

CS 314 Principles of Programming Languages

Lecture 20: Parallelism and Dependence Analysis

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Class Information

- Homework 7 is released.
- Project 2 deadline will be extended to 11/25 Sunday.

Review: Dependence Test

Given

do $i_1 = L_1, U_1$

...

do $i_n = L_n, U_n$

S1 : $A[f_1(i_1, \dots, i_n), \dots, f_m(i_1, \dots, i_n)] = \dots$

S2 : $\dots = A[g_1(i_1, \dots, i_n), \dots, g_m(i_1, \dots, i_n)]$

Let α & β be a vector of n integers within the ranges of the lower and upper bounds of the n loops.

Does $\exists \alpha, \beta$ in the loop iteration space, s.t.

$$f_k(\alpha) = g_k(\beta) \quad \forall k, 1 \leq k \leq m?$$

Integer Linear Programming (ILP) for Dependence Test

Does $\exists \alpha, \beta$ in the loop iteration space, s.t.

$$f_k(\alpha) = g_k(\beta) \quad \forall k, 1 \leq k \leq m?$$

```
for (i=1; i<=100; i++)  
  for (j=1; j<=100; j++){  
    S1: X[i,j] = X[i,j] + Y[i-1, j];  
    S2: Y[i,j] = Y[i,j] + X[i, j-1];  
  }
```

Consider the two memory references:

S1(α): **X[i₁, j₁]**, S2(β): **X[i₂, j₂-1]**

α : (i₁, j₁)

β : (i₂, j₂)

Access the same
memory location →

Loop bounds
constraint →

```
i1 = i2  
j1 = j2 - 1  
1 <= i1 <= 100  
1 <= j1 <= 100  
1 <= i2 <= 100  
1 <= j2 <= 100
```

Does there exist a solution to
this integer linear
programming (ILP) problem?

Integer Linear Programming (ILP) for Dependence Test

If we use the matrix vector notation $\langle F_1, f_1, B_1, b_1 \rangle$ and $\langle F_2, f_2, B_2, b_2 \rangle$ for two references at two iterations $\alpha: (i_1, j_1)$ and $\beta: (i_2, j_2)$

$$\begin{array}{l} i_1 = i_2 \\ j_1 = j_2 - 1 \\ 1 \leq i_1 \leq 100 \\ 1 \leq j_1 \leq 100 \\ 1 \leq i_2 \leq 100 \\ 1 \leq j_2 \leq 100 \end{array}$$

Memory reference $X[i_1, j_1]$

$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad f_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} i_1 \\ j_1 \end{bmatrix}$$

Memory reference $X[i_2, j_2 - 1]$

$$F_1 \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} + f_1 = F_2 \begin{bmatrix} i_2 \\ j_2 \end{bmatrix} + f_2$$

$$\begin{bmatrix} i_1 \\ j_1 \end{bmatrix} = \begin{bmatrix} i_2 \\ j_2 - 1 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad f_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_2 \\ j_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} i_2 \\ j_2 - 1 \end{bmatrix}$$

Integer Linear Programming (ILP) for Dependence Test

If we use the matrix vector notation $\langle F_1, f_1, B_1, b_1 \rangle$ and $\langle F_2, f_2, B_2, b_2 \rangle$ for two references at two iterations $\alpha: (i_1, j_1)$ and $\beta: (i_2, j_2)$

$$\begin{array}{l} i_1 = i_2 \\ j_1 = j_2 - 1 \\ 1 \leq i_1 \leq 100 \\ 1 \leq j_1 \leq 100 \\ 1 \leq i_2 \leq 100 \\ 1 \leq j_2 \leq 100 \end{array}$$

Loop bounds for (i_1, j_1)

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad b_1 = \begin{bmatrix} -1 \\ -1 \\ 100 \\ 100 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 100 \\ 100 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B_1 \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} + b_1 \geq 0$$

$$B_2 \begin{bmatrix} i_2 \\ j_2 \end{bmatrix} + b_2 \geq 0$$

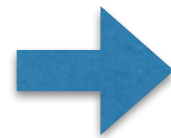
Loop bounds for (i_2, j_2)

$$B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad b_2 = \begin{bmatrix} -1 \\ -1 \\ 100 \\ 100 \end{bmatrix}$$

Putting Everything Together

If we use the matrix vector notation $\langle F_1, f_1, B_1, b_1 \rangle$ and $\langle F_2, f_2, B_2, b_2 \rangle$ for two references at two iterations $\alpha: (i_1, j_1)$ and $\beta: (i_2, j_2)$

$$\begin{aligned} i_1 &= i_2 \\ j_1 &= j_2 - 1 \\ 1 &\leq i_1 \leq 100 \\ 1 &\leq j_1 \leq 100 \\ 1 &\leq i_2 \leq 100 \\ 1 &\leq j_2 \leq 100 \end{aligned}$$



$$\begin{aligned} F_1 \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} + f_1 &= F_2 \begin{bmatrix} i_2 \\ j_2 \end{bmatrix} + f_2 \\ B_1 \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} + b_1 &\geq 0 \\ B_2 \begin{bmatrix} i_2 \\ j_2 \end{bmatrix} + b_2 &\geq 0 \end{aligned}$$

$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad f_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad f_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

B_1, b_1, B_2, b_2 see previous slides

Review: Parallelizing Affine Loops

Three spaces

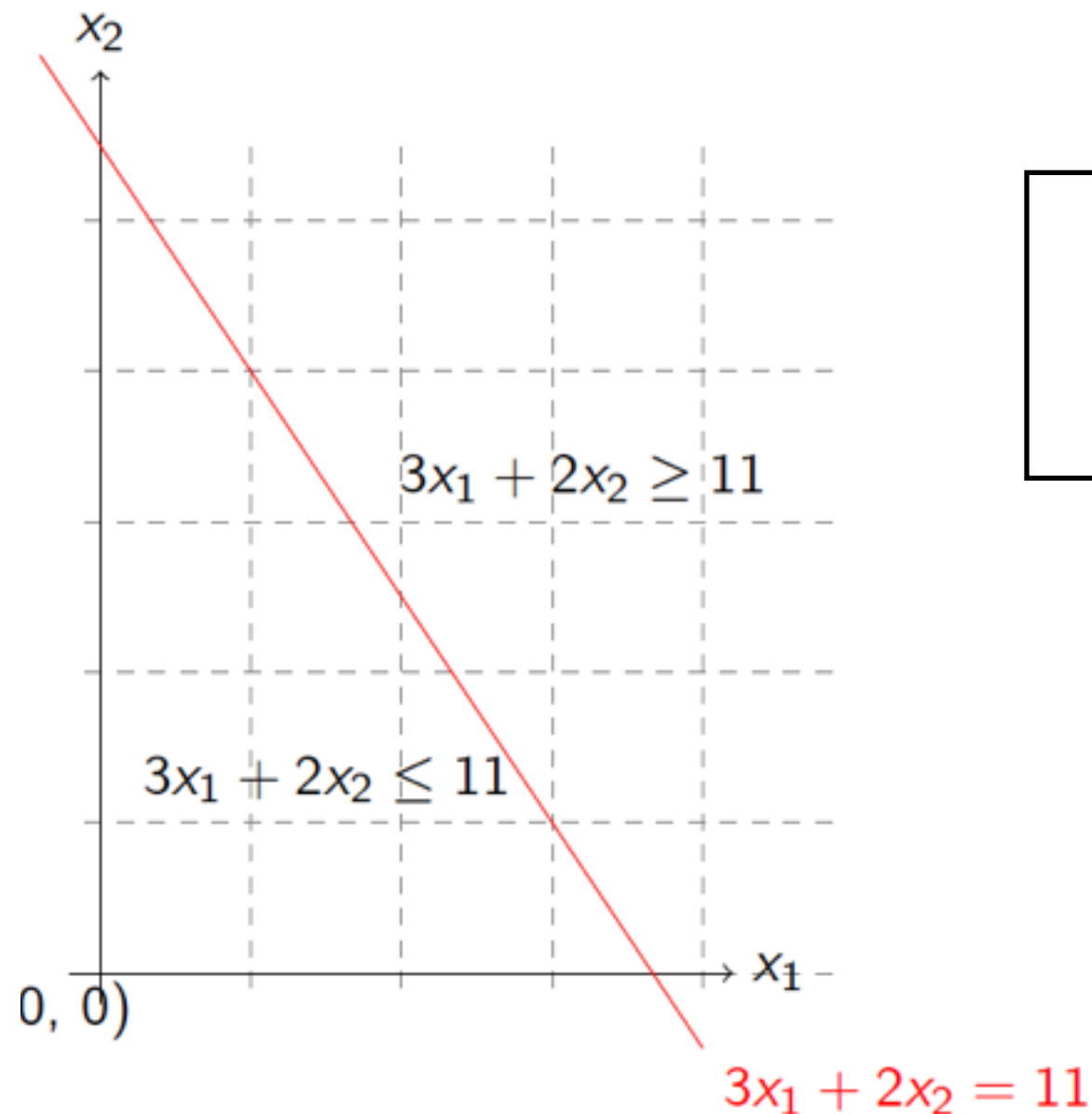
- Iteration space
 - The set of dynamic execution instances
For instance, the set of value vectors taken by loop indices
 - A k -dimensional space for a k -level loop nest
- Data space
 - The set of array elements accessed
 - An n -dimensional space for an n -dimensional array
- Processor space
 - The set of processors in the system
 - In analysis, we may pretend there are unbounded # of virtual processors

