Supplement for "SFQRA: Scaled Factor-augmented Quantile Regression with Aggregation in Conditional Mean Forecasting"

The supplement consists of two parts. Section S.1 provides the details to determine the optimal weights in quantile aggregation. Section S.2 presents some additional numerical analysis.

S.1 Additions on weights in quantile aggregation

We would like to point out that weighing different quantiles appropriately is crucial to SFQRA. Let us look at an experiment. Table S.1 below records the optimal weights corresponding to seven different quantiles τ_j when using SFQRA or FQRA. We construct FQRA by aggregating information from traditional FQR estimates according to (10). Five different idiosyncratic errors ϵ_{t+h} are used. One may observe that the weights of SFQRA are sensitive to error distributions, while those of FQRA are less susceptible. Moreover, symmetry over quantiles is not even satisfied for SFQRA. We think that it is due to the scaling step in our SFQRA, which captures the target information as well as the idiosyncratic errors. As a consequence, it produces different factors under distinct error distributions. While the same factors independent of the errors are used in FQRA. In addition, for both FQRA and SFQRA, we observe that contrary to intuitive imagination, the weights of extreme quantiles may be weighted more heavily than a more central one. This also reflects the special aspect of using quantile aggregation for mean prediction, where the median, which is close to the mean, does not dominate the aggregation process.

S.2 Additional numerical analysis

Table S.1: Optimal weights corresponding to five different error distributions

Method	Distribution	$\tau_1 = 0.2$	$\tau_2 = 0.3$	$\tau_3 = 0.4$	$\tau_4 = 0.5$	$\tau_5 = 0.6$	$\tau_6 = 0.7$	$\tau_7 = 0.8$
FQRA	$\mathcal{N}(0,1)$	0.2631	0.1289	0.0768	0.0427	0.0844	0.1405	0.2636
	t_2	0.2545	0.1725	0.0861	0.0440	0.0731	0.1142	0.2557
	t_4	0.2566	0.1379	0.0602	0.0364	0.0875	0.1518	0.2696
	Mixture1	0.2687	0.1295	0.1067	0.0352	0.0653	0.1479	0.2467
	Mixture2	0.2564	0.1401	0.0973	0.0307	0.0529	0.1560	0.2665
SFQRA	$\mathcal{N}(0,1)$	7.5372	-0.3923	-1.4769	-12.4376	-0.6677	1.4873	6.9500
	t_2	0.8743	-1.5161	2.8147	-2.3719	0.8409	-1.4347	1.7928
	t_4	4.1411	-2.4581	2.5449	-6.1794	-1.5974	1.0977	3.4512
	Mixture1	4.8853	2.0254	-4.2280	-4.8868	-0.7511	-0.3161	4.2714
	Mixture2	4.4252	-3.1909	0.0769	-3.7294	0.7155	2.3153	0.3873

Notes. The data generation process is the same as in Section 4.1 (DGP1). Average optimal weights over 5000 repetitions are recorded.

S.2.1 Numerical analysis of the SFQR method

To better illustrate the performance of the SFQR method, we consider the following data generation process:

$$X_{it} = \boldsymbol{\lambda}_i^\top \boldsymbol{f}_t + e_{it} \text{ and } y_{t+1} = \eta_{t+1} g_t + \epsilon_{t+1},$$

where $\boldsymbol{\lambda}_i = (\lambda_{i1}, \lambda_{i2}, \lambda_{i3})^\top$, $\boldsymbol{f}_t = (g_t, h_{t1}, h_{t2})^\top$ and $g_t = 0.8 g_{t-1} + \xi_{t1}$, $h_{t1} = 0.5 h_{t-1,1} + \xi_{t2}$, $h_{t2} = 0.2 h_{t-1,2} + \xi_{t3}$. The variables ξ_{t1}, ξ_{t2}, ξ_{t3}, λ_{i1}, λ_{i2} and λ_{i3} are all independent draws from $\mathcal{N}(0,1)$, $\eta_{t+1} \sim \mathcal{U}[1,2]$, and the quantile-dependent idiosyncratic errors $\epsilon_{t+1} \sim i.i.d.\ t_v$ for $v = 2, 3, 4$, where t_v denotes the Student t-distribution with v degrees of freedom. In this setting, there are N observable covariates X_{it} $(i = 1, 2, \cdots, N)$, that are loaded on both the target-relevant factor g_t and target-irrelevant factors (h_{t1}, h_{t2}), which can be regarded as the strong factor case.

