

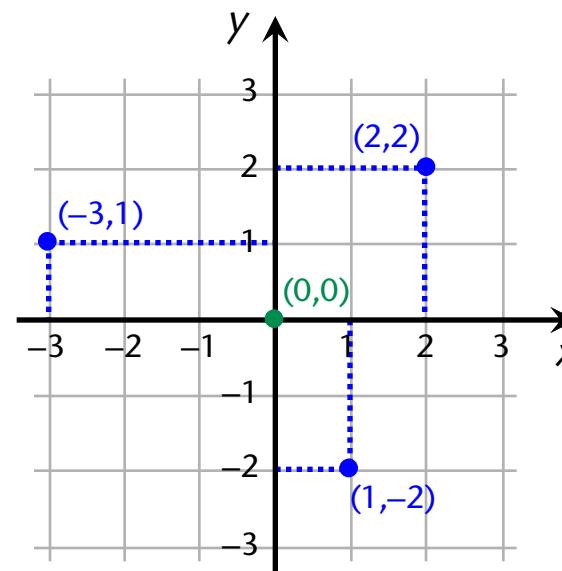
- The essence of barycentric coordinates
- Three-point coordinates
- Implementation
- Barycentric mappings
- Mesh deformations

- *The essence of barycentric coordinates*
- Three-point coordinates
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Cartesian coordinates



René Descartes
(1596–1650)



point (2,2) with

- x-coordinate: 2
- y-coordinate: 2

mathematically:

$$(2,2) = 2 \cdot (1,0) + 2 \cdot (0,1)$$

in general:

$$(x,y) = x \cdot (1,0) + y \cdot (0,1)$$

x- and y-coordinates

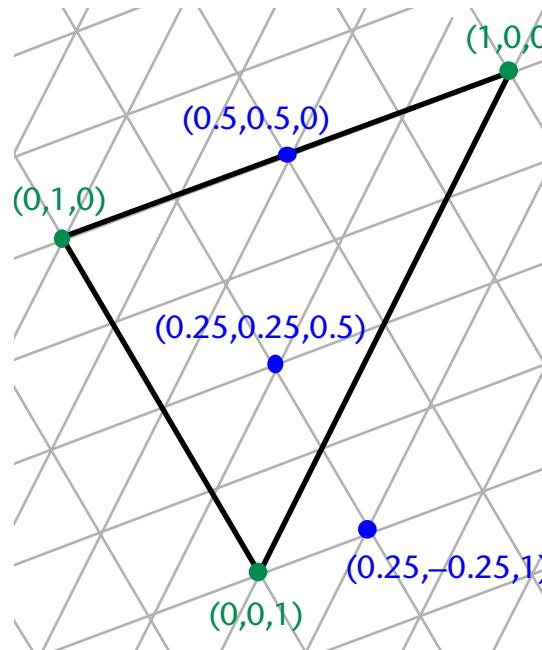
w.r.t. **base points**

(1,0) and (0,1)

Barycentric coordinates



August Ferdinand Möbius
(1790–1868)



point (a,b,c) with
3 coordinates w.r.t.
base points A, B, C

mathematically:

$$(a,b,c) = a \cdot A + b \cdot B + c \cdot C$$

where

$$A = (1,0,0)$$

$$B = (0,1,0)$$

$$C = (0,0,1)$$

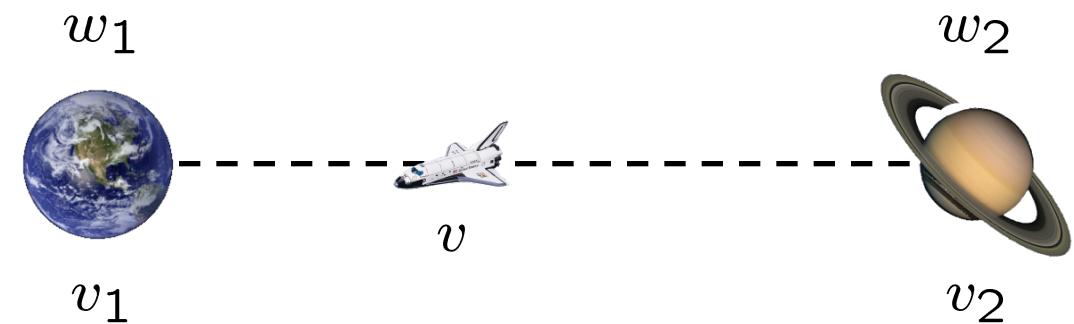
"D" and

$$a + b + c = 1$$

Barycentric coordinates

- system of masses w_i at positions v_i
- position of the system's *barycentre* :

$$v = \frac{\sum_i w_i v_i}{\sum_i w_i}$$



Zweites Capitel.
Der barycentrische Calcul.

§. 15. Bei Rechnungen, wie wir so eben (§. 11 und 12.) führten, bietet sich gleichsam von selbst eine kleine Abkürzung. Da nämlich die Glieder aller jenen

Gleichungen, welche man erhält, aus den Coefficienten der Endpunkte bestehen, so kann man diese Coefficienten in den Gleichungen durch die entsprechenden Abschläge ersetzen.

$+b.BB' - c.CC' = (a+b-c)SS'$, so schreibe man statt dessen: $aA + bB - cC = (a+b-c)S$.

Und wirklich könnte man auch nicht einfacher den blossen Satz, dass S der Schwerpunkt von A, B, C mit den Gewichten $a, b, -c$ sey, und dass man sich in S diese Gewichte vereinigt zu denken habe, durch die Zeichen der Algebra darstellen. Allein unsere Formel ist mehr, als ein blos abgekürzter Ausdruck dieses Satzes, in welchem Falle sie nur die Gestalt einer algebraischen Gleichung hätte, noch nicht aber algebraische Operationen mit sich vornehmen liesse. In-

dem man $A, B, C\dots$ nicht mehr als die blossen Punkte, sondern als die ihnen entsprechenden Abschnitte nimmt, — woran man aber im Verlauf der Rechnung nicht weiter zu denken braucht, — stellt jene Formel zugleich eine Haupteigenschaft des Schwerpunkts in der Sprache der Algebra dar, und wird dadurch eben der Behandlung, wie jede andere algebraische Gleichung, fähig.

§. 14. Die Rechnung mit solchen abgekürzten Formeln ist es nun, welche ich den barycentrischen, d. i. den aus dem Begriffe des Schwerpunkts abgeleiteten, Calcul genannt habe, einen Calcul, der es nicht

2) Dass von den Punkten A, B, C, D, \dots denen resp. die Coefficienten a, b, c, d, \dots zukommen, S der Schwerpunkt ist, wird ausgedrückt durch:

$$\text{I. } aA + bB + cC + dD + \dots = (a+b+c+d+\dots) S,$$

ihren Zeichen vorgesetzt. So heisst z. B. aA , oder $+aA$ im Zusammenhange, der Punkt A mit dem Coefficienten a ; $-bB$, der Punkt B mit dem Coefficienten $-b$. Ist der Coefficient die Einheit, so wird nur das Zeichen derselben dem Punkte vorgesetzt, als A oder $+A$, $-B$, d. i. A mit dem Coefficienten 1 , B mit dem Coefficienten -1 .

