

CS 314 Principles of Programming Languages

Lecture 17: Lambda Calculus

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Class Information

- Midterm exam 11/7 Wednesday 10:20am - 11:40am
- Extended hours are Posted
- No classes on 11/2 this Friday

Review: Lambda Calculus - Historical Origin

- Church's model of computing is called the *lambda calculus*

It is based on the notion of parameterized expressions (with each parameter introduced by an occurrence of the letter λ — hence the notation's name). Lambda calculus was the inspiration for functional programming: one uses it to compute by *substituting parameters into expressions*, just as one computes in a high level functional program by *passing arguments to functions*.

Review: Functional Programming

- Functional languages such as Lisp, Scheme, FP, ML, Miranda, and Haskell are an attempt to realize Church's lambda calculus in practical form as a programming language
- **The key idea: do everything by composing functions**
 - No mutable state
 - No side effects
 - Function as first-class values

Review: Lambda Calculus

λ -terms are inductively defined.

A **λ -term** is:

- a variable x
- $(\lambda x. M) \Rightarrow$ where x is a variable and λ is a λ -term (abstraction)
- $(M N) \Rightarrow$ where M and N are both λ -terms (application)

Review: λ -terms

The context-free grammar for λ -terms:

$\lambda\text{-term}$	\rightarrow	expr
expr	\rightarrow	$\text{name} \mid \text{number} \mid \lambda \text{name} . \text{expr} \mid \text{func arg}$
func	\rightarrow	$\text{name} \mid (\lambda \text{name} . \text{expr}) \mid \text{func arg}$
arg	\rightarrow	$\text{name} \mid \text{number} \mid (\lambda \text{name} . \text{expr}) \mid (\text{func arg})$

Example 1:

$\lambda \boxed{y} . \boxed{y \ x}$

\swarrow \searrow

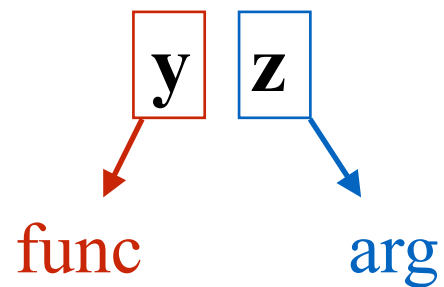
name (as parameter) expr (another λ -term)

Review: λ -terms

The context-free grammar for λ -terms:

λ -term	\rightarrow	expr
expr	\rightarrow	name number λ name . expr func arg
func	\rightarrow	name (λ name . expr) func arg
arg	\rightarrow	name number (λ name . expr) (func arg)

Example 2:



