

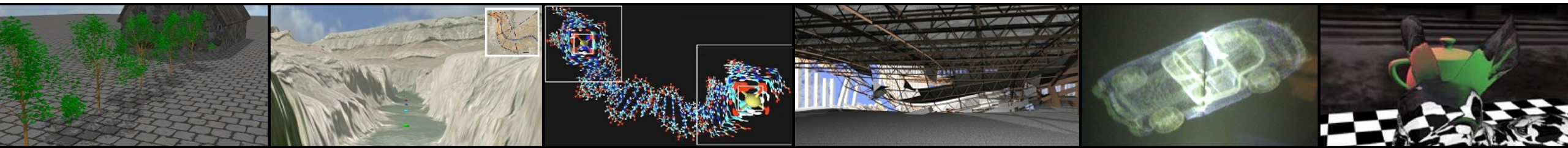
COT 4521: INTRODUCTION TO COMPUTATIONAL GEOMETRY



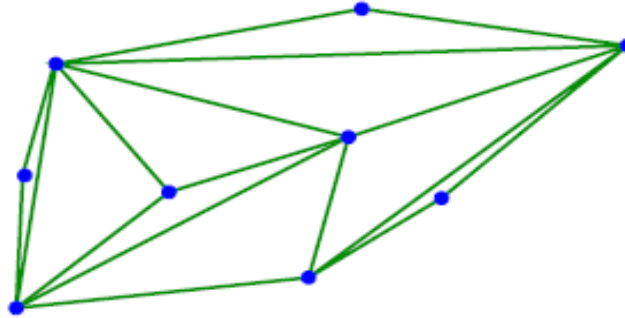
Delaunay Triangulation

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Some slides from Valentina Korzhova



TRIANGULATION OF A POINT-SET

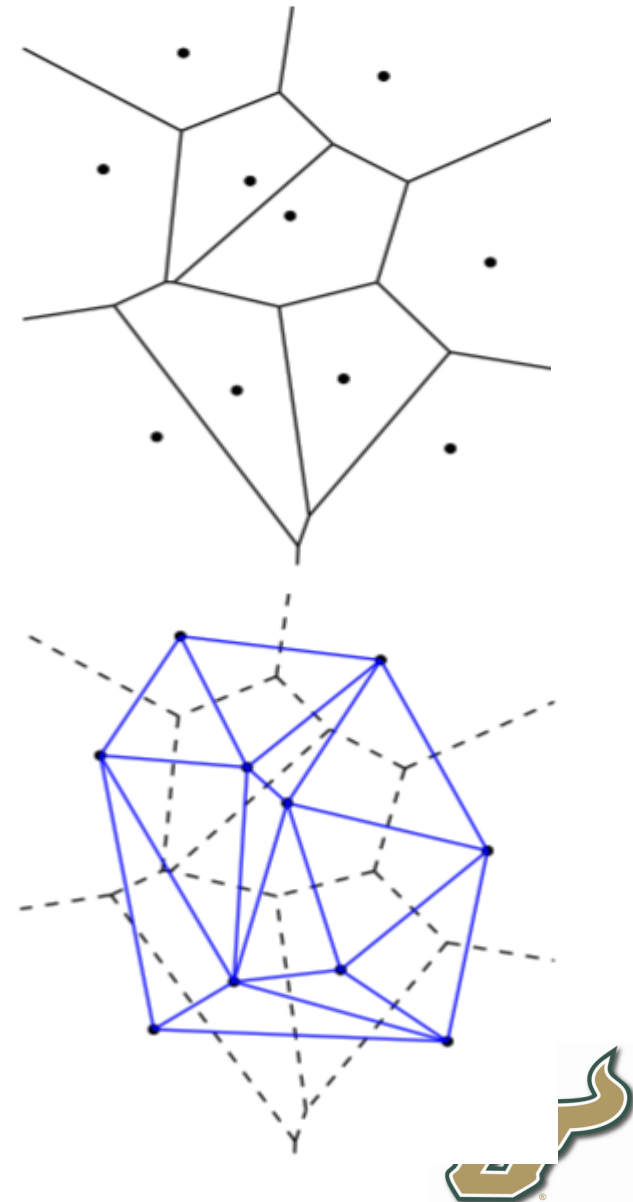


- DEFINITION (POINT-SET TRIANGULATION)
 - Given a set S of n points in \mathbb{R}^2 , a triangulation of S is a planar graph with vertex set S , such that all the bounded faces are triangles, and these faces form a partition of the convex hull $CH(S)$ of S .