Affine Half Space

Definition

An affine half-space of \mathbb{Z}^d is defined as the set of points

$$\{ \vec{x} \in \mathbb{Z}^d \mid \vec{a} * \vec{x} \leq b \}$$



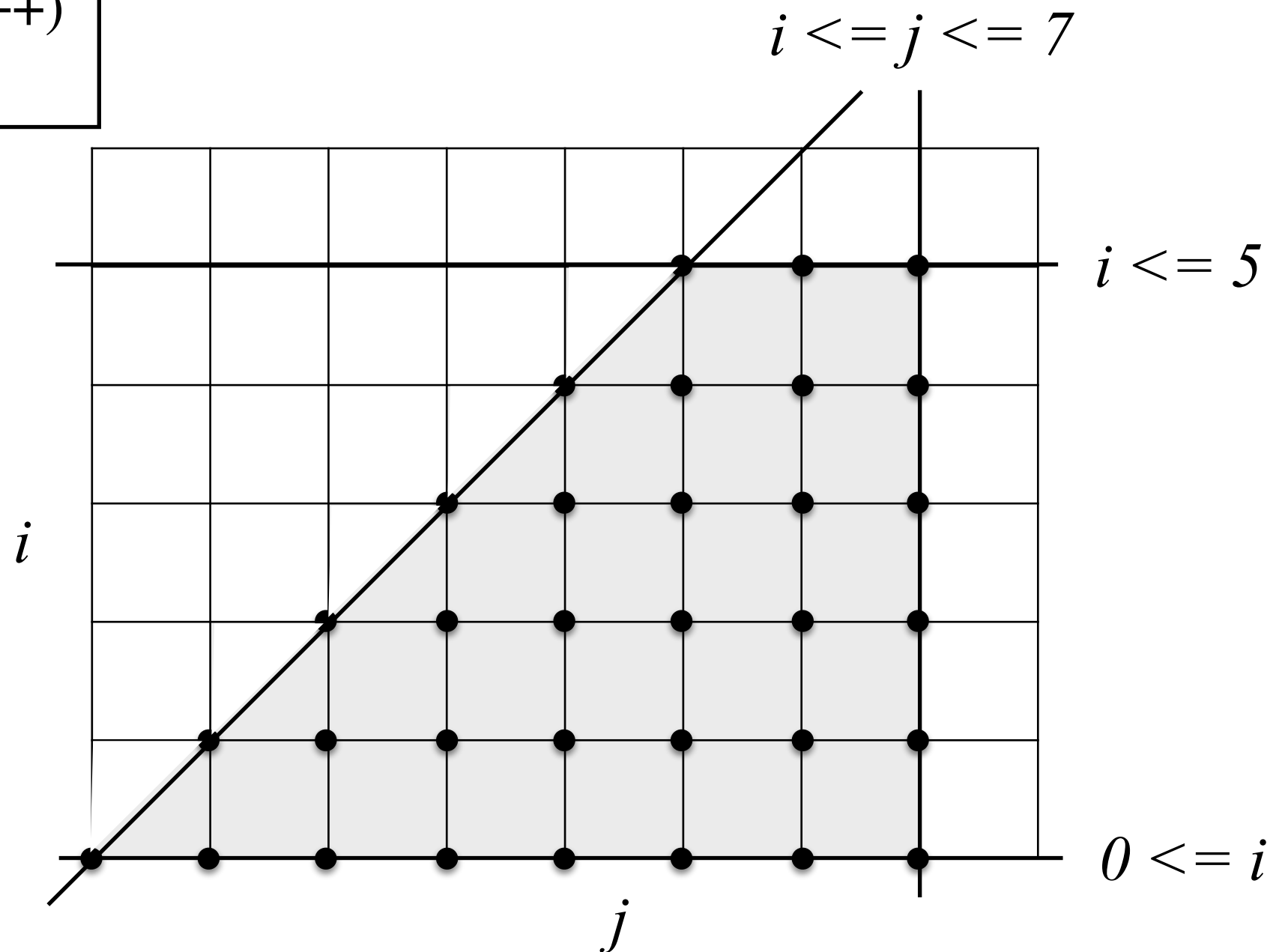
$\vec{a} * \vec{x} = b$ is the hyperplane that divides the d -dimensional space into two half-spaces.

Iteration Space

- Bounded by multiple hyperplanes

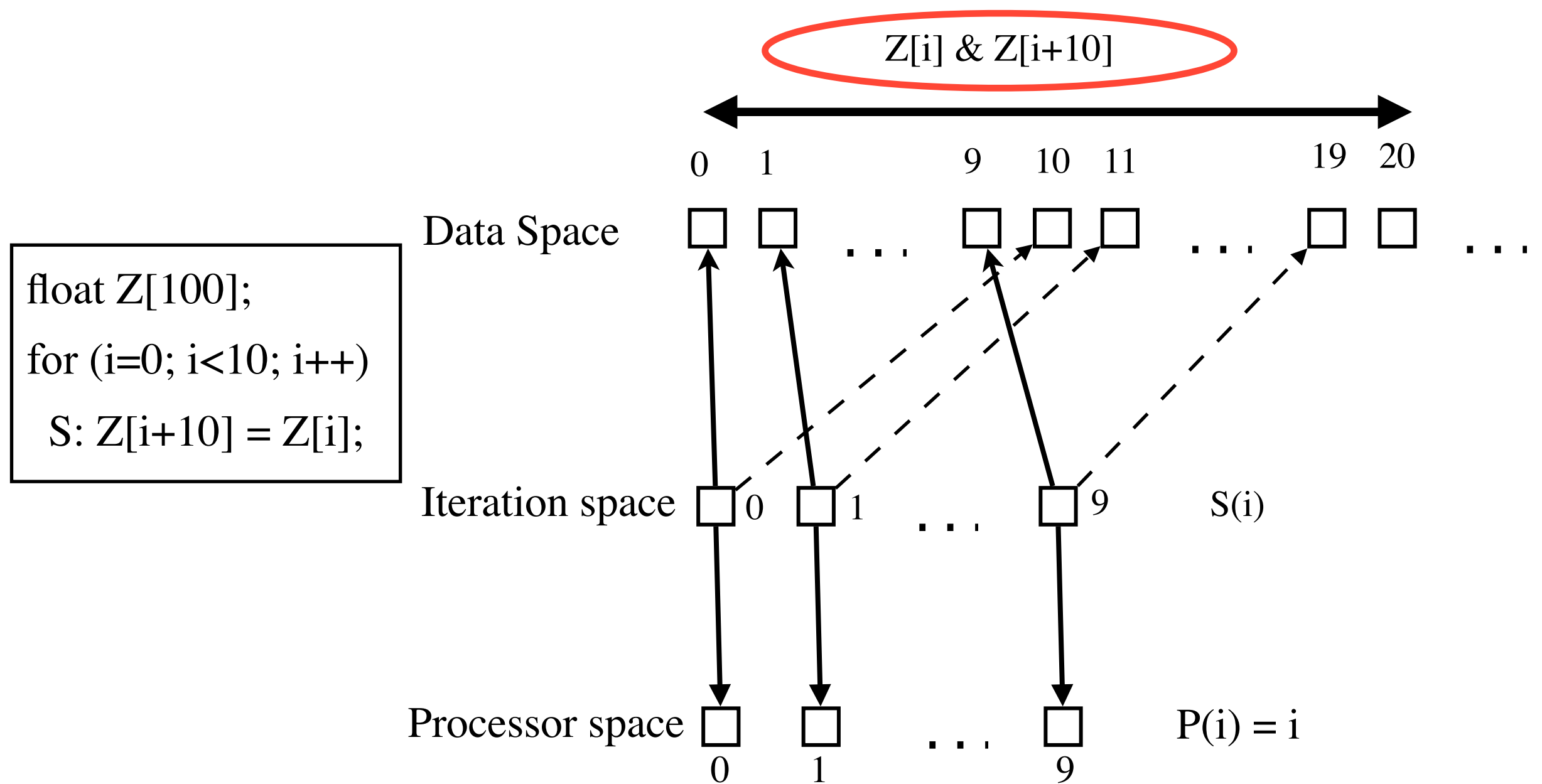
```
for (i=0; i<=5; i++)  
  for (j=i; j<=7; j++)  
    Z[j, i] = 0;
```

$$0 \leq i \leq 5$$
$$i \leq j \leq 7$$



Three Spaces

- Iteration space, data space, and processor space



Synchronization-free Parallelism

Parallelize an application **without** allowing any *communication* or *synchronization* among (logical) processors.