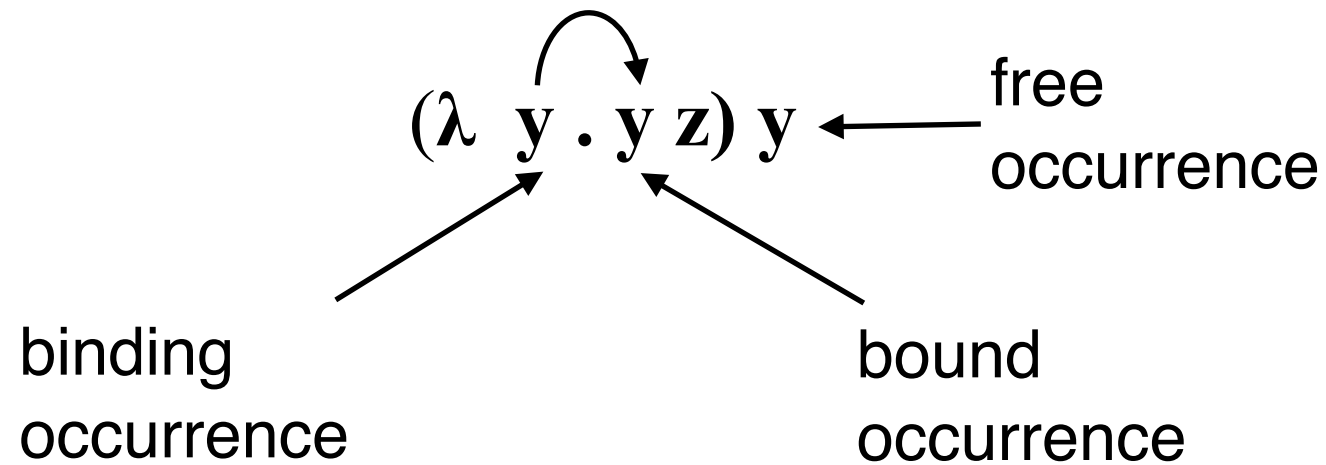
Review: Lambda Calculus

Associativity and Precedence

- Function application is left associative: $(f\ g\ z)$ is $((f\ g)\ z)$
- Function application has precedence over function abstraction.
“function body” extends as far to the right as possible:
 $(\lambda x.yz)$ is $(\lambda x.(yz))$
- Multiple arguments: $(\lambda xy.z)$ is $(\lambda x(\lambda y.z))$

Review: Free and Bound Variables

Abstraction $(\lambda x. M)$ “binds” variable x in “body” M . You can think of this as a declaration of variable x with scope M .

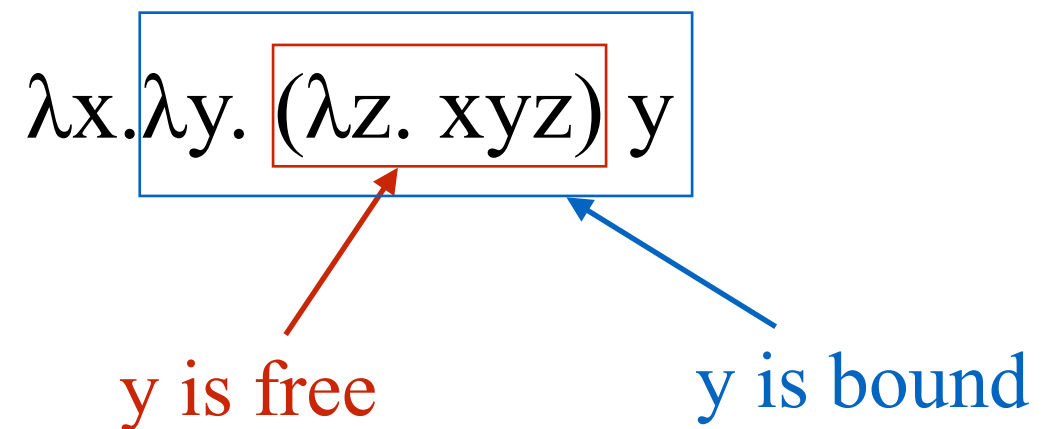


Review: Free and Bound Variables

Note:

A variable can occur **free** and **bound** in a λ -term.

Example:



“free” is relative to a λ -sub-term.

Review: Free and Bound Variables

Let M, N be λ -terms and x is a variable. The set of *free variable* of M , $\text{free}(M)$, is defined inductively as follows:

- $\text{free}(x) = \{x\}$
- $\text{free}(M N) = \text{free}(M) \cup \text{free}(N)$
- $\text{free}(\lambda x.M) = \text{free}(M) - \text{free}(x)$

Review: Function Application

Computation in lambda calculus is based on the concept of reduction. Simplify an expression until it can no longer be simplified.

β –reduction:

$$(\lambda x. E)y \rightarrow_{\beta} E[y/x]$$

1. Return function body E
2. Replace every free occurrence of x in E with y

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Example:

$$\begin{aligned} (\lambda a.\lambda b.a+b) \textcolor{red}{2} x &\rightarrow_{\beta} (\lambda b.2+b) x \\ &\rightarrow_{\beta} 2+x \end{aligned}$$

Function Application

Computation in lambda calculus is based on the concept of reduction. Simplify an expression until it can no longer be simplified.

