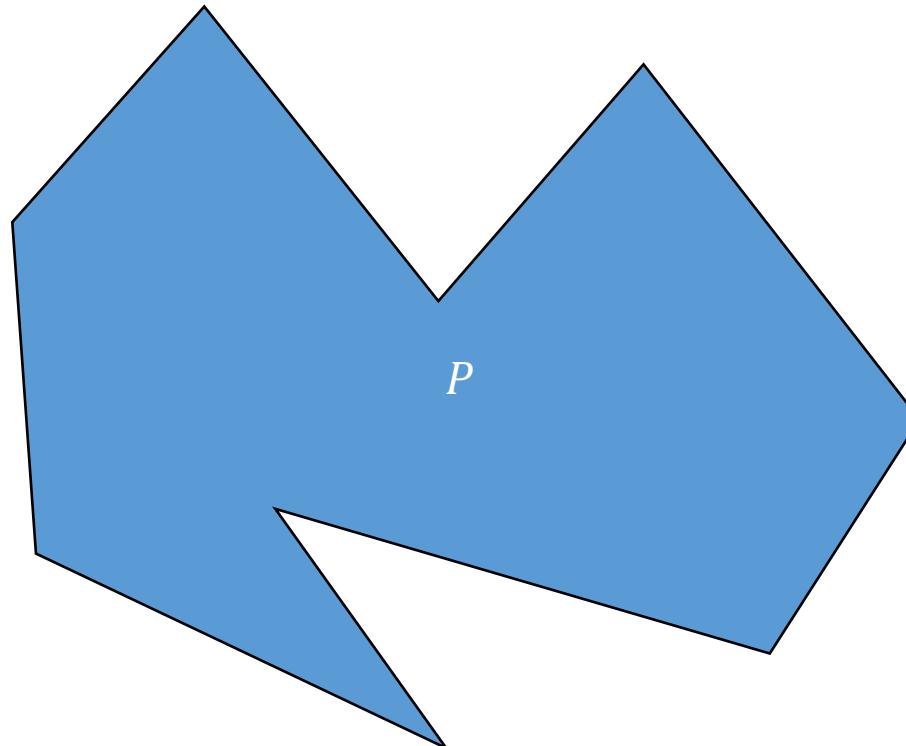
MOST COMMON POLYGONS

Number of Sides	Name
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
8	Octagon
9	Nonagon
10	Decagon
12	Dodecagon
n	n-gon



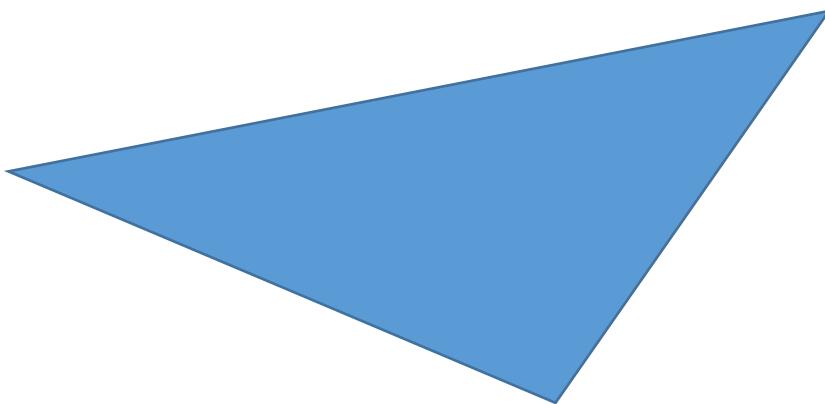
SUM OF ANGLES

- THE SUM OF THE INTERNAL ANGLES OF A POLYGON OF N VERTICES IS?



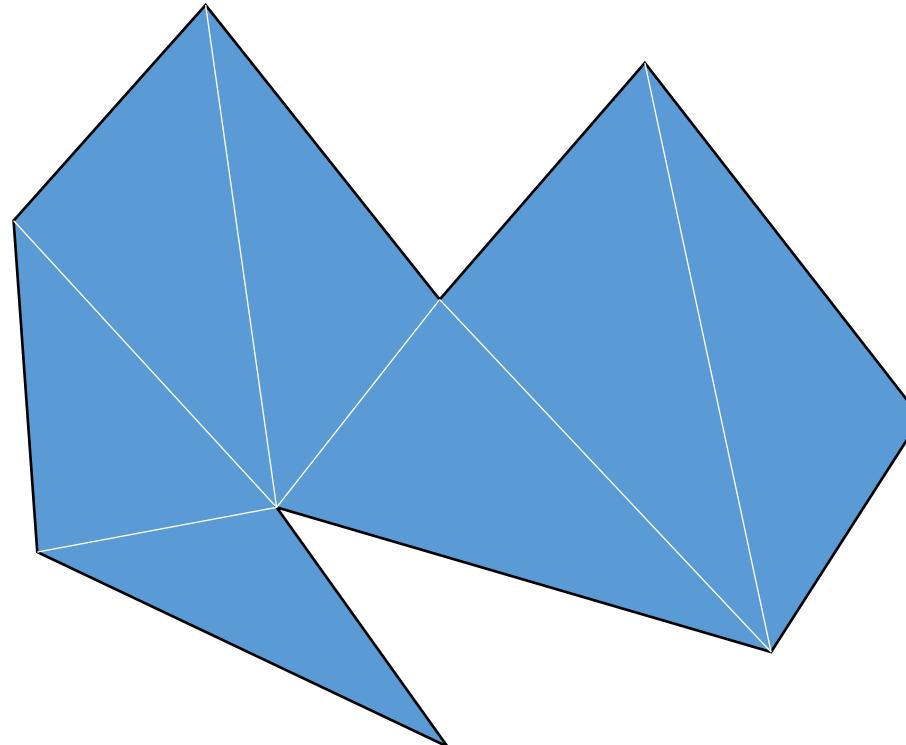
SUM OF ANGLES

- USE THE TRIANGLE ANGLE-SUM THEOREM TO FIND THE SUM OF THE MEASURES OF THE ANGLES OF A POLYGON.
- TRIANGLE ANGLE-SUM THEOREM
 - The sum of the measures of the angles of a triangle measure 180°



SUM OF ANGLES

- THEOREM: THE SUM OF THE MEASURES OF THE INTERNAL ANGLES OF AN N-GON IS $(N - 2) * 180$.