We focus on the mean quantile forecast error (MQFE), defined as follows:

$$\text{MQFE} = \frac{1}{T-h} \sum_{t=1}^{T-h} [\rho_\tau (y_{t+h} - \hat{\beta}_{\tau,0} - \hat{\boldsymbol{\beta}}_\tau^\top \boldsymbol{f}_t)],$$

under different quantile levels, forecasting with only one estimated factor. Tables S.2 and S.3 represent the in-sample and out-of-sample simulation results of SFQR and FQR methods for $N, T \in \{100, 200\}$ obtained from 5000 replications. Columns 3, 4, and 5 present the average values of MQFE under diverse idiosyncratic error types at the quantile $\tau = 0.2$. Meanwhile, columns 6, 7, and 8 provide corresponding results at the $\tau = 0.5$ level.

A cursory examination of these outcomes reveals that FQR's predictive performance is subpar when compared to the SFQR approach across a range of N and T values, as well as at various quantile levels τ. The lower MQFE of SFQR for both in-sample and out-of-sample

Table S.2: Simulation for the conditional quantile forecasting performance (in-sample)

	method	$\tau = 0.2$ d.f.= 2	$\tau = 0.2$ d.f.= 3	$\tau = 0.2$ d.f.= 4	$\tau = 0.5$ d.f.= 2	$\tau = 0.5$ d.f.= 3	$\tau = 0.5$ d.f.= 4
N = 100, T = 100	FQR	0.7911	0.6247	0.5779	0.7911	0.6247	0.5779
	SFQR	0.7443	0.5737	0.5252	0.7411	0.5704	0.5209
N = 100, T = 200	FQR	0.7741	0.6076	0.5558	0.7741	0.6076	0.5558
	SFQR	0.7423	0.5717	0.5182	0.7401	0.5692	0.5161
N = 200, T = 100	FQR	0.7680	0.6171	0.5593	0.7680	0.6171	0.5593
	SFQR	0.7298	0.5703	0.5146	0.7258	0.5669	0.5108
N = 200, T = 200	FQR	0.7518	0.5900	0.5382	0.7518	0.5900	0.5382
	SFQR	0.7288	0.5646	0.5123	0.7269	0.5621	0.5111

Table S.3: Simulation for the conditional quantile forecasting performance (out-of-sample)

	method	$\tau = 0.2$ d.f.= 2	$\tau = 0.2$ d.f.= 3	$\tau = 0.2$ d.f.= 4	$\tau = 0.5$ d.f.= 2	$\tau = 0.5$ d.f.= 3	$\tau = 0.5$ d.f.= 4
N = 100, T = 100	FQR	0.9329	0.7784	0.7161	0.9352	0.7756	0.7374
	SFQR	0.8939	0.7362	0.6671	0.8931	0.7310	0.6853
N = 100, T = 200	FQR	0.8849	0.7180	0.6864	0.8894	0.7236	0.6774
	SFQR	0.8599	0.6893	0.6553	0.8633	0.6915	0.6448
N = 200, T = 100	FQR	0.9399	0.7900	0.7274	0.9423	0.7797	0.7259
	SFQR	0.9092	0.7483	0.6900	0.9079	0.7336	0.6853
N = 200, T = 200	FQR	0.8831	0.7167	0.6908	0.8745	0.7147	0.6752
	SFQR	0.8661	0.6972	0.6706	0.8559	0.6937	0.6544