β –reduction:

$$(\lambda x.E)y \rightarrow_{\beta} E[y/x]$$

We should not perform β –reduction if y is a bound variable within E

Example:

$$\begin{array}{ccc} (\lambda a.\lambda b.a+b) \ b \ 2 & \rightarrow_{\beta} & (\lambda b.b+b) \ 2 \\ \downarrow & & \rightarrow_{\beta} \ 2+2 \\ & & \text{Incorrect} \end{array}$$

b is a bound variable within $\lambda a.\lambda b.a+b$

This is called capturing

Review: Function Application

Computation in lambda calculus is based on the concept of reduction. Simplify an expression until it can no longer be simplified.

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Review: Function Application

Computation in lambda calculus is based on the concept of reduction.
Simplify an expression until it can no longer be simplified.

α -reduction:

$$(\lambda x.E) \rightarrow_{\alpha} \lambda y.E[y/x]$$

Perform α -reduction first

Example: $(\lambda a.\underline{\lambda b.a+b})\ b\ 2 \rightarrow_{\alpha} (\lambda a.\underline{\lambda x.a+x})\ b\ 2$

$\rightarrow_{\beta} \lambda x.b+x\ 2$

$\rightarrow_{\beta} b+2$

Review: Programming in Lambda Calculus

Remember: Computation in the lambda calculus is a sequence of applications of reduction rules (mostly β -reductions).

Logical constants and operations (incomplete list):

true $\equiv \lambda a. \lambda b. a$

select-first

false $\equiv \lambda a. \lambda b. b$

select-second

cond $\equiv \lambda p. \lambda m. \lambda n. (p \ m \ n)$

not $\equiv \lambda x. (x \ \text{false} \ \text{true})$

and $\equiv \lambda x. \lambda y. (x \ y \ \text{false})$

or $\equiv \text{homework}$

Review: Programming in Lambda Calculus

What about data structures?

Data structures:

pairs can be represented as:

$$[M.N] \equiv \lambda z. (z M N)$$

first $\equiv \lambda x. (x \text{ true})$

(car)

second $\equiv \lambda x. (x \text{ false})$

(cdr)

build $\equiv \lambda x. \lambda y. \lambda z. (z x y)$

(cons)

Programming in Lambda Calculus

What about the encoding of arithmetic constants?

Church Numerals:

$$0 \equiv \lambda f x. x$$

$$1 \equiv \lambda f x. (f x)$$

$$2 \equiv \lambda f x. (f (f x))$$

...

$$n \equiv \lambda f x. (f (f (\dots (f x) \dots))) \equiv \lambda f x. (f^n x)$$

The natural number n is represented as a function that applies a function f n -times to x .

$$\mathbf{succ} \equiv \lambda m. (\lambda f x. (f (m f x)))$$

$$\mathbf{add} \equiv \lambda m n. (\lambda f x. ((m f) (n f x)))$$

$$\mathbf{mult} \equiv \lambda m n. (\lambda f x. ((m (n f)) x))$$

$$\mathbf{isZero?} \equiv \lambda m. (m \lambda x. \text{false} \text{ true})$$

Recursion in Lambda Calculus

Does this make sense?

$$f \equiv \dots f \dots$$

In lambda calculus, \equiv is “abbreviated as”, **but not an assignment.**

Recursion in Lambda Calculus

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$$\text{add} \equiv \lambda mn. (\text{if (isZero?n) m (add (m+1) (n-1)) })$$

↑
Incorrect! “add” is not defined

How about

$$\begin{array}{lcl} \lambda f. (\lambda mn. (\text{if (isZero?n) m (f (m+1) (n-1)) })) & \text{add} & \\ \downarrow & & \\ F & \lambda mn. (\text{if (isZero?n) m (add (m+1) (n-1)) }) & = \text{add} \\ & \downarrow & \\ & F \text{ add} & = \text{add} \end{array}$$

