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# Containment Control for a Directed Social Network with State-Dependent Connectivity

[1]Z. Kan, [1]J. Klotz, [2]E. L. Pasiliao Jr, [1]W. E. Dixon
[1]Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL 32611-6250, USA.
[2]Air Force Research Laboratory, Munitions Directorate, Eglin AFB, FL 32542, USA.
Email: kanzhen0322@ufl.edu, jklotz@ufl.edu, pasiliao@eglin.af.mil, wdixon@ufl.edu

***Abstract*— Social interactions influence our thoughts, opinions and actions. In this paper, social interactions are studied within a group of individuals composed of influential social leaders and follower groupies. Each person is assumed to maintain a social state, which can be an emotional state or an opinion on a social event. Followers update their own states based on the states of local neighbors, which are considered as reasonable, while the social leaders maintain a desired constant state. Social interactions are modeled as a directed graph where each directed edge represents an influence from one person to another. Motivated by the non-local property of fractional-order systems, the social response of individuals in the network are modeled by fractional-order dynamics whose states depend on influences from local neighbors and past experiences. A decentralized influence method is then developed to maintain existing social influence between individuals (i.e., without isolating peers in the group), and to influence the social group to a common desired state (i.e., within a convex hull spanned by social leaders). Mittag-Leffler stability methods are used to prove asymptotic stability of the networked fractional-order system.**

## I. INTRODUCTION[1]

Social interactions influence our thoughts, opinions, and actions. When making a decision or forming an opinion, individuals tend to communicate with parents, friends, or colleagues and take advice from those social peers. Some individuals, called leaders in this paper, in a social group may exhibit more powerful influences in others' decision making. The leader influence may either directly affect an individual or may percolate through a network of social peers and indirectly influence an individual. The influences which social leaders impose upon group behavior are prolific in the fields of politics, marketing and media, to name a few [1]–[3].

To study the social behavior of an individual, there is a growing interest in modeling psychological phenomena, including efforts to model the emotional response of different individuals [4]–[6]. In [4], a dynamic model of love is developed to describe the time-variation of emotions between individuals involved in a romantic relationship, and in [5] a set of differential equations are used to model the individual's happiness in response to external inputs. Since fractional-order differential equations exhibit a non-local property where the next state of a system not only depends upon its current state but also upon its historical states starting from the initial time [7], many researchers are motivated to explore various natural and social phenomena by using fractional-order systems. For instance, the work in [4] and [5] were revisited in [8] and [9], where the models of love and happiness were generalized to fractional-order dynamics by taking into account the fact that a person's emotional response is influenced by past experiences and memories. However, the models developed in [4], [5], [8], [9] only focus on an individual's emotional response, without considering the interaction with social peers where rapid and widespread influences from the social peers can prevail.

By modeling human emotional response as a fractional-order system, the influence of a person's emotions within a social network is studied, and emotion synchronization for a group of individuals is achieved in our recent work [10]. However, the emotion synchronization behavior in [10] only considers an undirected network structure: the one-sided influence of social leaders is not considered. A social leader (e.g., teacher in a class, star athletes, CEO in a company, or president of a country) may be able to influence a large number of people, while other individuals may have limited influence. Moreover, the interaction among individuals is generally not ubiquitous, and the underlying social network enables the influence to pass from powerful individuals to normal individuals. An interesting, less investigated, problem is how individuals will behave under the direct or indirect influence of social leaders in a network.

Containment control is a particular class of consensus problems for leader-follower networks, where the followers' states converge to a region determined by the leaders' states through local information exchange with neighboring agents. Some representative works are [11]–[13], where distributed containment control algorithms for agents with integer-order dynamics are developed such that the group of followers is driven to a convex hull spanned by multiple leaders' states under an undirected, directed or switching topology. However, a directed spanning tree is assumed to exist in [11]–[13], which indicates that network connectivity is assumed to be constant and the leaders can always influence the states of followers through the underlying network. Moreover, only constant weights on the influence from neighboring agents are considered.

