COT 4521: Introduction to Computational Geometry

Polygon Partitioning

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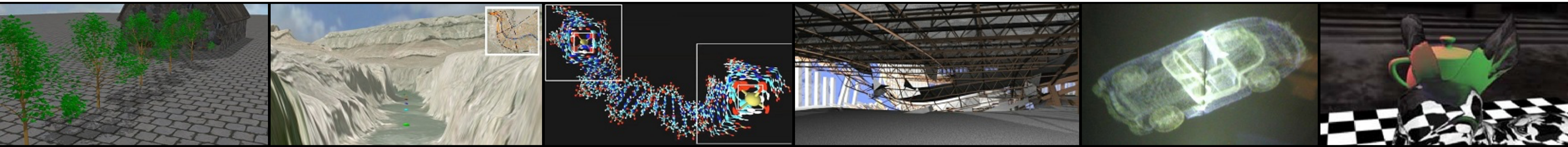
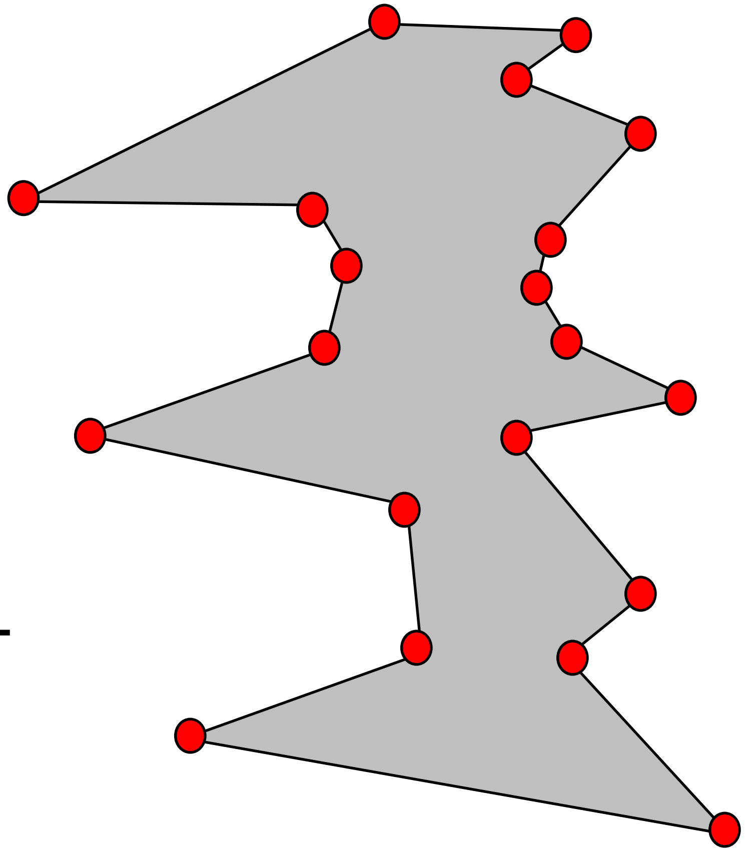


