COT 4521: Introduction to Computational Geometry

Polygons

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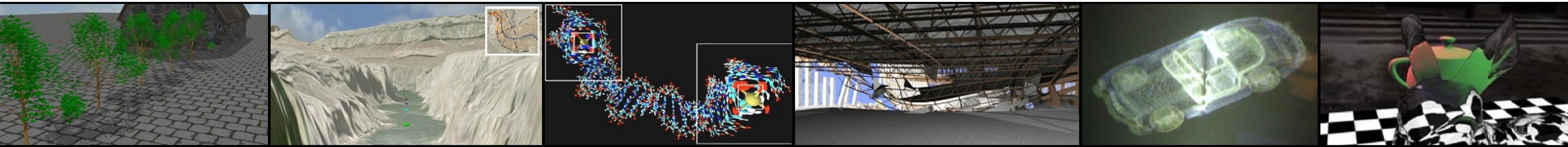
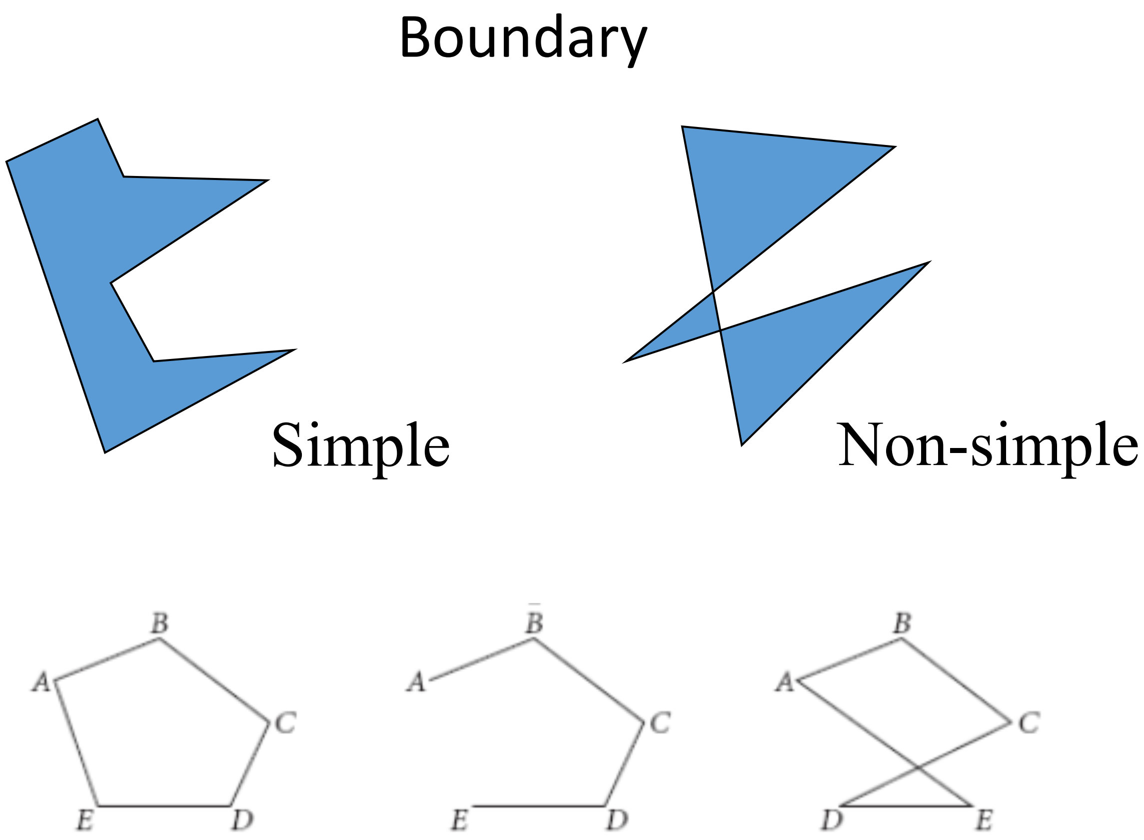
Some slides from Valentina Korzhova



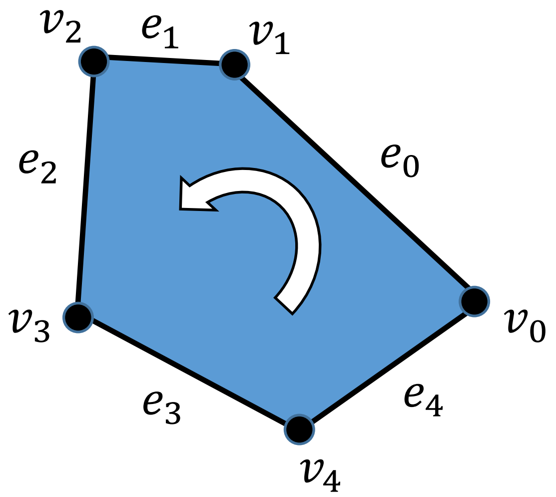
<u>POLYGON</u>

- **POLYGON** IS A <u>REGION</u> OF A PLANE BOUNDED BY A <u>FINITE COLLECTION</u> OF LINE SEGMENTS FORMING A <u>SIMPLE CLOSED</u> CURVE.



POLYGON

- EDGES – THE LINE SEGMENTS $(e_0, e_1, \cdots, e_{n-1})$
- VERTICES – THE POINTS WHERE ADJACENT EDGES MEET
  - Start at any vertex and list the vertices consecutively in a counterclockwise direction $(v_0, v_1, \ldots, v_{n-1})$
- ANGLES
  - Name by angle naming convention
  - $\measuredangle v_0, \measuredangle v_1, \ldots, \measuredangle v_{n-1}$

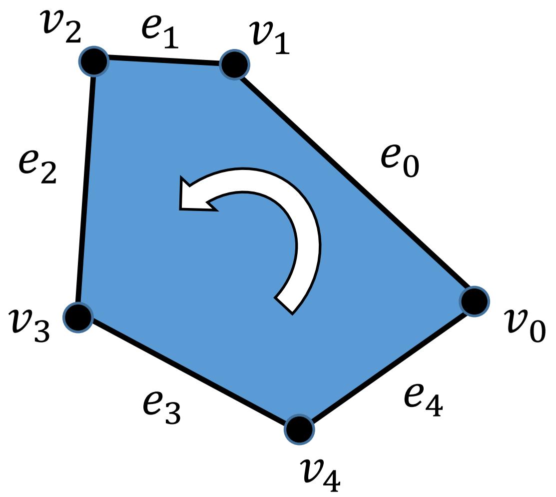


Properties of polygon

- The line segments that make-up a polygon (called sides or edges) meet only at their endpoints, called vertices (singular: vertex) or less formally "corners"
- Exactly two edges meet at every vertex
- The number of edges always equals the number of vertices.
- Two edges meeting at a corner are required to form an angle that is not straight (180°); otherwise, the line segments will be considered parts of a single edge

Formal Definition of a Simple Polygon

- Formally, we are given n vertices (i.e., points) $v0, v1, \dots, v_{n-1}$, the chain formed by $v0v1 \cdots v_{n-1}$ is a simple polygon iff
  - The segments $e0 = v0v1, \dots, e_{n-2} = v_{n-2}v_{n-1}$, and $e_{n-1} = v_{n-1}v0$ are disjoint in their interior
  - Consecutive segments intersect only in their endpoints. Namely $ei \cap e_{i+1} = v_{i+1}$ for $i = 0, \dots, n-2$ and $e_{n-1} \cap e0 = v0$
  - Non adjacent segments do not intersect $ei \cap ej = \emptyset$, for $\forall j \neq i + 1$
- We work $mod\ n$.
  - Namely $v_i = v_i\ mod\ n$



DIAGONALS

- DEFINITION: THE DIAGONAL OF A POLYGON IS A LINE SEGMENT LINKING TWO NON-ADJACENT VERTICES, INTERIOR TO THE POLYGON, AND NOT BLOCKED BY PORTION OF POLYGON'S BOUNDARY

- LEMMA: THE SEGMENT $s = v_i v_j$ IS A DIAGONAL OF P IFF
  1. For all edges e of P that are not incident to either $v_i or\ v_j$, s and e do not intersect: $s \cap e = \emptyset$;
  2. s is internal to P in neighborhood of v_i and v_j

DIAGONALS

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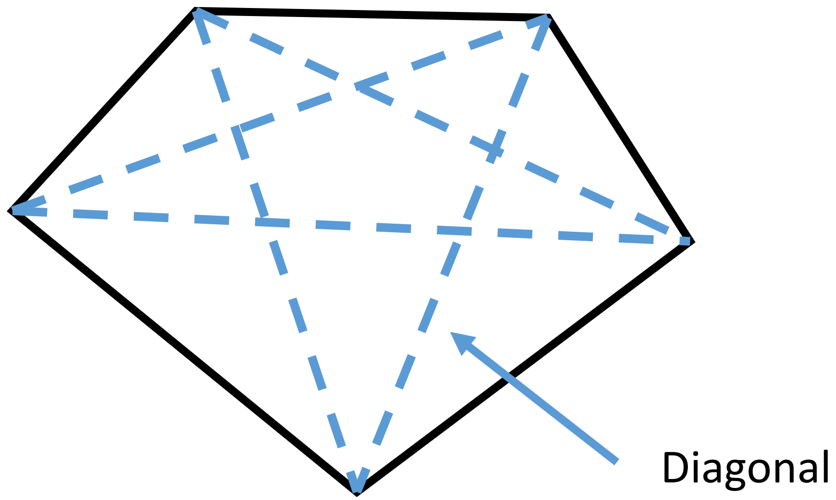
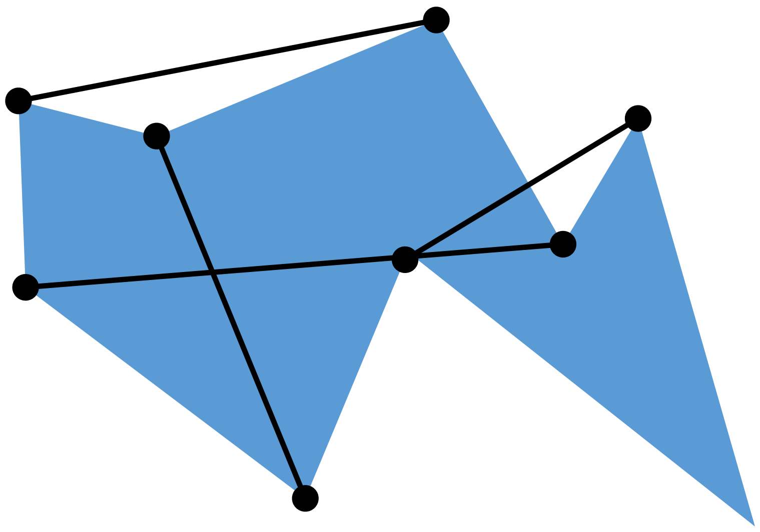


Diagonals

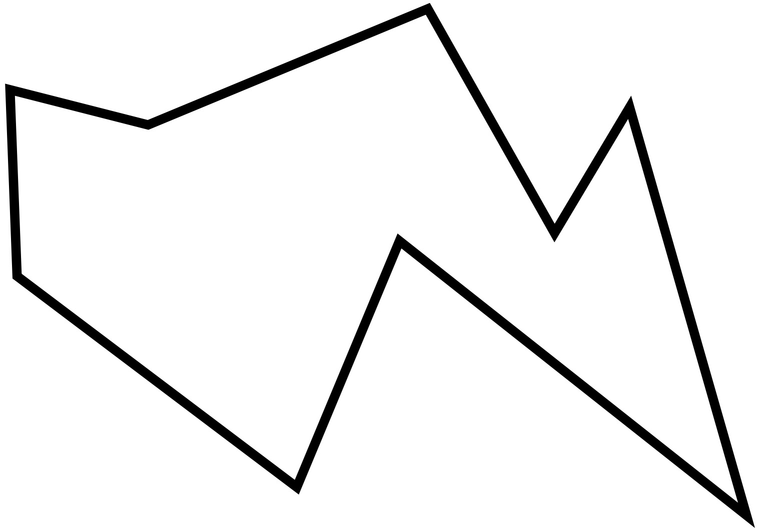
- Definition: The diagonal of a polygon is a line segment linking two non-adjacent vertices, interior to the polygon, and not blocked by portion of polygon's boundary



which are legal diagonals?