In this work, following our previous effort in [10], the social

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group is modeled as a networked fractional-order system, where social response of each individual is described by fractional-order dynamics whose states depend on influences from social peers, as well as past experiences. Due to the consideration of social leaders, the network topology in [10] is extended to a directed graph, where the directed edges indicate the influence capability between two individuals (e.g., the leaders can influence the followers' state, but not vice versa.). The goal in this work is to develop a decentralized influence algorithm where individuals within a social group update their beliefs by considering beliefs from social peers and the social group achieves a desired common belief (i.e., the finally achieved agreement converges to a convex hull spanned by social leaders). Since an individual generally only considers others' beliefs as reasonable if their beliefs differ by less than a threshold, social difference is introduced to measure the closeness of the beliefs between individuals. In contrast to the constant weights considered in [11]–[13], the social difference is a time-varying weight which depends on individuals' current states. Due to the time-varying nature of such social interaction, an individual may not be influenced by others if his belief is far different from others'. In fact, extremist opinions may result in social isolation. Instead of assuming network connectivity (i.e., there always exists a path of influence between people) such as in [11]–[13], a challenge here is to maintain network connectivity to ensure consistent interaction among individuals (i.e., individuals can always be influenced by social peers, instead of being isolated from the social group) within a graph with time-varying weights. Moreover, when modeled as a networked fractional-order system, the development of a containment algorithm can be more challenging compared to the integer-order dynamics in [11]–[13], which can be considered as a particular case of generalized fractional-order dynamics. In addition, due to the time-varying weights considered, previous stability analysis tools such as examining the eigenvalues of linear time-invariant fractional-order systems (cf. [9], [14], [15]) are not applicable to the time-varying networked fractional-order system in this work. To address those challenges, a decentralized influence function is developed to achieve containment control for a networked fractional-order system while preserving a continued social interaction among individuals. Asymptotic convergence of the social beliefs to the convex hull spanned by leaders' states in the social network is then analyzed via LaSalles's invariance theorem [16], convex properties [17] and a Mittag-Leffler stability [18] approach.

## II. Problem Formulation

### A. Individual Social Behavior

Consider a social network composed of $n$ individuals. Each individual $i$ maintains a state $q_i(t) \in \mathbb{R}^d$ in a social network, which can be opinions on social events, or emotional states such as happiness, love, anger or fear. It is assumed that the current state $q_i(t)$ of an individual $i$ can be detected from other members (i.e., social neighbors such as close friends or family) in the social network. Generally, the opinions or emotional states formed by individuals about economic, political, and social issues are not only influenced by the information gathered through communication with their friends, family, colleagues, and media sources, but also depend on the personal experiences of individuals. To capture the evolution of individual social states by taking into account not only peer influences but also their own character (e.g., past experience, memory), $q_i(t)$ is modeled as the solution to a fractional-order dynamic as

$$ {}_{0}^{C}D_t^{\alpha} q_i(t) = u_i(t), \ i = 1, \cdots, n, \tag{1}$$

where $u_i \in \mathbb{R}^d$ denotes an influence (i.e., control input) over the social state, and ${}_{0}^{C}D_t^{\alpha} q_i(t)$ is the $\alpha^{th}$ derivative of $q_i(t)$ with $\alpha \in (0, 1]$. In contrast to the integer derivative of a function, which is only related to nearby points, a fractional-order derivative involves all historical points. Caputo and Riemann-Liouville (R-L) fractional derivatives are the two most widely used fractional operators [7]. Since the R-L fractional operator requires a fractional-order initial condition, which can be difficult to interpret [19], the Caputo fractional operator is used in (1).

Note that the model in (1) is a heuristic approximation to a social response, which indicates that a person's social state has a direct relationship with external influence integrated over the history of a person's emotional states. On-going efforts by the scientific community are focused on the development of clinically derived models; yet, to date, there is no widely accepted model to describe a person's social response in a social network. Stability of the solutions to (1) are defined by the M-L function as follows [18].

*Definition 1:* (Mittag-Leffler Stability) Consider a fractional order system ${}_{t_0}^{C}D_t^{\alpha} x(t) = f(t, x)$ with initial condition $x(t_0)$, where $\alpha \in (0, 1]$ and $f(t, x)$ is piecewise continuous in $t$ and locally Lipschitz in $x$. The Mittag-Leffler (M-L) function given by $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}$, where $\alpha,\ \beta > 0$ and $z \in \mathbb{C}$, is frequently used in solutions of fractional-order systems. The solution to the fractional order system is said to be Mittag-Leffler stable if $\|x(t)\| \leq \{m[x(t_0)] E_{\alpha,1}(-\lambda(t-t_0)^{\alpha})\}^{b}$, where $t_0$ is the initial time, $\alpha \in (0, 1)$, $b > 0$, $\lambda > 0$, $m(0) = 0$, $m(x) \geq 0$, $m(x)$ is locally Lipschitz, and $E_{\alpha,1}$ is defined in $E_{\alpha,\beta}$ with $\beta = 1$.