data indicates its effectiveness. This suggests that the factor estimates derived using this method perform admirably in both model fitting and prediction. Interestingly, an increase in T at the same quantile level τ is associated with a decrease in MQFE. This observation is intuitive as larger sample sizes offer more robust parameter estimates, thereby increasing their accuracy. In parallel, an increase in N leads to a smaller in-sample MQFE but a larger out-of-sample MQFE. This is because larger sample sizes introduce higher dimensionality when using the estimated factors for quantile forecasting, thereby including additional noise during the factor extraction process. In summary, the various facets of this numerical analysis strongly demonstrate that our scale-based methodology yields state-of-the-art forecasting performance.

S.2.2 Additional numerical results of Section 4.2

In Section 4.2 of the main document, we only show the out-of-sample performance results for the eight methods, and here in Table S.4 we supplement the in-sample fitting results. The performance of the DGP2 case is similar to that of the DGP1 case. While OLS is prone to over-fitting, the MSEs of the other seven methods are significantly better. Among the remaining seven methods, FR and FQRA exhibit the poorest in-sample performance, due to their inability to incorporate any information about the target variable in the factor capturing process. Overall, the in-sample performance of SFQRA and t-SFQRA was also consistently better than that of the other methods, with t-SFQRA performing slightly better than SFQRA.

S.2.3 Numerical analysis of weak relevant factors

Different from DGP2 in Section 4.2, in this subsection, we consider a weak relevant factors framework, where each factor is correlated with only a subset of covariates, and only the relevant factors are embedded in the covariates. Specifically, the data generation process (DGP3) is:

$$\mathbf{X}_t = \mathbf{\Lambda} \boldsymbol{f}_t + \boldsymbol{e}_t \text{ and } y_{t+1} = f_{t1} + \epsilon_{t+1},$$

where $\mathbf{\Lambda} = (\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_N)^\top$, $\boldsymbol{\lambda}_i = (\lambda_{i1}, \lambda_{i2})^\top$ and $\boldsymbol{f}_t = (f_{t1}, f_{t2})^\top$. The matrix $\mathbf{\Lambda}$ has only the first $d_N = \lceil T/\log(T) \rceil$ nonzero rows and satisfies that $\lambda_{i1}, \lambda_{i2} \sim i.i.d.\ \mathcal{U}(-1, 1)$ for $i = 1, 2, \cdots, d_N$. In addition, $\boldsymbol{e}_t \sim i.i.d.\ \mathcal{N}(\mathbf{0}, \mathbf{I}_N)$, $f_{t1}, f_{t2} \sim i.i.d.\ \mathcal{N}(0, 1)$ and the same six types of idiosyncratic errors ϵ_{t+1} as in DGP1 are considered. In DGP3, there are d_N observable covariates X_{it} $(i = 1, \cdots, d_N)$ loaded on the factor $\boldsymbol{f}_t$, and the remaining $(N - d_N)$ observable covariates X_{it} $(i = d_N + 1, d_N + 2, \cdots, N)$ contain only the error term e_{it}. This can be considered as the weak factor case. We are still interested in both the in-sample and out-of-sample average MSEs for the eight methods.

The simulated results are reported in Tables S.5 and S.6. The phenomena they exhibit are typically similar to those discussed in Section 4.2. Comparing DGP2 with DGP3, one may notice that the relationship between y_{t+1} and the factor f_{t1} in DGP3 remains unchanged at each quantile level, while the one in DGP2 is different at each quantile level as η_t is a random variable instead of a constant. Thus it makes intuitive sense to assign a more complex forecasting model such as the quantile aggregation method to DGP2 and a general linear regression method to DGP3. However, our simulated results indicate that SFQRA and t-SFQRA not only achieve the most outstanding forecasting performance in DGP2, but also outperforms linear regression methods in DGP3, which implies that SFQRA would be an efficient forecasting method in general cases, no matter how the data generation progress would be.