“**add**” is a fixed point to function **F**

Function Fixed Points

The fixed point of a function g is the set of values

$$\{ x \mid x = g(x) \}$$

Examples:

function g	$\text{fix}(g)$
$\lambda x.6$	$\{6\}$
$\lambda x.(6 - x)$	$\{3\}$
$\lambda x.((x * x) + (x - 4))$	$\{-2, 2\}$
$\lambda x.x$	entire domain of function f
$\lambda x.(x + 1)$	$\{ \}$

Function Fixed Points

Is there a way to “compute” the fixed point of any function F

$$x = F(x)$$

YES. $x = YF$, and Y is called the fixed point combinator.

$$Y \equiv \lambda f.((\lambda x.f(x\ x)) (\lambda x.f(x\ x)))$$

$$YF = ((\lambda f.((\lambda x.f(x\ x)) (\lambda x.f(x\ x)))) F)$$

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$$Y \equiv \lambda f. ((\lambda x. f(x \ x)) (\lambda x. f(x \ x)))$$

$$\begin{aligned} YF &= ((\lambda f. ((\lambda x. \underline{f}(x \ x)) (\lambda x. \underline{f}(x \ x)))) \underline{F}) \\ &= (\lambda x. F(x \ x)) (\lambda x. F(x \ x)) \end{aligned}$$

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$$YF = F((\lambda x. F(x \ x)) (\lambda x. F(x \ x)))$$

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$$YF = (\lambda x. F(x \ x)) (\lambda x. F(x \ x))$$

$$YF = F((\lambda x. F(x \ x)) (\lambda x. F(x \ x)))$$

$$YF = F(YF)$$

The Y - Combinator Example (Cont.)

- Informally, the Y-Combinator allows us to get as many copies of the recursive procedure body as we need. The computation “unrolls” recursive procedure calls one at a time

$$Y \equiv \lambda f.((\lambda x.f(x\ x)) (\lambda x.f(x\ x)))$$

The Y - Combinator

Example:

$F \equiv \lambda f. (\lambda mn. \text{if } (\text{isZero? } n) \text{ then } m \text{ else } (f \text{ (succ } m) \text{ (pred } n))))$

$$\begin{aligned} (YF \ 3 \ 2) &= (((\lambda f. ((\lambda x. f(x \ x)) (\lambda x. f(x \ x)))) F) \ 3 \ 2) \\ &= ((F \ (YF)) \ 3 \ 2) \\ &= ((\lambda mn. \text{if } (\text{isZero? } n) \text{ then } m \text{ else } YF \text{ (succ } m) \text{ (pred } n)))) \ 3 \ 2) \\ &= \text{if } (\text{isZero? } 2) \text{ then } 3 \text{ else } YF \text{ (succ } 3) \text{ (pred } 2)) \\ &= (YF \ 4 \ 1) \\ &= ((F \ (YF)) \ 4 \ 1) \\ &= \text{if } (\text{isZero? } 1) \text{ then } 4 \text{ else } YF \text{ (succ } 4) \text{ (pred } 1)) \\ &= (YF \ 5 \ 0) \\ &= (F \ (YF) \ 5 \ 0) \\ &= \text{if } (\text{isZero? } 0) \text{ then } 5 \text{ else } (YF \text{ (succ } 5) \text{ (pred } 0)) \\ &= \mathbf{5} \end{aligned}$$

Lambda Calculus - Final Remarks

- We can express all computable functions in our λ -calculus.
- All computable functions can be expressed by the following two combinators, referred to as **S** and **K**.
 - $K \equiv \lambda xy.x$
 - $S \equiv \lambda xyz.xz(yz)$

Combinatoric logic is as powerful as Turing Machines.

Next Lecture

Reading:

- Scott, Chapter 11.1 - 11.3
- Scott, Chapter 11.7
- ALSU, Chapter 11.1 - 11.3