- PROOF BY INDUCTION



TRIANGULATION THEORY

- **THEOREM:** EVERY TRIANGULATION OF AN n -VERTEX POLYGON P USES $n - 3$ DIAGONALS AND CONSISTS OF $n - 2$ TRIANGLES.

- Proof by induction:

- Base case $N = 3$
- Assume true for any polygon $< N$ sides
- Given a N sided polygon partition it into two (N_1 and N_2) by adding a diagonal

Total number of diagonals:

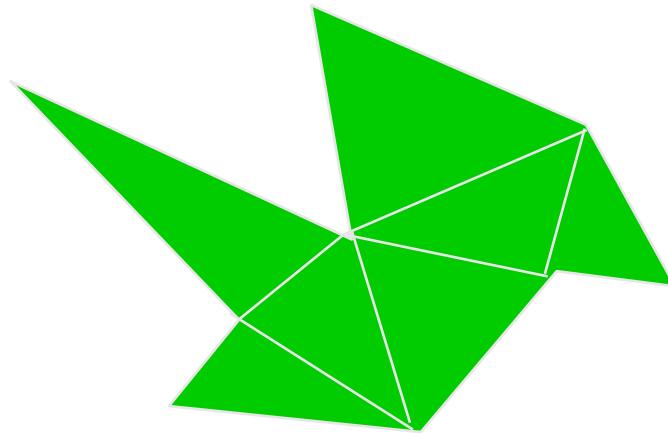
$$(N_1 - 3) + (N_2 - 3) + 1 = (N_1 + N_2 - 2) - 3 = N - 3$$

Total number of triangles:

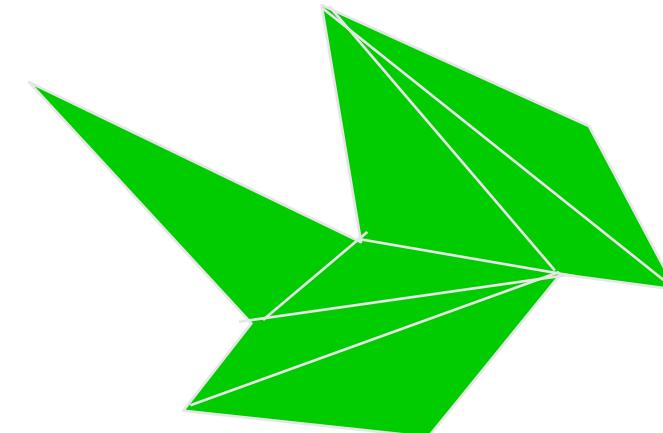
$$(N_1 - 2) + (N_2 - 2) = (N_1 + N_2) - 4 = N + 2 - 4 = N - 2$$