Efficient Diagonal Finding

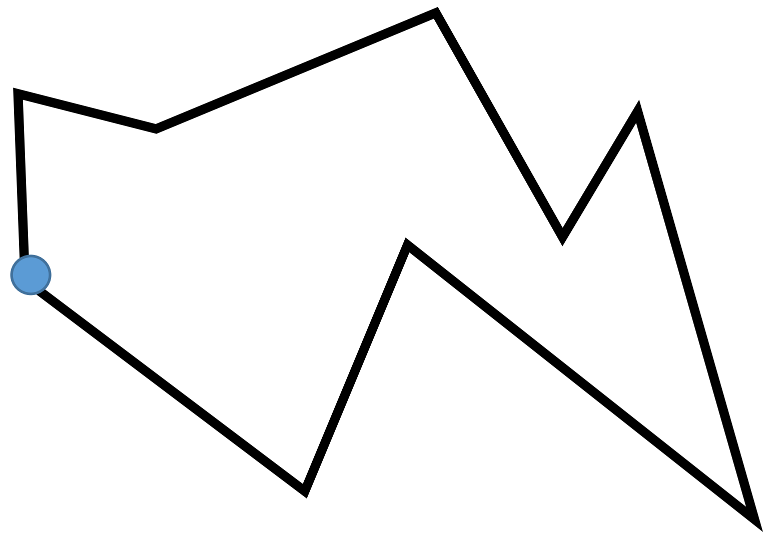
- Brute force
  - Generate each potential diagonal—$O(n^2)$ possible diagonals
  - Check each diagonal against the boundary to determine
    - If it is inside or outside
    - If it intersects the boundary
  - Total performance $O(n^3)$



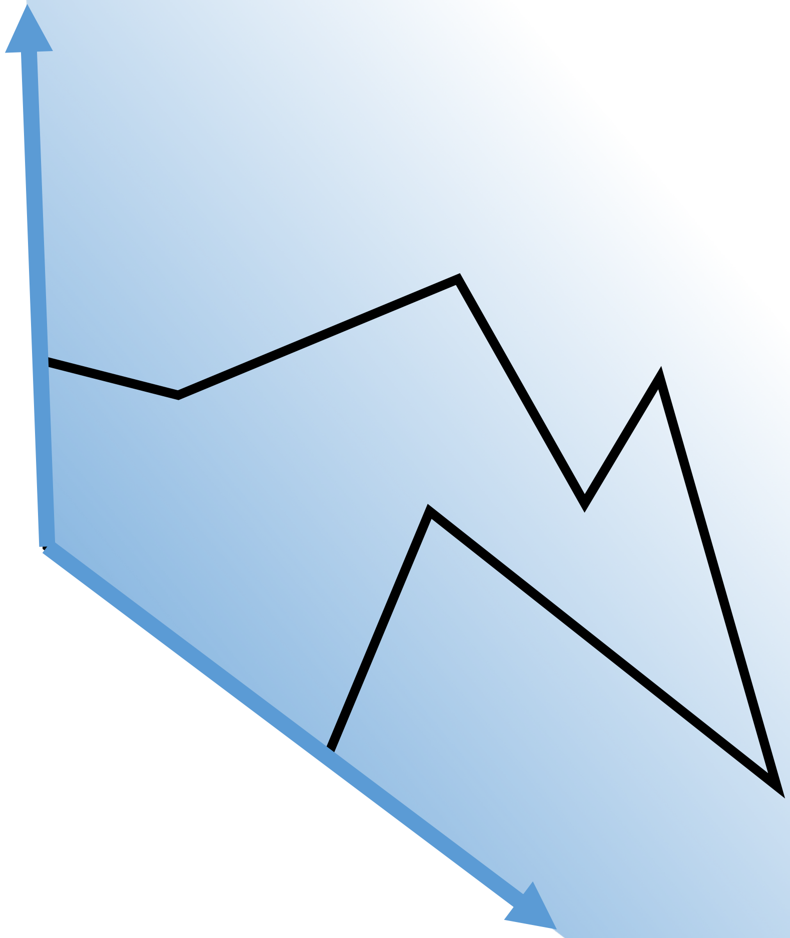
- Can we do Better?

EFFICIENT DIAGONAL FINDING

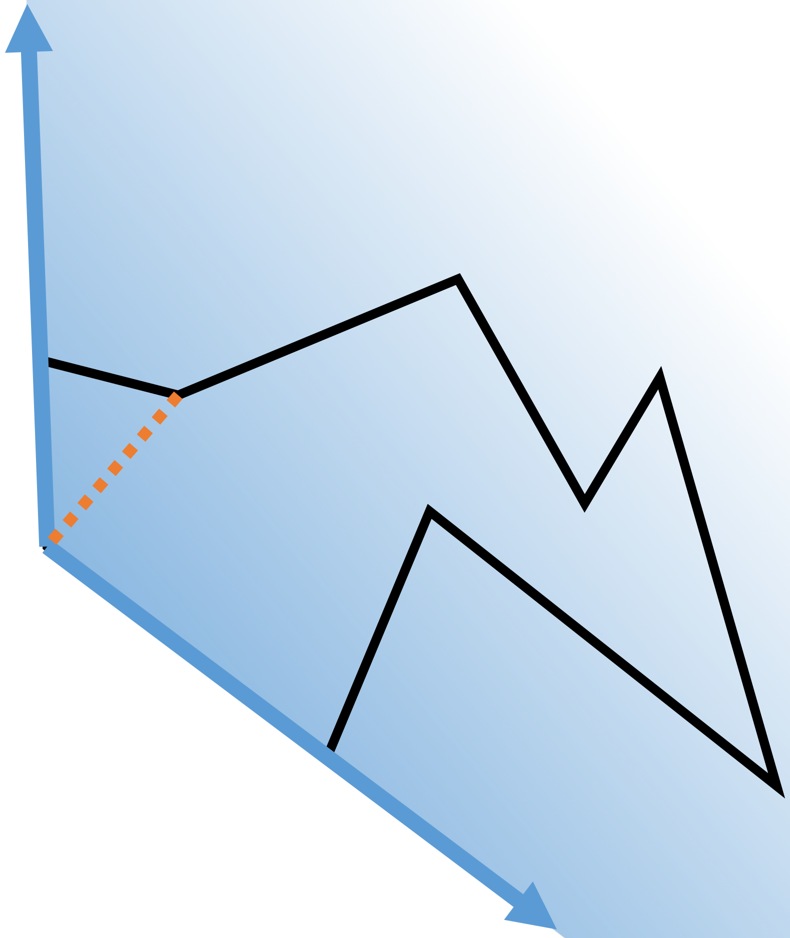
- VARIATION ON A SWEEP ALGORITHM
  - From a given vertex, sweep both clockwise and counterclockwise, ordered by the distance of points
  - Retain a notion of "valid space" for diagonals



