COT 4521: Introduction to Computational Geometry

The Art Gallery Problem

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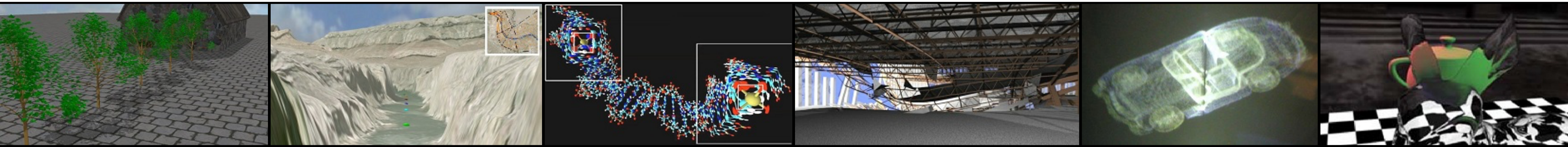
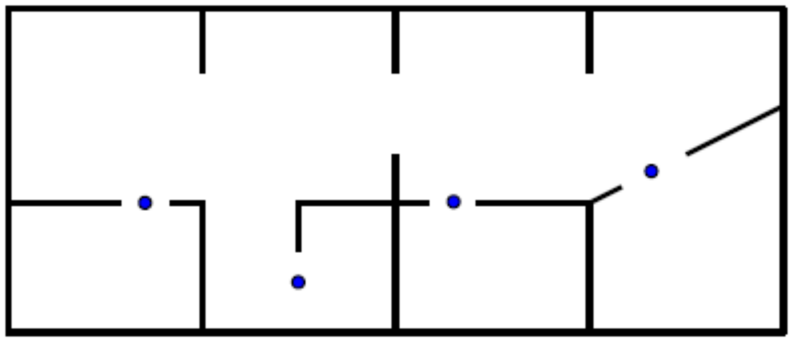


The Art Gallery Problem

- The Art Gallery Problem: how many cameras we need to guard a given gallery so that every point is seen, and how we decide to place them?



- In geometry terminology: How many points are needed in a simple polygon with n vertices so that every point in the polygon is seen?

The Art Gallery Problem

- This problem was posed by Victor Klee in 1973
- A guard of the gallery corresponds to a point on the polygonal floor plan.
- Guards can see in every direction, with a full range of visibility
- The optimization problem is computationally difficult

THE ART GALLERY PROBLEMS

- IN A SIMPLE POLYGON P, A POINT X IS SAID TO BE *VISIBLE* FROM A POINT Y (OR, VICE VERSA) WHENEVER THE LINE SEGMENT XY DOES NOT INTERSECT WITH THE EXTERIOR OF P

$$P: XY \subseteq P$$

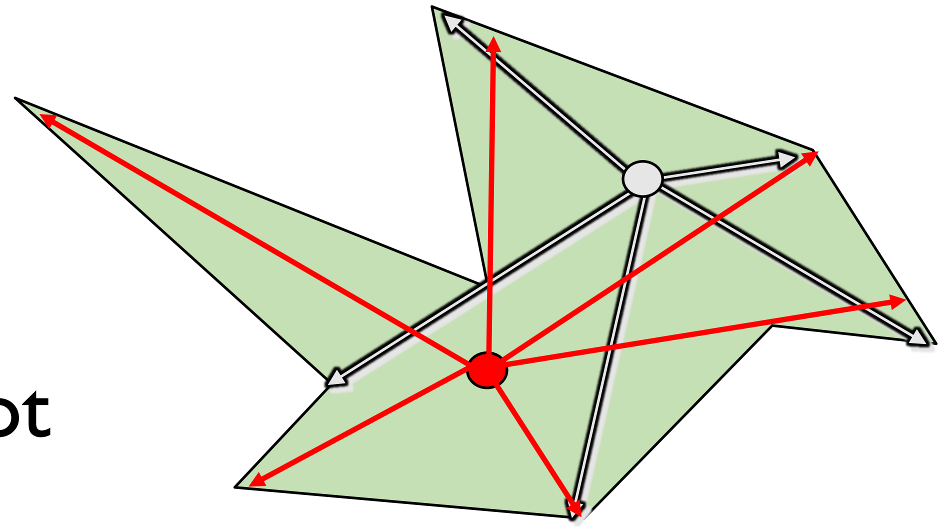
- VERTICES OF P ARE CONSIDERED NON-BLOCKERS OF VISIBILITY

- VISIBILITY: 2π RANGE

The Art Gallery Problems

- Consider a room whose floor is polygon of N vertices, how many point lights (cameras) are needed to light the whole room?

- A set of lights is said to <u>cover</u> a polygon if every point in the polygon is lighted.
  - Assume the lights themselves are not sources of shadows

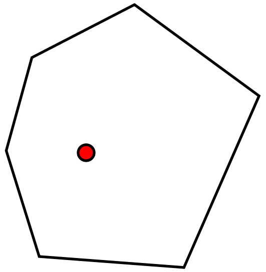
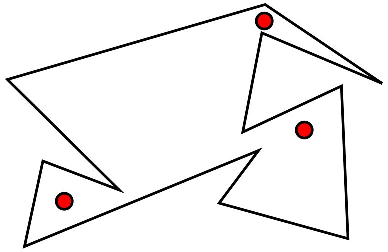
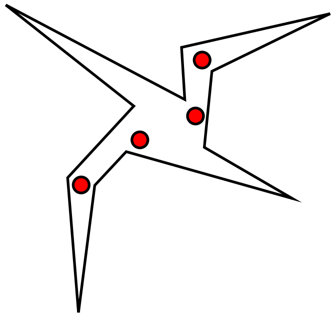


Guarding a simple polygon

- Given a simple polygon P with n vertices, find the *minimum number of guards* required for every point of P to be visible from some guard