2) Dass von den Punkten A, B, C, D, \dots denen resp. die Coefficienten a, b, c, d, \dots zukommen, S der Schwerpunkt ist, wird ausgedrückt durch:

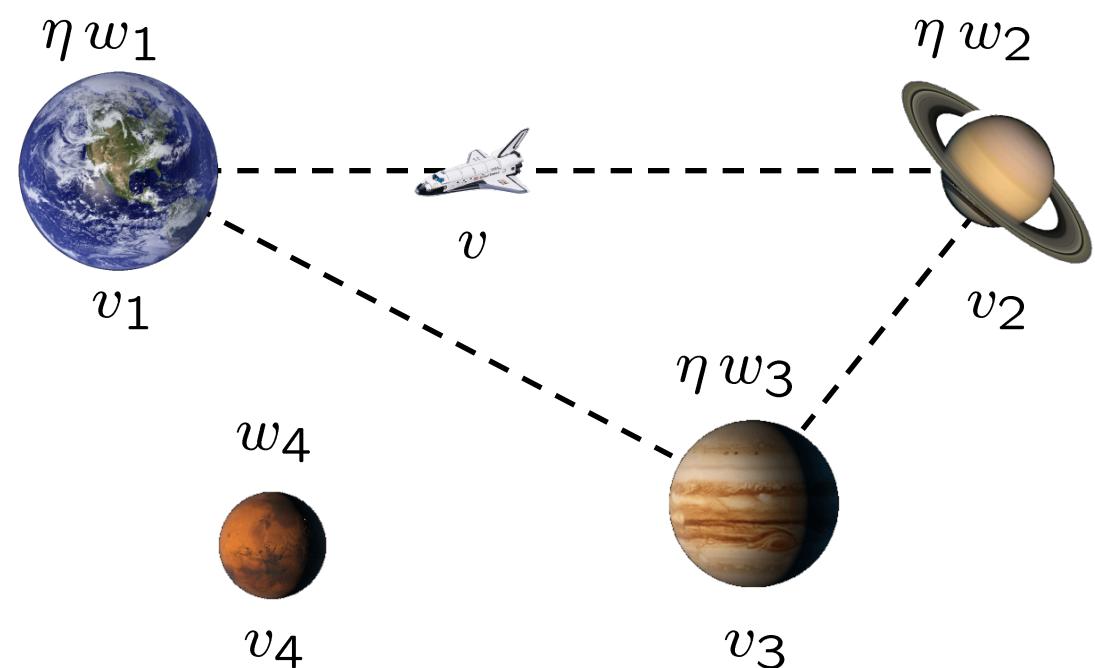
$$\text{I. } aA + bB + cC + dD + \dots = (a+b+c+d+\dots) S,$$

B

Barycentric coordinates

- system of masses w_i at positions v_i
- position of the system's **barycentre**:
- w_i are the **barycentric coordinates** of v
- **not unique**
- at least $d + 1$ points needed to span \mathbb{R}^d

$$v = \frac{\sum_i w_i v_i}{\sum_i w_i}$$



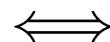
Barycentric coordinates

■ Theorem [Möbius, 1827]:

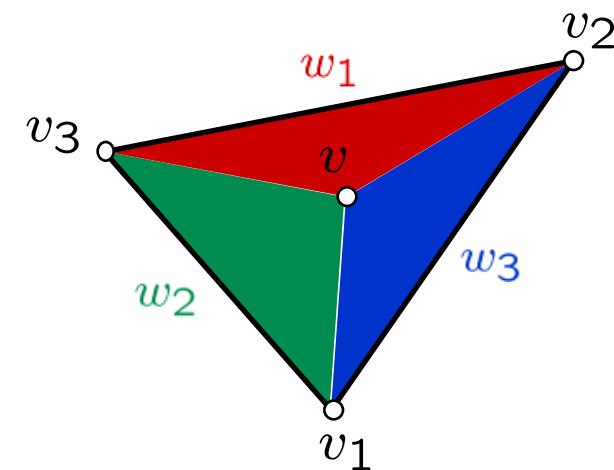
*The barycentric coordinates w_1, \dots, w_{d+1} of $v \in \mathbb{R}^d$ with respect to v_1, \dots, v_{d+1} are **unique** up to a common factor*

■ example: $d = 2$

$$v = \frac{w_1 v_1 + w_2 v_2 + w_3 v_3}{w_1 + w_2 + w_3}$$



$$w_i = \eta A(v, v_{i+1}, v_{i+2})$$



zweier anderer in derselben liegenden Punkte A und B betrachtet, deren Coëfficienten in dem durch die α , β , γ genseitig bestimmt $BC : CA$.

Alle raden auf Punkten finden als mehr be A und B gen, $\equiv \alpha A + (\beta B + \gamma C)$ genommen sind in derholen.

c. Aus $\alpha A + \beta B = (\alpha + \beta) A$ folgt $\alpha A + \beta B = (\alpha + \beta) C$ $\equiv 0$, und wenn wir $-(\alpha + \beta) = c$ setzen, $\alpha A + \beta B + cC \equiv 0$, $\beta B + cC \equiv A$, und hieraus eben so wie vorhin, $b : c \equiv CA : AB$. Unser Satz lässt sich daher noch symmetrischer so ausdrücken:

Ist $\alpha A + \beta B + cC \equiv 0$, so liegen A, B, C in gerader Linie, und es verhalten sich $a : b : c \equiv BC : CA : AB$. Auch gilt dieser Satz umgekehrt.

§. 23. Lehrsatze. Wenn $\alpha A + \beta B + cC \equiv D$, und A, B, C nicht in einer Geraden enthalten sind, so liegt D mit A, B, C in einer Ebene (§. 4. und 5.), und es verhalten sich:

$$a : b : c \equiv \text{die Dreiecke } DBC : DCA : DAB.$$

Beweis. Von den drei Summen, welche sich, je zwei der drei Coëfficienten α, β, γ zusammengenommen, bilden lassen, ist immer wenigstens eine nicht $\equiv 0$. Sey $\alpha + \beta$ diese Summe, und man setze I. $\alpha A + \beta B = (a+b) Z$, so wird II. $D \equiv (a+b) Z + cC$.

Wegen II. liegen nun (§. 21.) C, D, Z in gerader

ADZ

a.) AB ,

3. b.)