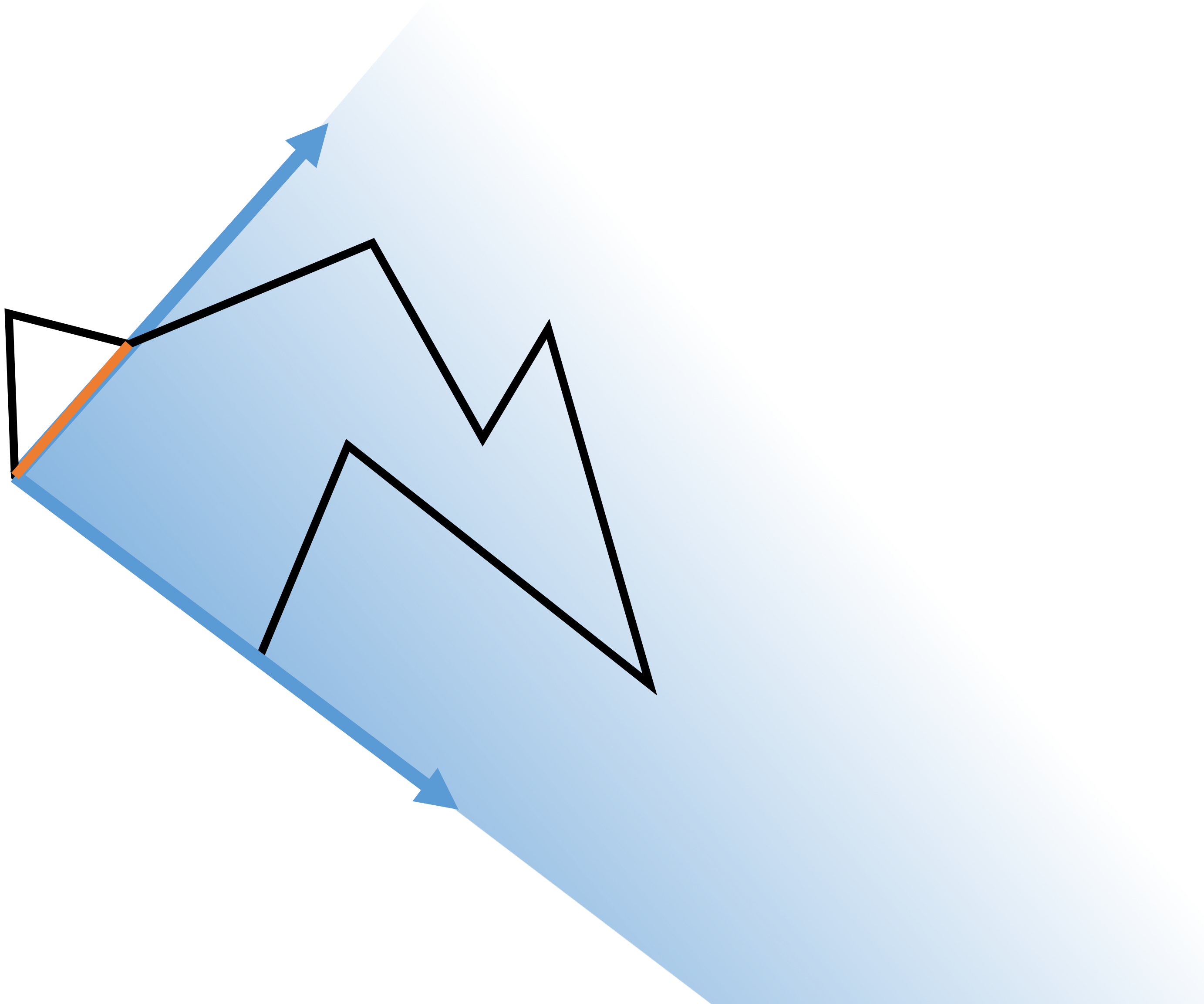
Efficient Diagonal Finding



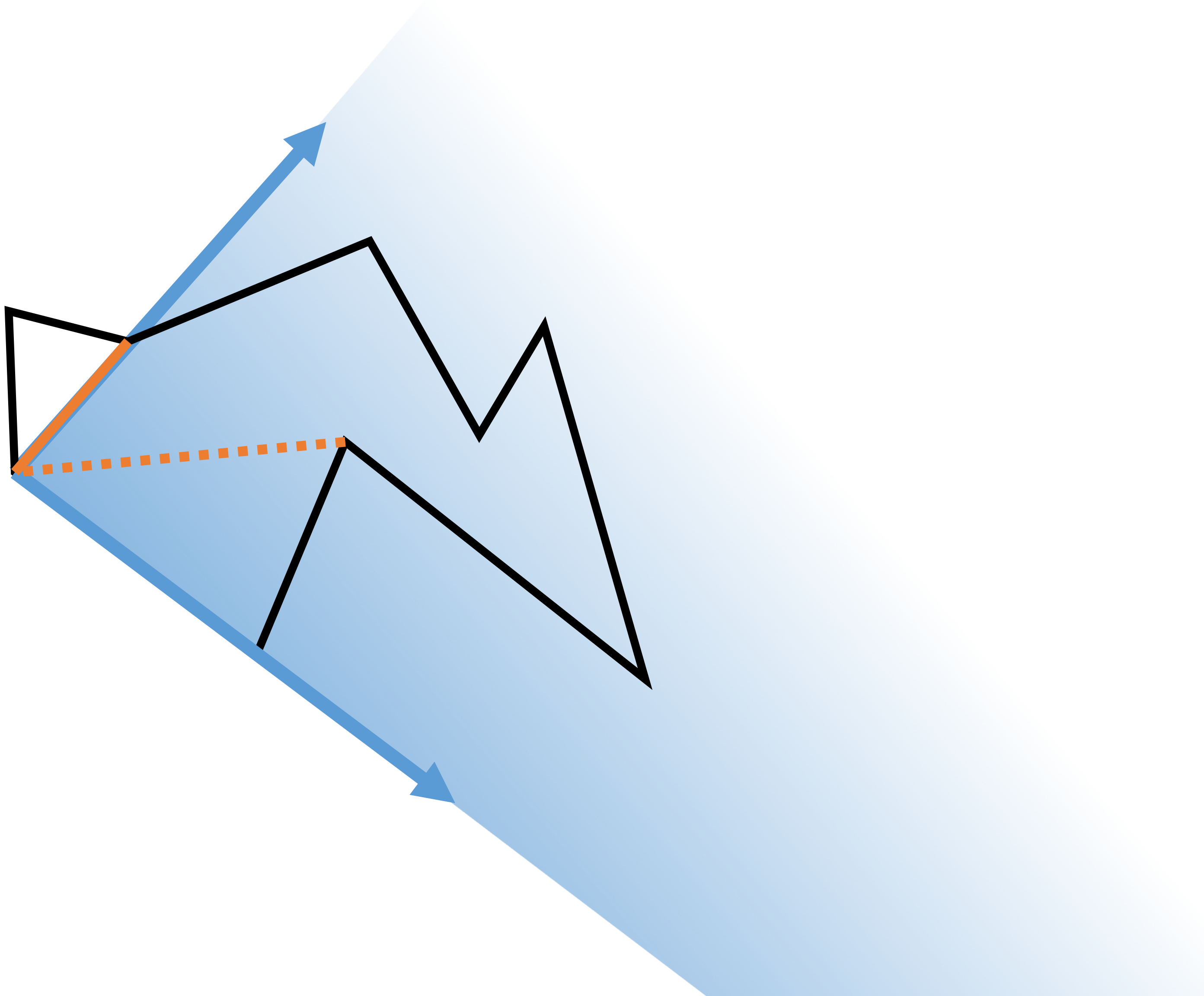
Efficient Diagonal Finding



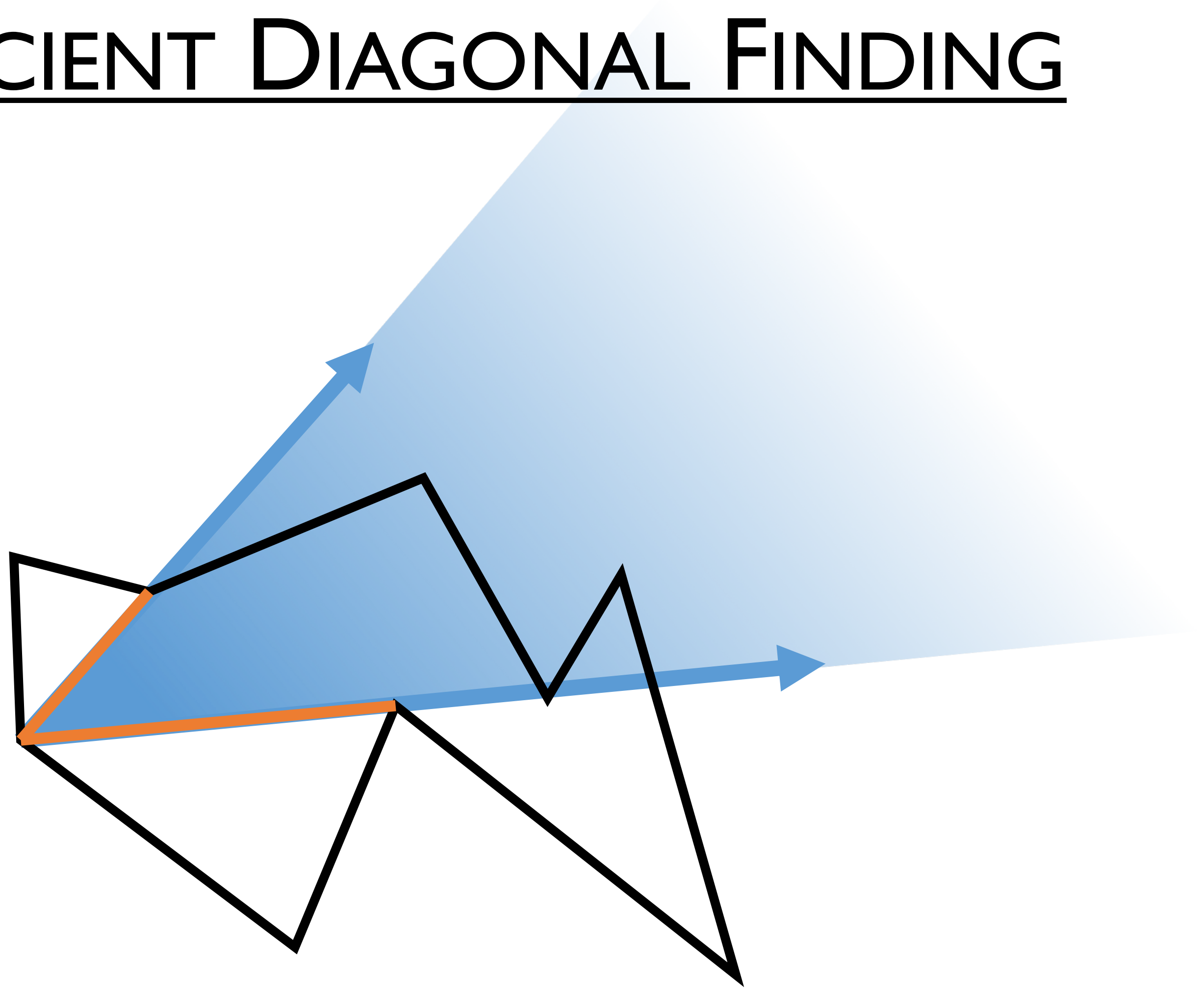
Efficient Diagonal Finding



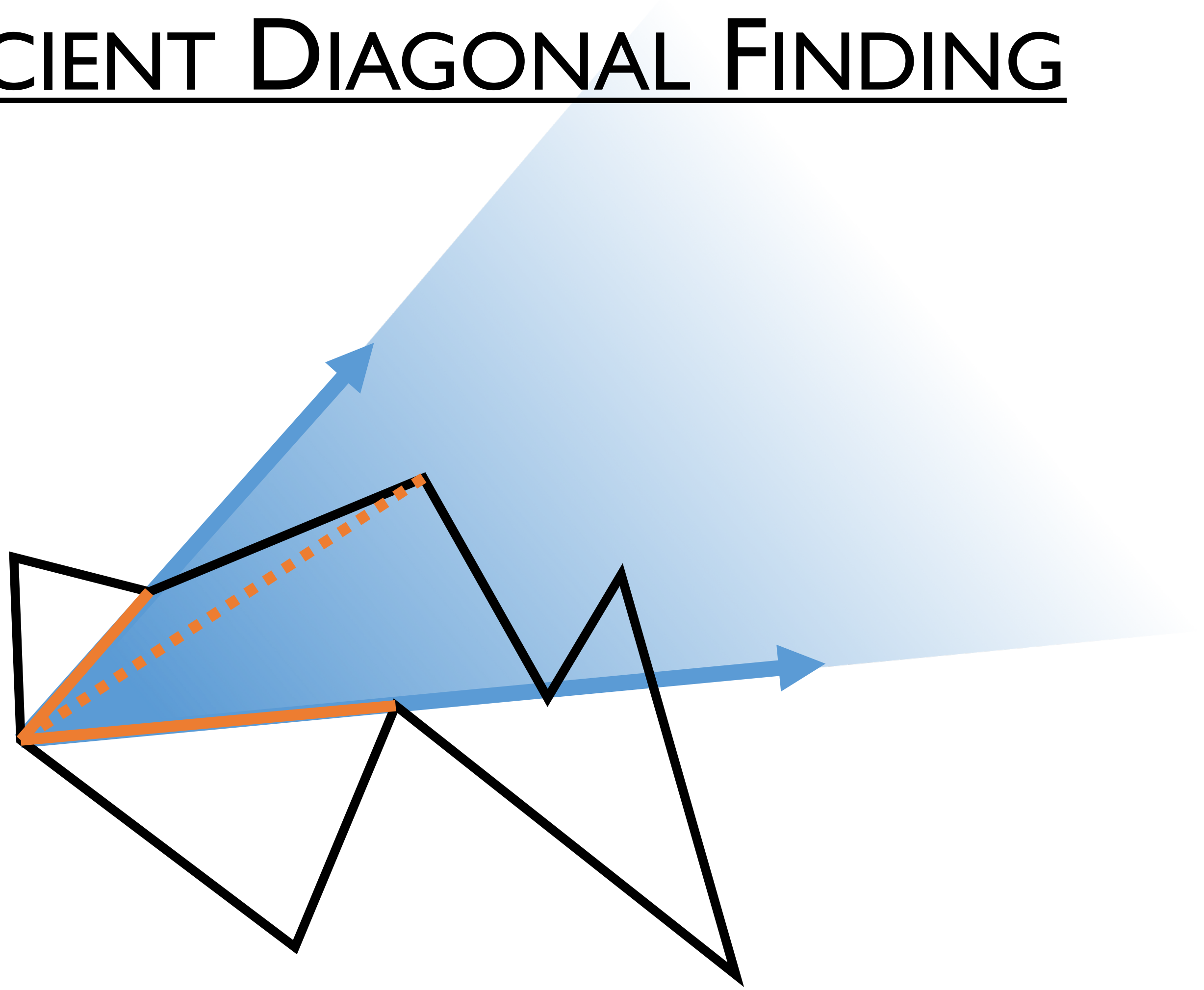
Efficient Diagonal Finding



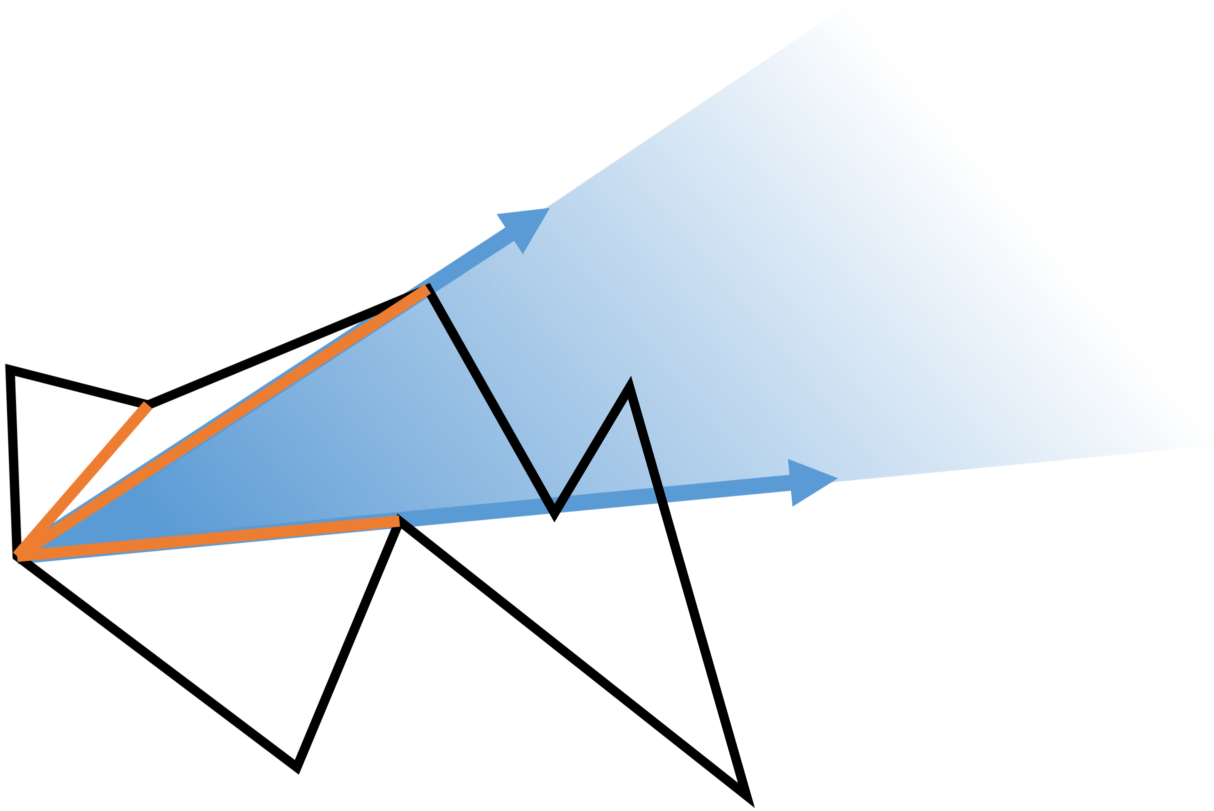
Efficient Diagonal Finding



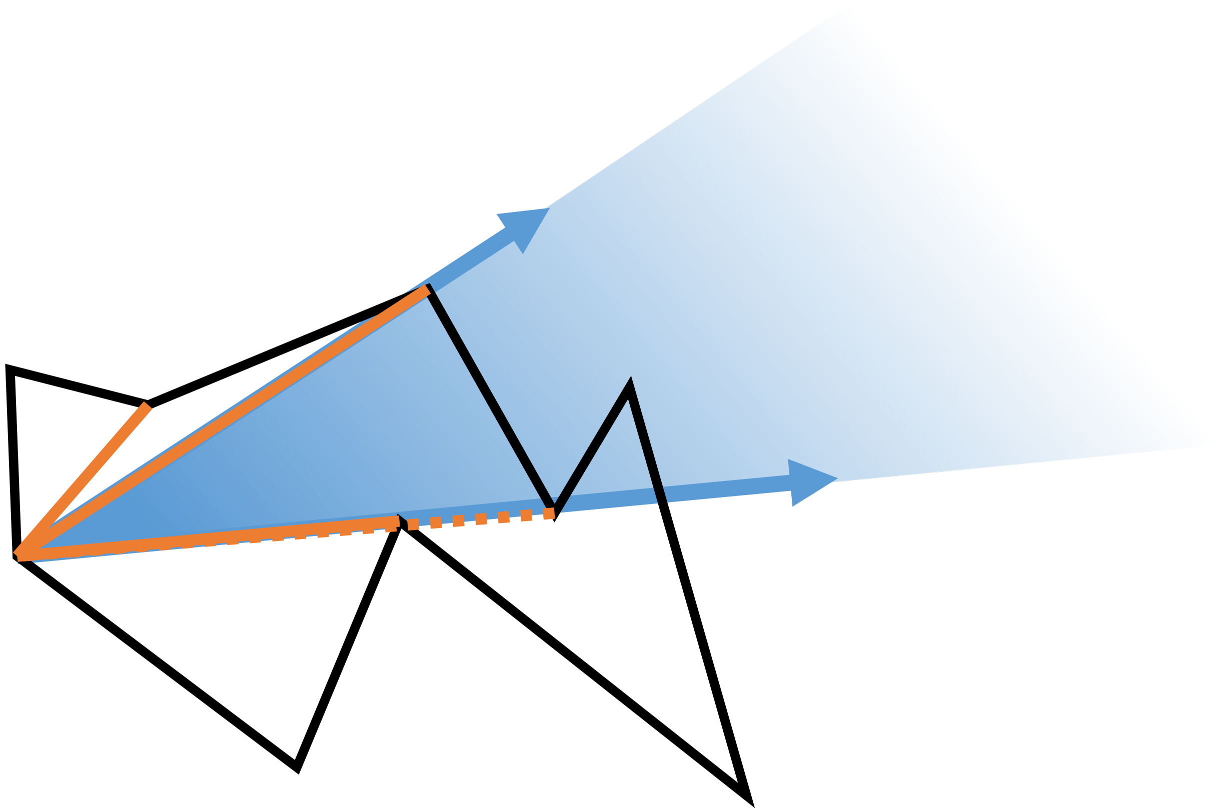
Efficient Diagonal Finding



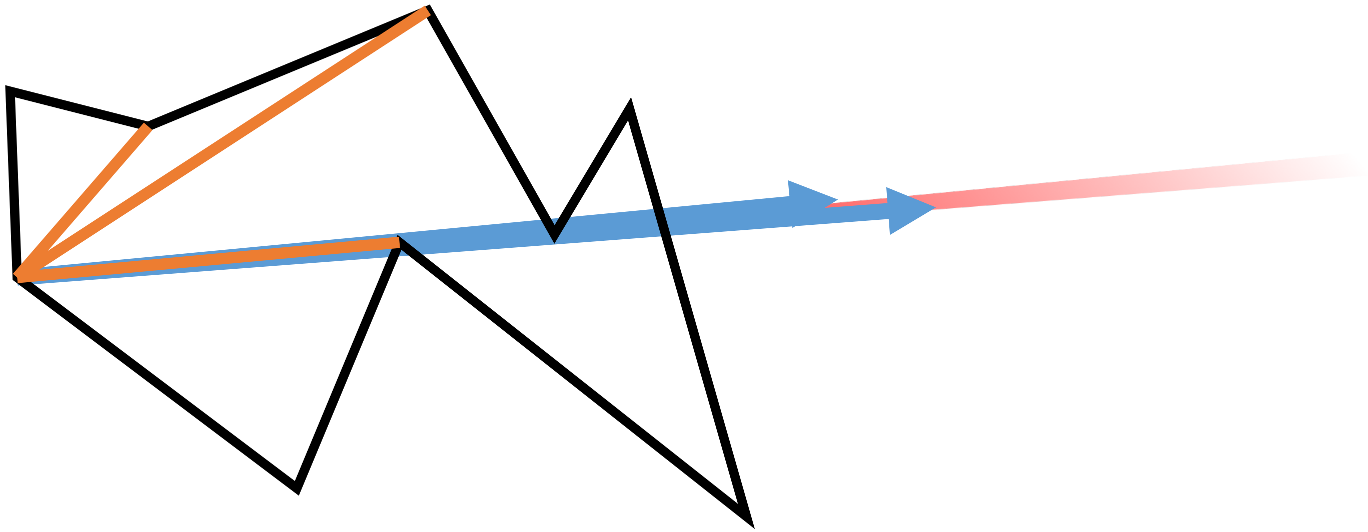
Efficient Diagonal Finding



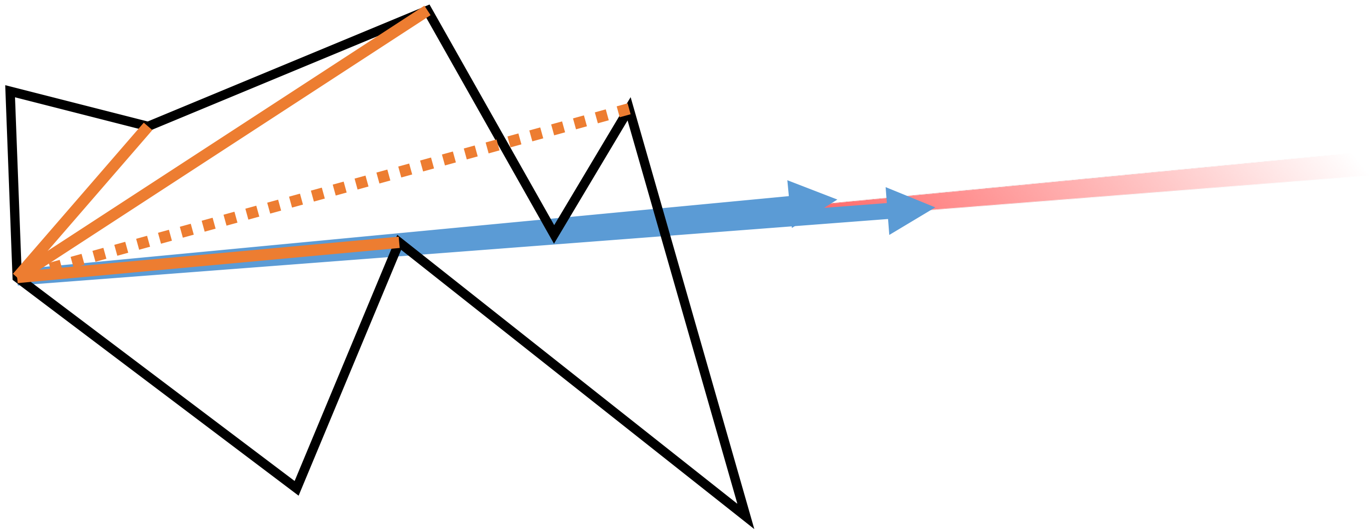
Efficient Diagonal Finding



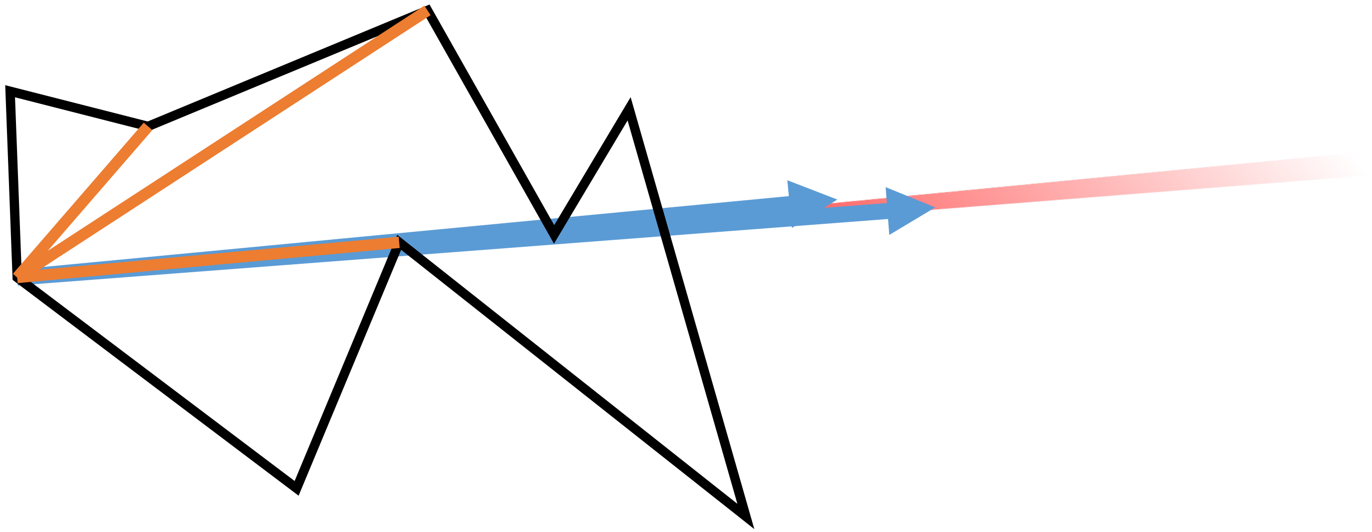
Efficient Diagonal Finding



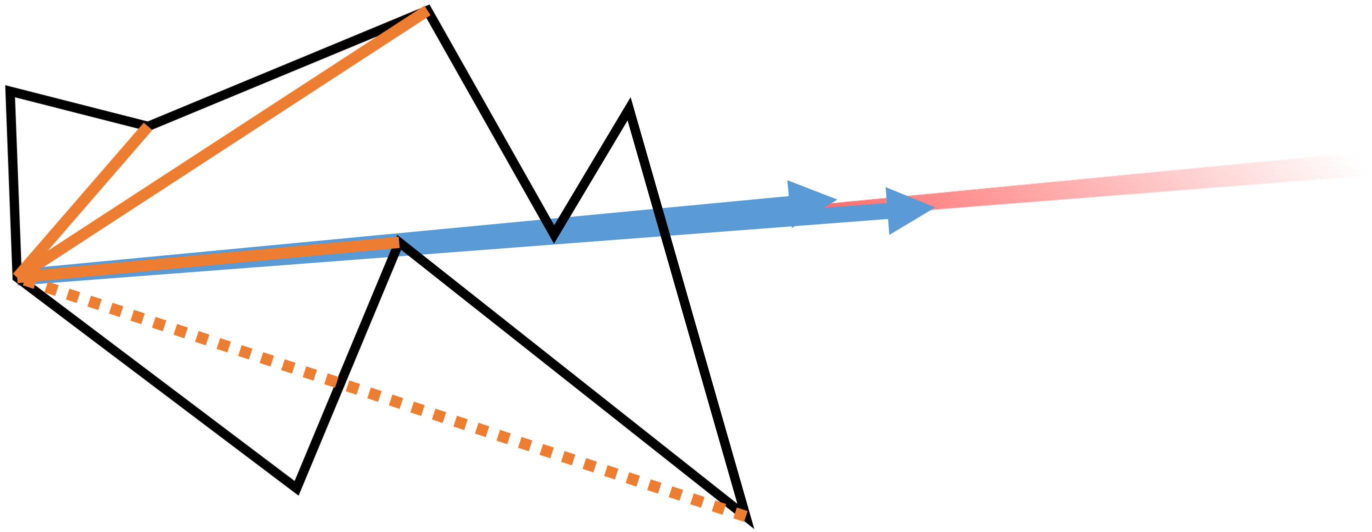