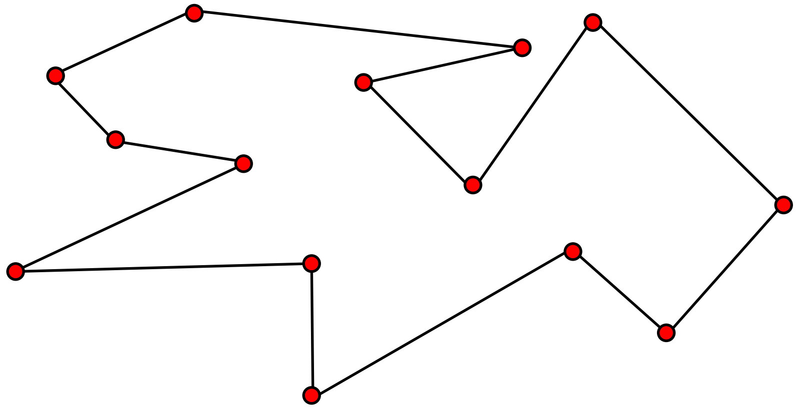
- Assume that every guard can view 360 degrees around it

- How many lights we need to place to guard a simple polygon?
  - One guard is both necessary and sufficient for any convex polygon

Sufficient number of guards for any polygon of n vertices

- How many guards are sufficient to cover any n-vertex simple polygon?
  - By placing a guard at every vertex, any n-vertex simple polygon can be trivially guarded with n guards — loose upper bound



Maximum over minimum formulation
Formal definition

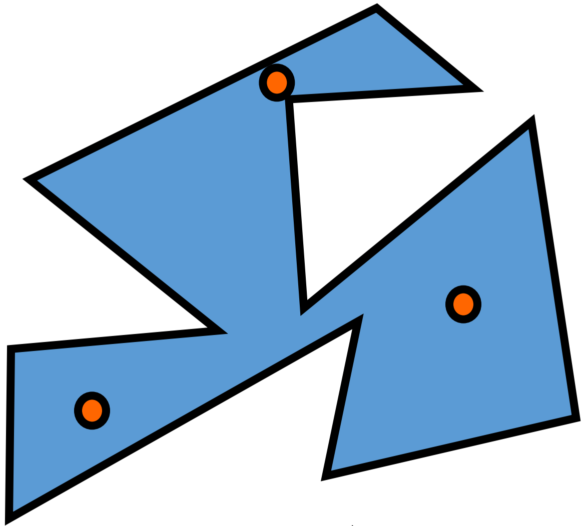
- Let $g(P_N)$ be the smallest number of lights need to cover a particular polygon of N sides.

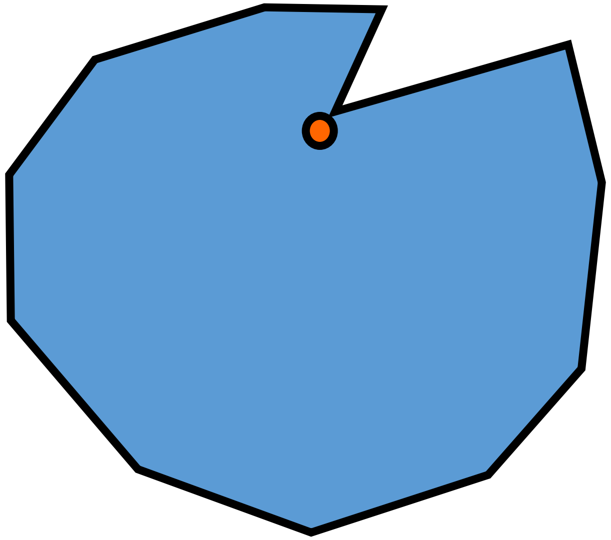
$$\boxed{g(P_N) = \min_S |\{S : S \text{ covers } P\}|}$$

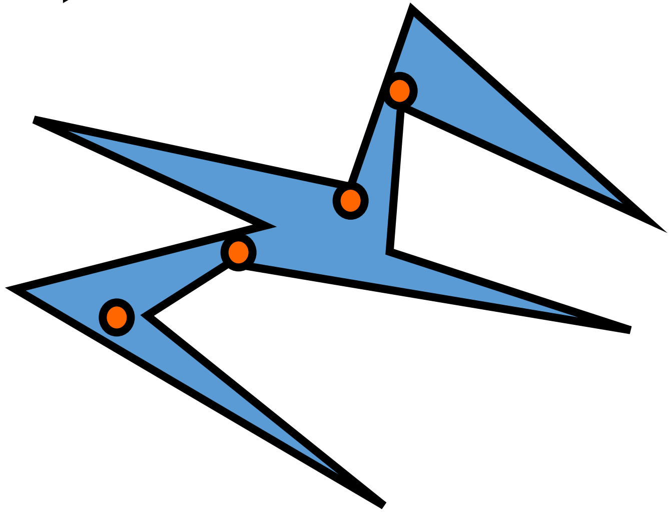
  - S is the set of points where the lights are located
- What is the max of $g(P_N)$ over all P_N?

$$\boxed{G(N) = \max_{P_N} g(P_N)}$$

HOW MANY LIGHTS ARE NEEDED?