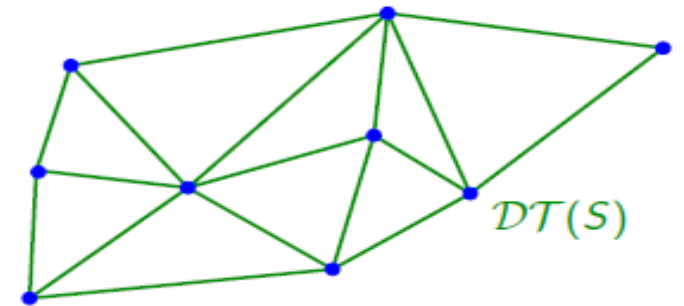
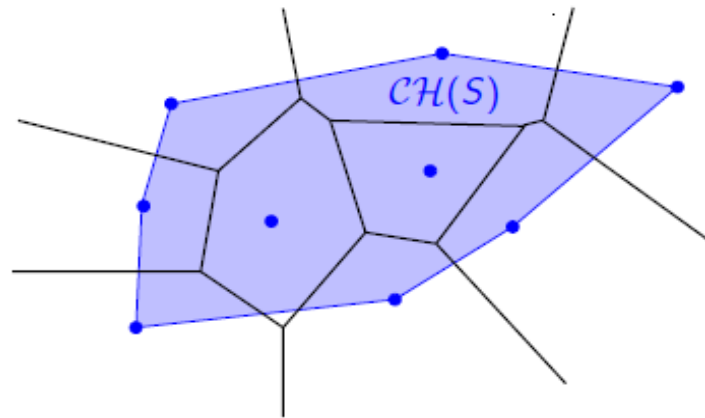
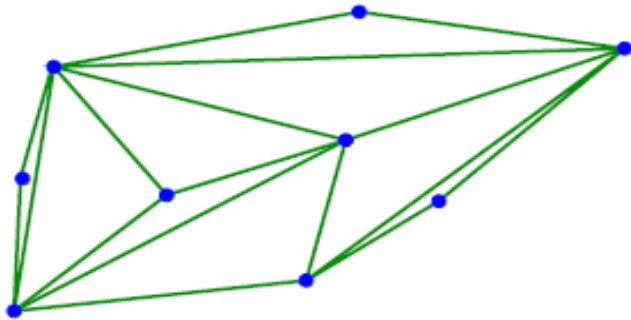
DELAUNAY TRIANGULATIONS

- THE VORONOI DIAGRAM $Vor(P)$ IS THE SUBDIVISION OF THE PLANE INTO VORONOI CELLS $V(p)$ FOR ALL $p \in P$
- IN 1934 DELAUNAY PROVED THAT WHEN THE DUAL GRAPH IS DRAWN WITH STRAIGHT LINES, IT PRODUCES A PLANAR TRIANGULATION OF THE VORONOI SITES P , NOW CALLED THE DELAUNAY TRIANGULATION



THE DELAUNAY TRIANGULATION

- THE DELAUNAY TRIANGULATION OF THE SAME SET.
- IT HAS MANY INTERESTING PROPERTIES.



THE DELAUNAY TRIANGULATION

- LET S BE A SET OF N POINTS IN R^2 . WE ASSUME GENERAL POSITION IN THE SENSE THAT NO 4 POINTS IN S ARE CO-CIRCULAR. THE DELAUNAY TRIANGULATION DT OF S IS THE DUAL GRAPH OF THE VORONOI DIAGRAM OF S SUCH THAT:
 - Each vertex $DT(s_i)$ is located at the corresponding site s_i
 - The edges of $DT(S)$ are straight line segments.