Efficient Diagonal Finding



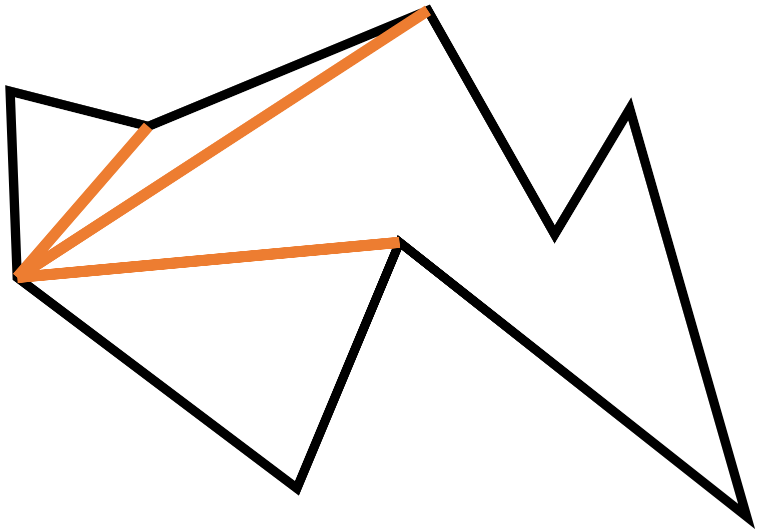
Efficient Diagonal Finding



Efficient Diagonal Finding

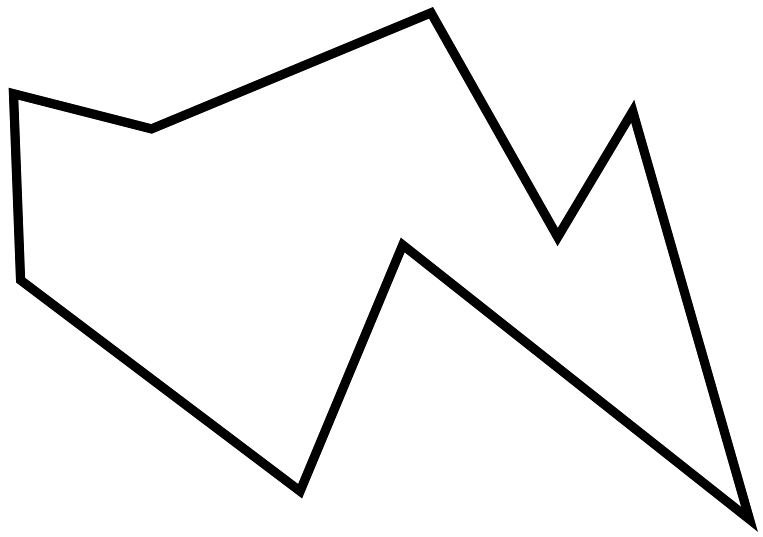


Efficient Diagonal Finding



EFFICIENT DIAGONAL FINDING

- ANALYSIS
  - Preprocessing: None
  - Query: Worst case $O(n^2)$; $O(n)$ per vertex
  - Storage: $O(n)$



Triangulation Theory: Properties

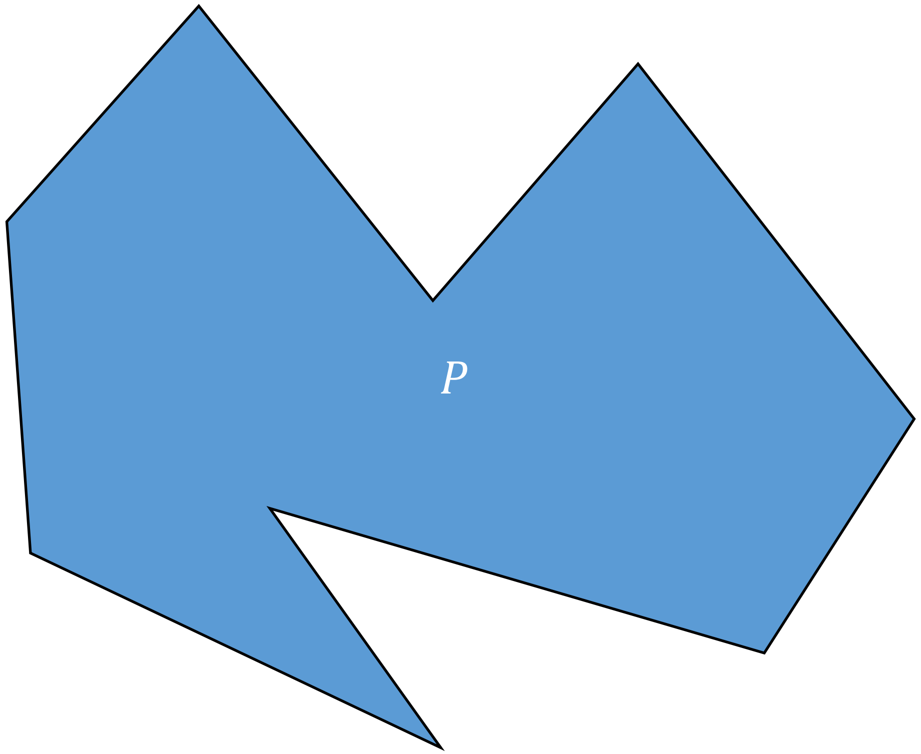
- Lemma: An internal diagonal exists between any two nonadjacent vertices of a polygon P if and only if P is convex polygon.
- Proof: The proof consists of two parts, both established by contradiction.

