COT 4521: Introduction to Computational Geometry

Polygon Triangulation

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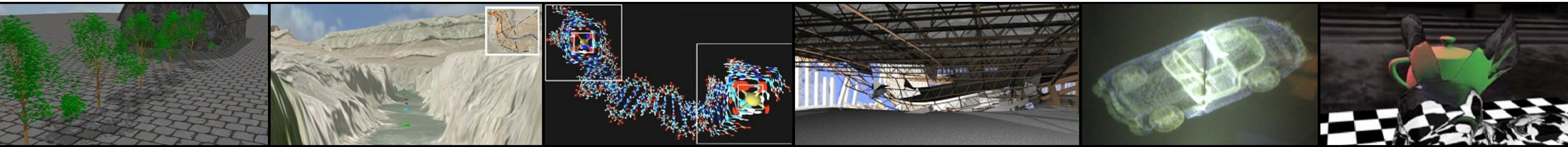
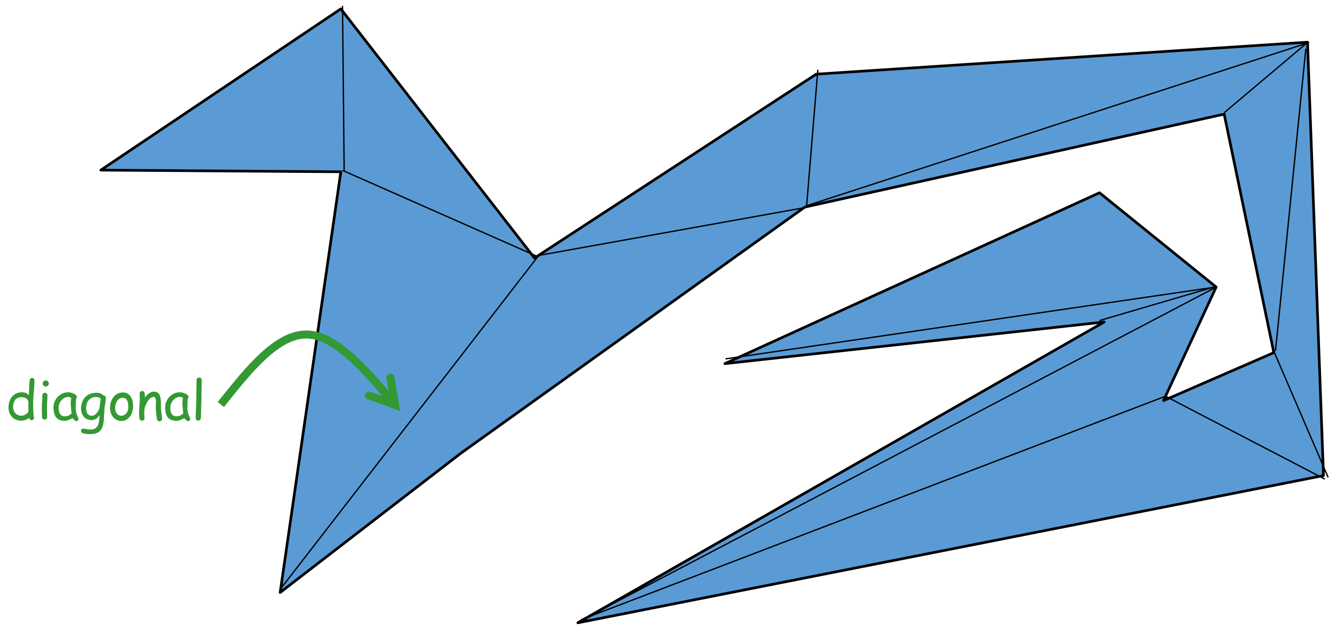