### B. Social Interaction

Graph theory (see cf. [20]) is used to describe the interaction among individuals in a social network. Let $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ denote a directed graph, where the node set $\mathcal{V} = \{1, \cdots, n\}$ represents individuals and the edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ represents the interactions between individuals in a social network. Suppose that there exist $m$ dynamic followers $\mathcal{V}_F = \{1, \cdots, m\}$ and $n - m$ stationary leaders $\mathcal{V}_L = \{m+1, \cdots, n\}$, satisfying $\mathcal{V}_L \cup \mathcal{V}_F = \mathcal{V}$. For each stationary leader, it is assumed that their states $q_i \in \mathbb{R}^d$, $i \in \mathcal{V}_L$, are desired and immutable. For each dynamic follower, its state $q_i(t)$, $i \in \mathcal{V}_F$, evolves according to the dynamics (1) under the influence from both followers and leaders directly or indirectly by the underlying network.

A directed edge $(i,j)\in\mathcal{E}$ in $\mathcal{G}$ represents the neighborhood of node $i$ and $j$, which indicates that node $i$ can be influenced by node $j$, but not vice versa. Each edge $(i,j)$ is associated with a time-varying weighting factor called social difference $S_{ij}\in\mathbb{R}^{+}$, which is defined as

$$S_{ij}=\left\|q_i-q_j\right\|^2. \quad (2)$$

Since individuals are assumed to be rational in the sense that they fail to incorporate the information provided by neighbors whose states are far from their own, the social difference designed in (2) aims to capture the closeness of the states between two neighboring nodes $i$ and $j$. It is also assumed that there exists a threshold $\delta\in\mathbb{R}^{+}$, and individuals $i$ and $j$ are able to influence each other only when their social difference $S_{ij}\leq\delta$. In other words, an edge $(i,j)$ in graph $\mathcal{G}$ does not exist if the social difference $S_{ij}$ is greater than the threshold $\delta$. The neighbors of individual $i$ in graph $\mathcal{G}$ are defined as $\mathcal{N}_i=\{j\mid S_{ij}\leq\delta\}$, which determines a set of individuals who can influence the social states of individual $i$. A directed path from node $v_1$ to node $v_k$ is a sequence of edges $(v_1,v_2)$, $(v_2,v_3)$, $\cdots$, $(v_i,v_k)$ in the directed graph. If graph $\mathcal{G}$ contains a directed tree, every node has exactly one parent node except for one node, called the root, and the root has directed paths to every other node in graph $\mathcal{G}$. The subsequent assumption is required.

**Assumption 1:** For each follower $i\in\mathcal{V}_F$, there exists at least one leader that has a directed path to the follower $i$ in the initial graph $\mathcal{G}(0)$.

Assumption 1 implies that there exists a directed tree for the initial graph $\mathcal{G}(0)$, where the set of leaders acting as the roots in the directed tree has an influence directly or indirectly on all followers through a series of directed paths in the network. Note that a connected graph (i.e., a directed tree structure) is only assumed in the initial graph, and the controller developed in the subsequent section will preserve the network connectivity to ensure consistent influence between social neighbors.

### C. Objectives

*Definition 2:* [17] For a set of points $X\triangleq\{x_1,\cdots,x_n\}$, the convex hull $Co(X)$ is defined as the minimal set containing all points in $X$, satisfying that $Co(X)\triangleq\left\{\sum_{i=1}^{n}\alpha_i x_i\middle|x_i\in X,\ \alpha_i>0,\ \sum_{i=1}^{n}\alpha_i=1\right\}$.

*Definition 3:* [16] A state $x(t)$ approaches a set $M$ as $t$ goes to infinity (i.e., $x(t)\to M$ as $t\to\infty$), if for each $\varepsilon>0$ there exists a $T>0$ such that $dist(x(t),M)<\varepsilon$ for $t>T$, where $dist(p,\ M)$ denotes the distance from a point $p$ to a set $M$. More precisely, $dist(p,M)=\inf_{y\in M}\|p-y\|$, which is the smallest distance from $p$ to any point in $M$.

