Assignment 6

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`https://github.com/USI-Projects-Collection/Computer_Vision.git`

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Problem 1 [5 points]

1. Separate the purely quadratic part

Write the polynomial in vector form

$$F(x,y) = \underbrace{\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix}}_{\text{quadratic}} + 2g^T \begin{bmatrix} x \\ y \end{bmatrix} + f, \qquad A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix},\ g = \frac{1}{2} \begin{bmatrix} d \\ e \end{bmatrix}.$$

2. Definiteness of A

Because $\delta = b^2 - 4ac < 0$,

$$\det A = a\,c - \left(\tfrac{b}{2}\right)^2 = \frac{-\delta}{4} > 0.$$

A symmetric 2×2 matrix with positive determinant is **definite**; its sign is the sign of its trace:

- If $a + c > 0 \Rightarrow A$ is **positive definite**.
- If $a + c < 0 \Rightarrow A$ is **negative definite**.

3. Translate to the centre of the conic

The gradient of F vanishes at

$$\mathbf{x}_0 = -A^{-1}g.$$

Completing the square gives the standard form

$$F(x,y) = (\mathbf{x} - \mathbf{x}_0)^T A (\mathbf{x} - \mathbf{x}_0) + \Big(f - g^T A^{-1} g\Big). \tag{$\star$}$$

4. Express the constant term via $\det C$

For a block matrix $C = \begin{pmatrix} A & g \\ g^T & f \end{pmatrix}$ the Schur complement formula gives

$$\boxed{\det C = \det A \left(f - g^T A^{-1} g\right)} \quad \Longrightarrow \quad f - g^T A^{-1} g = \frac{\det C}{\det A}.$$

Insert this into $(\star)$:

$$F(x, y) = (\mathbf{x} - \mathbf{x}_0)^T A (\mathbf{x} - \mathbf{x}_0) \; + \; \frac{\det C}{\det A}. \tag{$\star\star$}$$

5. Compare the two terms in $(\star\star)$

Recall

- $\det A > 0$ (Step 2),
- A is definite with the same sign as $a + c$,
- assumption $(a + c) \det C > 0$ says $\det C$ **has the same sign as** A.

Case 1: A **positive definite** $(a + c > 0)$

- $\det C > 0 \Rightarrow \frac{\det C}{\det A} > 0$.
- The first term $(\mathbf{x} - \mathbf{x}_0)^T A (\mathbf{x} - \mathbf{x}_0)$ is **non-negative**, vanishing only at $\mathbf{x} = \mathbf{x}_0$.

Hence $F(x, y) \geq \frac{\det C}{\det A} > 0$ for every $(x, y) \in \mathbb{R}^2$; the equation $F = 0$ has **no real solution**.

Case 2: A **negative definite** $(a + c < 0)$

- $\det C < 0 \Rightarrow \frac{\det C}{\det A} < 0$.
- The quadratic term is **non-positive**, again zero only at $\mathbf{x}_0$.

Thus $F(x, y) \leq \frac{\det C}{\det A} < 0$ for all (x, y); the equation $F = 0$ is likewise **unsatisfiable**.

6. Conclusion

In both cases the sign of the constant term in $(\star\star)$ matches the sign of the definite quadratic part, so their sum **never vanishes** on $\mathbb{R}^2$.

$$\boxed{\mathcal{C} = \varnothing}$$

Therefore the conic determined by C is *imaginary* (has no real points) whenever

$$\delta = b^2 - 4ac < 0, \qquad (a + c) \det C > 0, \qquad \det C \neq 0.$$

Problem 2 [5 points]

The goal of this task was to rectify a 300×400 grayscale image A (“`homework6.pgm`”) and generate a 300×370 output image B by applying a projective transformation h. This transformation maps the quadrilateral defined by the following points in image A:

$$
\begin{aligned}
p_1 &= (244, 263), \\
p_2 &= (238, 353), \\
p_3 &= (199, 350), \\
p_4 &= (201, 262)
\end{aligned}
$$

to the following points in image B:

$$
\begin{aligned}
q_1 &= (232, 216), \\
q_2 &= (232, 311), \\
q_3 &= (197, 311), \\
q_4 &= (197, 216)
\end{aligned}
$$

To accomplish this, the inverse of the homography matrix H that maps destination points q_i to source points p_i was computed using `cv2.findHomography`. This inverse transformation was then used to warp the original image using bilinear interpolation via `cv2.warpPerspective`, thereby generating image B. The output image has a fixed resolution of 300×370 as required.

Resulting image:

Figure 1: Rectified image B with a resolution of 300×370