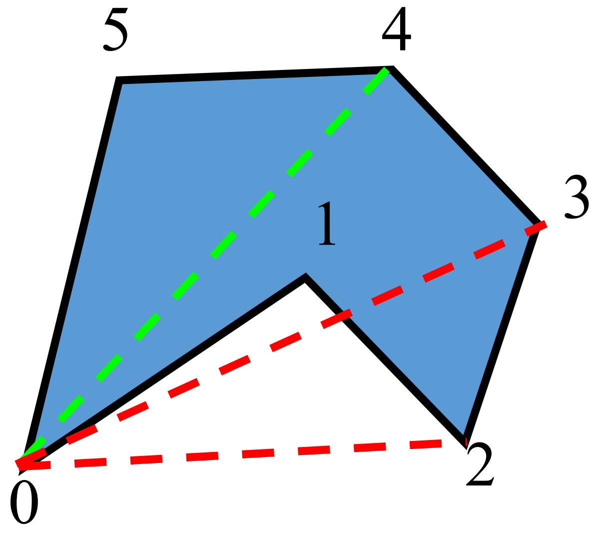
Triangulation of polygons

- Decompose the polygon into shapes that are easier to handle: triangles
- A triangulation of a polygon P is a decomposition of P into triangles whose vertices are vertices of P. In other words, a triangulation is a maximal set of non-crossing diagonals.



Diagonal-Based Triangulation

- Diagonal Test
- The segment $s = v_i v_j$ is a diagonal of P iff
  - for all edges e of P that are not incident to either v_i and v_j, s and e do not intersect.
  - s is internal to P in the neighborhood of v_i and v_j.



- Algorithm: Diagonal Triangulation

  Repeat n-3 times

  for each candidate diagonal

  test each of neighborhoods

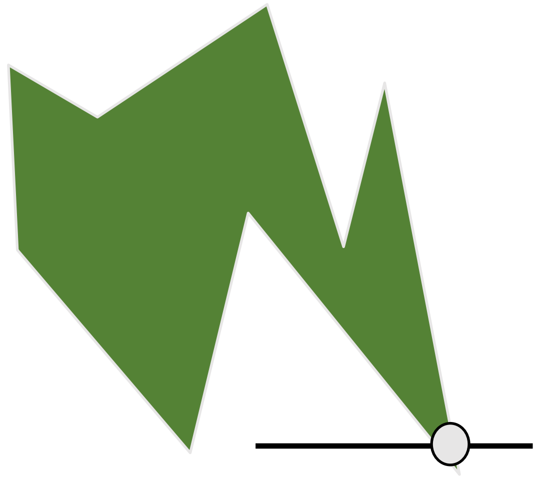
  output Proper diagonal

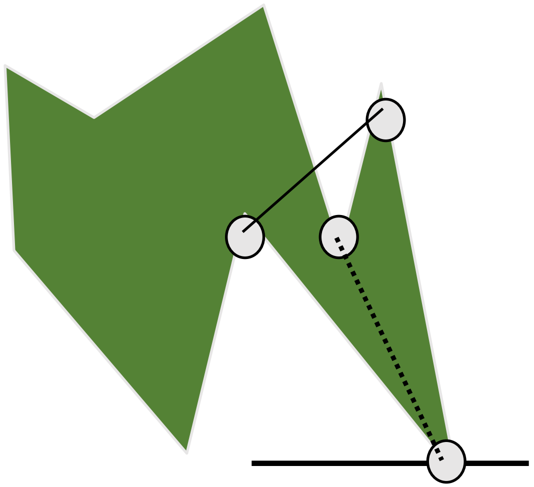
Triangulation Theory: Existence of a Diagonal

- Every polygon must have ≥ 1 *strictly* convex vertex (no collinearity)

- Every polygon of $n \geq 4$ vertices has a diagonal

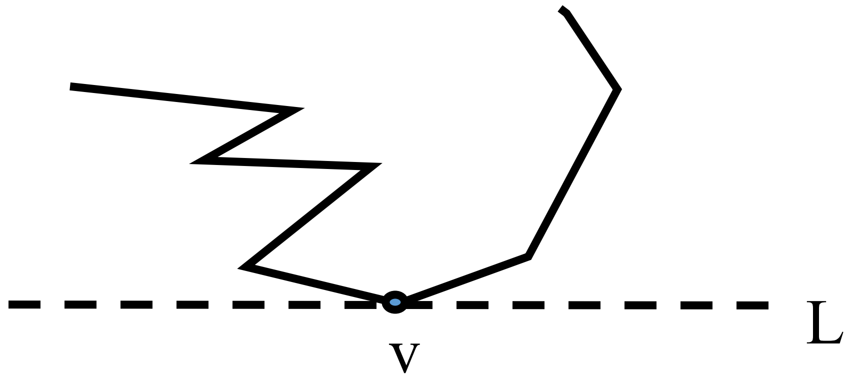
- Every n-vertex polygon P may be partitioned into triangles by adding (≥ 0) diagonals [proof by induction using diagonals]