TRIANGULATION EXAMPLE

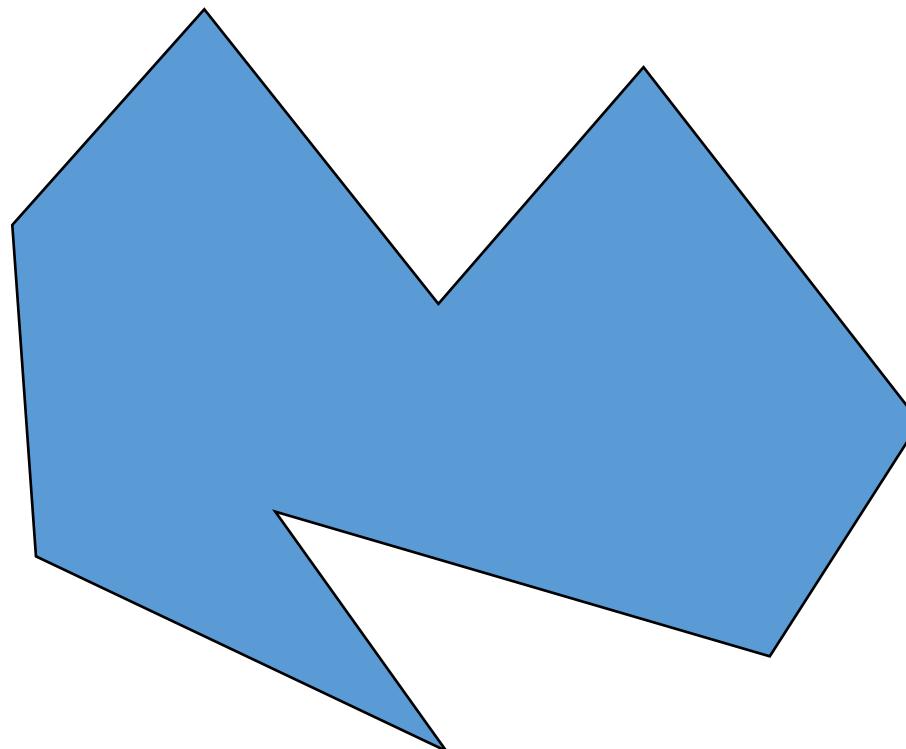


9 vertices
7 triangles
6 diagonals

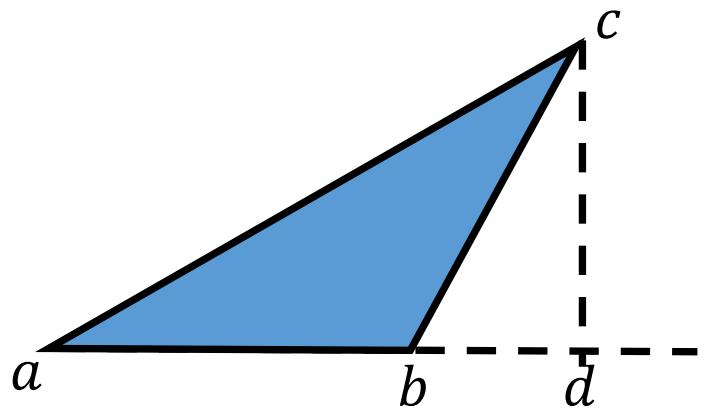


AREA OF A POLYGON

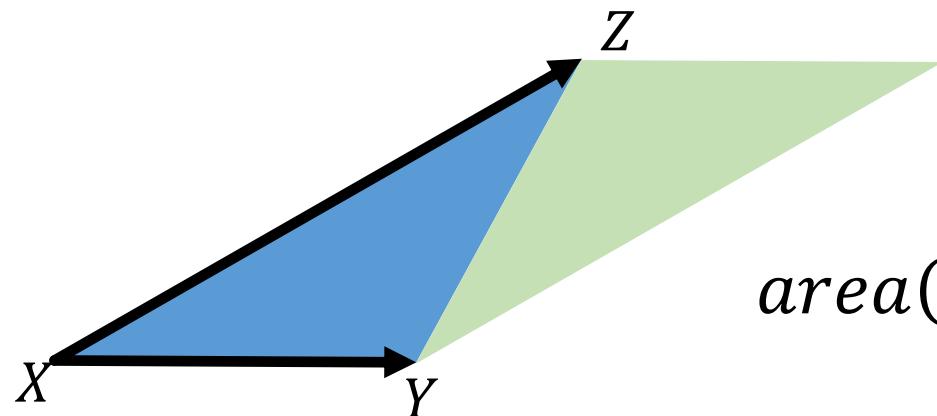
- WHAT IS THE AREA OF THE POLYGON



AREA OF A TRIANGLE



$$\text{area} = 0.5|a - b||c - d|$$

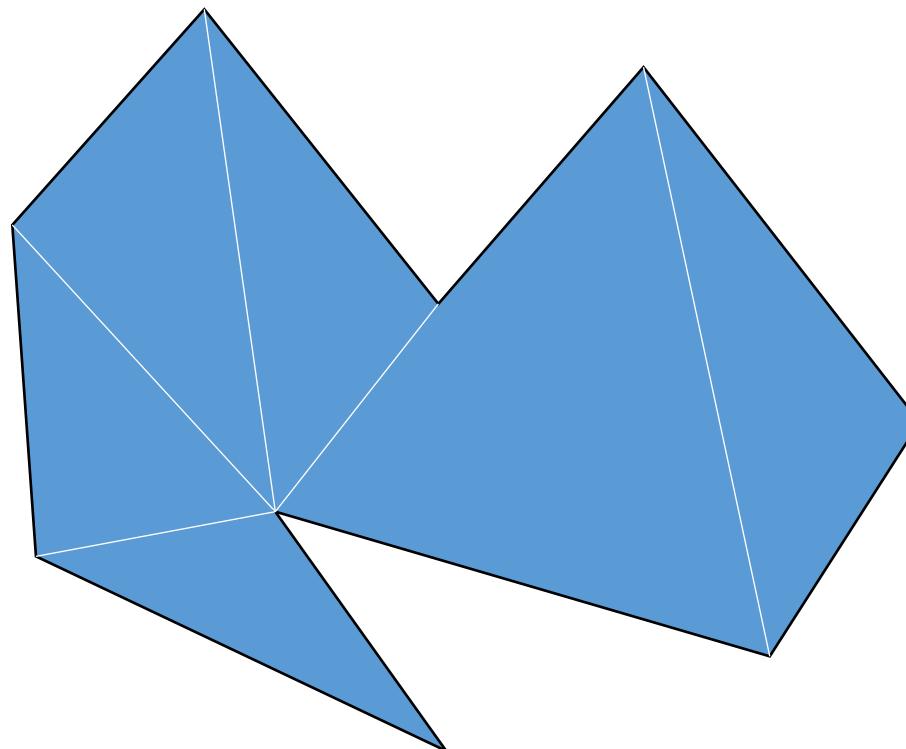


$$\text{area(blue + green)} = |\overrightarrow{(b-a)} \times \overrightarrow{(c-a)}|$$



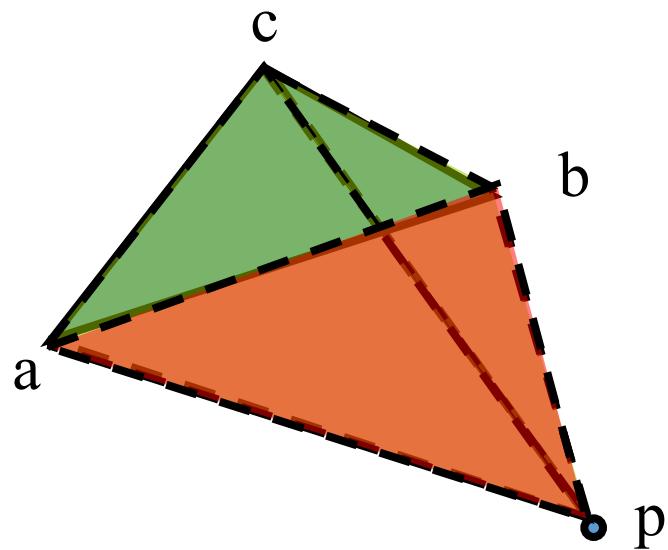
SUM OF AREAS

- AREA OF THE POLYGON CAN BE FOUND BY ADDING THE AREAS OF A TRIANGULATION



ANOTHER WAY OF COMPUTING AREA

$$\text{Area}(T) = \text{Area}(p, b, c) + \text{Area}(p, c, a) + \text{Area}(p, a, b)$$