Classification of Polygons by the Number of Edges
Most Common Polygons

Number of Sides	Name
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
8	Octagon
9	Nonagon
10	Decagon
12	Dodecagon
n	n-gon

Sum of Angles

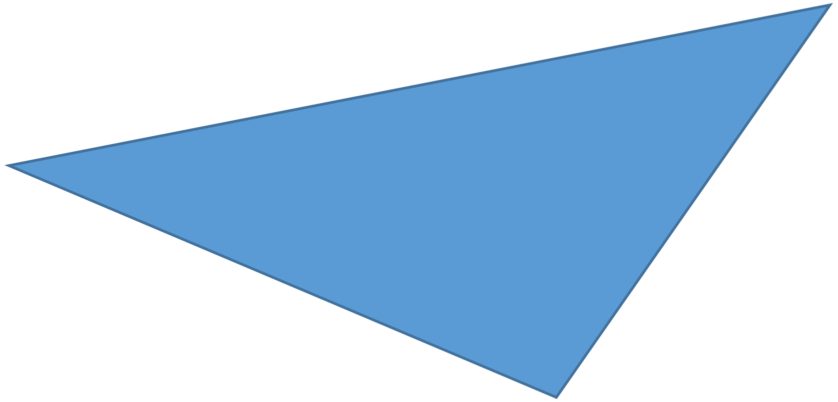
- The sum of the internal angles of a polygon of n vertices is?



Sum of Angles

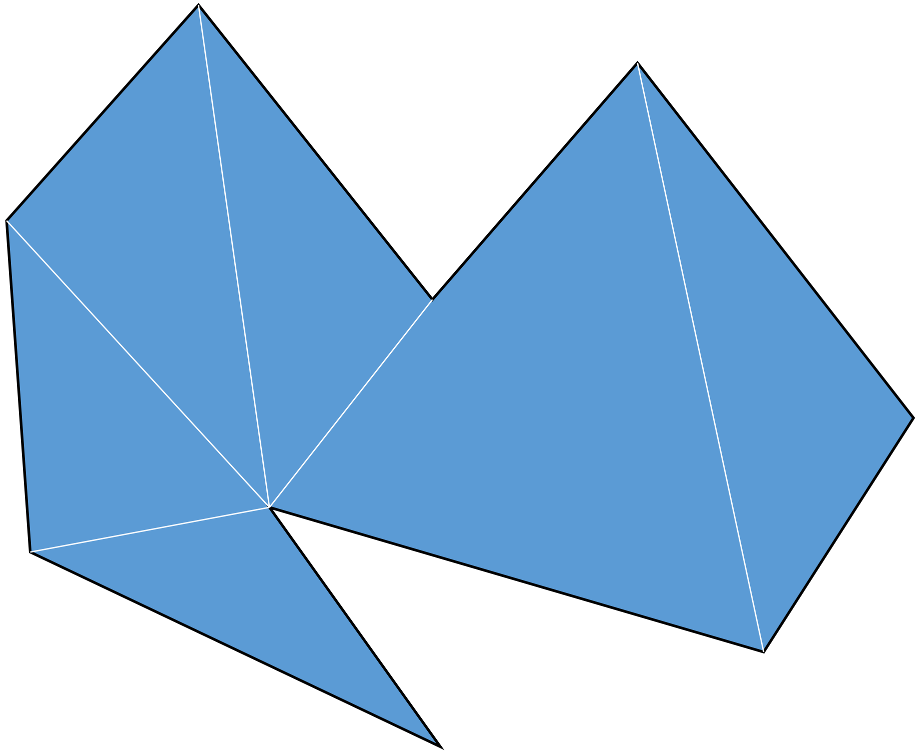
- Use the Triangle Angle-Sum Theorem to find the sum of the measures of the angles of a polygon.

- Triangle Angle-Sum Theorem
  - The sum of the measures of the angles of a triangle measure 180°



Sum of Angles

- Theorem: The sum of the measures of the internal angles of an n-gon is (n - 2) *180.