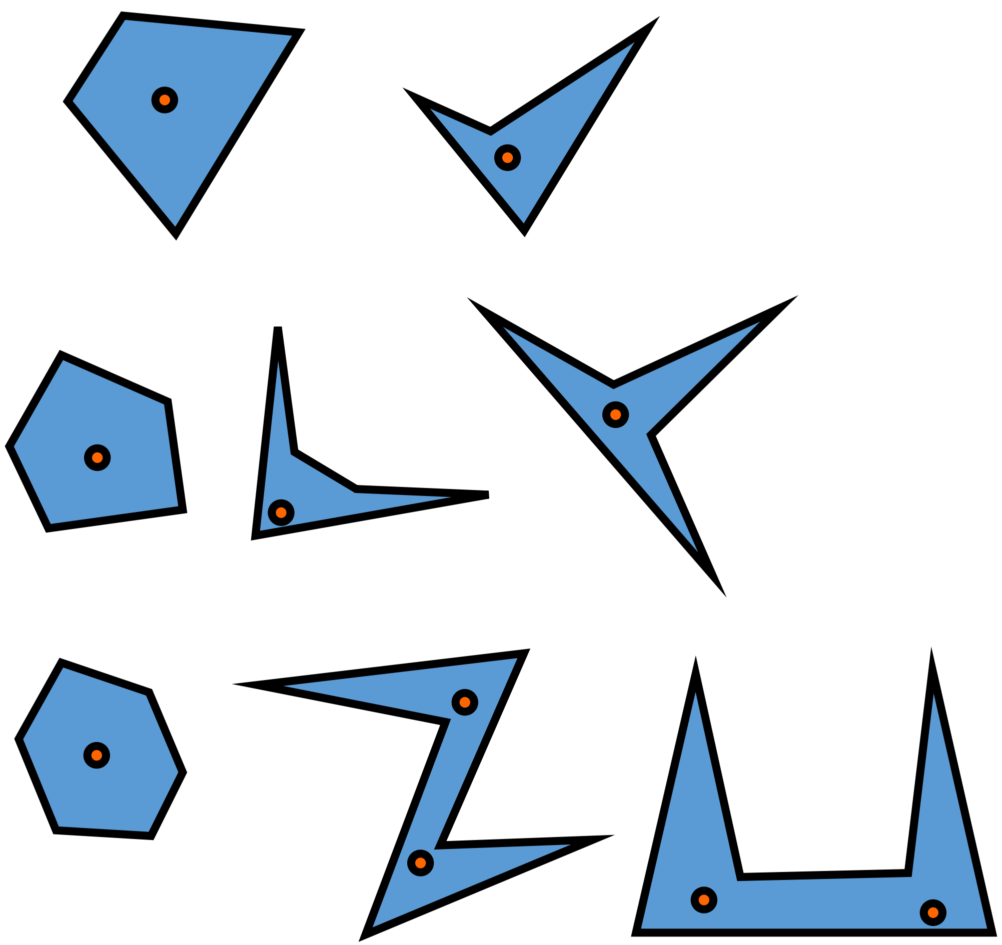






What is the maximum of the minimum number of lights needed to cover a 12-sided polygon?

$G(N) = ?$



$$1 \le G(N) \le N$$

$$G(3) = 1$$

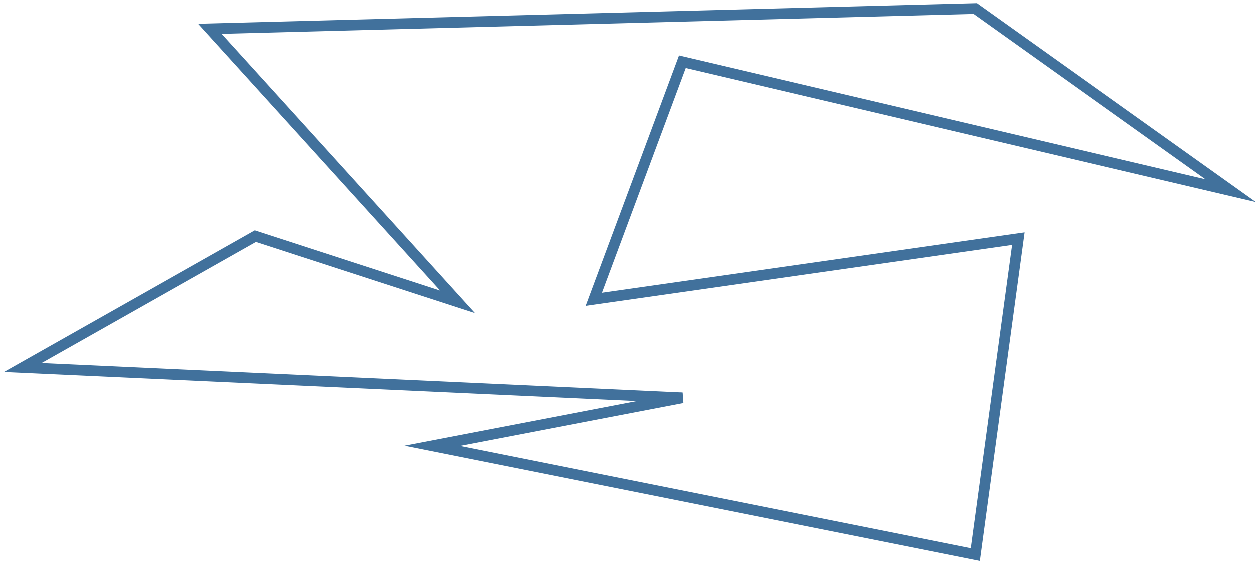
$$G(4) = 1$$

$$G(5) = 1$$

$$G(6) = 2$$

Maximum over minimum formulation

- How many lights (cameras) needed (n=12)



Maximum over minimum formulation: Quiz

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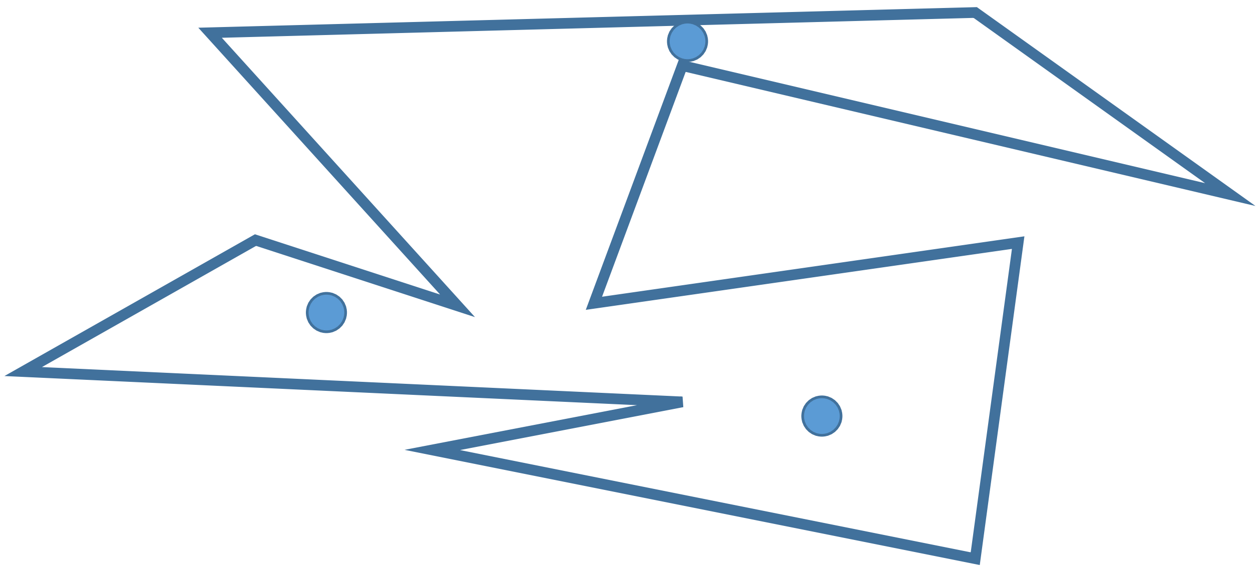