AREA OF POLYGON

- THEOREM: LET A POLYGON (CONVEX OR NON-CONVEX) P HAVE VERTICES v_0, v_1, \dots, v_{n-1}

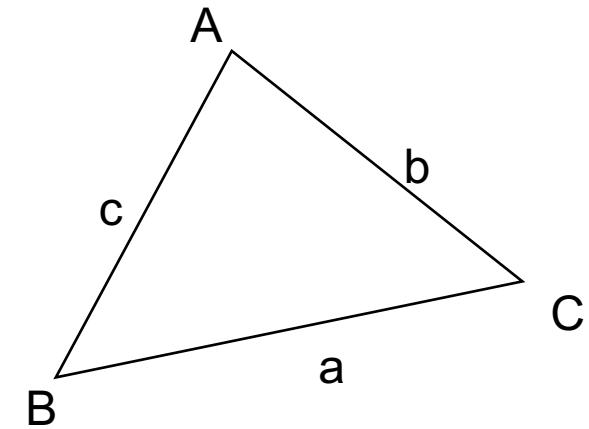
$$\begin{aligned}Area(P) &= A(p, v_0, v_1) + A(p, v_1, v_2) + A(p, v_2, v_3) \\&\quad + \cdots + A(p, v_{n-2}, v_{n-1}) + A(p, v_{n-1}, v_0)\end{aligned}$$

- PROOF IS BY INDUCTION



AREA OF TRIANGLE

- HERON'S FORMULA:
 - $\text{Area} = \sqrt{s(s - a)(s - b)(s - c)}$
 - where $s = (a + b + c)/2$ is the semiperimeter



AREA OF TRIANGLE

- WHAT IF ONLY THE VERTICES OF THE TRIANGLE ARE GIVEN?
- GIVEN 3 VERTICES $(x_1, y_1), (x_2, y_2), (x_3, y_3)$

$$Area = \frac{|x_1 * y_2 + x_2 * y_3 + x_3 * y_1 - x_2 * y_1 - x_3 * y_2 - x_1 * y_3|}{2}$$

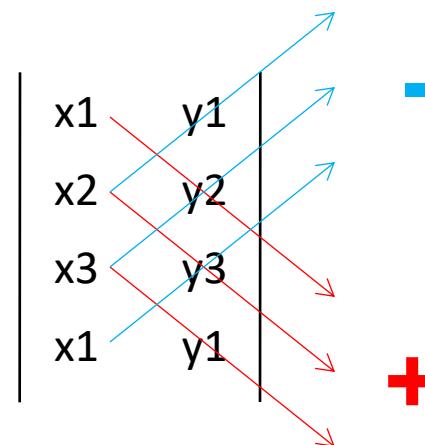
- Note: abs can be omitted if the vertices are in **countrerclockwise** order. If the vertices are in clockwise order, the difference evaluates to a negative quantity



AREA OF TRIANGLE

- THAT HARD-TO-MEMORIZE EXPRESSION CAN BE WRITTEN THIS WAY:

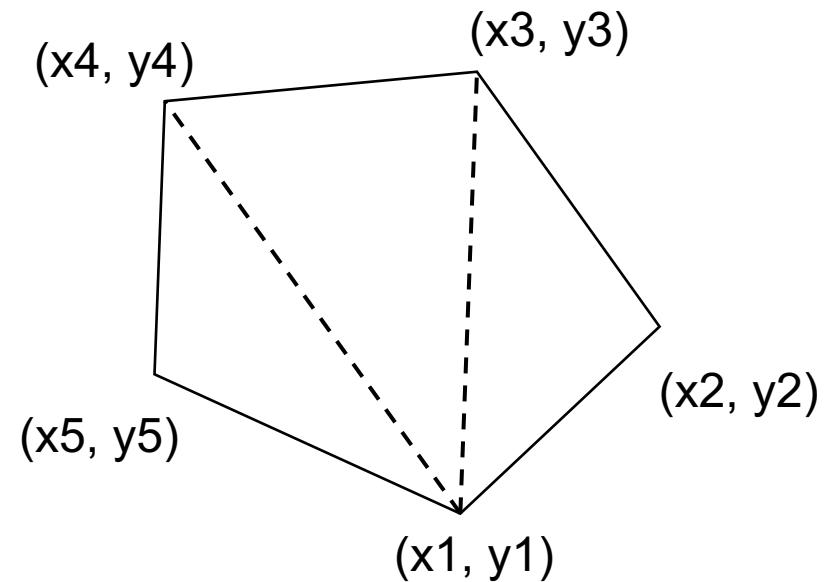
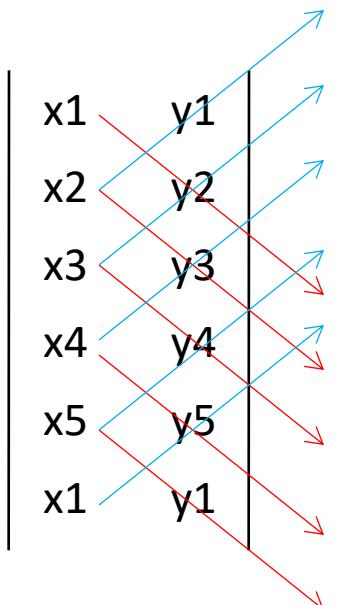
- AREA = $\frac{1}{2} *$