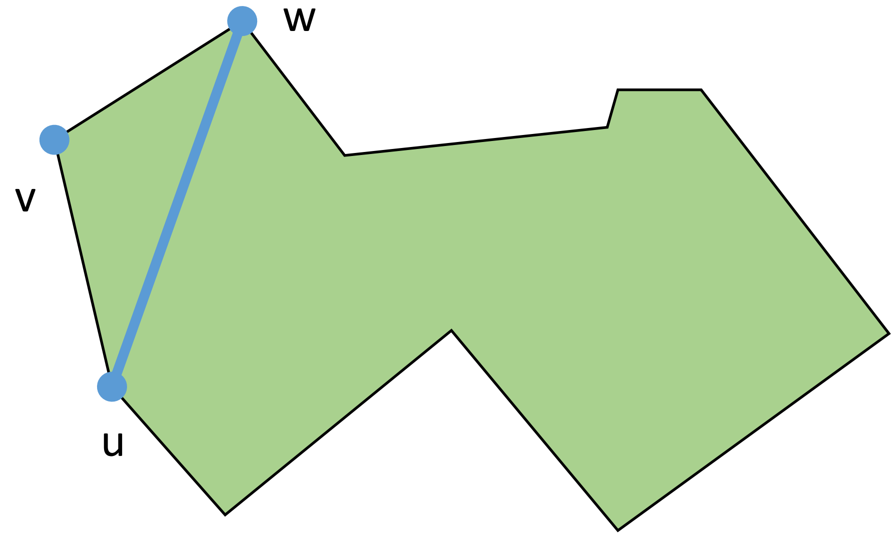
Triangulation Theory of Polygon

- Lemma: Every polygon must have at least one strictly convex vertex.
- Proof:
  - If the edges of polygon oriented in a counter-clockwise traversal, then a convex vertex is a left turn, and reflex vertex is right turn and interior of the polygon is always to the left
  - Let L is the line through the lowest vertex v (y-coordinate)
    - The interior of the polygon must be above
    - The edges following v must be above L
    - The walker make the left turn at v, thus v is convex



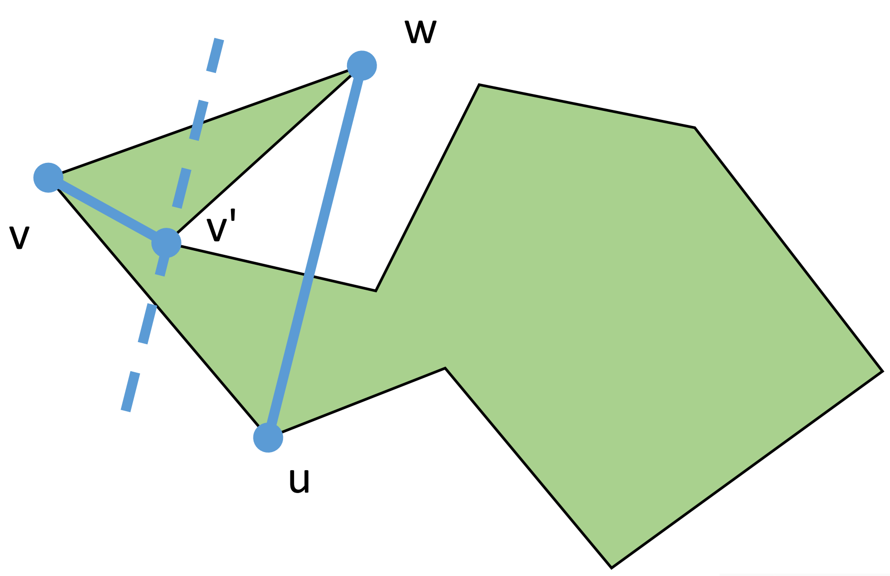
Existence of a Diagonal

- Lemma: every polygon P with more than three vertices has a diagonal
- Proof:
  - Let v be the leftmost vertex of P.
  - Let u and w be its neighbors.
  - If uw is a diagonal we are done



Existence of a Diagonal

- If uw is not a diagonal, let v' be the vertex in triangle (u, v, w) that is farthest from uw
- Then vv' is a diagonal: if an edge was crossing it, one of its endpoints would be farther from uw and inside (u, v, w)



Triangulation Theory: Properties

- Lemma: An internal diagonal exists between any two nonadjacent vertices of a polygon P if and only if P is convex polygon.
- Proof: The proof consists of two parts, both established by contradiction.