Example:

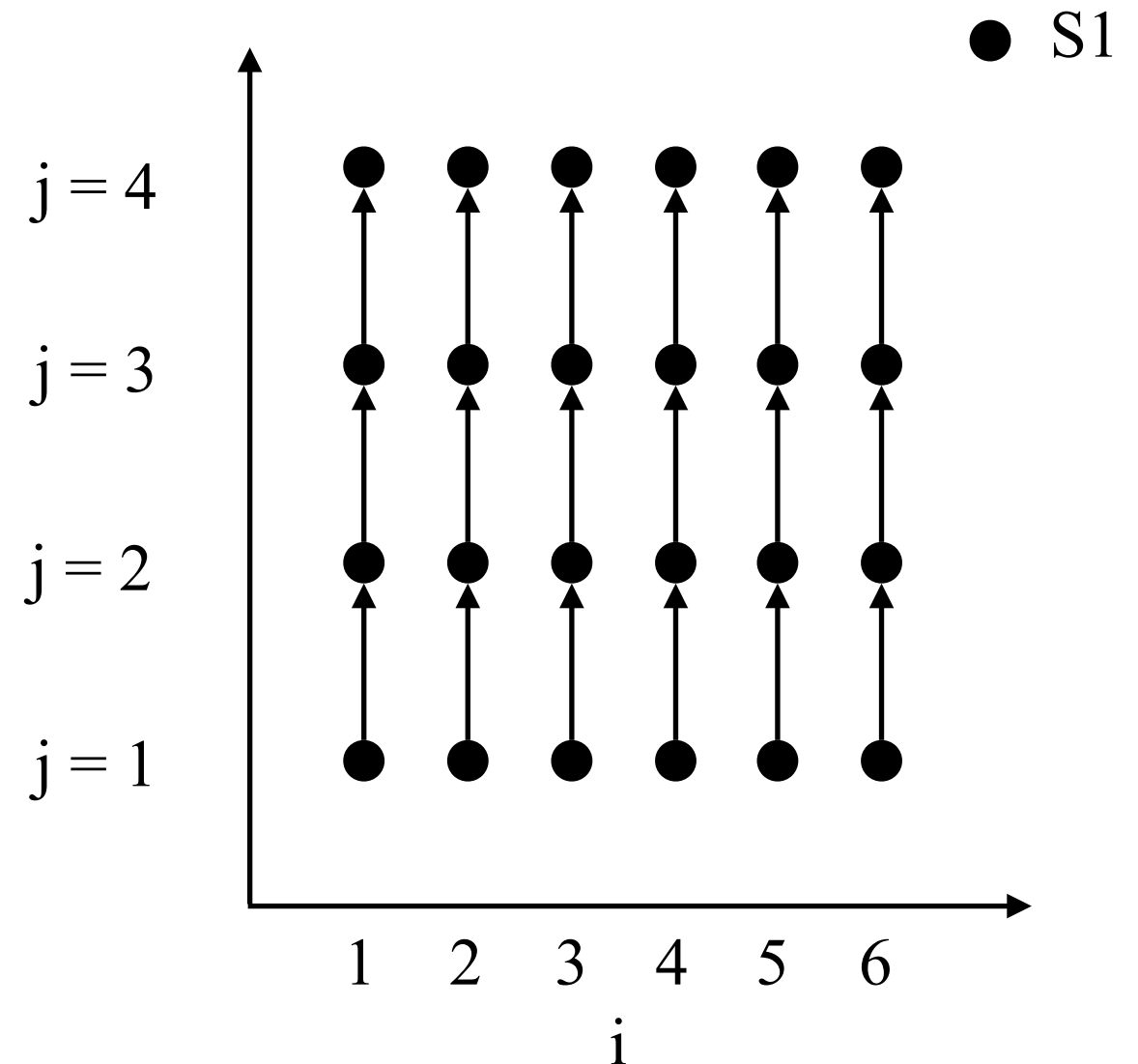
```
do i = 1, N
  do j = 1, N
    S1: A[i, j] = A[i, j - 1]
```

Write in $S_1(1,1)$ to Read in $S_1(1,2)$



Write in $S_1(i, j)$ to Read in $S_1(i, j+1)$

Dependence from S1 to S1



Synchronization-free Parallelism

Parallelize an application **without** allowing any *communication* or *synchronization* among (logical) processors.

Example:

```
do i = 1, N
  do j = 1, N
    S1: A[i, j] = A[i, j - 1]
```

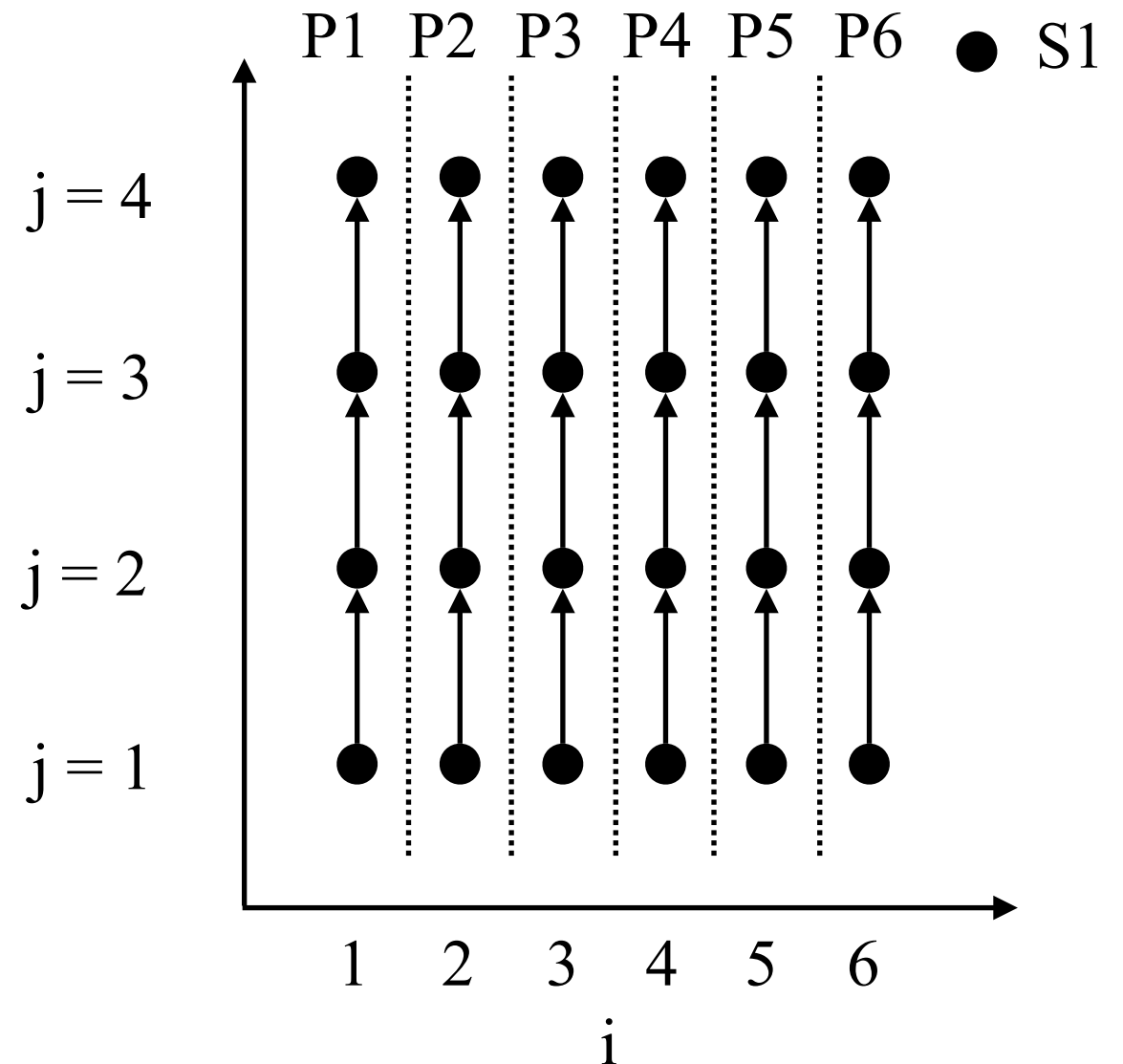
Write in $S_1(1,1)$ to Read in $S_1(1,2)$



Write in $S_1(i, j)$ to Read in $S_1(i, j+1)$

Dependence from S1 to S1

Communication is limited to the iterations on one processor.



Synchronization-free Parallelism

Parallelize an application **without** allowing any *communication* or *synchronization* among (logical) processors.