S.2.4 Additional real data results

In Figure S.1, we incorporate the t-SFQRA approach of the four different targets 'GDPC1', 'INDPRO', 'HWIURATIOx' and 'DFSARG3Q086SBEA' for the FRED-QD dataset when the number of factors is changed. It can be seen that the prediction performance of t-SFQRA and SFQRA is relatively close and performs better than the other three methods. It is noteworthy that t-SFQRA uses only some of the macroeconomic variables that have a large aggregated quantile correlation with the target, fewer but more efficient covariates, which is more explanatory than SFQRA. This is because, from a practical point of view, we have good reasons to believe that not all variables are suitable to be put together to estimate the factors.

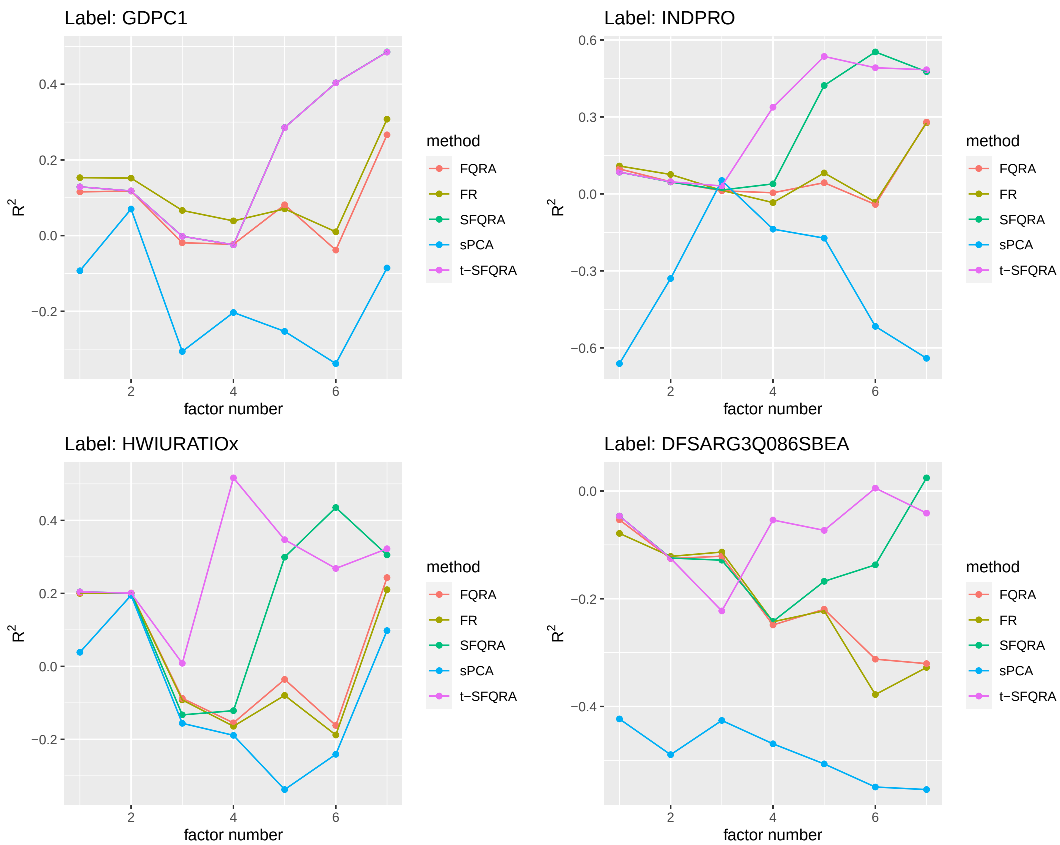

Figure S.1: Out-of-sample forecasting performance with different factor numbers for FRED-QD database. (In the top left plot, the two curves representing methods SFQRA and t-SFQRA almost overlap.)