AREA OF CONVEX POLYGON

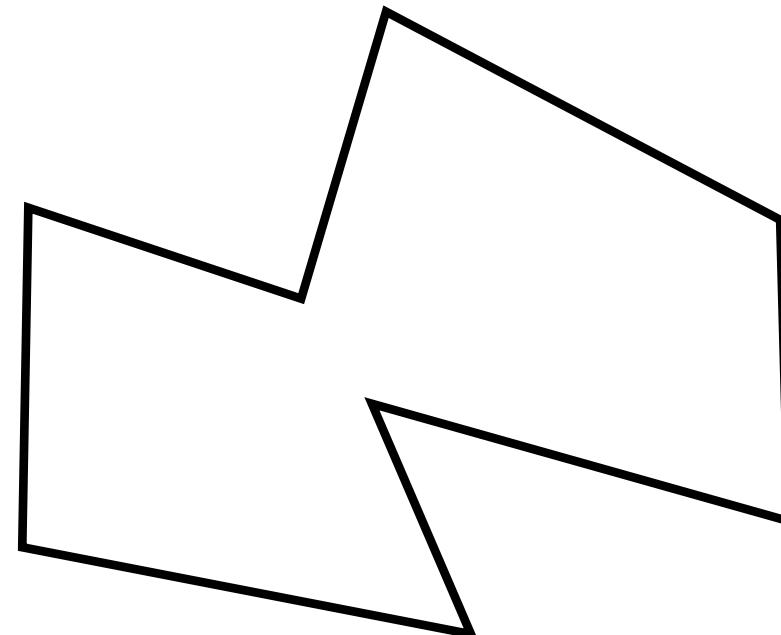
- IT TURNS OUT THE PREVIOUS FORMULA STILL WORKS!

- AREA = $\frac{1}{2} *$



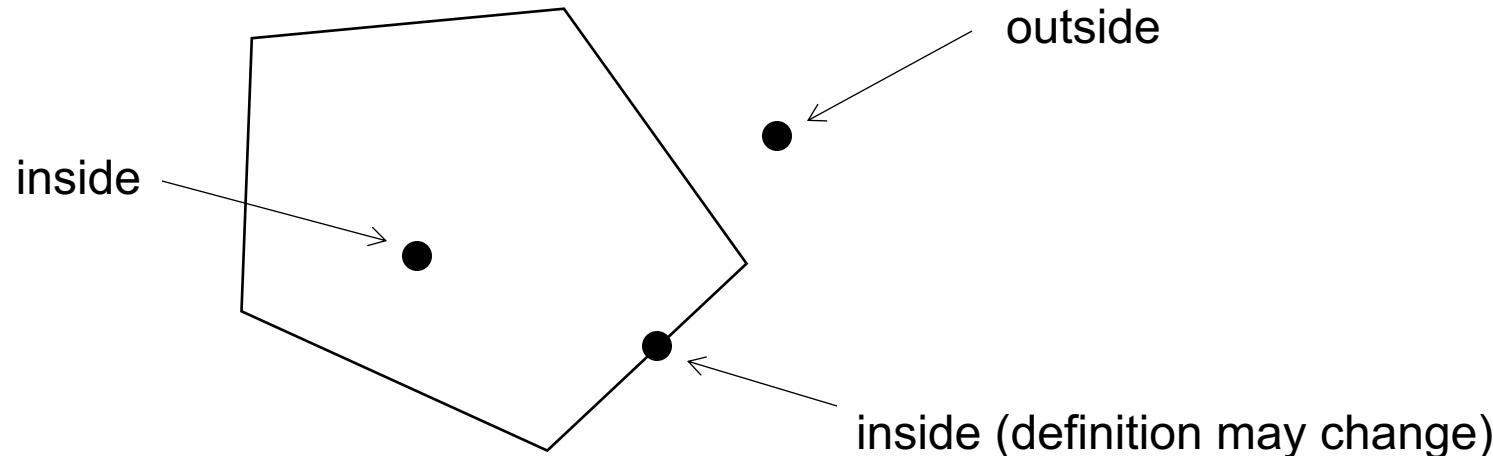
AREA OF (NON-CONVEX) POLYGON

- MIRACULOUSLY, THE SAME FORMULA STILL HOLDS FOR NON-CONVEX POLYGONS!
- AREA = $\frac{1}{2} * \dots$



POINT INSIDE CONVEX POLYGON?

- GIVEN A CONVEX POLYGON AND A POINT, IS THE POINT CONTAINED INSIDE THE POLYGON?
 - Assume the vertices are given in **counterclockwise** order for convenience



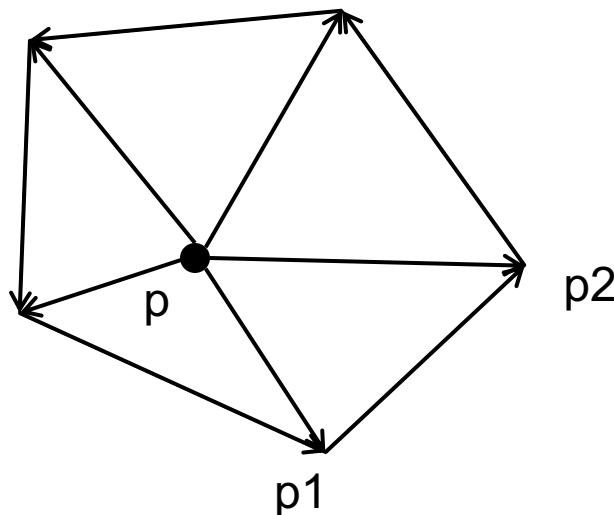
DETOUR – IS POLYGON CONVEX?