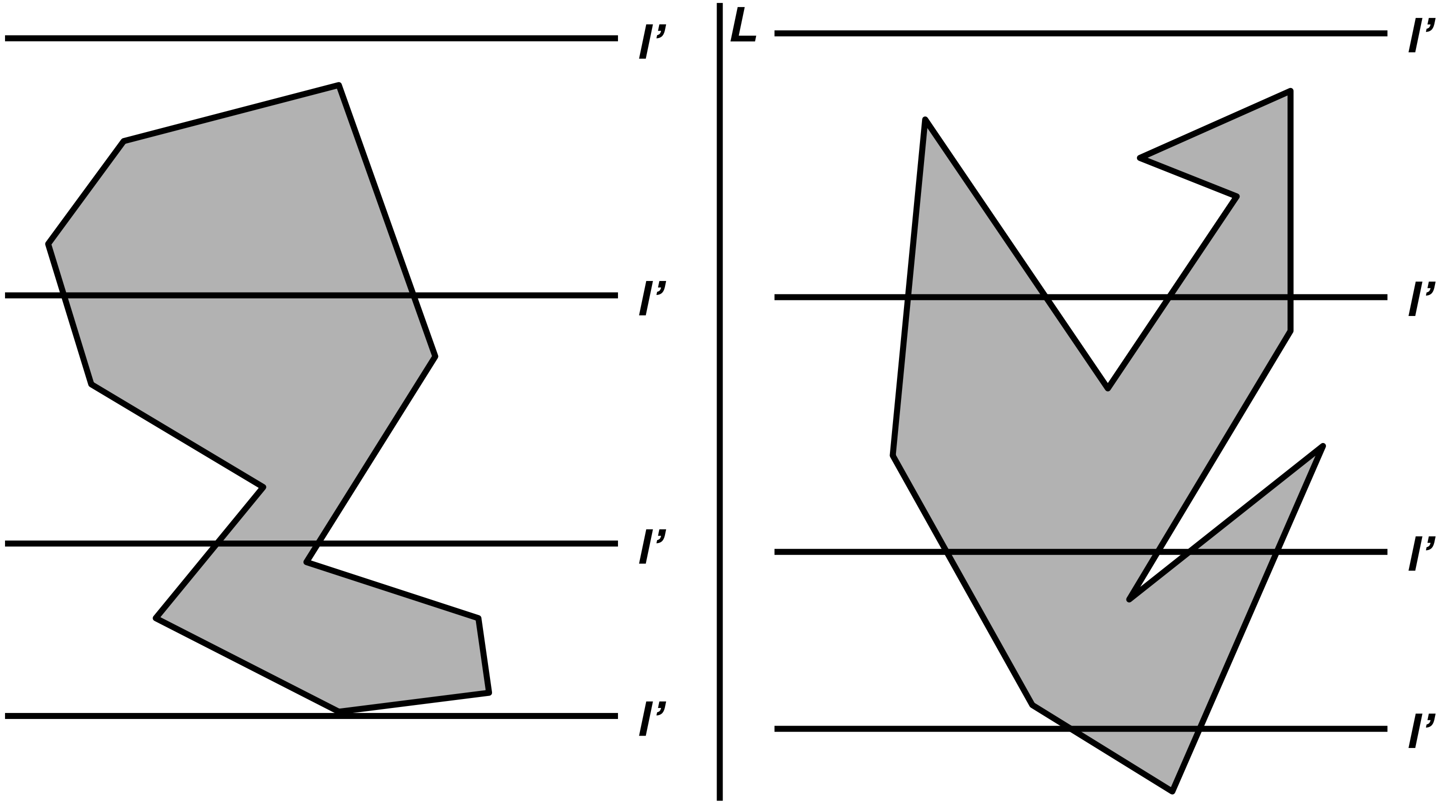
Monotone Partitioning

- A **chain** is **monotone** with respect to a line L if every line orthogonal to L intersects the chain in at most 1 point
- **Polygon** is **monotone** with respect to a line L if boundary of P can be split into 2 polygonal chains A and B such that each chain is monotone with respect to L
- Monotonicity implies sorted order with respect to L
- Monotone polygon can be (greedily) triangulated in O(n) time!



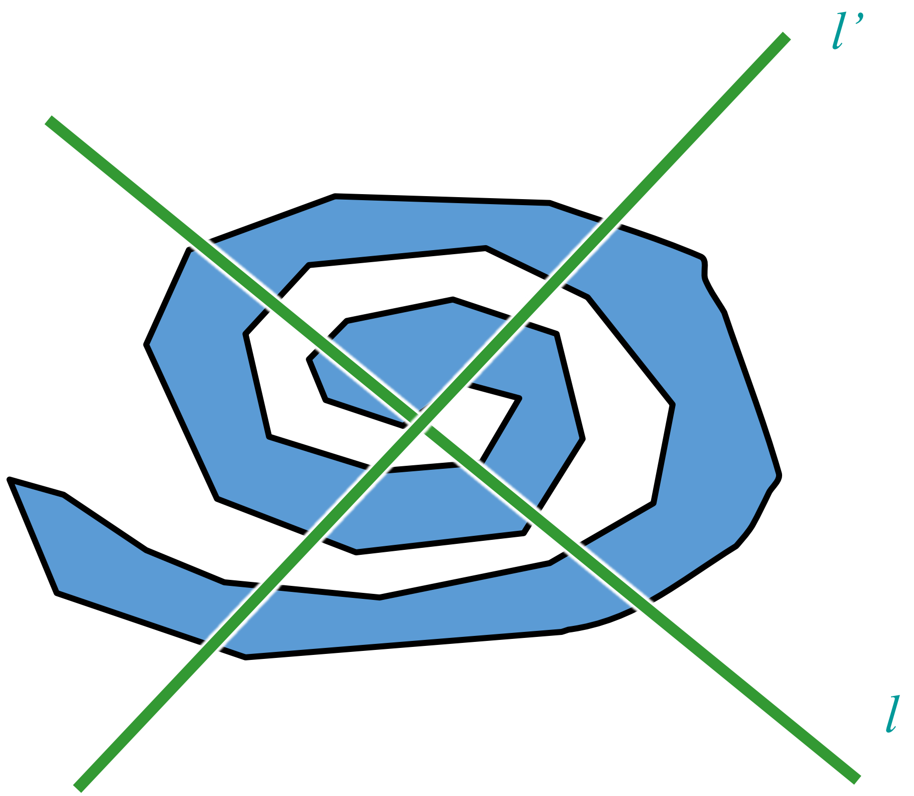
MONOTONE POLYGON

Monotone
w.r.t. line L



not monotone
w.r.t. L

Monotone Polygon



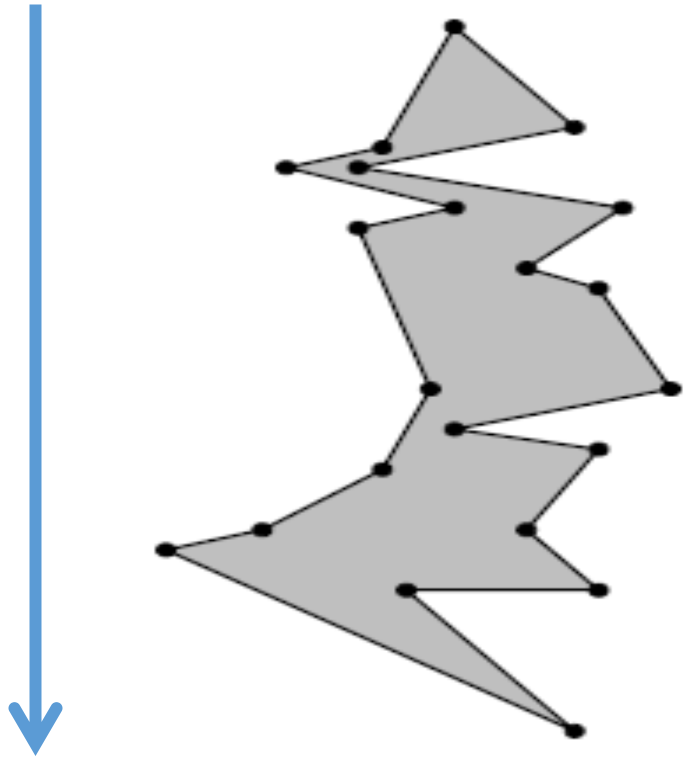
not monotone
w.r.t. any line

POLYGON TRIANGULATION

- **ALGORITHM**: POLYGON TRIANGULATION: MONOTONE POLYGON WITH RESPECT TO Y-LINE
  - Partition into monotone polygons
  - Triangulate each monotone polygon