- Proof by induction

Triangulation Theory

- **Theorem**: Every triangulation of an n-vertex polygon P uses $n - 3$ diagonals and consists of $n - 2$ triangles.
  - Proof by induction:
    - Base case $N = 3$
    - Assume true for any polygon $< \ N$ sides
    - Given a N sided polygon partition it into two ($N1$ and $N2$) by adding a diagonal

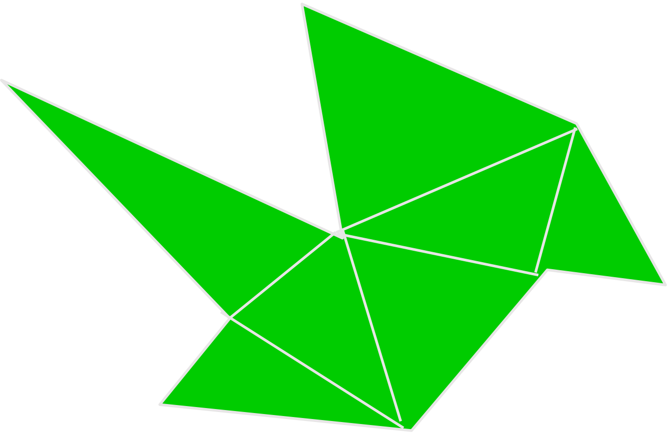
      Total number of diagonals:

      $(N1 - 3) + (N2 - 3) + 1 = (N1 + N2 - 2) - 3 = N - 3$

      Total number of triangles:

      $(N1 - 2) + (N2 - 2) = (N1 + N2) - 4 = N + 2 - 4 = N - 2$

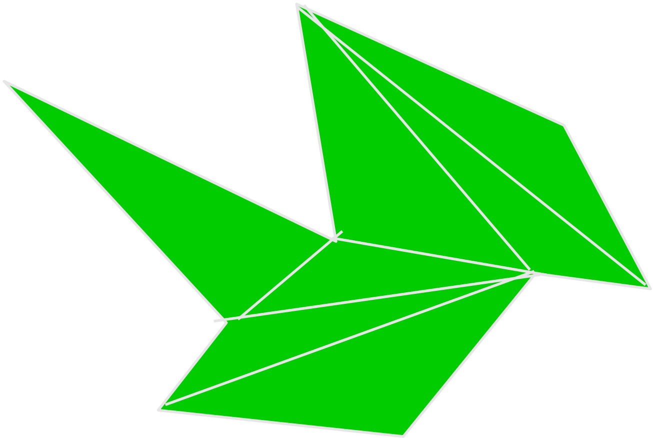
Triangulation Example



9 vertices

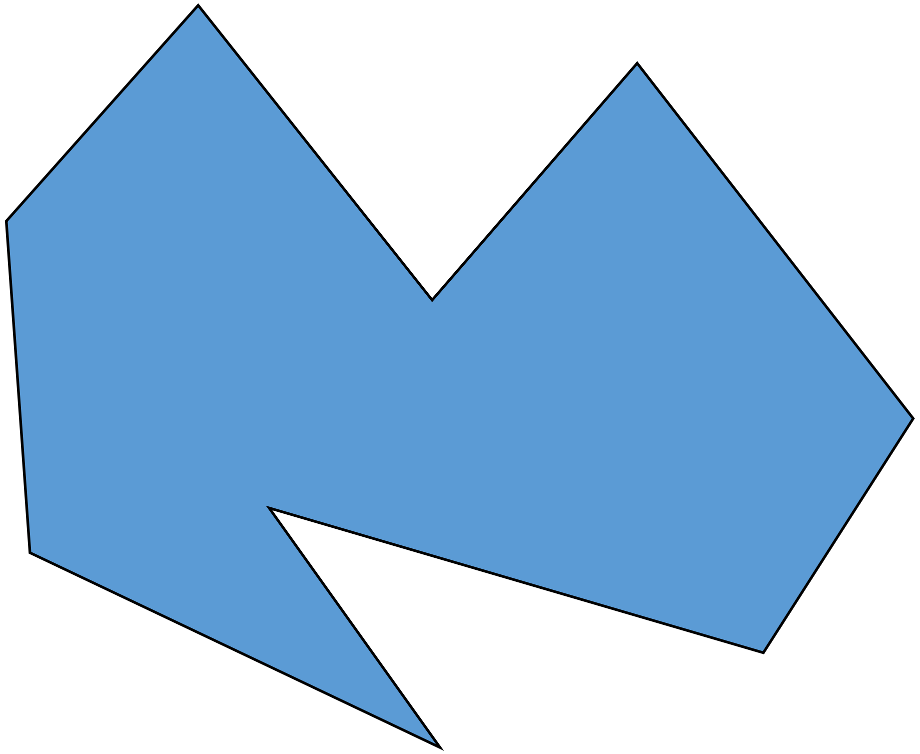
7 triangles

6 diagonals

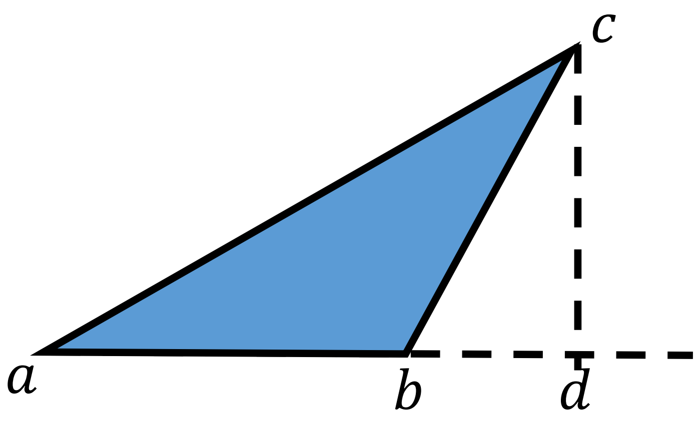


Area of a Polygon

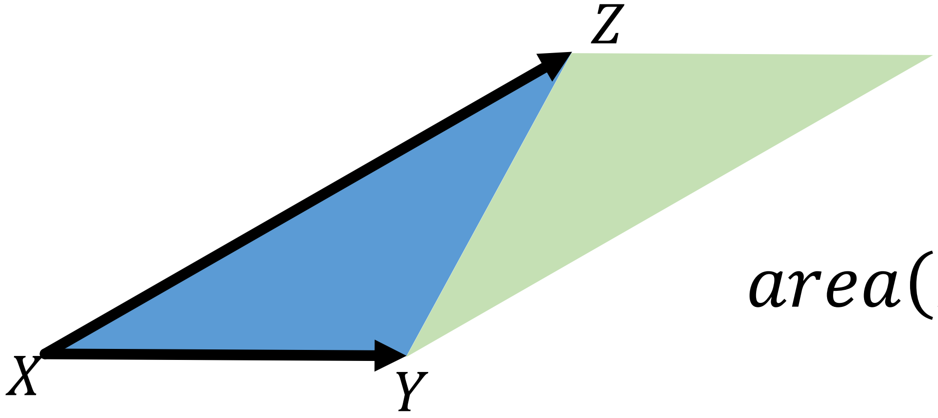
- What is the area of the polygon



Area of a Triangle



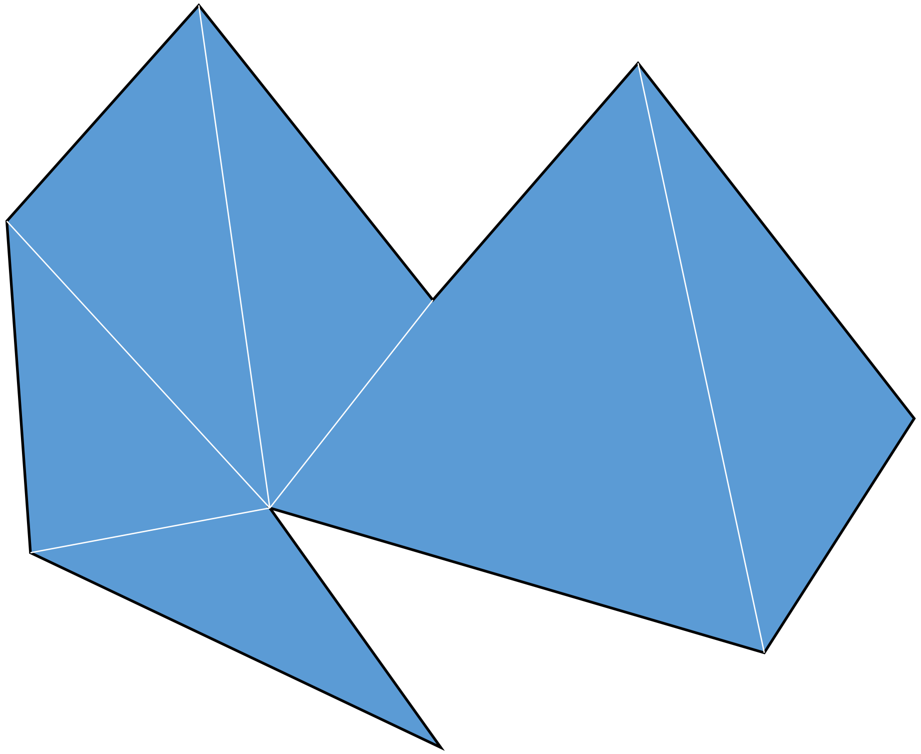
$$area = 0.5|a - b||c - d|$$



$$area(blue + green) = \left|\left(\overrightarrow{b - a}\right) \times \left(\overrightarrow{c - a}\right)\right|$$

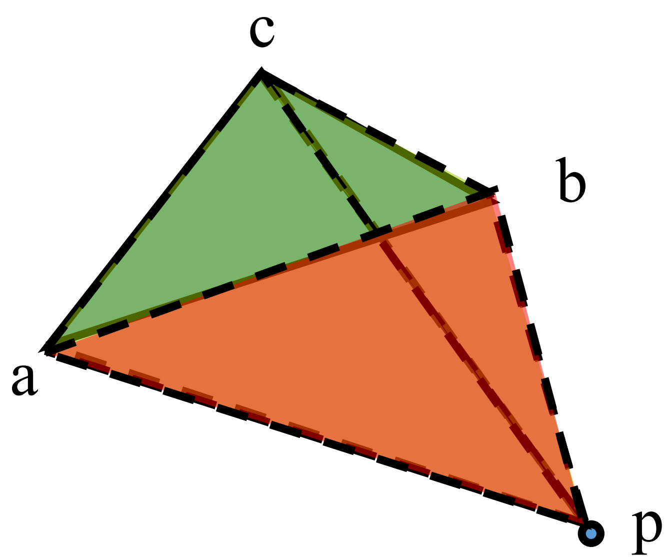
Sum of Areas

- Area of the polygon can be found by adding the areas of a triangulation



Another way of computing area

$$Area\ (T) = Area(p, b, c) + Area(p, c, a) + Area(p, a, b)$$



Area of Polygon

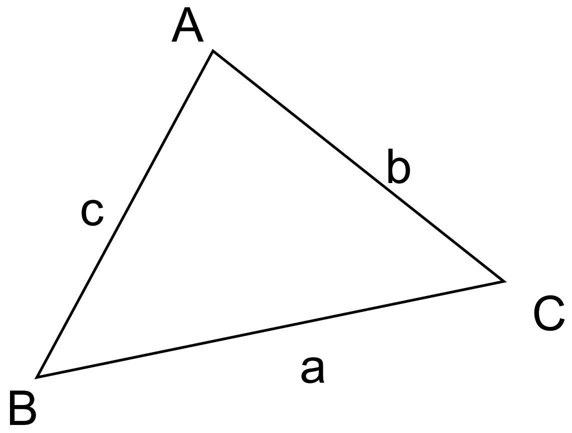
- Theorem: Let a polygon (convex or non-convex) P have vertices $v_0, v_1, \dots, v_{n-1}$

$$Area(P) = A(p, v_0, v_1) + A(p, v_1, v_2) + A(p, v_2, v_3) + \cdots + A(p, v_{n-2}, v_{n-1}) + A(p, v_{n-1}, v_0)$$

- Proof is by induction

Area of triangle

- Heron's formula:
  - $Area = sqrt\big(s(s-a)(s-b)(s-c)\big)$
  - where $s = (a+b+c)/2$ is the semiperimeter



AREA OF TRIANGLE

- WHAT IF ONLY THE VERTICES OF THE TRIANGLE ARE GIVEN?
- GIVEN 3 VERTICES $(x_1, y_1), (x_2, y_2), (x_3, y_3)$

$$Area = \frac{|x_1 * y_2 + x_2 * y_3 + x_3 * y_1 - x_2 * y_1 - x_3 * y_2 - x_1 * y_3|}{2}$$

- Note: abs can be omitted if the vertices are in **counterclockwise** order. If the vertices are in clockwise order, the difference evaluates to a negative quantity

Area of triangle

- That hard-to-memorize expression can be written this way:

- Area = ½ *

$$\begin{array}{cc} \begin{vmatrix} x1 & y1 \\ x2 & y2 \\ x3 & y3 \\ x1 & y1 \end{vmatrix} & \begin{array}{c} - \\ \\ \\ + \end{array} \end{array}$$

Area of convex polygon

- It turns out the previous formula still works!