§. 23. Lehrsatz. Wenn $\alpha A + \beta B + cC \equiv D$, und A, B, C nicht in einer Geraden enthalten sind, so liegt D mit A, B, C in einer Ebene (§. 4. und 5.), und es verhalten sich:

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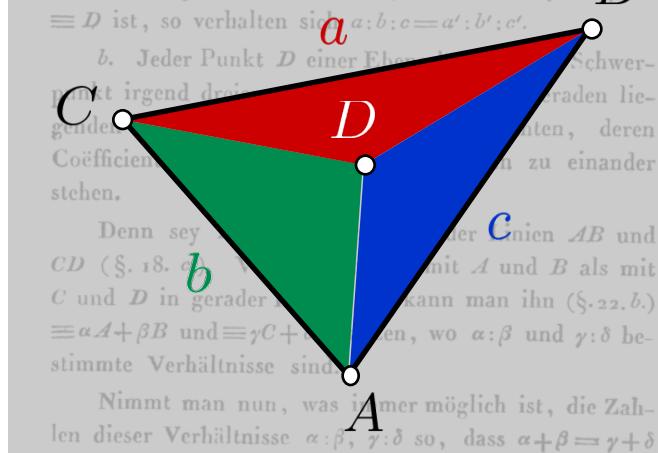
und hieraus in Verbindung mit 4) die Proportion des Lehrsatzes.

§. 24. Zusätze. a. Wenn A, B, C nicht in einer Geraden liegen und $\alpha A + \beta B + cC \equiv D$, $\alpha' A + \beta' B + c'C \equiv D$ ist, so verhalten sich $a : b : c \equiv a' : b' : c'$.

b. Jeder Punkt D einer Ebene ABC hat einen Schwerpunkt irgend dreier Dreiecke ABC , welche in Geraden liegenden Punkten A, B, C enthalten, deren Coëfficienten α, β, γ so bestimmt sind, dass sie zu einander stehen.

Denn sey $\alpha A + \beta B + cC \equiv D$, $\alpha' A + \beta' B + c'C \equiv E$ und $\alpha'' A + \beta'' B + c''C \equiv F$ die drei Dreiecke ABC mit A und B als mit C und D in gerader Linie liegenden Punkten. Man kann ihn (§. 22. b.) $\equiv \alpha A + \beta B$ und $\equiv \gamma C + \delta D$ schreiben, wo $\alpha : \beta$ und $\gamma : \delta$ bestimmte Verhältnisse sind.

Nimmt man nun, was immer möglich ist, die Zahlen dieser Verhältnisse $\alpha : \beta, \gamma : \delta$ so, dass $\alpha + \beta = \gamma + \delta$

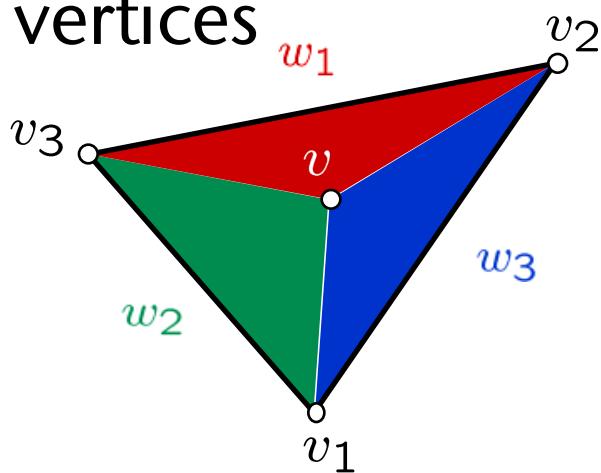


Computing areas

- area of triangle $\triangle_1 = [v, v_2, v_3]$ with vertices

$$v = (x, y), \quad v_2 = (x_2, y_2), \quad v_3 = (x_3, y_3)$$

$$w_1 = 2A(v, v_2, v_3)$$



- similar for the triangles $\triangle_2 = [v, v_3, v_1]$

$$\text{and } \triangle_3 = [v, v_1, v_2]$$

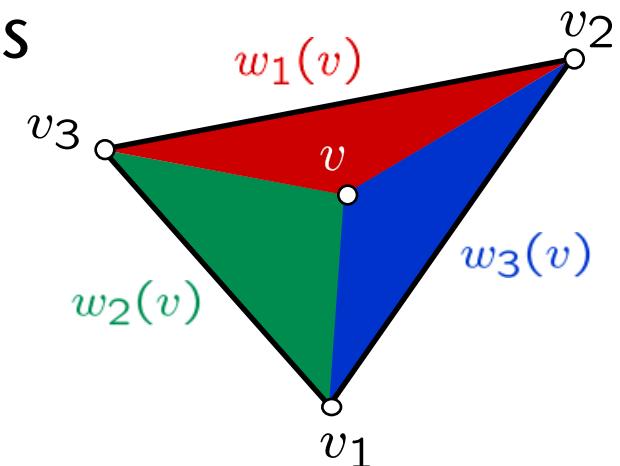
$$w_2 = 2A(v, v_3, v_1) = (x_3 - x)(y_1 - y) - (x_1 - x)(y_3 - y)$$

$$w_3 = 2A(v, v_1, v_2) = (x_1 - x)(y_2 - y) - (x_2 - x)(y_1 - y)$$

Barycentric coordinates for triangles

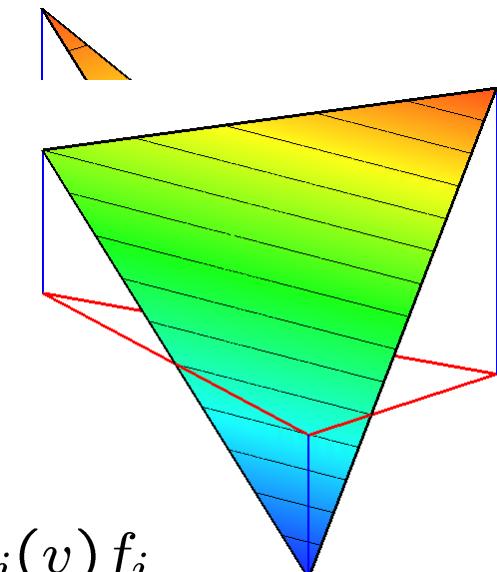
- normalized barycentric coordinates

$$b_i(v) = \frac{w_i(v)}{w_1(v) + w_2(v) + w_3(v)}$$



- properties

- partition of unity $\sum_i b_i(v) = 1$
- reproduction $\sum_i b_i(v)v_i = v$
- positivity $b_i(v) > 0, \quad v \in \overset{\circ}{\triangle}$
- Lagrange property $b_i(v_j) = \delta_{ij}$