Example:

```
do i = 1, N
  do j = 1, N
    S1: A[i, j] = A[i, j - 1]
```

Write in $S_1(1,1)$ to Read in $S_1(1,2)$

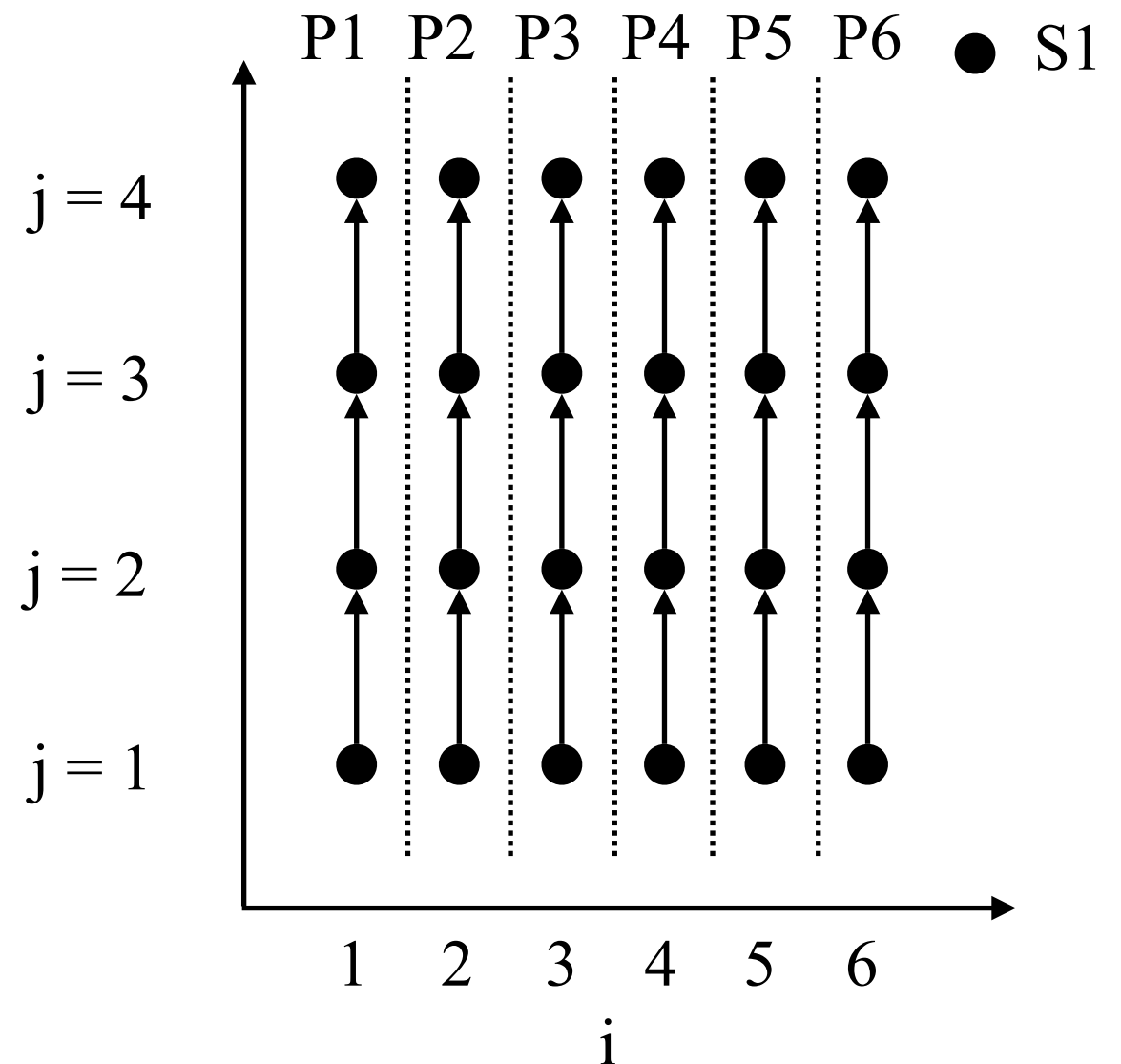


Write in $S_1(i, j)$ to Read in $S_1(i, j+1)$

Which loop can be parallelized?

The “i” loop or the “j” loop?

Answer: the “i” loop



Synchronization-free Parallelism

Parallelize an application **without** allowing any *communication* or *synchronization* among (logical) processors.

Example 1:

```
doall i = 1, N
  do j = 1, N
    S1: A[i, j] = A[i, j - 1]
```

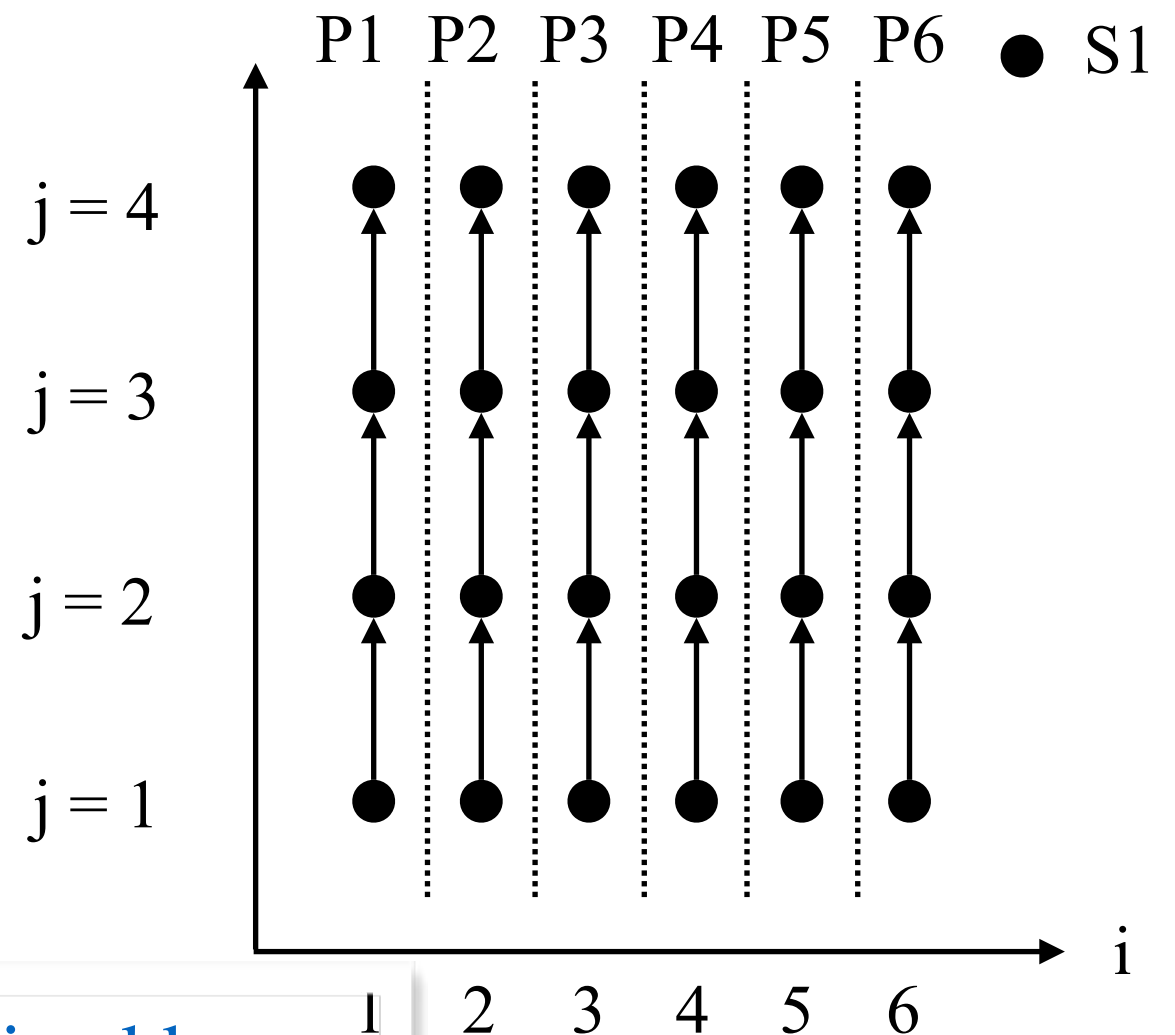
Write in $S_1(1,1)$ to Read in $S_1(1,2)$



Write in $S_1(i, j)$ to Read in $S_1(i, j+1)$

Dependence from **S₁(1,1)** to **S₂(1,2)**

The dependence chain is characterized by a **hyperplane**. In this case it is “ $i = \text{constant}$ ”.



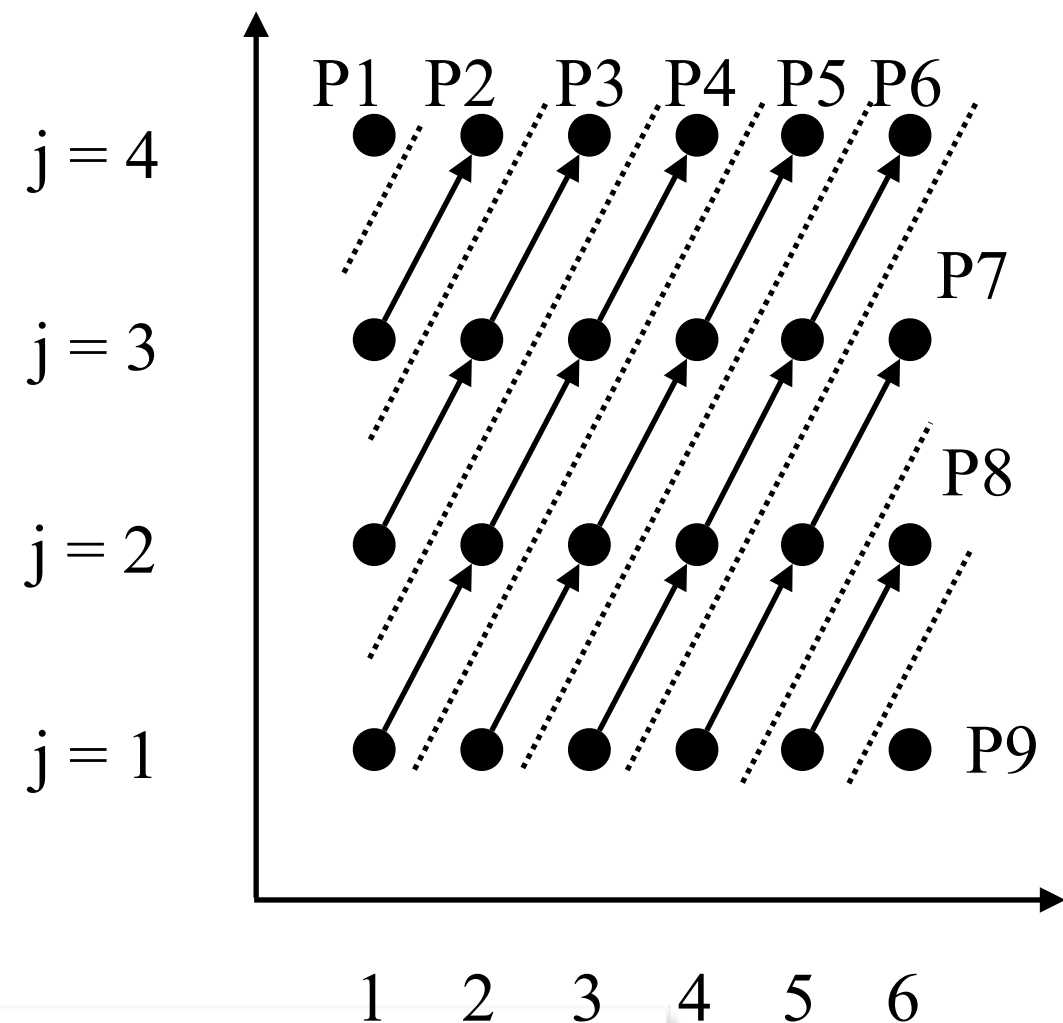
Synchronization-free Parallelism

Parallelize an application **without** allowing any *communication* or *synchronization* among (logical) processors.