Let $\mathbf{q}(t)$, $\mathbf{q}^F(t)$, and $\mathbf{q}^L(t)$ denote the stacked vector of all states $q_i(t)$, $i\in\mathcal{V}$, the followers' states $q_i(t)$, $i\in\mathcal{V}_F$, and the leaders' states $q_i(t)$, $i\in\mathcal{V}_L$, respectively. The convex hull spanned by the states of leaders, and all states (i.e., both leaders and followers), are then represented as $Co\left(\mathbf{q}^L(t)\right)$ and $Co(\mathbf{q}(t))$, respectively. Since the leaders' states are assumed to be static (i.e., they are pushing an agenda), the convex hull $Co\left(\mathbf{q}^L(t)\right)$ is constant, while the convex hull $Co(\mathbf{q}(t))$ is time varying and depends on the states of the followers. After formulating the social network as a networked fractional-order system described by (1), one objective is to regulate the states of followers to a desired region, which is a convex hull spanned by all stationary leaders' states (i.e., $q_i(t)\to Co(\mathcal{V}_L)\ \forall i\in\mathcal{V}_F$). To ensure that each individual is able to be influenced by social leaders through a path of directed edges by communication with their local neighbors only, another goal is to preserve the network connectivity for the underlying social network (i.e., maintain the social difference $S_{ij}\leq\delta$ so that peers remain peers) when given an initially connected graph $\mathcal{G}$. Since the systems in (1) along different directions are decoupled, for the simplicity of presentation, only a scalar system ($d=1$), that is $q_i(t)\in\mathbb{R}$, is considered in the following analysis. However, the results are valid for a $d$ dimensional case by the introduction of the Kronecker product.

## III. Distributed Influence Design

The artificial potential field based approach is one of the most widely used methods in the control of multi-agent systems, which consists of an attractive potential encoding the control objective and a repulsive potential representing the motion constraints (cf. [21], [22]). To apply the potential field based approach to a social network problem, following a similar idea in our previous efforts in [10], [23], a decentralized potential function $\varphi_i:\mathbb{R}^d\to[0,1]\ \forall i\in\mathcal{V}_F$ is developed to influence the followers' states to a desired end as

$$\varphi_i=\frac{\gamma_i}{\left(\gamma_i^k+\beta_i\right)^{1/k}},\ i\in\mathcal{V}_F \quad (3)$$

where $k\in\mathbb{R}^{+}$ is a tuning parameter, $\gamma_i:\mathbb{R}^d\to\mathbb{R}^{+}$ is the goal function, and $\beta_i:\mathbb{R}^d\to\mathbb{R}^{+}$ is a constraint function. The goal function in (3) is designed as

$$\gamma_i=\sum\nolimits_{j\in\mathcal{N}_i}\frac{1}{2}\left\|q_i-q_j\right\|^2, \quad (4)$$

which aims to achieve consensus for node $i$ with its neighbors $j\in\mathcal{N}_i$. To ensure consistent influence from neighbors (i.e., maintain the social difference $S_{ij}\leq\delta$), the constraint function in (3) is designed as

$$\beta_i=\prod\nolimits_{j\in\mathcal{N}_i}\frac{1}{2}b_{ij}, \quad (5)$$

where $b_{ij}=\delta-S_{ij}\in\mathbb{R}^{+}$ with $S_{ij}$ defined in (2), and $b_{ij}=0$ if $S_{ij}\geq\delta$. For an existing interaction between individuals $i$ and $j$, the potential function $\varphi_i$ in (3) will approach its maximum whenever the constraint function $\beta_i$ decreases to 0 (i.e., the social difference $S_{ij}$ increases to the threshold of $\delta$).

Based on the definition of the potential function in (3), the distributed influence algorithm for each follower is designed as

$$u_i=-K_i\nabla_{q_i}\varphi_i,\ i\in\mathcal{V}_F \quad (6)$$

where $K_i$ is a positive gain and $\nabla_{q_i}\varphi_i$ denotes the gradient of $\varphi_i$ with respect to $q_i$. Applying (6) to (1), the closed-loop dynamics of social response for all individuals in a social network can be obtained as

$$\begin{cases} {}_0^C D_t^\alpha q_i(t) = -K_i \nabla_{q_i}\varphi_i & i \in \mathcal{V}_F \\ {}_0^C D_t^\alpha q_i(t) = 0 & i \in \mathcal{V}_L \end{cases}. \tag{7}$$

In (7), since leaders' states are stationary, the input to leaders is zero, and $\nabla_{q_i}\varphi_i$ can be computed as

$$\nabla_{q_i}\varphi_i = \frac{k\beta_i \nabla_{q_i}\gamma_i - \gamma_i \nabla_{q_i}\beta_i}{k(\gamma_i^k + \beta_i)^{\frac{1}{k}+1}}. \tag{8}$$