- application

- linear interpolation of data $F(v) = \sum_{i=1}^3 b_i(v)f_i$

Arbitrary polygons

- barycentric coordinates $w_1(v), \dots, w_n(v)$

$$v = \frac{\sum_{i=1}^n w_i(v)v_i}{\sum_{j=1}^n w_j(v)}$$

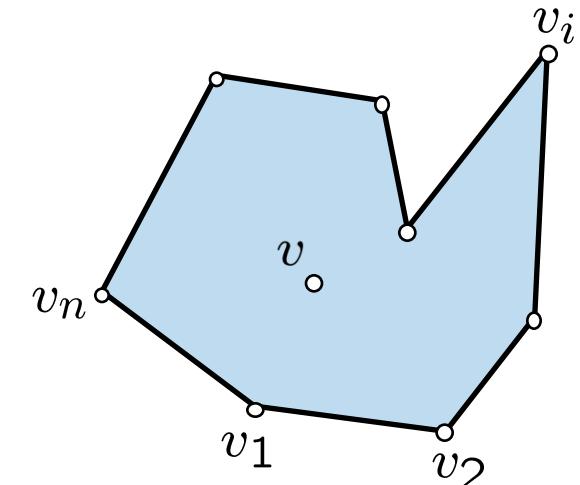
- normalized coordinates

$$b_i(v) = \frac{w_i(v)}{\sum_{j=1}^n w_j(v)}$$

- properties

- partition of unity $\sum_{i=1}^n b_i(v) = 1$
- reproduction $\sum_{i=1}^n b_i(v)v_i = v$

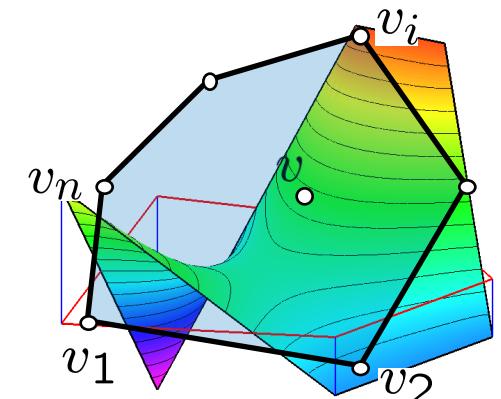
linear precision $\Rightarrow \sum_{i=1}^n b_i(v)\phi(v_i) = \phi(v)$
for all $\phi \in \pi_1$



Convex polygons

[Floater, H. & Kós 2006]

- **Theorem:** If all $w_i(v) > 0$, then
 - positivity $b_i(v) > 0$
 - Lagrange property $b_i(v_j) = \delta_{ij}$
 - linear along boundary $b_i|_{[v_i, v_{i+1}]} \in \pi_1$
- Application $F(v) = \sum_{i=1}^n b_i(v) f_i$
 - interpolation of data given at the vertices
 - $F(v)$ inside the convex hull of the f_i
 - direct and efficient evaluation



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- **Wachspress** (WP) coordinates

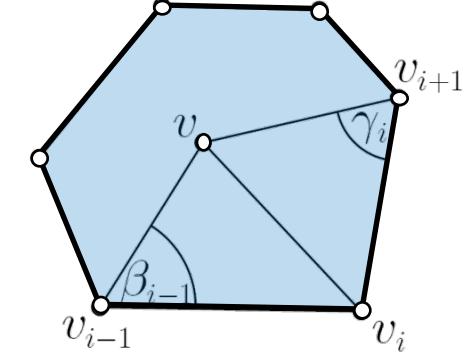
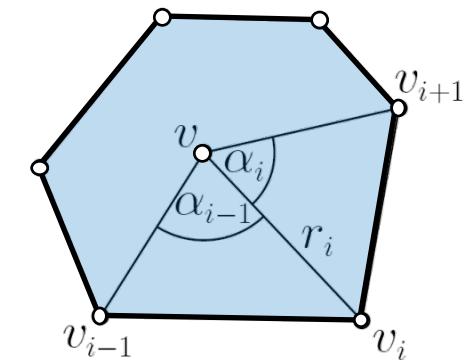
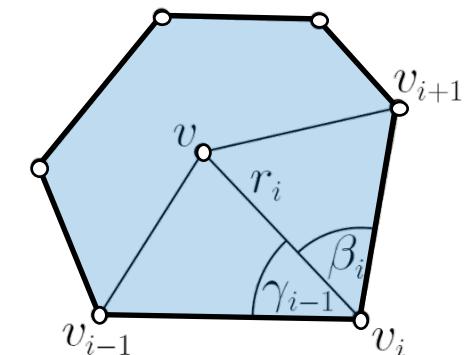
$$w_i = \frac{\cot \gamma_{i-1} + \cot \beta_i}{r_i^2}$$

- **mean value** (MV) coordinates

$$w_i = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{r_i}$$

- **discrete harmonic** (DH) coordinates

$$w_i = \cot \beta_{i-1} + \cot \gamma_i$$



[Floater, H. & Kós 2006]

- **Theorem:** All barycentric coordinates can be written as

$$w_i = \frac{c_{i+1}A_{i-1} - c_iB_i + c_{i-1}A_i}{A_{i-1}A_i}$$

with certain real functions c_i

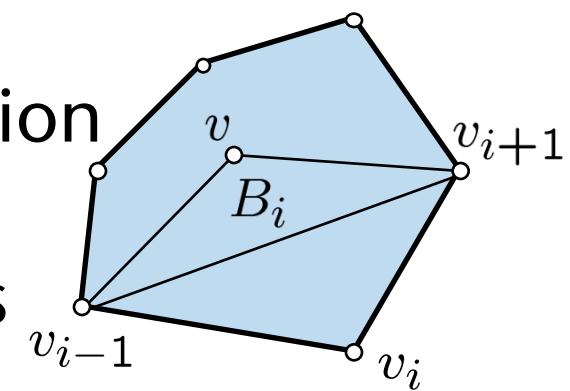
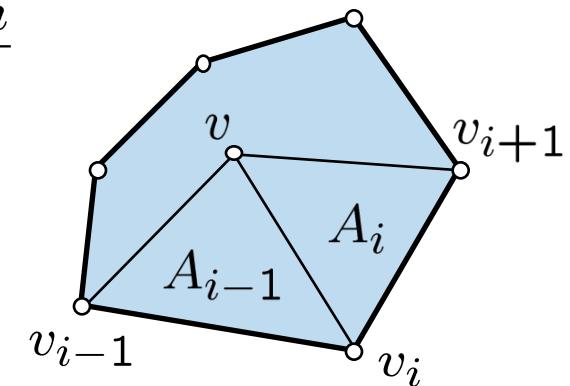
- ***three-point coordinates***

