Assignment 6

Paolo Deidda (paolo.deidda@usi.ch)
Raffaele Perri (raffaele.perri@usi.ch)
https://github.com/USI-Projects-Collection/Computer_Vision.git

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Problem 1 [5 points]

Show that the conic represented by a non-singular, symmetric matrix

$$C = \begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{pmatrix},$$

with $\delta = b^2 - 4ac < 0$ and $(a+c)\det C > 0$, is imaginary, i.e.

$$\{(x,y) \in \mathbf{R}^2 : ax^2 + bxy + cy^2 + dx + ey + f = 0\} = \emptyset.$$

Solution.

1. *Rewrite as a quadratic form.* Set

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}, \quad v = \begin{pmatrix} d/2 \\ e/2 \end{pmatrix}.$$

Then

$$ax^2 + bxy + cy^2 + dx + ey + f = X^T A\, X + 2\, v^T X + f,$$

and

$$C = \begin{pmatrix} A & v \\ v^T & f \end{pmatrix} \implies \det C = \det A\,\big(f - v^T A^{-1} v\big).$$

2. *Definiteness of A.* Since $\delta = b^2 - 4ac < 0$, we have

$$\det A = ac - \tfrac{b^2}{4} = \frac{4ac - b^2}{4} > 0.$$

Moreover $\operatorname{tr} A = a + c$. Hence a real symmetric 2×2 matrix with positive determinant has two real eigenvalues of the same sign:

$$\begin{cases} a + c > 0 \implies A \text{ is positive-definite,} \\ a + c < 0 \implies A \text{ is negative-definite.} \end{cases}$$

3. *Complete the square.* Define

$$Q(X) = X^T A\, X + 2\, v^T X + f.$$

Its unique critical point is

$$\nabla Q = 2A\, X + 2v = 0 \quad \Longrightarrow \quad X_0 = -A^{-1} v.$$

Substituting back,

$$Q(X) = (X - X_0)^T A\, (X - X_0) \ + \ \underbrace{\big(f - v^T A^{-1} v\big)}_{S},$$

and one checks $\det C = \det A \cdot S$, so

$$S = \frac{\det C}{\det A}.$$

4. *Sign analysis and conclusion.* We know $\det A > 0$. The hypothesis $(a + c) \det C > 0$ forces:

$$\begin{cases} a + c > 0 \implies \det C > 0 \implies S > 0 & \text{and } A \text{ positive-definite,} \\ a + c < 0 \implies \det C < 0 \implies S < 0 & \text{and } A \text{ negative-definite.} \end{cases}$$

In the first case,

$$Q(X) \ = \ (X - X_0)^T A (X - X_0) + S \ \geq S > 0 \quad \forall X,$$

and in the second,

$$Q(X) \ = \ (X - X_0)^T A (X - X_0) + S \ \leq S < 0 \quad \forall X.$$

Thus in *either* case $Q(X)$ never vanishes on $\mathbf{R}^2$, so $\{Q = 0\} = \emptyset$. Hence the conic is *imaginary*.

Problem 2 [5 points]