Furthermore, based on Figure 4 in the main document, we speculate that SFQRA is an appropriate choice for analyzing the FRED-QD dataset, and the t-SFQRA method would be more suitable for the FRED-MD dataset. To illustrate, we present in Figure S.2 the magnitude of the aggregated quantile correlations of 'DPCERA3M086SBEA' and 'HWI'[1]

[1]'DPCERA3M086SBEA' denotes Real Personal Consumption Expenditures; 'HWI' denotes Help-Wanted

with the remaining 119 variables in the FRED-MD dataset. The upper panel shows the full values, while the lower panel is truncated and hides those values that are less than 0.11. It can be found that they all have a high correlation mainly with the 'Consumption, Orders, and Inventories', 'Output and Income' and 'Labor Market' categories of macroeconomic indicators.

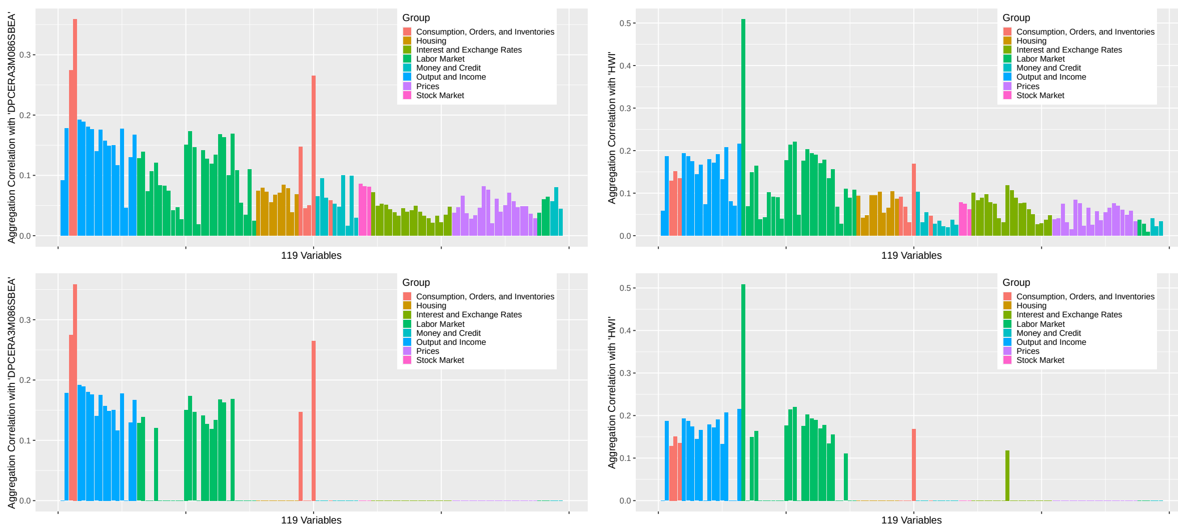

Figure S.2: Aggregated Quantile Correlation for FRED-MD database. Upper panel: all values; lower panel: truncated with small values hidden.

Finally, a selection of four variables from the block, which showcases a notably high correlation at the center of Figure 4's left panel in main document, has been further evaluated. The out-of-sample performance of these variables is demonstrated in Figure S.3. Upon reviewing these results, we find a striking similarity with our previous findings, thereby strengthening the recommendation for method t-SFQRA. Therefore, before adopting methods SFQRA or t-SFQRA, we may perform a simple analysis of the data to decide which method to use.

Index for United States.