- A QUICK QUESTION – HOW TO TELL IF A POLYGON IS CONVEX?
- ANSWER: IT IS CONVEX IF AND ONLY IF EVERY TURN (AT EVERY VERTEX) IS A LEFT TURN
 - Whether a “straight” turn is allowed depends on the problem definition
- OUR CROSSPROD FUNCTION IS SO USEFUL



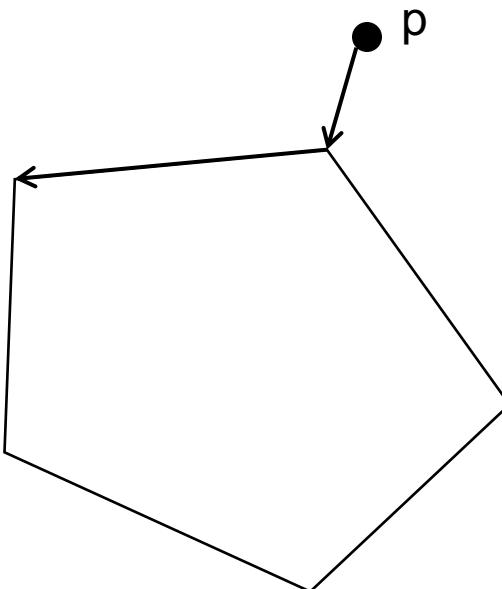
POINT INSIDE CONVEX POLYGON?

- CONSIDER THE TURN $P \rightarrow P_1 \rightarrow P_2$
- IF P DOES LIE INSIDE THE POLYGON, THE TURN MUST **NOT** BE A RIGHT TURN
- ALSO HOLDS FOR OTHER EDGES (MIND THE DIRECTIONS)



POINT INSIDE CONVEX POLYGON?

- CONVERSELY, IF P WAS OUTSIDE THE POLYGON, THERE WOULD BE A RIGHT TURN FOR SOME EDGE



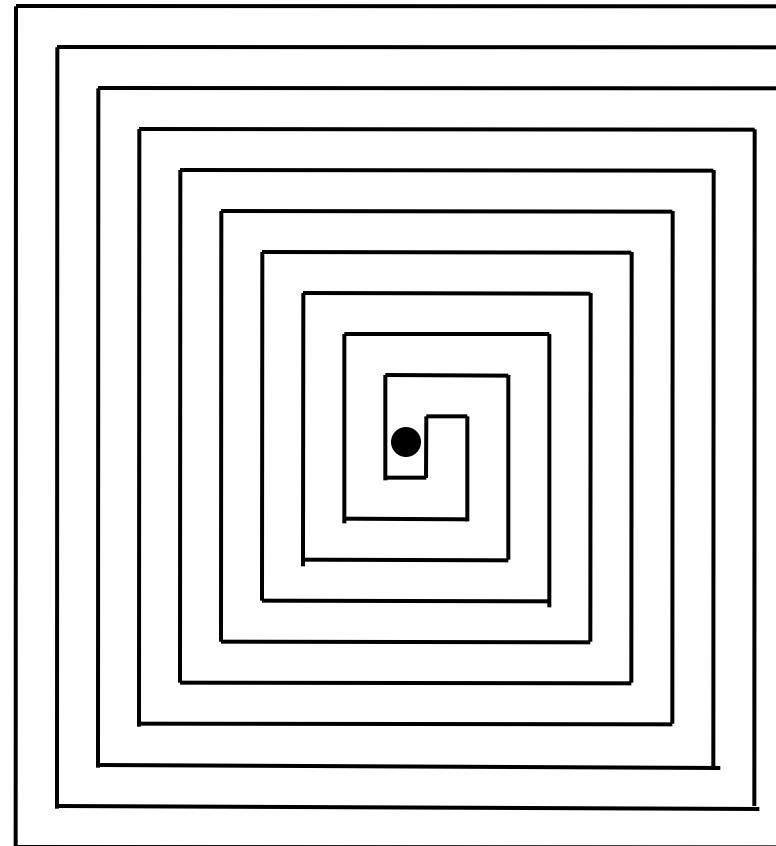
POINT INSIDE CONVEX POLYGON

- CONCLUSION: P IS INSIDE THE POLYGON IF AND ONLY IF IT MAKES A **NON-LEFT TURN** FOR **EVERY** EDGE (IN THE COUNTERCLOCKWISE DIRECTION)