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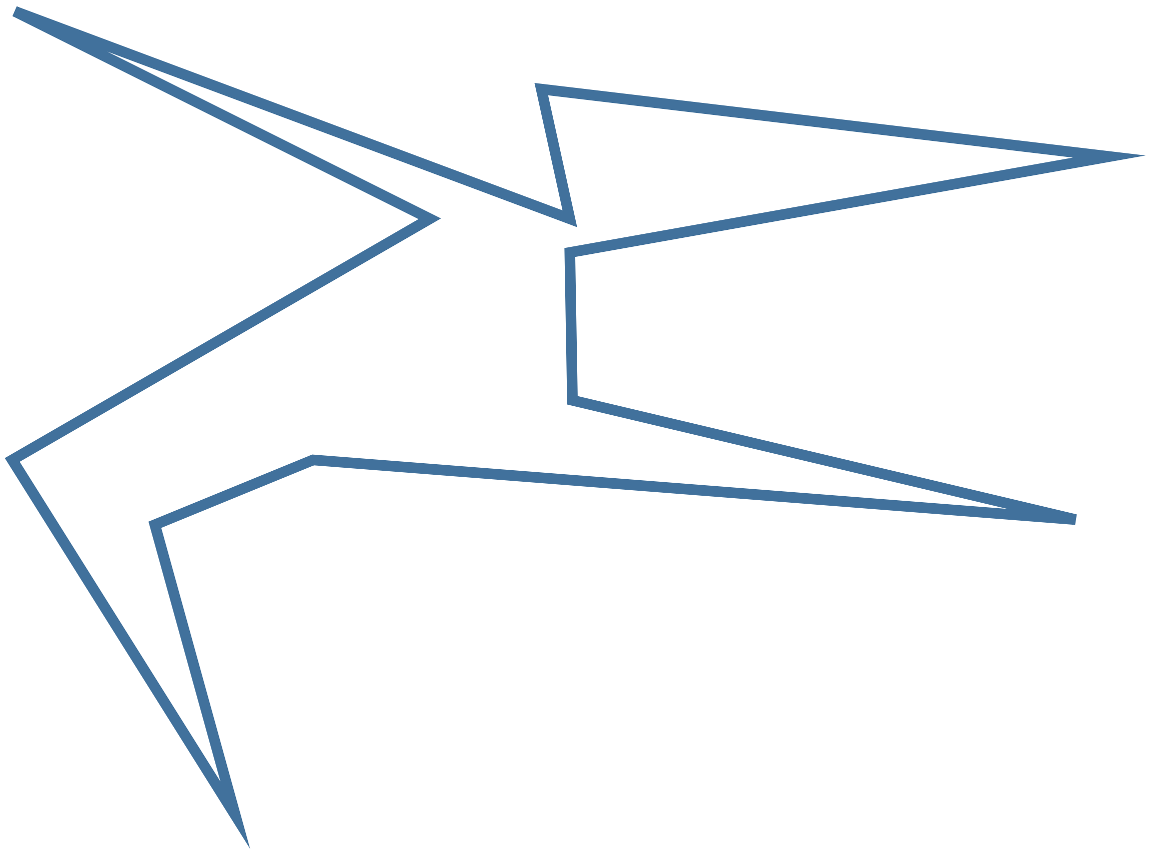

