Assignment 5

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https://github.com/USI-Projects-Collection/Computer_Vision.git

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Problem 1 [2 points]

In the projective plane $\mathbb{P}^2$, a line is represented by a homogeneous vector $(a, b, c)^T$, defining the equation $ax + by + cz = 0$, where $(x, y, z)^T$ are the homogeneous coordinates of points on the line. The intersection of two lines is found using the cross product of their homogeneous vectors.

Step 1: Define the Line Equations

- For line $l = (2, 4, 8)^T$, the equation is:

$$2x + 4y + 8z = 0$$

- For line $m = (-2, 3, -1)^T$, the equation is:

$$-2x + 3y - z = 0$$

Step 2: Compute the Cross Product

The homogeneous coordinates of the intersection point are given by $p = l \times m$. For vectors $l = (l_1, l_2, l_3) = (2, 4, 8)$ and $m = (m_1, m_2, m_3) = (-2, 3, -1)$, the components are:

$$p_x = l_2 m_3 - l_3 m_2 = 4 \cdot (-1) - 8 \cdot 3 = -4 - 24 = -28$$
$$p_y = l_3 m_1 - l_1 m_3 = 8 \cdot (-2) - 2 \cdot (-1) = -16 - (-2) = -16 + 2 = -14$$
$$p_z = l_1 m_2 - l_2 m_1 = 2 \cdot 3 - 4 \cdot (-2) = 6 - (-8) = 6 + 8 = 14$$

Thus, the homogeneous coordinates are:

$$p = (-28, -14, 14)^T$$

Since homogeneous coordinates are defined up to scale, divide by 14:

$$p = \left(\frac{-28}{14}, \frac{-14}{14}, \frac{14}{14} \right) = (-2, -1, 1)^T$$

Step 3: Convert to Cartesian Coordinates

For a point $(x, y, z)^T$ with $z \neq 0$, the Cartesian coordinates are $(x/z, y/z)$. Here, $z = 1$, so:

$$x = \frac{-2}{1} = -2, \quad y = \frac{-1}{1} = -1$$

Thus, the Cartesian coordinates are $(-2, -1)$.

Step 4: Verification

Substitute $(x, y, z) = (-2, -1, 1)$ into both equations:

- Line l: $2(-2) + 4(-1) + 8(1) = -4 - 4 + 8 = 0$
- Line m: $-2(-2) + 3(-1) - 1 = 4 - 3 - 1 = 0$

Both are satisfied, confirming the solution.

Final Answer

The Cartesian coordinates of the intersection point are:

$$\boxed{(-2, -1)}$$

Problem 2 [3 points]

We aim to prove:

$$\text{The points } \mathbf{x}, \mathbf{y}, \mathbf{z} \text{ are collinear} \iff \det(M) = 0$$

(⇒) Only If:
Assume the three points are collinear in projective space $\mathbb{P}^2$. This means they lie on the same line. In projective geometry, three points are collinear if one of them can be expressed as a linear combination of the other two. That is, there exist scalars $\lambda, \mu \in \mathbb{R}$ such that:

$$\mathbf{z} = \lambda \mathbf{x} + \mu \mathbf{y}$$

This implies that the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are linearly dependent. By a basic result in linear algebra, if the columns of a matrix are linearly dependent, then the determinant of that matrix is zero:

$$\Rightarrow \det(M) = 0$$

(⇐) If:
Now assume that $\det(M) = 0$. Then the columns of M, i.e., the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$, are linearly dependent. Therefore, there exist scalars α, β, γ, not all zero, such that:

$$\alpha \mathbf{x} + \beta \mathbf{y} + \gamma \mathbf{z} = \mathbf{0}$$

This means one of the vectors can be written as a linear combination of the other two. For example, if $\gamma \neq 0$, then:

$$\mathbf{z} = -\frac{\alpha}{\gamma}\mathbf{x} - \frac{\beta}{\gamma}\mathbf{y}$$

Thus, the points $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are collinear in projective space.

Conclusion:

$$\boxed{\mathbf{x}, \mathbf{y}, \mathbf{z} \text{ are collinear } \iff \det(M) = 0}$$

Problem 3 [5 points]

The code is provided in the separate file **source.py**.

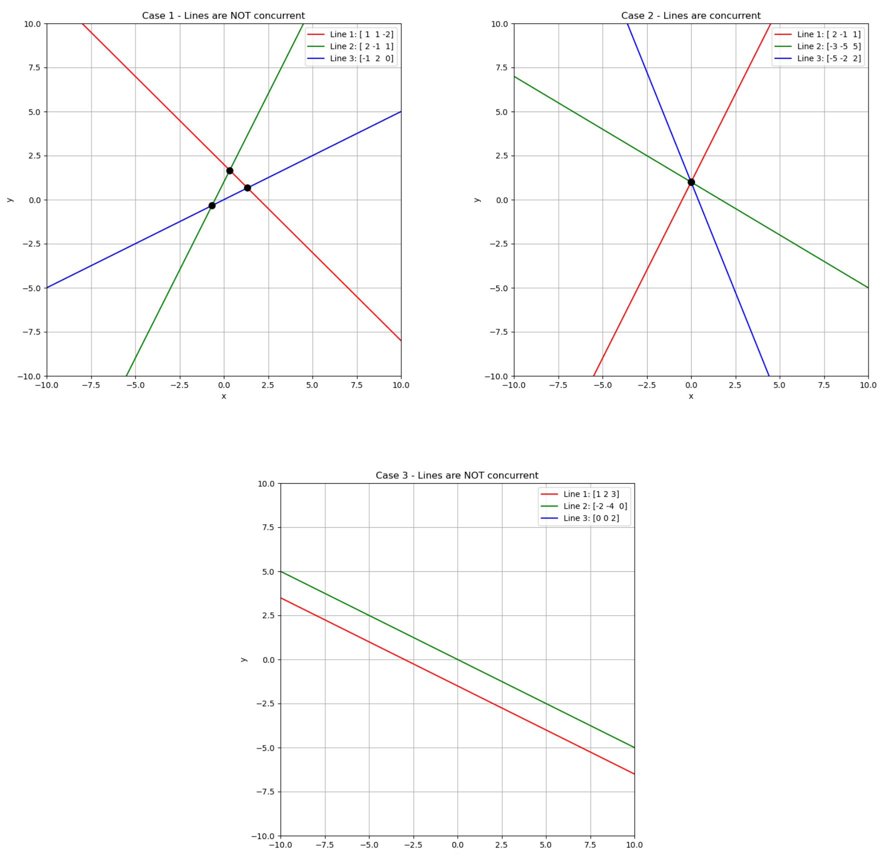

Figure 1: Intersection of lines l and m

In the third case lines 2 and 3 are parallel, so the intersection point is not defined.