- $c_i = f(r_i)$ with $r_i = \|v - v_i\|$

- **Theorem:** Such a generating function

$$f: \mathbb{R}^+ \rightarrow \mathbb{R}$$

exists for all three-point coordinates



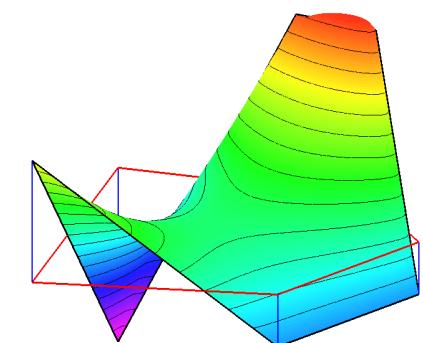
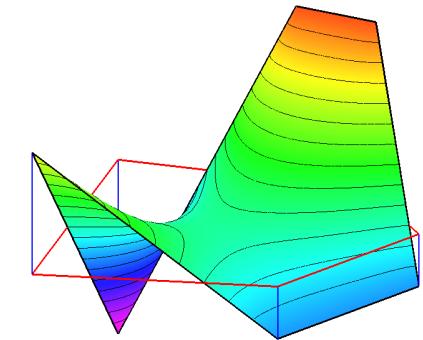
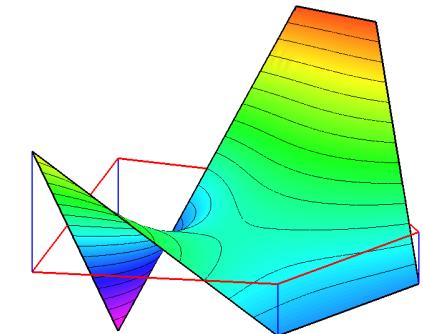
Three-point coordinates

- **Theorem:** $w_i(v) > 0$ if and only if f is

- positive $f(r) > 0$
- monotonic $f'(r) \geq 0$
- convex $f''(r) \geq 0$
- sub-linear $f'(r) \leq f(r)/r$

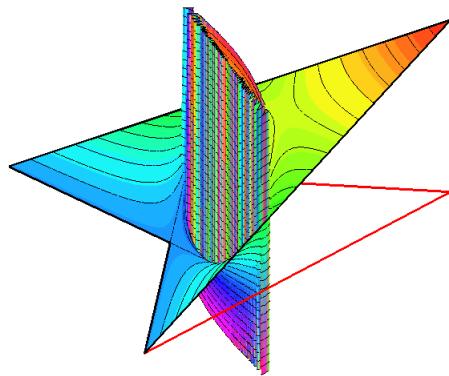
- examples

- WP coordinates $f(r) = 1$
- MV coordinates $f(r) = r$
- DH coordinates $f(r) = r^2$

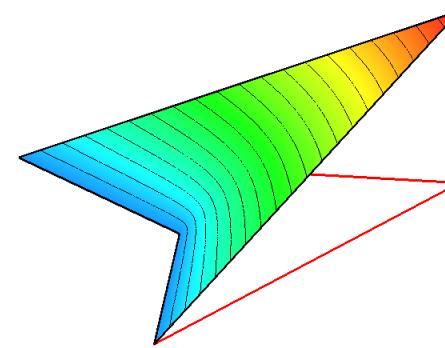


Non-convex polygons

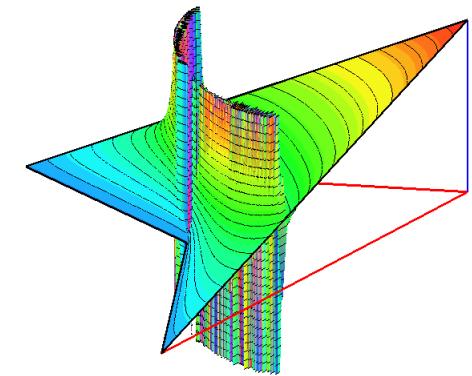
Wachspress



mean value



discrete harmonic



$$f(r) = 1$$

$$f(r) = r$$

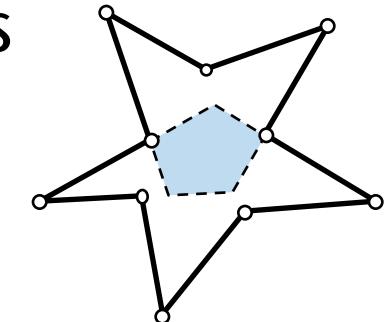
$$f(r) = r^2$$

- poles, if $W(v) = \sum_{j=1}^n w_j(v) = 0$, since $b_i(v) = \frac{w_i(v)}{W(v)}$

Star-shaped polygons

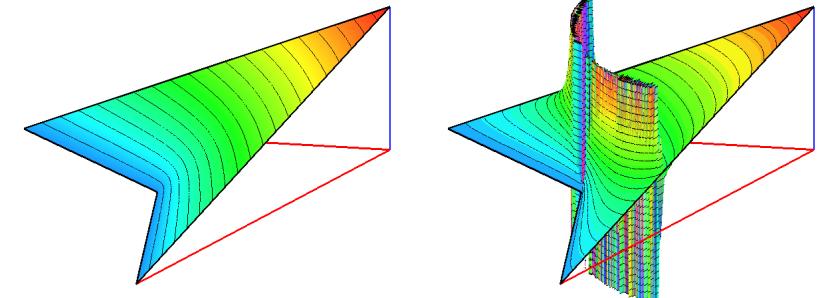
- **Theorem:** $W(v) \neq 0$ if and only if f is

- positive $f(r) > 0$
- super-linear $f'(r) \geq f(r)/r$



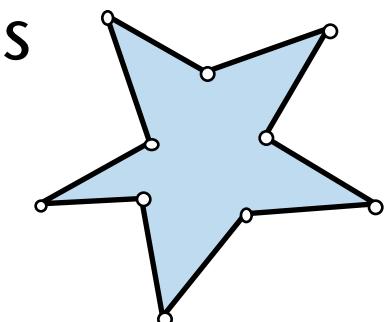
- examples

- MV coordinates $f(r) = r$
- DH coordinates $f(r) = r^2$



- **Theorem:** $W(v) = 0$ for some v if f is

- strictly super-linear $f'(r) > f(r)/r$

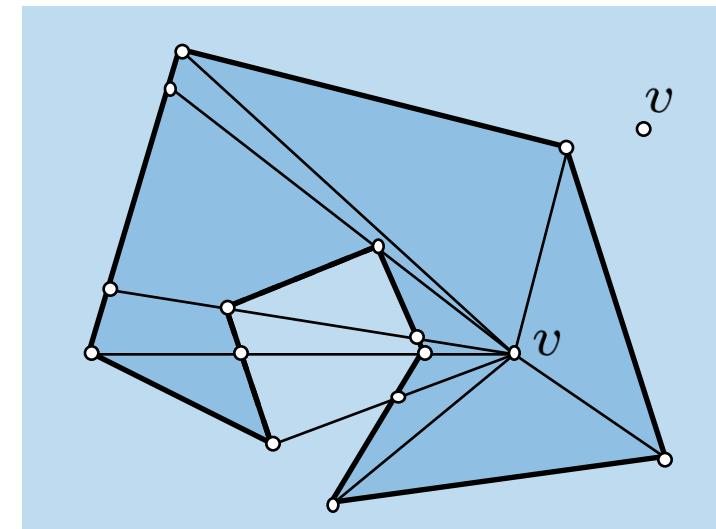
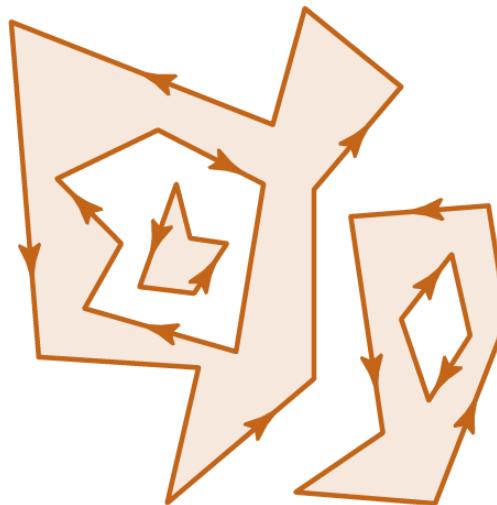