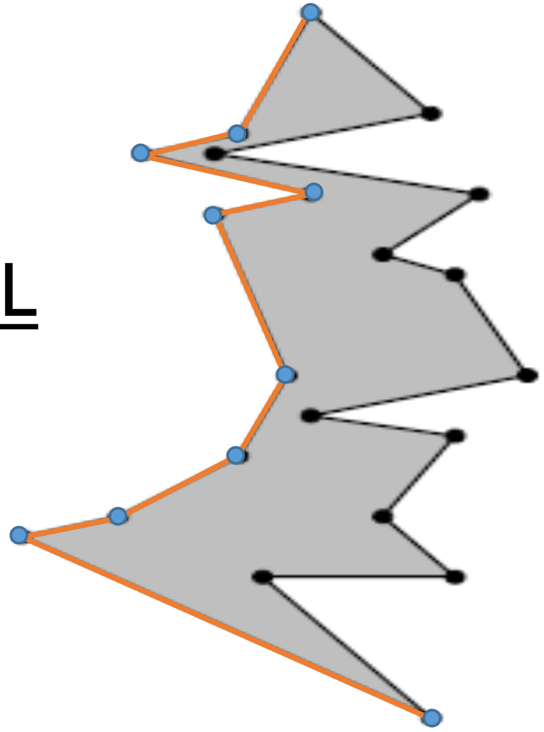
Monotone Polygon

- A y-monotone polygon has a top vertex, a bottom vertex, and two y-monotone chains between top and bottom as its boundary



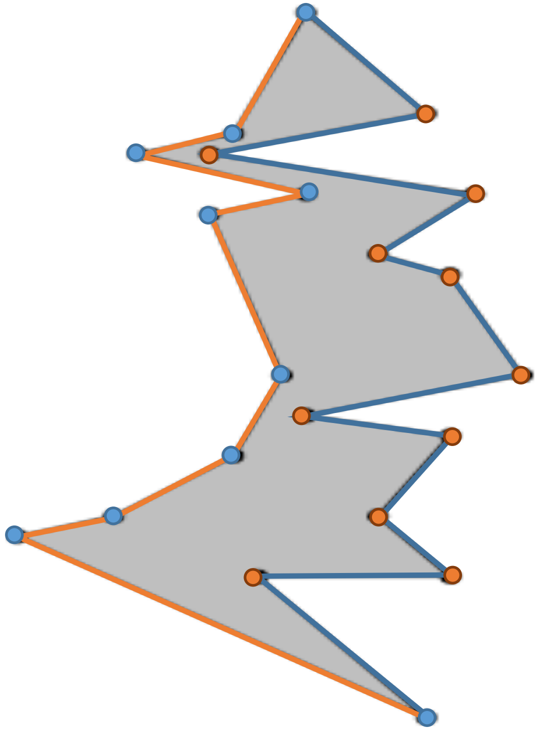
MONOTONE POLYGON

- A POLYGONAL CHAIN C IS STRICTLY MONOTONE W.R.T. L' IF EVERY L ORTHOGONAL TO L' MEETS C IN AT MOST ONE POINT.
  - Simply monotone if L∩C has at most one connected line segment.



Monotone Polygon

- A polygon P is said to be monotone w.r.t. a line L if ∂P can be split into two monotone chains w.r.t. L



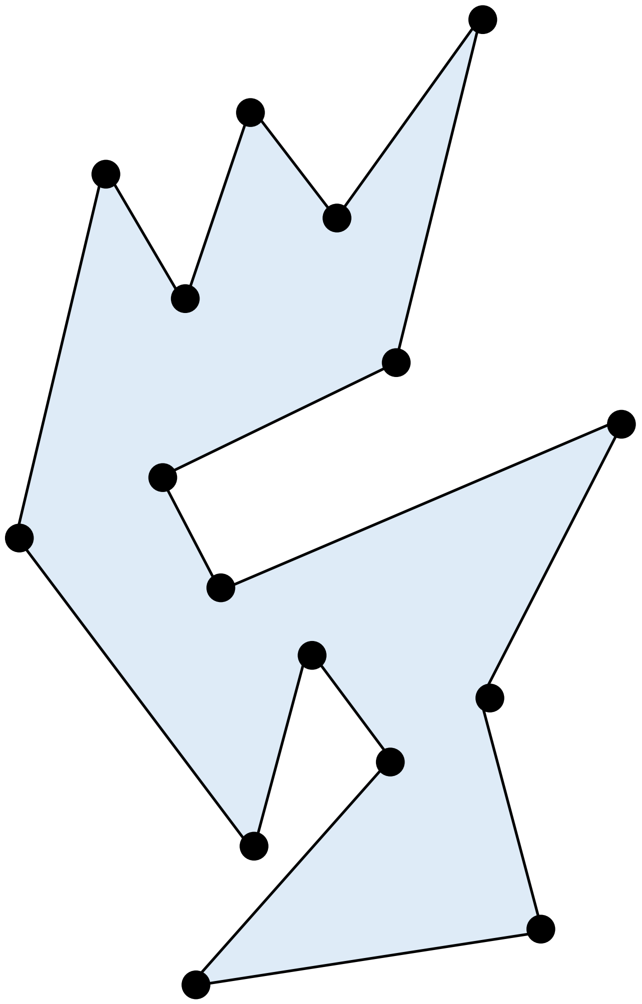
Monotone Polygon

- The following property is characteristic for y-monotone polygons:
  - If we walk from a topmost to a bottommost vertex along the left (or the right) boundary chain, then we always move downwards or horizontally, never upwards.