Example 2:

```
do i = 1, N
  do j = 1, N
    S1: A[i, j] = A[i - 1, j - 1]
```

Dependence from **S₁(i,j)** to **S₁(i+1,j+1)**



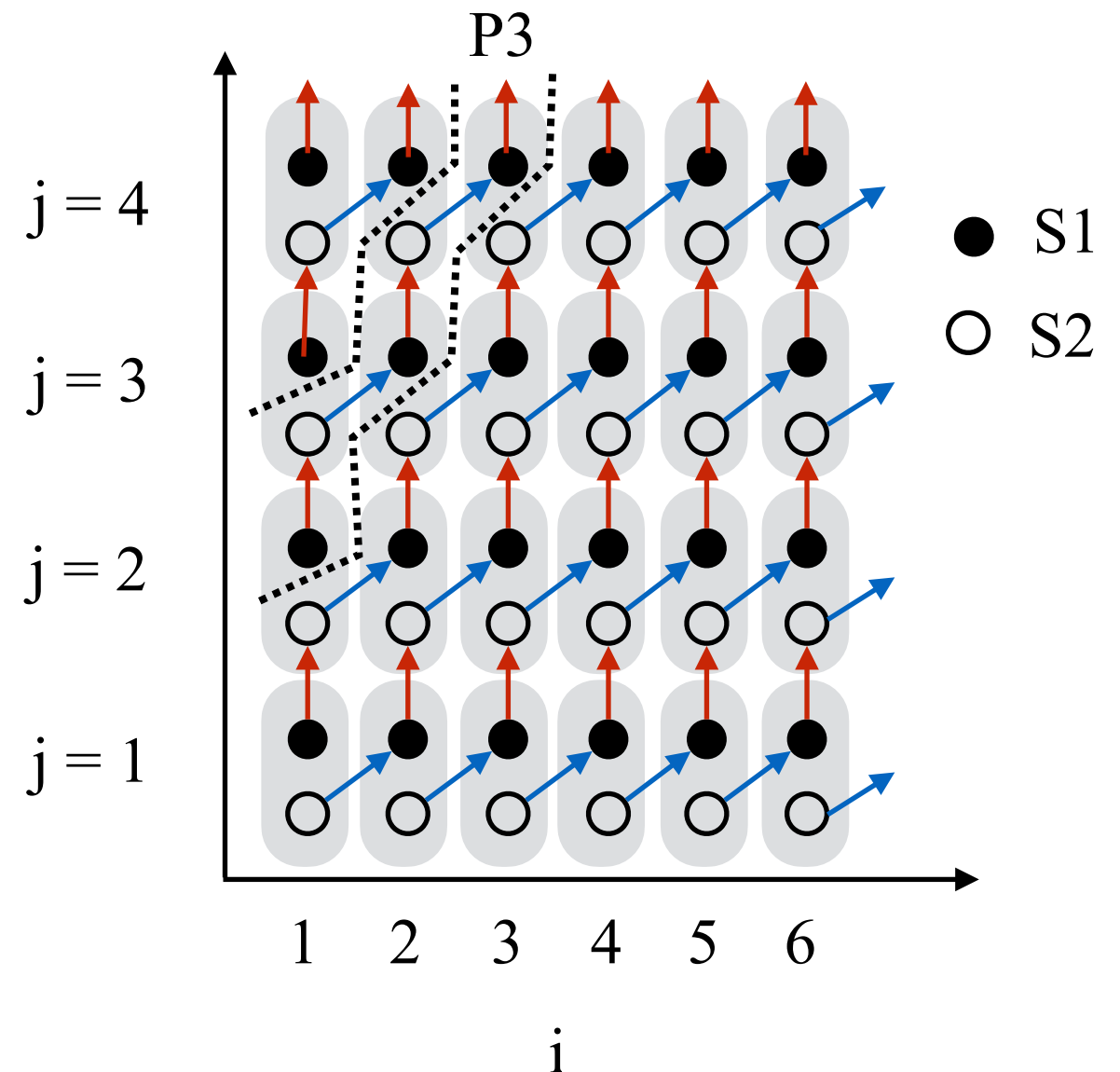
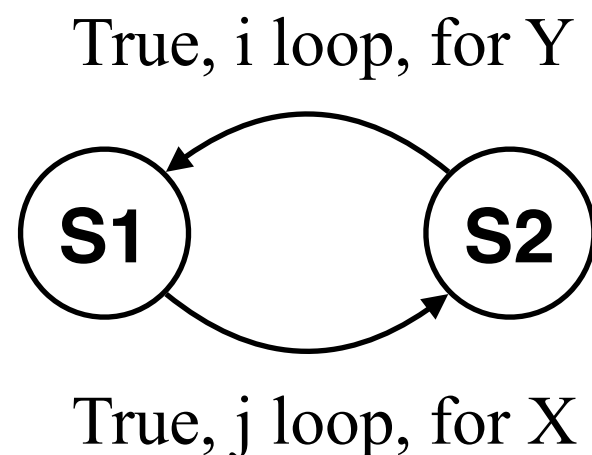
The dependence chain is characterized by a **hyperplane**. In this case it is “ $j - i + \text{constant} = 0$ ”.

Synchronization-free Parallelism

Parallelize an application **without** allowing any *communication* or *synchronization* among (logical) processors.

Example 3:

```
for (i=1; i<=100; i++)  
  for (j=1; j<=100; j++){  
    S1: X[i,j] = X[i,j] + Y[i-1, j];  
    S2: Y[i,j] = Y[i,j] + X[i, j-1];  
  }
```



Dependence from **S1(1,1)** to **S2(1,2)**

Dependence from **S2(1,1)** to **S1(2,1)**

Dependence and Parallelization

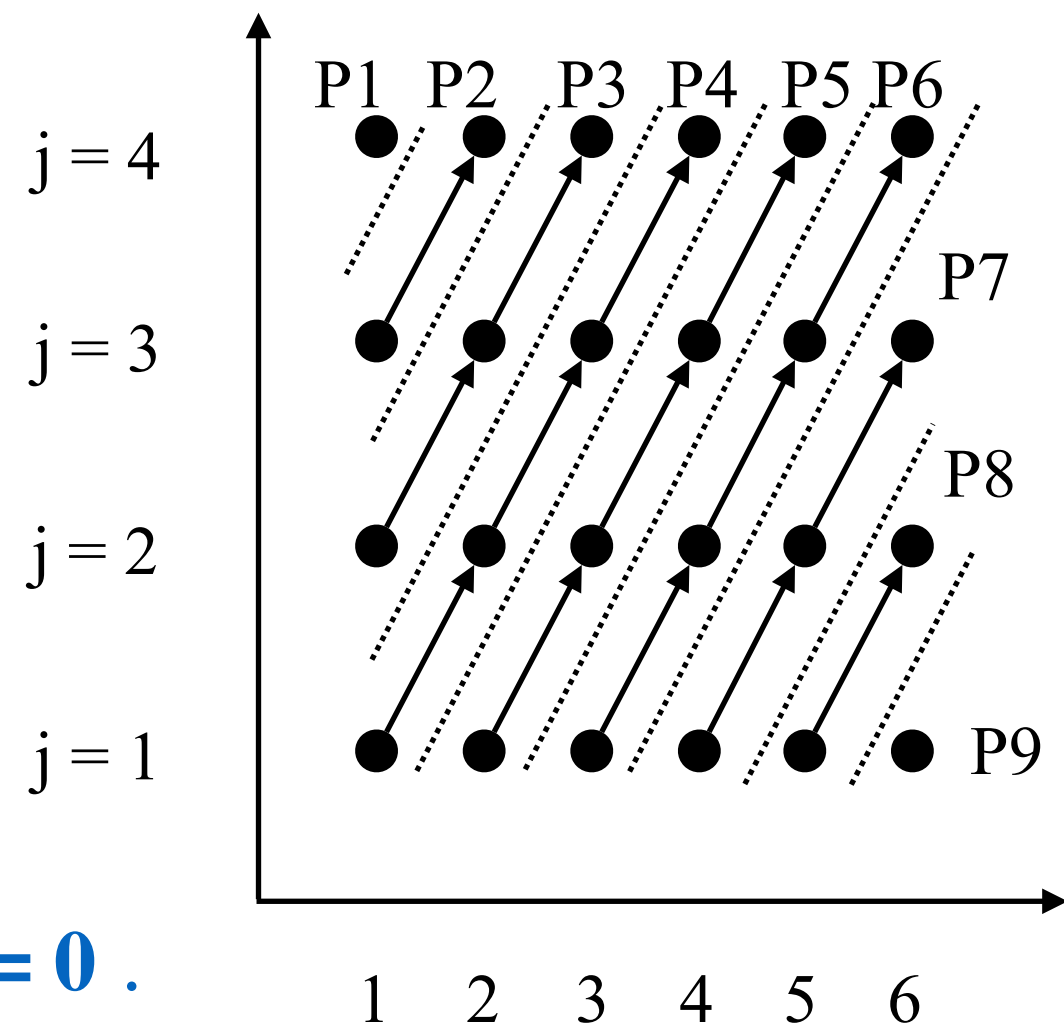
- Dependence chain in affine loops modeled as a hyperplane.
- Iterations along the same hyperplane must execute sequentially.
- **Iterations on different hyperplanes can execute in parallel.**

Example:

```
do i = 1, N
  do j = 1, N
    S1: A[i, j] = A[i - 1, j - 1]
```

Dependence from **S₁(i,j)** to **S₁(i+1,j+1)**

The hyperplane is **$j - i + \text{constant} = 0$** .