- Area = ½ * $\begin{vmatrix} x1 & y1 \\ x2 & y2 \\ x3 & y3 \\ x4 & y4 \\ x5 & y5 \\ x1 & y1 \end{vmatrix}$

-

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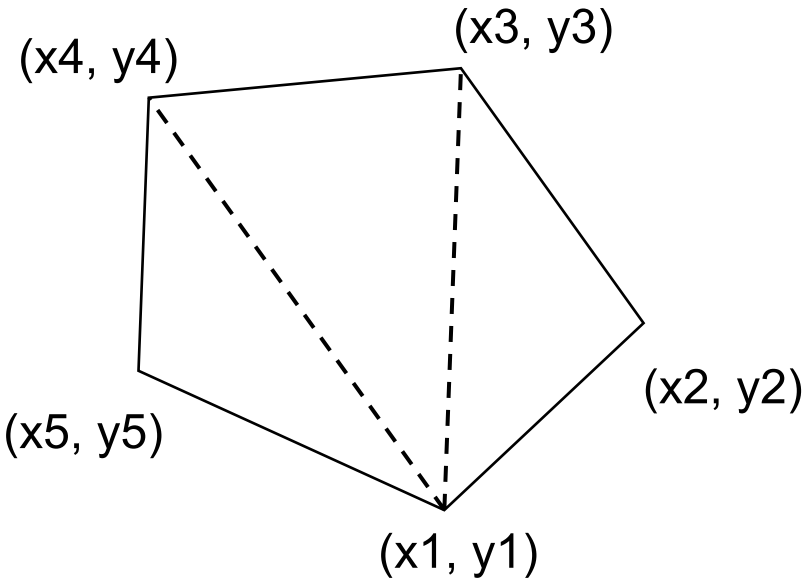
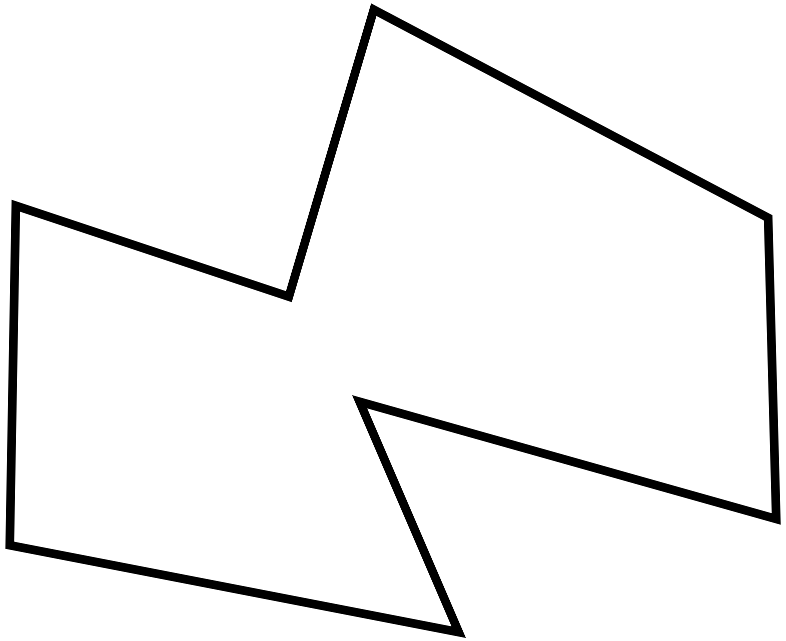


AREA OF (NON-CONVEX) POLYGON

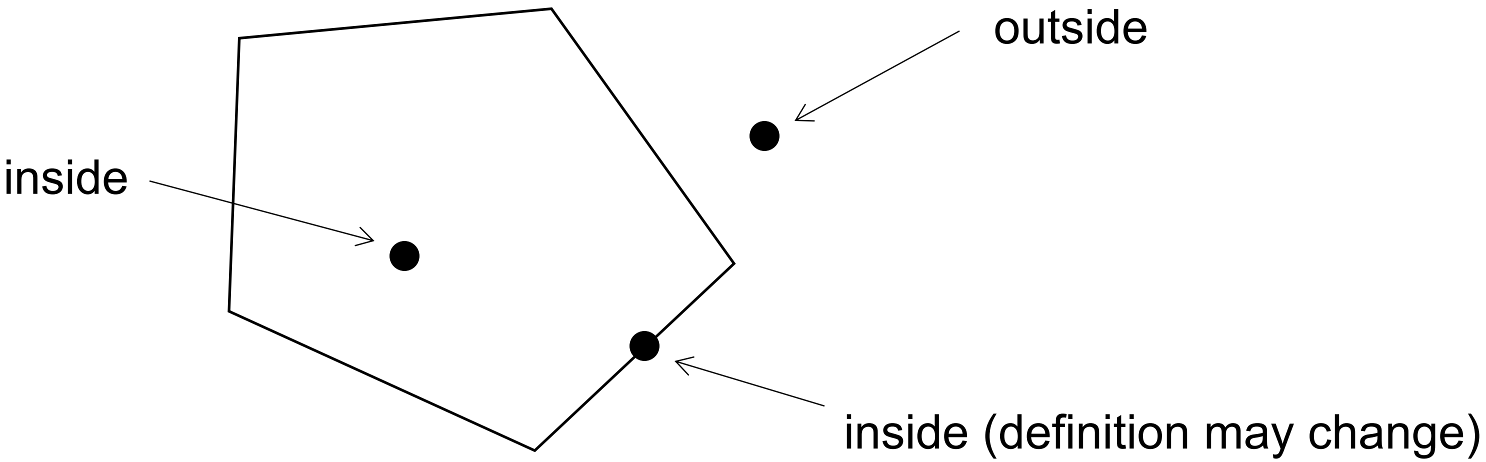
- MIRACULOUSLY, THE SAME FORMULA STILL HOLDS FOR NON-CONVEX POLYGONS!

- AREA = ½ * …



Point inside convex polygon?

- Given a convex polygon and a point, is the point contained inside the polygon?
  - Assume the vertices are given in **counterclockwise** order for convenience

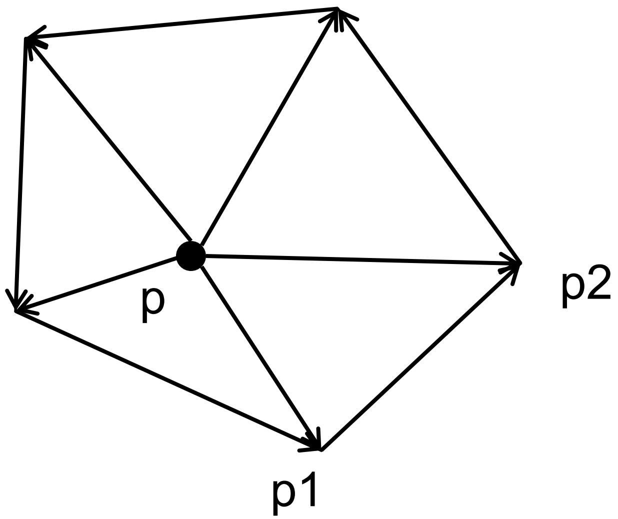


Detour – Is polygon convex?

- A quick question – how to tell if a polygon is convex?
- Answer: It is convex if and only if every turn (at every vertex) is a left turn
  - Whether a "straight" turn is allowed depends on the problem definition
- Our crossProd function is so useful

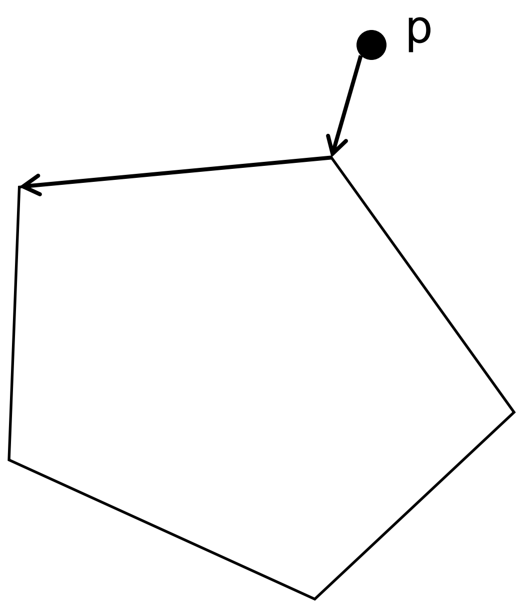
Point inside convex polygon?

- Consider the turn p → p1 → p2
- If p does lie inside the polygon, the turn must **not** be a right turn
- Also holds for other edges (mind the directions)



Point inside convex polygon?

- Conversely, if p was outside the polygon, there would be a right turn for some edge

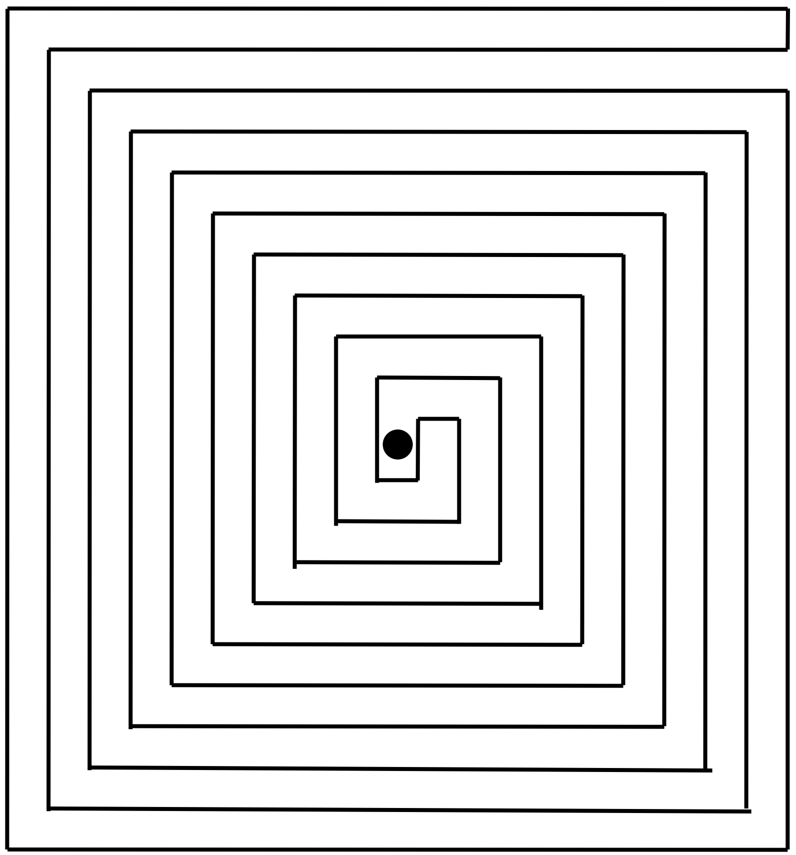


Point inside convex polygon

- Conclusion: p is inside the polygon if and only if it makes a non-left turn for every edge (in the counterclockwise direction)