Triangulation of Monotone Polygon

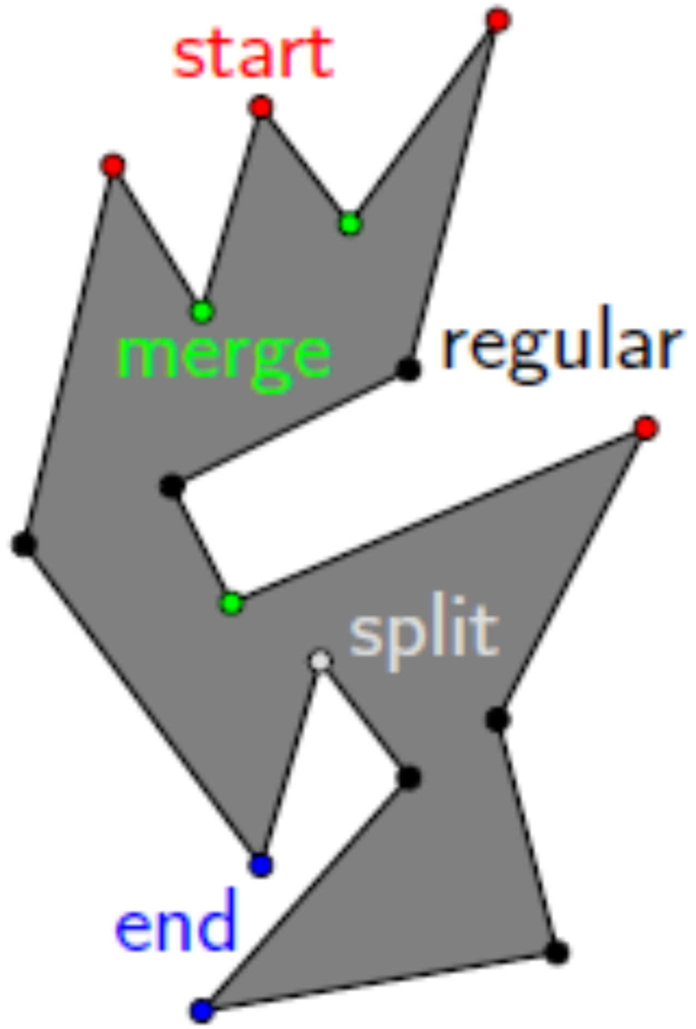
- The algorithm to triangulate a monotone polygon depends on its monotonicity.
- Developed in 1978 by Garey, Johnson, Preparata, and Tarjan
- described in both
  - Preparata pp. 239-241 (1985)
  - Laszlo pp. 128-135 (1996)
  - The former uses y-monotone polygons, the latter uses x-monotone.

What kind of vertices does a Non-Y-Monotone Polygon have with respect to sweep of y?