Processing Space: Affine Partition Schedule

- Map an iteration to a processor using $\langle \mathbf{C}, \mathbf{d} \rangle$

\mathbf{C} is a n by m matrix

- $m = d$ (the loop level)
- n is the dimension of the processor grid

$\vec{\mathbf{d}}$ is a n -element constant vector

$\vec{\mathbf{p}} = \mathbf{C} \vec{\mathbf{x}} + \vec{\mathbf{d}}$, where $\vec{\mathbf{x}}$ is an iteration vector

Processing Space: Affine Partition Schedule

- Map an iteration to a processor using $\langle \mathbf{C}, \mathbf{d} \rangle$

$$\vec{p} = \mathbf{C} \vec{x} + \vec{d}, \text{ where } \vec{x} \text{ is an iteration vector}$$

- Example

```
for (i=1; i<=N; i++)  
  S: Y[i] = Z[i];
```

$$\mathbf{C} = [1], \mathbf{d} = [0]$$

$$\begin{aligned} \vec{p}(S(i)) &= 1 * i + 0 \\ &= i \end{aligned}$$

Map iteration **i** to Processor **i**

Synchronization-free Parallelism

- Two memory references as $\langle F_1, f_1, B_1, b_1 \rangle$ and $\langle F_2, f_2, B_2, b_2 \rangle$ such that $\langle F_2, f_2, B_2, b_2 \rangle$ at iteration $\alpha: (i_1, j_1)$ depends on $\langle F_1, f_1, B_1, b_1 \rangle$ at iteration $\beta: (i_2, j_2)$
 - F_1 is a matrix and f_1 is a vector.
The affine memory access index is $F_1 * \alpha + f_1$.
 - B_1 is a matrix and b_1 is a vector.
The affine loop bounds can be expressed as $B_1 * \alpha + b_1 \geq 0$
- Let $\langle C_1, d_1 \rangle$ and $\langle C_2, d_2 \rangle$ represent the respective processor schedule, to have synchronization-free parallelism,

$$C_1 * \alpha + d_1 = C_2 * \beta + d_2$$

These two memory references must execute on the same processor (sequentially).

Synchronization-free Parallelism

- Two memory references as $\langle F_1, f_1, B_1, b_1 \rangle$ and $\langle F_2, f_2, B_2, b_2 \rangle$ such that $\langle F_2, f_2, B_2, b_2 \rangle$ at iteration (i_1, j_1) depends on $\langle F_1, f_1, B_1, b_1 \rangle$ at iteration (i_2, j_2)

```

for (i=1; i<=100; i++)
  for (j=1; j<=100; j++){
    S1: X[i,j] = X[i,j] + Y[i-1, j];
    S2: Y[i,j] = Y[i,j] + X[i, j-1];
  }
    
```



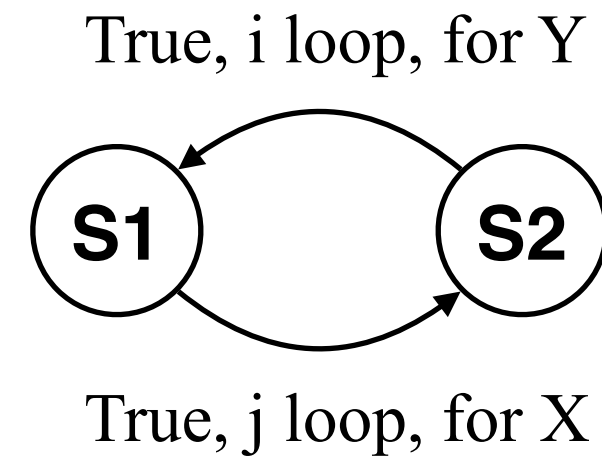
$$\begin{aligned}
 \mathbf{F}_1 \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} + f_1 &= \mathbf{F}_2 \begin{bmatrix} i_2 \\ j_2 \end{bmatrix} + f_2 \\
 \mathbf{B}_1 \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} + b_1 &\geq 0 \\
 \mathbf{B}_2 \begin{bmatrix} i_2 \\ j_2 \end{bmatrix} + b_2 &\geq 0
 \end{aligned}$$

- We want to find processor schedule $\langle \mathbf{C}_1, \mathbf{d}_1 \rangle$ and $\langle \mathbf{C}_2, \mathbf{d}_2 \rangle$ such that

$$\begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \end{bmatrix} \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} + \mathbf{d}_1 = \begin{bmatrix} \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \begin{bmatrix} i_2 \\ j_2 \end{bmatrix} + \mathbf{d}_2$$

Synchronization-free Parallelism

```
for (i=1; i<=100; i++)  
  for (j=1; j<=100; j++){  
    S1: X[i,j] = X[i,j] + Y[i-1, j];  
    S2: Y[i,j] = Y[i,j] + X[i, j-1];  
  }
```



$$\begin{aligned} 1 \leq i_1 \leq 100, \quad 1 \leq j_1 \leq 100, \\ 1 \leq i_2 \leq 100, \quad 1 \leq j_2 \leq 100, \\ i_1 = i_2, \quad j_1 = j_2 - 1, \end{aligned}$$

$$\begin{bmatrix} C_{11} & C_{12} \end{bmatrix} \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} + [d_1] = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix} \begin{bmatrix} i_2 \\ j_2 \end{bmatrix} + [d_2]$$



$$\begin{bmatrix} C_{11} - C_{21} & C_{12} - C_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} + [d_1 - d_2 - C_{22}] = 0$$

S1 to S2 dependence

$$C_{11} - C_{21} = 0$$

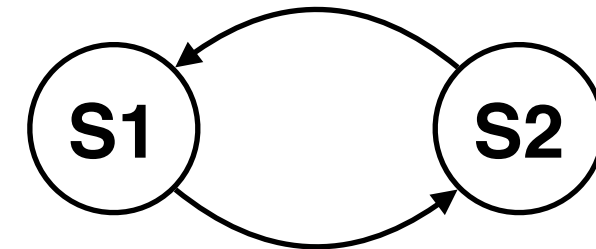
$$C_{12} - C_{22} = 0$$

$$d_1 - d_2 - C_{22} = 0$$

Synchronization-free Parallelism

```
for (i=1; i<=100; i++)
  for (j=1; j<=100; j++){
    S1: X[i,j] = X[i,j] + Y[i-1, j];
    S2: Y[i,j] = Y[i,j] + X[i, j-1];
  }
```

True, i loop, for Y



True, j loop, for X

$$1 \leq i_3 \leq 100, \quad 1 \leq j_3 \leq 100,$$

$$1 \leq i_4 \leq 100, \quad 1 \leq j_4 \leq 100,$$

$$i_3 - 1 = i_4, \quad j_3 = j_4,$$

$$\begin{bmatrix} C_{11} & C_{12} \end{bmatrix} \begin{bmatrix} i_3 \\ j_3 \end{bmatrix} + [d_1] = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix} \begin{bmatrix} i_4 \\ j_4 \end{bmatrix} + [d_2]$$



$$\begin{bmatrix} C_{11} - C_{21} & C_{12} - C_{22} \end{bmatrix} \begin{bmatrix} i_3 \\ j_3 \end{bmatrix} + [d_1 - d_2 + C_{21}] = 0$$

S2 to S1 dependence

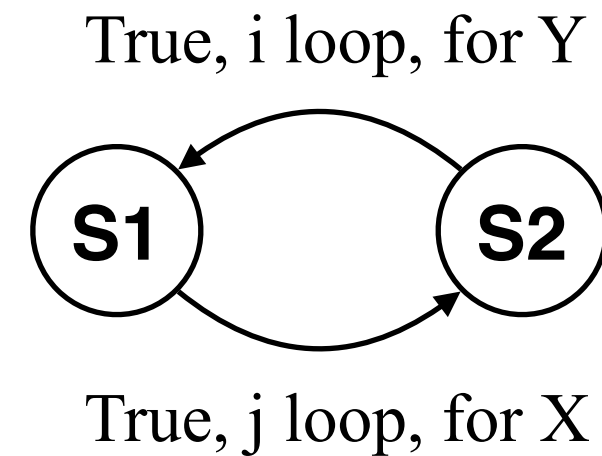
$$C_{11} - C_{21} = 0$$

$$C_{12} - C_{22} = 0$$

$$d_1 - d_2 + C_{21} = 0$$

Synchronization-free Parallelism

```
for (i=1; i<=100; i++)  
  for (j=1; j<=100; j++){  
    S1: X[i,j] = X[i,j] + Y[i-1, j];  
    S2: Y[i,j] = Y[i,j] + X[i, j-1];  
  }
```