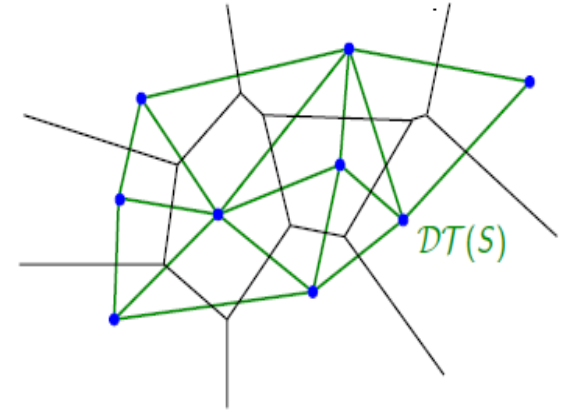
PROPERTIES OF DELAUNAY TRIANGULATIONS

- THE PLANAR VORONOI DIAGRAM AND THE DELAUNAY TRIANGULATION ARE DUALS IN A GRAPH THEORETICAL SENSE
 - Voronoi vertices correspond to Delaunay triangles
 - Node of $DT(P)$ corresponds to Voronoi regions
 - Edges of both types correspond by definition.



THE DELAUNAY TRIANGULATION

- DT OVER VD
 - Face of $DT(S)$ \Leftrightarrow vertex of $VD(S)$
 - Node of $DT(S)$ \Leftrightarrow sites of $VD(S)$
 - Edges of $DT(S)$ \Leftrightarrow edges of $VD(S)$
 - Boundary of $DT(S)$: convex hull
 - Interior of each $DT(S)$ face do not contain any cite



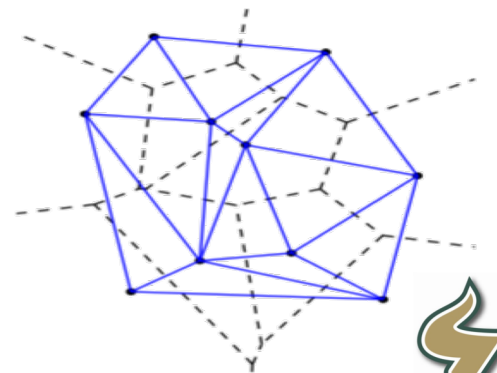
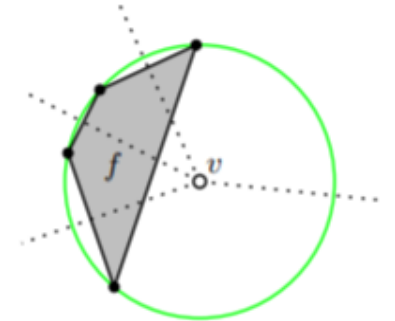
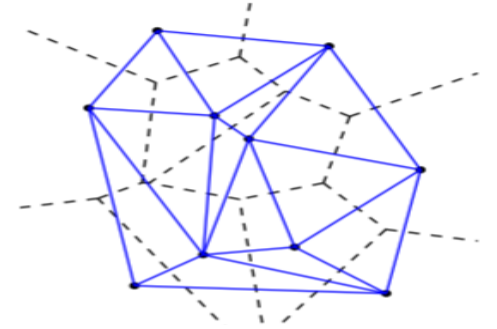
PROPERTIES OF DELAUNAY TRIANGULATIONS

- THE DUALITY IMMEDIATELY IMPLIES UPPER BOUNDS OF $3n - 6$ AND OF $2n - 5$ ON THE NUMBER OF DELAUNAY EDGES AND TRIANGLES, RESPECTIVELY.
- THE DELAUNAY TRIANGULATION AND ITS DUALITY TO VORONOI DIAGRAMS GENERALIZE TO HIGHER DIMENSIONS IN AN OBVIOUS WAY.