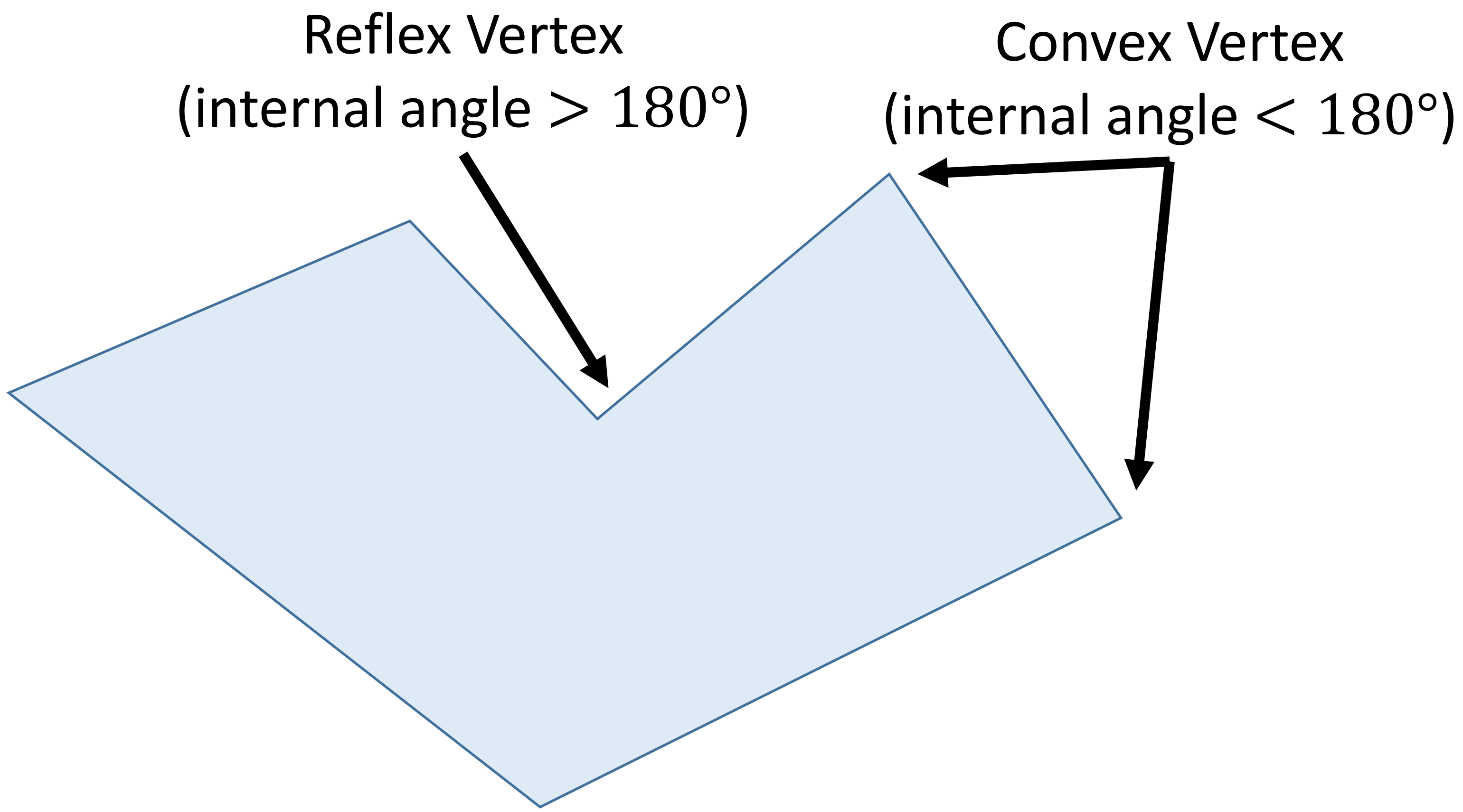


Properties of monotone polygon

- If a polygon P has no interior cusps, then it is monotone
- What types of vertices does a simple polygon have?
  - start
  - end
  - split
  - merge
  - regular



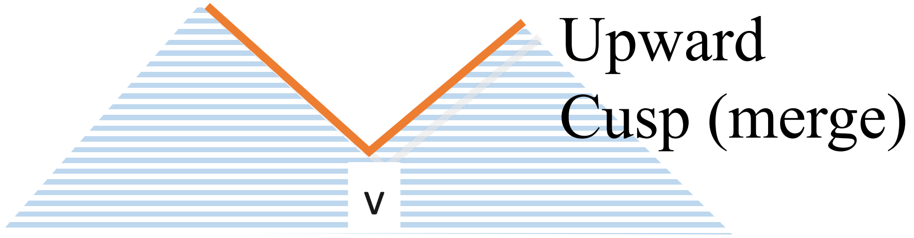
REMINDER

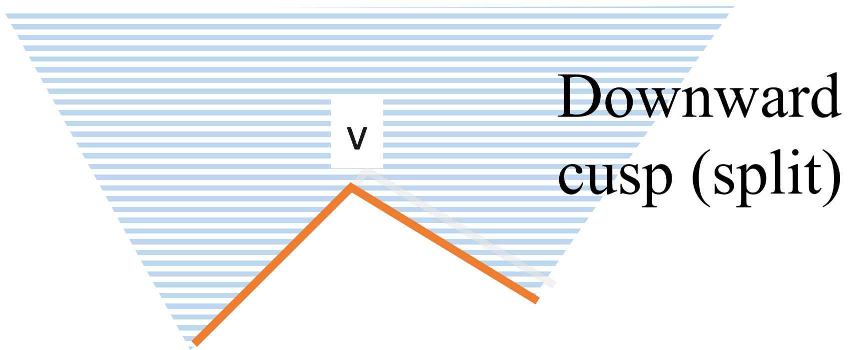

Reflex Vertex
(internal angle $> 180^\circ$)
Convex Vertex
(internal angle $< 180^\circ$)

PROPERTIES OF MONOTONE POLYGON

- REFLEX VERTEX WHOSE ADJACENT VERTICES ARE EITHER BOTH AT OR ABOVE V, OR BOTH AT OR BELOW IT.
- IF A POLYGON HAS NO INTERIOR CUSPS THEN IT IS MONOTONE WITH RESPECT TO THE VERTICAL LINE