Maximum over minimum formulation: Quiz

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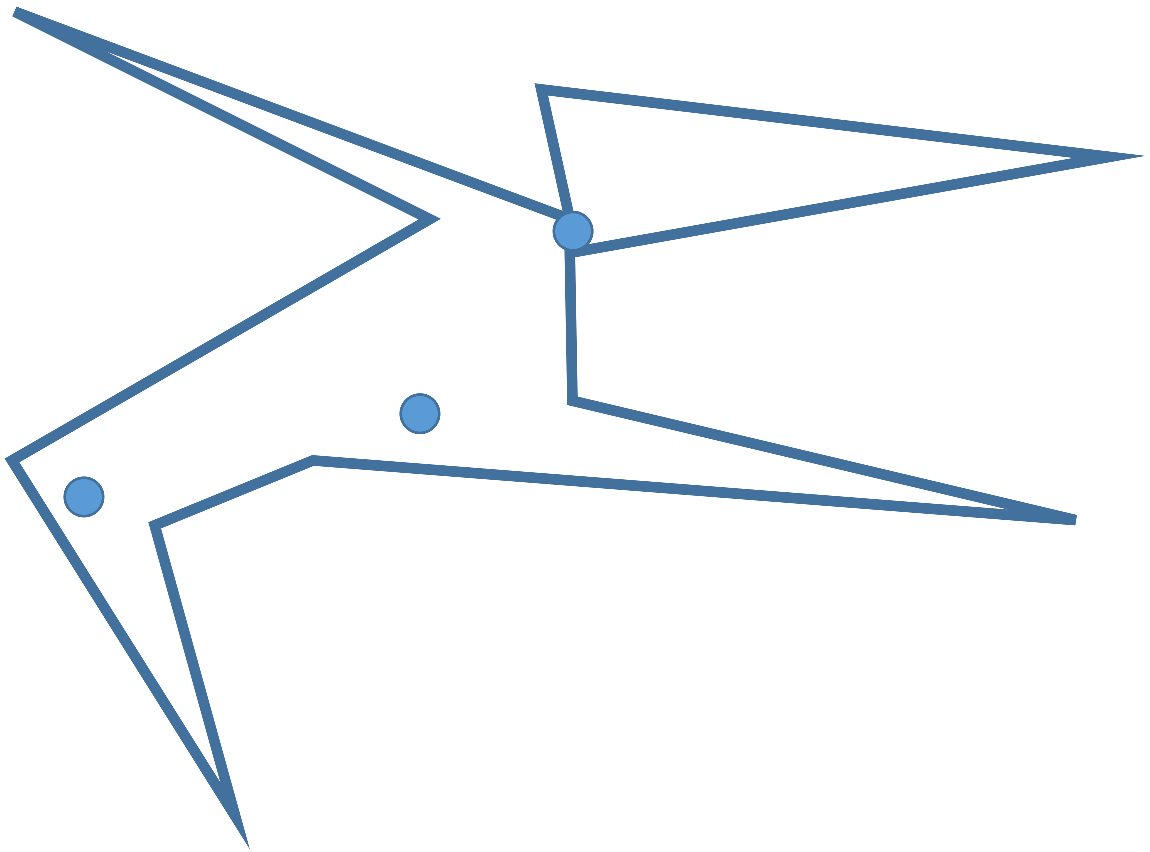
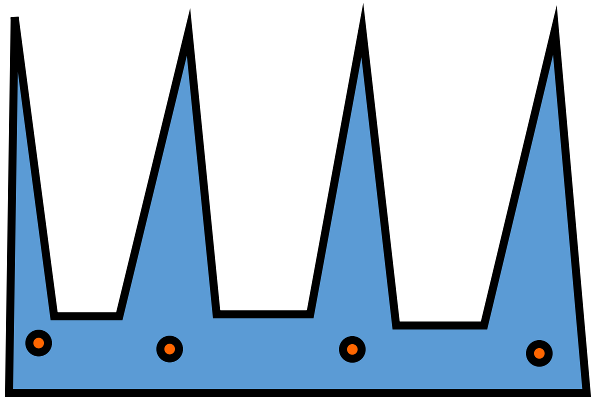


$G(N) = \ldots$

- Chvatal's comb
  - $G(12) = 4$

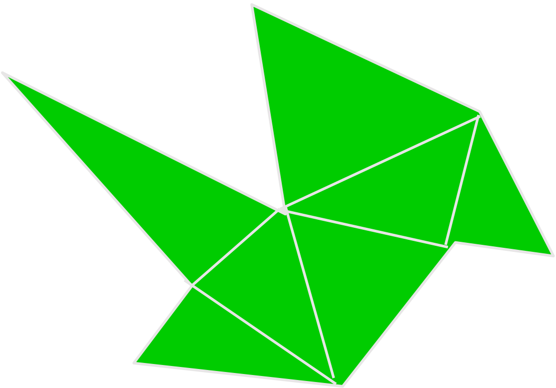


- Can it be that $G(N) = \left\lfloor \frac{N}{3} \right\rfloor$?

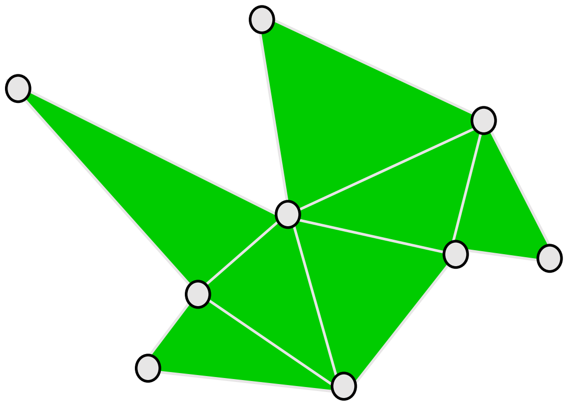
MAXIMUM OVER MINIMUM FORMULATION

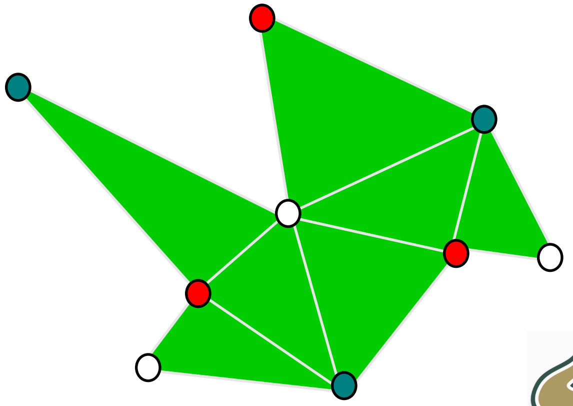
- **THEOREM** (ART GALLERY THEOREM). FOR A SIMPLE POLYGON WITH N VERTICES, $\lfloor n/3 \rfloor$ CAMERAS ARE OCCASIONALLY NECESSARY AND ALWAYS SUFFICIENT TO HAVE EVERY POINT IN THE POLYGON VISIBLE FROM AT LEAST ONE OF THE CAMERAS
  - Sufficiency of n
    - Certainly at least one camera is needed—lower bound on $G(n)$: $1 \leq G(n)$
    - An upper bound on $G(n)$: $G(n) \leq n$
  - The first proof that $G(n) = \lfloor n/3 \rfloor$ was due to Ghvatal (1975)
  - We will present Fiske's proof of sufficiency of $\lfloor n/3 \rfloor$ guards for any n-sided polygon

FISKE’ PROOF



- GIVEN ARBITRARY N-VERTEX P:
  - Triangulate P
  - Color the vertices of triangulation graph G
  - G can be 3-colored
  - Place lights at same colored nodes
  - Guaranteed to light the whole polygon P