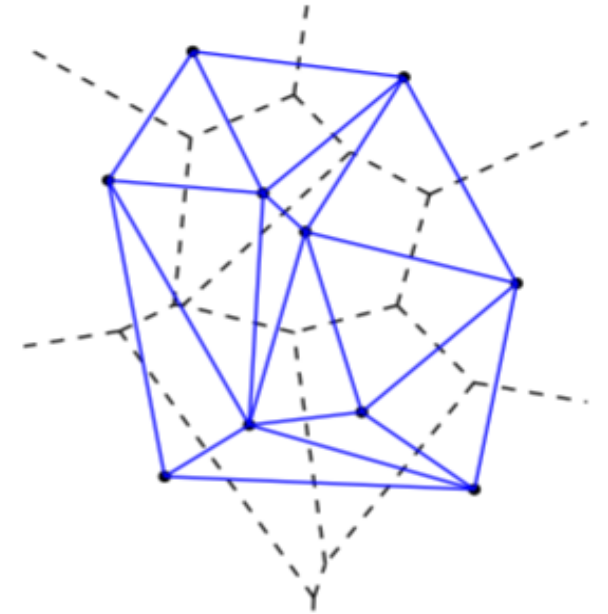
PROPERTIES OF DELAUNAY TRIANGULATIONS

- $DT(P)$ IS THE STRAIGHT-LINE DUAL OF $VD(P)$. THIS IS BY DEFINITION.
- $DT(P)$ IS A TRIANGULATION IF NO FOUR POINTS OF P ARE CO-CIRCULAR: EVERY FACE IS A TRIANGLE. THIS IS A DELAUNAY'S THEOREM. THE FACES OF $DT(P)$ ARE CALLED DELAUNAY TRIANGLES
- EACH FACE (TRIANGLE) OF $DT(P)$ CORRESPONDS TO A VERTEX OF $VD(P)$
- EACH EDGE OF $DT(P)$ CORRESPONDS TO AN EDGE OF $VD(P)$



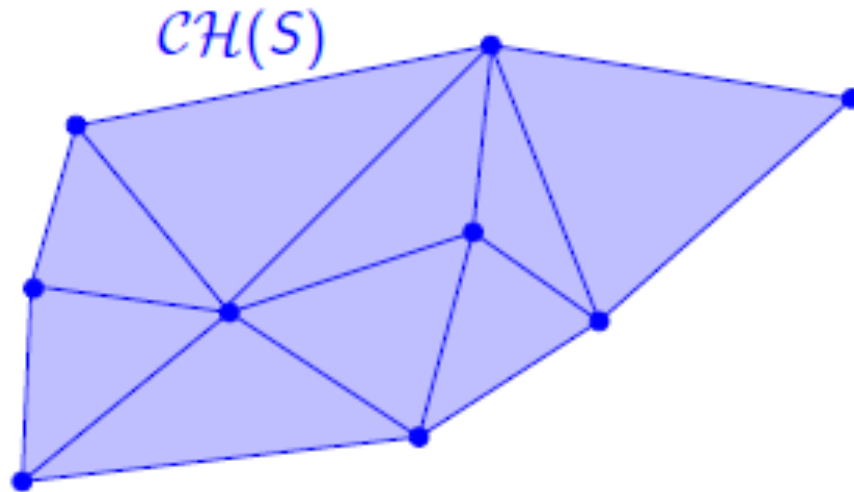
PROPERTIES OF DELAUNAY TRIANGULATIONS

- EACH NODE OF $DT(P)$ CORRESPONDS TO A REGION OF $VD(P)$
- THE BOUNDARY OF $DT(P)$ IS THE CONVEX HULL OF SITES
- THE INTERIOR OF EACH (TRIANGLE) FACE OF $DT(P)$ CONTAINS NO SITES



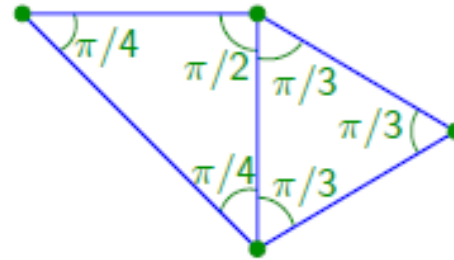
CONVEX HULL

- THE BOUNDARY OF $DT(S)$ IS CONVEX HULL OF THE SITES.