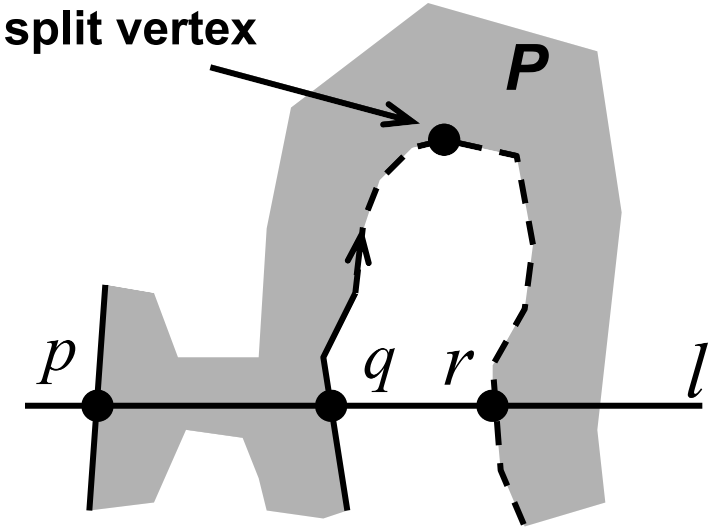
Interior cusps





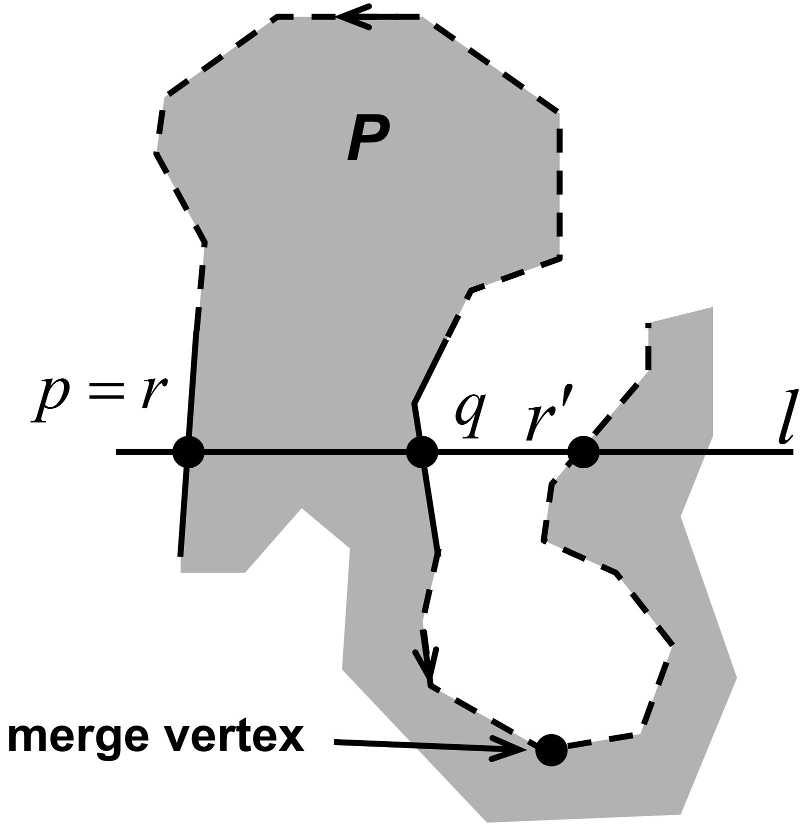
Properties of monotone polygon

- **Lemma**: A polygon is y-monotone if it has neither split vertices nor merge vertices
- **Proof**: If P is not monotone, there must exist a line l intersecting P in more than a single segment.
  Let [p,q] be its leftmost sub segment.
- Follow the boundary of P starting at q, where P is on the left. At some point r we must cross l.
- If r ≠ p then the highest vertex must be a split one.



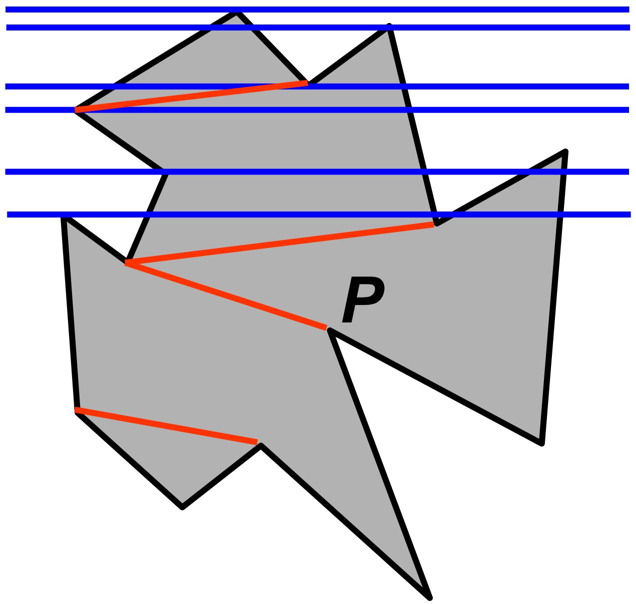
Properties of monotone polygon

- If r = p we follow the boundary from q in opposite direction.
- At some point r' we must cross l. $r' \neq p$ as otherwise it contradicts that P is not monotone.
- This implies that the lowest encountered vertex must be a merge one.



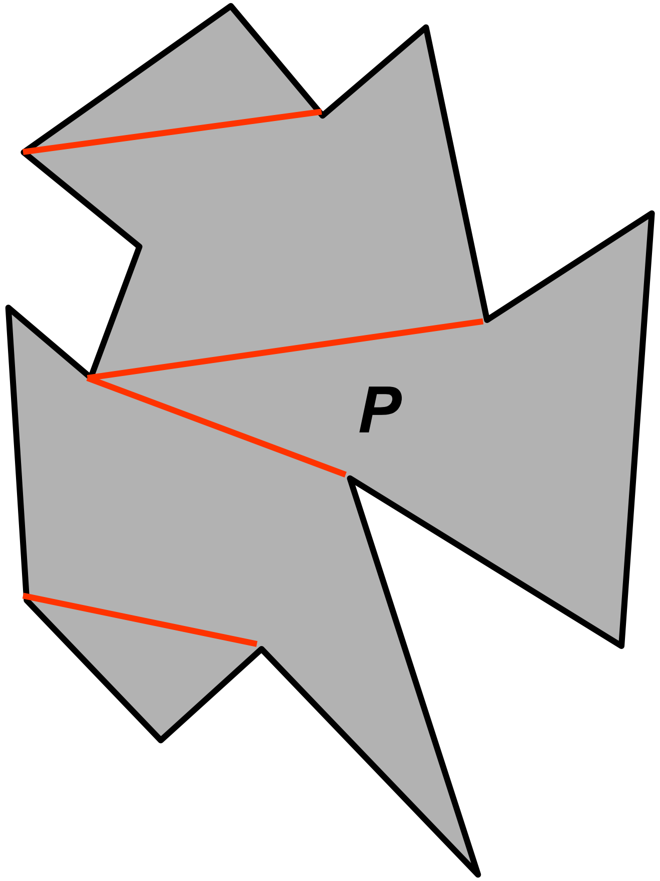
POLYGON TRIANGULATION

- ALGORITHM: POLYGON TRIANGULATION: MONOTONE PARTITION
  - **Partition into monotone polygons**
  - Triangulate each monotone polygon