Triangulation Theory: Properties

- **Theorem:** The number of distinct triangulations of a convex polygon with n vertices is the Catalan number

$$C_n = \frac{1}{n-1}\binom{2(n-2)}{n-2}$$

Proof: Let P_n be a convex polygon with vertices labeled from 1 to n counterclockwise. Let τ_n be the set of triangulation of P_n with t_n elements.

Let ϕ be the map from τ_n to τ_{n-1}

Triangulation Theory: Properties

- Theorem: Let P be a simple polygon with n vertices. The number of triangulations of P is between 1 and C_n.

Triangulation Theory

- Every polygon P of N vertices can be partitioned into triangles by the addition of zero or more diagonals. (Induction proof)
  - Base case: N = 3 (triangle)
  - Assumption: Let it be true for < N sided polygon
  - Any N sided polygon can be partitioned into two polygons of less then N sides each by adding a diagonal, each of which can be partitioned by using premise 2 above
  - Thus, it is true for all N.

TRIANGULATION THEORY

- ANY DIAGONAL CUTS P INTO TWO SIMPLE SUBPOLYGONS P_1 AND P_2
- LET m_1 BE THE NUMBER OF VERTICES OF P_1 AND m_2 THE NUMBER OF VERTICES OF P_2
- BOTH m_1 AND m_2 MUST BE SMALLER THAN n
  - So by induction P_1 and P_2 can be triangulated
  - Hence, P can be triangulated as well

TRIANGULATION THEORY

- ANY TRIANGULATION OF P CONSISTS OF $n - 2$ TRIANGLES.
  - Consider an arbitrary diagonal in some triangulation T_P
  - The diagonal cuts P into two subpolygons with m_1 and m_2 vertices
  - Every vertex of P occurs in exact1y one of the two subpolygons, except for the vertices defining the diagonal, which occur in both subpolygons. Hence, $m_1 + m_2 = n + 2$.
  - By induction, any triangulation of P_i consists of $m_i - 2$ triangles, which implies that T_P consists of $(m_1 - 2) + (m_2 - 2) = n - 2$ triangles.

Brute Force Triangulation

- **Theorem**: Every polygon P of n vertices can be partitioned into triangle by the addition of (zero or more) diagonals.
- Complexity of diagonal-based algorithm:
  - $O(n^2)$ - # of diagonal candidates
  - $O(n)$ testing each of neighborhoods
  - Repeating this $O(n^3)$ computation for each of the $n-3$ diagonals yields $O(n^4)$
  - Can be made $O(n^3)$

Ear Based Idea….

- Locate an ear
- Output diagonal
- Clip the ear
- Repeat until a triangle is left

Ear Based Idea….

- Definition of Ear: Three consecutive vertices, A, B, C form an ear if AC is a diagonal

- Meisters' Two Ears Theorem: Every polygon $(N \geq 4)$ has at least two non-overlapping ears.

Ear Based Idea….

- Proof: Consider a triangulation of an n-polygon, with $n > 3$. The triangulation consists of $n - 2$ triangles. Since the polygon has n edges but there are only $n - 2$ triangles, by the Pigeonhole Principle, there are at least two triangles with two polygon's edges. These are the ears.

- Another proof: It is known that a simple polygon can always be triangulated. Leaves in the dual-tree of the triangulated polygon correspond to ears and every tree of two or more nodes must have at least two leaves.

TRIANGULATION: IMPLEMENTATION

Algorithm: **TRIANGULATION**

Initialize the ear tip status of each vertex.

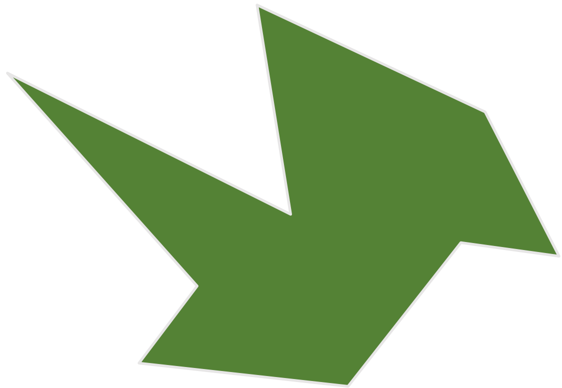
while n > 3 do

Locate an ear tip v_2.

Output diagonal v_1 v_3.

Delete v_2.

Update the ear tip status of v_1 and v_3.