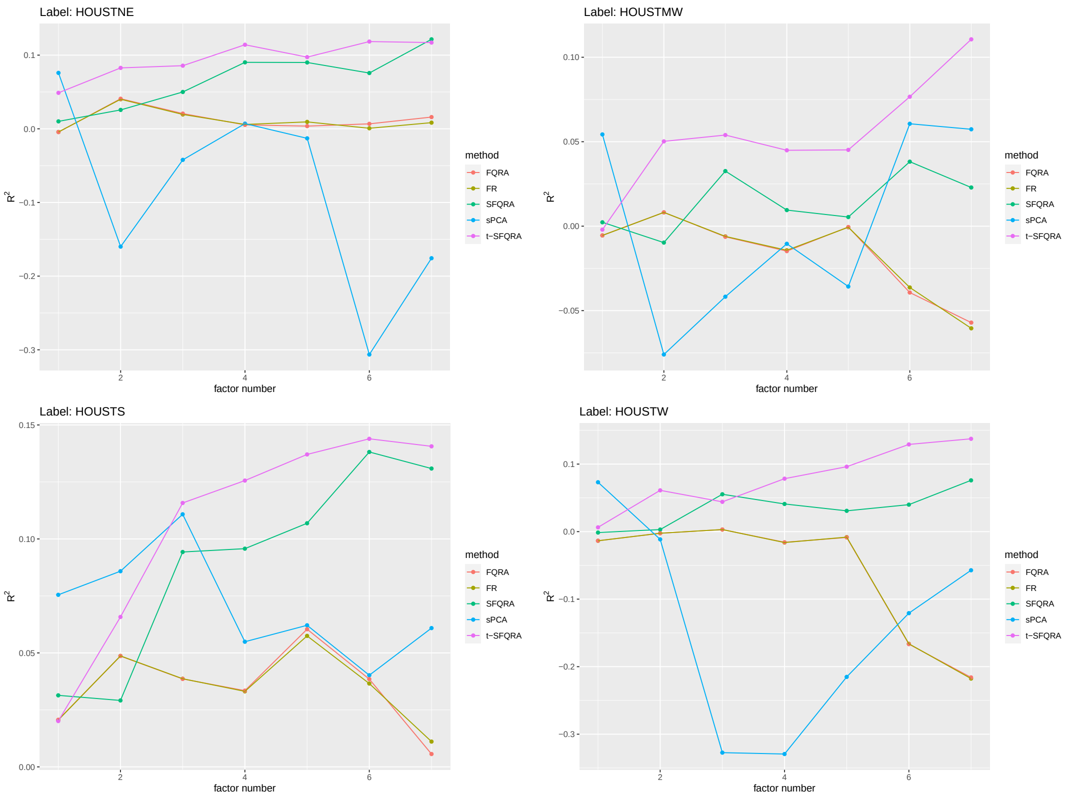

Figure S.3: Out-of-sample forecasting performance of the other four targets on housing starts with different factor numbers for FRED-MD database. (In the bottom right plot, the two curves representing methods FQRA and FR almost overlap.)

Table S.4: Simulation for the fitting performance in DGP2 framework (in-sample)

	distribution	$\mathcal{N}(0,1)$	t_3	t_4	t_5	Mixture1	Mixture2
	OLS	0	0	0	0	0	0
	FR	3.5920	5.6651	4.6984	4.2718	4.9122	3.1230
	sPCA	1.5570	3.5404	2.5597	2.1859	2.7872	1.0615
N = 100	t-PCA	3.7376	3.5295	2.7164	3.0843	3.7931	1.1590
T = 100	FQRA	3.5920	5.6651	4.6984	4.2718	4.9121	3.1230
	t-qcPCA	1.5441	3.5302	2.5450	2.1696	2.7715	1.0464
	SFQRA	1.0871	3.0501	2.0742	1.7039	2.3014	0.6055
	t-SFQRA	1.0855	3.0521	2.0711	1.7029	2.2986	0.6035
	OLS	0.5622	1.5513	1.0621	0.8851	1.1937	0.3085
	FR	2.4049	4.4470	3.4566	3.0767	3.7765	1.9703
	sPCA	1.5997	3.5807	2.6190	2.2456	2.8732	1.0971
N = 100	t-PCA	1.5759	4.7175	4.8576	4.4675	2.8503	1.0751
T = 200	FQRA	2.4049	4.4466	3.4565	3.0766	3.7764	1.9703
	t-qcPCA	1.5752	3.5577	2.5941	2.0809	2.8498	1.0739
	SFQRA	1.1239	3.0796	2.1234	1.7639	2.3761	0.6254
	t-SFQRA	1.1206	3.0805	2.1230	1.6203	2.3755	0.6209
	OLS	0	0	0	0	0	0
	FR	4.4513	6.6789	5.4839	5.0945	5.7094	4.1164
	sPCA	1.5589	3.6769	2.5885	2.2371	2.8379	1.0708
N = 200	t-PCA	2.4327	6.8151	4.1629	4.7377	2.7918	1.2067
T = 100	FQRA	4.4511	6.6789	5.4838	5.0945	5.7094	4.1162
	t-qcPCA	1.5242	3.6154	2.5064	2.1932	2.7900	1.0413
	SFQRA	1.0825	3.1445	2.0434	1.7333	2.3213	0.6029
	t-SFQRA	1.0748	3.1376	2.0299	1.7245	2.3097	0.5956
	OLS	0	0	0	0	0	0
	FR	4.2054	6.1543	5.1728	4.9363	5.5039	3.7714
	sPCA	1.5903	3.5089	2.5533	2.2511	2.8684	1.0958
N = 200	t-PCA	1.5675	3.8002	3.3112	2.2291	2.8469	1.4406
T = 200	FQRA	4.2053	6.1543	5.1728	4.9361	5.5039	3.7712
	t-qcPCA	1.5673	3.4844	2.5277	2.2288	2.8447	1.0720
	SFQRA	1.1102	2.9988	2.0735	1.7721	2.3747	0.6303
	t-SFQRA	1.1070	3.0031	2.0719	1.7705	2.3726	0.6236