From (4) and (5), $\nabla_{q_i}\gamma_i$ and $\nabla_{q_i}\beta_i$ in (8) can be determined as

$$\nabla_{q_i}\gamma_i = \sum\nolimits_{j\in\mathcal{N}_i} (q_i - q_j) \tag{9}$$

and

$$\nabla_{q_i}\beta_i = -\sum\nolimits_{j\in\mathcal{N}_i} \bar{b}_{ij} (q_i - q_j), \tag{10}$$

respectively, where $\bar{b}_{ij} \triangleq \prod_{l\in\mathcal{N}_i, l\neq j} b_{il} \in \mathbb{R}^+$. Substituting (9) and (10) into (8), $\nabla_{q_i}\varphi_i$ is rewritten as

$$\nabla_{q_i}\varphi_i = -\sum\nolimits_{j\in\mathcal{N}_i} m_{ij} (q_i - q_j), \tag{11}$$

where

$$m_{ij} = \frac{k\beta_i + \bar{b}_{ij}\gamma_i}{k(\gamma_i^k + \beta_i)^{\frac{1}{k}+1}} \tag{12}$$

is non-negative, based on the definition of $\gamma_i$, $\beta_i$, $k$, and $\bar{b}_{ij}$.

## IV. Convergence Analysis

To show that the followers in the fractional-order network converge to a convex hull spanned by the static leaders' states, the following analysis is segregated into three proofs. The first proof shows that the existing interaction between individuals is maintained by the influence function designed in (6) (i.e., maintain the social difference $S_{ij} \leq \delta$ for all time), and thus the connectivity for the social network is preserved. The second proof shows the exponential convergence to the convex hull for an integer-order representation of the dynamic system in (1), which is then used to establish the asymptotic convergence to the equilibrium set of consensus states for the fractional-order system by using Mittag-Leffler stability analysis in the third proof.

### A. Maintenance of Social Influence

A tree structure in the directed graph $\mathcal{G}(t)$ ensures that each follower is directly or indirectly influenced by at least one leader in the network. If $\mathcal{G}(t)$ does not have a directed tree, there must exist a follower to which no leader have a path to influence the follower's states. Hence, the state of the follower is independent of the influence of leaders, and thus can not converge to the stationary convex hull spanned by leaders. To ensure the continued influence from leaders to all followers, a directed tree structure must be maintained. The following theorem shows that, given an initial graph in Assumption 1, the tree structure will be preserved under the influence function in (6) (i.e., network connectivity is maintained and social peers do not become isolated from the social group).

*Theorem 1: The influence function in (6) guarantees a directed tree structure in $\mathcal{G}$ for all time.*

*Proof:* To show every existing edge in the directed tree assumed in $\mathcal{G}(0)$ is preserved, consider a state $q_0$ for individual $i$, where the interaction between individual $i$ and neighbor $j \in \mathcal{N}_i$ decreases to zero (i.e., $b_{ij}(q_0, q_j) \to 0$), which indicates that their social difference is too large to influence each others' opinion, and the associated edge is about to break. From (5), $\beta_i = 0$ when $b_{ij} = 0$, and the navigation function $\varphi_i$ achieves its maximum value from (3). Since $\varphi_i$ is maximized at $q_0$, no open set of initial conditions can be attracted to $q_0$ under the negative gradient control law designed in (6). Therefore, the social bond between individual $i$ and $j$ is maintained less than $\delta$ by (6), and the associated edge is also maintained. Repeating this argument for every pair, every edge in $\mathcal{G}$ is maintained and the directed tree structure is preserved. ∎

### B. Convergence Analysis

*Lemma 1:* [24] *Consider a linear system*

$$\dot{x}(t) = A(t)\,x(t), \tag{13}$$

*and a Lyapunov function* $V(x) = \max\{x_1, \cdots, x_n\} - \min\{x_1, \cdots, x_n\}$, *where* $x(t) = [x_1, \ldots, x_n]^T \in R^n$ *is a $n$ dimensional state. If the time-varying matrix $A(t) \in R^{n\times n}$ in (13) is a piecewise continuous function of time with bounded elements, $A(t)$ is a Metzler matrix*[2] *with zero row sums, and the time-varying graph corresponding to $A(t)$ is connected for all $t \geq 0$, then $\dot{V} < 0$ and $V(x)$ decreases exponentially fast.*

To establish the exponential convergence to the convex hull $Co(\mathbf{q}^L(t))$ for the fractional-order dynamics in (1), an integer-order system $\dot{q}_i(t) = u_i(t)$ with $\alpha = 1$ in (7) is considered first in the following theorem.