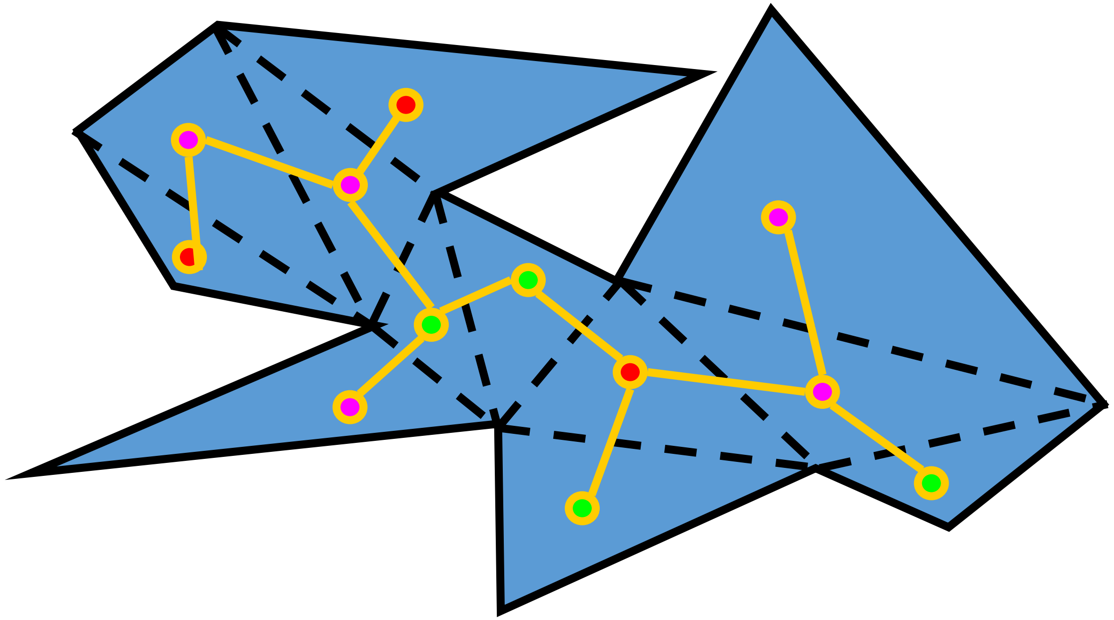


Brute Force Triangulation

- Triangulate P using a diagonal-based approach
- **Theorem:** Every polygon P of n vertices can be partitioned into triangle by the addition of (zero or more) diagonals.
  - Complexity of diagonal-based algorithm:
    - $O(n^2)$ - # of diagonal candidates
    - $O(n)$ testing each of neighborhoods
    - Repeating this $O(n^3)$ computation for each of the n-3 diagonals yields $O(n^4)$

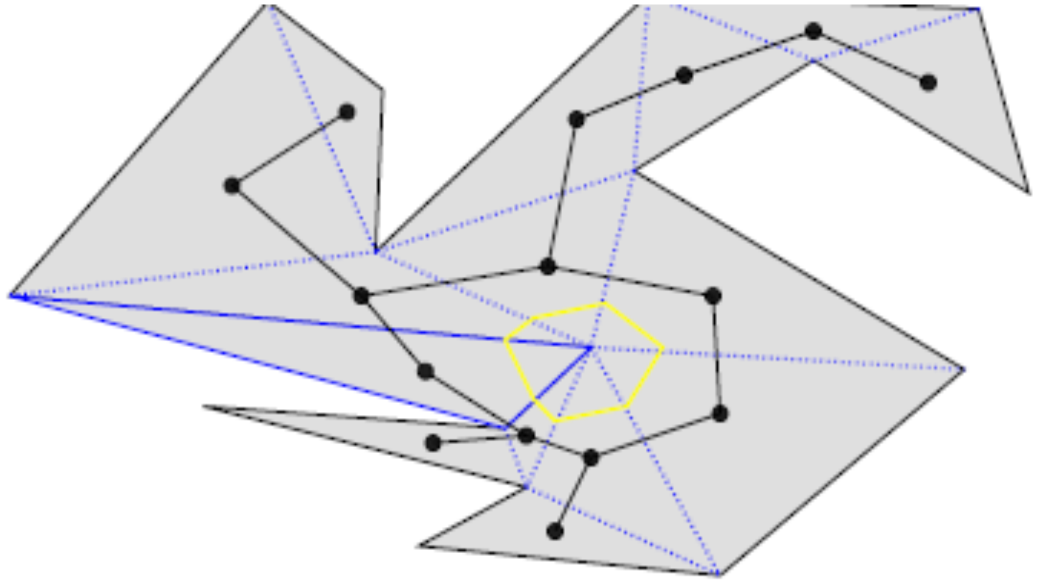
Triangulation Dual

- The dual T of a triangulation is a <u>tree</u>, with each node of <u>degree at most three</u>.
- Dual graph: each face gives a node; two nodes are connected if the faces are adjacent



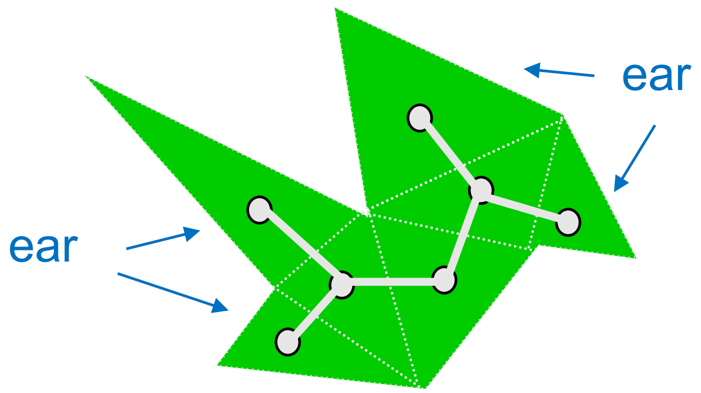
PROPERTIES OF TRIANGULATIONS

- PROOF:
  - The degree three is immediate from the fact that every triangle have three sides.
  - If there is a cycle C in T it is easy to verify that...
  - There must be a vertex inside the polygon...