Table S.5: Simulation for the fitting performance in the weak relevant factor framework (in-sample)

	distribution	$\mathcal{N}(0,1)$	t_3	t_4	t_5	Mixture1	Mixture2
	OLS	0	0	0	0	0	0
	FR	1.4426	3.3286	2.3949	2.0561	2.7168	0.9515
	sPCA	1.0118	2.8657	1.9417	1.6237	2.2564	0.5298
N = 100	t-PCA	1.5937	3.2107	2.0034	1.8615	2.4432	0.7809
T = 100	FQRA	1.4426	3.3285	2.3948	2.0561	2.7167	0.9515
	t-qcPCA	1.0102	2.8639	1.9351	1.6136	2.2474	0.5155
	SFQRA	0.9507	2.7692	1.8649	1.5489	2.1649	0.4748
	t-SFQRA	0.9456	2.7633	1.8612	1.5463	2.1620	0.4718
	OLS	0.4970	1.5177	0.9816	0.8249	1.1250	0.2489
	FR	1.4689	3.4973	2.4443	2.1239	2.7289	0.9821
	sPCA	1.0160	3.0173	1.9741	1.6658	2.2732	0.5210
N = 100	t-PCA	1.1515	3.5109	2.6673	2.2388	2.3092	0.5314
T = 200	FQRA	1.4688	3.4973	2.4442	2.1238	2.7288	0.9820
	t-qcPCA	0.9948	3.0002	1.9619	1.6557	2.2589	0.5106
	SFQRA	0.9802	2.9669	1.9304	1.6229	2.2203	0.4907
	t-SFQRA	0.9778	2.9591	1.9240	1.6196	2.2179	0.4865
	OLS	0	0	0	0	0	0
	FR	1.4519	3.3291	2.4480	2.1182	2.6901	0.9661
	sPCA	1.0093	2.8585	1.9788	1.6729	2.2284	0.5319
N = 200	t-PCA	1.2765	3.3383	2.2276	1.8743	2.3875	0.6682
T = 100	FQRA	1.4519	3.3290	2.4480	2.1180	2.6900	0.9661
	t-qcPCA	0.9912	2.8471	1.9663	1.6599	2.2134	0.5194
	SFQRA	0.9529	2.7795	1.9084	1.6117	2.1465	0.4784
	t-SFQRA	0.9496	2.7744	1.8989	1.6073	2.1402	0.4773
	OLS	0	0	0	0	0	0
	FR	1.4724	3.6746	2.4816	2.1295	2.7146	0.9768
	sPCA	1.0158	3.1921	2.0136	1.6771	2.2571	0.5201
N = 200	t-PCA	1.1227	3.2199	2.1324	1.8743	2.3577	0.6614
T = 200	FQRA	1.4723	3.6745	2.4816	2.1294	2.7145	0.9768
	t-qcPCA	1.0095	3.1787	2.0003	1.6612	2.2431	0.5006
	SFQRA	0.9808	3.1486	1.9729	1.6344	2.2089	0.4899
	t-SFQRA	0.9777	3.1453	1.9687	1.6298	2.2023	0.4845