TRIANGULATION MAXIMIZING THE MINIMUM ANGLE

- LET T BE A TRIANGULATION OF S
- ANGLE SEQUENCE $\theta(T)$: SEQUENCE OF ALL THE ANGLES OF THE TRIANGLE OF T IN NON-DECREASING ORDER
- EXAMPLE: ANGLE SEQUENCE



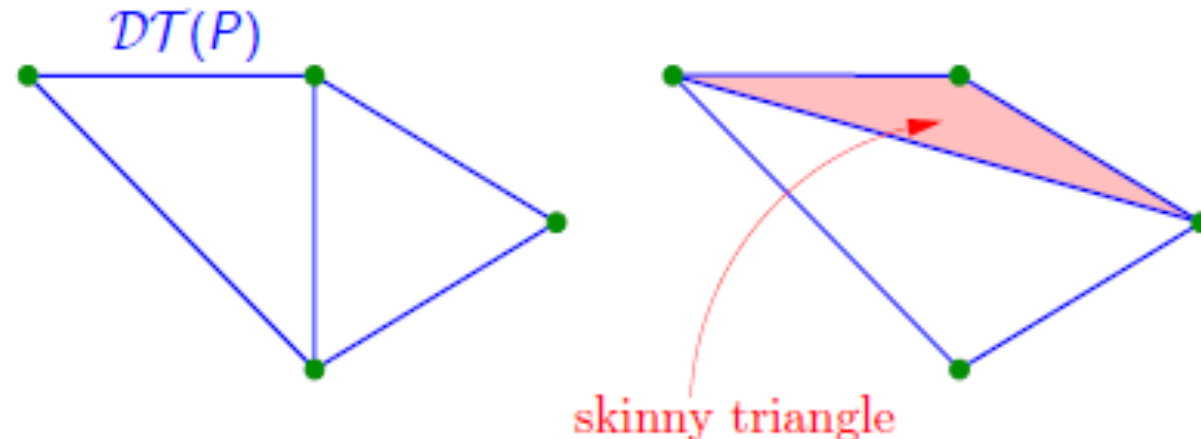
$$\Theta(T) = (\pi/4, \pi/4, \pi/3, \pi/3, \pi/3, \pi/2)$$

- COMPARISON: LET T AND T' BE TWO TRIANGULATION OF S . WE COMPARE $\theta(T)$ and $\theta(T')$ IN LEXICOGRAPHICAL ORDER
- EXAMPLE: $\{1, 1, 3, 4, 5\} < \{1, 2, 5, 6, 7\}$



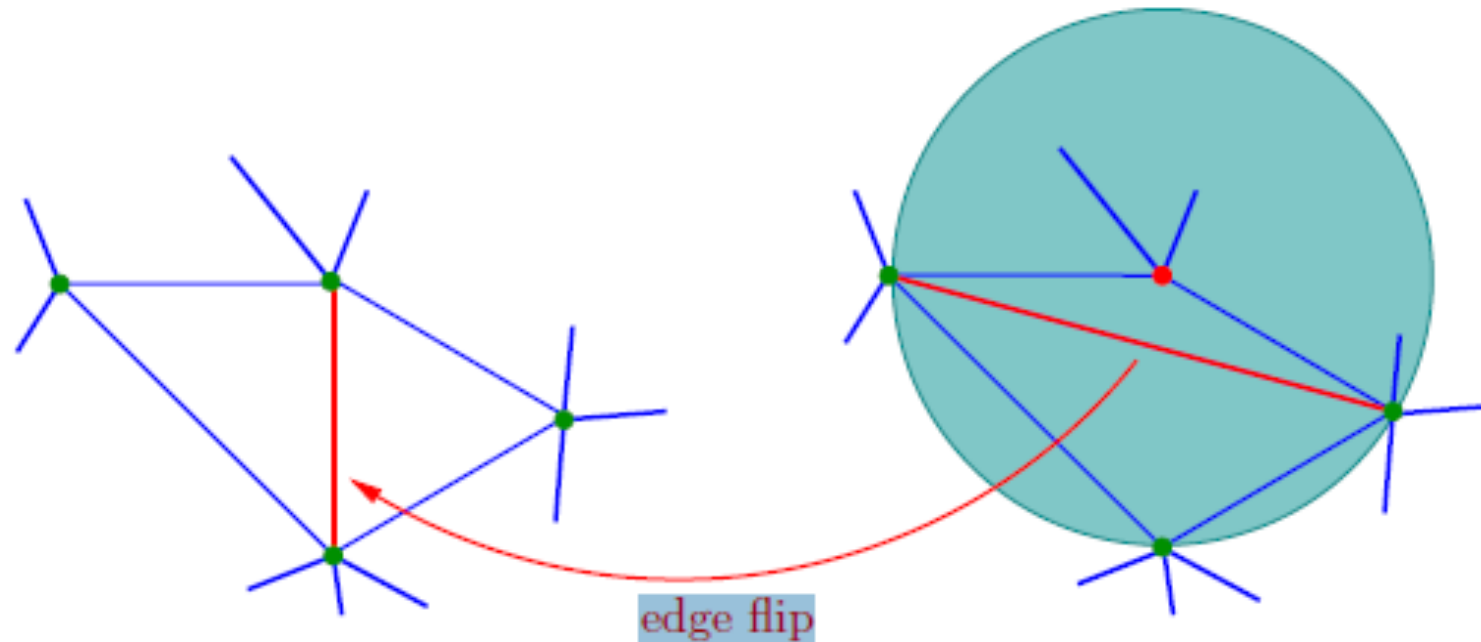
OPTIMALITY OF THE DELAUNAY TRIANGULATION