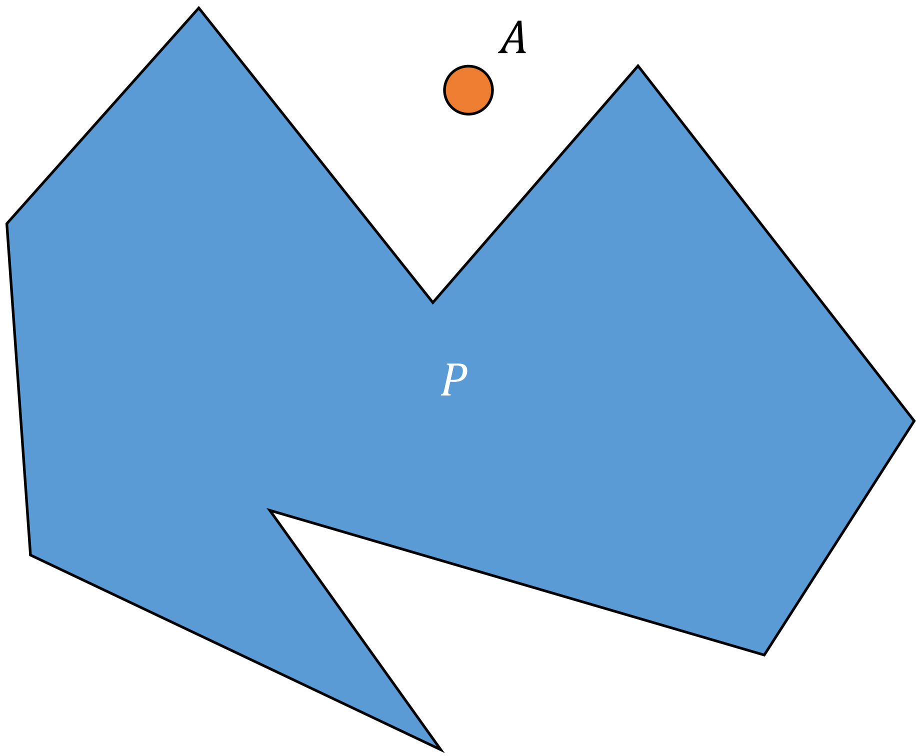
Point inside (non-convex) polygon

- Such a pain



Point inside a polygon test

- How can we tell if point A is inside of polygon P?



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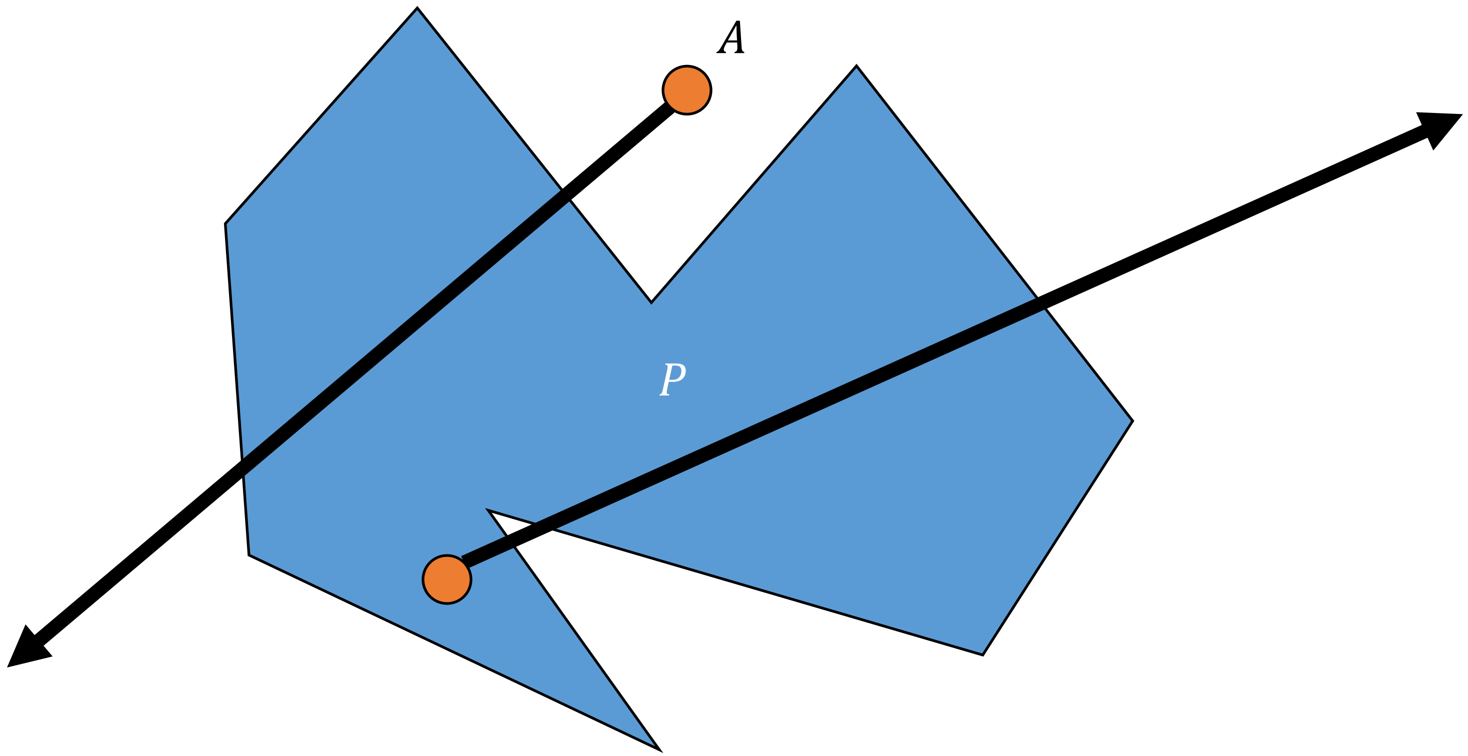
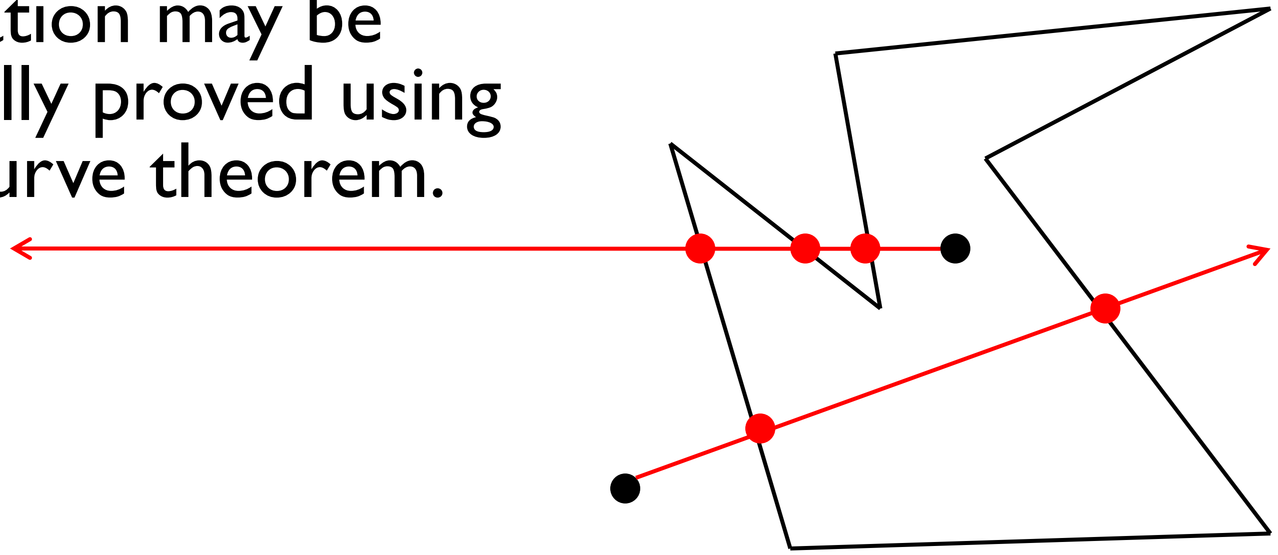


POINT INSIDE POLYGON

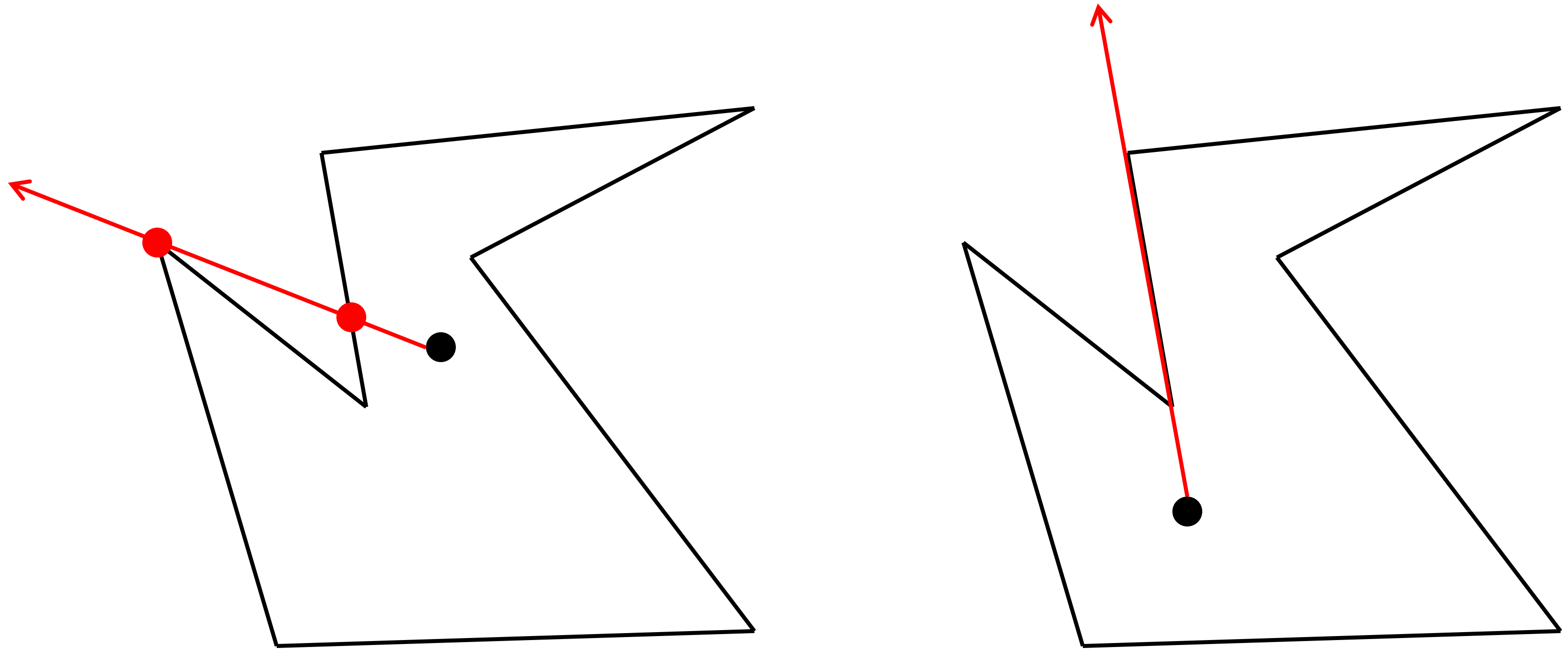
- EVEN–ODD RULE ALGORITHM
  - If a point A moves along a ray from infinity to A
  - If it crosses the boundary of a polygon, possibly several times, then it alternately goes from the outside to inside to outside
  - After every two "border crossings" the moving point goes outside.

  - This observation may be mathematically proved using the Jordan curve theorem.



Point inside polygon

- Problematic cases: Degenerate intersections



- Solution: Pick a random direction (i.e. random slope). If the ray hits a vertex of the polygon, pick a new direction. Repeat.

Point inside polygon

- Solution: Pick a random direction (i.e. random slope). If the ray hits a vertex of the polygon, pick a new direction. Repeat.