Table S.6: Simulation for the forecasting performance in the weak relevant factor framework (out-of-sample)

	distribution	$\mathcal{N}(0,1)$	t_3	t_4	t_5	Mixture1	Mixture2
N = 100 T = 100	OLS	1.4×10^5	1.3×10^5	1.3×10^6	3.2×10^4	4.7×10^5	5.1×10^5
	FR	1.5453	3.5834	2.5584	2.2096	2.8671	1.0233
	sPCA	1.1076	3.1798	2.1204	1.7897	2.4354	0.5849
	t-PCA	1.1084	3.1771	2.1117	1.7756	2.4213	0.5708
	FQRA	1.5452	3.5834	2.5583	2.2095	2.8670	1.0232
	t-qcPCA	1.1123	3.2198	2.1045	1.7961	2.5120	0.5844
	SFQRA	1.0767	3.1908	2.1062	1.7743	2.4314	0.5438
	t-SFQRA	1.0721	3.1910	2.1009	1.7723	2.4287	0.5401
N = 100 T = 200	OLS	2.0323	6.1163	4.0312	3.4102	4.6320	1.0135
	FR	1.5291	3.5484	2.5212	2.1908	2.8007	1.0244
	sPCA	1.0640	3.0939	2.0640	1.7358	2.3604	0.5515
	t-PCA	1.0598	3.0882	2.0529	1.7277	2.3530	0.5396
	FQRA	1.5290	3.5483	2.5212	2.1907	2.8006	1.0243
	t-qcPCA	1.0652	3.0928	2.0627	1.7366	2.3619	0.5512
	SFQRA	1.0361	3.0864	2.0466	1.7131	2.3502	0.5252
	t-SFQRA	1.0322	3.0869	2.0433	1.7095	2.3486	0.5231
N = 200 T = 100	OLS	3.0×10^4	2.4×10^6	2.1×10^5	1.2×10^5	2.9×10^7	1.4×10^4
	FR	1.5450	3.6851	2.5755	2.2109	2.8411	1.0357
	sPCA	1.1001	3.2884	2.1420	1.7912	2.4229	0.5899
	t-PCA	1.0954	3.2912	2.1376	1.7884	2.4231	0.5651
	FQRA	1.5449	3.6850	2.5755	2.2108	2.8411	1.0356
	t-qcPCA	1.0945	3.2842	2.1397	1.7825	2.4233	0.5801
	SFQRA	1.0688	3.2923	2.1254	1.7732	2.4227	0.5494
	t-SFQRA	1.0655	3.2930	2.1202	1.7698	2.4186	0.5433
N = 200 T = 200	OLS	1.3×10^7	5.7×10^5	1.1×10^5	3.1×10^5	5.4×10^5	3.5×10^4
	FR	1.5165	3.7482	2.5834	2.1904	2.8070	1.0150
	sPCA	1.0593	3.3083	2.1095	1.7351	2.3727	0.5495
	t-PCA	1.0597	3.3109	2.1032	1.7293	2.3686	0.5443
	FQRA	1.5164	3.7482	2.5834	2.1903	2.8069	1.0150
	t-qcPCA	1.0508	3.3006	2.1011	1.7234	2.3664	0.5417
	SFQRA	1.0353	3.2931	2.0918	1.7103	2.3593	0.5226
	t-SFQRA	1.0311	3.2929	2.0883	1.7077	2.3564	0.5201