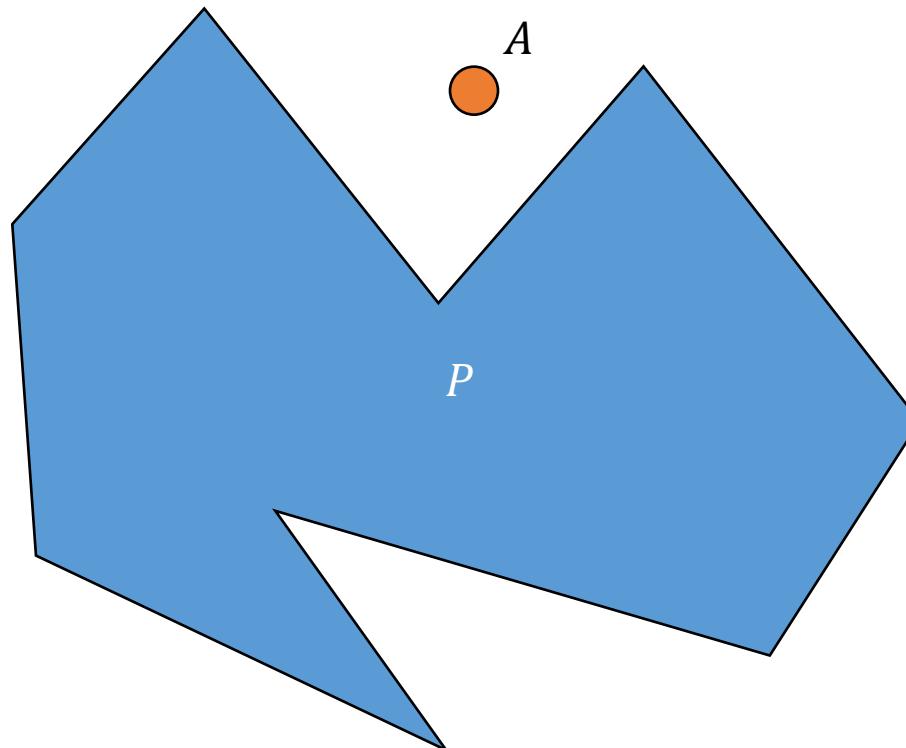
POINT INSIDE (NON-CONVEX) POLYGON

- SUCH A PAIN



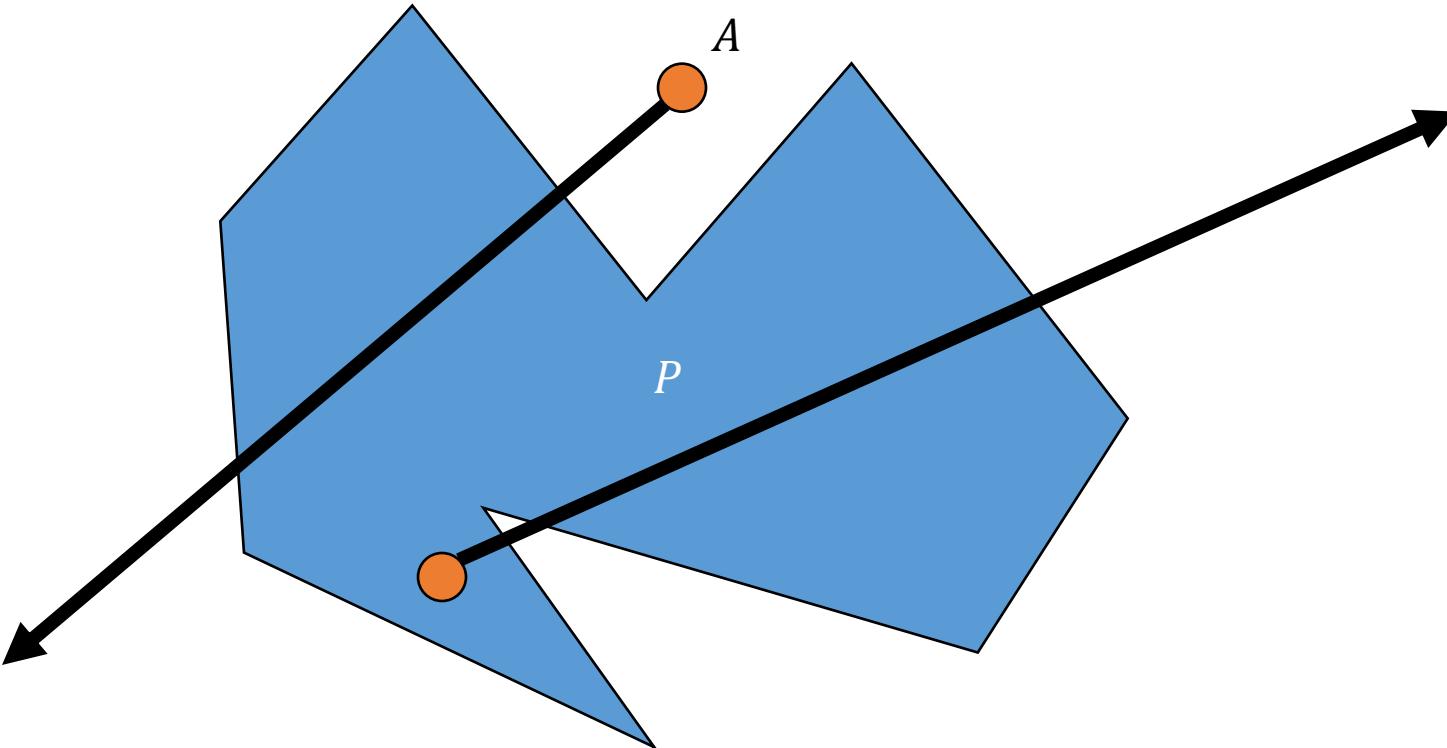
POINT INSIDE A POLYGON TEST

- HOW CAN WE TELL IF POINT A IS INSIDE OF POLYGON P ?