*Theorem 2: Consider a network composed of stationary leaders and dynamic followers described by (7). The followers $i \in V_F$ exponentially converge to the equilibrium points (i.e., a convex hull $Co(\mathbf{q}^L(t))$ spanned by the leaders states only), if there always exists at least one leader $j \in V_L$ that has a directed path to any follower $i$ (i.e., a directed tree is maintained).*

*Proof:* This theorem is proven by using LaSalle's invariance principle and convex properties. Let $V(\mathbf{q}(t))$ be the volume of the convex hull $Co(\mathbf{q}(t))$ formed by all leaders' and followers' states. First, we show that there exists a compact set $\Omega$ such that if $q_i(0) \in \Omega$ for $\forall i \in \mathcal{V}$, then $q_i(t) \in \Omega$ for all $t \geq 0$, which implies that $\Omega$ is a positively invariant set. Second, let $E$ be the set of all points in $\Omega$ where $\dot{V} = 0$ (i.e., the volume of $Co(\mathbf{q}(t))$ stays constant). It is then shown that $M$ is the largest invariant set, where $M$ is the set of points in the convex hull $Co(\mathbf{q}^L(t))$ formed by stationary leaders only.

[2]Metzler matrix is defined as a $n \times n$ matrix with positive or zero off-diagonal elements.

For $\alpha = 1$ in (7), substituting (11) into (7) yields the following closed-loop emotion dynamics

$$\begin{cases} \dot{q}_i(t) = -\sum_{j \in \mathcal{N}_i} K_i m_{ij} \left(q_i\left(t\right) - q_j\left(t\right)\right) & i \in \mathcal{V}_F \\ \dot{q}_j(t) = 0 & j \in \mathcal{V}_L \end{cases}, \quad (14)$$

which can be rewritten in a compact form of a time-varying linear system as

$$\dot{\mathbf{q}}\left(t\right) = \begin{bmatrix} \dot{\mathbf{q}}^F\left(t\right) \\ \dot{\mathbf{q}}^L\left(t\right) \end{bmatrix} = \begin{bmatrix} \pi\left(t\right) \\ \mathbf{0}_{(n-m)\times n} \end{bmatrix} \mathbf{q}\left(t\right), \quad (15)$$

where $\mathbf{0}_{(n-m)\times n}$ denotes the $(n-m) \times n$ matrix with all zeros, and the elements of $\pi\left(t\right) \in \mathbb{R}^{m\times n}$ are defined as

$$\pi_{ik}\left(t\right) = \begin{cases} -\sum_{j\in\mathcal{N}_i} K_i m_{ij} & i = k \\ K_i m_{ij} & j \in \mathcal{N}_i, i \neq k \\ 0, & j \notin \mathcal{N}_i, i \neq k. \end{cases} \quad (16)$$

Each follower $i \in \mathcal{V}_F$ in (14) evolves according to the dynamics:

$$\dot{q}_i(t) = -\sum_{j\in\mathcal{N}_i} \pi_{ij}\left(t\right)\left(q_i\left(t\right) - q_j\left(t\right)\right). \quad (17)$$

To facilitate the analysis, (17) can be written in discrete time as

$$\frac{q_i(t+1) - q_i(t)}{T} = -\left(\sum_{j\in\mathcal{N}_i} \pi_{ij}\right) q_i\left(t\right) + \sum_{j\in\mathcal{N}_i} \pi_{ij} q_j\left(t\right),$$

where $T$ is a sufficiently small sampling period. The follower state $q_i(t+1)$ at the next time instant can then be obtained as

$$q_i(t+1) = \left(1 - T\sum_{j\in\mathcal{N}_i} \pi_{ij}\right) q_i\left(t\right) + T\sum_{j\in\mathcal{N}_i} \pi_{ij} q_j\left(t\right). \quad (18)$$

From (18) and Definition 2, it is clear that $q_i(t+1)$ is a convex combination of its current state $q_i\left(t\right)$ and its neighbors' states $q_j\left(t\right)$, $j \in \mathcal{N}_i$, which implies that the follower $i$ moves towards the convex hull spanned by itself and its neighborhood set $\mathcal{N}_i$. Since the leaders' states are stationary and the followers' states are evolving within the convex hull, $V\left(\mathbf{q}\left(t\right)\right)$ is uniformly non-increasing and thus $V\left(\mathbf{q}\left(0\right)\right)$ will serve as the compact set $\Omega$.