TRIANGULATION: IMPLEMENTATION

$O(n^2)$

+

$O(n)$ iterations

$O(n)$ iterations

Algorithm: **TRIANGULATION**

Initialize the ear tip status of each vertex.

while $n > 3$ do

Locate an ear tip v_2.

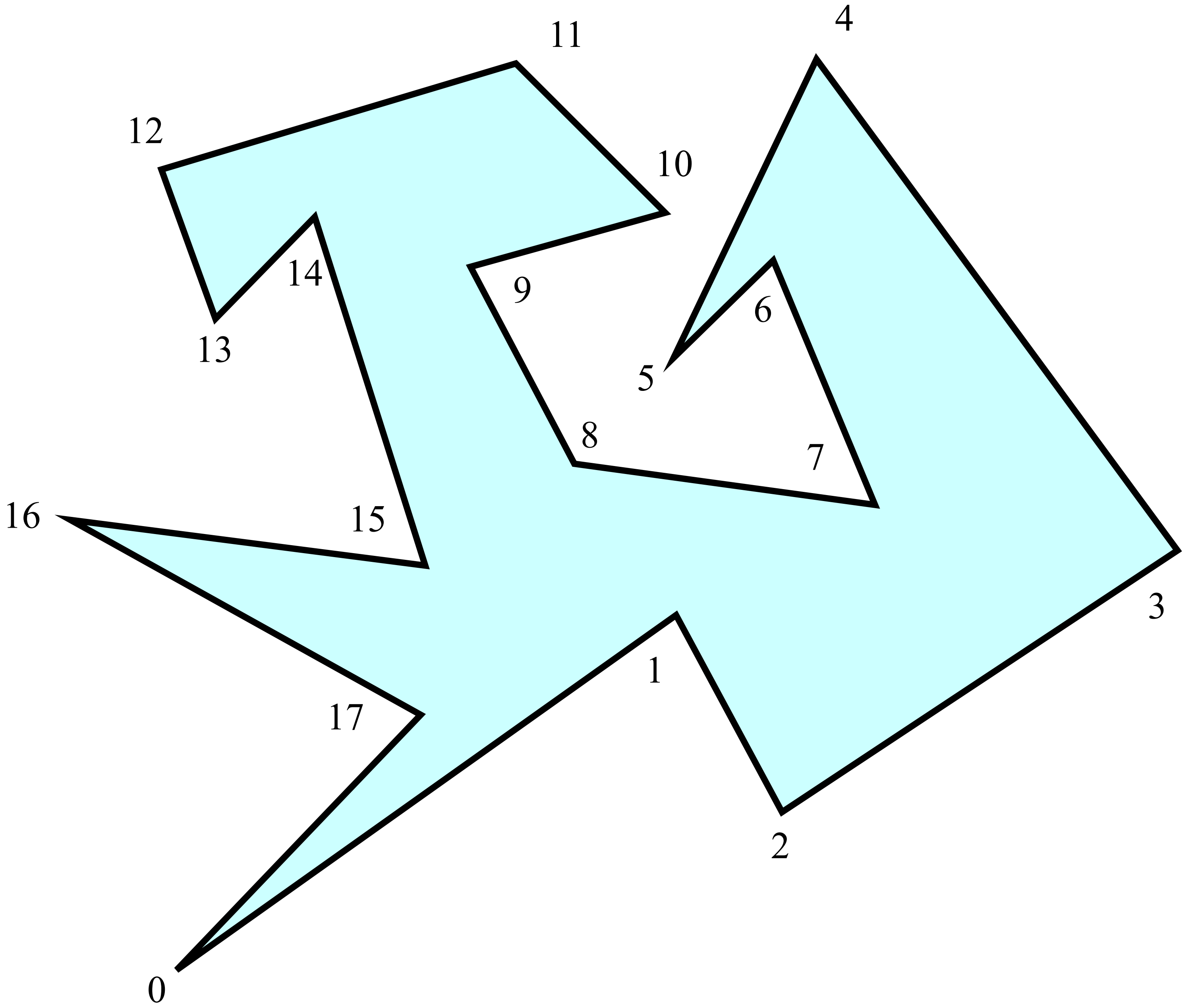
Output diagonal $v_1 v_3$.

Delete v_2.

$O(n)$ → Update the ear tip status of v_1 and v_3.

Total: $O(n^3)$

Can be made: $O(n^2)$



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