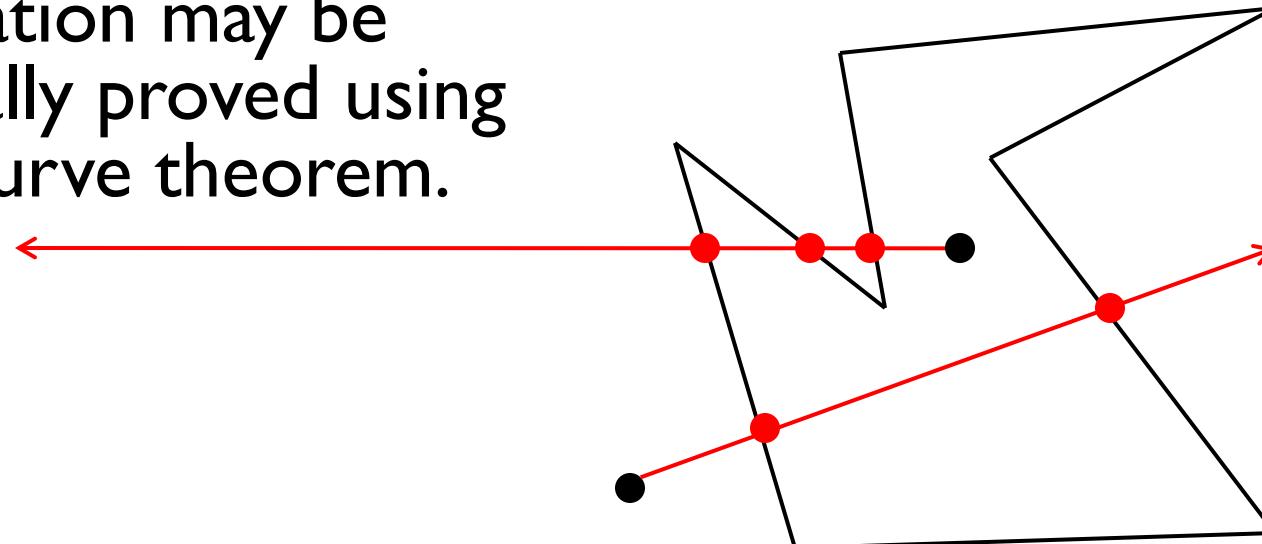
POINT INSIDE A POLYGON TEST

- HOW CAN WE TELL IF POINT A IS INSIDE OF POLYGON P ?



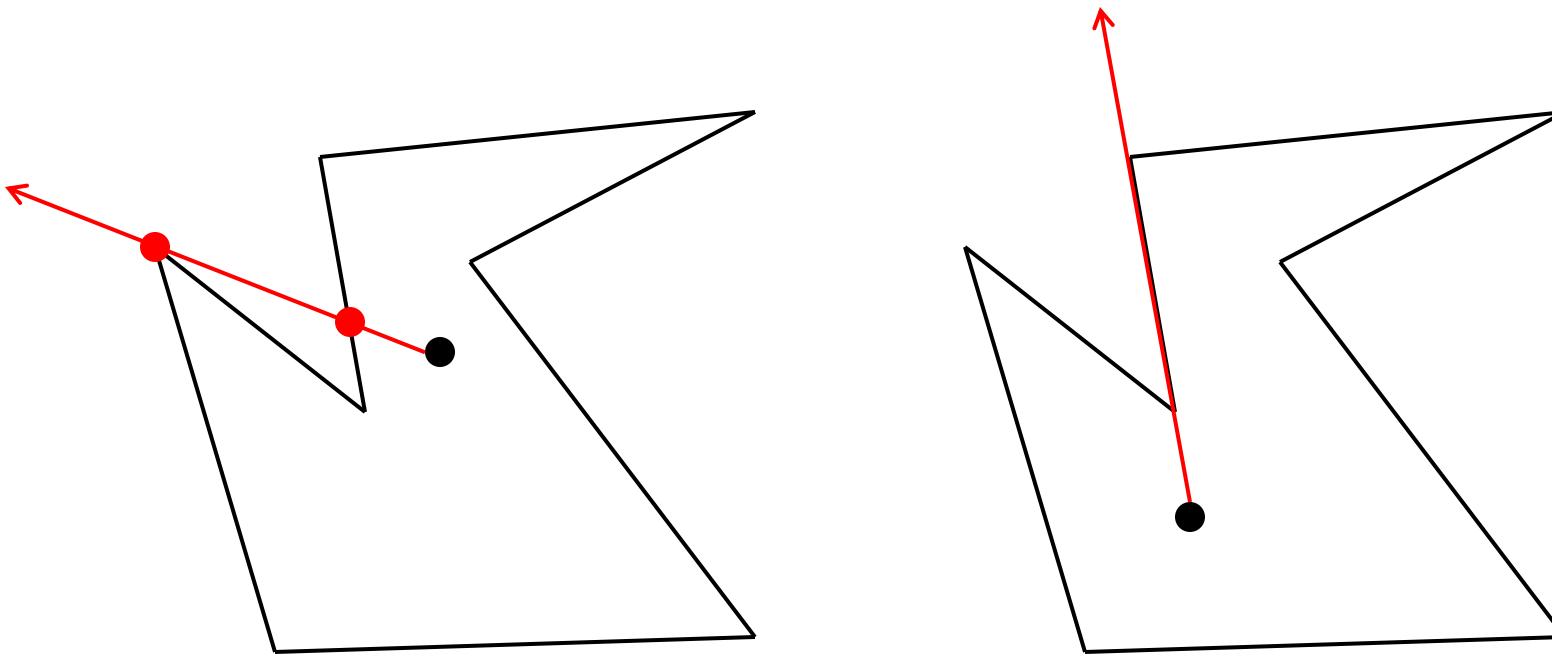
POINT INSIDE POLYGON

- EVEN–ODD RULE ALGORITHM
 - If a point A moves along a ray from infinity to A
 - If it crosses the boundary of a polygon, possibly several times, then it alternately goes from the outside to inside to outside
 - After every two "border crossings" the moving point goes outside.
- This observation may be mathematically proved using the Jordan curve theorem.



POINT INSIDE POLYGON

- PROBLEMATIC CASES: DEGENERATE INTERSECTIONS



- SOLUTION: PICK A RANDOM DIRECTION (I.E. RANDOM SLOPE). IF THE RAY HITS A VERTEX OF THE POLYGON, PICK A NEW DIRECTION. REPEAT.



POINT INSIDE POLYGON

- SOLUTION: PICK A RANDOM DIRECTION (I.E. RANDOM SLOPE). IF THE RAY HITS A VERTEX OF THE POLYGON, PICK A NEW DIRECTION. REPEAT.