The next step is to show that all followers' states will asymptotically converge to their equilibrium points. To see that the equilibrium points are indeed the stationary convex hull $Co\left(\mathbf{q}^L\left(t\right)\right)$, let $q_{i,eq}$ be an equilibrium for a follower $i \in \mathcal{V}_F$. For an equilibrium point, it must have $\dot{q}_{i,eq} = 0$, which yields that

$$q_{i,eq} = \frac{1}{-\pi_{ii}\left(t\right)} \sum_{j\in\mathcal{N}_i} \pi_{ij}\left(t\right) q_{j,eq}\left(t\right) \quad (19)$$

by using (17). Due to $-\pi_{ii}\left(t\right) = \sum_{j\in\mathcal{N}_i} \pi_{ij}\left(t\right) \in \mathbb{R}^+$ from (16), (19) indicates that the equilibrium point $q_{i,eq}$ lies in a convex hull spanned by its neighbors' states (i.e., leaders and/or followers). Since every follower ends up in a convex hull spanned by its neighbors' states and the leaders' states are static, every follower will end up in a convex hull spanned by the leaders states only (i.e., $Co\left(\mathbf{q}^L\left(t\right)\right)$). To show the asymptotic convergence to the equilibrium points for all followers, using the fact that $m_{ij}$ is non-negative from (12) and $K_i$ is a positive constant gain in (6), $\begin{bmatrix} \pi\left(t\right) \\ \mathbf{0}_{(n-m)\times n} \end{bmatrix}$ in (15) is a Metzler matrix with zero row sums. According to Lemma 1 and following a similar procedure in [24], it is determined that the convex hull $Co\left(\mathbf{q}\left(t\right)\right)$ is shrinking (i.e., $\dot{V}\left(\mathbf{q}\left(t\right)\right) < 0$), since the difference of the extremes $\max\left\{\, x_1, \ \cdots, \ x_n \,\right\}$ and $\min\left\{\, x_1, \ \cdots, \ x_n \,\right\}$ is decreasing. Indeed, if all followers states are initially within the convex hull $Co\left(\mathbf{q}^L\left(t\right)\right)$, the states will always stay within $Co\left(\mathbf{q}^L\left(t\right)\right)$ (i.e., $\dot{V} = 0$), since the convex combination in (18) for the states within a convex hull will still be a point within the same convex hull.

The last step is to show that $M$ (i.e., $Co\left(\mathbf{q}^L\left(t\right)\right)$) is the largest invariant set. To prove this by contradiction, let $M'$ be a larger invariant set in $E$, such as $M \subset M'$. Suppose that there is a follower whose state $q_i\left(0\right) \notin M$ and $q_i\left(0\right)$ is on the boundary of $M'$. Since $M' \subset E$, the volume of the set $M'$ stays constant. The only way for the volume of $M'$ to stay constant is that $q_i\left(0\right) = q_i\left(t\right)$ for all $t \geq 0$. However, for this to happen, we must have $\pi_{ij}\left(t\right) = 0$ for $\forall j \in \mathcal{N}_i$ from (17), which indicates that the follower $i$ is isolated from the group. This gives a contradiction with the network connectivity. Hence, $M$ is the largest invariant set. Also, note that for linear systems, uniform asymptotic stability is equivalent to exponential stability. Now it can be concluded that the followers exponentially converge to the largest invariant set $M$ (i.e., the equilibrium points $Co\left(\mathbf{q}^L\left(t\right)\right)$) by using LaSalle's invariance principle in [16]. ■

Since exponential stability for the integer-order system (14) is proven in Theorem 2, a similar proof procedure in our recent work [10] can be followed to show the asymptotic stability for the fractional order system (7) by using Mittag-Leffler stability analysis and the converse Lyapunov theorem from [25]. To be self-contained, the proof in [10] is also included.

*Lemma 2:* [18] *Let $x = 0$ be an equilibrium point for the system ${}_{t_0}^{C}D_t^{\alpha} x\left(t\right) = f\left(t, x\left(t\right)\right)$, where $f\left(t, x\left(t\right)\right)$ is a piecewise continuous, and $D \subset R^n$ be a domain containing the origin. Let $V\left(t, x\left(t\right)\right) : (0, \infty] \times D \to R$ be a continuously differentiable function and locally Lipschitz with respect to $x$ such that $k_1 \left\|x\right\|^a \leq V\left(t, x\right) \leq k_2 \left\|x\right\|^{ab}$ and ${}_{0}^{C}D_t^{\beta} V\left(t, x\right) \leq -k_3 \left\|x\right\|^{ab}$, where $x \in D$, $\beta \in (0, 1)$, $k_1$, $k_2$, $k_3$, $a$ and $b$ are arbitrary positive constants. Then $x = 0$ is Mittag-Leffler stable.*