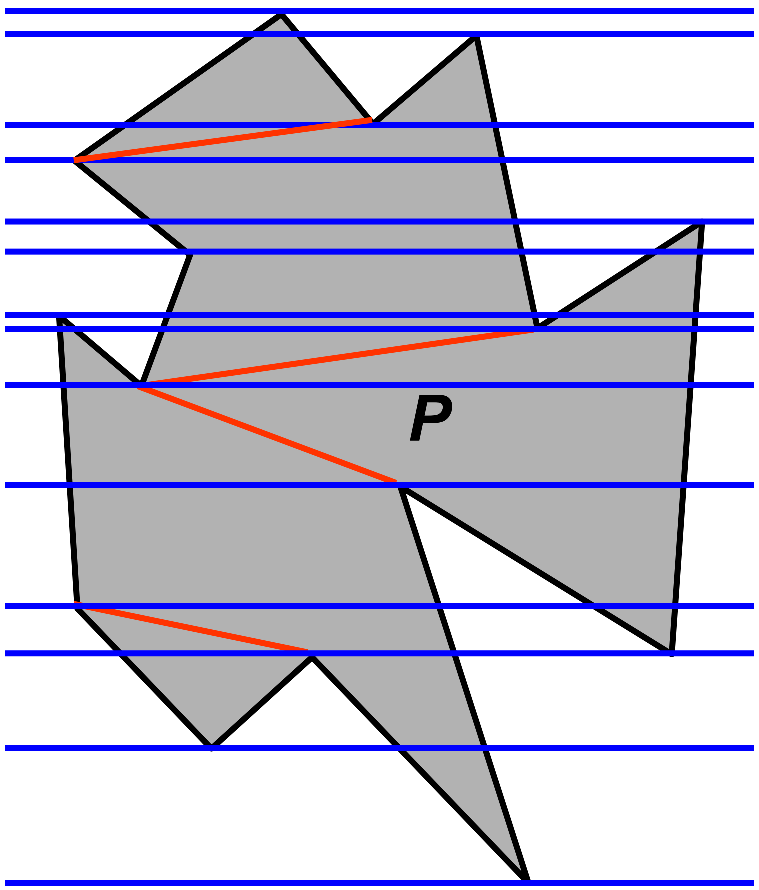
Getting Rid of Split and Merge Vertices

- Sort P's vertices from top to bottom
  - takes $O(n \log n)$ time.
- Scan from top to bottom to encounter vertices.
- Diagonals are introduced at split and merge vertices.



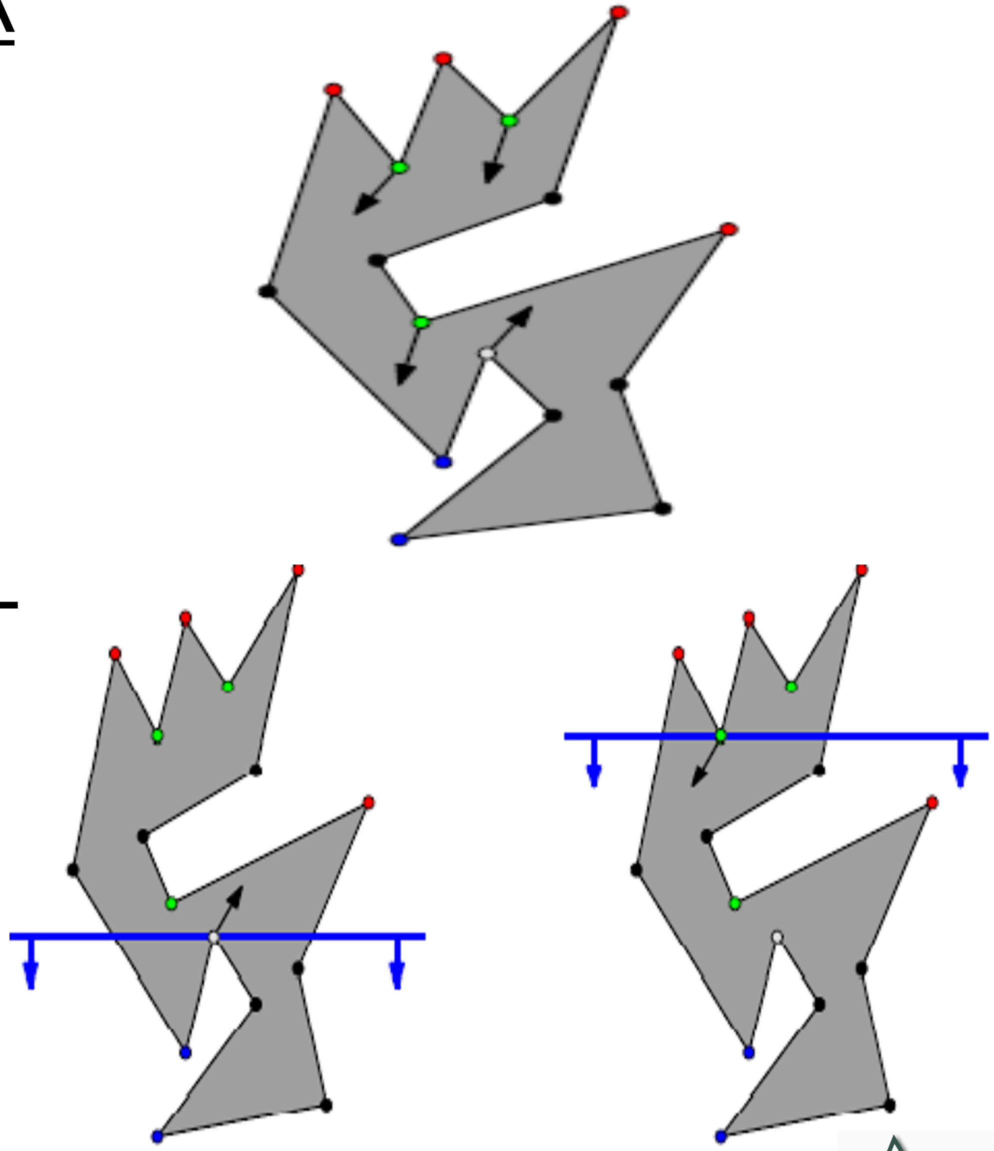
TRAPEZOIDALIZATION

- "DRAW" HORIZONTAL LINE THROUGH EACH VERTEX
  - Consider only the connected segment inside the polygon containing the vertex
  - Two supporting vertices – top and bottom
- IF AN "INTERIOR" SUPPORTING VERTEX IS AN INTERIOR CUSP, BREAK IT
  - Connect downward for a upward cusp
  - Connect upward for an downward cusp
  - These connections partitions the polygon into monotone parts.