S1 to S2 dependence

$$C_{11} - C_{21} = 0$$

$$C_{12} - C_{22} = 0$$

$$d_1 - d_2 - C_{22} = 0$$

S2 to S1 dependence

$$C_{11} - C_{21} = 0$$

$$C_{12} - C_{22} = 0$$

$$d_1 - d_2 + C_{21} = 0$$

$$C_{11} = C_{21} = -C_{22} = -C_{12} = d_2 - d_1$$

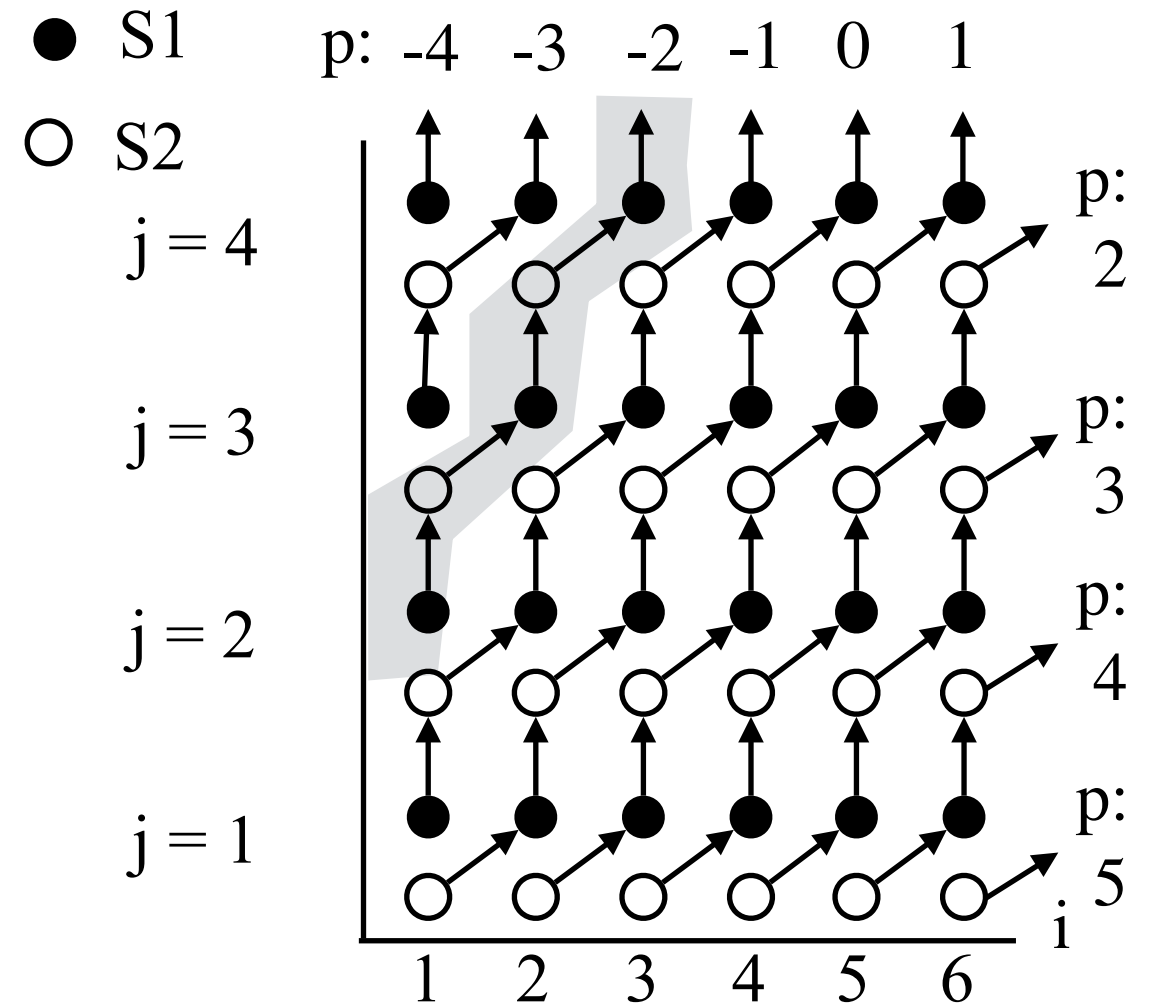
Synchronization-free Parallelism

Example:

```

for (i=1; i<=100; i++)
  for (j=1; j<=100; j++){
    S1:  $X[i,j] = X[i,j] + Y[i-1,j];$ 
    S2:  $Y[i,j] = Y[i,j] + X[i,j-1];$ 
  }
    
```

$$C_{11} = C_{21} = -C_{22} = -C_{12} = d_2 - d_1$$



One Potential Solution:

Affine schedule for S1, $p(S1)$: $[C_{11} \ C_{12}] = [1 \ -1]$, $d_1 = -1$

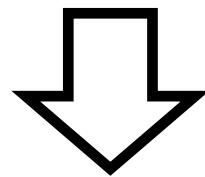
i.e. (i, j) iteration of S1 to processor $p = i - j - 1$;

Affine schedule for S2, $p(S2)$: $[C_{21} \ C_{22}] = [1 \ -1]$, $d_2 = 0$

i.e. (i, j) iteration of S2 to processor $p = i - j$.

Code Generation

```
for (i=1; i<=6; i++)  
  for (j=1; j<=4; j++){  
    X[i,j] = X[i,j] + Y[i-1, j];  /* S1 */  
    Y[i,j] = Y[i,j] + X[i, j-1];  /* S2 */  
  }
```



```
forall (p=-4; p<=5; p++)  
  for (i=1; i<=6; i++)  
    for (j=1; j<=4; j++){  
      if (p== i-j-1)  
        X[i,j] = X[i,j] + Y[i-1, j];  /* S1 */  
      if (p== i-j)  
        Y[i,j] = Y[i,j] + X[i, j-1];  /* S2 */  
    }
```

**S1(i, j): processor $p = i-j-1$;
S2(i, j): processor $p = i-j$.**

- Step 1: find processor ID ranges
 - S1: $-4 \leq p \leq 4$
 - S2: $-3 \leq p \leq 5$
 - Union: $-4 \leq p \leq 5$
- Step 2: generate code

Naive Code Generation

```
forall (p=-4; p<=5; p++)  
  for (i=1; i<=6; i++)  
    for (j=1; j<=4; j++){  
      if (p== i-j-1)  
        X[i,j] = X[i,j] + Y[i-1, j];  /* S1 */  
      if (p== i-j)  
        Y[i,j] = Y[i,j] + X[i, j-1];  /* S2 */  
    }
```

What are the issues with this code?

- Idle iterations
- Lots of tests in loop body

Remove Idle Iterations

Some iterations have idle operations

For example, when $p=-4$, only 1 of the 24 iterations has useful operations, $i = 1, j = 4$.

```
forall (p=-4; p<=5; p++)  
  for (i=1; i<=6; i++)  
    for (j=1; j<=4; j++){  
      if (p== i-j-1)  
        X[i,j] = X[i,j] + Y[i-1,j];  /* S1 */  
      if (p== i-j)  
        Y[i,j] = Y[i,j] + X[i,j-1];  /* S2 */  
    }
```

Make Loop Bounds Tighter

$$-4 \leq p \leq 5$$

$$1 \leq i \leq 6$$

$$1 \leq j \leq 4$$

$$i - p - 1 = j$$

$$-4 \leq p \leq 5$$

$$1 \leq i \leq 6$$

$$1 \leq j \leq 4$$

$$i - p = j$$

↓ Fourier-Motzkin Elimination

S1

$$j: \quad i - p - 1 \leq j \leq i - p - 1$$

$$1 \leq j \leq 4$$

$$i: \quad p + 2 \leq i \leq p + 5 \quad \leftarrow \text{Eliminate } j$$

$$1 \leq i \leq 6$$

$$p: \quad -4 \leq p \leq 4 \quad \leftarrow \text{Eliminate } i$$

$$j = i - p - 1$$

$$1 \leq i - p - 1 \leq 4$$

$$p + 1 + 1 \leq i \leq 4 + p + 1$$

$$p + 2 \leq i \leq p + 5$$

S2

$$j: \quad i - p \leq j \leq i - p$$

$$1 \leq j \leq 4$$

$$i: \quad p + 1 \leq i \leq 4 + p$$

$$1 \leq i \leq 6$$

$$p: \quad -3 \leq p \leq 5$$

Make Loop Bounds Tighter

S1

j: $i-p-1 \leq j \leq i-p-1$
 $1 \leq j \leq 4$
i: $p+2 \leq i \leq p+5$
 $1 \leq i \leq 6$
p: $-4 \leq p \leq 4$

S2

j: $i-p \leq j \leq i-p$
 $1 \leq j \leq 4$
i: $p+1 \leq i \leq 4+p$
 $1 \leq i \leq 6$
p: $-3 \leq p \leq 5$



Union result:

j: $i-p-1 \leq j \leq i-p$
 $1 \leq j \leq 4$
i: $p+1 \leq i \leq 5+p$
 $1 \leq i \leq 6$
p: $-4 \leq p \leq 5$