Mean value coordinates

[H. & Floater 2006]

- **Theorem:** MV coordinates have no poles in \mathbb{R}^2

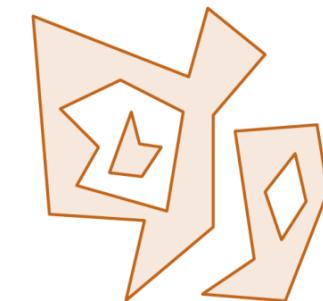
$$W(v) = \sum w_j(v) = \sum \kappa_i(v) > 0 \neq 0$$



Mean value coordinates

■ properties

- well-defined everywhere in \mathbb{R}^2
- Lagrange property $b_i(v_j) = \delta_{ij}$
- linear along boundary $b_i|_{[v_i, v_{i+1}]} \in \pi_1$
- linear precision $\sum_i b_i(v) \phi(v_i) = \phi(v)$ for $\phi \in \pi_1$
- smoothness C^0 at v_i , otherwise C^∞
- similarity invariance $b_i = \hat{b}_i \circ \psi$ for $\hat{\Omega} = \psi(\Omega)$



■ application

- direct interpolation of data

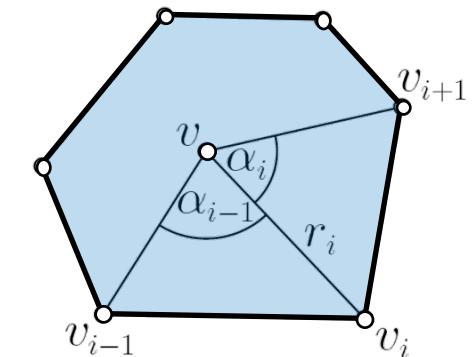
$$F(v) = \sum_{i=1}^n b_i(v) f_i$$

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Implementation

■ Mean Value coordinates

$$w_i = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{r_i}$$



$$\tan(\alpha_i/2)$$

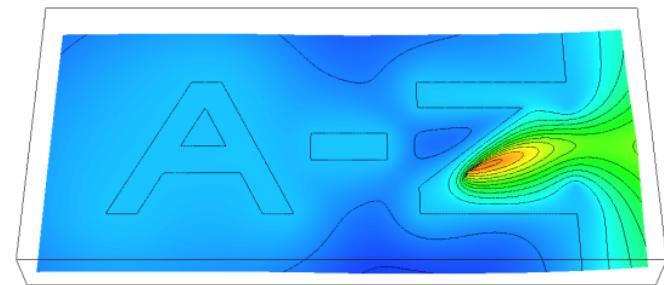
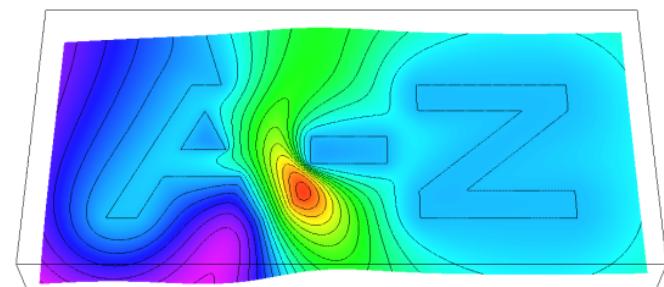
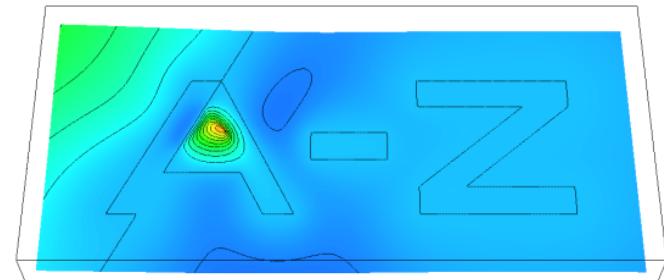
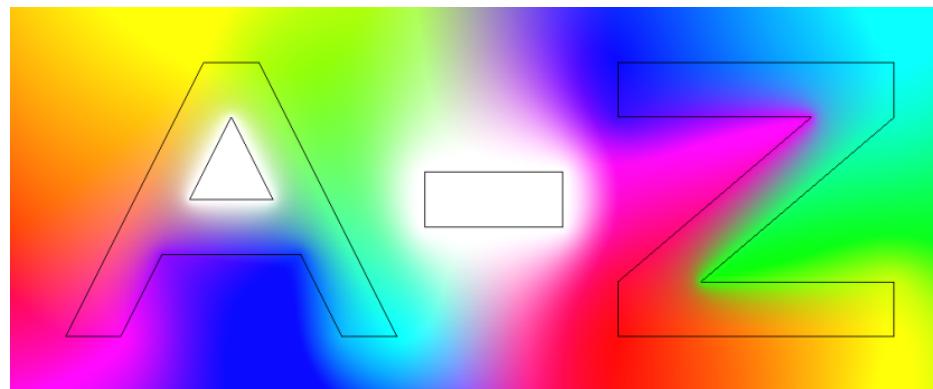
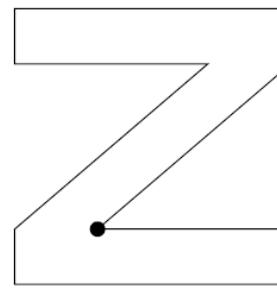
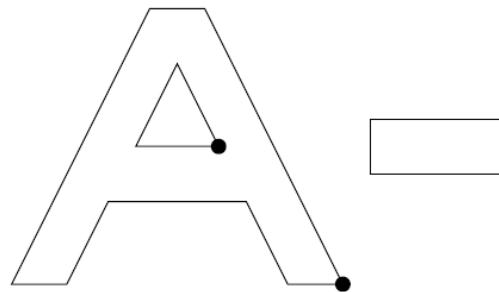
$$s_i = v_i - v$$

$$w_i = \frac{r_{i-1} - \langle s_{i-1}, s_i \rangle / r_i}{\det(s_{i-1}, s_i)} + \frac{r_{i+1} - \langle s_i, s_{i+1} \rangle / r_i}{\det(s_i, s_{i+1})}$$