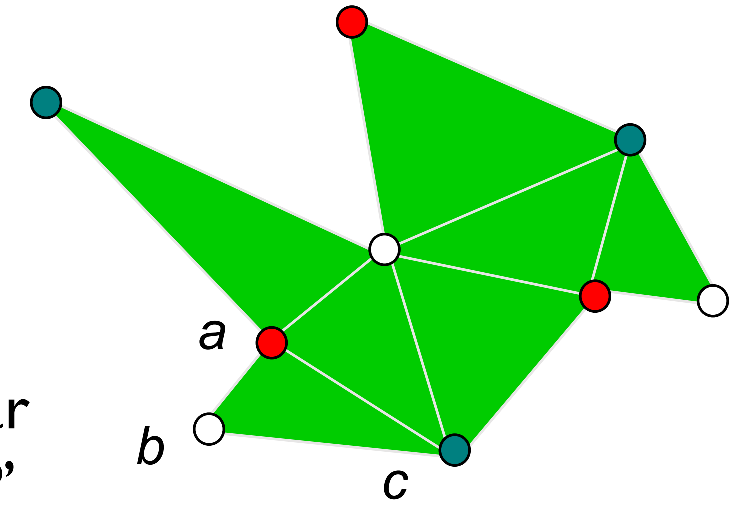
MEISTER'S TWO EARS THEOREM

- THREE CONSECUTIVE VERTICES, A, B, C FORM AN EAR IF AC IS A DIAGONAL
- "2-EARS" THEOREM: EVERY POLYGON OF $n \geq 4$ VERTICES HAS AT LEAST 2 NON-OVERLAPPING EARS.
  - The triangulation dual has at least 2 nodes
  - A tree of more than 2 nodes has at least 2 leaf nodes
  - Each leaf node corresponds to an ear.



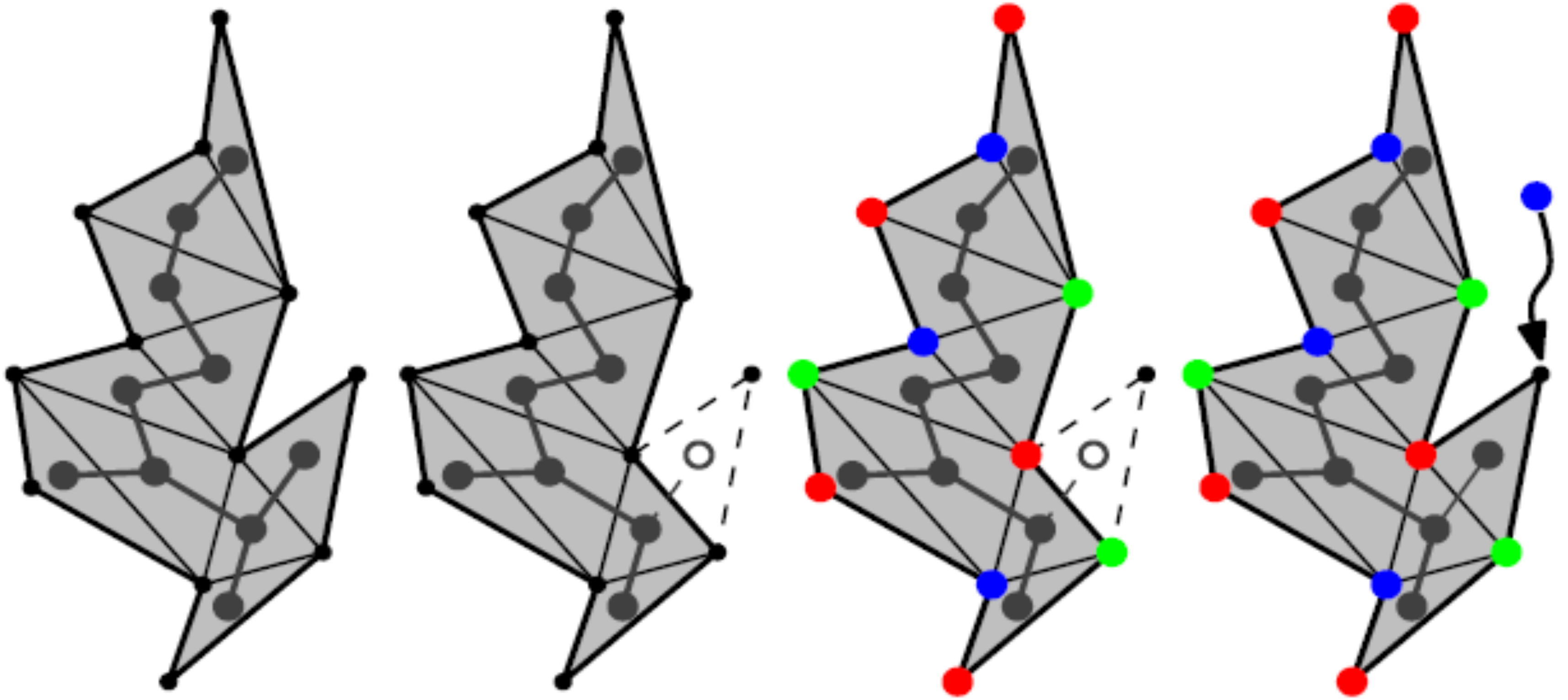
Triangulation Theory: 3-Coloring

- "2-Ears" Theorem can be used to easily prove 3-colorability of triangulation graphs
  - Induction on n
    - Base case: $n = 3$
    - For $n \geq 4$: 2-ears theorem guarantees that an ear abc exists apply inductive hypothesis to polygon P' without ear "reattaching" ear adds back in one vertex (w.l.o.g. b) color b whatever color a and c don't use result is a 3-coloring of P