- **THEOREM:** LET S BE A SET OF POINTS IN GENERAL POSITION. THEN THE ANGLE SEQUENCE OF $DT(S)$ IS MAXIMAL AMONG ALL TRIANGULATIONS OF S .
 - So the Delaunay triangulation maximizes the minimum angle.
 - Intuition: Avoids skinny triangles.



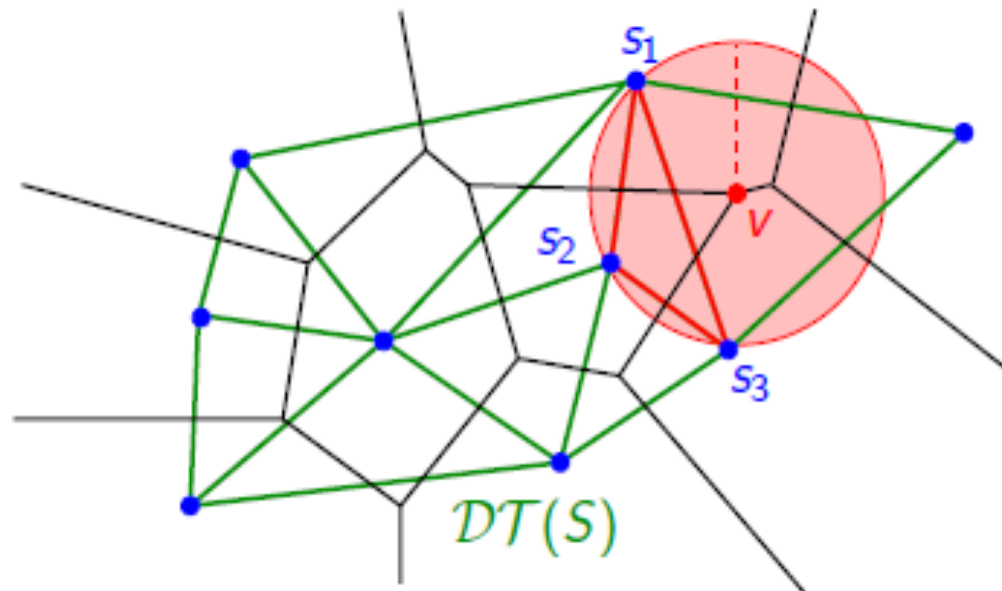
OPTIMALITY OF THE DELAUNAY TRIANGULATION

- PROOF: IDEA
 - Flip edges to ensure the circumcircle property.
 - It increases the angle sequence.