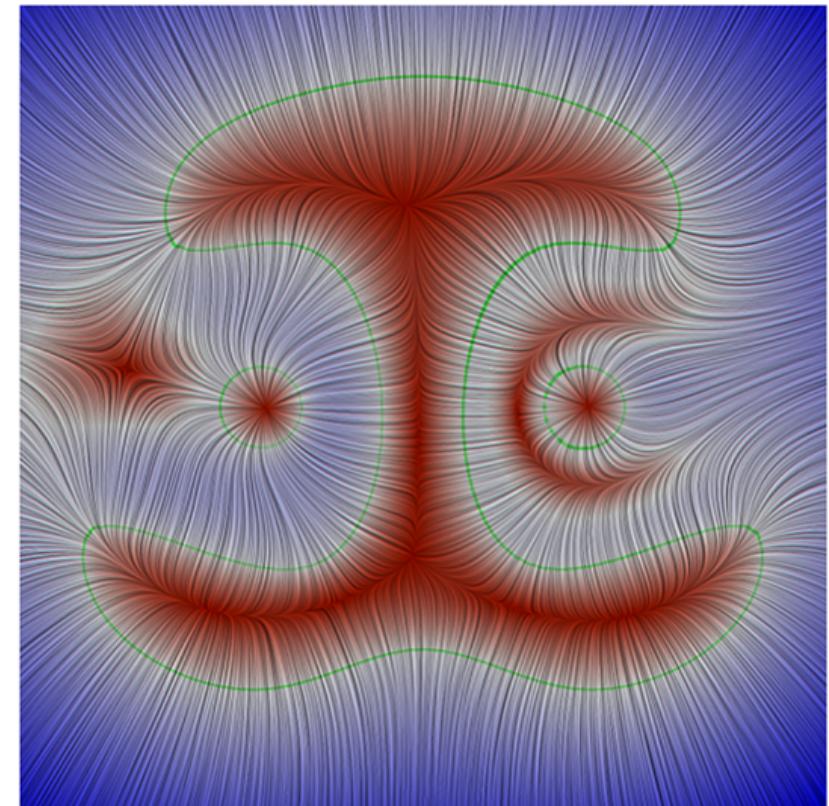
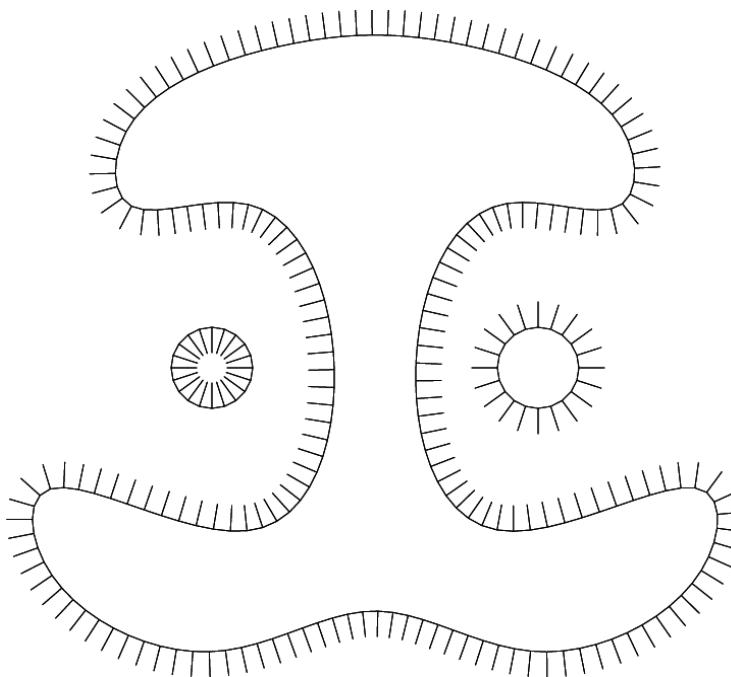
Implementation

```
for i=1 to n do          f = 0
    s[i] = v[i]-p
    r[i] = ||s[i]||
    if r[i]==0 then
        return f[i]
    for i=1 to n
        w = (t[i-1]+t[i])/r[i]
        f = f + wf[i]
        W = W + w
    for i=1 to n do
        A[i] = det(s[i],s[i+1])      return f/W
        D[i] = <s[i],s[i+1]>
        if A[i]==0 and D[i] < 0 then
            return
            (r[i+1]f[i]+r[i]f[i+1])/(
            (r[i]+r[i+1])
t[i]=A[i]/(r[i]r[i+1]+D[i])
```

Colour interpolation



Vector fields

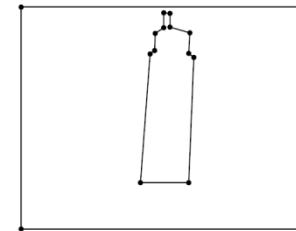


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Image warping



original image



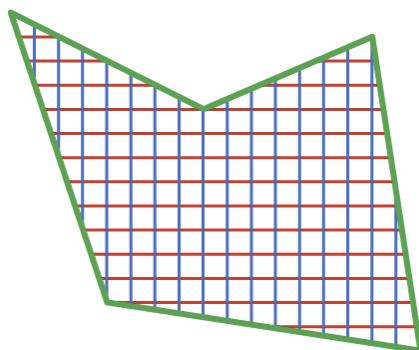
mask



warped image

Barycentric mappings

source polygon

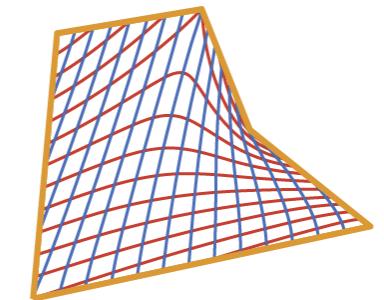


for $v \in :$

$$f(v) = \sum_{j=1}^n b_j(v) \hat{v}_j$$



target polygon



$$P = [v_1, \dots, v_n]$$

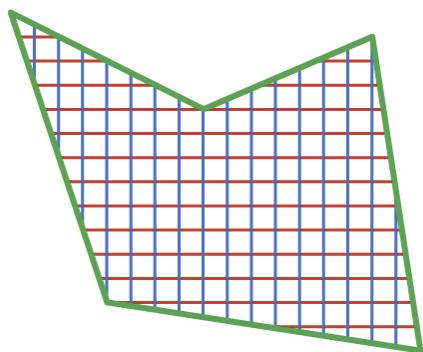
$$v_j \in \mathbb{R}^2$$

$$\hat{P} = [\hat{v}_1, \dots, \hat{v}_n]$$

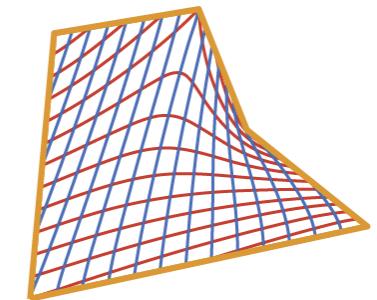
$$\hat{v}_j \in \mathbb{R}^2$$

Complex barycentric mappings

source polygon



target polygon



for $z \in :$

$$g(z) = \sum_{j=1}^n c_j(z) \hat{z}_j$$



with **complex**
barycentric coordinates

$$P = [z_1, \dots, z_n]$$

$$z_j \in \mathbb{C}$$

$$c_j : \rightarrow \mathbb{C}$$

$$\hat{P} = [\hat{z}_1, \dots, \hat{z}_n]$$

$$\hat{z}_j \in \mathbb{C}$$

Complex barycentric coordinates

[Weber et al. 2009]