Make Loop Bounds Tighter

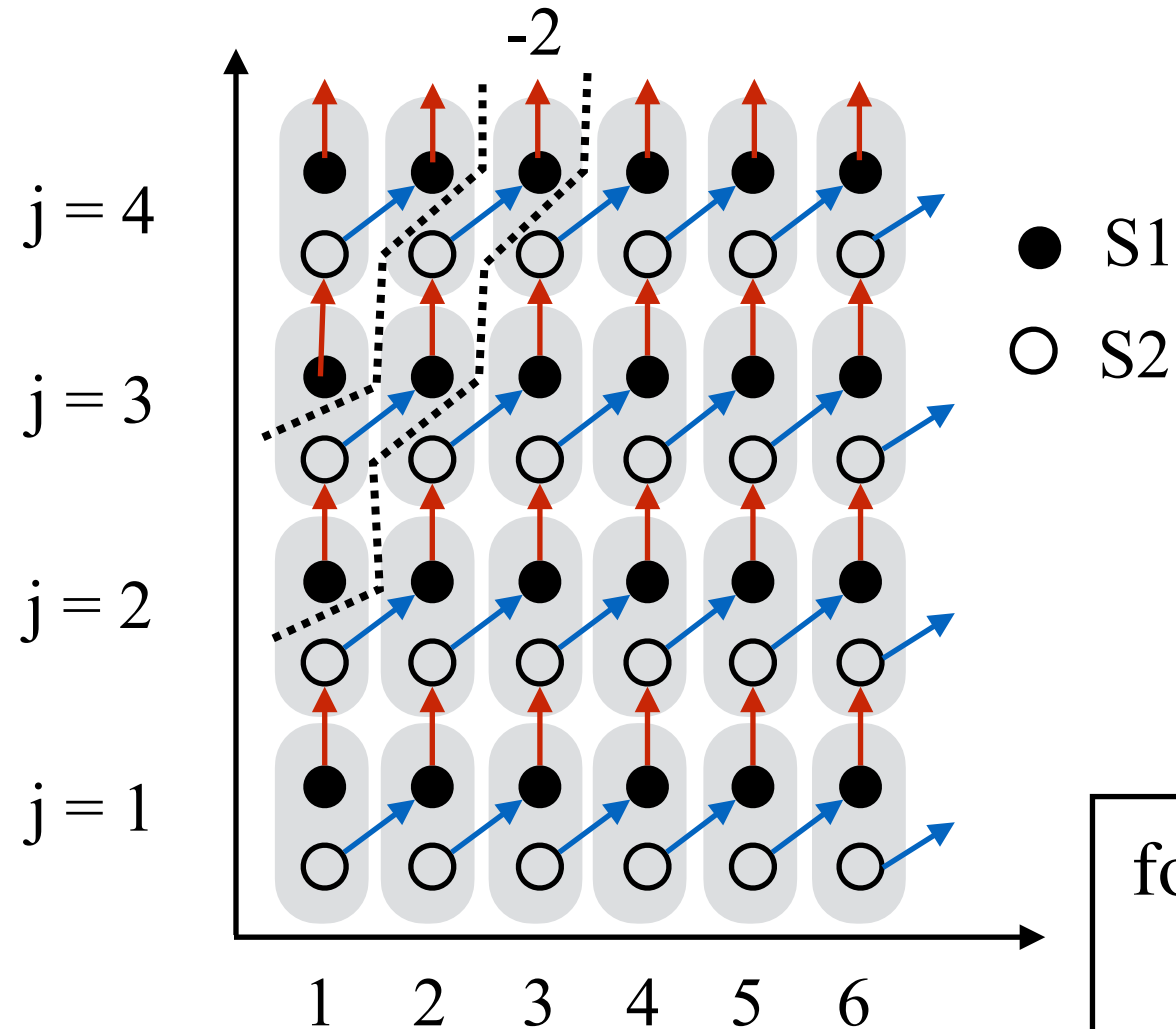
```
forall (p=-4; p<=5; p++)  
  for (i=1; i<=6; i++)  
    for (j=1; j<=4; j++){  
      if (p== i-j-1)  
        X[i,j] = X[i,j] + Y[i-1, j];  /* S1 */  
      if (p== i-j)  
        Y[i,j] = Y[i,j] + X[i, j-1];  /* S2 */  
    }
```

Union result:

j: $i-p-1 \leq j \leq i-p$
 $1 \leq j \leq 4$
i: $p+1 \leq i \leq 5+p$
 $1 \leq i \leq 6$
p: $-4 \leq p \leq 5$

```
for (p=-4; p<=5; p++)  
  for (i=max(1,p+1); i<=min(6,5+p); i++)  
    for (j=max(1,i-p-1); j<=min(4,i-p); j++){  
      if (p== i-j-1)  
        X[i,j] = X[i,j] + Y[i-1, j];  /* S1 */  
      if (p== i-j)  
        Y[i,j] = Y[i,j] + X[i, j-1];  /* S2 */  
    }
```


Make Loop Bounds Tighter



When $p = -2$,
 $i: [1, 3]$
 $j: [i+1, \min(4, i+2)]$

```
for (p=-4; p<=5; p++)
  for (i=max(1,p+1); i<=min(6,5+p); i++)
    for (j=max(1,i-p-1); j<=min(4,i-p); j++)
    {
      if (p== i-j-1)
        X[i,j] = X[i,j] + Y[i-1, j];  /* S1 */
      if (p== i-j)
        Y[i,j] = Y[i,j] + X[i, j-1];  /* S2 */
    }
```

Remove Tests

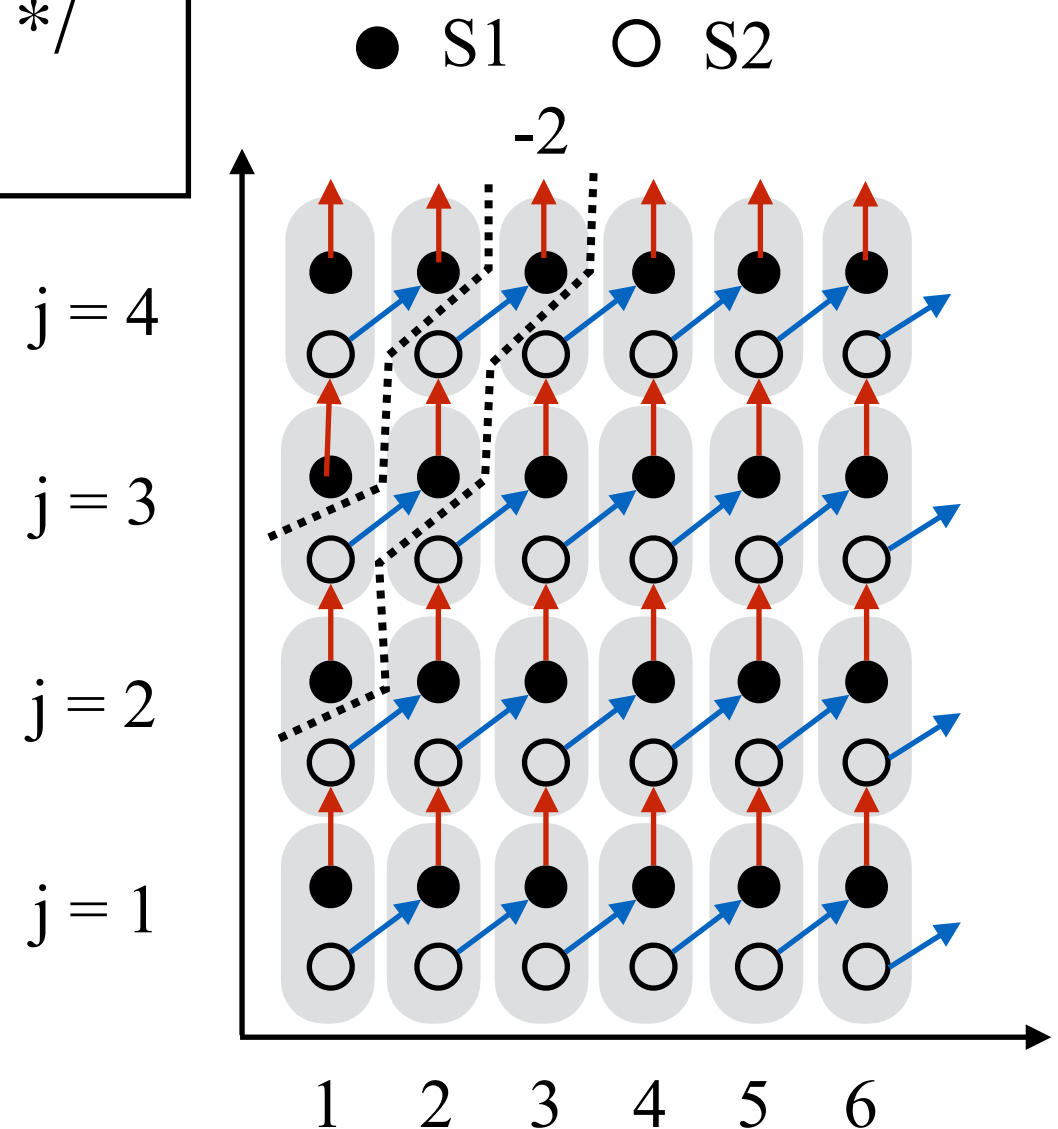
```

for (p=-4; p<=5; p++)
  for (i=max(1,p+1); i<=min(6,5+p); i++)
    for (j=max(1,i-p-1); j<=min(4,i-p); j++){
      if (p== i-j-1)
        X[i,j] = X[i,j] + Y[i-1, j];  /* S1 */
      if (p== i-j)
        Y[i,j] = Y[i,j] + X[i, j-1];  /* S2 */
    }
  
```

↓ ?

```

for (p=-4; p<=5; p++)
  for (i=max(1,p+1); i<=min(6,5+p); i++)
    j=i-p-1;
    X[i,j] = X[i,j] + Y[i-1, j];  /* S1 */
    j=i-p;
    Y[i,j] = Y[i,j] + X[i, j-1];  /* S2 */
  
```



Next Class

Reading

- ALSU, Chapter 11.1 - 11.7