*Theorem 3: The follower $i \in V_F$ with closed-loop fractional-order dynamics in (7) with $\alpha \in (0, 1)$ asymptotically converges to the convex hull spanned by stationary leaders if at least one leader $j \in V_L$ has a directed path to the follower $i$.*

*Proof:* Let $x_i\left(t\right) \triangleq q_i\left(t\right) - q_{i,eq}$, and $\mathbf{x}^F\left(t\right) \triangleq \mathbf{q}^F\left(t\right) - \mathbf{q}_{eq}^F$, where $\mathbf{q}_{eq}^F$ denotes the stacked vector of $q_{i,eq}$. Since the leaders states are constant, the closed-loop fractional-order dynamics in (7) for all followers can be written in a compact form as

$$ {}_0^C D_t^{\alpha} \mathbf{x}^F(t) = g(\mathbf{x}^F), \tag{20} $$

where $g(\mathbf{x}^F)$ is a function of follower states. Since the stability of a fractional-order system is given in Definition 1 and Mittag-Leffler stability implies asymptotic convergence as discussed in [18], the following development aims to show that (20) is Mittag-Leffler stable.

Since $\gamma_i$ and $\beta_i$ are not zero simultaneously, and $\gamma_i$, $\beta_i$ and their partial derivatives are bounded from (9) and (10), $g(\mathbf{x}^F)$ in (20) is bounded. Assuming that $g(\mathbf{x}^F)$ is bounded by a constant $l \in \mathbb{R}^+$, the Lipschitz condition for $g(\mathbf{x}^F)$ in (20) is

$$ \frac{\left\| g(\mathbf{x}^F) \right\|}{\left\| \mathbf{x}^F \right\|} \leq l. \tag{21} $$

Since Theorem 2 states an exponential convergence for the integer-order system of (18), the converse Lyapunov theorem, Theorem 4.9 in [25], is invoked to establish that there exists a function $V\left(t, \mathbf{x}^F\right) : (0, \infty] \times \mathbb{R}^m \rightarrow \mathbb{R}$ and strictly positive constants $k_1$, $k_2$, and $k_3$ such that

$$ k_1 \left\| \mathbf{x}^F \right\| \leq V\left(t, \mathbf{x}^F\right) \leq k_2 \left\| \mathbf{x}^F \right\|, \tag{22} $$

$$ \dot{V} \leq -k_3 \left\| \mathbf{x}^F \right\|. \tag{23} $$

Let $\beta = 1 - \alpha \in (0, 1)$. From Theorem 8 in [18], (21) and (23), the Caputo fractional derivative of $V$ is computed as

$$ {}_0^C D_t^{\beta} V\left(t, \mathbf{x}^F\right) = {}_0^C D_t^{-\alpha} \dot{V} \leq -\frac{k_3}{l} \left\| \mathbf{x}^F \right\|. \tag{24} $$

Mittag-Leffler stability of system (20) with $\alpha \in (0, 1)$ can be obtained as $\mathbf{x}^F(t) \leq \frac{V\left(0, \mathbf{x}^F(0)\right)}{k_1} E_{1-\alpha}\left(-\frac{k_3}{k_2 l} t^{1-\alpha}\right)$, by applying Lemma 2 to (22) and (24) with $a = b = 1$, which implies that the equilibrium points $\mathbf{q}_{eq}^F$ for the followers in the closed-loop fractional-order system in (20) are asymptotically stable. ■

## V. Conclusion

In this work, containment control is studied for a group of individuals in a social network with state-dependent time-varying influence. By modeling the group social response as a networked fractional-order system, a decentralized potential field-based influence algorithm is developed to ensure that all individuals' states achieve consensus asymptotically to a desired convex hull spanned by the stationary leaders' states, while maintaining consistent influence between individuals (i.e., network connectivity). Future work will extend the current work to dynamic leaders, where the social leaders can drive a group of individuals to a set of desired social agreements by dynamically changing their own states. Moreover, future effort will also consider different influence capabilities between individuals. The current development only examines a constant and common threshold for social difference for all individuals, without considering the different tolerance of social difference between various relationships. For instance, a person tends to have a larger tolerance about the difference on opinions for a certain social event in a close friend than some random person, and thus, can be more easily influenced by the close friend. Hence, future work is being considered to explore the relationship between individuals and the associated different levels of influence.

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