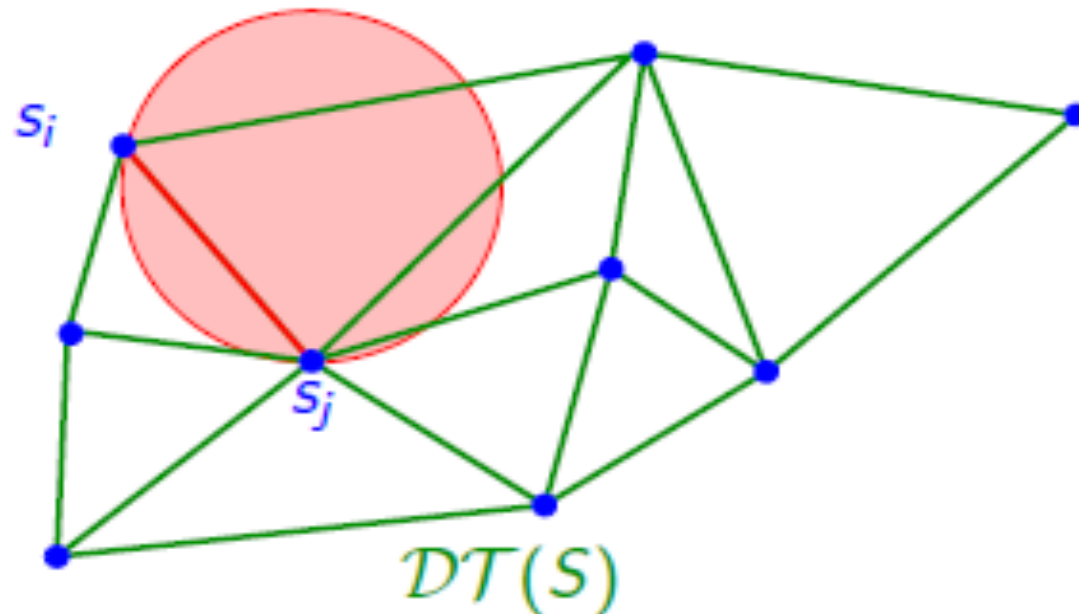
CIRCUMCIRCLE PROPERTY

- PROPERTY (CIRCUMCIRCLE)
 - The circumcircle of any triangle in $DT(S)$ is empty. (It contains no site s_i in its interior.)
- PROOF: LET $s_1s_2s_3$ BE A TRIANGLE IN $DT(S)$, LET v BE THE CORRESPONDING VORONOI VERTEX. PROPERTY OF VORONOI VERTICES: THE CIRCLE CENTERED AT v THROUGH $s_1s_2s_3$ IS EMPTY.



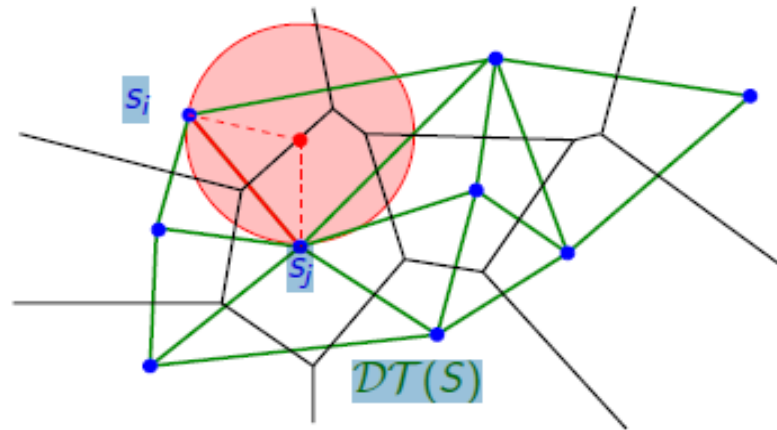
EMPTY CIRCLE PROPERTY

- PROPERTY (EMPTY CIRCLE)
 - $(s_i s_j)$ is an edge of $DT(S)$ iff there is an empty circle through s_i and s_j .



EMPTY CIRCLE PROPERTY

- PROOF :
 - If $\overline{s_i, s_j}$ is a Delaunay edge then $V(s_i)$ and $V(s_j)$ share the positive edge $e \in V(P)$.
 - Put the circle $C(x)$ with the center x on the interior e with the radius equal to the distance to s_i or s_j . If circle is not empty then x would be in $V(c)$, but we know that x is in $V(s_i)$ or $V(s_j)$



DELAUNAY-IZATION OF A TRIANGULATION

- ANY TRIANGULATION OF THE CONVEX HULL CAN BE CONVERTED INTO A DELAUNAY TRIANGULATION BY REPEATEDLY TEST THE EMPTY CIRCLE PROPERTY.
- IF ANY EDGE “FAILS” THE TEST, IT IS SWAPPED WITH A NEW EDGE BETWEEN THE CONNECTING TRIANGLES