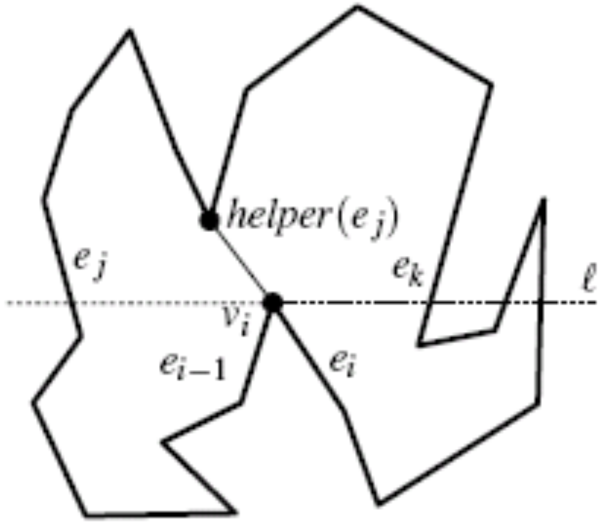
SWEEP IDEA

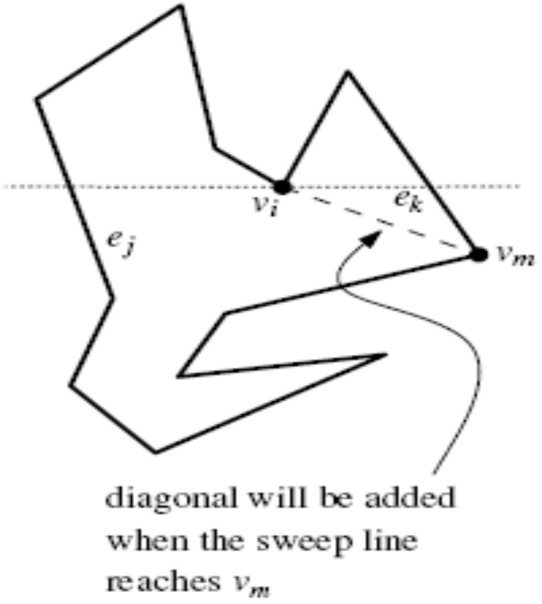
- FIND DIAGONALS FROM EACH MERGE VERTEX DOWN, AND FROM EACH SPLIT VERTEX UP
- A SIMPLE POLYGON WITH NO SPLIT OR MERGE VERTICES CAN HAVE AT MOST ONE START AND ONE END VERTEX, SO IT IS Y-MONOTONE



Sweep idea

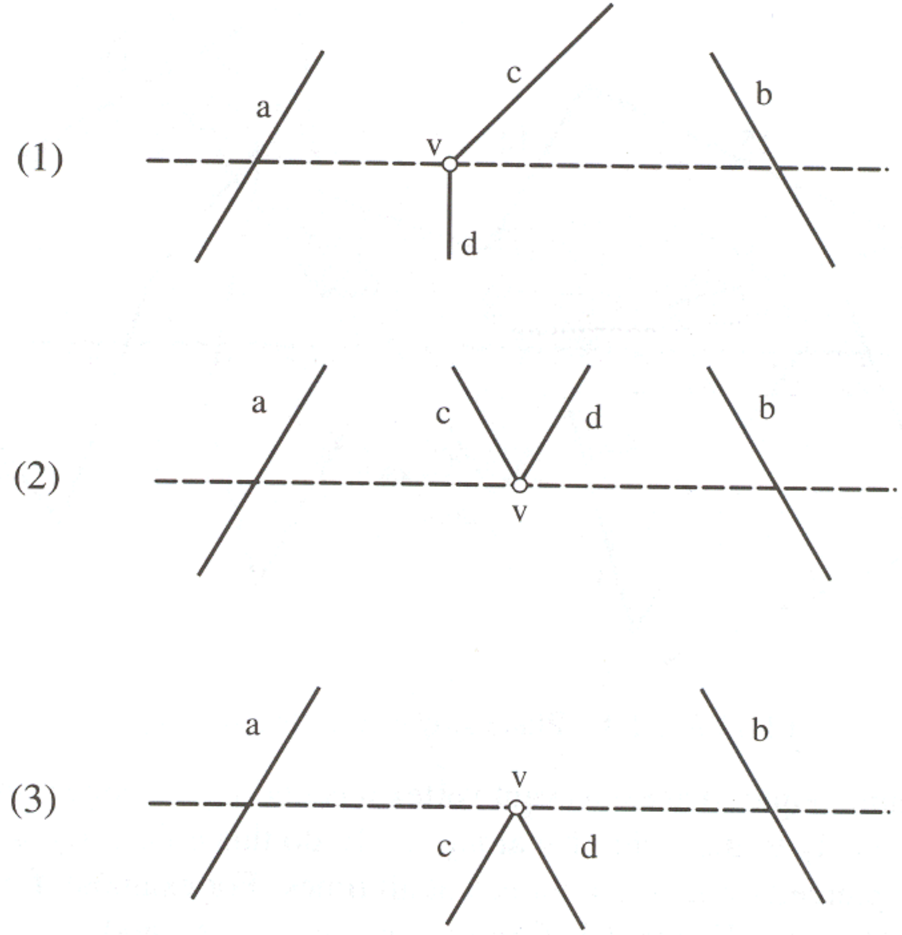
- For vertex of interest v, find the closest vertex (in the y direction) that is between the edges to the left and right of v





FORMING TRAPEZOIDS

- MAINTAIN A LIST OF SIDES INTERSECTED BY THE SWEEPING LINE, SORTED BY THE X-COORD OF INTERSECTION
- AT EACH EVENT, UPDATE THE LIST
  - Can be done in $O(\log N)$ if the list is maintained as a balanced binary tree
- OVERALL: $O(N \log N)$



Line Sweep Example