Fiske' proof

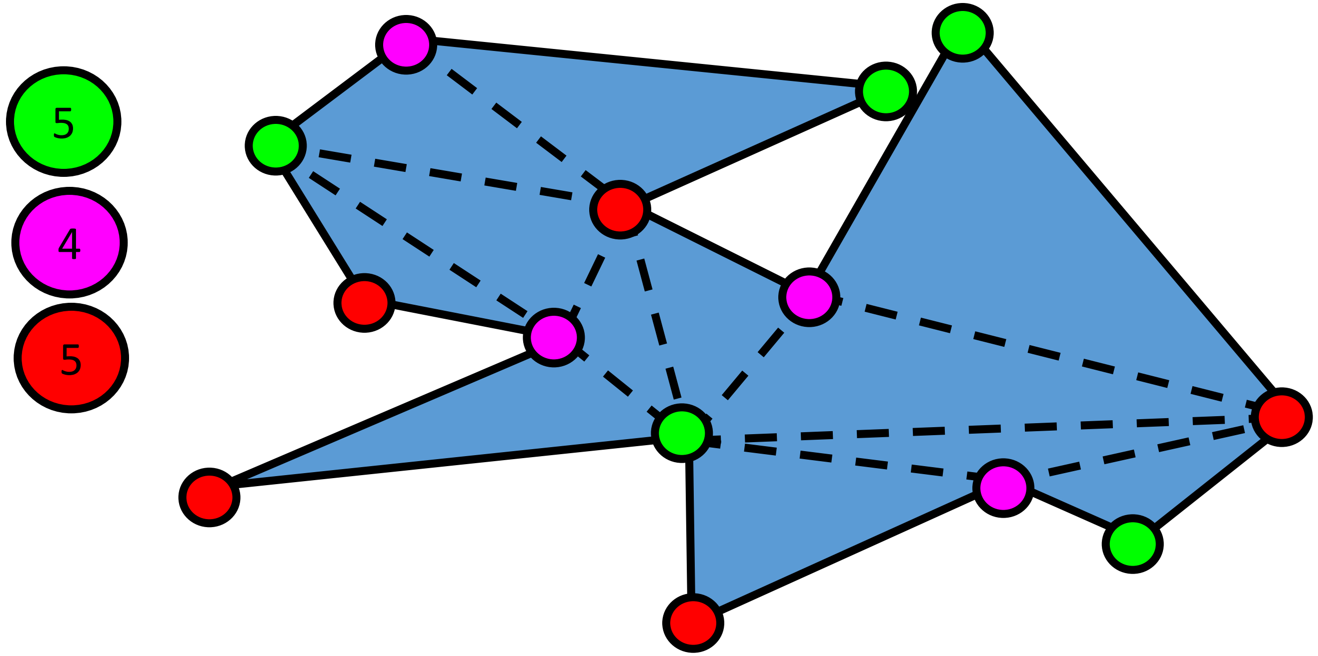
- 3 colors



Fiske' proof

- Apply the "pigeon-hole principle" – If n objects are placed into k pigeon holes, then at least one hole must contain no more than n/k objects

3 COLORS SUFFICE…



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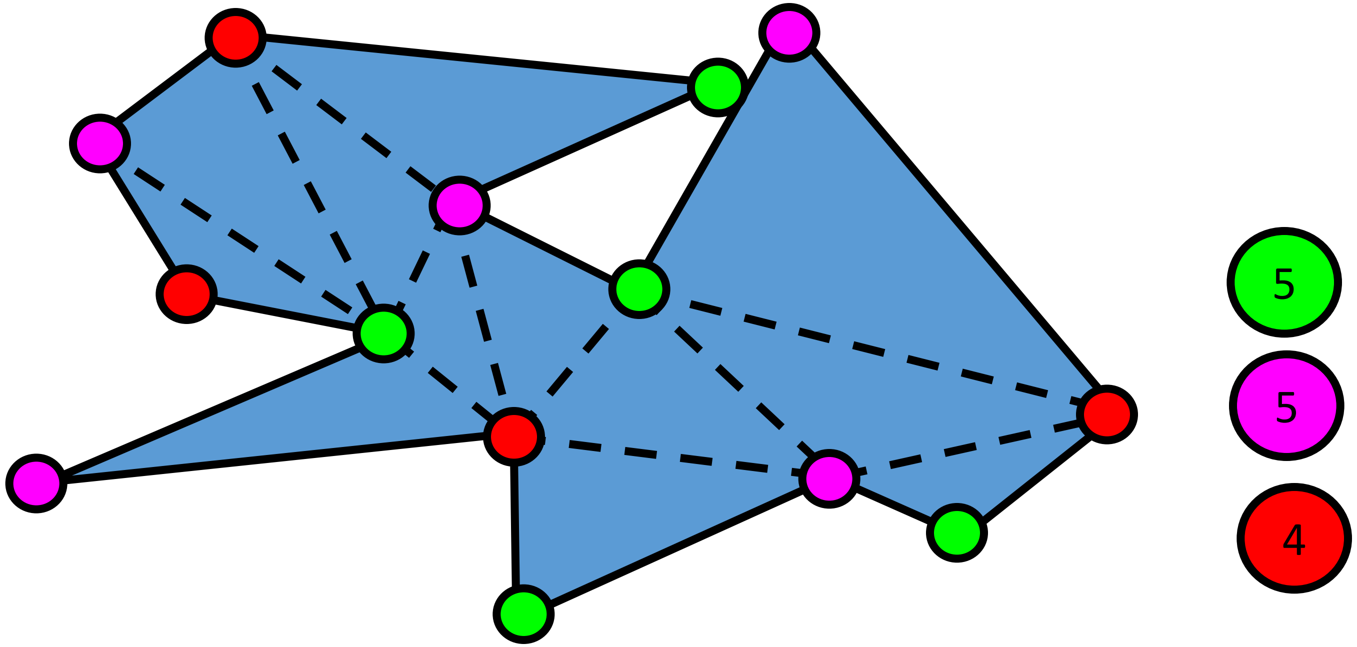

Pigeon Hole Principle

- 3 holes (colors) and 14 pigeons (vertices) to go into them.
- There will always be one hole with less or equal to 14/3 pigeons
- Generalizing: for 3 colors and N vertices there will be a color that is used at most N/3 times. Place the light at those colors.

Example