- **normal form** of complex barycentric coordinates

- edge weight functions $\gamma_j: \rightarrow \mathbb{C}$
- homogeneous coordinates

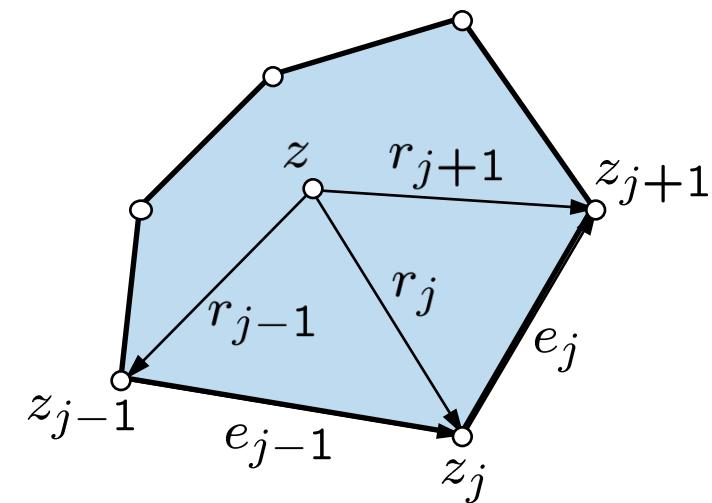
$$\tilde{c}_j = \gamma_j \frac{r_{j+1}}{e_j} - \gamma_{j-1} \frac{r_{j-1}}{e_{j-1}}$$

- normalized coordinates

$$c_j = \frac{\tilde{c}_j}{\sum_{k=1}^n \tilde{c}_k}$$

- example

- $\gamma_j = \log(r_{j+1}/r_j)$ gives **Cauchy–Green coordinates**

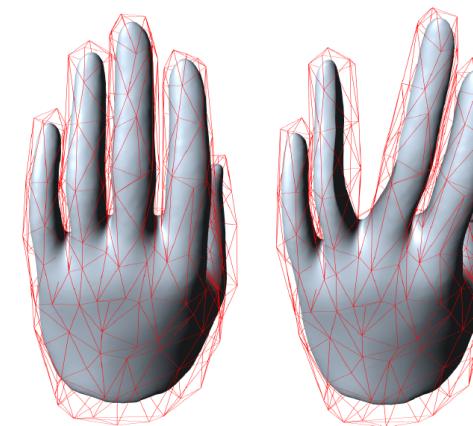
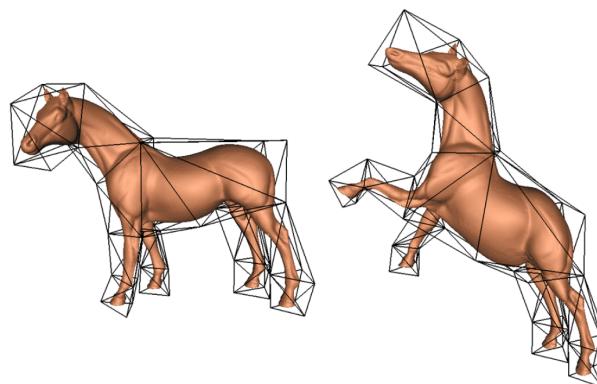


- The essence of barycentric coordinates
- Three-point coordinates
- Implementation
- Barycentric mappings
- *Mesh deformations*

Mesh warping

- MV coordinates in 3D

[Ju et al. 2005]

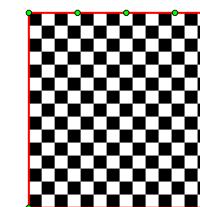


MVC

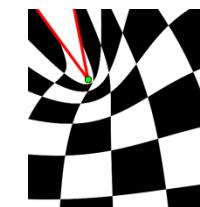
PMVC

- negative inside the domain
- positive MV coordinates
 - only C^0 -continuous
 - no closed form

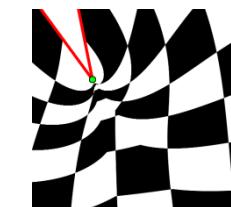
[Lipman et al. 2007]



MVC



MVC



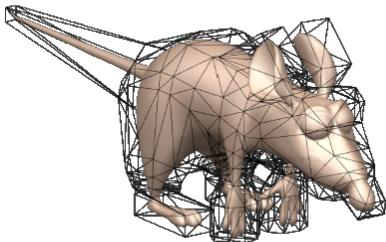
PMVC

Harmonic coordinates

- define normalized coordinate b_i as solution of PDE

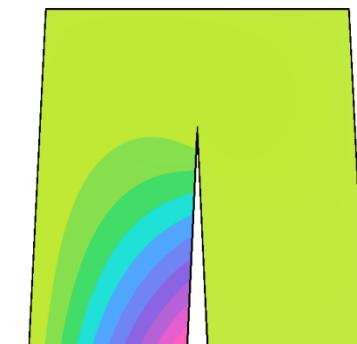
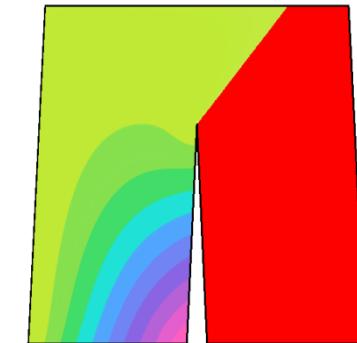
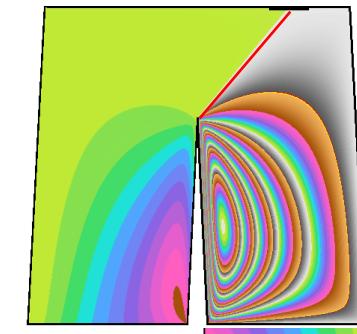
$$\Delta b_i = 0 \quad \text{subject to } b_i(v_j) = \delta_{ij}, \quad b_i|_{[v_i, v_{i+1}]} \in \pi_1$$

- Lagrange property ✓ well-defined ✓ smooth ✓
- linear precision ✓ positivity ✓ efficient ✗
- animation for Ratatouille [Joshi et al. 2007]



Positive barycentric coordinates

- drawbacks so far ...
- mean value coordinates
 - negative
- positive mean value coordinates
 - not smooth (only C^0)
- harmonic coordinates
 - rather expensive to compute
 - not smooth in practice



Maximum entropy coordinates

[H. & Sukumar 2008]

- based on maximizing the *Shannon-Jaynes entropy*
- Lagrange property ✓ well-defined ✓ smooth (✓)
- linear precision ✓ positivity ✓ efficient (✓)

